The geometry of differential constraints for a class of evolution PDEs

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Abstract

The problem of computing differential constraints for a family of evolution PDEs is discussed from a constructive point of view. A new method, based on the existence of generalized characteristics for evolution vector fields, is proposed in order to obtain explicit differential constraints for PDEs belonging to this family. Several examples, with applications in non-linear stochastic filtering theory, stochastic perturbation of soliton equations and non-isospectral integrable systems, are discussed in detail to verify the effectiveness of the method.

1 Introduction

The method of differential constraints is a well known and general method for determining particular explicit solutions to a partial differential equation (PDE) reducing the PDE to a system of ordinary differential equations (ODEs) through a suitable ansatz on the form of the solution. In particular, given a system of evolution equations of the form $\partial_t(u^k) = F^k(x,t,u,u_\sigma)$, where $(x^i, u^k) \in M \times N$ and u^k_σ are the derivatives of u^k with respect to x^j the number of times defined by the multi-index σ, we can look for solutions of the form

$$u(x,t) = K(x, w^1(z), ..., w^L(z)), \tag{1}$$

where $K : M \times \mathbb{R}^L \to N$ and $z : M \times \mathbb{R} \to \mathbb{R}$ are smooth functions. Replacing this ansatz in the initial evolution equation, we may obtain a system of ODEs for the functions w^i with respect to the variable z. We remark that, for general functions K and z, the system of ODEs for w^i is overdetermined and has no solutions. When the system for the functions w^i admits solutions, the ansatz (1) is said compatible with the equation and K is called a *differential constraint* for $\partial_t(u^k) = F^k(x,t,u,u_\sigma)$. This method appears with different names in several papers and books (see, e.g., [12, 23, 31, 35, 37, 39, 41]) and is equivalent to append to the original equation a suitable overdetermined system of PDEs of the form $\mathcal{I} = \{I_1(x,t,u,u_\sigma) = 0, ..., I_K(x,t,u,u_\sigma) = 0\}$. Indeed, if $\mathcal{I}$ admits a finite dimensional solution, this can be described by a function of the form (1) and the requirement that $\mathcal{I}$ and $\partial_t(u^k) = F^k(x,t,u,u_\sigma)$ have common solutions can be interpreted as the compatibility condition for the ansatz. In this situation the system $\mathcal{I}$ is called differential constraint as well and, in order to distinguish between the two approaches, some Authors refer to

direct differential constraints for the formulation with the function K and to *indirect differential constraints* when the system $\mathcal{I}$ is considered (see, e.g. [39, 41]).
In general, checking that the ansatz (1) or the overdetermined system $\mathcal{I}$ are compatible with the evolution equations $\partial_t(u^k) = F^k(x,t,u,u_\sigma)$ is a difficult task. Even in the simplest case $M = \mathbb{R}$, in order to verify that $\mathcal{I}$ is a differential constraint, we have to solve a system of strongly non-linear PDEs for the unknown functions I_k. For this reason many Authors look for differential constraints imposing some restrictions on the form of the functions K and z, or, equivalently, on the form of the constraints I_k or, finally, on the form of the equation $\partial_t(u^k) = F^k(x,t,u,u_\sigma)$.

In this paper we are interested in evolution equations with a particular form of the functions F^k and we make suitable assumptions on the form of the differential constraints. In particular we deal with the problem of finding differential constraints for evolution PDEs of the form

$$\partial_t(u^k) = \sum_{i=1}^{s} c^i(t) F_i^k(x,u,u_\sigma), \tag{2}$$

where the functions F_i^k do not depend on t. Furthermore we chose the new variable $z(x,t) = t$ and we look for differential constraints that are independent of any possible choice of the smooth functions $c^i(t)$.
The choice of equations of the form (2) is not a matter of computational convenience, but is triggered by many theoretical and applied problems arising in different branches of mathematics.

First of all, evolution equations of the form (2) appear in the theory of stochastic processes and in particular in the study of finite dimensional solutions to stochastic partial differential equations (SPDEs). More precisely, the problem of finding finite dimensional solutions to SPDEs can be reduced to the problem of finding differential constraints for evolution equations of the form (2) for any choice of $c^k(t)$, since the functions $c^k(t)$ can be seen as the derivatives of some stochastic process which can assume any possible values. The given justification is only formal, since the typical processes considered in stochastic analysis, e.g. the Brownian motions, are not derivable.

The research of finite dimensional solutions to SPDEs has many applications: the first one is in the stochastic filtering theory, where the main object is a linear equation of the form (2) called Zakai equation (see [3]). Looking for finite dimensional solutions to Zakai equation, and so for finite dimensional filter, is (formally) equivalent to search finite dimensional differential constraints for (2). We want to point out that the relation between the Lie algebra generated by the operators F_i and the existence of finite dimensional filter is not new, but can be found in the classical literature on the subject (see [4, 10, 49]). Indeed the necessary conditions obtained in Proposition 3.10 are, in the case of Zakai equation, equivalent to the conditions obtained for the existence of finite dimensional filters. Finally we note that our theory provides an extension to the general non-linear case of the methods used in [13, 14] for studying Zakai equation.
A second application of finite dimensional solutions to SDPEs of the form (2) is to the case of the Heath-Jarrow-Morton (HJM) equation appearing in the study of interest rate in mathematical finance (see [28, 8, 20]). Indeed, our approach is deeply inspired by the works of Filipovic, Tappe and Teichmann about finite dimensional solutions to HJM equation (see [19, 21, 22, 45]). In particular, Theorem 5.5 and Theorem 5.7 are reformulations of [22], where the use of the convenient setting of global analysis ([34]) is replaced by the infinite jet bundle geometry and the characteristics of subsection 4.2.
A third and final application is to the study of stochastic soliton equations. In addition to the pioneering work of Wadati on the KdV stochastic equation preserving soliton solutions (see [46, 48]),

we have also been inspired by the growing recent interest in the study of variational stochastic systems of hydrodynamic type (see e.g. [2, 29, 16]). In particular it is important to mention [30], where Holm and Tyranowski found many families of finite dimensional soliton type solutions to a physically important stochastic perturbation of Camassa-Holm equation.

We remark that the reduction of all the previous stochastic problems to their deterministic counterparts is a completely probabilistic matter. On the other hand, the methods proposed in this paper in order to find differential constraints for equations of the form (2) are completely geometric. For this reason we postpone the applications of our results to stochastic problems to a following paper.

Equations of the form (2) have also interesting applications to integrable system theory, and our results turn out to be useful for dealing with non-isospectral deformations of integrable systems (see [6, 11, 25]). In fact the local non-isospectral deformation of KdV hierarchy can be reduced to the usual isospectral KdV hierarchy by means of a time dependent transformation (see e.g. [26] and the examples of Subsections 6.2 and 6.5).
Finally, evolution equations of the form (2) can be considered in infinite dimensional control theory (see [5, 36]). In this framework our method can be interpreted as an application of usual methods of geometric control theory to the explicit computation of the reachable sets of some particular point (see [1] for the finite dimensional case and [32] for the infinite dimensional one).
It is worth to remark that, although the problem of finding differential constraints for equations of the form (2) has been faced many times, to the best of our knowledge this is the first time that the abstract form (2) of the problem has been recognized (since the previous Authors consider particular forms of the functions F^i_k) and that the problem has been tackled by using the geometrical framework of differential constraints for PDEs. In particular, the use of the differential constraints method permits to consider, from a theoretical point of view, a very general form for the functions F^k_i and allows us to obtain a useful algorithm for the explicit computation of the solutions to equations (2) (see Subsection 6.1).

On the other hand, the problem of finding differential constraints for PDEs of the form (2) is an interesting challenge in itself. In order to address this issue we provide a geometrical framework for the description of differential constraints method which allows us to simplify the formulation of the problem. In particular, we associate with any autonomous evolution equation of the form $\partial_t(u^k) = F^k_i(x, u, u_\sigma)$ an evolution vector field V_{F_i} on the space of infinite jets $J^\infty(M, N)$ of the functions from M into N. In this setting the time independent overdetermined system $\mathcal{I}$, or equivalently the time independent function K, can be described as a particular finite dimensional submanifold $\mathcal{K}$ of $J^\infty(M, N)$ and we prove that $\mathcal{I}$ is a differential constraint for the system (2) for any $c^i(t)$ if and only if $V_{F_i} \in T\mathcal{K}$ $\forall i = 1, \dots s$.
This geometrical reformulation gives new insight into the problem of finding differential constraints. First of all it provides a general and powerful method for dealing with many different evolution equations which have been previously faced with different techniques and in different frameworks. The second important result following by our general approach is the derivation of some necessary conditions for the existence of differential constraints for systems of the form (2). These conditions form an infinite dimensional analogue of necessary condition of the well-known Frobenius theorem. Indeed the existence of a differential constraint for (2) ensures that the functions F_i, restricted on some subset $\mathcal{K}$ of $J^\infty(M, N)$, form a module with respect to the Lie brackets $[F_i, F_j]$ induced by the vector fields Lie brackets $[V_{F_i}, V_{F_j}]$. This imposes severe conditions on F_i: in particular, if $\mathcal{K} = J^\infty(M, N)$, F_i have to be generators of a Lie algebra on $J^\infty(M, N)$.
Moreover, in order to obtain a sufficient condition for the existence of differential constraints for systems of the form (2), we introduce the notion of characteristic flow for a general evolution vector

field. In particular this definition, generalizing to higher order the standard notion of characteristic of a first order scalar evolution equation, is an infinity dimensional analogue of the flow of a vector field in the finite dimensional framework. Thereafter we divide the functions F_i in two sets H_i and G_j so that the functions G_j form a Lie algebra and their evolution vector fields V_{G_j} admit generalized characteristic flow. Under these assumptions we prove that, if the vector fields V_{H_i} admit a differential constraint $\mathcal{H}$, then the complete set of V_{F_i} admits a differential constraint $\mathcal{K}$ that can be explicitly computed starting from $\mathcal{H}$ and using the characteristic flows of G_i. In addition we provide a generalization of this theorem to the case of H_i, G_i forming a finite dimensional Lie algebra on a real analytic submanifold $\mathcal{H}$ of $J^\infty(M,N)$ which is also a differential constraint for V_{H_i}.
The proofs of both these results are constructive so that we can compute explicitly the differential constraints in many interesting examples. In particular some of the examples have been chosen in order to show the flexibility and the effectiveness of our geometrical approach with respect to the standard differential constraints method (see Subsections 6.2, 6.4 and also [17] for other examples on the same topics). Other examples instead have been proposed for their relevance in applied mathematical problems such as non-linear stochastic filtering theory (see Subsection 6.3), stochastic perturbation of integrable equations (see Subsections 6.4 and 6.5) and non-isospectral deformation of integrable systems (see Subsection 6.2 and 6.5).

Finally, we would like to specify that the aim of the paper is not finding differential constraints for a large class of equations (this kind of problem has been better faced in many other paper and books on differential constraints) but using differential constraints in order to solve a family of problems which naturally arise in stochastic analysis and in other applications involving equations of the form (2). We think that the infinite jet bundle geometry provides a natural geometrical framework, strongly inspired by the literature on finite dimensional solutions to SPDEs, for the study of equation (2). Indeed, statements similar to Theorem 5.5 and Theorem 5.7 have been proved in [22] by means of infinite dimensional methods similar to those used here.
Since the theory of differential constraints provides an effective algorithm to explicitly solve equations of the form (2), the reader mainly interested in finding explicit solutions to (2) can skip all the theoretical part and go directly to Section 6.

The paper is organized as follows: after recalling some basic facts on the geometry of $J^\infty(M,N)$ in Section 2, in Section 3 we provide a geometric characterization of differential constraints for systems of evolution PDEs of the form (2). Hence, in Section 4, we discuss the problem of characteristics in $J^\infty(M,N)$ and in Section 5 we apply previous results to the explicit construction of differential constraints (or reduction functions) for evolution PDEs of the form (2). Finally, in Section 6, we apply the results of the previous sections to several explicit examples.

2 Preliminaries

In this section we collect some basic facts about (infinite) jet bundles in order to provide the necessary geometric tools for our aims.

Given the trivial fiber bundle $M\times N \to M$, where M,N are open sets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, we denote by x^i the cartesian coordinate system of M and by x^i, u^j the cartesian coordinate system of $M\times N$. The coordinates x^i, u^j induce a global coordinate system x^i, u^j, u^j_σ on the k-order jet bundle $J^k(M,N)$, where $\sigma=(\sigma_1,...,\sigma_m)\in\mathbb{N}_0^m, |\sigma|\leq k$ is a multi-index denoting the number σ_l of derivatives of u^j with respect x^l.

