Münster J. of Math. **13** (2020), 317–352
DOI 10.17879/90169661145
urn:nbn:de:hbz:6-90169661585

**Münster Journal of Mathematics**


# Diagonal classes and the Bloch–Kato conjecture

Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci

(Communicated by Guido Kings)

*Dedicated to Christopher Deninger on the occasion of his 60th birthday*

**Abstract.** The aim of this note is twofold. Firstly, we prove an explicit reciprocity law for certain diagonal classes in the étale cohomology of the triple product of a modular curve, stated in [8] and used there as a crucial ingredient in the proof of the main results. Secondly, we apply the aforementioned reciprocity law to address the rank-zero case of the equivariant Bloch–Kato conjecture for the self-dual motive of an elliptic newform of weight $k \geqslant 2$. In the special case $k = 2$, our result gives a self-contained and simpler proof of the main result of [15].

## 1. Introduction

Let $p \geqslant 5$ be a rational prime and let $N \geqslant 1$ be an integer. Fix algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$, respectively, embeddings $i_\infty\colon \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p\colon \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and a finite extension $L$ of $\mathbf{Q}_p(\mu_N)$. For each positive integers $n$ and $u$, denote by $M_u(n,\chi)_L$ the space of complex modular forms of weight $u$, level $\Gamma_1(n)$, character $\chi\colon (\mathbf{Z}/n\mathbf{Z})^* \to L^*$ and Fourier coefficients in $\bar{\mathbf{Q}} \cap L$, and by $S_u(n,\chi)_L$ the subspace of cuspidal modular forms.

In the rest of the introduction, assume that $p \nmid N$ and consider three (nonzero) cusp forms

$$f \in S_k(N,\chi_f)_L, \quad g \in S_l(N,\chi_g)_L \quad \text{and} \quad h \in S_m(N,\chi_h)_L$$

of weights $k \geqslant 2$, $l \geqslant 1$ and $m \geqslant 1$, respectively, which are eigenvectors for the Hecke operator $T_\ell$ for each prime $\ell$ which does not divide $N$, and satisfy the *self-duality condition*

$$\chi_f \cdot \chi_g \cdot \chi_h = 1. \tag{1}$$

Denote by $\mathtt{D}(f)$ the Deligne $p$-adic representation of (the primitive form associated with) $f$, and by $V(f)$ the tensor product of $\mathtt{D}(f)$ with the $f$-isotypic component of $S_k(N,\chi_f)_L$. If $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, the $L[G_{\mathbf{Q}}]$-module $V(f)$ is then (non-canonically) isomorphic to the direct sum of a finite number of copies of $\mathtt{D}(f)$. If $\xi$ denotes either $g$ or $h$, define similarly $V(\xi)$, after replacing $\mathtt{D}(\xi)$ with the Deligne–Serre representation $\mathtt{DS}(\xi)$ if the weight of $\xi$ is equal to one. Equation (1) implies that $k+l+m$ is even and that the $G_{\mathbf{Q}}$-representation

$$V(f,g,h)=V(f)\otimes_L V(g)\otimes_L V(h)\otimes_{\mathbf{Z}_p}\mathbf{Z}_p\big((k+l+m-2)/2\big)$$

is Kummer self-dual, viz. it is isomorphic to its $L$-linear dual representation twisted by $\mathbf{Z}_p(1)$.

1.1. **The geometric and balanced case.** Assume in this section that the triple $(k,l,m)$ is *geometric* and *balanced*, that is, $l\geqslant 2, m\geqslant 2$ and $k,l$ and $m$ are the lengths of the sides of a triangle. In this setting [8] associates to $(f,g,h)$ a *diagonal class* $\kappa(f,g,h)$ in the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ of the $G_{\mathbf{Q}}$-representation $V(f,g,h)$. (Its construction is recalled in Section 2.) The first aim of this note is to prove Theorem A below, a generalization of the *explicit reciprocity law* for $\kappa(f,g,h)$ stated as Proposition 3.5 in [8] and used as a crucial ingredient in the proof of the main results of [8] and [9].

We first introduce the relevant notations. Assume that $p$ does not divide $N$, and denote by $\xi$ one of $f,g$ and $h$. Let $\alpha_\xi$ and $\beta_\xi$ be the roots of the Hecke polynomial $h_{p,\xi}(X)=X^2-\lambda_p(\xi)\cdot X+\chi_\xi(p)p^{u-1}$, where $T_p\xi=\lambda_p(\xi)\cdot\xi$ and $u$ is the weight of $\xi$. Enlarging $L$ if necessary, assume it contains $\mathbf{Q}_p(\alpha_\xi,\beta_\xi,\mu_N)$. Assume in the rest of the paper that

$$\alpha_\xi\neq\beta_\xi.$$

Assume moreover that $\mathrm{ord}_p(\alpha_\xi)<k-1$. Denote by $V_{\mathrm{dR}}(f,g,h)$ the filtered $L$-module $D_{\mathrm{dR}}(V(f,g,h))$ associated by Fontaine to $V(f,g,h)$. The Faltings comparison isomorphism and (a suitably twisted) Poincaré duality identify the Bloch–Kato $p$-adic logarithm of (the restriction at $p$ of) $\kappa(f,g,h)$ with a linear functional

$$\log_p(\kappa(f,g,h))\colon \mathrm{Fil}^0V_{\mathrm{dR}}(f,g,h)\to L$$

(cp. Section 3.1.2). The $L$-module $\mathrm{Fil}^0V_{\mathrm{dR}}(f,g,h)$ has dimension four, and contains a distinguished class

$$\eta_f^\alpha\otimes\omega_g\otimes\omega_h\in\mathrm{Fil}^0V_{\mathrm{dR}}(f,g,h).$$

Here $\omega_\xi$ is de Rham class in $V_{\mathrm{dR}}(\xi)=D_{\mathrm{cris}}(V(\xi))$ corresponding to $\xi$ under the Faltings comparison isomorphism and $\eta_f^\alpha$ is a natural element in $V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$ associated with $f$, where $\varphi$ is the crystalline Frobenius. (We refer to Section 3.1.3 for precise definitions.) The explicit reciprocity law relates the value of $\log_p(\kappa(f,g,h))$ at $\eta_f^\alpha\otimes\omega_g\otimes\omega_h$ to a *$p$-adic period* $I_p(f,g,h)$ which we now define.

Let $f^w=w_Nf$ in $M_k(N,\bar{\chi}_f)_L$ be the image of $f$ under the Atkin–Lehner operator $w=w_N$. One has $T_pf^w=\bar{\chi}(p)\lambda_p(f)\cdot f^w$, so that $\bar{\chi}(p)\cdot\alpha_f$ and $\bar{\chi}(p)\cdot\beta_f$ are the roots of the $p$-th Hecke polynomial $h_{p,f^w}(X)$. Define

$$f_\alpha^w\in S_k(Np,\bar{\chi}_f)_L \tag{2}$$

to be the $p$-stabilizations of $f^w$ satisfying $U_p f^w_\alpha = \bar\chi_f(p)\alpha_f \cdot f^w_\alpha$. Regard $g$ and $h$ as $p$-adic modular forms and let

$$\Xi_k(g,h) = d^{(k-l-m)/2} g^{[p]} \times h,$$

where $g^{[p]}$ and $d^{(k-l-m)/2} g^{[p]}$ are defined as follows. If $g$ has $q$-expansion $\sum_{n\geqslant 0} a_n(g)\cdot q^n$, then its *$p$-depletion* $g^{[p]}$ is the weight-$l$ $p$-adic modular form with $q$-expansion $\sum_{n\nmid p} a_n(g)\cdot q^n$ (cp. Equation (15)). Let $d = q\frac{d}{dq}$ be Serre's derivative operator on $L[\![q]\!]$, which sends (the $q$-expansion of) a $p$-adic modular form of weight $u$ to a $p$-adic modular form of weight $u+2$. For each integer $n$ (not necessarily positive), the sequence of $p$-adic modular forms $d^{n+(p-1)p^m} g^{[p]}$, then converges, for $m \to \infty$, to a $p$-adic modular form $d^n g^{[p]}$ of weight $l+2n$. It follows that $\Xi_k(g,h)$ defines a $p$-adic modular form of weight $k$. As proved in Section 4.7 (see in particular Equation (46)) the form $\Xi_k(g,h)$ belongs to the space $M_k^{\text{n-o}}(N,L)$ of nearly-overconvergent forms of weight $k$ defined over $L$ (cp. Section 3.3 or [41, 14]). Under the additional assumption $\mathrm{ord}_p(\alpha_f) < k-1$, the work of Coleman defines a natural *$f^w_\alpha$-isotypic projection*

$$e_{f^w_\alpha} \colon M_k^{\text{n-o}}(N,L) \to S_k(Np,L)_{f^w_\alpha},$$

where $S_k(N,L)_{f^w_\alpha}$ is the $f^w_\alpha$-isotypic component of $S_k(Np,\chi_f)_L$ (cp. Section 3.3). In this case define

$$I_p(f,g,h) = \frac{(f^w_\alpha, e_{f^w_\alpha}\cdot \Xi_k(g,h))_{Np}}{(f^w_\alpha, f^w_\alpha)_{Np}},$$

where $(\zeta,\xi)_M = \int_{Y_1(M)} \zeta(z)\bar\xi(z) y^{u-2}\,dx\,dy$ is the Petersson scalar product on $S_u(M,\mathbf{C})$.[1] It is easily seen that the $p$-adic period $I_p(f,g,h)$ is algebraic and belongs to $L$.

**Theorem A.** *Assume that $p\nmid N$ and that $\mathrm{ord}_p(\alpha_f) < k-1$. Then*

$$\log_p(\kappa(f,g,h))(\eta_f^\alpha \otimes \omega_g \otimes \omega_h)$$

*is equal to*

$$\frac{(-1)^k N^{c-2}(c-k)!(1-\frac{\beta_f}{\alpha_f})(1-\frac{\beta_f}{p\alpha_f})}{(1-\frac{\beta_f\alpha_g\alpha_h}{p^c})(1-\frac{\beta_f\alpha_g\beta_h}{p^c})(1-\frac{\beta_f\beta_g\alpha_h}{p^c})(1-\frac{\beta_f\beta_g\beta_h}{p^c})}\cdot I_p(f,g,h),$$

*where $c = c(k,l,m)$ denotes the positive integer $(k+l+m-2)/2$.*

The proof of Theorem A is given in Section 4. It uses the work of Bannai, Bannai–Kings, Besser, Nekovář, Nizioł [1, 2, 10, 11, 31, 35, 36, 33] in an essential way. See also [5, 4, 14, 6, 7, 26] for related results.

[1] If $f^w_\alpha$ is ordinary (i.e., $\mathrm{ord}_p(\alpha_f)=0$), there is no need to prove that $\Xi_k(g,h)$ is nearly-overconvergent in order to define $e_{f^w_\alpha}\cdot\Xi_k(g,h)$ and $I_p(f,g,h)$. In this case Hida [22] defines an ordinary projector $e_{\text{ord}}$ from the space $\mathbf{M}_k(N,L)$ of weight-$k$ $p$-adic modular forms over $L$ to the space $M_k^{\text{ord}}(Np,L)$ of classical $p$-ordinary modular forms. The composition of $e_{\text{ord}}$ with the natural projection $M_k^{\text{ord}}(Np,L)\to S_k(Np,L)_{f^w_\alpha}$ onto the $f^w_\alpha$-isotypic component is an extension of the Coleman morphism $e_{f^w_\alpha}$ to $\mathbf{M}_k(N,L)$.

1.2. **Applications to the Bloch–Kato conjecture.** Throughout this section, $(f, g, h)$ is a triple of *newforms* of weights $(k, l, m) = (k, 1, 1)$ and conductors $(N_f, N_g, N_h)$. The following assumption is in force.

**Assumption 1.3.**

1. The product of $\chi_f, \chi_g$ and $\chi_h$ is the trivial character.
2. $p$ does not divide $N_f \cdot N_g \cdot N_h$ and $(N_f, N_g, N_h) = 1$.
3. For $\xi = g, h$ the $p$-th Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$ is separable.
4. $f$ is $p$-ordinary (that is its $p$-th Fourier coefficient is a $p$-adic unit).

Let

$$\mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \hookrightarrow H^1(\mathbf{Q}, V(f, g, h))$$

be the Bloch–Kato Selmer group of the $G_{\mathbf{Q}}$-representation $V(f, g, h)$ and let

$$H^1_{\mathrm{str}}(\mathbf{Q}, V(f, g, h)) = \ker\bigl(\mathrm{res}_p \colon \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \to H^1(\mathbf{Q}_p, V(f, g, h))\bigr)$$

be its *strict* Selmer subgroup. Write $L(f \otimes g \otimes h, s)$ for the complex $L$-series of the tensor product of the motives of $f, g$ and $h$. Under Assumptions 1.3.1 and 1.3.2, it admits an analytic continuation and satisfies a functional equation with sign $+1$ at the central critical point $s = k/2$. The following theorem (proved in Section 5) is the main result of this note.

**Theorem B.** *If* $L(f \otimes g \otimes h, s)$ *does not vanish at* $s = k/2$, *then the Selmer group* $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ *is equal to the strict Selmer group* $H^1_{\mathrm{str}}(\mathbf{Q}, V(f, g, h))$.

The Bloch–Kato conjecture predicts that the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ is trivial if (and only if) the $L$-series $L(f \otimes g \otimes h, s)$ does not vanish at the central critical point $s = k/2$. As explained below, the methods of this paper fall short of proving this conjecture. Nonetheless, the previous result provides strong evidence in support of it.

When $k = 2$, Theorem B gives a significantly simpler proof of the main result proved by Darmon and Rotger in [15] (cp. Section 1.3.1 below) and has important applications to the equivariant Birch and Swinnerton-Dyer conjecture. Let $A$ be an elliptic curve defined over the rationals and let $L = L_\varrho$ be the splitting field of the tensor product $\varrho = \varrho_1 \otimes \varrho_2$ of two irreducible, odd Artin representations satisfying $\det(\varrho_1) = \det(\varrho_2)^{-1}$. Then Theorem B and the Serre modularity conjecture prove that the non-vanishing of the $L$-series $L(A, \varrho, s)$ at $s = 1$ implies the triviality of the $\varrho$-isotypic component $A(L)^\varrho = (A(L) \otimes_{\mathbf{Z}} V_\varrho)^{\mathrm{Gal}(L/\mathbf{Q})}$ of the Mordell–Weil group of $A$ over $L$. Indeed, $L(A, \varrho, s) = L(f \otimes g \otimes h, s)$, where $f, g$ and $h$ are the cusp forms associated with $A$, $\varrho_1$ and $\varrho_2$ by modularity, and a non-torsion element of $A(L)^\varrho$ gives rise, via the $p$-adic Kummer map, to a class in $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ with nontrivial restriction at $p$, id est not in $H^1_{\mathrm{str}}(\mathbf{Q}, V(f, g, h))$. One can then apply Theorem B with any (carefully choosen) prime $p$ for which Assumption 1.3 is satisfied.

More generally, let $f$ be a newform of weight $k \geqslant 2$ and let $\varrho = \varrho_1 \otimes \varrho_2$ be as above. The representation $V(f)$ can be realized in the middle cohomology $\mathscr{V}_k = H^{k-1}_{\text{ét}}(\mathcal{E}^{k-2} \otimes_{\mathbf{Q}} \bar{\mathbf{Q}}, \mathbf{Q}_p)$ of the $i$-fold fibre product $\mathcal{E}^i = \mathcal{E}_1(N)^i$ of the universal elliptic curve $\mathcal{E}_1(N) \to Y_1(N)$ over the open modular curve of level

$\Gamma_1(N)$ over $\mathbf{Q}$. The $p$-adic Abel–Jacobi map and the $f$-isotypic projection $\mathscr{V}_k \to V(f)$ gives a morphism

$$r_p\colon \mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0 \to \mathrm{Sel}(L, V_f),$$

where $\mathcal{E}_L^i = \mathcal{E}^i \otimes_{\mathbf{Q}} L$, the $\mathrm{Gal}(L/\mathbf{Q})$-module $\mathrm{CH}^i(\,\cdot\,)_0$ is the Chow group of homologically trivial codimension $i$ cycles on $\cdot$ modulo rational equivalence and $V_f$ denotes the $k/2$-th Tate twist of $V(f)$. If (Assumption 1.3 is satisfied and) $L(f,\varrho,s) = L(f\otimes g\otimes h, s)$ does not vanish at $s = k/2$, Theorem B proves that $r_p$ maps the $\varrho$-component $\mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0^{\varrho} = H^0(\mathrm{Gal}(L/\mathbf{Q}), \mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0 \otimes_{\mathbf{Z}} V_\varrho)$ to the restricted Selmer group $H^1_{\mathrm{str}}(\mathbf{Q}, V(f,g,h))$. In contrast with the weight two case, when $k > 2$, this is far from proving the (conjectural) vanishing of the $f$-isotypic component of $\mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0^{\varrho}$, as the injectivity of the Abel–Jacobi maps is arguably the deepest aspect of the Beilinson–Bloch–Kato conjectures. Despite this, Theorem B still provides strong evidence in support of the Bloch–Kato conjecture for the $\varrho$-twist of the self-dual motive associated with $f$.

1.3.1. *Outline of the proof and comparison with* [15]. The general strategy underlying the proof of Theorem B dates back to Kato's work on the cyclotomic main conjecture, as revisited and extended in a series of recent works, including [14, 4, 28, 7, 42, 15, 27]. It can be summarized as follows. (We refer the reader to Section 5 for the actual proof of Theorem B.)

For $\xi = g, h$, fix a root $\alpha_\xi$ of the Hecke polynomial $X^2 - a_p(\xi)\cdot X + \chi_\xi(p)$ and write $\xi_\alpha(q) = \xi(q) - (\chi_\xi(p)/\alpha)\cdot \xi(q^p)$ for the corresponding $p$-stabilization of $\xi$. According to a result of Wiles, there exist Hida families $\boldsymbol{g} = \boldsymbol{g}_\alpha$ and $\boldsymbol{h} = \boldsymbol{h}_\alpha$ specializing, respectively, to $g_\alpha$ and $h_\alpha$ in weight one. For each integer $u$ in a dense subset of a small $p$-adic disc $U$ centered at one, the constructions outlined in the previous section associate to $f$ and the weight-$u$ specializations $\boldsymbol{g}_u$ and $\boldsymbol{h}_u$ an algebraic number $I_p(f, \boldsymbol{g}_u, \boldsymbol{h}_u)$. A method due to Hida (cp. [23]) shows that these algebraic numbers are $p$-adically interpolated by an analytic function $\mathscr{L}_p(f, \boldsymbol{gh})$ on $U$. Thanks to the proof by Harris–Kudla of a conjecture of Jacquet, the value of $\mathscr{L}_p(f, \boldsymbol{gh})$ at $u = 1$ is related to the complex special value $L(f\otimes g\otimes h, k/2)$. The key technical step in the proof of Theorem B consists in showing that there exists a class $\kappa(f, \boldsymbol{gh})$, in a suitable big Selmer group with coefficients in the Tate algebra of analytic functions on $U$, such that

$$\mathscr{L}_p(f, \boldsymbol{gh}) = \mathcal{L}\big(\mathrm{res}_p(\kappa(f, \boldsymbol{gh}))\big), \tag{3}$$

where $\mathcal{L}$ is a branch of the appropriate Perrin-Riou big logarithm map. (We refer to Theorem 5.3 for a precise statement of this result.) Once this is proved, the previous discussion relates $L(f\otimes g\otimes h, k/2)$ to the value at $u = 1$ of the right-hand side of Equation (3), which in turn is related by results of Colmez–Perrin-Riou to the Bloch–Kato dual exponential of the specialization $\kappa(f, g_\alpha, h_\alpha)$ of $\kappa(f, \boldsymbol{gh})$ at $u = 1$. Assuming that $L(f\otimes g\otimes h, s)$ does not vanish at $s = k/2$, this produces a *ramified class* $\kappa(f, g_\alpha, h_\alpha)$ in the relaxed-at-$p$ Selmer group of $V(f,g,h)$ over $\mathbf{Q}$. Under Assumption 1.3.3, one actually produces *four* ramified classes $\kappa(f, g_i, h_j)$, one for each choice of the roots $i$ and $j$ of the $p$-th Hecke polynomials of $g$ and $h$. The $p$-adic residues of these classes are easily seen to be linearly independent, hence Theorem B follows from an application of Poitou–Tate duality.