It is well known that $J^k(M,N)$ admits a natural structure of finite-dimensional smooth vector bundle on M and, considering the natural projections $\pi_{k,h} : J^k(M,N) \to J^h(M,N)$, it is possible to define the inverse limit $J^\infty(M,N)$ of the sequence

$$M \stackrel{\pi_0}{\leftarrow} M \times N = J^0(M,N) \stackrel{\pi_{1,0}}{\leftarrow} J^1(M,N) \stackrel{\pi_{2,1}}{\leftarrow} ... \stackrel{\pi_{k,k-1}}{\leftarrow} J^k(M,N) \stackrel{\pi_{k+1,k}}{\leftarrow} ...$$

Unfortunately the space $J^\infty(M,N)$ is not a finite-dimensional manifold, being the inverse limit of a sequence of spaces of increasing dimension. From a topological point of view $J^\infty(M,N)$ is a Fréchet manifold modeled on $\mathbb{R}^\infty$ (see [43]) and any open set U of $J^\infty(M,N)$ contains a set of the form $U' = \pi_k^{-1}(V)$ for some $k \in \mathbb{N}$ and some open set $V \subset J^k(M,N)$. This means that, for any σ with $|\sigma| > k$, the coordinates u^i_σ vary in the whole $\mathbb{R}$.
Furthermore, the differential structure of $J^k(M,N)$ induces a natural differential structure in $J^\infty(M,N)$ (see [33, 9] for a complete description).
Hereafter we denote by $\mathcal{F}_k$ the algebra of real-valued smooth functions defined on $J^k(M,N)$ and we deduce the differential structure of $J^\infty(M,N)$ from the geometric smooth algebra $\mathcal{F}$ defined as the direct limit of the sequence

$$C^\infty(M) \stackrel{\pi_0^*}{\to} \mathcal{F}_0 \stackrel{\pi_{1,0}^*}{\to} ... \stackrel{\pi_{k,k-1}^*}{\to} \mathcal{F}_k \stackrel{\pi_{k+1,k}^*}{\to} ...$$

Let $\mathcal{G} \subset \mathcal{F}$ be a finitely generated subalgebra of $\mathcal{F}$, which means that there are a finite number of functions $g_1, ..., g_l \in \mathcal{G}$ such that any $g \in \mathcal{G}$ is of the form $g = G(g_1, ..., g_l)$ for a unique smooth function G. It is possible to associate with $\mathcal{G}$ in a unique way a finite dimensional manifold $M_\mathcal{G}$ (see [38]). For this reason in the following we identify the subalgebra $\mathcal{G}$ with the manifold $M_\mathcal{G}$ such that $\mathcal{G} = C^\infty(M_\mathcal{G})$. The inclusion $i : \mathcal{G} \to \mathcal{F}$ induces a unique projection $\tilde{\pi} : J^\infty(M,N) \to M_\mathcal{G}$ such that $\tilde{\pi}^* = i$.
The algebra $\mathcal{F}$ is a graded algebra and a vector field X on $J^\infty(M,N)$ is a derivation on the space $\mathcal{F}$ which respects the order.
It is well known that the Cartan distribution $\mathcal{C}$ on $J^\infty(M,N)$ generated by the vector fields

$$D_i = \partial_{x^i} + \sum_{k,\sigma} u^k_{\sigma+1_i} \partial_{u^k_\sigma}$$

defines an integrable connection on $J^\infty(M,N)$. Hence, for any vector field X on $J^\infty(M,N)$, we can write

$$X = X_v + X_h,$$

where $X_h \in \mathcal{C}$ and X_v is a vertical vector field i.e. $X_v(x^i) = 0$ for any $i = 1, ..., m$.
A vector field X on $J^\infty(M,N)$ is a symmetry of the Cartan distribution if $[X, \mathcal{C}] \subset \mathcal{C}$. We remark that if $X = X_v + X_h$ is a symmetry of $\mathcal{C}$ then also X_h and X_v are symmetries of $\mathcal{C}$.

Definition 2.1 *A vertical vector field X that is a symmetry of $\mathcal{C}$ is called* evolution vector field. *If X is an evolution vector field there exists a unique smooth function $F : J^\infty(M,N) \to \mathbb{R}^n$ such that*

$$X = \sum_{i,\sigma} D^\sigma(f^i) \partial_{u^i_\sigma}, \tag{3}$$

where $F = (f^1, ..., f^n)$ and $D^\sigma = (D_1)^{\sigma_1} ... (D_m)^{\sigma_m}$. We call F the generator of the evolution vector field X *and we write $X = V_F$.*

If V_F and V_G are two evolution vector fields, then $[V_F, V_G]$ is also an evolution vector field and there exists an unique $H \in \mathcal{F}^n = \mathcal{F} \times ... \times \mathcal{F}$ such that $[V_F, V_G] = V_H$. Therefore the commutator between evolution vector fields induces a commutator in $\mathcal{F}^n$ and we define $[F, G] = H$ when

$[V_F, V_G] = V_H$.

We conclude this section recalling that a subset $\mathcal{E}$ of $J^\infty(M,N)$ is a submanifold of $J^\infty(M,N)$ if for any $p \in \mathcal{E}$ there exists a neighborhood U_p of p such that $\pi_h(\mathcal{E} \cap U_p)$ is a submanifold of $J^h(M,N)$ for $h > H_p$.
If, for any $p \in \mathcal{E}$, all the submanifolds $\pi_h(\mathcal{E} \cap U_p)$ with $h > H_p$ have the same dimension L, we say that $\mathcal{E}$ is an L-dimensional submanifold of $J^\infty(M,N)$. In particular, given an L-dimensional manifold B and a smooth immersion $K : B \to J^\infty(M,N)$, for any point $y \in B$ there exists a neighborhood V of p such that $K(V)$ is a finite dimensional submanifold of $J^\infty(M,N)$.

Definition 2.2 *A submanifold $\mathcal{E}$ of $J^\infty(M,N)$ such that $\mathcal{C} \subset T\mathcal{E}$ is said* canonical submanifold*. Any canonical submanifold $\mathcal{E}$ can be locally described as the set of zeros of a finite number of smooth independent functions $f_1, ..., f_L$ and of all their differential consequences $D^\sigma(f_i)$.*

3 Differential constraints and PDEs reduction

In this section we propose a geometric reformulation of differential constraints method for a family of evolution PDEs. In particular we introduce the notion of reduction function and we discuss its relation with differential constraints seen as finite dimensional submanifolds of $J^\infty(M,N)$.

3.1 Differential constraints: from the reduction function to the submanifold

Let us consider a system of evolution PDEs of the form

$$\partial_t(u^k) = \sum_{i=1}^{s} c^i(t) F_i^k(x, u, u_\sigma), \tag{4}$$

where $F_i \in \mathcal{F}^n$ and $k = 1, \ldots, n$.

Definition 3.1 *Given a system of evolution PDEs of the form (4) and an L-dimensional manifold B, let*

$$K : M \times B \to N$$

be a smooth function. We say that K is a reduction function *for (4) if there are smooth functions f^j such that*

$$U(x,t) = K(x, b^1(t), ..., b^L(t))$$

is a solution to the system (4) for any $c^1(t), ..., c^s(t)$ if and only if $b^j(t)$ are solutions to the system of ODEs

$$\partial_t(b^j) = f^j(t, b, c^1(t), ..., c^s(t)).$$

In this framework it is natural to associate with K a function $R^K : M \times B \to J^\infty(M,N)$ that is the lift of K to $J^\infty(M,N)$. In particular, if $\pi_0 : J^\infty(M,N) \to J^0(M,N) = M \times N$ denotes the natural projection of $J^\infty(M,N)$ onto $J^0(M,N)$, the function R^K satisfies

$$\pi_0 \circ R^K(x,b) = (x, K(x,b))$$
$$R^K_*(\partial_{x^i}) = D_i,$$

and in coordinates we have

$$(u^i_\sigma \circ R^K)(x,b) = \partial^\sigma_x(K^i(x,b)).$$

If R^K is an immersion, then $\mathcal{K} = R^K(M \times B)$ is (possibly restricting B) a finite dimensional submanifold of $J^\infty(M,N)$. Furthermore the following theorem holds.

Theorem 3.2 *Let $K : M \times B \to N$ be a smooth function and $\mathcal{K} = R^K(M \times B)$. Then K is a reduction function for the system (4) if and only if $V_{F_i} \in T\mathcal{K}$, $\forall i = 1, \ldots, s$.*

Proof. If K is a reduction function for the system (4), we have

$$\sum_j f^j(x,b,c)\partial_{b^j}(K^l)(x,b) = \sum_{i=1}^s c^i(t)F^l_i(x, K(x,b), \partial^\sigma_x(K)(x,b)).$$

Choosing $c^i = \delta_{1,i}$ and applying ∂^σ_x to both sides of the previous equation we get

$$\sum_j f^j\partial_{b^j}(u^l_\sigma \circ (R^K)) = D^\sigma(F^l_1) \circ R^K.$$

Since $R^K_*(\partial_{b^i}) = \partial_{b^i}(R^K) \in T\mathcal{K}$ and $V_{F_1}(u^l_\sigma)|_\mathcal{K} = D^\sigma(F^l_1)|_\mathcal{K}$, we have $V_{F_1} \in T\mathcal{K}$. Choosing $c^i = \delta_{p,i}$ we obtain $V_{F_p} \in T\mathcal{K}$.
Conversely suppose that $V_{F_i} \in T\mathcal{K}$. Since V_{F_i} are vertical and the vertical vector fields of $T\mathcal{K}$ are generated by $\partial_{b^i}(R^K)$ there exist suitable functions $g^j_i : M \times B \to \mathbb{R}$ such that

$$V_{F_j} = \sum_i g^i_j \partial_{b^i}(R^K).$$

It is easy to show that the functions g^j_i do not depend on $x \in M$ being $R^K_*(\partial_{x^i}) = D_i$ so the thesis follows choosing

$$f^j(b, c^1, ..., c^N) = \sum_i c^i g^j_i(b).$$

■

Since the submanifold $\mathcal{K} = R^K(M \times B)$ is a finite dimensional canonical submanifold, $\mathcal{K}$ can be locally described as the set of zeros of a finite number of smooth independent functions $f_1, ..., f_L$ and of all their differential consequences $D^\sigma(f_i)$. Therefore a necessary and sufficient condition for $V_F \in T\mathcal{K}$ is $V_F(D^\sigma(f_i))|_\mathcal{K} = 0$ but, since D_i and V_F commute and $D_i \in T\mathcal{K}$, it is sufficient to check that $V_F(f_i)|_\mathcal{K} = 0$.

Remark 3.3 *In the proof of Theorem 3.2 the hypothesis that $\mathcal{K}$ is a submanifold of $J^\infty(M,N)$ is not necessary. Indeed we prove that $V_{F_i} \in \mathrm{Image}(TR^K)$ even if R^K is not an immersion (and so $\mathcal{K}$ is not a submanifold). Even so, for the sake of simplicity, in the following we always consider submanifolds $\mathcal{K}$ of $J^\infty(M,N)$.*
In particular, if K is a real analytic function with respect the x^i variables, a necessary and sufficient condition for $\mathcal{K}$ to be a submanifold is that $\partial_{b^i}(K(\cdot,b))$ are linearly independent as functions from M into N. In the smooth case it can happen that $\partial_{b^i}(K(\cdot,b))$ are linearly independent but R^K is not an immersion. However, this situations can be considered as exceptional: indeed the set of K such that R^K is an immersion is an open everywhere dense subset of $C^\infty(M,N)$ with respect the Whitney topology (see [24]).

Remark 3.4 *An interesting consequence of Theorem 3.2 is that, if $s = 1$, any solution $U(x,t) : M \times \mathbb{R} \to N$ to the system (4) for $c_1 = 1$ is a reduction function. Indeed in this case we have that*

$$\partial_b(U(x,b)) = F_1(x, U, U_\sigma),$$

so $\partial_x^{\sigma'}(\partial_b(U(x,b))) = D_{\sigma'}(F_1)(x, U, U_\sigma)$. Hence $R^U_*(\partial_b) = V_{F_1}$ and equation (4) with $s = 1$ becomes and ODE for b of the form

$$\partial_t(b)(t) = c^1(t).$$

3.2 Differential constraints: from the submanifold to the reduction function

In this section we discuss the problem of computing the reduction function K starting from the knowledge of a suitable canonical submanifold $\mathcal{K}$. The resulting algorithm will be used in the examples of Section 6.