Theorem 5.3 (or better its proof) shows that Equation (3) can be deduced directly from Theorem A and a simple density argument. More precisely, take a sequence $u_i$ of integers congruent to 1 modulo $p-1$, which converges to infinity in the ordinary topology and to 1 in the $p$-adic topology (e.g., take $u_i = 1 + (p-1)p^i$). We prove that the *existence* of a class $\kappa(f, \boldsymbol{gh})$ satisfying Equation (3) is a direct consequence of the explicit reciprocity law at each crystalline weight-$u_i$ specialization $(f, \boldsymbol{g}_{u_i}, \boldsymbol{h}_{u_i})$ of the triple $(f, \boldsymbol{g}, \boldsymbol{h})$. For this strategy to work, it is crucial to use the good integrality properties enjoyed by the diagonal classes introduced in [8] (cp. Section 2 and the proof of Theorem 5.3). This simple method applies to the study of the analytic rank-zero case of the equivariant Bloch–Kato conjecture in many other interesting settings (e.g., the one considered in [7]).

In the significant special case $k = 2$, Theorem B recasts the main result of [15]. The proof of the latter follows a different pattern. More precisely, *loc. cit.* constructs an explicit class $\kappa(f, \boldsymbol{gh})$ satisfying the identity (3) by using delicate geometric arguments. For each positive integer $s$, a *twisted diagonal cycle* is defined in the Chow group of codimension two cycles in the triple product of the modular curve $X_1(Np^s)$ of level $\Gamma_1(Np^s)$ over $\mathbf{Q}$. The $p$-adic Abel–Jacobi images of these cycles satisfy certain compatibilities under the natural maps from $X_1(Np^{s+1})$ to $X_1(Np^s)$, from which $\kappa(f, \boldsymbol{gh})$ arises as the inverse limit of classes in the ordinary parts of the middle étale cohomology with constant coefficients of the cubes of the curves $X_1(Np^s)$. Once $\kappa(f, \boldsymbol{gh})$ is constructed, reciprocity laws for its specializations at triples of the form $(f, \boldsymbol{g}_{2,\chi}, \boldsymbol{h}_{2,\chi^{-1}})$ are proved, where $\boldsymbol{g}_{2,\chi}$ denotes the *non-crystalline* specialization of $\boldsymbol{g}$ at an arithmetic point of weight 2 and character $\chi$ of conductor divisible by $p$. This entails working on varieties with *bad* reduction at $p$, which makes it harder to obtain the reciprocity laws directly. In this special setting, Equation (3) follows from these reciprocity laws and the properties of the Perrin-Riou logarithm.

## 2. Diagonal classes

This section recalls the definition of the diagonal classes introduced in [8], to which we refer for more details.

Let $N \geqslant 3$ be a positive integer and let $Y_1(N)$ be the affine modular curve of level $\Gamma_1(N)$ over $\mathbf{Z}[1/N]$, classifying isomorphism classes of pairs $(E, P)$, where $E$ is an elliptic curve over a $\mathbf{Z}[1/N]$-scheme $S$ and $P$ is a section in $E(S)$ of exact order $N$. Let $R$ be a $\mathbf{Z}[1/N]$-algebra, let $Y = Y_1(N)_R$ be the base change of $Y_1(N)$ to $R$ and let $v\colon E \to Y$ be the universal elliptic curve over $Y$. There is a natural functor $\cdot_{\text{ét}}$ from the category of $p$-adic representations of $\mathrm{GL}_2(\mathbf{Z}_p)$ to the category of $p$-adic étale sheaves on $Y$. If St denotes the standard representation of $\mathrm{GL}_2(\mathbf{Z}_p)$, then $\mathscr{S} = \mathrm{St}_{\text{ét}}$ is equal to the relative étale cohomology $R^1 v_* \mathbf{Z}_p$ of $E$ over $Y$. In particular, one has $\det_{\text{ét}} = \mathbf{Z}_p(-1)$ for the determinant det of St (see [8, Section 3] and the references therein, in particular, [19, Prop. A I.8] for more details). For each nonnegative integer $u$, denote by $S_u = \mathrm{Symm}^u_{\mathbf{Z}_p}(\mathrm{St})$ the symmetric quotient of the $u$-fold tensor power of St and by $\mathscr{S}_u = \mathrm{Symm}^u_{\mathbf{Z}_p}\mathscr{S}$ the étale sheaf corresponding to $S_u$ under $\cdot_{\text{ét}}$. Write $H^{\cdot}_{\text{ét}}(Y, \mathscr{S}_u)$ for the continuous étale cohomology groups (in the sense of Janssen [24]) of $Y$ with coefficients in $\mathscr{S}_u$.

**Notation.** In this rest of this section $Y = Y_1(N)_{\mathbf{Q}}$ denotes the modular curve over $\mathbf{Q}$. We also fix a rational prime $p > 3$.

Let $(k, l, m)$ be a *balanced* triple in $(\mathbf{Z}_{\geqslant 2})^3$ such that $k + l + m$ is *even*. (Balanced means that $k, l$ and $m$ are the lengths of the sides of a triangle.) The Clebsch–Gordan decomposition of classical invariant theory gives a canonical generator $\mathtt{Det}_{\boldsymbol{r}}$ of $H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r})$, where $\boldsymbol{r} = (r_1, r_2, r_3)$ is equal to $(k-2, l-2, m-2)$, $r$ is equal to $(r_1 + r_2 + r_3)/2$ and $S_{\boldsymbol{r}}$ is a shorthand for $S_{r_1} \otimes_{\mathbf{Z}_p} S_{r_2} \otimes_{\mathbf{Z}_p} S_{r_3}$. After setting $\mathscr{S}_{\boldsymbol{r}} = \mathscr{S}_{r_1} \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2} \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}$, the invariant $\mathtt{Det}_{\boldsymbol{r}}$ corresponds (under $\cdot_{\text{ét}}$) to a global section

$$\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}} = \mathtt{Det}^{\text{ét}}_{N,\boldsymbol{r}} \in H^0_{\text{ét}}(Y, \mathscr{S}_{\boldsymbol{r}}(r)).$$

Let $d \colon Y \hookrightarrow Y^3$ be the diagonal embedding and let

$$\mathscr{S}_{[\boldsymbol{r}]} = \mathscr{S}_{r_1} \boxtimes \mathscr{S}_{r_2} \boxtimes \mathscr{S}_{r_3},$$

so that $d^* \mathscr{S}_{[\boldsymbol{r}]} = \mathscr{S}_{\boldsymbol{r}}$. The push-forward of $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ along $d$ gives a class in $H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$, and the Hochschild–Serre spectral sequence yields a natural map $\mathtt{HS}_{\text{ét}}$ from $H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$ to the global Galois cohomology group $H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}})$ of the lattice

$$\mathtt{W}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2)$$

in the $p$-adic representation $W_{N,\boldsymbol{r}} = \mathtt{W}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The class

$$\kappa_{N,\boldsymbol{r}} = \mathtt{HS}_{\text{ét}} \circ d_*(\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}) \tag{4}$$

is called the *diagonal class* of level $N$ and weights $(k, l, m)$. The results of [33] imply that (after inverting $p$) $\kappa_{N,\boldsymbol{r}}$ belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, W_{N,\boldsymbol{r}})$ of $W_{N,\boldsymbol{r}}$ over $\mathbf{Q}$ (cp. [8] and Section 4.1 below).

Let $L$ be a finite extension of $\mathbf{Q}_p$ and consider a triple of modular forms

$$f \in S_k(N, \chi_f)_L, \quad g \in S_l(N, \chi_g)_L \quad \text{and} \quad h \in S_m(N, \chi_h)_L,$$

where $(k, l, m)$ is a balanced triple with $k, l, m \geqslant 2$ and $k + l + m$ even. Assume that $f, g$ and $h$ are (nonzero) eigenforms for the Hecke operator $T_\ell$ with eigenvalues $\lambda_\ell(f), \lambda_\ell(g)$ and $\lambda_\ell(h)$, for each prime $\ell$ not dividing $N$. As in the introduction, assume in addition that they satisfy the self-duality condition Equation (1), namely, that the product of the characters of $f, g$ and $h$ is the trivial character modulo $N$. Let

$$\mathrm{pr}_{fgh} \colon W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \to V(f, g, h)$$

be the maximal $L$-quotient of $W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$ on which the Hecke operator $T_\ell \otimes \mathrm{id} \otimes \mathrm{id}$ (resp., $\mathrm{id} \otimes T_\ell \otimes \mathrm{id}$, $\mathrm{id} \otimes \mathrm{id} \otimes T_\ell$) acts as multiplication by $\lambda_\ell(f)$ (resp., $\lambda_\ell(g)$, $\lambda_\ell(h)$) for each prime $\ell$ not dividing $N/p^{\mathrm{ord}_p(N)}$, and $\langle d_1 \rangle \otimes \langle d_2 \rangle \otimes \langle d_3 \rangle$ acts as multiplication by $\chi_f(d_1) \cdot \chi_g(d_2) \cdot \chi_h(d_3)$ for each $d_i$ in $(\mathbf{Z}/N\mathbf{Z})^*$. The $L[G_{\mathbf{Q}}]$-module $V(f,g,h)$ is a direct summand of $W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$, isomorphic to the direct sum of a finite number of copies of the $(r+2)$-th Tate twist of the tensor product of the $L$-adic Deligne representations of $f, g$ and $h$. Define

$$\kappa(f,g,h) = \mathrm{pr}_{fgh*}(\kappa_{N,\boldsymbol{r}}) \in \mathrm{Sel}(\mathbf{Q}, V(f,g,h))$$

to be the image of $\kappa_{N,\boldsymbol{r}}$ under the map induced in cohomology by $\mathrm{pr}_{fgh}$.

## 3. Cohomology and modular forms

This section briefly recalls the needed facts on the de Rham and rigid cohomology of modular curves over $\mathbf{Z}_p$. We refer to [25, 39, 13, 2, 5] for the details.

**Notation.** In this section $Y = Y_1(N)_{\mathbf{Q}_p}$ and $X = X_1(N)_{\mathbf{Q}_p}$ denote the open and compact modular curves of level $\Gamma_1(N)$ over $\mathbf{Q}_p$. Let $C = X - Y$ and let $u\colon E \to Y$ be the universal elliptic curve. Let $L$ be a finite extension of $\mathbf{Q}_p(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$.

3.1. **De Rham cohomology.** Let $\boldsymbol{\omega} = u_*\Omega^1_{E/Y}$ and $\mathscr{S}_{\mathrm{dR}} = \mathbf{R}^1 u_* \Omega^\bullet_{E/Y}$ denote, respectively, the line bundle of relative differentials and the first relative de Rham cohomology of $E/Y$, extended to vector bundles on $X$ as in [39, Section 2.3]. For $i \geqslant 0$, set $\mathscr{S}_{\mathrm{dR},i} = \mathrm{Symm}^i_{\mathscr{O}_X} \mathscr{S}_{\mathrm{dR}}$ and $\boldsymbol{\omega}^i = \boldsymbol{\omega}^{\otimes i}$; one has a natural isomorphism between $\boldsymbol{\omega}^2$ and $\Omega^1_X(\log C)$, called the Kodaira–Spencer isomorphism. For $0 \leqslant q \leqslant i$, denote by $\mathrm{Fil}^q \mathscr{S}_{\mathrm{dR},i} = \boldsymbol{\omega}^q \otimes_{\mathscr{O}_X} \mathscr{S}_{\mathrm{dR},i-q}$ the $q$-th step in the Hodge filtration and by $\mathscr{S}^{\cdot}_{\mathrm{dR},i} = \mathscr{S}^{\cdot}_{\mathrm{dR},i}(X)$ the logarithmic de Rham complex of $X$:

$$\mathscr{S}^{\cdot}_{\mathrm{dR},i} = \left[\nabla\colon \mathscr{S}_{\mathrm{dR},i} \to \mathscr{S}_{\mathrm{dR},i} \otimes_{\mathscr{O}_X} \Omega^1_X(\log C)\right]$$

(concentrated in degrees zero and one), where $\nabla$ is the Gauß–Manin connection. For each open subscheme $U$ of $X$, write $\mathscr{S}^{\cdot}_{\mathrm{dR},i}(U)$ for the restriction of $\mathscr{S}^{\cdot}_{\mathrm{dR},i}$ to $U$. Write

$$H_{\mathrm{dR}}(Y, \mathscr{S}_i) = H\Gamma(Y, \mathscr{S}^{\cdot}_{\mathrm{dR},i}(Y)) \tag{5}$$

for the de Rham cohomology of $Y$ with values in $(\mathscr{S}_{\mathrm{dR},i}(Y), \mathrm{Fil}^\bullet, \nabla)$. According to [16, Cor. II.3.15], this is naturally isomorphic to the de Rham cohomology $H_{\mathrm{dR}}(X, \mathscr{S}_i) = H_{\mathrm{dR}}(X, \mathscr{S}^{\cdot}_{\mathrm{dR},i})$, viz. to the cohomology groups of the derived complex $\mathbf{R}\Gamma(X, \mathscr{S}^{\cdot}_{\mathrm{dR},i})$. The Hodge filtration and the Kodaira–Spencer isomorphism then give a natural isomorphism

$$M_{i+2}(N, L) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y, \mathscr{S}_i)_L,$$

where $M_i(N, L) = \Gamma(X, \boldsymbol{\omega}^i)_L$ is the space of weight-$i$ modular forms of level $\Gamma_1(N)$ defined over $L$.

3.1.1. *Comparison with étale cohomology.* Let $k \geqslant 2$ and let $f$ in $S_k(N,\chi_f)_L$ be an eigenvector for the Hecke operator $T_\ell$, with eigenvalue $\lambda_\ell(f)$, for each prime $\ell$ not dividing $N_o = N/p^{\mathrm{ord}_p(N)}$. Denote by

$$V_{\mathrm{dR}}(f) = H^0(\mathbf{Q}_p, B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V(f))$$

the de Rham module of the restriction to $G_{\mathbf{Q}_p}$ of the $G_{\mathbf{Q}}$-representation $V(f)$ defined in the introduction. The comparison isomorphism between étale and de Rham cohomology proved by Faltings–Tsuji [18, 40] yields a natural isomorphism of filtered modules

$$V_{\mathrm{dR}}(f) \cong H^1_{\mathrm{dR}}(Y, \mathscr{S}_{k-2})_f, \tag{6}$$

where the right-hand side is the direct summand of $H^1_{\mathrm{dR}}(Y,\mathscr{S}_{k-2})_L$ on which the Hecke operator $T_\ell$ (resp., diamond operator $\langle d\rangle$) acts as multiplication by $\lambda_\ell(f)$ (resp., $\chi_f(d)$) for each prime $\ell$ not dividing $N_o$ (resp., each unit $d$ in $\mathbf{Z}/N\mathbf{Z}$). We identify $V_{\mathrm{dR}}(f)$ with a direct summand of $H^1_{\mathrm{dR}}(Y,\mathscr{S}_{k-2})_L$ under the previous isomorphism, so that the $f$-isotypic component $S_k(N,L)_f$ of $M_k(N,L)$ becomes identified with $\mathrm{Fil}^1 V_{\mathrm{dR}}(f)$. Define

$$\omega_f \in \mathrm{Fil}^1 V_{\mathrm{dR}}(f)$$

to be the element corresponding to the modular form $f$ in $M_k(N,L)_f$ under these identifications.

If $(f,g,h)$ is a triple of modular forms as in Section 2, the isomorphism (6) and the Künneth decomposition for de Rham cohomology induce a natural isomorphism of filtered modules (considered as an equality)

$$V_{\mathrm{dR}}(f,g,h) \cong H^3_{\mathrm{dR}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]})_{fgh} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[r+2]. \tag{7}$$

Here $V_{\mathrm{dR}}(f,g,h) = H^0(\mathbf{Q}_p, V(f,g,h) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ and $\mathbf{Q}_p[n] = D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ for each $n$ in $\mathbf{Z}$. The filtered vector bundle with connection $\mathscr{S}_{[\boldsymbol{r}],\mathrm{dR}}$ on $Y^3$ is defined by $\mathscr{S}_{\mathrm{dR},k-2} \boxtimes \mathscr{S}_{\mathrm{dR},l-2} \boxtimes \mathscr{S}_{\mathrm{dR},m-2}$. Finally, the $fgh$-isotypic component $H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{\boldsymbol{r}})_{fgh}$ of $H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{[\boldsymbol{r}]})_L = H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{\mathrm{dR},[\boldsymbol{r}]})_L$ is defined as in Section 2.

3.1.2. *Duality.* Let

$$(\,\cdot\,,\cdot\,)\colon \mathscr{S}_{\mathrm{dR}} \otimes_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR}} \to \mathscr{O}_Y(-1)$$

be the perfect relative Poincaré duality pairing, arising from the dualities $(\,\cdot\,,\cdot\,)_x\colon H^1_{\mathrm{dR}}(E_x/k) \otimes_{\mathbf{Q}_p} H^1_{\mathrm{dR}}(E_x/k) \to k$ on the fibres at $x\colon \mathrm{Spec}(k) \to Y$ (with $k$ a field extension of $\mathbf{Q}_p$). Here $\mathscr{O}_Y(n)$ (for $n$ in $\mathbf{Z}$) denotes the sheaf $\mathscr{O}_Y$, equipped with the trivial connection and with the filtration $\mathrm{Fil}^\bullet \mathscr{O}_Y(n)$, given by $\mathrm{Fil}^q \mathbf{Q}_p(n) = \mathscr{O}_Y$ for $q \leqslant -n$ and $\mathrm{Fil}^q \mathscr{O}_Y(n) = 0$ for $q \geqslant 1-n$. For each $i \geqslant 0$, the pairing $(\,\cdot\,,\cdot\,)$ induces a duality

$$(\,\cdot\,,\cdot\,)_i\colon \mathscr{S}_{\mathrm{dR},i} \otimes_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR},i} \to \mathscr{O}_Y(-i), \tag{8}$$

whose restriction to the fibre at $x\colon \mathrm{Spec}(k) \to Y$ is given by

$$(\boldsymbol{\alpha},\boldsymbol{\beta})_{i,x} = \frac{1}{i!} \sum_{\sigma \in S_i} (\alpha_1, \beta_{\sigma(1)})_x \cdots (\alpha_i, \beta_{\sigma(i)})_x \tag{9}$$

for each $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_i$ and $\boldsymbol{\beta} = \beta_1 \cdots \beta_i$ in $\mathrm{Symm}^i_k H^1_{\mathrm{dR}}(E_x/k)$. This in turn induces a perfect duality

(10) $(\cdot\,,\cdot)_i\colon H^1_{\mathrm{dR}}(Y,\mathscr{S}_i)\otimes_{\mathbf{Q}_p} H^1_{\mathrm{dR},c}(Y,\mathscr{S}_i) \to H^2_{\mathrm{dR},c}(Y,\mathscr{O}_Y(-i)) \cong \mathbf{Q}_p[-i-1].$

Let $(f,g,h)$ be as in Section 2 and (as in the introduction) set $\xi^w = w_N\xi$, for $\xi$ equal to $f$, $g$ and $h$. As $\xi^w$ is cuspidal, the morphism $H^1_{\mathrm{dR},c} \to H^1_{\mathrm{dR}}$ maps the $\xi^w$-isotypic component of $H^1_{\mathrm{dR},c}(Y,\mathscr{S}_i)_L$ isomorphically onto $V_{\mathrm{dR}}(\xi^w)$ (cp. Equation (6)), and $(\cdot\,,\cdot)_{u+2}$ induces a perfect pairing

$$(\cdot\,,\cdot)_\xi\colon V_{\mathrm{dR}}(\xi)\otimes_L V_{\mathrm{dR}}(\xi^w) \to L[1-u], \tag{11}$$

where $u$ is the weight of $\xi$. With a slight abuse of notation, write again

$$w_N\colon H^1_{\mathrm{dR},\cdot}(Y,\mathscr{S}_i) \to H^1_{\mathrm{dR},\cdot}(Y,\mathscr{S}_i)$$

for the geometric Atkin–Lehner isomorphism (cp. [8, Section 2.3.1]), which induces an isomorphism $w_N\colon V_{\mathrm{dR}}(\xi) \to V_{\mathrm{dR}}(\xi^w)$. The composition of $(\cdot\,,\cdot)_\xi$ and $\mathrm{id}\otimes w_N$ then yields a perfect duality

$$\langle\cdot\,,\cdot\rangle_\xi\colon V_{\mathrm{dR}}(\xi)\otimes_L V_{\mathrm{dR}}(\xi) \to L[1-u],$$

under which $S_u(N,L)_\xi = \mathrm{Fil}^1 V_{\mathrm{dR}}(\xi)$ is the orthogonal complement of itself.