Definition 3.5 *A finite dimensional canonical submanifold $\mathcal{K}$ of $J^\infty(M,N)$ is a* differential constraint *for equation (4) if $V_{F_i} \in T\mathcal{K}$*

In order to prove that with any differential constraint $\mathcal{K}$ for (4) it is possible to associate a reduction function K, we need to recall the following technical result.

Theorem 3.6 *Let $\mathcal{H}$ be an m-dimensional canonical submanifold of $J^\infty(M,N)$ (i.e. $T\mathcal{H} = \mathcal{C}$). Denoting by π the canonical projection $\pi : J^\infty(M,N) \to M$, if $\pi(\mathcal{H}) = V \subset M$, then there exists a unique smooth function $U : V \to N$ such that $R^U(V) = \mathcal{H}$.*

Proof. A proof of this result can be found in [9], Chapter 4, Proposition 2.3. ∎

Theorem 3.7 *Let $\mathcal{K}$ be a finite dimensional canonical submanifold of $J^\infty(M,N)$ which is a differential constraint for equation (4). Then, for any point $p \in \mathcal{K}$, there exist a neighborhood $U \subset J^\infty(M,N)$ of p and a function $K : V \times B \to N$, where $V \subset \pi(U)$, such that $R^K(V \times B) = \mathcal{K} \cap U \cap \pi^{-1}(V)$ and K is a reduction function for the system (4).*

Proof. The manifold $\mathcal{K}$ with respect the projection π is a finite dimensional fibred manifold with base M. Moreover, since $D_i \in \mathcal{K}$, the Cartan distribution $\mathcal{C}$ is a finite dimensional flat connection of $(\mathcal{K}, \pi, M)$ and, for any $p_0 \in \mathcal{K}$, there exist a neighborhood $B \subset \pi^{-1}(x_0)$ of p_0 (where $x_0 = \pi(p_0)$) and a local trivialization $R : V \times B \to \mathcal{K} \subset J^\infty(M,N)$ of $\mathcal{C}$. Therefore Theorem 3.6 ensures that there exists a smooth function $K : V \times B \to N$ such that $R^K = R$ and Theorem 3.2 guarantees that K is also a reduction function for the system (4). ∎

Remark 3.8 *There are two obstructions for a global version of Theorem 3.7. The first one is that, if M is not simply connected, $\mathcal{C}$ may admit only a local trivialization and not a global one and the second is that, if $\mathcal{C}$ is a non-linear connection on $\mathcal{K}$, it may not admit a global trivialization, since non-linear ODEs can blow-up. Obviously, if M is simply connected and $\mathcal{C}|_{\mathcal{K}}$ has at most linear grow for some coordinate system on $\mathcal{K}$, Theorem 3.7 admits a global version.*

Given an L-dimensional canonical submanifold $\mathcal{K}$ which is a differential constraint for (4), Theorem 3.7 provides an explicit construction procedure for the function K and for the system of ODEs for the parameters b^i.
Indeed let (x^i, y^j) be an adapted coordinate system for the fibred manifold $(\mathcal{K}, \pi, M)$, which means that x^i is the standard coordinate system on M and y^j is an adapted coordinate system for the fiber $\pi^{-1}(x)$, with $j = 1, ..., L - m$. The coordinates y^i can be chosen among the functions u^k, u^l_σ and,

in general, it is possible to find smooth functions $f^k(x,y), f^l_\sigma(x,y)$ such that $u^k = f^k(x,y), u^l_\sigma = f^l_\sigma(x,y)$.
In the coordinate system (x^i, y^j) the vector fields D_i have the form

$$D_i = \partial_{x^i} + \sum_j \Psi^j_i(x,y)\partial_{y^j} \qquad (i = 1, \dots m)$$

and, $\forall x_0 \in M$ and $y_{x_0} \in \pi^{-1}(x_0)$, the solution $y^j = \tilde{K}^j(x, y_{x_0})$ to the system

$$\begin{aligned} \partial_{x^i}(\tilde{K}^j(x, y_{x_0})) &= \Psi^j_i(x, y_{x_0}) \\ \tilde{K}^j(x_0, y_{x_0}) &= y^j_{x_0} \end{aligned}$$

provides the local trivialization of the flat connection $\mathcal{C} = \text{span}\{D_i\}$. The explicit expression of the function K can be obtained by rewriting u^l as functions of (x^i, y^j) leading to

$$K^l(x, y_{x_0}) = f^l(x, \tilde{K}(x, y_{x_0})).$$

Moreover the system of ODEs for the parameters $y^j_{x_0}$ can be obtained expressing V_{F_i} in the coordinates (x^i, y^j)

$$V_{F_i} = \sum_j V_{F_i}(y^j)\partial_{y^j} = \sum_j \Phi^j_i(x,y)\partial_{y^j},$$

so that

$$\frac{dy^j_{x_0}}{dt} = \sum_i c^i(t)\Phi^j_i(x_0, y_{x_0}(t)).$$

3.3 A necessary condition for existence of differential constraints

The main goal of the general theory of differential constraints is to find a reduction function for a system of the form (4) for $s = 1$. As proven in [39] (see also Remark 3.4) this problem admits an infinite number of solutions. On the other hand, the problem of existence of reduction functions (or differential constraints) for $s > 1$ is completely different: actually, in the general case, there are no reduction functions at all.
In this section we address the problem of existence of a reduction function for a system of the form (4) starting from the following remark.

Remark 3.9 *If a system of evolution PDEs of the form* (4) *admits a differential constraint $\mathcal{K}$, the set*

$$\mathcal{S} = \text{span}\{V_{F_1}, ..., V_{F_s}\},$$

is a finite dimensional module on $\mathcal{K}$.

The following Proposition provides a useful characterization for the vector fields V_{F_i}.

Proposition 3.10 *Let $V_{F_1}, ..., V_{F_s}$ be evolution vector fields in $J^\infty(M,N)$ such that $\mathcal{S}$ is an s-dimensional (formally) integrable distribution on a submanifold $\mathcal{K}$ of $J^\infty(M,N)$. If*

$$[V_{F_i}, V_{F_j}] = \sum_h \lambda^h_{i,j} V_{F_h}$$

then $D_l(\lambda^h_{i,j}) = 0$ on $\mathcal{K}$.

Proof. The proof is given for the case $N = M = \mathbb{R}$ and $\mathcal{H} = J^\infty(M,N)$; the general case is a simple generalization of this one.
Since $\mathcal{S}$ is s-dimensional, for any point $p \in J^\infty(M,N)$ there exist a neighborhood U of p and an integer $h \in \mathbb{N}_0$ such that the matrix $A = (D_x^{h+j-1}(F_i))|_{i,j=1,\dots,s}$ is non-singular. Moreover, since the commutator of two evolution vector fields is an evolution vector field, there exist some $F_{i,j} \in \mathcal{F}$ such that $[V_{F_i}, V_{F_j}] = V_{F_{i,j}}$ and, by the definition of evolution vector field, we have

$$D_x^r(F_{i,j}) = \sum_h \lambda_{i,j}^h D_x^r(F_h). \tag{5}$$

Deriving with respect to x the previous relations we obtain

$$D_x^{r+1}(F_{i,j}) = \sum_h D_x(\lambda_{i,j}^h) D_x^r(F_h) + \sum_h \lambda_{i,j}^h D_x^{r+1}(F_h) \tag{6}$$

and combining (5) and (6) we find

$$\sum_h D_x(\lambda_{i,j}^h) D_x^r(F_h) = 0.$$

Since the matrix A is non-singular we get $D_x(\lambda_{i,j}^h) = 0$. ■

In Section 5 we will consider two particular cases for the functions $F_i \in \mathcal{F}^n$.
In the first case V_{F_i} form a finite dimensional module of constant dimension on all $J^\infty(M,N)$. In this case Proposition 3.10 ensures that V_{F_i} form a Lie algebra, since the only functions f in $J^\infty(M,N)$ such that $D_i(f) = 0$ are the constants.
In the second case we suppose that V_{F_i} form a finite dimensional module on a real analytic finite dimensional submanifold $\mathcal{H}$ of $J^\infty(M,N)$.

4 Characteristic vector fields in $J^\infty(M,N)$

In this section we define the notion of generalized characteristic flow for an evolution vector field and we discuss the connection with the usual characteristic flow for scalar first order evolution PDEs. These results will play a central role in the explicit construction of differential constraints in Section 5.

4.1 Characteristics of scalar first order evolution PDEs

It is well known that, if $N = \mathbb{R}$ and $F \in \mathcal{F} \backslash \mathcal{F}_0$, the evolution vector field V_F is not the prolongation of a vector field on $J^0(M,N)$ and does not admit flow in $J^\infty(M,N)$, which is why the equation

$$\partial_t(u) = F(x, u, u_\sigma) \tag{7}$$

may not admit solutions even for smooth initial data, or may admit infinite solutions for any smooth initial data. For this reason the problem of finding solutions to evolution PDEs is usually solved only in specific situations (for example the linear or semilinear cases) where it is possible to use the powerful techniques of analysis.
Anyway, a classical geometric approach to scalar first order evolution PDEs (see, e.g., [15]) shows that something can be done in order to solve equation (7) even when V_F does not admit flow in $J^\infty(M,N)$. Indeed given a first order scalar autonomous PDE

$$\partial_t(u) = F(x^j, u, u_i) \tag{8}$$

it is possible to solve (8) considering the following system of ODEs on $J^1(M,N)$

$$\begin{aligned}
\frac{dx^i}{da} &= -\partial_{u_i}(F)(x^j,u,u_k)\\
\frac{du}{da} &= F(x^j,u,u_k) - \sum_h u_h\partial_{u_h}(F)(x^j,u,u_k)\\
\frac{du_i}{da} &= \partial_i(F)(x^j,u,u_k) + u_i\partial_u(F)(x^j,u,u_k).
\end{aligned}$$

If Φ_a is flow of the vector field on $J^1(M,N)$ corresponding to the previous system and we define $\phi^i_a = (\Phi^*_a(x^i))$ and $\eta_a = \Phi^*_a(u)$, the solution $U(x,t)$ to the PDE (8) with initial data $U(x,0) = f(x)$ is given by

$$U(x,t) = \eta_t(\bar{\phi}_t^{-1}(x), f(\bar{\phi}_t^{-1}(x)), \partial_{x^i}(f)(\bar{\phi}_t^{-1}(x)))$$

where $\bar{\phi}_a(x) = \phi_a(x, f(x), \partial_j(f)(x))$.
Moreover it is possible to uniquely extend the flow Φ_a to $J^k(M,N)$ as the solution to the following system of ODEs

$$\frac{du_\sigma}{da} = D^\sigma(F)(x,u,u_\sigma) - \sum_i u_{\sigma+1_i}\partial_{u_i}(F)(x,u,u_\sigma).$$

Defining $\psi_{\sigma,a} = \Phi^*_a(u_\sigma)$ we have

$$\partial^\sigma(U)(x,t) = \psi_{\sigma,t}(\bar{\phi}_t^{-1}(x), f(\bar{\phi}_t^{-1}(x)), \partial^\sigma(f)(\bar{\phi}_t^{-1}(x))),$$

and the vector field corresponding to the flow Φ_a on $J^\infty(M,N)$ is given by

$$\bar{V}_F := \partial_a(\Phi_a)|_{a=0} = V_F - \sum_i \partial_{u_i}(F)D_i.$$

We call Φ_a the *characteristic flow* of F and $\bar{V}_F$ its *characteristic vector field.*

4.2 Characteristics in the general setting

In this section we propose an extension of the notion of characteristic vector field and characteristic flow to multidimensional and higher order case. This extension is based on the geometric analysis of $J^\infty(M,N)$ presented in [33]. We start by recalling the definition of one-parameter group of local diffeomorphisms on $J^\infty(M,N)$ which reduces to the classical one in the finite dimensional setting.

Definition 4.1 *A map $\Phi_a : U_a \to J^\infty(M,N)$ is a one-parameter group of local diffeomorphisms if Φ_a are smooth maps, U_a are open sets $\forall a$ (with $U_0 = J^\infty(M,N)$) and $\forall p \in U_{a+b} \subset U_b \cap \Phi_b^{-1}(U_a)$ (with $ab \geq 0$) we have $\Phi_a \circ \Phi_b(p) = \Phi_{a+b}(p)$.*
The one-parameter group Φ_a of local diffeomorphisms is the flow of the vector field X if

$$\partial_a(\Phi^*_a(f)(p))|_{a=0} = X(f)(p)$$

for any $f \in \mathcal{F}$.

Definition 4.2 *Given an evolution vector field V_F, we say that V_F (or its generator F)* admits characteristics *if there exist suitable smooth functions $h^1,...,h^n \in \mathcal{F}$ such that the vector field*

$$\tilde{V}_F = V_F - \sum_i h^i D_i,$$

admits flow on $J^\infty(M,N)$.

If we restrict to the scalar case ($N = \mathbb{R}$), which is discussed in Subsection 4.1, the following Theorem provides a complete characterization of evolution vector fields admitting characteristics.

Theorem 4.3 *An evolution vector field V_F on $J^\infty(M,\mathbb{R})$ with generator F admits characteristic if an only if $F \in \mathcal{F}_1$.*

Proof. The proof that any $F \in \mathcal{F}_1$ admits characteristic flow is given in Subsection 4.1. The proof of the converse can be found in [33]. ■

Remark 4.4 *Theorem 4.3 does not hold if, instead of requiring that $\tilde{V}_F$ admits flow on the whole $J^\infty(M,\mathbb{R})$, we restrict to a submanifold of $J^\infty(M,\mathbb{R})$. For example if we consider $M = \mathbb{R}^2$ with coordinates (x,y) and $F = u_{xy}$, Theorem 4.3 ensures that $F = u_{xy}$ does not admit characteristics on $J^\infty(\mathbb{R}^2,\mathbb{R})$ but, considering the submanifold $\mathcal{E} \subset J^\infty(\mathbb{R}^2,\mathbb{R})$ generated by the equation $u_{yy} = 0$, it is easy to prove that $V_F \in T\mathcal{E}$ and that V_F admits characteristics on $\mathcal{E}$.*

If we do not restrict to the scalar case the situation becomes more complex and, to the best of our knowledge, a complete theory of characteristics in $J^\infty(M,N)$ for $N \neq \mathbb{R}$ has not been developed.
Indeed in this case we can find $F \notin \mathcal{F}_1^n$ such that V_F admits characteristics. For example if we consider $M = \mathbb{R}$ and $N = \mathbb{R}^2$ (with coordinates x and (u,v) respectively) and $F = (v_{xx}, 0) \in \mathcal{F}_2$, the flow of the vector field V_F is given by the following transformation

$$\begin{array}{rcl} x_a &=& x \\ u_a &=& u + av_{xx} \\ u_{x,a} &=& u_x + av_3 \\ u_{xx,a} &=& u_{xx} + av_4 \\ & \cdots & \\ v_a &=& v \\ v_{x,a} &=& v_x \\ & \cdots & \end{array}$$

In this paper, in order to deal with the general case, we propose a stronger definition of characteristics that, although imitating in some respects the scalar case, is weak enough to include many cases of interest.
Given an open subset $U \subset J^\infty(M,N)$ we denote by

$$\mathcal{F}|_U = \bigcup_k \mathcal{F}_k|_U$$

the set of smooth functions defined on U, that is the union of the sets of smooth functions defined on $\pi_k(U) \subset J^k(M,N)$.
Given a subalgebra $\mathcal{G}_0 \subset \mathcal{F}|_U$, we denote by $\mathcal{G}_k$ the algebra generated by smooth composition of functions of the form $D^\sigma(f)$, where $f \in \mathcal{G}_0$ and σ is a multi-index with $|\sigma| \leq k$.

Definition 4.5 *A subalgebra $\mathcal{G}_0 \subset \mathcal{F}|_U$ generates $\mathcal{F}|_U$ if $x^i \in \mathcal{G}_0$ and*

$$\mathcal{F}|_U = \bigcup_k \mathcal{G}_k.$$

Definition 4.6 *An evolution vector field V_F with generator F admits strong characteristics if there exists an open set $U \subset J^\infty(M,N)$, a finitely generated subalgebra $\mathcal{G}_0$ of $\mathcal{F}|_U$ generating $\mathcal{F}|_U$ and $g^1,...,g^n \in \mathcal{F}$ such that the vector field $\bar{V}_F = V_F - \sum_i g^i D_i$ satisfies*

$$\bar{V}_F(\mathcal{G}_0) \subset \mathcal{G}_0.$$

In the scalar case an evolution vector field admits characteristics if and only if admits strong characteristics: indeed in this case $\bar{V}_F(\mathcal{F}_1) \subset \mathcal{F}_1$. Moreover, if we consider the evolution vector field V_F of the previous example (with generator $F = (v_{xx}, 0)$), it is easy to check that V_F admits strong characteristics. In fact the subalgebra $\mathcal{G}_0$ generated by x, u, v, v_x, v_{xx} is such that $V_F(\mathcal{G}_0) \subset \mathcal{G}_0$. Actually we do not know any example of evolution vector field admitting characteristics which are not strong characteristics.

Remark 4.7 *In Definition 4.6 we consider a general subalgebra $\mathcal{G}_0$ generating $\mathcal{F}$ instead of restricting to the case $\mathcal{G}_0 = \mathcal{F}_k$ for some $k \in \mathbb{N}$. This is a crucial point because, in the vector case $N = \mathbb{R}^n$ (with $n > 1$), condition $\bar{V}_F(\mathcal{F}_k) \subset \mathcal{F}_k$ implies $\bar{V}_F(\mathcal{F}_0) \subset \mathcal{F}_0$ (see [33]) and the vector fields $\bar{V}_F$ satisfying $\bar{V}_F(\mathcal{F}_k) \subset \mathcal{F}_k$ turn out to be tangent to the prolongations of infinitesimal transformations in $J^0(M,N)$.*
A well-known consequence of this fact is that, in the vector case, the only infinitesimal symmetries of a PDE which can be defined using finite jet spaces $J^k(M,N)$ are Lie-point symmetries. On the other hand, if we allow $\mathcal{G}_0$ to be a general subalgebra generating $\mathcal{F}$, we obtain a larger and non-trivial class of evolution vector fields admitting strong characteristics.

Theorem 4.8 *With the notations of Definition 4.6, if an evolution vector field admits strong characteristics then it admits characteristics, and $\bar{V}_F$ is its characteristic vector field.*

Proof. The vector field $\bar{V}_F$ admits flow on the space of functions $\mathcal{G}_0$ since $\mathcal{G}_0$ is finite dimensional. In order to show that $\bar{V}_F$ admits flow on all $\mathcal{F}|_U$ and so (since U depends on a generic point) on $\mathcal{F}$ we prove by induction that $\bar{V}_F(\mathcal{G}_k) \subset \mathcal{G}_k$.
By hypothesis $\bar{V}_F(\mathcal{G}_0) \subset \mathcal{G}_0$. Suppose that $\bar{V}_F(\mathcal{G}_{k-1}) \subset \mathcal{G}_{k-1}$. Since $\bar{V}_F$ is a symmetry of the Cartan distribution, there exist some functions $h^j_i \in \mathcal{F}$ such that

$$[\bar{V}_F, D_i] = \sum_j h^j_i D_j$$

where $h^i_j \in \mathcal{G}_1$, being $\bar{V}_F(\mathcal{G}_0) \subset \mathcal{G}_0$ and $x^i \in \mathcal{G}_0$.
We recall that $\mathcal{G}_k$ is generated by functions of the form $D_i(g)$ with $g \in \mathcal{G}_{k-1}$. So

$$\bar{V}_F(D_i(g)) = D_i(\bar{V}_F(g)) + \sum_j h^j_i D_j(g) \in \mathcal{G}_k$$

since $\bar{V}_F(g) \in \mathcal{G}_{k-1}$ and $h^i_j \in \mathcal{G}_1$. Hence $\bar{V}_F$ admits flow on $\mathcal{G}_k$ and the flow on $\mathcal{G}_k$ is compatible with the flow on $\mathcal{G}_{k-1}$ being $\mathcal{G}_{k-1} \subset \mathcal{G}_k$.
The problem of the previous construction is that in general the domain U_k of the flow in $\mathcal{G}_k$ depends on k. This means that, if we denote with $P_{h,k}$ the natural projection of $\mathcal{G}_h$ on $\mathcal{G}_k$ with $h > k$, it might happen that $P^{-1}_{h,k}(U_k) \neq U_h$. But this is not actually the case. Indeed since $\mathcal{G}_0$ generates $\mathcal{F}|_U$, then $\mathcal{F}_0|_U \subset \mathcal{G}_0$ and so $\mathcal{F}_k|_U \subset \mathcal{G}_k$. In particular $u^i_\sigma \in \mathcal{G}_k$ if $|\sigma| \leq k$. But by Remark 7.2 and Corollary 7.3 (see Appendix) $\Phi_a(u^i_\sigma)$ is polynomial in $u^j_{\sigma'}$ for $|\sigma'|$ sufficiently large. This means that $u^j_{\sigma'}$ can vary in all $\mathbb{R}$ and so the domain of definition of Φ_a in $U \subset J^\infty(M,N)$ is not empty and is of the form $U' = \pi^{-1}_{\infty,k}(U_k)$ for k sufficiently large. Since U' is an open subset of $J^\infty(M,N)$ this concludes the proof. ■

Definition 4.9 *Let $y^1, ..., y^l, ... \in \mathcal{F}|_U$ be a sequence of functions defined in an open set U. We say that $Y = \{y^i\}|_{i\in\mathbb{N}}$ is a local adapted coordinate system with respect to a subalgebra $\mathcal{G}_0$ generating $\mathcal{F}|_U$, if there exists a sequence $k_1, ..., k_l, ... \in \mathbb{N}$, with $k_i < k_{i+1}$, such that $y^1, ..., y^{k_i}$ is a coordinate system for $\mathcal{G}_i$.*

Remark 4.10 *The flow of a vector field with strong characteristics solves a triangular infinite dimensional system of ODEs. Indeed if we consider an adapted coordinate system with respect to a subalgebra $\mathcal{G}_0$ we have $\bar{V}_F(y^i) = f(y^1, .., y^{k_1})$ for $i = 1, ..., k_1$, $\bar{V}_F(y^i) = f(y^1, .., y^{k_2})$ for $i = k_1 + 1, ..., k_2$ and so on. So we can start by solving the system for $i = 1, ..., k_1$ and then solve the system for $i = k_1 + 1, ..., k_2$, since the system is of triangular type.*

The main trouble when working with a family of evolution vector fields admitting characteristic flows is that the sum or the Lie brackets of two of them usually do not admit characteristic flow. In order to overcome this problem we give the following Definition.

Definition 4.11 *A set of evolution vector fields $V_{F_1}, ..., V_{F_s}$ with strong characteristics admits a* common filtration *if $\forall p \in J^\infty(M, N)$ there exist a neighborhood U of p and a subalgebra $\mathcal{G}_0 \subset \mathcal{F}|_U$ such that $\mathcal{G}_0$ is the subalgebra required in Definition 4.6 for $\bar{V}_{F_1}, ..., \bar{V}_{F_s}$.*

If $F_1, ..., F_s$, correspond to evolution vector fields with strong characteristics admitting a common filtration, then also $cF_i + dF_j$ (where $c, d \in \mathbb{R}$) and $[F_i, F_j]$ correspond to vector fields with strong characteristics. Furthermore $cF_i + dF_j$ and $[F_i, F_j]$ admit the same common filtration of $F_1, ..., F_k$.

5 Building differential constraints

In this section we consider a system of PDEs of the form (4) such that some of the evolution vector fields V_{F_i} admit strong characteristics and a common filtration. In this setting we show how it is possible to construct a differential constraint for the system (4) starting from the knowledge of a suitable submanifold of $J^\infty(M, N)$. The construction, which is completely explicit, take the cue from of the moving frame method (see [18]).