Define the perfect duality

$$\langle\cdot\,,\cdot\rangle_{fgh}\colon V_{\mathrm{dR}}(f,g,h)\otimes_L V_{\mathrm{dR}}(f,g,h) \to L[1] \tag{12}$$

to be the tensor product of the pairings $\langle\cdot\,,\cdot\rangle_\xi$ for $\xi = f,g,h$. As easily checked, the Bloch–Kato exponential gives an isomorphism $\exp_p$ between the tangent space $\mathrm{tg}_{\mathrm{dR}}(f,g,h)$ of $V_{\mathrm{dR}}(f,g,h)$ and the finite part $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V^*(f,g,h))$ of the local cohomology group $H^1(\mathbf{Q}_p, V(f,g,h))$. After identifying $\mathrm{tg}_{\mathrm{dR}}(f,g,h)$ with the $L$-linear dual of $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the perfect duality $\langle\cdot\,,\cdot\rangle_{fgh}$, the inverse of $\exp_p$ then gives rise to an $L$-linear isomorphism

$$\log_p\colon H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \cong \mathrm{Hom}_L\big(\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h), L\big).$$

In particular, the image under $\log_p$ of (the restriction at $p$ of) the Selmer class $\kappa(f,g,h)$ yields a functional

$$\log_p(\kappa(f,g,h))\colon \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h) \to L. \tag{13}$$

3.1.3. *The class* $\eta_f^\alpha$. Assume in this section $\mathrm{ord}_p(N) \leqslant 1$ and let $f$ be as in Section 3.1.1. Assume in addition that $p$ does not divide the conductor of the character of $f$. Then $V(f)$ is a semi-stable representation of $G_{\mathbf{Q}_p}$. As a consequence, $V_{\mathrm{dR}}(f) = H^0(\mathbf{Q}_p, B_{\mathrm{st}}\otimes_{\mathbf{Q}_p} V(f))$ is equipped with a semi-stable Frobenius endomorphism $\varphi$. As in the introduction, let $\alpha_f$ and $\beta_f$ be the roots of the $p$-th Hecke polynomial $h_{f,p}(X) = X^2 - \lambda_p(f)\cdot X + \chi_f(p)p^{k-1}$ and assume that $L$ contains $\mathbf{Q}_p(\alpha_f,\beta_f)$. Under the assumptions of Section 1.1 the characteristic polynomial of $\varphi$ is a power of $h_{f,p}(X)$ and $V_{\mathrm{dR}}(f)$ is the direct sum of $\mathrm{Fil}^1 V_{\mathrm{dR}}(f) = S_k(N,L)_f$ and the $\varphi$-eigenspace $V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$ (cp. [38]). It follows from this and Section 3.1.2 that there exists a unique de Rham class

$$\eta_f^\alpha \in V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$$

such that, for each $\xi$ in $S_k(N,L)_f$, one has (cp. the introduction)

$$\langle \eta_f^\alpha, \omega_\xi\rangle_f = \frac{(\xi^w, f^w)_N}{(f^w, f^w)_N}.$$

If $(f,g,h)$ is a triple of modular forms as in Section 2, the Künneth product of $\eta_f^\alpha, \omega_g$ and $\omega_h$ defines a class

$$\eta_f^\alpha \otimes \omega_g \otimes \omega_h \in \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h). \tag{14}$$

(To show that the class $\eta_f^\alpha \otimes \omega_g \otimes \omega_h$ indeed belongs to the zeroth step of the Hodge filtration of $V_{\mathrm{dR}}(f,g,h)$, note that $\mathrm{Fil}^1 V_{\mathrm{dR}}(\xi) = \mathrm{Fil}^{u-1} V_{\mathrm{dR}}(\xi)$ for a modular form $\xi$ of weight $u$ and recall that the triple $(k,l,m)$ is *balanced*.)

3.2. ***p*-adic modular forms.** Let $X^{\mathrm{rig}}$ and $Y^{\mathrm{rig}}$ be the rigid analytic varieties over $\mathbf{Q}_p$ associated with $X$ and $Y$, respectively, and let $X^{\mathrm{ord}}$ and $Y^{\mathrm{ord}}$ be their ordinary loci. Let $L$ be a finite extension of $\mathbf{Q}_p(\boldsymbol{\mu}_N)$ and fix a generator $\zeta_N$ of $\boldsymbol{\mu}_N(L)$. For each integer $s$, denote by

$$\mathbf{M}_s(N,L) = \Gamma(X^{\mathrm{ord}}, \boldsymbol{\omega}^s)_L$$

the space of Katz $p$-adic modular forms of weight $s$ and level $\Gamma_1(N)$ defined over $L$. Let $R_N = \mathcal{O}_L[\![q]\!] \otimes_{\mathbf{Z}} \mathbf{Q}$ and let $\mathrm{Tate}(q) = (\mathbf{G}_m/q^{\mathbf{Z}}, \zeta_N)$ be the Tate generalized elliptic curve with $\Gamma_1(N)$-level structure over $R_N$. As $\mathrm{Tate}(q)$ is defined by a global affine equation $y^2 + xy = x^3 + b(q)\cdot x + c(q)$ over $\mathbf{Z}[\![q]\!]$, the invertible sheaf $\boldsymbol{\omega}|_{\mathrm{Tate}(q)} = \imath^*\boldsymbol{\omega}$ has a canonical generator $\omega_{\mathrm{can}} = dx/(2y+x)$ (cp. [25, Section A.1.2]). Given a section $\omega$ of $\boldsymbol{\omega}^s$ over a neighborhood of $\mathrm{Tate}(q)$, its restriction $\omega|_{\mathrm{Tate}(q)}$ to $\mathrm{Tate}(q)$ is then of the form $f_\omega \cdot \omega_{\mathrm{can}}^s$ for a unique element $f_\omega$ in $R_N$, called the $q$-expansion of $\omega$. The $q$-expansion map indeed gives an *injective* morphism

$$\mathbf{M}_s(N,L) \hookrightarrow R_N,$$

which we consider as an inclusion. If $f$ in $R_N$ is the $q$-expansion of a $p$-adic modular form of weight $s$, we write $\omega_f$ for the corresponding section of $\boldsymbol{\omega}^{s-2}$ over the ordinary locus (so that $\omega = \omega_{f_\omega}$).

The module $\mathbf{M}_s(N,L)$ is equipped with the action of the Hecke operator $U = U_p$ and of the Verschiebung $V$, defined on $q$-expansions by

$$U\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_{np}\cdot q^n \quad \text{and} \quad V\Big(\sum_{n\geqslant 0} a_n\cdot q^n\Big) = \sum_{n\geqslant 0} a_n \cdot q^{np},$$

respectively. In particular, for each $p$-adic modular form $f = \sum_{n\geqslant 0} a_n(f)\cdot q^n$ in $\mathbf{M}_s(N,L)$, its *p-depletion*

$$f^{[p]} = (1 - VU)f = \sum_{p\nmid n} a_n(f)\cdot q^n \tag{15}$$

is again a $p$-adic modular form of weight $s$. The derivation $d = q\frac{d}{dq}$ on $R_N$ restricts to *Serre's operator*

$$d\colon \mathbf{M}_s(N,L) \to \mathbf{M}_{s+2}(N,L).$$

In addition, $\mathbf{M}_s(N,L)$ is equipped with the Hecke operators $T_\ell$ and $\langle d\rangle$ for primes $\ell$ not dividing $Np$ and units $d$ in $\mathbf{Z}/N\mathbf{Z}$, which restrict to the usual Hecke operators on the space $M_s(N,L)$ of *classical* modular forms if $s \geqslant 0$.

3.3. **Rigid cohomology.** In this section $p$ does not divide $N$, so that $Y_1(N)_{\mathbf{Z}_p}$ and $X_1(N)_{\mathbf{Z}_p}$ are smooth models of $Y$ and $X$, respectively, over $\mathbf{Z}_p$.

Denote by $\imath\colon Y^{\mathrm{rig}} \hookrightarrow X^{\mathrm{rig}}$ and by $\jmath\colon X^{\mathrm{ord}} \hookrightarrow X^{\mathrm{rig}}$ the natural inclusions and by $\imath^\dagger$ and $\jmath^\dagger$ the corresponding Berthelot functors from the category of abelian sheaves on $X^{\mathrm{rig}}$ to itself [3]. If $\mathscr{F}$ is a coherent sheaf on $X$ and $\kappa = \imath, \jmath$, we write $\kappa^\dagger\mathscr{F}$ for the image of the analytic sheaf $\mathscr{F}|_{X^{\mathrm{rig}}}$ under $\kappa^\dagger$. Set

$$\mathscr{S}^{\cdot}_{\mathrm{rig},i} = \imath^\dagger \mathscr{S}^{\cdot}_{\mathrm{dR},i}$$

and denote again by $\mathrm{Fil}^{\cdot}$ and $\nabla$ the filtration and connection on

$$\mathscr{S}_{\mathrm{rig},i} = \mathscr{S}^{0}_{\mathrm{rig},i}$$

induced by the corresponding structures on $\mathscr{S}_{\mathrm{dR},i}$. The abelian sheaf $\mathscr{S}_{\mathrm{rig},i}$ is also equipped with a Frobenius endomorphism $\varphi$, such that $(\mathscr{S}_{\mathrm{rig},i}, \mathrm{Fil}^{\cdot}, \nabla, \varphi)$ is an overconvergent filtered $\varphi$-isocrystal on the special fibre $Y_{\mathbf{F}_p}$ of $Y_1(N)_{\mathbf{Z}_p}$ (cp. [2, Appendix A]). According to a result of Dwork [25, Thm. A2.3.6], the restriction of $\mathscr{S}_{\mathrm{rig}} = \mathscr{S}_{\mathrm{rig},1}$ to the ordinary locus admits a unique $\varphi$-equivariant splitting $\mathrm{spl}^{ur}\colon \mathscr{S}_{\mathrm{rig}}|_{Y^{\mathrm{ord}}} \to \mathrm{Fil}^1\mathscr{S}_{\mathrm{rig}}|_{Y^{\mathrm{ord}}} = \boldsymbol{\omega}|_{Y^{\mathrm{ord}}}$ of the Hodge filtration such that the Frobenius $\varphi$ acts invertibly on its kernel. Write again

$$\mathrm{spl}^{ur}\colon \mathscr{S}_{\mathrm{rig},i}|_{Y^{\mathrm{ord}}} \to \boldsymbol{\omega}^i|_{Y^{\mathrm{ord}}}$$

for the map induced on the $i$-th symmetric powers, called the *unit root splitting.*

The cohomology of $\mathbf{R}\Gamma(X^{\mathrm{rig}}, \imath^\dagger\mathscr{S}^{\cdot}_{\mathrm{dR},i})$ and $\mathbf{R}\Gamma(X^{\mathrm{rig}}, \jmath^\dagger\mathscr{S}^{\cdot}_{\mathrm{dR},i})$ compute the rigid cohomology groups

$$H_{\mathrm{rig}}(Y_{\mathbf{F}_p}, \mathscr{S}_i) = H_{\mathrm{rig}}(Y_{\mathbf{F}_p}/\mathbf{Q}_p, \imath^\dagger\mathscr{S}_{\mathrm{dR},i})$$

and

$$H_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i) = H_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}/\mathbf{Q}_p, \jmath^\dagger\mathscr{S}_{\mathrm{dR},i}),$$

respectively, where $Y_{\mathbf{F}_p} = Y_1(N)_{\mathbf{F}_p}$ and $Y^{\mathrm{ord}}_{\mathbf{F}_p}$ is the complement in $Y_{\mathbf{F}_p}$ of the finitely many $\mathbf{F}_{p^2}$-rational supersingular points. Theorem 5.4 of [13] proves that the Hodge filtration induces an isomorphism

$$(16) \qquad [\,\cdot\,]_{i+2}\colon \frac{M^\dagger_{i+2}(N,L)}{d^{i+1}M^\dagger_{-i}(N,L)} \cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)_L.$$

Here $M^\dagger_s(N,L) = \Gamma(X^{\mathrm{rig}}, \jmath^\dagger\boldsymbol{\omega}^s)_L$ is the space of overconvergent modular forms of level weight $s \in \mathbf{Z}$ and level $\Gamma_1(N)$ defined over $L$, and $d^{i+2}$ is the $(i+2)$-th iterate of the Serre derivative operator $d$ (denote by $\theta$ in *loc. cit.*). The $L$-submodule $M^\dagger_s(N,L)$ of $\mathbf{M}_s(N,L)$ is invariant under the action of the Hecke operators $U$, $T_\ell$ for primes $\ell$ not dividing $Np$, $\langle d\rangle$ for units $d$ in $(\mathbf{Z}/N\mathbf{Z})^*$, and under the action of the Verschiebung $V$. Loc. cit. proves that the isomorphism

$[\,\cdot\,]_{i+2}$ intertwines the action of the rigid Frobenius $\varphi$ on $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)$ with that of $p^{i+1}\langle p\rangle V$ on overconvergent modular forms, that is,

$$\varphi \circ [\,\cdot\,]_{i+2} = [\,\cdot\,]_{i+2} \circ p^{i+1}\langle p\rangle V. \tag{17}$$

(Note that our model $Y_1(N)$ of the modular curve of level $\Gamma_1(N)$, in which $\mathrm{Tate}(q)$ is *not* defined over $\mathbf{Q}$, differs from the one used in [13]. This explains the appearance of the diamond operator $\langle p\rangle$ in the previous equation.)

The restriction of the unit-root splitting to the global sections of $\mathscr{S}_{\mathrm{rig},i}$ and the Kodaira–Spencer isomorphism induce an injective map

$$\mathrm{spl}^{ur}\colon \Gamma\big(X^{\mathrm{rig}}, \mathscr{S}^1_{\mathrm{rig},i}\big)_L \hookrightarrow \mathbf{M}_{i+2}(N,L).$$

Its image $M^{\text{n-o}}_{i+2}(N,L)$ is called the space of nearly-overconvergent modular forms. The composition of the inverse of $[\,\cdot\,]_{i+2}$ with the natural map

$$\Gamma\big(X^{\mathrm{rig}}, \mathscr{S}^1_{\mathrm{rig},i}\big) \to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)$$

then yields a morphism

$$e^{\dagger}\colon M^{\text{n-o}}_{i+2}(N,L) \to M^{\dagger}_{i+2}(N,L)/d^{i+1}M^{\dagger}_{-i}(N,L). \tag{18}$$

Let $f$ in $S_k(N,\chi_f)_L$ be a cusp form of weight $k \geqslant 2$, level $\Gamma_1(N)$, character $\chi_f\colon (\mathbf{Z}/N\mathbf{Z})^* \to L^*$ and Fourier coefficients in $L$. Assume that $f$ is an eigenvector of the Hecke operator $T_\ell$, with eigenvalue $a_\ell(f)$, for each prime $\ell$ not dividing $N$. Let $\alpha_f, \beta_f$ and $f^w_\alpha \in S_k(Np,\bar{\chi}_f)_L$ be as in Section 1.1 (see in particular Equation (2)). Define

$$H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_L \twoheadrightarrow H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha} \tag{19}$$

to be the maximal quotient on which

$$\varphi = \bar{\chi}_f(p)\cdot\beta_f, \quad T_\ell = \bar{\chi}_f(p)\cdot a_\ell(f) \quad \text{and} \quad \langle d\rangle = \bar{\chi}_f(d)$$

for each prime $\ell$ not dividing $Np$ and each unit $d$ in $\mathbf{Z}/N\mathbf{Z}$. According to Equation (17), the inclusion $S_k(Np,L) \hookrightarrow M^{\dagger}_k(N,L)$ and the Coleman isomorphism $[\,\cdot\,]_k$ defined in Equation (16) induce a morphism

$$[\,\cdot\,]^{\alpha}_f\colon S_k(Np,L)_{f^w_\alpha} \to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha},$$

where $S_k(Np,L)_{f^w_\alpha}$ is the $f^w_\alpha$-isotypic quotient of $S_k(Np,L)$.

If one further assumes that $f^w_\alpha$ has *small slope*, viz. $\mathrm{ord}_p(\alpha_f) < k-1$, then $[\,\cdot\,]^{\alpha}_f$ is an *isomorphism*:

$$[\,\cdot\,]^{\alpha}_f\colon S_k(Np,L)_{f^w_\alpha} \cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha}. \tag{20}$$

Indeed, [13, Thm. 6.1 and Lem. 6.3] proves that the natural map

$$S_k(Np,L) \to M^{\dagger}_k(N,L)/d^{k-1}M^{\dagger}_{2-k}(M,L)$$

induces an isomorphism on the $f^w_\alpha$-isotypic quotients, provided that $f^w_\alpha$ has small slope. In this case, define

$$e_{f^w_\alpha}\colon M^{\text{n-o}}_k(N,L) \to S_k(Np,L)_{f^w_\alpha} \tag{21}$$

to be the composition of the morphism $e^{\dagger}$ defined in Equation (18) with the projection to the $f_{\alpha}^{w}$-isotypic quotient. The morphism $e_{f_{\alpha}^{w}}$ is the (*Coleman*) $f_{\alpha}^{w}$*-isotypic projector* mentioned in Section 1.1.

3.4. **Explicit formulas (cp.** [2, Section 4]**).** Let $\tilde{\mathcal{Y}} \to Y^{\mathrm{ord}}$ be the affine formal scheme over $\mathbf{Z}_p$ which classifies trivialized elliptic curves with $\Gamma_1(N)$-level structure defined over $p$-rings. (We recall that a *trivialization* on an elliptic $E \to S$ is an $S$-isomorphism between the formal multiplicative group $\hat{\mathbf{G}}_m$ over $S$ and the formal completion $\hat{E}$ of $E$ along the zero section.) Let $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ be the coordinate ring of $\tilde{\mathcal{Y}}$, the space of *Katz generalized p-adic modular forms* of level $\Gamma_1(N)$. Write $\tilde{R}_N$ for the $p$-adic completion of $\mathbf{Z}_p[\zeta_N](\!(q)\!)$. Evaluation at the Tate curve $\mathrm{Tate}(q)$ over $\tilde{R}_N$ gives a $q$-expansion map

$$\tilde{\mathbf{M}}(N, \mathbf{Z}_p) \hookrightarrow \tilde{R}_N,$$

which we consider as an inclusion. Then $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ is invariant under the action on $\tilde{R}_N$ of the Hecke operator $U$, of the Verschiebung $V$ and of Serre's derivative operator $d = q\frac{d}{dq}$.

Denote by $\tilde{\boldsymbol{\omega}}$ and $\tilde{\mathscr{S}}_{\mathrm{rig},i}$ the restrictions of $\boldsymbol{\omega}$ and $\mathscr{S}_{\mathrm{rig},i}$, respectively, to $\tilde{\mathcal{Y}}$. These are *free* $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$-modules. More precisely, let $\mathcal{E} \to \tilde{\mathcal{Y}}$ be the universal elliptic curve with trivialization $\psi\colon \hat{\mathbf{G}}_m \cong \hat{\mathcal{E}}$. The line bundle $\tilde{\boldsymbol{\omega}}$ is then generated by the global section $\tilde{\omega}_{\mathrm{can}}$ satisfying $\psi^*\tilde{\omega}_{\mathrm{can}} = dT/(1+T)$ (with $\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[T, T^{-1}])$), which specializes to $\omega_{\mathrm{can}}$ on $\mathrm{Tate}(q)$. Let $\tilde{\Omega}$ be the module of Kähler differentials of the $\mathbf{Z}_p$-algebra $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ and denote by $\tilde{\delta}_{\mathrm{can}}$ the differential in $\tilde{\Omega}$ corresponding to $\tilde{\omega}_{\mathrm{can}}^2$ under the Kodaira–Spencer isomorphism. The derivation of $\tilde{M}(N, \mathbf{Z}_p)$ corresponding to $\tilde{\delta}_{\mathrm{can}}$ is Serre's operator $d$. After setting $\tilde{\eta}_{\mathrm{can}} = \nabla_d(\tilde{\omega}_{\mathrm{can}})$, one has

$$\tilde{\mathscr{S}}_{\mathrm{rig}} = \tilde{\mathbf{M}}(N, \mathbf{Z}_p) \cdot \tilde{\omega}_{\mathrm{can}} \oplus \tilde{\mathbf{M}}(N, \mathbf{Z}_p) \cdot \tilde{\eta}_{\mathrm{can}},$$

and the action of the Gauß–Manin connection $\nabla$ is described by the formula

$$\nabla(f \cdot \tilde{\omega}_{\mathrm{can}} + g \cdot \tilde{\eta}_{\mathrm{can}}) = \big(df \cdot \tilde{\omega}_{\mathrm{can}} + (f + dg) \cdot \tilde{\eta}_{\mathrm{can}}\big) \otimes \tilde{\delta}_{\mathrm{can}}. \tag{22}$$

The action of the Frobenius $\varphi$ can also be described explicitly (paying some attention to the fact that $\mathrm{Tate}(q)$ is not defined over $\mathbf{Q}$). In particular,

$$\varphi \begin{pmatrix} \tilde{\omega}_{\mathrm{can}} \\ \tilde{\eta}_{\mathrm{can}} \end{pmatrix} = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega}_{\mathrm{can}} \\ \tilde{\eta}_{\mathrm{can}} \end{pmatrix}. \tag{23}$$

Let $i$ be an integer, let $L$ be a finite extension of $\mathbf{Q}_p[\zeta_N]$ and write $\tilde{\mathbf{M}}(N, L)$ for the base change of $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ to $L$. Identify $\Gamma(\tilde{\mathcal{Y}}, \tilde{\boldsymbol{\omega}}^i)$ with $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ via $\tilde{\omega}_{\mathrm{can}}$, and $\boldsymbol{\omega}^2$ with $\Omega_X^1(\log C)$ under the Kodaira–Spencer isomorphism. Then restriction to $\tilde{\mathcal{Y}}$ gives an injective map $\mathbf{M}_i(N, L) \hookrightarrow \tilde{\mathbf{M}}(N, L)$ compatible with the $q$-expansion maps, which we consider as an inclusion. As the pullback of $\tilde{\delta}_{\mathrm{can}}$ to the Tate curve is equal to $dq/q$, one deduces that the restriction to $\tilde{\mathcal{Y}}$ of a classical modular form $f$ in $M_{i+2}(N, L)$ is given by $f(q) \cdot \tilde{\omega}_{\mathrm{can}}^i \otimes \tilde{\delta}_{\mathrm{can}}$.

## 4. Proof of Theorem A

This section proves Theorem A stated in Section 1.1.

**Notation.** Let the notations and assumptions be as in *loc. cit.* In particular, $N \geqslant 1$ is a positive integer not divisible by $p$ and $(k, l, m)$ is a geometric balanced triple in $(\mathbf{Z}_{\geqslant 2})^3$. Throughout this section one writes $Y = Y_1(N)_{\mathbf{Z}_p}$ and $X = X_1(N)_{\mathbf{Z}_p}$ for the open and closed modular curves over $\mathbf{Z}_p$, respectively. Moreover, (as in Section 2), $\boldsymbol{r} = (r_1, r_2, r_3)$ equals $(k-2, l-2, m-2)$ and $r$ denotes the nonnegative integer $(r_1 + r_2 + r_3)/2$. To ease notation, in this section only we write $\mathscr{S}_\cdot = \mathscr{S}_{\text{ét},\cdot}$ for the $\mathbf{Q}_p$-linear extensions of the $p$-adic étale sheaves denoted by the same symbol in Section 2. (For example, the étale cohomology groups $H^\cdot_{\text{ét}}(Y, \mathscr{S}_i) = H^\cdot_{\text{ét}}(Y, \mathscr{S}_{\text{ét},i})$ are $\mathbf{Q}_p$-vector spaces).

4.1. **Syntomic and finite polynomial cohomology.** This section recalls the needed facts on rigid syntomic and finite polynomial cohomology. We use [12] and [2, Appendix A] as main references.

For each smooth pair $\mathscr{U} = (U, \bar{U})$ over $\mathbf{Z}_p$, write $S(\mathscr{U})$ for the category of *admissible* filtered overconvergent $\varphi$-isocrystals on $\mathscr{U}$ defined in [2, Def. A.2]. We also call an element of $S(\mathscr{U})$ a *syntomic sheaf* on $\mathscr{U}$. For each syntomic sheaf $\mathscr{F}$ on $\mathscr{U}$ and each polynomial $P(t)$ in $1 + t \cdot L[t]$, denote by $H^\cdot_P(\mathscr{U}, \mathscr{F})$ the Besser rigid finite-polynomial cohomology groups of $\mathscr{U}$ with values in $\mathscr{F}$. In the special case $P(t) = 1 - t$, these are the *syntomic* cohomology groups defined in *loc. cit.* and denoted by $H^\cdot_{\text{syn}}(\mathscr{U}, \mathscr{F})$. The definition given there readily generalizes to the more general setting considered here (cp. [10, 12]). Moreover, one can define finite polynomial cohomology groups with compact support $H^\cdot_{P,c}(\mathscr{U}, \mathscr{F})$ as in [12].

4.1.1. *Syntomic sheaves I: the case $\mathscr{U} = \mathbf{Z}_p$.* Write $\mathbf{Z}_p$ for the smooth pair $(\mathrm{Spec}(\mathbf{Z}_p), \mathrm{Spec}(\mathbf{Z}_p))$ and let $P(t) = \prod_i (1 - \alpha_i t)$ and $Q(t) = \prod_j (1 - \beta_j t)$ be polynomials in $1 + t \cdot L[t]$ (with $\alpha_i, \beta_j$ in $\bar{\mathbf{Q}}_p$).

The category $S(\mathbf{Z}_p)$ of syntomic sheaves on $\mathbf{Z}_p$ is simply the one of filtered $\varphi$-modules over $\mathbf{Q}_p$. For $F$ in $S(\mathbf{Z}_p)$ consider on $F_L = F \otimes_{\mathbf{Q}_p} L$ the (induced filtration and the) $L$-linear endomorphism $\varphi = \varphi \otimes_{\mathbf{Q}_p} L$. Then the finite polynomial cohomology group $H^i_P(\mathbf{Z}_p, F)$ vanishes when $i \neq 0, 1$ and one has

$$H^0_P(\mathbf{Z}_p, F) = F_L^{P(\varphi)=0} \cap \mathrm{Fil}^0 F_L \quad \text{and} \quad H^1_P(\mathbf{Z}_p, F) = F_L / P(\varphi) \cdot \mathrm{Fil}^0 F_L$$

(where $F_L^{P(\varphi)=0}$ denotes the kernel of $P(\varphi)$.) Let $P \star Q(t) = \prod_{i,j} (1 - \alpha_i \beta_j t)$ and let $a(x, y)$ and $b(x, y)$ be any pair of two-variable polynomials satisfying

$$P \star Q(xy) = a(x, y) \cdot P(x) + b(x, y) \cdot Q(y).$$

Let $F$, $G$ and $H$ be filtered $\varphi$-modules, let $\gamma\colon F \otimes_{\mathbf{Q}_p} G \to H$ be a morphism of filtered $\varphi$-modules and let $i, j$ be nonnegative integers which sum to one. Define the cup-product pairing

$$\cup_{\text{fp}} \colon H^i_P(\mathbf{Z}_p, F) \otimes_L H^j_Q(\mathbf{Z}_p, G) \to H^1_{P \star Q}(\mathbf{Z}_p, H)$$

by $cl(f) \cup_{\mathrm{fp}} g = cl(\gamma(a(x,y) \cdot f \otimes g))$ when $i = 1$, respectively, $f \cup_{\mathrm{fp}} cl(g) = cl(\gamma(b(x,y) \cdot f \otimes g))$ when $j = 1$, for each $f$ in $F$ and $g$ in $G$, where the variables $x$ and $y$ act on $F \otimes_{\mathbf{Q}_p} G$ as $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$, respectively.

4.1.2. *Syntomic sheaves II: the general case.* Let $\mathscr{U}$ be a smooth pair over $\mathbf{Z}_p$. A syntomic sheaf $\mathscr{F}$ in $S(\mathscr{U})$ admits (and is characterized by) de Rham and rigid realisations $\mathscr{F}_{\mathrm{dR}}$ and $\mathscr{F}_{\mathrm{rig}}$. The de Rham realization $\mathscr{F}_{\mathrm{dR}}$ is a filtered coherent $\mathscr{O}_{\bar{U}_{\mathbf{Q}_p}}$-module equipped with an integrable connection with logarithmic singularities along $\bar{U} - U$. Write $H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F})$ for the de Rham cohomology groups $H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR}}) \cong H^{\cdot}_{\mathrm{dR}}(\bar{U}_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR}})$ (cp. [2, Def. A.2] and the discussion surrounding Equation (5)). The rigid realization $\mathscr{F}_{\mathrm{rig}}$ is an overconvergent filtered $\varphi$-isocrystal (in the sense of Berthelot) on the special fibre $U_{\mathbf{F}_p}$ of $U$. (If $\jmath\colon \mathcal{U}_{\mathbf{Q}_p} \hookrightarrow \bar{\mathcal{U}}_{\mathbf{Q}_p}$ is the natural inclusion of the Raynaud generic fibre of the $p$-adic completion of $U$ into that of $\bar{U}$, then $\mathscr{F}_{\mathrm{rig}} = \jmath^{\dagger}(\mathscr{F}_{\mathrm{dR}}|U^{\mathrm{rig}})$ as a coherent $\jmath^{\dagger}\mathscr{O}_{U^{\mathrm{rig}}}$-module with connection, where $U^{\mathrm{rig}}$ is the rigid space over $\mathbf{Q}_p$ associated with $U_{\mathbf{Q}_p}$. See *loc. cit.* for more detials.) Denote by $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ the Berthelot rigid cohomology groups $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}/\mathbf{Q}_p, \mathscr{F}_{\mathrm{rig}})$. By the admissibility of $\mathscr{F}$, the natural map from de Rham to rigid cohomology gives an isomorphism

$$H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F}) \cong H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F}),$$

which allows us to view $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ as a filtered $\varphi$-module, i.e., an element of $S(\mathbf{Z}_p)$. Indeed, $H^{i}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ is the $i$-th direct image $R^i\pi_*\mathscr{F}$ of $\mathscr{F}$ under the structural morphism $\pi\colon \mathscr{U} \to \mathbf{Z}_p$, and the Leray spectral sequence

$${}^{\mathrm{syn}}E_2^{p,q} = H^p_{\mathrm{syn}}(\mathbf{Z}_p, H^q_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) \Longrightarrow H^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{F})$$

degenerates into the short exact sequences

$$(24) \qquad 0 \to H^1_{\mathrm{syn}}(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) \xrightarrow{\mathbf{i}_{\mathrm{syn}}} H^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{F}) \xrightarrow{\mathbf{p}_{\mathrm{syn}}} H^0_{\mathrm{syn}}(\mathbf{Z}_p, H^i_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) \to 0.$$

More generally, for any polynomial $P(t)$ in $1 + t \cdot L[t]$ one has short exact sequences

$$(25) \qquad 0 \to H^1_P(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig},\cdot}(U_{\mathbf{F}_p}, \mathscr{F})_L) \xrightarrow{\mathbf{i}_P} H^i_{P,\cdot}(\mathscr{U}, \mathscr{F}) \xrightarrow{\mathbf{p}_P} H^0_P(\mathbf{Z}_p, H^i_{\mathrm{rig},\cdot}(U_{\mathbf{F}_p}, \mathscr{F})_L) \to 0,$$

(where $\blacktriangle_L = \blacktriangle \otimes_{\mathbf{Q}_p} L$ and "$,\cdot$"$= \varnothing$, "$,c$"). If $P$ is clear from the context, we simply write $\mathbf{i} = \mathbf{i}_P$ and $\mathbf{p} = \mathbf{p}_P$.

Let $P$ and $Q$ be polynomials in $1 + t \cdot L[t]$ and let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be syntomic sheaves on $\mathscr{U}$. To a morphism $\mathscr{F} \otimes \mathscr{G} \to \mathscr{H}$ in $S(\mathscr{U})$, one associates as in [12, Section 2] finite polynomial cup product pairings

$$\cup_{\mathrm{fp}}\colon H^i_P(\mathscr{U}, \mathscr{F}) \otimes_L H^j_{Q,c}(\mathscr{U}, \mathscr{G}) \to H^{i+j}_{P\star Q,c}(\mathscr{U}, \mathscr{F} \otimes \mathscr{G}).$$

These are compatible with the Leray spectral sequence, viz. the diagram

$$\begin{array}{ccccc} H^1_P(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) & \otimes_L & H^0_Q(\mathbf{Z}_p, H^j_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G})) & \xrightarrow{\cup_{\mathrm{fp}}} & H^1_{P\star Q}(\mathbf{Z}_p, H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H})) \\ \downarrow{\scriptstyle \mathbf{i}_P} & & \uparrow{\scriptstyle \mathbf{p}_Q} & & \downarrow{\scriptstyle \mathbf{i}_{P\star Q}} \\ H^i_P(\mathscr{U}, \mathscr{F}) & \otimes_L & H^j_{Q,c}(\mathscr{U}, \mathscr{G}) & \xrightarrow{\cup_{\mathrm{fp}}} & H^{i+j}_{P\star Q,c}(\mathscr{U}, \mathscr{H}) \\ \downarrow{\scriptstyle \mathbf{p}_P} & & \uparrow{\scriptstyle \mathbf{i}_Q} & & \uparrow{\scriptstyle \mathbf{i}_{P\star Q}} \\ H^0_P(\mathbf{Z}_p, H^{i}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) & \otimes_L & H^1_Q(\mathbf{Z}_p, H^{j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G})) & \xrightarrow{\cup_{\mathrm{fp}}} & H^1_{P\star Q}(\mathbf{Z}_p, H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H})) \end{array} \tag{26}$$

commutes, where the top and bottom cup-products $\cup_{\mathrm{fp}}$ are the ones associated in Section 4.1.1 with

$$\cup_{\mathrm{rig}} \colon H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F}) \otimes_{\mathbf{Q}_p} H^j_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G}) \to H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H}).$$

For each integer $n$, denote by $\mathbf{Q}_p(n)$ the $n$-th Tate object in $S(\mathscr{U})$. The de Rham realization of $\mathbf{Q}_p(n)$ is the free rank-one $\mathscr{O}_{\bar{U}}$-module $\mathscr{O}_{\bar{U}} \cdot t_n$, with trivial connection and decreasing filtration given by $\mathrm{Fil}^{1-n}\mathbf{Q}_p(n) = 0$ and $\mathrm{Fil}^{-n}\mathbf{Q}_p(n) = \mathbf{Q}_p(n)$, and the Frobenius on $\mathbf{Q}_p(n)_{\mathrm{rig}}$ is defined by $\varphi(t_n) = p^{-n} \cdot t_n$. (When $\mathscr{U} = \mathbf{Z}_p$ the filtered $\varphi$-module $\mathbf{Q}_p(1)$ is then equal to $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$.) If $U$ is geometrically connected of relative dimension $d$ over $\mathbf{Z}_p$, the trace $\mathrm{tr}_{\mathrm{rig}}$ in rigid cohomology gives an isomorphism between $H^{2d}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathbf{Q}_p(d+1))$ and $\mathbf{Q}_p(1)$ and $\mathbf{i}_P$ is an isomorphism between $H^{2d}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathbf{Q}_p(d+1))_L$ and $H^{2d+1}_{P,c}(\mathscr{U}, \mathbf{Q}_p(d+1))$. Assuming that $P(t)$ does not vanish at $t = p^{-1}$, define the (normalized) trace isomorphism

$$\mathrm{tr}_P = P(p^{-1})^{-1} \cdot \mathrm{tr}_{\mathrm{rig}} \circ \mathbf{i}_P^{-1} \colon H^{2d+1}_{P,c}(\mathscr{U}, \mathbf{Q}_p(d+1)) \cong L(1).$$

Given a morphism $\mathscr{F} \otimes_{\mathbf{Q}_p} \mathscr{G} \to \mathbf{Q}_p(d+1)$ in $S(\mathscr{U})$ and polynomials $P$ and $Q$ in $1 + t \cdot L[t]$ such that $P \star Q(t)$ does not vanish at $t = p^{-1}$, the composition of $\cup_{\mathrm{fp}}$ and $\mathrm{tr}_{P\star Q}$ then yields cup-product pairings

$$\langle \cdot , \cdot \rangle_{\mathscr{U}} \colon H^i_P(\mathscr{U}, \mathscr{F}) \otimes_L H^{2d+1-i}_{Q,c}(\mathscr{U}, \mathscr{G}) \to L(1).$$

4.1.3. *Syntomic sheaves III: modular curves.* We are mainly interested in the smooth pairs

$$\mathscr{Y} = (Y, X) \quad \text{and} \quad \mathscr{Y}^{\mathrm{ord}} = (Y^{\mathrm{ord}}, X),$$

where $Y^{\mathrm{ord}} = Y_1(N)^{\mathrm{ord}}_{\mathbf{Z}_p}$ is the open subscheme of $Y$ on which the Hasse invariant $E_{p-1}$ is invertible. For $i \geqslant 0$, the sheaves $\mathscr{S}_{\mathrm{dR},i}$ and $\mathscr{S}_{\mathrm{rig},i}$ arise as the de Rham and rigid realisations of a syntomic sheaf $\mathscr{S}_{\mathrm{syn},i}$ on $\mathscr{Y}$ (cp. [2]). More precisely, let $\mathscr{E}^i$ denote the smooth pair $(\mathcal{E}^i, \bar{\mathcal{E}}^i)$ over $\mathbf{Z}_p$, where $\mathcal{E}^i$ is the $i$-fold fibre product of the universal elliptic curve $\mathcal{E} \to Y$ and $\bar{\mathcal{E}}^i$ is the corresponding Kuga–Sato variety (viz. Deligne's canonical desingularization of the $i$-fold fibre product of the universal generalized elliptic curve $\bar{\mathcal{E}} \to X$). Then

$$\mathscr{S}_{\mathrm{syn},i} = R^1(\mathscr{E}^i \to \mathscr{Y})_* \mathbf{Q}_p$$

is the first higher direct image of the trivial syntomic sheaf on $\mathscr{E}^i$ under the smooth proper morphism $\mathscr{E}^i \to \mathscr{Y}$ attached to the structural map $\bar{\mathcal{E}}^i \to X$. We denote by the same symbol $\mathscr{S}_{\mathrm{syn},i}$ its restriction to $\mathscr{Y}^{\mathrm{ord}}$.