Definition 5.1 *Let $\mathcal{H} \subset J^\infty(M, N)$ be a submanifold and U be an open neighborhood of $p \in \mathcal{H}$. Given a sequence of independent functions $f^i \in \mathcal{F}|_U$ ($i \in \mathbb{N}$) such that $\mathcal{H} \cap U$ is the annihilator of f^i, we say that a distributions $\Delta = \mathrm{span}\{V_{G_1}, \ldots, V_{G_h}\}$ is transversal to $\mathcal{H}$ in U if there exist $r_1, \ldots, r_h$ such that the matrix $(\bar{V}_{G_i}(f^{r_j}))|_{i,j=1,\ldots,h}$ has maximal rank in U. In the following the sequence f^i will be chosen so that $r_j = j$ and f^i is a local coordinate system adapted with respect to the filtration $\mathcal{G}_k$ for k sufficiently large.*

Lemma 5.2 *Let $G_1, ..., G_h$ be a subalgebra of $\mathcal{F}^n$ admitting strong characteristics and a common filtration. Let $\Phi^i_{a^i}$ be the characteristic flow of G_i and $\mathcal{H}$ be a canonical finite dimensional submanifold of $J^\infty(M, N)$ such that the distribution $T\mathcal{H} \oplus \mathrm{span}\{V_{G_1}, ..., V_{G_h}\}$ has constant rank and the distribution $\Delta = \mathrm{span}\{V_{G_1}, \ldots, V_{G_h}\}$ is transversal to $\mathcal{H}$. Then there exists a suitable neighborhood of the origin $\mathcal{V} \subset \mathbb{R}^h$ such that*

$$\mathcal{K} = \bigcup_{(a^1,\ldots,a^h)\in\mathcal{V}} \Phi^h_{a^h}(...(\Phi^1_{a^1}(\mathcal{H}))...) \tag{9}$$

is a finite dimensional submanifold of $J^\infty(M, N)$.

Proof. In the following, for the sake of clarity, we write

$$\mathbf{\Phi}^*_\alpha(f) = \Phi^{1*}_{a^1}(...\Phi^{h*}_{a^h}(f)...),$$

where $\alpha = (a^1, ..., a^h) \in \mathbb{R}^h$. Given a sequence of independent functions f^i ($i \in \mathbb{N}$) such that $\mathcal{H}$ is the annihilator of f^i, for any point $p \in \mathcal{H}$ there exists a neighborhood U such that the matrix $(\bar{V}_{G_i}(f^j))|_{i,j=1,...,h}$ has maximal rank in U. Therefore, considering the submanifold $\tilde{\mathcal{H}}$ defined as the annihilator of the functions $f^i \in \mathcal{F}|_U$ ($i = 1, \ldots, h$), the equations

$$\mathbf{\Phi}^*_\alpha(f^i) = 0 \qquad i = 1, \ldots h \tag{10}$$

can be solved with respect to α. This means that, possibly restricting the open set U, there exist a smooth function $A(p) = (A^1(p), ..., A^h(p))$ defined on U such that $\mathbf{\Phi}^*_{A(p)}(f^i)(p) = 0$ (for $i = 1, \ldots h$), i.e. $\Phi^h_{A^h}(...(\Phi^1_{A^1}(p))...) \in \tilde{\mathcal{H}}$. In the following we prove that $\mathcal{K}$ is the annihilator of the functions

$$K^j(p) = \mathbf{\Phi}^*_{A(p)}(f^j)(p), \qquad j > h \tag{11}$$

and, since K^j are independent and adapted with respect to the filtration $\mathcal{G}_k$ for k sufficiently large, $\mathcal{K}$ is a submanifold of $J^\infty(M, N)$.
We start by proving that if $p_0 \in \mathcal{K} \cap U$, then $K^j(p_0) = 0$ (for $j > h$). Indeed, if $p_0 \in \mathcal{K} \cap U$, the point p_0 can be reached starting from $p \in \mathcal{H}$ by means of composition of suitable flows $\Phi^i_{a^i}$. On the other hand, for any $p_0 \in \mathcal{K} \cap U$, there exists $A(p_0) = (A^1, \ldots, A^h)$ such that $\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...) \in \tilde{\mathcal{H}}$. Since $\mathcal{H} \subset \tilde{\mathcal{H}}$ and the transversality condition ensures that equation (10) have a unique solution, we have $\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...) \in \mathcal{H}$. Therefore

$$K^j(p_0) = \mathbf{\Phi}^*_{A(p_0)}(f^j)(p_0) = f^j(\Phi^h_{A^h}(...\Phi^1_{A^1}(p_0)...) = 0$$

for any j and in particular for $j > h$. In order to prove the other inclusion we have to ensure that p_0 can be reached starting from a point $p \in \mathcal{H}$ by means of the flows $\Phi^i_{a^i}$. Given $p \in \tilde{\mathcal{H}}$ such that $\Phi^h_{A^h}(...(\Phi^1_{A^1}(p_0))...) = p$, the definition of $A(p_0)$ ensures that $f^i(p) = 0$ for $i = 1, \ldots h$ whereas by hypothesis we have

$$K^j(p_0) = \mathbf{\Phi}^*_{A(p_0)}(f^j)(p_0) = f^j(p) = 0 \qquad j > h.$$

Hence $f^i(p) = 0$ $\forall i \in \mathbb{N}$ and $p \in \mathcal{H}$. ■

Lemma 5.3 *In the hypotheses and with the notations of Lemma 5.2, $\bar{V}_{G_j} \in T\mathcal{K}$ and $D_i \in T\mathcal{K}$.*

Proof. We recall that a vector field $V \in T\mathcal{K}$ if and only if $V(K^j) = 0$, where K^j are given by (11). Since for any j (with $j > h$) there exists a suitable k such that $f^1, \ldots f^j \in \mathcal{G}_k$, it is possible to chose as coordinates in $\mathcal{K} \cap U \cap \mathcal{G}_k$ the functions f^i ($i = 1, \ldots h$) and some functions $y^1, \ldots, y^r$ (with $r = \dim(\mathcal{G}_k) - h$) such that $\bar{V}_{G_i}(y^l) = 0$. In particular, for any $j > h$, there exists a smooth function L^j such that

$$f^j(p) = L^j(f^1(p), ..., f^h(p), y^1(p), ...y^r(p)).$$

Since $f^1, \ldots, f^h$ vanish on $\tilde{\mathcal{H}}$ we have

$$K^j(p) = L^j(0, ..., 0, y^1(p), ..., y^r(p)),$$

and so $\bar{V}_{G_i}(K^j) = \bar{V}_{G_i}(L^j(0, ..., 0, y^1(p), ..., y^r(p))) = 0$.
In order to prove that $D_i \in T\mathcal{K}$, we consider

$$D^\alpha_i = \mathbf{\Phi}^*_\alpha(D_i).$$

By definition, being $D_i \in T\mathcal{H}$, we have that $D_i^\alpha \in T\mathcal{K}$ and, by Theorem 7.1 (see Appendix), there exist smooth functions $C^i_j(\alpha, p)$ such that

$$D_i^\alpha = \sum_j C_i^j(\alpha, p) D_j.$$

Moreover, since $\Phi^i_{a^i}$ are diffeomorphisms, $\text{span}\{D_1^\alpha, ..., D_m^\alpha\}$ and $\text{span}\{D_1, ..., D_m\}$ have the same dimension. Hence the matrix C^i_j is invertible for any α, ensuring that $D_i \in T\mathcal{K}$. ■

Remark 5.4 *The functions K^j defined by* (11) *are a set of independent invariants for the vector fields $\bar{V}_{G_i}$. Furthermore, since $\mathcal{K}$ is finite dimensional, it is possible to add a finite number of functions z^i such that (z^i, K^j) form an adapted coordinate system with respect the filtration $\mathcal{G}_k$ for k sufficiently large.*

Theorem 5.5 *In the hypotheses and with the notations of Lemma 5.2, let V_F be an evolution vector field such that $V_F \in T\mathcal{H}$,* $\dim(\text{span}\{V_F, V_{G_1}, ..., V_{G_h}\}) = h + 1$ *and*

$$[G_i, F] = \mu_i F + \sum_k \lambda_i^k G_k \qquad \mu_i, \lambda_i^j \in \mathbb{R}$$

Then $V_F \in T\mathcal{K}$.

Proof. Given the $(m + h + 1)$-dimensional distribution

$$\Delta := \text{span}\{D_1, ..., D_m, V_{G_1}, ..., V_{G_h}, V_F\},$$

we have $\Delta|_\mathcal{H} \subseteq T\mathcal{H} \oplus \text{span}\{V_{G_1}, \ldots, V_{G_h}\} \subseteq T\mathcal{K}|_\mathcal{H}$ and, by hypothesis, $[\bar{V}_{G_i}, \Delta] \subseteq \Delta$. If we prove that

$$\Phi^i_{a*}(\Delta) = \Delta,$$

we have $\Delta|_\mathcal{K} \subset T\mathcal{K}$ and, in particular, $V_F \in T\mathcal{K}$.
Considering the coordinate system z^i, K^j of Remark 5.4 we can suppose, possibly relabeling some invariant z^i with K^j for some j, that we have exactly h coordinates z^i. Eliminating some element of the form ∂_{K^j}, the sequence $V_F, \bar{V}_{G_j}, D_k, \partial_{K^l}$ form a basis of $TJ^\infty(M, N)$ and for any vector field $X \in TJ^\infty(M, N)$ there exist suitable functions b, c^i, d^j, e^l depending on a and $p \in U_a$ such that

$$X_a := \Phi^i_{a*}(X) = b(a, p)V_F + \sum_{j,k,l} c^j(a, p)\bar{V}_{G_j} + d^k(a, p)D_k + e^l(a, p)\partial_{K^l}.$$

From the definition of X_a and using $[\bar{V}_{G_i}, \Delta] \subset \Delta$ and $[\bar{V}_{G_i}, \partial_{K^l}] \in \Delta$, we obtain that the functions e^l must solve the equations

$$\partial_a(e^l) = -\bar{V}_{G_i}(e^l).$$

Moreover, since $X_0 = X \in \Delta$, $e^l(0, p) = 0$ and, from the previous equation, we get $e^l(a, p) = 0$ for any a, which ensures $X_a \in \Delta$ for any a. ■

Remark 5.6 *Theorem 5.5 still holds if we consider r functions $F_i \in \mathcal{F}$ such that* $\dim(\text{span}\{V_{F_1}, ..., V_{F_r}, V_{G_1}, ..., V_{G_h}\}) = r + h$, *$V_{F_i} \in T\mathcal{H}$ for any $i = 1, ..., r$ and*

$$[G_i, F_j] = \sum_{k,l} (\mu^k_{i,j} F_k + \lambda^l_{i,j} G_l)$$

for some constants $\lambda^l_{i,j}, \mu^k_{i,j} \in \mathbb{R}$.

Theorem 5.7 *In the hypotheses and with the notations of Lemma 5.2, if F, G_i are real analytic, $\mathcal{H}$ is defined by real analytic functions and, denoting by $L = \langle F, G_1, \dots, G_h \rangle$ the Lie algebra generated by F and G_i, we have*

$$L|_{\mathcal{H}} \subset T\mathcal{H} \oplus \text{span}\{V_{G_1}, ..., V_{G_h}\},$$

then $V_F \in T\mathcal{K}$.

Proof. We note that the functions K^i defined by (11) are real analytic if the vector fields $\bar{V}_{G_i}$ and the submanifold $\mathcal{H}$ are real analytic.
The vector field V_F is in $T\mathcal{K}$ if for any $p_0 \in \mathcal{K}$ and any K^i we have

$$V_F(K^i)(p_0) = 0.$$

We know that if $p_0 \in \mathcal{K}$ there exists an $\alpha = (a^1, ..., a^h) \in \mathbb{R}^h$ and $p_1 \in \mathcal{H}$ such that

$$p_0 = \Phi^h_{a^h}(...(\Phi^1_{a^1}(p_1))...).$$

Moreover, being K^i invariants of $\Phi^j_{a^j}$ we have

$$\begin{aligned} V_F(K^i)(p_0) &= \mathbf{\Phi}^*_\alpha(V_F(K^i))(p_1) \\ &= \mathbf{\Phi}^*_\alpha(V_F)(K^i)(p_1). \end{aligned}$$

Since the previous expression is real analytic it is sufficient to prove that any derivative of any order with respect a^i evaluated in $(a^1, ..., a^h) = 0$ is zero. It is easy to verify that

$$\partial^{k_1}_{a^1}(...\partial^{k_h}_{a^h}(\tilde{\mathbf{\Phi}}^*_\alpha(V_F))...)|_{\alpha=0} = \bar{V}^{k_h}_{G_h}(...(\bar{V}^{k_1}_{G_1}(V_F))...),$$

where we use the notation

$$\bar{V}^k_{G_i}(X) = \underbrace{[\bar{V}_{G_i}, [...[\bar{V}_{G_i}, X]...]]}_{k \text{ times}}.$$