Define the syntomic sheaves $\mathscr{S}_{\mathrm{syn},\boldsymbol{r}}$ and $\mathscr{S}_{\mathrm{syn},[\boldsymbol{r}]}$ on $\mathscr{Y}$ and $\mathscr{Y}^3=(Y^3,X^3)$, respectively, as in Section 2. Set $\mathscr{E}^{\boldsymbol{r}}=\mathscr{E}^{r_1}\times_{\mathbf{Z}_p}\mathscr{E}^{r_2}\times_{\mathbf{Z}_p}\mathscr{E}^{r_3}$. The Leray spectral sequences associated with $\mathscr{E}^{2r}\to\mathscr{Y}$ and $\mathscr{E}^{\boldsymbol{r}}\to\mathscr{Y}^3$ induce, respectively, natural isomorphisms ("Lieberman's trick", cp. the proof of [17, Lem. 5.3])

$$H^i_{\mathrm{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(j))=H^{i+2r}_{\mathrm{syn}}(\mathscr{E}^{2r},\mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}) \tag{27}$$

and

$$H^i_{\mathrm{syn}}(\mathscr{Y}^3,\mathscr{S}_{[\boldsymbol{r}]}(j))=H^{i+2r}_{\mathrm{syn}}(\mathscr{E}^{\boldsymbol{r}},\mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}),$$

where $\cdot(\varepsilon_{\boldsymbol{r}})$ are defined as follows. Let $S_i$ denote the symmetric group on $i$ letters. The semi-direct product $\mathfrak{S}_i=S_i\rtimes\mu_2^i$ acts naturally as a group of automorphisms of $\mathscr{E}^i$ (the nontrivial element of the $i$-th factor of $\mu_2$ acting as multiplication by $-1$ on the $i$-th factor $\mathscr{E}$ of $\mathscr{E}^i$). As a consequence, the subgroup $\mathfrak{S}_{\boldsymbol{r}}=\mathfrak{S}_{r_1}\times\mathfrak{S}_{r_2}\times\mathfrak{S}_{r_3}$ of $\mathfrak{S}_{2r}$ acts by automorphisms on both $\mathscr{E}^{2r}$ and $\mathscr{E}^{\boldsymbol{r}}$. For any $\mathbf{Q}[\mathfrak{S}_{\boldsymbol{r}}]$-module $\cdot$, one defines $\cdot(\varepsilon_{\boldsymbol{r}})$ to be the submodule of elements of $\cdot$ on which $\mathfrak{S}_{\boldsymbol{r}}$ acts via the character $\varepsilon_{\boldsymbol{r}}=\varepsilon_{r_1}\times\varepsilon_{r_2}\times\varepsilon_{r_3}$, where $\varepsilon_i\colon\mathfrak{S}_i\to\mu_2$ maps $\sigma\rtimes(s_1,\dots,s_i)$ to $\mathrm{sign}(\sigma)\cdot s_1\cdots s_i$. Similarly, in $p$-adic étale cohomology there are natural isomorphisms

$$H^i_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(j))=H^{i+2r}_{\text{ét}}(\mathcal{E}^{2r}_{\mathbf{Q}_p},\mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}) \tag{28}$$

and

$$H^i_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{[\boldsymbol{r}]}(j))=H^{i+2r}_{\text{ét}}(\mathcal{E}^{\boldsymbol{r}}_{\mathbf{Q}_p},\mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}).$$

One has analogues of the isomorphisms (27) and (28) after replacing $(\mathscr{Y},\mathscr{E}^{\cdot})$ with $(\mathscr{X},\bar{\mathscr{E}}^{\cdot})$, where $\mathscr{X}$ and $\bar{\mathscr{E}}$ denote the proper smooth pairs $(X,X)$ and $(\bar{\mathcal{E}},\bar{\mathcal{E}})$ over $\mathbf{Z}_p$, respectively.

The Hecke correspondences on $X$ and $\bar{\mathcal{E}}$ equip the syntomic and finite polynomial cohomology groups which appear in this section with the action of Hecke operators away from $Np$, which make the exact sequences (24)–(25) and the isomorphisms (27) Hecke equivariant.

4.1.4. *Comparison with étale cohomology.* Let $\mathscr{U}=(U,\bar{U})$ be a smooth pair over $\mathbf{Z}_p$. The work of Nekovář and Nizioł [35, 36, 31, 33] gives *comparison morphisms*

$$\varrho_{\mathrm{syn}}\colon H^i_{\mathrm{syn}}(\mathscr{U},\mathbf{Q}_p(n))\to H^i_{\text{ét}}(U_{\mathbf{Q}_p},\mathbf{Q}_p(n)),$$

satisfying the following properties. (See [11, Section 9] and the references quoted there for more details):

- The maps $\varrho_{\mathrm{syn}}$ are compatible with pullbacks and proper pushforwards.
- If $U$ is proper over $\mathbf{Z}_p$, then the following diagram commutes.

$$\begin{array}{ccccc} & & H^1_{\mathrm{syn}}(\mathbf{Z}_p,H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p},\mathbf{Q}_p(n))) & \xrightarrow{\ \mathrm{i}_{\mathrm{syn}}\ } & F^1H^i_{\mathrm{syn}}(\mathscr{U},\mathbf{Q}_p(n)) \\ & \overset{1-\varphi}{\nearrow} & & & \downarrow{\scriptstyle \varrho_{\mathrm{syn}}} \\ \mathrm{tg}(H^{i-1}_{\mathrm{dR}}(U_{\mathbf{Q}_p},\mathbf{Q}_p(n))) & & & & \\ & \underset{\exp_p}{\searrow} & & & \\ & & H^1(\mathbf{Q}_p,H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\mathbf{Q}_p(n))) & \xleftarrow{\ \mathrm{HS}_{\text{ét}}\ } & F^1H^i_{\text{ét}}(U_{\mathbf{Q}_p},\mathbf{Q}_p(n)) \end{array} \tag{29}$$

Here $F^1H^i_{\text{ét}}(U_{\mathbf{Q}_p},\cdot\,)$ is the kernel of $H^i_{\text{ét}}(U_{\mathbf{Q}_p},\cdot\,) \to H^i_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,)$ and $F^1H^i_{\text{syn}}(\mathscr{U},\cdot\,)$ is the kernel of $\mathbf{p}_{\text{syn}}$ (that is the image of $\mathbf{i}_{\text{syn}}$, cp. Equation (24)). Moreover, $\exp_p$ denotes the composition

$$\operatorname{tg}\big(H^{i-1}_{\text{dR}}(U_{\mathbf{Q}_p},\cdot\,)\big) \to D_{\text{dR}}(H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,))/\text{Fil}^0 \to H^1(\mathbf{Q}_p, H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,))$$

of Faltings' comparison isomorphism and the Bloch–Kato exponential.

In light of Equations (27)–(28) and the first property above, the maps $\varrho_{\text{syn}}$ for $\mathscr{U} = \mathscr{E}^r$ and $\mathscr{U} = \mathscr{E}^{\boldsymbol{r}}$ induce, respectively, Hecke equivariant comparison morphisms (denoted again by the same symbol)

$$\varrho_{\text{syn}}\colon H^i_{\text{syn}}(\mathscr{Y},\mathscr{S}_r) \to H^i_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_r) \tag{30}$$

and

$$\varrho_{\text{syn}}\colon H^i_{\text{syn}}(\mathscr{Y}^3,\mathscr{S}_{[\boldsymbol{r}]}) \to H^i_{\text{ét}}(Y^3_{\mathbf{Q}_p},\mathscr{S}_{[\boldsymbol{r}]}),$$

which are compatible with the pullback $d^*$ and pushforward $d_*$ along the diagonal $d\colon \mathscr{Y} \to \mathscr{Y}^3$. (There are similar comparison morphisms for $\mathscr{X}$ and $\mathscr{X}^3$ in place of $\mathscr{Y}$ and $\mathscr{Y}^3$, induced, respectively, by the maps $\varrho_{\text{syn}}$ for $\mathscr{U} = \bar{\mathscr{E}}^r$ and $\mathscr{U} = \bar{\mathscr{E}}^{\boldsymbol{r}}$, cp. Section 4.1.3.) In particular,

$$\varrho_{\text{syn}}\colon H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) \to H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r))$$

is an isomorphism, given by the composition of the canonical isomorphisms

$$\begin{aligned} H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) &= \text{Fil}^0 H^0_{\text{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{\boldsymbol{r}}(r))^{\varphi=1} \\ &= \text{Fil}^0 D_{\text{cris}}(H^0_{\text{ét}}(Y_{\bar{\mathbf{Q}}_p},\mathscr{S}_{\boldsymbol{r}}(r)))^{\varphi=1} \\ &= H^0(\mathbf{Q}_p, H^0_{\text{ét}}(Y_{\bar{\mathbf{Q}}_p},\mathscr{S}_{\boldsymbol{r}}(r))) \\ &= H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r)), \end{aligned}$$

where the first equality arises from $\mathbf{p}_{\text{syn}}$, the second is the comparison isomorphism, the third follows from the well-known equality $\text{Fil}^0 B_{\text{cris}} \cap B^{\varphi=1}_{\text{cris}} = \mathbf{Q}_p$ and the forth is defined by the inverse of the base change along the morphism $\text{Spec}(\bar{\mathbf{Q}}_p) \to \text{Spec}(\mathbf{Q}_p)$ (i.e., by the Hochschild–Serre spectral sequence). Let

$$\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}} \in H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) \quad \text{and} \quad \mathtt{Det}^{\text{rig}}_{\boldsymbol{r}} \in \text{Fil}^0 H^0_{\text{rig}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r))^{\varphi=1} \tag{31}$$

be defined by the identities $\varrho_{\text{syn}}(\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}}) = \mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ and $\mathbf{p}_{\text{syn}}(\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}}) = \mathtt{Det}^{\text{rig}}_{\boldsymbol{r}}$, respectively. (Here we write again $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r))$ for the $\mathbf{Q}_p$-base change of the Clebsch–Gordan invariant $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y_1(N)_{\mathbf{Q}},\mathscr{S}_{\boldsymbol{r}}(r))$.)

4.2. **The syntomic Abel–Jacobi map.** Because $Y^3_{\mathbf{Q}_p}$ is a smooth affine threefold, the de Rham cohomology group $H^4_{\text{dR}}(Y^3_{\mathbf{Q}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))$ vanishes. As a consequence the inverse of $\mathbf{i}_{\text{syn}}$ gives an isomorphism

$$\mathtt{HS}_{\text{syn}}\colon H^4_{\text{syn}}(\mathscr{Y}^3,\mathscr{S}_{[\boldsymbol{r}]}(r+2)) \cong H^1_{\text{syn}}(\mathbf{Z}_p, H^3_{\text{rig}}(Y^3_{\mathbf{F}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))_L).$$

After setting $V^*_{\text{dR}}(f,g,h) = V_{\text{dR}}(f^w,g^w,h^w)$, composing $\mathtt{HS}_{\text{syn}}$ with the map induced by the natural projection

$$\text{pr}_{f^w g^w h^w}\colon H^3_{\text{rig}}(Y^3_{\mathbf{F}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))_L \twoheadrightarrow V^*_{\text{dR}}(f,g,h)$$

(arising from the comparison isomorphism between rigid and de Rham cohomology) gives a surjective map

$$H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \twoheadrightarrow H^1_{\mathrm{syn}}(\mathbf{Z}_p, V^*_{\mathrm{dR}}(f,g,h)) = \frac{V^*_{\mathrm{dR}}(f,g,h)}{(1-\varphi)\cdot \mathrm{Fil}^0 V^*_{\mathrm{dR}}(f,g,h)},$$

which we denote by $\mathtt{HS}^{fgh}_{\mathrm{syn}}$. As $p \nmid N$, the Ramanujan–Petersson conjecture implies that $1-\varphi$ is an automorphism of $V^*_{\mathrm{dR}}(f,g,h)$. Denote by $\mathrm{tg}^*_{\mathrm{dR}}(f,g,h)$ the tangent space of $V^*_{\mathrm{dR}}(f,g,h)$ and define the *syntomic Abel–Jacobi map*

$$\mathrm{AJ}^{fgh}_{\mathrm{syn}} \colon H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \twoheadrightarrow \mathrm{tg}^*_{\mathrm{dR}}(f,g,h)$$

to be the composition of $\mathtt{HS}^{fgh}_{\mathrm{syn}}$ with the inverse of $1-\varphi$. Then the following diagram commutes:

$$\begin{array}{ccc} H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) & \xrightarrow{\mathrm{AJ}^{fgh}_{\mathrm{syn}}} & \mathrm{tg}^*_{\mathrm{dR}}(f,g,h) \\ \downarrow{\scriptstyle \varrho_{\mathrm{syn}}} & & \downarrow{\scriptstyle \exp_p} \\ H^4_{\mathrm{\acute{e}t}}(Y^3_{\mathbf{Q}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) & \xrightarrow{\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}}} & H^1(\mathbf{Q}_p, V^*(f,g,h)), \end{array} \tag{32}$$

where $\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}} = \mathrm{pr}_{f^w g^w h^w *} \circ \mathtt{HS}_{\mathrm{\acute{e}t}}$ (cp. Section 2), $V^*(f,g,h) = V(f^w, g^w, h^w)$ and $\exp_p$ is the composition of the Faltings comparison isomorphism

$$\mathrm{tg}^*_{\mathrm{dR}}(f,g,h) \cong D_{\mathrm{dR}}(V^*(f,g,h))/\mathrm{Fil}^0$$

with the Bloch–Kato exponential. This is a consequence of Equation (29) for $i=4$ and $\mathscr{U} = \bar{\mathscr{E}}^{\boldsymbol{r}}$ (so that $U = \bar{\mathcal{E}}^{\boldsymbol{r}}$ is smooth and proper over $\mathbf{Z}_p$). Indeed, by construction, the map $\mathrm{AJ}^{fgh}_{\mathrm{syn}}$ (resp., $\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}}$) factors through the $(f^w, g^w, h^w)$-isotypic component of $H^4_{\mathrm{syn}}(\mathscr{Y}^3, \cdot)$ (resp., $H^4_{\mathrm{\acute{e}t}}(Y^3_{\mathbf{Q}_p}, \cdot)$), which is naturally isomorphic to that of $H^4_{\mathrm{syn}}(\mathscr{X}^3, \cdot)$ (resp., $H^4_{\mathrm{\acute{e}t}}(X^3_{\mathbf{Q}_p}, \cdot)$), since $f, g$ and $h$ are cuspidal forms. Similarly, $V^*(f,g,h)$ and $V^*_{\mathrm{dR}}(f,g,h)$ can be realized, respectively, in the étale and de Rham cohomology of the Kuga–Sato variety $\bar{\mathcal{E}}^{\boldsymbol{r}}$ (via Equation (28) and its analog for the de Rham cohomology). By the definition of the maps $\varrho_{\mathrm{syn}}$ (cp. Equation (30)), the previous diagram can then be rewritten in terms of cohomology groups of $\bar{\mathscr{E}}^{\boldsymbol{r}}$, and once this is done its commutativity is a direct consequence of Equation (29) and the definitions.

The commutative diagram (32) and the compatibility of $\varrho_{\mathrm{syn}}$ with $d_*$ (cp. Equation (30)) yield the equality

$$\log_p(\kappa(f,g,h)) = N^r \cdot \mathrm{AJ}^{fgh}_{\mathrm{syn}}\big(d_*(\mathtt{Det}^{\mathrm{syn}}_{\boldsymbol{r}})\big) \tag{33}$$

of $L$-valued linear forms on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$, cp. Equations (13) and (31). More precisely, we remind that the left-hand side of the previous equation is identified with an $L$-linear form on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the *twisted* Poincaré duality $\langle \cdot, \cdot \rangle_{fgh}$ introduced in Equation (12). On the other hand, we identify the right-hand side of the previous equation with a linear functional on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the perfect duality

$$(\cdot,\cdot)_{fgh} \colon V^*_{\mathrm{dR}}(f,g,h) \otimes_L V_{\mathrm{dR}}(f,g,h) \to L(1)$$

induced by the pairings $(\,\cdot\,,\cdot\,)_i$ defined in Equation (10). Equation (33) then follows from Equations (32), because (as easily checked)

$$N^r \cdot \kappa(f^w, g^w, h^w) = \mathrm{AJ}^{fgh}_{\text{ét}}(d_*(\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}})) \in H^1(\mathbf{Q}, V^*(f,g,h))$$

is the image of the diagonal class

$$\kappa(f,g,h) \in H^1(\mathbf{Q}, V(f,g,h))$$

under the map induced in cohomology by the $G_{\mathbf{Q}}$-equivariant isomorphism

$$w_N^{\otimes 3} \colon V(f,g,h) \cong V^*(f,g,h).$$

Here $w_N^{\otimes 3}$ arises from the Künneth decomposition and the product of the geometric Atkin–Lehner automorphisms $w_N$ of $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_i)$, for $i+2$ equal to $k$, $l$ and $m$. (Recall that $\chi_f \cdot \chi_g \cdot \chi_h$ is equal to the trivial character.)

Because $H^2_{\mathrm{rig},c}(Y^3_{\mathbf{F}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) = 0$, each class

$$\omega \in \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h) \subset \mathrm{Fil}^0 H^3_{\mathrm{dR},c}(Y^3_{\mathbf{Q}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)),$$

which is killed by a polynomial $P_\omega(T) \in 1 + T \cdot L[T]$ has a unique lift $\tilde{\omega}$ in the $(f,g,h)$-isotypic component of the finite-polynomial cohomology group $H^3_{P_\omega,c}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$. Assuming that $P_\omega(p^{-1})$ is nonzero (so that the trace on $H^7_{P_\omega,c}(\mathscr{Y}^3, \mathbf{Q}_p(4))$ is defined), the compatibility of the finite polynomial cup-product with the Leray spectral sequence, viz. Equation (26), gives the following identity of functionals on $H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$:

$$\mathrm{AJ}^{fgh}_{\mathrm{syn}}(\,\cdot\,)(\omega) = \langle\,\cdot\,, \tilde{\omega}\rangle_{\mathscr{Y}^3}. \tag{34}$$

Here the finite polynomial cup product pairing

$$\begin{aligned}\langle\,\cdot\,,\cdot\,\rangle_{\mathscr{Y}^3} \colon H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) &\otimes_L H^3_{P_\omega,c}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \\ &\to H^7_{P_\omega,c}(\mathscr{Y}^3, \mathbf{Q}_p(4)) \cong L(1)\end{aligned}$$

is the one arising from the perfect relative Poincaré dualities of syntomic sheaves (cp. Equations (8))

$$(\,\cdot\,,\cdot\,)_i \colon \mathscr{S}_{\mathrm{syn},i} \otimes_{\mathbf{Q}_p} \mathscr{S}_{\mathrm{syn},i} \to \mathbf{Q}_p(-i).$$

(Unless otherwise stated, all the cup-product pairings which appear below arise from the dualities $(\,\cdot\,,\cdot\,)_i$.) Since the pullback $d^* = d^*_{\mathrm{syn}}$ and push-forward $d_* = d_{\mathrm{syn},*}$, associated with the diagonal embedding $d$ in finite polynomial cohomology, satisfy the projection formula, Equations (33) and (34) yield

$$\log_p(\kappa(f,g,h))(\omega) = N^r \cdot \langle \mathtt{Det}^{\mathrm{syn}}_{\boldsymbol{r}}, d^*(\tilde{\omega})\rangle_{\mathscr{Y}}. \tag{35}$$

Take $\omega$ equal to the class $\eta_f^\alpha \otimes \omega_g \otimes \omega_h$ defined in Equation (14) and $P_\omega$ equal to

$$P_{fgh}(T) = \Big(1 - \frac{p^{r+2}T}{\alpha_f\alpha_g\alpha_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\alpha_g\beta_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\beta_g\alpha_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\beta_g\beta_h}\Big).$$

As by assumption $\chi_f\chi_g\chi_h$ is the trivial character, a direct computation shows that $P_{fgh}(p^{-1})$ equals

$$\mathcal{E}(f,g,h) = \Big(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\Big),$$

which is nonzero by the Ramanujan–Petersson conjecture under the current hypothesis $p\nmid N$.

Let $\xi$ denote either $g$ or $h$ and set

$$P_f(T) = 1-\frac{p^{r_1-r}T}{\alpha_f} \quad\text{and}\quad P_\xi(T) = \Big(1-\frac{p^{u+1}T}{\alpha_\xi}\Big)\Big(1-\frac{p^{u+1}T}{\beta_\xi}\Big),$$

so that $P_{fgh} = P_f\star P_g\star P_h$. Let

(36) $\quad \tilde\eta_f^\alpha \in H^1_{P_f,c}(\mathscr{Y},\mathscr{S}_{r_1}(r_1-r)),\quad$ resp. $\tilde\omega_\xi\in H^1_{P_\xi}(\mathscr{Y},\mathscr{S}_u(u+1))$

(with $u+2$ the weight of $\xi$), denote the unique lift of

$$\eta_f^\alpha \in \mathrm{Fil}^0 H^1_{\mathrm{dR},c}(Y_{\mathbf{Q}_p},\mathscr{S}_{r_1}(r_1-r))^{P_f(\varphi)=0},$$

resp. a lift of

$$\omega_\xi \in \mathrm{Fil}^0 H^1_{\mathrm{dR}}(Y_{\mathbf{Q}_p},\mathscr{S}_u(u+1))^{P_\xi(\varphi)=0},$$

under $\mathbf{p}$. Equation (35) can then be rewritten as

(37) $\quad \log_p(\kappa(f,g,h))\big(\eta_f^\alpha\otimes\omega_g\otimes\omega_h\big) = N^r\cdot\langle\tilde\eta_f^\alpha,\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{syn}}\cup\tilde\omega_g\cup\tilde\omega_h\rangle_{\mathscr{Y}}.$

Write $\mathscr{S}_{gh} = \mathscr{S}_{\mathrm{syn},r_2}\otimes\mathscr{S}_{\mathrm{syn},r_3}(r_2+r_3+2)$ and $P_{gh} = P_g\star P_h$. After noting that $H^2_{\mathrm{dR}}(Y_{\mathbf{Q}_p},\mathscr{S}_{gh})$ vanishes, let

$$\Phi\in H^1_{P_{gh}}\big(\mathbf{Z}_p, H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{gh})\big)$$

be the class defined by the identity

$$\mathbf{i}(\Phi) = \tilde\omega_g\cup\tilde\omega_h.$$

Equation (37) and a direct computation using Equation (26) then prove the following (cp. Equation (31)).

**Proposition 4.3.** *One has*

$$\log_p(\kappa(f,g,h))\big(\eta_f^\alpha\otimes\omega_g\otimes\omega_h\big) = \frac{N^r}{\mathcal{E}(f,g,h)}\cdot\big\langle\eta_f^\alpha(r_1-r),\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{rig}}\cup\Phi\big\rangle_{Y,\mathrm{rig}}.$$

4.4. **Restriction to the ordinary locus.** Given a $\mathbf{Q}_p$-vector space $V$, a $\mathbf{Q}_p$-linear endomorphism $e$ of $V$ and a nonzero element $a$ of $L$, denote by $V_{e=a}$ (resp., $V^{e=a}$) the maximal $L$-quotient (resp., $L$-submodule) of $V\otimes_{\mathbf{Q}_p}L$ on which $e$ acts as multiplication by $a$. As explained in the proof of Proposition III.1.4 of [34], the restriction map $\cdot_{\mathrm{ord}}\colon H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{r_1})\to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{r_1})$ induces an isomorphism

$$H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{r_1})_{\varphi=\bar\chi_f(p)\cdot\beta_f}\cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{r_1})_{\varphi=\bar\chi_f(p)\cdot\beta_f},$$

which commutes with the action of the Hecke operators $T_\ell$ and $\langle d\rangle$ for $\ell\nmid Np$ and $d\in(\mathbf{Z}/N\mathbf{Z})^*$. (This follows from weight considerations, recalling that the square of $\beta_f$ has complex absolute value $k-1$ under the running assumption

$p \nmid N$.) Taking the duals and using Poincaré duality, this induces an isomorphism

$$\cdot^{\mathrm{ord}}\colon H^1_{\mathrm{rig},c}(Y_{\mathbf{F}_p}, \mathscr{S}_{r_1})^{\varphi=\alpha_f} \cong H^1_{\mathrm{rig},c}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})^{\varphi=\alpha_f}.$$

After setting

$$\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} = (\mathtt{Det}^{\mathrm{rig}}_{\boldsymbol{r}})_{\mathrm{ord}} \otimes t_{-2-r} \in H^0_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{\boldsymbol{r}}(-2)) \tag{38}$$

(so that $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}$ belongs to $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$), Proposition 4.3 then gives the following.

**Proposition 4.5.** *One has*

$$\log_p(\kappa(f,g,h))(\eta^\alpha_f \otimes \omega_g \otimes \omega_h) = \frac{N^r}{\mathcal{E}(f,g,h)} \cdot \left\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}} \right\rangle_{Y^{\mathrm{ord}},\mathrm{rig}}.$$

The linear form

$$\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \cdot \rangle_{Y^{\mathrm{ord}},\mathrm{rig}}\colon H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_L \to L$$

factors through the quotient

$$H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_L \twoheadrightarrow H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_{f^w_\alpha}$$

defined in Equation (19). As by assumption $f^w_\alpha = (f^w)_\alpha$ has small slope (i.e., $\mathrm{ord}_p(\alpha_f) < k-1$), Equation (20) shows that the latter is isomorphic to $S_k(Np,L)_{f^w_\alpha}$ under the Coleman map $[\,\cdot\,]^\alpha_f$. Let

$$\Xi \in S_k(Np,L)_{f^w_\alpha}$$

be the cusp form satisfying

$$[\Xi]^\alpha_f = \left[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}\right]_{f^w_\alpha}, \tag{39}$$

where $[\,\cdot\,]_{f^w_\alpha}$ denotes the projection of $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$ onto $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_{f^w_\alpha}$.

**Proposition 4.6.** *After setting $\mathcal{E}^*(f) = 1 - \frac{\beta_f}{\alpha_f}$, one has*

$$\log_p(\kappa(f,g,h))(\eta^\alpha_f \otimes \omega_g \otimes \omega_h) = \frac{N^r \mathcal{E}^*(f)}{\mathcal{E}(f,g,h)} \frac{(f^w_\alpha, \Xi)_{Np}}{(f^w_\alpha, f^w_\alpha)_{Np}}.$$

*Proof.* One has $\Xi = (1 - \bar{\chi}_f(p)\beta_f \cdot V) \cdot \xi$ for a cusp form $\xi \in S_k(N,L)$. Let

$$\omega_\xi \in H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p}, \mathscr{S}_{r_1})$$

be the class associated with $\xi$ and let $\omega_{\xi,\mathrm{ord}} \in H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$ be the restriction of $\omega_\xi$ to the ordinary locus. Then

$$[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}]_{f^w_\alpha} = [\Xi]^\alpha_f = \left[\left(1 - \frac{\beta_f \cdot \varphi}{p^{k-1}}\right) \cdot \omega_{\xi,\mathrm{ord}}\right]_{f^w_\alpha} = \mathcal{E}^*(f) \cdot [\omega_{\xi,\mathrm{ord}}]_{f^w_\alpha},$$

hence

$$\begin{aligned}\left\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}} \right\rangle_{Y^{\mathrm{ord}},\mathrm{rig}} &= \mathcal{E}^*(f) \cdot \left\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \omega_{\xi,\mathrm{ord}} \right\rangle_{Y^{\mathrm{ord}},\mathrm{rig}} \\ &= \mathcal{E}^*(f) \cdot \left\langle \eta^\alpha_f(r_1+2), \omega_\xi \right\rangle_{Y,\mathrm{rig}} \\ &= \mathcal{E}^*(f) \cdot (\xi, f^w)_N / (f^w, f^w)_N,\end{aligned}$$

by the definitions of $\eta_f^\alpha$ and $\eta_f^{\alpha,\mathrm{ord}}$ As easily checked $(\xi, f^w)_N/(f^w, f^w)_N$ is equal to $(\Xi, f_\alpha^w)_{Np}/(f_\alpha^w, f_\alpha^w)_{Np}$. The statement then follows from the previous equation and Proposition 4.5. □

4.7. **Conclusion of the proof.** This section concludes the proof of Theorem A.

Let $\xi$ in $M_{u+2}(N, L)$ denote either $g$ or $h$ and let

$$\omega_\xi \in \mathrm{Fil}^0 H^1_{\mathrm{dR}}(Y, \mathscr{S}_u(u+1))_L$$

be the corresponding de Rham class. With a slight abuse of notation, denote by $\omega_\xi$ in $\Gamma(X, \boldsymbol{\omega}^u(u+1) \otimes \Omega^1(C))_L$ also the section representing $\omega_\xi$, so that

$$\omega_\xi|_{\tilde{\mathcal{Y}}} = \xi \cdot \tilde{\omega}^u_{\mathrm{can}} \otimes \tilde{\delta}_{\mathrm{can}} \otimes t_{u+1}$$

in $\Gamma(\tilde{\mathcal{Y}}, \tilde{\boldsymbol{\omega}}^u(u+1) \otimes \tilde{\Omega}^1)_L$ (cp. Section 3.4).

Let $\tilde{\omega}_{\xi,\mathrm{ord}}$ in $H^1_{P_\xi}(\mathscr{Y}^{\mathrm{ord}}, \mathscr{S}_u(u+1))$ be the restriction to the ordinary locus of $\tilde{\omega}_\xi$ (cp. Equation (36)). By construction $\tilde{\omega}_{\xi,\mathrm{ord}}$ is a lift under $\mathbf{p}$ of the restriction of $\omega_\xi$ to the ordinary locus. (If $u \geqslant 1$ such a lift is unique, cp. [2, Lem. 4.2]). According to [2, Prop. A.16], the class $\tilde{\omega}_{\xi,\mathrm{ord}}$ is uniquely represented by (the restriction to the ordinary locus $\mathscr{Y}^{\mathrm{ord}}$ of) a pair $(F_\xi, \omega_\xi)$, where the overconvergent section

$$F_\xi \in \Gamma\big(X^{\mathrm{rig}}_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{rig},u}(u+1)\big)_L \quad \text{satisfies } \nabla F_\xi = P_\xi(\varphi) \cdot \omega_\xi.$$

As explained in [5, Sections 3.6–3.8] (see in particular Proposition 3.24), one can, and will, choose $\tilde{\omega}_\xi$ in such a way that $\tilde{\omega}_{\xi,\mathrm{ord}}$ is represented by the pair $(F_\xi, \omega_\xi)$ with

$$F_\xi|_{\tilde{\mathcal{Y}}} = \sum_{j=0}^{u} (-1)^j j! \binom{u}{j} d^{-1-j} \xi^{[p]}(q) \cdot \tilde{\omega}^{u-j}_{\mathrm{can}} \tilde{\eta}^j_{\mathrm{can}} \otimes t_{u+1} \tag{40}$$

in $\Gamma(\tilde{\mathcal{Y}}, \tilde{\mathscr{S}}_{u,\mathrm{rig}}(u+1))_L$. (The equality $\nabla F = P_\xi(\varphi) \cdot \omega_\xi$ over $\tilde{\mathcal{Y}}$ can be easily checked using Equations (22) and (23). Note that the lift $\tilde{\omega}_\xi$ of $\omega_\xi$, and then $F_\xi$, is unique if the weight of $\xi$ is strictly greater than two, cp. [2, Lem. 4.2].)

The finite polynomial cup product $\tilde{\omega}_{g,\mathrm{ord}} \cup \tilde{\omega}_{h,\mathrm{ord}} = (\tilde{\omega}_g \cup \tilde{\omega}_h)_{\mathrm{ord}}$ is represented by any 2-cocycle of the form

$$\bigcup\big(a(x,y) \cdot F_g \otimes \omega_h - b(x,y) \cdot \omega_g \otimes F_h, \omega_g \otimes \omega_h\big), \tag{41}$$

where $a(x,y)$ and $b(x,y)$ are polynomials in $L[x,y]$ satisfying

$$P_{gh}(xy) = a(x,y) \cdot P_g(x) + b(x,y) \cdot P_h(y)$$

and $x$ and $y$ act via $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$, respectively (cp. [10, Rem. 4.3]). Proposition 5.2.5 of [29] shows that one can take $a(x,y)$ and $b(x,y)$ of the form

$$a(x,y) = 1 - \chi_f(p) p^{r_2+r_3+2} \cdot x^2 y^2 + y \cdot a_o(x,y) \quad \text{and} \quad b(x,y) = x \cdot b_o(x,y), \tag{42}$$

with $a_o(x,y)$ and $b_o(x,y)$ in $L[x,y]$. (Recall that $\chi_g \chi_h$ equals $\chi_f^{-1}$.)

Let $F_{g,j} \in \tilde{\mathbf{M}}(N,L)$ be the $\tilde{\omega}_{\mathrm{can}}^{j}\tilde{\eta}_{\mathrm{can}}^{r_2-j}$-coefficient of $F_g|_{\tilde{\mathcal{Y}}}$. The section $F_{g,j}$ is $p$-depleted, viz. the $n$-th Fourier coefficient of its $q$-expansion is zero if $p$ divides $n$ (cp. Equation (40)). On the other hand, the $n$-th Fourier coefficient of the $q$-expansion $\sum_{n\geqslant 0} a_n(h)\cdot q^{pn}$ of $V(h)$ is zero if $p$ does not divide $n$. It follows that $F_{g,j}\cdot V(h)$ is $p$-depleted, hence so is each coefficient of $F_g\otimes\varphi(\omega_h)$ (as the restriction of $\varphi(\omega_h)$ to $\tilde{\mathcal{Y}}$ is a multiple of $V(h)\cdot\tilde{\omega}_{\mathrm{can}}^{r_3}\otimes\tilde{\delta}_{\mathrm{can}}\otimes t_{r_3+1}$). This implies that $U_p$ kills the class in $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{gh})_L$ represented by $F_g\cup\varphi(\omega_h)$. Because $U_p$ is an isomorphism, one deduces that the section $F_g\cup\varphi(\omega_h)$ is exact. Similarly, one proves that $\varphi(\omega_g)\cup F_h$ is exact. Together with Equations (41) and (42) this proves that $(\tilde{\omega}_g\cup\tilde{\omega}_h)_{\mathrm{ord}}$ is represented by

$$\big((1-\chi_f(p)p^{r_2+r_3+2}\cdot\varphi^2)\cdot F_g\cup\omega_h,0\big).$$

As $\Phi_{\mathrm{ord}}$ is characterized by the equality $\mathbf{i}(\Phi_{\mathrm{ord}})=(\tilde{\omega}_g\cup\tilde{\omega}_h)_{\mathrm{ord}}$, the previous equation then yields

$$\Phi_{\mathrm{ord}} = \text{class of } (1-\chi_f(p)p^{r_2+r_3+2}\cdot\varphi^2)\cdot F_g\cup\omega_h. \tag{43}$$

Identify the $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)$-module of global sections of $\tilde{\mathscr{S}}_{\mathrm{rig},r_i}$ with the set of two-variable homogeneous polynomials of degree $r_i$ in $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)[x_i,y_i]$ via $x_i^j y_i^{r_i-j}=\tilde{\omega}_{\mathrm{can}}^{j}\tilde{\eta}_{\mathrm{can}}^{r_i-j}$. Then $\tilde{\mathscr{S}}_{\mathrm{rig},\boldsymbol{r}}=\tilde{\mathscr{S}}_{\mathrm{rig},r_1}\otimes\tilde{\mathscr{S}}_{\mathrm{rig},r_2}\otimes\tilde{\mathscr{S}}_{\mathrm{rig},r_3}$ becomes identified with a submodule of $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)[x_i,y_i\colon 1\leqslant i\leqslant 3]$ and (cp. Equation (38))

$$\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}|_{\tilde{\mathcal{Y}}}=(x_1y_2-y_1x_2)^{r-r_3}\cdot(x_1y_3-y_1x_3)^{r-r_2}\cdot(x_2y_3-y_2x_3)^{r-r_1}\otimes t_{-2} \tag{44}$$

in $\Gamma(\tilde{\mathcal{Y}},\tilde{\mathscr{S}}_{\mathrm{rig},\boldsymbol{r}}(-2))_L$. Note that the rigid Frobenius acts on $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}$ as multiplication by $p^{2+r}$, hence (cp. Equation (39))

$$\begin{aligned}[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup\Phi_{\mathrm{ord}}]_{f^w_\alpha} &= \Big[\Big(1-\frac{\chi_f(p)\cdot\varphi^2}{p^{r_1+2}}\Big)\cdot\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h\Big]_{f^w_\alpha}\\ &=\Big(1-\frac{\beta_f}{p\alpha_f}\Big)\cdot[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h]_{f^w_\alpha},\end{aligned} \tag{45}$$

by Equation (43). According to Equations (40) and (44) the restriction of $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h$ to $\tilde{\mathcal{Y}}$ is equal to

$$\begin{aligned}\sum_{i_1,i_2,i_3,j}&(-1)^{r-i_1-i_2-i_3+j}j!\binom{r-r_3}{i_1}\binom{r-r_2}{i_2}\binom{r-r_1}{i_3}\binom{r_2}{j}\\ &\cdot d^{-1-j}g^{[p]}\cdot h\cdot x_1^{i_1+i_2}y_1^{r_1-i_1-i_2}\otimes\tilde{\delta}_{\mathrm{can}}\\ &\otimes x_2^{r-r_3-i_1+i_3}y_2^{r-r_1-i_3+i_1}x_3^{r_3-i_2-i_3}y_3^{i_2+i_3}\cup x_2^{r_2-j}y_2^{j}x_3^{r_3}\otimes t_{r_2+r_3},\end{aligned}$$

where the sum runs over the tuples $(i_1,i_2,i_3,j)$, with $0\leqslant j\leqslant r_2$ and $0\leqslant i_s\leqslant r_s$ for $s=1,2,3$. The only contribution to the $x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}$-component comes from $(i_1,i_2,i_3,j)=(r-r_3,r-r_2,r-r_1,r-r_1)$ and is equal to (cp.

Equation (9))

$$(-1)^{r-r_1}(r-r_1\binom{r_2}{r-r_1}\cdot d^{-1-r+r_1}g^{[p]}\cdot h\cdot x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}$$
$$\otimes x_2^{r-r_1}y_2^{r-r_3}y_3^{r_3}\cup x_2^{r-r_3}y_2^{r-r_1}x_3^{r_3}\otimes t_{r_2+r_3}.$$

As

$$x_2^{r-r_1}y_2^{r-r_3}y_3^{r_3}\cup x_2^{r-r_3}y_2^{r-r_1}x_3^{r_3}=(-1)^r\binom{r_2}{r-r_1}^{-1}\cdot t_{-r_2-r_3},$$

one deduces

$$\mathrm{spl}^{ur}(\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{ord}}\cup F_g\cup\omega_h)|_{\tilde{\mathcal{Y}}}=(-1)^{r_1}(r-r_1)!\cdot d^{-1-r+r_1}g^{[p]}(q)\cdot h(q)\cdot x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}.$$

This proves that (as claimed in the discussion preceding the statement of Theorem A) the $p$-adic modular form

$$d^{-1-r+r_1}g^{[p]}\cdot h=\mathrm{spl}^{ur}\big((-1)^{r_1}(r-r_1)!^{-1}\cdot\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{ord}}\cup F_g\cup\omega_h\big)\tag{46}$$

is nearly-overconvergent, and (after unwinding the definitions, cp. Equations (21), (39) and (45)) yields the identity

$$(-1)^{r_1}(r-r_1)!\Big(1-\frac{\beta_f}{p\alpha_f}\Big)\cdot e_{f_\alpha^w}(d^{-1-r+r_1}g^{[p]}\cdot h)=\Xi.$$

Theorem A follows from Proposition 4.6 and the previous equation.