By hypothesis $\bar{V}^{k_h}_{G_h}(...(\bar{V}^{k_1}_{G_1}(V_F))...) \in T\mathcal{K}|_{\mathcal{H}}$ and so for any $p_1 \in \mathcal{H}$ and any K^i

$$\bar{V}^{k_1}_{G_1}(...(\bar{V}^{k_{h'}}_{G_{h'}}(V_F))...)(K^i)(p_1) = 0.$$

■

Remark 5.8 *If $\bar{V}_{G_i}, V_F$ are real analytic and $\mathcal{H}$ is defined by real analytic equations, Theorem 5.5 implies Theorem 5.7. On the other hand Theorem 5.5 turns out to be very useful when we consider smooth (not analytic) invariant manifolds $\mathcal{H}$.*

Remark 5.9 *It is important to note that Theorems 5.5 and 5.7 hold also if $\mathcal{H}$ is a manifold with boundary. In this case if $V_{G_1}, ..., V_{G_h} \in T(\partial\mathcal{H})$ we obtain that $\mathcal{K}$ is also a local manifold with boundary.*

6 Examples

6.1 The general algorithm

For the convenience of the reader we start this subsection describing the general algorithm we use in the examples. Given a PDE of the form

$$\partial_t(u^i) = c^0(t)F^i(x, u, u_\sigma) + \sum_{k=1}^{h} c^k(t)G^i_k(x, u, u_\sigma), \tag{12}$$

where $G_k \in \mathcal{F}^n$ admit characteristic flows, the first step is to compute the characteristic flows of G_k integrating the characteristic vector fields $\bar{V}_{G_k}$ and obtaining the local diffeomorphisms Φ^i_a. Then, considering the differential constraint $\mathcal{H}$ for F given by the equations $f^i(x, u, u_\sigma) = 0$ and their differential consequences $D^\sigma(f^i) = 0$, we choose h independent functions g^i between the functions $f^i, D^\sigma(f^i)$ such that the matrix $(\bar{V}_{G_k}(g^i))$ is non-singular and we solve the equations

$$\mathbf{\Phi}^*_\alpha(g^i)(x, u, u_\sigma) = 0, \tag{13}$$

where $\alpha = (a^1, ..., a^h)$, obtaining $a^i = A^i(x, u, u_\sigma)$. Hence, the new differential constraint $\mathcal{K}$ is obtained by replacing α with $(A^1(x, u, u_\sigma), ..., A^h(x, u, u_\sigma))$ in the expression

$$\mathbf{\Phi}^*_{(A^1(x,u,u_\sigma),...,A^h(x,u,u_\sigma))}(h^i) = 0,$$

where h^i are all the other functions $f^j, D^\sigma(f^k)$.

We remark that, in order to integrate the system of ODEs representing the evolution equation on $\mathcal{K}$ and the connection $\mathcal{C}|_\mathcal{K}$ representing the reduction function $K(x, b)$, we have to compute the coordinate expressions for the vector fields V_F, V_{G_k}, D_i restricted to $\mathcal{K}$.

For this purpose we choose a coordinate system given by x^i, some coordinate system y^i on $\mathcal{H}$ and the functions $a^i = A^i(x, u, u_\sigma)$.

Using coordinates (x^i, y^j, a^k) the vector fields V_F, V_{G_k}, D_i have a rather simple expression. Obviously $D_i(x^j) = \delta^j_i$, but we have also $D_i(a^k)|_\mathcal{K} = 0$. Furthermore, if the hypotheses of Theorem 5.5 hold, $V_{G_k}(a^k), V_F(a^k)$ depend only on the coordinates a^k. Indeed on $\mathcal{H}$, or equivalently on the submanifold $a^k = 0$ in $\mathcal{K}$, we have $\partial_{a^k} = -\bar{V}_{G_k}$ (the minus sign owing to the fact that we use the pull-back in (13)). This means that for $\alpha = (a^1, ..., a^h)$ we have

$$\partial_{a^k} = \mathbf{\Phi}^*_\alpha(-\bar{V}_{G_k})$$

and it is easy to prove that $\partial_{a^k} = \sum_j \tilde{C}^j_k(\alpha)\bar{V}_{G_j}$ and so

$$\bar{V}_{G_k} = \sum_j C^j_k(\alpha)\partial_{a^j},$$

where C is the inverse matrix of $\tilde{C}$.

Since $\bar{V}_{G_k} = V_{G_k} - \sum_l \tilde{h}^l D_l$ (for some functions $\tilde{h}^l \in \mathcal{F}$) and $D_i(a^k) = 0$, the expression $V_{G_k}(a^l)$ depends only on $a^1, ..., a^h$. The situation is completely similar for V_F.

Remark 6.1 *It is important to note that, in the hypotheses of Theorem 5.5, $V_{G_k}(a^l)$ and $V_F(a^l)$ do not depend on the choice of $\mathcal{H}$ and on the functions g^i but only on the order we choose to apply the pull-back in equation (13).*

Once we have the expressions of V_F, D_i, V_{G_k} in coordinates (x^i, y^j, a^k)

$$\begin{aligned} V_F &= \sum_j \phi^j_0 \partial_{y^j} + \sum_k \psi^k_0 \partial_{a_k} \\ V_{G_h} &= \sum_j \phi^j_h \partial_{y^j} + \sum_k \psi^k_h \partial_{a_k} \\ D_i &= \partial_{x^i} + \sum_j \tilde{\phi}^j_i \partial_{y^j} \end{aligned}$$

the reduced system for the unknown functions $Y^i(x,t), \tilde{A}^l(x,t)$ is

$$\begin{aligned}
\partial_t(Y^i) &= \sum_{k=0}^{h} c^k(t)\phi_k^i(Y(x,t),\tilde{A}(x,t))\\
\partial_t(\tilde{A}^l) &= \sum_{k=0}^{h} c^k(t)\psi_k^i(\tilde{A}(x,t))\\
\partial_{x^j}(Y^i) &= \tilde{\phi}_j^i(Y(x,t),\tilde{A}(x,t))\\
\partial_{x^j}(\tilde{A}^l) &= 0.
\end{aligned}$$

6.2 A non-linear transport equation with dissipation

Let us consider the following equation

$$\partial_t(u) = c^1(t)uu_x + c^2(t)u. \qquad (14)$$

Since, for $c^1(t) = 1, c^2(t) = 0$, equation (14) is a non-linear transport equation $\partial_t(u) = uu_x$ (see for example [47]), if $c^2(t) \leq 0$ and $u \geq 0$ the term u can be considered as a dissipation factor. Furthermore being equation (14) for $c^1(t) = 1, c^2(t) = 0$ the first equation of an Hamiltonian hierarchy (see, e.g., [40]), the complete equation (14) can be considered as a non-isospectral perturbation. We can see (14) as an evolution PDE of the form (12) with

$$\begin{aligned}
F &= 0\\
G_1 &= uu_x\\
G_2 &= u.
\end{aligned}$$

In this case, since V_F, V_{G_1}, V_{G_2} are not linearly independent, Theorem 5.5 can not be applied. However Lemma 5.2 and 5.3 are enough to provide a differential constraint $\mathcal{K}$ for (14), since $V_F \in T\mathcal{K}$ being $V_F = 0$. The characteristic vector fields for G_1 and G_2 are $\bar{V}_{G_1} = V_{G_1} - uD_x$ and $\bar{V}_{G_2} = V_{G_2}$ with corresponding characteristic flows

$$\begin{aligned}
\Phi_a^1(x) &= x - au\\
\Phi_a^1(u) &= u\\
\Phi_a^1(u_x) &= \frac{u_x}{1-au_x}\\
\Phi_a^1(u_k) &= \frac{D_x(\Phi_a^1(u_{k-1}))}{1-au_x}\\
\Phi_b^2(x) &= x\\
\Phi_b^2(u_k) &= e^b u_k.
\end{aligned}$$

In order to construct a differential constraint for (14) we consider the differential constraint $\mathcal{H}$ for V_F defined by $f^1 := u - x^2 = 0$ and all its differential consequences $f^2 := u_x - 2x = 0$, $f^3 := u_{xx} - 2 = 0$ and $f^k := u_{k-1} = 0$ for $k > 2$. From equations $\mathbf{\Phi}_\alpha^*(f^1) = \mathbf{\Phi}_\alpha^*(f^2) = 0$ we obtain

$$\begin{aligned}
a &= -\frac{u_x(xu_x - 2u)}{4(xu_x - u)^2}\\
b &= \log\left(\frac{4(xu_x - u)^2}{uu_x^2}\right)
\end{aligned}$$

and conversely we have

$$
\begin{aligned}
u &= \frac{2ax + 1 - \sqrt{4ax+1}}{2a^2 e^b} \\
u_x &= -\frac{1 - \sqrt{4ax+1}}{ae^b \sqrt{4ax+1}}.
\end{aligned}
$$

Using the previous expressions for a, b in $\mathbf{\Phi}^*_\alpha(f^3) = 0$ we find that $\mathcal{K}$ is defined by the vanishing of the function

$$2u^2 u_{xx} - xu_x^3 + uu_x^2$$

and all its differential consequences. Choosing the coordinate system (x, a, b) on $\mathcal{K}$ we have

$$
\begin{aligned}
V_{G_1} &= -e^{-b}\partial_a \\
V_{G_2} &= -\partial_b \\
D_x &= \partial_x.
\end{aligned}
$$

Hence the function $A(x, t)$ and $B(x, t)$ have to satisfy

$$
\begin{aligned}
\partial_t(A) &= -c^1(t)e^{-B} \\
\partial_t(B) &= -c^2(t) \\
\partial_x(A) &= 0 \\
\partial_x(B) &= 0
\end{aligned}
$$

and we get

$$A(x,t) = A(t) = a_0 - \int_0^t c^1(s) e^{-b_0 + \int_0^s c^2(\tau)d\tau} ds \tag{15}$$

$$B(x,t) = B(t) = b_0 - \int_0^t c^2(s) ds. \tag{16}$$

Therefore, using the expression of u in terms of x, a, b we obtain

$$U(x,t) = \frac{2A(t)x + 1 - \sqrt{4A(t)x+1}}{2A(t)^2 e^{B(t)}}.$$

Indeed Remark 6.1 provides a more general result. In fact, if we consider $A(t), B(t)$ given by equations (15) and (16), and we chose any manifold $\mathcal{H}$ of the form $u = H(x)$ (with $H(x)$ a non-linear function) we have that any function $U(x,t)$ solution to

$$U(x,t) - e^{-B(t)} H(x - A(t)e^{B(t)}U(x,t)) = 0 \tag{17}$$

is a solution to equation (14). Hence in this case the use of the implicit form for the constraint turns out to be more effective than the direct use of the reduction function K.

6.3 Modified heat equation

Let us consider the following equation

$$u_t = c^0 u_{xx} + c^1(t)xu + c^2(t)x^2 u,$$

which can be seen as an equation of the form (12) with

$$\begin{aligned} F &= u_{xx} \\ G_1 &= xu \\ G_2 &= x^2 u. \end{aligned}$$

This equation (with c^1, c^2 constants) has already been studied in [42, 44] and coincides with the Zakai equation of the simplest Kalman filter in one dimension if $c^1 = 1/2$ and c^2 is the derivative of a Brownian motion process (see [3]).
The vector fields V_{G_1}, V_{G_2} form an abelian Lie algebra and admit strong characteristics. Furthermore F, G_1, G_2, $\tilde{G}_1 = u_x$, $\tilde{G}_2 = xu_x$, $\tilde{G}_3 = u$ form a Lie algebra.
Let $\mathcal{H}$ be the submanifold of $J^\infty(\mathbb{R}, \mathbb{R})$ defined by $f^1 := u_x = 0$ and all its differential consequences $f^{k+1} := D_x^k(f^1) = 0$ for any $k \in \mathbb{N}$. It is easy to prove that $V_F, V_{\tilde{G}_i} \in T\mathcal{H}$ for any $i = 1, 2, 3$. In this situation F, G_1, G_2 satisfy the hypotheses of Theorem 5.7 on $\mathcal{H}$ and, in order to find the equations of $\mathcal{K}$, we start by computing the characteristic flows of G_1 and G_2

$$\begin{gathered} \Phi_a^{1*}(x) = \Phi_b^{2*}(x) = x \\ \Phi_a^{1*}(u) = e^{ax} u \\ \Phi_b^{2*}(u) = e^{bx^2} u. \end{gathered}$$

By Theorem 7.1 we obtain the characteristic flows for the derivatives of an order

$$\begin{aligned} \mathbf{\Phi}_\alpha^*(u_x) &= e^{ax+bx^2}((a+2bx)u + u_x) \\ \mathbf{\Phi}_\alpha^*(u_{xx}) &= e^{ax+bx^2}(((a+2bx)^2 + 2b)u + 2(a+2bx)u_x + u_{xx}). \end{aligned}$$

and, using equations $\mathbf{\Phi}_\alpha^*(u_x) = \mathbf{\Phi}_\alpha^*(u_{xx}) = 0$, we get

$$\begin{aligned} u_x &= -(a+2bx)u \\ u_{xx} &= u((2bx+a)^2 - 2b). \end{aligned}$$

Therefore we can express a, b as functions of u_x, u_{xx} as follows

$$\begin{aligned} a &= \frac{(uu_{xx} - u_x^2)x - uu_x}{u^2} \\ b &= \frac{u_x^2 - uu_{xx}}{2u^2} \end{aligned}$$

and equation $\Phi_\alpha(u_{xxx}) = 0$ defining (together with all its differential consequences) the manifold $\mathcal{K}$ is

$$u_{xxx} = \frac{3u_x u_{xx}}{u} - \frac{2u_x^3}{u^2}.$$

Since the manifold $\mathcal{K}$ is four dimensional, we use coordinates (x, u, a, b) on it and, computing the components of the vector fields V_{F_i}, V_{G_j} and D_x, we get

$$\begin{aligned} V_F &= u((2bx+a)^2 - 2b)\partial_u - 4ab\partial_a - 4b^2\partial_b \\ V_{G_1} &= xu\partial_u - \partial_a \\ V_{G_2} &= x^2 u\partial_u - \partial_b \\ D_x &= \partial_x - u(a+2bx)\partial_u. \end{aligned}$$

Hence the equations for U, A, B are

$$\begin{aligned}
\partial_t(U) &= U[c^0((2Bx+A)^2 - 2B) + c^1(t)x + c^2(t)x^2] \\
\partial_t(A) &= -4c^0AB - c^1(t) \\
\partial_t(B) &= -4c^0B^2 - c^2(t) \\
\partial_x(U) &= -U(A+2Bx) \\
\partial_x(A) &= 0 \\
\partial_x(B) &= 0.
\end{aligned}$$

The previous system has a unique solution such that $U(x_0,t_0) = u_0, A(x_0,t_0) = a_0, B(x_0,t_0) = b_0$. Without loss of generality we can suppose that $x_0 = 0$ and, in order to simplify computation, we also suppose $c^1, c^2 \in \mathbb{R}$. The equations for A, B, U in t derivative form a triangular system (linear in A, U and with Riccati form not dependent on time in B) and can be solved explicitly getting some functions $U^0(t) = U(0,t), A^0(t) = A(0,t), B^0(t) = B(0,t)$ satisfying $U^0(t_0) = u_0, A^0(t_0) = a_0$ and $B^0(t_0) = b_0$.
Hence we can explicitly integrate the equations for x and we get

$$\begin{aligned}
A(x,t) &= A^0(t) \\
B(x,t) &= B^0(t) \\
U(x,t) &= e^{-A^0(t)x - B^0(t)x^2}U^0(t).
\end{aligned}$$

The function $U(x,t)$ is the well known Gaussian solution for the modified heat equation.

6.4 An integrable two dimensional system

Let us consider the following system

$$\begin{aligned}
u_t &= c^0u_{xx} + \sum_{i,j\leq N} c^{i,j}(t)v_iv_j \\
v_t &= v_{xx},
\end{aligned}$$

where $v_k = D_x^k(v)$ for $k > 0$ and $v_0 = v$. In the case $c^{i,j} = 0$ for $i, j \geq 2$ and $c^{i,j} = cost$ for $i, j < 2$, this system admits an infinite number of higher order symmetries and a recursion operator (see for example [7]). Furthermore denoting by

$$\begin{aligned}
F &= \begin{pmatrix} c^0u_{xx} \\ v_{xx} \end{pmatrix}, \\
G_{i,j} &= \begin{pmatrix} v_iv_j \\ 0 \end{pmatrix},
\end{aligned}$$

it is easy to prove that $V_F, V_{G_{i,j}}$ form a pro-finite Lie algebra so that this provides a toy-model for the pro-finite Lie algebras used in non-linear filtering problem (see [27]). If we consider the submanifold $\tilde{\mathcal{H}} \subset J^\infty(\mathbb{R}, \mathbb{R}^2)$ given by the equation

$$g = v_k - \sum_{i<k} d_iv_i = 0 \qquad d_i \in \mathbb{R}$$

and its differential consequences, obviously $D_x \in T\tilde{\mathcal{H}}$. Furthermore we have $V_F, V_{G_{i,j}} \in T\tilde{\mathcal{H}}$ and $L_1 = \text{span}\{V_{G_{i,j}}\}$ restricted on $\tilde{\mathcal{H}}$ is finite dimensional. So putting $\tilde{L} = \text{span}\{V_{G_{i,j}}|i, j < k\}$, if

$\mathcal{H}$ is a finite dimensional submanifold of $\tilde{\mathcal{H}}$ such that $\tilde{L}$ has maximal rank on $\mathcal{H}$, $D_x \in T\mathcal{H}$ and $V_F \in T\mathcal{H}$.
The hypotheses of Theorem 5.7 are satisfied: indeed, denoting by $L = \mathrm{span}\{V_F, V_{G_{i,j}}\}$, we have

$$[L, L]|_{\tilde{\mathcal{H}}} = L_1|_{\tilde{\mathcal{H}}} = \tilde{L}|_{\tilde{\mathcal{H}}}.$$

Hereafter, in order to simplify computation, we take $c^{i,j} = 0$ for $i, j > 1$, $c^0 = 1$ and $g = v_{xx} - \beta v$. In this case we choose as submanifold $\mathcal{H}$ of $\tilde{\mathcal{H}}$ the set of zeros of $h = u_x - \gamma u$. Hence, writing $V_1 = V_{G_{0,0}}, V_2 = V_{G_{1,1}}, V_3 = V_{G_{0,1}}$ and denoting by Φ^i the corresponding characteristic flows, we have

$$\begin{aligned}
\Phi^1_a \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u + av^2 \\ v \end{pmatrix}, \\
\Phi^2_b \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u + bv_x^2 \\ v \end{pmatrix}, \\
\Phi^3_c \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u + cvv_x \\ v \end{pmatrix}.
\end{aligned}$$

So, from $\boldsymbol{\Phi}^*_\alpha(h) = \boldsymbol{\Phi}^*_\alpha(D_x(h)) = \boldsymbol{\Phi}^*_\alpha(D_x^2(h)) = 0$, we obtain that $\mathcal{K}$ is defined by

$$u_4 = (u_{xxx} - 4\beta u_x)\gamma + 4\beta u_{xx}.$$

On $\mathcal{K}$ we use the natural coordinate system (x, u, v, v_x, a, b, c), where

$$\begin{aligned}
a &= \frac{((2\beta u - u_{xx})v_x^2 + 2\beta u_x v v_x - 2\beta^2 u v^2)\gamma^3 + (u_{xxx} - 4\beta u_x)v_x^2\gamma^2 +}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
&\quad + \frac{((4\beta u_{xx} - 8\beta^2 u)v_x^2 - 2\beta u_{xxx} v v_x + 8\beta^3 u v^2)\gamma + (8\beta^2 u_x - 2\beta u_{xxx})v_x^2}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
&\quad + \frac{(2\beta^2 u_{xxx} - 8\beta^3 u_x)v^2}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
b &= -\frac{(2uv_x^2 - 2u_x v v_x + (u_{xx} - 2\beta u)v^2)\gamma^3 + (4\beta u_x - u_{xxx})v^2\gamma^2}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
&\quad - \frac{(-8\beta u v_x^2 + 2u_{xxx} v v_x + (8\beta^2 u - 4\beta u_{xx})v^2)\gamma + (8\beta u_x - 2u_{xxx})v_x^2}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
&\quad - \frac{+(2\beta u_{xxx} - 8\beta^2 u_x)v^2}{(2v_x^4 - 4\beta v^2 v_x^2 + 2\beta^2 v^4)\gamma^3 + (-8\beta v_x^4 + 16\beta^2 v^2 v_x^2 - 8\beta^3 v^4)\gamma} \\
c &= -\frac{(u_x v_x^2 - u_{xx} v v_x + \beta u_x v^2)\gamma^2 + (u_{xxx} - 4\beta u_x) v v_x \gamma - u_{xxx} v_x^2}{(v_x^4 - 2\beta v^2 v_x^2 + \beta^2 v^4)\gamma^2 - 4\beta v_x^4 + 8\beta^2 v^2 v_x^2 - 4\beta^3 v^4} \\
&\quad - \frac{4\beta u_{xx} v v_x - \beta u_{xxx} v^2}{(v_x^4 - 2\beta v^2 v_x^2 + \beta^2 v^4)\gamma^2 - 4\beta v_x^4 + 8\beta^2 v^2 v_x^2 - 4\beta^3 v^4}
\end{aligned}$$

In this coordinate system we have

$$\begin{aligned}
V_F &= \left((bv_x^2 + cvv_x + av^2 + u)\gamma^2 - (2b\beta + 2a)v_x^2 - 4\beta c v v_x - (2b\beta^2 + 2a\beta)v^2\right)\partial_u \\
&\quad + \beta v \partial_v + \beta v_x \partial_{v_x} + 2\beta^2 b \partial_a + 2a\partial_b + 2\beta c \partial_c \\
V_1 &= v^2\partial_u - \partial_a \\
V_2 &= v_x^2\partial_u - \partial_b \\
V_3 &= vv_x\partial_u - \partial_c \\
D_x &= \partial_x + \left((bv_x^2 + cvv_x + av^2 + u)\gamma - cv_x^2 - (2b\beta + 2a)vv_x - \beta c v^2\right)\partial_u \\
&\quad + v_x\partial_v + \beta v \partial_{v_x}.
\end{aligned}$$

Fixing (t_0, x_0) and the initial conditions $U(t_0, x_0) = u_0, V(t_0, x_0) = v_0...$, the functions U, V, V_x, A, B, C must satisfy the following overdetermined system of equations

$$\begin{aligned}
\partial_t(A) &= 2\beta^2 B - c^{0,0}(t)\\
\partial_t(B) &= 2A - c^{1,1}(t)\\
\partial_t(C) &= 2\beta C - c^{0,1}(t)\\
\partial_t(U) &= \gamma^2 U + (BV_x^2 + CVV_x + AV^2)\gamma^2 - (2B\beta + 2A)V_x^2 - 4\beta CVV_x\\
&\quad -(2B\beta^2 + 2A\beta)V^2 + c^{0,0}(t)V^2 + c_{1,1}(t)V_x^2 + c^{0,1}(t)VV_x\\
\partial_t(V) &= \beta V\\
\partial_t(V_x) &= \beta V_x\\
\partial_x(A) &= 0\\
\partial_x(B) &= 0\\
\partial_x(C) &= 0\\
\partial_x(U) &= \gamma U + (BV_x^2 + CVV_x + AV^2)\gamma - CV_x^2 - (2B\beta + 2A)VV_x - \beta CV^2\\
\partial_x(V) &= V_x\\
\partial_x(V_x) &= \beta V.
\end{aligned}$$

In the part of system with the t derivative, the equations for A, B, C do not depend on the other variables and are linear and non-homogeneous with respect to A, B, C. So, considering the matrix

$$S(t) = \begin{pmatrix} \frac{1}{2}\cosh(2\beta t) & \frac{\beta}{2}\sinh(2\beta t) & 0\\ \frac{1}{2\beta}\sinh(2\beta t) & \frac{1}{2}\cosh(2\beta t) & 0\\ 0 & 0 & e^{2\beta t} \end{pmatrix},$$

we have

$$\begin{pmatrix} A^0(t)\\ B^0(t)\\ C^0(t) \end{pmatrix} = S(t-t_0)\cdot \begin{pmatrix} a_0\\ b_0\\ c_0 \end{pmatrix} + S(t-t_0)\int_{t_0}^{t} S(-s+t_0)\cdot \begin{pmatrix} c^{0,0}(s)\\ c^{1,1}(s)\\ c^{0,1}(s) \end{pmatrix} ds,$$

where $A^0(t) = A(x_0, t), etc.$. Moreover, since the equations in t for V, V_x are linear, we have

$$\begin{aligned}
V^0(t) &= v_0 e^{\beta(t-t_0)}\\
V_x^0(t) &= v_{x,0} e^{\beta(t-t_0)}
\end{aligned}$$

and, being also the equation for U linear in U and depending on $v^0, v_x^0, ...$, we obtain

$$\begin{aligned}
U^0(t) &= u_0 e^{\gamma^2(t-t_0)} + e^{\gamma^2(t-t_0)}\left(\int_{t_0}^{t} e^{-\gamma^2(s-t_0)}\gamma^2(B^0(s)V_x^0(s)^2 + C^0(s)V^0(s)V_x^0(s))ds\right.\\
&\quad + \int_{t_0}^{t} e^{\gamma^2(s-t_0)}(A^0(s)V^0(s)^2\gamma^2 + (-2\beta B^0(s) - 2A^0(s))V_x^0(s)^2)ds\\
&\quad + \int_{t_0}^{t} e^{-\gamma^2(s-t_0)}((-4\beta C^0(s)V^0(s)V_x^0(s) - 2\beta^2 B^0(s) - 2\beta A^0(s))V^0(s)^2)ds\\
&\quad \left. + \int_{t_0}^{t} e^{\gamma^2(s-t_0)}(c_{0,0}(s)V^0(s)^2 + c_{1,1}(s)V_x^0(s)^2 + c_{0,1}(s)V^0(s)V_x^0(s))ds\right).
\end{aligned}$$