## 5. Proof of Theorem B

This section proves Theorem B stated in Section 1.2. Let the notations and assumptions be as in *loc. cit.*

### 5.1. Hida theory.

Let $L$ be a finite extension of $\mathbf{Q}_p$ and let $U$ be an $L$-rational affinoid disc in the weight space $\mathcal{W}$ over $\mathbf{Q}_p$, centered at an integer $u_o\geqslant 1$. Let $\mathcal{O}(U)$ denote the ring of analytic functions on $U$. It can be identified with a subring of $L[\![\boldsymbol{u}-u_o]\!]$, where $\boldsymbol{u}-u_o$ is a uniformiser at $u_o$. Write $U^{\mathrm{cl}}$ for the set of positive integers in $U$ which are congruent to $u_o$ modulo $2(p-1)$, and let $\chi$ be an $L$-valued Dirichlet character modulo $N$. Denote by $S_U^{\mathrm{ord}}(N,\chi)$ the set of formal $q$-expansions $\boldsymbol{\xi}=\sum_{n\geqslant 0}r_n\cdot q^n$ in $\mathcal{O}(U)[\![q]\!]$ satisfying the following property: For each *classical point* $u$ in $U^{\mathrm{cl}}\cap\mathbf{Z}_{\geqslant 2}$, the *weight-$u$ specialization* $\boldsymbol{\xi}_u=\sum_{n\geqslant 0}r_n(u)\cdot q^n$ is the $q$-expansion of a cusp form in $S_u(Np,\chi)_L$, which is an eigenvector for the Hecke operator $T_\ell$, for each prime $\ell$ not dividing $Np$, and for the Hecke operator $U_p$ with eigenvalue a *$p$-adic unit* in $L$. For each classical point $u>2$, the form $\boldsymbol{\xi}_u$ is indeed the ordinary $p$-stabilization of a $p$-ordinary eigenform $\xi_u$ in $S_k(N,\chi)_L$. If $u=2$, then either $p$ divides the level of $\boldsymbol{\xi}_u$, in which case one sets $\xi_u=\boldsymbol{\xi}_u$, or $\boldsymbol{\xi}_u$ is the $p$-stabilization of a $p$-ordinary eigenform $\xi_u$ of level $\Gamma_1(N)$.

An element of $S_U^{\mathrm{ord}}(N,\chi)$ (for some $U$ as above) is called a (cuspidal) Hida family of tame level $N$, character $\chi$ and center $u_o$. One says that $\boldsymbol{\xi}$ is *primitive* if $\xi_u$ is a primitive form of conductor $N$ for all classical points $u>2$. Let $\boldsymbol{\xi}^\sharp$ be a primitive Hida family in $S_U^{\mathrm{ord}}(N_o,\chi)$ and let $N$ be a multiple of $N_o$.

A level-$N$ test vector for $\boldsymbol{\xi}^\sharp$ is a Hida family $\boldsymbol{\xi}$ in $S_U^{\mathrm{ord}}(N,\chi)$ such that, for all $u \geqslant 2$ in $U^{\mathrm{cl}}$, the specializations $\boldsymbol{\xi}_u^\sharp$ and $\boldsymbol{\xi}_u$ have the same eigenvalues under the action of the Hecke operators $U_p$ and $T_\ell$, for all primes $\ell$ not dividing $Np$.

Let $N$ denote the least common multiple of $N_f, N_g$ and $N_h$. For $\xi = f, g, h$, write $\alpha_\xi$ and $\beta_\xi$ for the roots of the $p$-th Hecke polynomial

$$X^2 - a_p(\xi)\cdot X + \chi_\xi(p)p^{u-1}$$

of $\xi$, where $u$ is the weight of $\xi$. Assume that $L$ contains $\alpha_\xi$ and $\beta_\xi$ and order $\alpha_f$ and $\beta_f$ in such a way that $\alpha_f$ is a $p$-adic unit. This is possible by Assumption 1.3.4. According to a theorem of Wiles [43], there exist primitive Hida families

$$\boldsymbol{g}^\sharp = \sum_{n\geqslant 0} b_n(\boldsymbol{u})\cdot q^n \in S_U^{\mathrm{ord}}(N_g,\chi_g) \quad \text{and} \quad \boldsymbol{h}^\sharp = \sum_{n\geqslant 0} c_n(\boldsymbol{u})\cdot q^n \in S_U^{\mathrm{ord}}(N_h,\chi_h)$$

of levels $N_g$ and $N_h$, common center $u_o = 1$ and tame characters $\chi_g$ and $\chi_h$, specializing, respectively, to the $p$-stabilized cusp forms $g_\alpha$ and $h_\alpha$ at weight one, namely, satisfying

$$\boldsymbol{g}_1^\sharp = g_\alpha \quad \text{and} \quad \boldsymbol{h}_1^\sharp = h_\alpha.$$

(Recall that $\xi_\alpha(q) = \xi(q) - \beta_\xi\cdot\xi(q^p)$ is an eigenvector for $U_p$ with eigenvalue $\alpha_\xi$.) Note that $\boldsymbol{g}^\sharp = \boldsymbol{g}_\alpha^\sharp$ and $\boldsymbol{h}^\sharp = \boldsymbol{h}_\alpha^\sharp$ depend on the choice of the roots $\alpha_g$ and $\alpha_h$ of the $p$-th Hecke polynomials of $g$ and $h$, respectively.

Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be level-$N$ test vectors for $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$, respectively. Moreover, let $\boldsymbol{f}_k$ be the *ordinary* $p$-stabilization of a cusp form $f_k$ in $S_k(N,\chi_f)_L$, which is an eigenvector of the Hecke operator $T_\ell$, with the same eigenvalue $a_\ell(f)$ as $f$, for each prime $\ell$ not dividing $N$. (We call $\boldsymbol{f}_k$ and $f_k$ *level-$N$ test vectors* for $f$.) For each $u \geqslant 2$ in $U^{\mathrm{cl}}$, set

$$\mathtt{W}_{Np}(u) = H^3_{\mathrm{\acute{e}t}}(Y_1(Np)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2}\boxtimes\mathscr{S}_{u-2}\boxtimes\mathscr{S}_{u-2})\otimes_{\mathbf{Z}_p}\mathscr{O}_L(k/2+u-1).$$

Denote by

$$\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_u\boldsymbol{h}_u} : \mathtt{W}_{Np}(u) \to \mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u) \tag{47}$$

the maximal $\mathscr{O}_L$-quotient on which the Hecke operators $U_p\otimes 1\otimes 1$, $1\otimes U_p\otimes 1$ and $1\otimes 1\otimes U_p$ (resp., $T_\ell\otimes 1\otimes 1, 1\otimes T_\ell\otimes 1, 1\otimes 1\otimes T_\ell$ and $\langle d_1\rangle\otimes\langle d_2\rangle\otimes\langle d_3\rangle$) act as multiplication by $\alpha_f, b_p(u)$ and $c_p(u)$ (resp., $a_\ell(f), b_\ell(u), c_\ell(u)$ and $\chi_f(d_1)\cdot\chi_g(d_2)\cdot\chi_h(d_3)$ for any prime $\ell\nmid Np$ and units $d_i \in (\mathbf{Z}/N\mathbf{Z})^*$), and set

$$V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u) = \mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\otimes_{\mathbf{Z}}\mathbf{Q}.$$

Note that $V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ depends only on the level $N$ and on the primitive forms $f, g_u^\sharp$ and $h_u^\sharp$.

Let $\|\cdot\|_U$ be the supremum norm on $\mathcal{O}(U)$ and let $\Lambda = \Lambda_U$ be the corresponding unit ball. The work of Hida, Perrin-Riou et al. yields a free $\Lambda$-module $\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}\boldsymbol{h})$, equipped with a continuous $\Lambda$-linear action of $G_{\mathbf{Q}}$, satisfying the following properties (cp. [8, Sections 4 and 6]).

- For each $u \geqslant 2$ in $U^{\mathrm{cl}}$, evaluation at $u$ on $\Lambda$ induces a natural isomorphism of $\mathscr{O}_L[G_{\mathbf{Q}}]$-modules

$$\rho_u \colon \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}) \otimes_u \mathscr{O}_L \cong \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u). \tag{48}$$

The representation $V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ is isomorphic to

$$\bigoplus_{i=1}^{a} V(f) \otimes_L V(g_u^\sharp) \otimes_L V(h_u^\sharp)((k/2+u-1)),$$

where $V(\,\cdot\,) = \mathtt{D}(\,\cdot\,)$ is the $L$-adic Deligne representation $\cdot$ and the positive integer $a = a_N$ is independent of $u$. If $u = 1$, the previous formula holds with $\mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ isomorphic to a lattice in $V(f, g, h)^a$.

- Let $U^{\mathrm{bal}}$ be the set of $u \geqslant 2$ in $U^{\mathrm{cl}}$ with $k < 2u$. There exists a $\Lambda[G_{\mathbf{Q}_p}]$-submodule

$$i_{\mathrm{bal}} \colon \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \to \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}),$$

free of rank $\frac{1}{2}\mathrm{rank}_\Lambda \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})$ over $\Lambda$, such that for all $u$ in $U^{\mathrm{bal}}$, the Bloch–Kato finite subspace

$$H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$$

of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ is equal to the image of the map

$$H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u L) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \tag{49}$$

induced by $\rho_u$.

The morphism induced in cohomology by $i_{\mathrm{bal}}$ is injective, and its image $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}))$ is called the *balanced* subspace. Similarly, for $u$ in $U^{\mathrm{cl}}$, one defines the *balanced* subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ of $H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ as the image of $H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u \mathscr{O}_L)$ under the morphism induced by $\rho_u$. The *balanced Selmer group*

$$H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,)) \hookrightarrow H^1(\mathbf{Q}, \mathtt{V}(\,\cdot\,))$$

is the module of global classes which are balanced at $p$ and unramified at any prime $\ell \neq p$. Set $H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) = H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

- There exists a (unique) morphism of $\mathcal{O}(U)$-modules

$$\mathcal{L} = \mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}} \colon H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{gh})) \to \mathcal{O}(U)$$

such that, for each $u \geqslant 1$ in $U^{\mathrm{cl}}$ and $\mathfrak{z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{gh}))$, one has

$$\mathcal{L}(\mathfrak{z}, u) = \frac{\left(1 - \frac{\beta_{\boldsymbol{f}_k} \alpha_{\boldsymbol{g}_u} \alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\right)}{\left(1 - \frac{\alpha_{\boldsymbol{f}_k} \beta_{\boldsymbol{g}_u} \beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\right)} \cdot \begin{cases} \frac{(-1)^{u-k/2-1}}{(u-k/2-1)!} \log_p(\mathfrak{z}_u)_f & \text{if } k < 2u, \\ (k/2-u)! \exp_p^*(\mathfrak{z}_u)_f & \text{if } k \geqslant 2u, \end{cases} \tag{50}$$

where the notations are as follows. One writes $\alpha_{\boldsymbol{f}_k}$ for the unit root of the $p$-th Hecke polynomial of $f$ and $\beta_{\boldsymbol{f}_k} = p^{k-1}/\alpha_{\boldsymbol{f}_k}$. Similarly, $\alpha_{\boldsymbol{g}_u} = b_p(u)$, $\alpha_{\boldsymbol{h}_u} = c_p(u)$, $\beta_{\boldsymbol{g}_u} = \frac{\chi_g(p) \cdot p^{u-1}}{\alpha_{\boldsymbol{g}_u}}$ and $\beta_{\boldsymbol{h}_u} = \frac{\chi_h(p) \cdot p^{u-1}}{\alpha_{\boldsymbol{h}_u}}$. The class $\mathfrak{z}_u$ is the image of $\mathfrak{z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ under the morphism induced by $\rho_u$, so that $\mathfrak{z}_u$ belongs to $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ if $u$ is in $U^{\mathrm{bal}}$ (cp. Equation (49)). One writes

$$\log_p\colon H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)^\vee$$

(where $\vee$ denotes the $L$-linear dual) and

$$\exp^*_p\colon H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)^\vee$$

for the Bloch–Kato logarithm and dual exponential, respectively, and $\log_p(\,\cdot\,)_f$ and $\exp^*_p(\,\cdot\,)_f$ for their evaluations on the class

$$\mho_u = \eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_u} \otimes \omega_{\boldsymbol{h}_u}.$$

When $u \geqslant 2$, this is the class defined in Section 3.1.3, which belongs to $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ if $u$ is balanced, i.e., $k < 2u$ (cp. Equation (14)). Moreover, in the definition of $\log_p$ and $\exp^*_p$, we identify $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ with its $L$-linear dual under the product of the $w_N$-twisted Poincaré dualities $(\,\cdot\,, w_N(\,\cdot\,))_\xi$ for $\xi$ equal to $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ (cp. Equation (11), noting that here $N$ is the tame level of the relevant modular curves).

When $u = 1$, the differential $\mho_1$ in $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is defined as above, using a suitable canonical generator $\omega_{\boldsymbol{\xi}_1}$ of $D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1))^{\varphi = \beta_\xi}$, for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$. The latter is the weight-1 specialization of a *big* differential $\omega_{\boldsymbol{\xi}}$ interpolating $\omega_{\boldsymbol{\xi}_u}$ at weight $u \geqslant 2$. Similarly, in the definition of $\log_p$ and $\exp^*_p$, we identify $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ with its dual under a suitable perfect canonical pairing $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{f}_k \boldsymbol{g}_1 \boldsymbol{h}_1}$, arising as the weight-1 specialization of a twisted Poincaré duality on $V(\boldsymbol{\xi})$. We refer to [8, Section 6.3] and its references for the details.

5.2. ***p*-adic *L*-functions and reciprocity laws.** The notations and assumptions are as in the previous section. Hida's method (cp. [23]) shows that the $p$-adic periods (cp. Section 1.1)

$$I_p(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) = I_p(f_k, g_u, h_u),$$

for $u$ in $U^{\mathrm{cl}}$, are interpolated by an analytic function $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ in $\mathcal{O}(U)$.

**Theorem 5.3.** *Shrinking $U$ if necessary, there exists a global balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$ such that*

$$\mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}}\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{gh}))\big) = \mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh}).$$

*Proof.* *Step 1.* There exist an integer $A \geqslant 0$ and, for each *balanced* point $u$ in $U^{\mathrm{bal}}$, a global cohomology class $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ in $H^1(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$, such that $p^A \cdot \kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ and

$$\log_p\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))\big)(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_u} \otimes \omega_{\boldsymbol{h}_u})$$

is equal to

$$(-1)^{u-k/2-1}(u - k/2 - 1)!\frac{(1 - \frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})}{(1 - \frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}})} \cdot I_p(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u). \tag{51}$$

*Proof of Step 1.* Denote by $\kappa_{Np}(u)$ the diagonal class of level $Np$ and weights $(k,u,u)$ (cp. Equation (4)). Let

$$\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\in H^1(\mathbf{Q}(\mu_p),\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$$

be the image of $\frac{\bar{\chi}_f(p)}{N^r}\cdot\kappa_{Np}(u)$ under the composition (cp. Equation (47))

$$\mathtt{W}_{Np}(u)\xrightarrow{w_p'\otimes\mathrm{id}\otimes\mathrm{id}}\mathtt{W}_{Np}(u)\xrightarrow{\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_u\boldsymbol{h}_u}}\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u),$$

where $w_p'$ is the dual $p$-th Atkin–Lehner endomorphism of $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}},\mathscr{S}_{k-2})$ as defined in [8, Section 2.3.1].

The image $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\otimes 1$ of $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ in $H^1(\mathbf{Q}(\mu_p),V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ is a *Selmer* class (cp. Section 2), invariant under the action of $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ (as $\boldsymbol{f}_k$ is $p$-old), hence can be identified with a class in the balanced Selmer group $H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ by Equation (49).

The explicit computations carried out in Proposition 7.3 and Lemma 7.4 of [8] prove that

$$\log_p\big(\mathrm{res}_p(\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))\big)(\eta^{\alpha}_{\boldsymbol{f}_k}\otimes\omega_{\boldsymbol{g}_u}\otimes\omega_{\boldsymbol{h}_u})$$

is equal to the product of

$$\frac{(1-\frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}})(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})}{N^r(1-\frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}})(1-\frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}})}$$

and

$$\log_p\big(\mathrm{res}_p(\kappa(f_k,g_u,h_u))\big)(\eta^{\alpha}_{f_k}\otimes\omega_{g_u}\otimes\omega_{h_u}).$$

According to the explicit reciprocity law Theorem A, this product is in turn equal to

$$(u-k/2-1)!\Big(1-\frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\Big)^{-1}\cdot I_p(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u).$$

As $\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}$ is in $p^{k/2+u-1}\mathscr{O}_L$ for $u$ in $U^{\mathrm{bal}}$, it follows that the class

$$\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)=(-1)^{u-k/2-1}\Big(1-\frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\Big)\cdot\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$$

belongs to $H^1(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ and that

$$\log_p\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))\big)(\eta^{\alpha}_{\boldsymbol{f}_k}\otimes\omega_{\boldsymbol{g}_u}\otimes\omega_{\boldsymbol{h}_u})$$

is equal to the expression displayed in Equation (51).

It remains to prove that there exists a nonnegative integer $A\geqslant 0$ such that $p^A\cdot\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ for each $u$ in $U^{\mathrm{bal}}$. Because $\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ is an $\mathscr{O}_L$-multiple of $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ and $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\otimes 1$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$, it is sufficient to exhibit a constant $A\geqslant 0$ such that $p^A$ kills the torsion subgroup of $H^1(\mathbf{Q}_p,\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh})/\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh})_{\mathrm{bal}}\otimes_u\mathscr{O}_L)$ for each balanced point $u$ in $U^{\mathrm{bal}}$.

Set $\mathtt{M}_u = \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})/\mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u \mathscr{O}_L$. There is then an exact sequence

$$0 \to \mathtt{M}_u^+ \to \mathtt{M}_u \to \mathtt{M}_u^- \to 0,$$

where (for some positive integer $a \geqslant 1$)

$$\mathtt{M}_u^+ = \mathscr{O}_L(\chi_{\mathrm{cyc}}^{k/2-u+1} \cdot \psi_{u,f})^a \oplus \mathscr{O}_L(\chi_{\mathrm{cyc}}^{1-k/2} \cdot \psi_{u,g})^a \oplus \mathscr{O}_L(\chi_{\mathrm{cyc}}^{1-k/2} \cdot \psi_{u,h})^a$$

and

$$\mathtt{M}_u^- = \mathscr{O}_L(\chi_{\mathrm{cyc}}^{2-u-k/2} \cdot \psi_u)^a,$$

and where the characters $\psi_\cdot$ are unramified and take on an arithmetic Frobenius $\sigma$ in $G_{\mathbf{Q}_p}$ the values

$$\psi_{u,f}(\sigma) = \frac{\chi_f(p)\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{\alpha_{\boldsymbol{f}_k}}, \quad \psi_{u,g}(\sigma) = \frac{\chi_g(p)\alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{h}_u}}{\alpha_{\boldsymbol{g}_u}}, \quad \psi_{u,h}(\sigma) = \frac{\chi_h(p)\alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}}{\alpha_{\boldsymbol{h}_u}}$$

and

$$\psi_u(\sigma) = \alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}.$$

It follows that the torsion subgroup of $H^1(\mathbf{Q}_p, \mathtt{M}_u)$ is killed by

$$\mu(u) = \prod_{\xi = f,g,h,\varnothing} (1 - \psi_{u,\xi}(\sigma)).$$

The values $\mu(u)$, for $u$ in $U^{\mathrm{cl}}$, are interpolated by an analytic function $\mu$ in $\Lambda$. Moreover, $\mu(1)$ is nonzero, as by assumption $p$ does not divide the conductor of $\boldsymbol{f}_k$. Shrinking $U$ if necessary, one can then assume that $\mathrm{ord}_p(\mu(u))$ equals the nonnegative integer $\mathrm{ord}_p(\mu(1))$ for all $u$ in $U$. Taking $A = e(L/\mathbf{Q}_p) \cdot \mathrm{ord}_p(\mu(1))$ concludes the proof.

*Step 2.* There exist a *finite* subset $\mathscr{E}^{\mathrm{cl}}$ of $U^{\mathrm{cl}}$ and a constant $B \geqslant 0$ satisfying the following property: For each $u$ in $\mathscr{U}^{\mathrm{cl}} = U^{\mathrm{cl}} - \mathscr{E}^{\mathrm{cl}}$, the isomorphism $\rho_u$ (cp. Equation (48)) induces a short exact sequence of $\mathscr{O}_L$-modules

$$0 \to H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})) \otimes_u \mathscr{O}_L \to H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to \mathtt{Err}_u \to 0,$$

where $\mathtt{Err}_u$ is a finite $\mathscr{O}_L$-module killed by $p^B$.

*Proof of Step 2.* This follows from the general base-change results for Selmer complexes proved in [32, 37].

*Step 3.* One has $\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{gh}))) = \mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ for a balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$.

*Proof of Step 3.* The statement is clear if $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ is zero. Assume that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ is nonzero and let $e_p$ be its order of vanishing at $\boldsymbol{u} = 1$. As $\mathcal{O}(U)$ is a principal ideal domain, the image of $\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}$ is a principal ideal, say generated by an analytic function $\mathscr{G}_{\mathrm{bal}}$ with order of vanishing $e_{\mathrm{bal}}$ at $\boldsymbol{u} = 1$. (By convention $e_{\mathrm{bal}} = +\infty$ if $\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}$ is the zero map.) According to the Weierstraß preparation theorem, shrinking $U$ if necessary one can assume that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh}) = (\boldsymbol{u} - 1)^{e_p} \cdot \mathscr{L}_p^*$ and $\mathscr{G}_{\mathrm{bal}} = (\boldsymbol{u} - 1)^{e_{\mathrm{bal}}} \cdot \mathscr{G}_{\mathrm{bal}}^*$, with $\mathscr{L}_p^*$ and $\mathscr{G}_{\mathrm{bal}}^*$

units in $\mathcal{O}(U)$ (and $(\boldsymbol{u}-1)^{e_{\rm bal}}$ equal to zero if $e_{\rm bal}=+\infty$). In order to prove the theorem, it is sufficient to show that

$$e_{\rm bal} \leqslant e_p. \tag{52}$$

Let $\mathscr{U}^{\rm cl}$ be as in Step 2. Without loss of generality, assume that $\mathscr{U}^{\rm cl}$ is contained in $U^{\rm bal}$ and that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ does not vanish at any point of $\mathscr{U}^{\rm cl}$. Let $A$ and $B$ be the constants which appear in Steps 1 and 2. Take $C \geqslant A+B$ such that $\|\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}(\mathfrak{z})\|_U \leqslant p^C$ for any class $\mathfrak{z}$ in $H^1_{\rm bal}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}))$. (This is possible since $H^1_{\rm bal}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}))$ is a finitely generated $\Lambda$-module.) According to Steps 1 and 2, for each $u$ in $\mathscr{U}^{\rm cl}$, there exists a global balanced class

$$\tilde{\kappa}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) \in H^1_{\rm bal}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}))$$

such that (cp. Equations (50) and (51))

$$\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}\big(\mathrm{res}_p\big(\tilde{\kappa}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)\big), u\big) = p^C \cdot \mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})(u) \neq 0. \tag{53}$$

In particular, $\mathscr{G}_{\rm bal}$ is nonzero, hence $e_{\rm bal}$ is a nonnegative integer.

Let $\{u_j\}_{j\geqslant 1}$ be a sequence in $\mathscr{U}^{\rm cl}$ which converges to 1. For each $j \geqslant 1$, define $\gamma_j \in \mathcal{O}(U)$ by the equation

$$\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}\big(\mathrm{res}_p\big(\tilde{\kappa}(\boldsymbol{f}_k, \boldsymbol{g}_{u_j}, \boldsymbol{h}_{u_j})\big)\big) = \gamma_j \cdot \mathscr{G}_{\rm bal}.$$

Because $\|\gamma_j \cdot \mathscr{G}_{\rm bal}\|_U \leqslant p^C$ for any $j \geqslant 1$, the sequence $\|\gamma_j\|_U$ is bounded, say by $p^D$ for some $D \geqslant 0$. Equation (53) and the Weierstraß preparation theorem show that for $j \gg 0$,

$$p^{-C}\xi_j \cdot |u_j - 1|_p^{e_p} = |\gamma_j(u_j)|_p \cdot |u_j - 1|_p^{e_{\rm bal}} \leqslant p^D \cdot |u_j - 1|_p^{e_{\rm bal}},$$

where $\{\xi_j\}_{j\gg 0}$ converges to the *positive* rational number $|\mathscr{L}_p^*(1)|_p / |\mathscr{G}^*_{\rm bal}(1)|_p$. Equation (52) follows. ☐

5.4. **Conclusion of the proof.** This section concludes the proof of Theorem B. Write $H^1_{\rm rel}(\mathbf{Q}, V(f,g,h))$ for the *relaxed* Selmer group of $V(f,g,h)$ over $\mathbf{Q}$, that is the set of global classes in $H^1(\mathbf{Q}, V(f,g,h))$ which are unramified at every rational prime $\ell \neq p$. Let $\boldsymbol{g}^\sharp = \boldsymbol{g}^\sharp_\alpha$, $\boldsymbol{h}^\sharp = \boldsymbol{h}^\sharp_\alpha$, $\boldsymbol{g}$ and $\boldsymbol{h}$ be as in the previous sections.

Let $\xi$ denote either $g$ or $h$ and let $\mathrm{Frob}_p$ be an arithmetic Frobenius in $G_{\mathbf{Q}_p}$. By Assumption 1.3, the restriction to $G_{\mathbf{Q}_p}$ of the Artin representation $V(\xi)$ is unramified and splits as the direct sum of the (distinct) $\mathrm{Frob}_p$-eigenspaces

$$V(\xi)_\alpha = V(\xi)^{\mathrm{Frob}_p = \alpha_\xi/\chi_\xi(p)} \quad \text{and} \quad V(\xi)_\beta = V(\xi)^{\mathrm{Frob}_p = \beta_\xi/\chi_\xi(p)}.$$

As a consequence, the $G_{\mathbf{Q}_p}$-representation $V(f,g,h)$ decomposes as

$$V(f,g,h) = V(f)_{\alpha\alpha} \oplus V(f)_{\alpha\beta} \oplus V(f)_{\beta\alpha} \oplus V(f)_{\beta\beta}, \tag{54}$$

where $V(f)_{ij} = V(f) \otimes_L V(g)_i \otimes V(h)_j \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(k/2)$. Similarly, for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, one has $V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}_1)_\alpha \oplus V(\boldsymbol{\xi}_1)_\beta$ and $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) = \bigoplus_{ij} V(\boldsymbol{f}_k)_{ij}$.

For each $p$-adic representation $V$ of $G_{\mathbf{Q}_p}$, let $V^+$ be the submodule on which the inertia subgroup of $G_{\mathbf{Q}_p}$ acts via the $k/2$-th power of the $p$-adic cyclotomic character and set $V^- = V/V^+$. A class in $H^1_{\rm rel}(\mathbf{Q}, V(f,g,h))$ belongs to the

Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ precisely if its restriction at $p$ is in the kernel of

$$p^- : H^1(\mathbf{Q}_p, V(f,g,h)) \to H^1(\mathbf{Q}_p, V(f,g,h)^-), \tag{55}$$

and belongs to the balanced Selmer group $H^1_{\mathrm{bal}}(\mathbf{Q}, V(f,g,h))$ precisely if its restriction at $p$ is in the kernel of the natural map

$$\begin{aligned} H^1(\mathbf{Q}_p, V(f,g,h) \to H^1(\mathbf{Q}_p, V(f)^-_{\alpha\beta}) \oplus H^1(\mathbf{Q}_p, V(f)^-_{\beta\alpha}) \\ \oplus H^1(\mathbf{Q}_p, V(f)_{\alpha\alpha}) \end{aligned} \tag{56}$$

(where $V(f)^-_{\cdot}$ is a shorthand for $(V(f)_{\cdot})^-$). A similar discussion applies with $(f,g,h)$ replaced by $(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ everywhere. After these preliminaries, we can begin the actual proof of Theorem B, which is divided in three steps.

*Step 1.* There exist level-$N$ test vectors $(\boldsymbol{f}_k, \boldsymbol{g}, \boldsymbol{h})$ for $(f, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ and a *nonzero* scalar $\mathcal{E}$ in $L^*$ such that

$$\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})(1) = \mathcal{E} \cdot \frac{L(f \otimes g \otimes h, k/2)}{\pi^{2k-2}(f,f)_N}.$$

*Proof.* Under the running Assumption 1.3, this follows by the special value formulas proved by Garrett and Harris–Kudla [20, 21] (cp. [14, Section 4]). □

*Step 2.* Assume that $L(f \otimes g \otimes h, s)$ does not vanish at $s = k/2$. Then there exists a global class $\kappa(f,g,h)_{\alpha\alpha}$ in the relaxed Selmer group $H^1_{\mathrm{rel}}(\mathbf{Q}, V(f,g,h))$ such that (cp. Equations (54) and (55))

$$p^-\big(\mathrm{res}_p(\kappa(f,g,h)_{\alpha\alpha})\big) \quad \text{is a } \textit{nonzero} \text{ element in } H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta}).$$

*Proof.* Step 1 implies that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ does not vanish at $\boldsymbol{u} = 1$ for some triple of level-$N$ test vectors $(\boldsymbol{f}_k, \boldsymbol{g}, \boldsymbol{h})$. Theorem 5.3 then yields a global balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$ such that

$$\exp_p^*\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))\big)(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}) \neq 0. \tag{57}$$

Here $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the image of $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))$ under the morphism induced in cohomology by $\rho_1$ (cp. Equation (48)) and one uses Assumption 1.3.2 to guarantee that the Euler factors which appear in Equation (50) are nonzero.

The projection $p^-$ induces a canonical isomorphism

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) \cong D_{\mathrm{cris}}(V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-),$$

which we consider as an equality. Then $\exp_p^*$ is equal to the composition

$$\begin{aligned} H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)) \xrightarrow{p^-} H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_2)^-) \\ \xrightarrow{\exp^*} D_{\mathrm{cris}}(V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-), \end{aligned}$$

where $\exp^*$ is the dual exponential for $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-$. Similarly, the inclusion $V(\boldsymbol{f}_k)(k/2)^+ \to V(\boldsymbol{f}_k)(k/2)$ induces a natural isomorphism

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k)(k/2)^+) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)^{\varphi=\alpha_f} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[k/2].$$

After recalling that $\omega_{\boldsymbol{\xi}_1}$, for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, is a nonzero element of

$$D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1))^{\varphi=\beta_{\boldsymbol{\xi}_1}} = D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1)_\alpha),$$

we can then identify $\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}$ with an element $\mho_1$ of the crystalline Dieudonné module of the direct summand $V(\boldsymbol{f}_k)(k/2)^+ \otimes_L V(\boldsymbol{g}_1)_\alpha \otimes_L V(\boldsymbol{h}_1)_\alpha$ of $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^+$. Equation (57) can then be rewritten as

$$\exp^*\big(\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}\big)(\mho_1) \neq 0,$$

where $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}$ is the $\beta\beta$-component of

$$\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) = p^-\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))\big).$$

On the other hand, since $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the specialization of a *balanced* class, it follows that $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) = \kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}$ belongs to $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k)^-_{\beta\beta})$ (cp. the discussion around Equation (56)). In particular, $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is an element of the relaxed Selmer group $H^1_{\mathrm{rel}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))$ such that $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is a nonzero element of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k)^-_{\beta\beta})$. Because the $G_{\mathbf{Q}}$-representation $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the direct sum of a finite number of copies of $V(f, g, h)$, the statement follows. □

*Step 3.* Set $V = V(f, g, h)$. Then there is an exact sequence of $L$-modules

$$0 \to \mathrm{Sel}(\mathbf{Q}, V) \to H^1_{\mathrm{rel}}(\mathbf{Q}, V) \xrightarrow{\partial} H^1(\mathbf{Q}, V^-) \to \mathrm{Sel}(\mathbf{Q}, V)^{dual} \to H^1_{\mathrm{str}}(\mathbf{Q}, V)^{dual} \to 0,$$

where $\partial$ is the composition of $p^-$ and $\mathrm{res}_p$ and $\cdot^{dual}$ denotes the $L$-linear dual.

*Proof.* As $V$ is Kummer self-dual, this is an instance of global Poitou–Tate duality (cp. [30, Ch. 1]). □

Varying the choices of the roots $\alpha_g$ and $\alpha_h$ (cp. Assumption 1.3.3), Step 2 yields four classes (namely, $\kappa(f, g, h)_\cdot$ for $\cdot$ in $\{\alpha, \beta\}^2$) in $H^1_{\mathrm{rel}}(\mathbf{Q}, V)$, whose images under the morphism $\partial$ are linearly independent over $L$. Theorem B then follows from Step 3, after noting that $H^1(\mathbf{Q}_p, V^-)$ has dimension four over $L$ under Assumption 1.3.2.

## REFERENCES


[1] K. Bannai, Syntomic cohomology as a $p$-adic absolute Hodge cohomology, Math. Z. **242** (2002), no. 3, 443–480. MR1985460

[2] K. Bannai and G. Kings, $p$-adic elliptic polylogarithm, $p$-adic Eisenstein series and Katz measure, Amer. J. Math. **132** (2010), no. 6, 1609–1654. MR2766179

[3] P. Berthelot, Finitude et pureté cohomologique en cohomologie rigide, Invent. Math. **128** (1997), no. 2, 329–377. MR1440308

[4] M. Bertolini and H. Darmon, Kato's Euler system and rational points on elliptic curves I: A $p$-adic Beilinson formula, Israel J. Math. **199** (2014), no. 1, 163–188. MR3219532

[5] M. Bertolini, H. Darmon, and K. Prasanna, Generalized Heegner cycles and $p$-adic Rankin $L$-series, Duke Math. J. **162** (2013), no. 6, 1033–1148. MR3053566

[6] M. Bertolini, H. Darmon, and V. Rotger, Beilinson–Flach elements and Euler systems I: Syntomic regulators and $p$-adic Rankin $L$-series, J. Algebraic Geom. **24** (2015), no. 2, 355–378. MR3311587

[7] M. Bertolini, H. Darmon, and V. Rotger, Beilinson–Flach elements and Euler systems II: the Birch–Swinnerton-Dyer conjecture for Hasse–Weil–Artin $L$-series, J. Algebraic Geom. **24** (2015), no. 3, 569–604. MR3344765

[8] M. Bertolini, M. A. Seveso, and R. Venerucci, Reciprocity laws for balanced diagonal classes, preprint (2018).

[9] M. Bertolini, M. A. Seveso, and R. Venerucci, Balanced diagonal classes and rational points on elliptic curves, preprint (2019).

[10] A. Besser, A generalization of Coleman's $p$-adic integration theory, Invent. Math. **142** (2000), no. 2, 397–434. MR1794067

[11] A. Besser, Syntomic regulators and $p$-adic integration. I. Rigid syntomic regulators, Israel J. Math. **120** (2000), part B, 291–334. MR1809626

[12] A. Besser, On the syntomic regulator for $K_1$ of a surface, Israel J. Math. **190** (2012), 29–66. MR2956231

[13] R. F. Coleman, Classical and overconvergent modular forms, Invent. Math. **124** (1996), no. 1-3, 215–241. MR1369416

[14] H. Darmon and V. Rotger, Diagonal cycles and Euler systems I: A $p$-adic Gross–Zagier formula, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 779–832. MR3250064

[15] H. Darmon and V. Rotger, Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin $L$-functions, J. Amer. Math. Soc. **30** (2017), no. 3, 601–672. MR3630084

[16] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970. MR0417174

[17] P. Deligne, Formes modulaires et représentations $l$-adiques, in *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, Exp. 355, 139–172, Lecture Notes in Math., 175, Springer, Berlin, 1971. MR3077124

[18] G. Faltings, $p$-adic Hodge theory, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299. MR0924705

[19] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, translated from the German by Betty S. Waterhouse and William C. Waterhouse, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 13, Springer-Verlag, Berlin, 1988. MR0926276

[20] P. B. Garrett, Decomposition of Eisenstein series: Rankin triple products, Ann. of Math. (2) **125** (1987), no. 2, 209–235. MR0881269

[21] M. Harris and S. S. Kudla, The central critical value of a triple product $L$-function, Ann. of Math. (2) **133** (1991), no. 3, 605–672. MR1109355

[22] H. Hida, A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms. I, Invent. Math. **79** (1985), no. 1, 159–195. MR0774534

[23] M.-L. Hsieh, Hida families and $p$-adic triple product $L$-functions, preprint (2017), available at `www.math.sinica.edu.tw/mlhsieh/research.htm`.

[24] U. Jannsen, Continuous étale cohomology, Math. Ann. **280** (1988), no. 2, 207–245. MR0929536

[25] N. M. Katz, $p$-adic properties of modular schemes and modular forms, in *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, 69–190. Lecture Notes in Mathematics, 350, Springer, Berlin, 1973. MR0447119

[26] G. Kings, D. Loeffler, and S. L. Zerbes, Rankin–Eisenstein classes and explicit reciprocity laws, to appear in: Amer. J. Math., available at `arxiv.org/abs/1501.03289` (2015).

[27] G. Kings, D. Loeffler, and S. L. Zerbes, Rankin–Eisenstein classes and explicit reciprocity laws, Camb. J. Math. **5** (2017), no. 1, 1–122. MR3637653

[28] A. Lei, D. Loeffler, and S. L. Zerbes, Euler systems for Rankin–Selberg convolutions of modular forms, Ann. of Math. (2) **180** (2014), no. 2, 653–771. MR3224721

[29] D. Loeffler, C. Skinner, and S. Livia Zerbes, Syntomic regulators of Asai–Flach classes, arXiv:1608.06112v2 (2017).

[30] J. S. Milne, *Arithmetic duality theorems*, second edition, BookSurge, LLC, Charleston, SC, 2006. MR2261462

[31] J. Nekovář, Syntomic cohomology and $p$-adic regulators, preprint (2004), available at `webusers.imj-prg.fr/~jan.nekovar/pu/syn.pdf`.

[32] J. Nekovář, Selmer complexes, Astérisque No. 310 (2006), viii+559 pp. MR2333680

[33] J. Nekovář and W. Nizioł, Syntomic cohomology and $p$-adic regulators for varieties over $p$-adic fields, Algebra Number Theory **10** (2016), no. 8, 1695–1790. MR3556797

[34] M. Niklas, *Rigid syntomic regulators and the $p$-adic $L$-function of a modular form*, PhD Thesis Regensburg, 2010.

[35] W. Nizioł, On the image of $p$-adic regulators, Invent. Math. **127** (1997), no. 2, 375–400. MR1427624

[36] W. Nizioł, Cohomology of crystalline smooth sheaves, Compositio Math. **129** (2001), no. 2, 123–147. MR1863299

[37] J. Pottharst, Analytic families of finite-slope Selmer groups, Algebra Number Theory **7** (2013), no. 7, 1571–1612. MR3117501

[38] T. Saito, Modular forms and $p$-adic Hodge theory, Invent. Math. **129** (1997), no. 3, 607–620. MR1465337

[39] A. J. Scholl, Modular forms and de Rham cohomology; Atkin–Swinnerton-Dyer congruences, Invent. Math. **79** (1985), no. 1, 49–77. MR0774529

[40] T. Tsuji, $p$-adic étale cohomology and crystalline cohomology in the semi-stable reduction case, Invent. Math. **137** (1999), no. 2, 233–411. MR1705837

[41] E. Urban, Nearly overconvergent modular forms, in *Iwasawa theory 2012*, 401–441, Contrib. Math. Comput. Sci., 7, Springer, Heidelberg, 2014. MR3586822

[42] R. Venerucci, Exceptional zero formulae and a conjecture of Perrin–Riou, Invent. Math. **203** (2016), no. 3, 923–972. MR3461369

[43] A. Wiles, On ordinary $\lambda$-adic representations associated to modular forms, Invent. Math. **94** (1988), no. 3, 529–573. MR0969243


Received February 24, 2919, accepted June 17, 2019

Massimo Bertolini
Universität Duisburg-Essen, Essen, Germany
E-mail: `massimo.bertolini@uni-due.de`

Marco Seveso
Università degli Studi di Milano, Milano, Italy
E-mail: `seveso.marco@gmail.com`

Rodolfo Venerucci
Universität Duisburg-Essen, Essen, Germany
E-mail: `rodolfo.venerucci@uni-due.de`