Finally, integrating the equations for x we get

$$\begin{aligned}
A(x,t) &= A^0(t)\\
B(x,t) &= B^0(t)\\
C(x,t) &= C^0(t)\\
V(x,t) &= \frac{V_x^0(t)}{\sqrt{\beta}}\sinh(\sqrt{\beta}(x-x_0)) + V^0(t)\cosh(\sqrt{\beta}(x-x_0))\\
V_x(x,t) &= \sqrt{\beta}V^0(t)\sinh(\sqrt{\beta}(x-x_0)) + V_x^0(t)\cosh(\sqrt{\beta}(x-x_0))\\
U(x,t) &= U^0(t)e^{\gamma(x-x_0)} + e^{\gamma(x-x_0)}\left(\int_{x_0}^x e^{-\gamma(y-x_0)}B^0(t)V_x(y,t)^2\gamma dy\right.\\
&\int_{x_0}^x e^{-\gamma(y-x_0)}((C^0(t)V(y,t)V_x(y,t) + a^0(t)V(y,t)^2 - C^0(t)V_x(y,t)^2)\gamma dy\\
&\left.+\int_{x_0}^x e^{-\gamma(y-x_0)}((-2\beta B^0(t) - 2A^0(t))V(y,t)V_x(y,t) - \beta c^0(t)V(y,t)^2)dy\right).
\end{aligned}$$

6.5 Perturbed KdV equation

Let us consider the following equation

$$\partial_t(u) = (u_{xxx} + uu_x) + c^1(t) + c^2(t)(xu_x + 2u), \tag{18}$$

corresponding to (12) with

$$\begin{aligned}
F &= u_{xxx} + uu_x\\
G_1 &= 1\\
G_2 &= xu_x + 2u.
\end{aligned}$$

If $c^2 = 0$ and c^1 is the derivative of a Brownian motion, equation (18) can be seen as a stochastic perturbation of KdV equation (see [46, 48]) whereas in all the other cases (18) can be interpreted as a non-isospectral perturbation of KdV equation (see e.g.[11, 26]). As submanifold $\mathcal{H}$ we consider the annihilator of $g = u_{xx} + \frac{1}{2}u^2 - \beta_0 u$ (where $\beta_0 \in \mathbb{R}_+$) which contains the one soliton solution to the KdV equation with velocity β_0.
The flows of $V_1 = V_{G_1}$ and of $\bar{V}_2 = V_{G_2} - xD_x$ are given by

$$\begin{aligned}
\Phi_a^1(u) &= u + a\\
\Phi_b^2(u_k) &= e^{(k+2)b}u_k,
\end{aligned}$$

where $u_k = D^k(u)$ and $u_0 = u$. Hence, solving $\mathbf{\Phi}_\alpha^*(g) = \mathbf{\Phi}_\alpha^*(D_x(g)) = 0$, we obtain

$$\begin{aligned}
b &= \frac{1}{4}\log\left(\frac{\beta_0^2 u_x^2}{u_{xxx}^2 + 2u_x^2 u_{xx}}\right)\\
a &= \sqrt{\frac{u_{xxx}^2 + 2u_x^2 u_{xx}}{u_x^2}} - \frac{u_{xxx}}{u_x} - u
\end{aligned}$$

and $\mathcal{K}$ is given by the zero set of

$$u_{xxxx} - \frac{(u_{xxx}u_{xx} - u_x^3)}{u_x}.$$

In order to simplify computation we introduce the coordinate system $(x, \tilde{u}, a, \beta, \gamma)$ on $\mathcal{K}$, where

$$\begin{aligned}
\beta &= e^{2b}\\
\gamma &= \frac{1}{2}u_x^2 + \frac{1}{6}u^3 - \frac{1}{2}\left(\frac{\beta_0}{\beta} - a\right)u^2 - \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)u\\
\tilde{u} &= \begin{cases} \int_d^u \frac{1}{\sqrt{2\left(\gamma - \frac{1}{6}z^3 + \frac{1}{2}\left(\frac{\beta_0}{\beta} - a\right)z^2 + \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)z\right)}}dz & \text{if} \quad u_x > 0\\ -\int_d^u \frac{1}{\sqrt{2\left(\gamma - \frac{1}{6}z^3 + \frac{1}{2}\left(\frac{\beta_0}{\beta} - a\right)z^2 + \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)z\right)}}dz & \text{if} \quad u_x < 0 \end{cases}
\end{aligned}$$

(here $d \in \mathbb{R}$ is such that $\gamma - \frac{1}{6}d^3 + \frac{1}{2}(\frac{\beta_0}{\beta} - a)d^2 + \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)d > 0$). Using this coordinate system it is easy to verify that

$$\begin{aligned}
V_F &= \left(\frac{\beta_0}{\beta} - a\right)\partial_{\tilde{u}}\\
V_{G_1} &= \left(\pm\frac{1}{\sqrt{\gamma - \frac{1}{6}d^3 + \frac{1}{2}\left(\frac{\beta_0}{\beta} - a\right)d^2 + \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)d}}\right)\partial_{\tilde{u}} - \partial_a - \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)\partial_\gamma\\
V_{G_2} &= \left(x \pm \frac{2d}{\sqrt{\gamma - \frac{1}{6}d^3 + \frac{1}{2}\left(\frac{\beta_0}{\beta} - a\right)d^2 + \left(\frac{a\beta_0}{\beta} - \frac{a^2}{2}\right)d}} - \tilde{u}\right)\partial_{\tilde{u}} + 2a\partial_a - 2\beta\partial_\beta + 6\gamma\partial_\gamma\\
D_x &= \partial_x + \partial_{\tilde{u}},
\end{aligned}$$

where in V_{G_1} and V_{G_2} we choose the plus sing if $u_x > 0$ and the minus sign if $u_x < 0$.
The equations for $\tilde{U}, A, B, \Gamma$ are

$$\begin{aligned}
\partial_t(\tilde{U}) &= -c^2(t)\tilde{U} + \left(\frac{\beta_0}{B} - A\right) +\\
&\quad \pm\frac{c^1(t) + 2dc^2(t)}{\sqrt{\Gamma - \frac{1}{6}d^3 + \frac{1}{2}\left(\frac{\beta_0}{B} - A\right)d^2 + \left(\frac{A\beta_0}{B} - \frac{A^2}{2}\right)d}}\\
\partial_t(B) &= -2c^2(t)B\\
\partial_t(A) &= 2c^2(t)A - c^1(t)\\
\partial_t(\Gamma) &= 6c^2(t)\Gamma - \left(\frac{A\beta_0}{B} - \frac{A^2}{2}\right)c^1(t)\\
\partial_x(\tilde{U}) &= 1\\
\partial_x(B) &= 0\\
\partial_x(A) &= 0\\
\partial_x(\Gamma) &= 0
\end{aligned}$$

and this system can be solved as in the previous example.
If we consider the particular case $x_0 = 0$ and $A(0, t_0) = 0, B(0, t_0) = 1, \Gamma(0, t_0) = 0$ we can

explicitly compute

$$\begin{aligned} B(x,t) &= B^0(t) = e^{-2C^2(t)} \\ A(x,t) &= A^0(t) = -(B^0(t))^{-1}\left(\int_{t_0}^t B^0(s)c^1(s)ds\right) \\ \Gamma(x,t) &= \Gamma^0(t) = \frac{1}{2}\beta_0(B^0(t))^{-1}(A^0(t))^2 - \frac{1}{6}(A^0(t))^3, \end{aligned}$$

where $C^2(t) = \int_{t_0}^t c^2(s)ds$. Expressing $\tilde{U}$ as a function of $A^0(t), B^0(t), \Gamma^0(t), u$ we get

$$\begin{aligned} \tilde{U}(A^0(t), B^0(t), \Gamma^0(t), u) &= \pm\int_d^u \frac{dz}{\sqrt{2\left(-\frac{(z+A^0(t))^3}{6} + \frac{\beta_0}{B^0(t)}\frac{(z+A^0(t))^2}{2}\right)}} \\ &= \mp 2\sqrt{\frac{B^0(t)}{\beta_0}}\left(\operatorname{acosh}\left(\sqrt{\frac{3\beta_0}{B^0(t)(u+A^0(t))}}\right)\right. \\ &\qquad \left. - \operatorname{acosh}\left(\sqrt{\frac{3\beta_0}{B^0(t)(d+A^0(t))}}\right)\right) \end{aligned}$$

and we obtain

$$\begin{aligned} U(x,t) &= 3\beta_0 e^{2C^2(t)}\left(\cosh\left(\mp\frac{\sqrt{\beta_0}e^{C^2(t)}\tilde{U}(x,t)}{2}\right.\right. \\ &\qquad \left.\left. + \operatorname{acosh}\left(\sqrt{\frac{3\beta_0}{B^0(t)(d+A^0(t))}}\right)\right)\right)^{-2} - A^0(t). \end{aligned}$$

If we solve the equations for $\tilde{U}(x,t)$ we find

$$\tilde{U}(x,t) = x + e^{-C^2(t)}\left(\int_0^t \left(\frac{\beta_0}{B^0(s)} - A^0(s)\right)ds \pm \frac{2}{\sqrt{\beta_0}}\operatorname{acosh}\left(\sqrt{\frac{3\beta_0}{B^0(t)(d+A^0(t))}}\right)\right)$$

and we have

$$\begin{aligned} U(x,t) &= 3\beta_0 e^{2C^2(t)}\left(\cosh\left(\frac{\sqrt{\beta_0}}{2}\left(e^{C^2(t)}x + \int_{t_0}^t \beta_0 e^{3C^2(s)}ds\right.\right.\right. \\ &\qquad \left.\left.\left. + \int_{t_0}^t e^{3C^2(s)}\left(\int_{t_0}^s e^{-2C^2(\tau)}c^1(\tau)d\tau\right)ds\right)\right)\right)^{-2} + \\ &\qquad + e^{2C^2(t)}\int_{t_0}^t e^{-2C^2(s)}c^1(s)ds, \end{aligned}$$

where we use the parity of the function cosh to eliminate $\mp$ sign.

7 Appendix

In this section we discuss the behavior of the Cartan distribution $\mathcal{C}$ under the action of the characteristic flow Φ_a associated with an evolution vector field V_G. An important consequence of the following Theorem is that $\Phi_a^*(u_\sigma^i)$ is a polynomial function with respect the variable $u_{\sigma'}^k$ if $|\sigma'|$ is sufficiently large.

Theorem 7.1 *Let V_G be an evolution vector field admitting characteristics and let Φ_a be the corresponding characteristic flow. Denoting by A the $n \times n$ matrix*

$$A = (A_i^j) := (D_i(\Phi_a^*(x^j)))|_{i,j},$$

and by $B = (B_j^i)$ the inverse matrix of A, then

$$\Phi_a^*(D_i) = \sum_j B_i^j D_j \tag{19}$$

and, for any $f \in \mathcal{F}$, we have

$$\Phi_a^*(D_i(f)) = \sum_j B_i^j D_j(\Phi_a^*(f)). \tag{20}$$

Proof. Let $\tilde{V}_G = V_G - \sum_i h^i D_i$ be the characteristic vector field of V_G. Since

$$[\tilde{V}_G, D_i] = \sum_j D_i(h^j) D_j,$$

the vector field $D_i^a = \Phi_a^*(D_i)$ solves the equation

$$\partial_a(D_i^a) = \Phi_a^*([\tilde{V}_G, D_i] = \sum_j D_i^a(\Phi_a^*(h^j))D_j^a. \tag{21}$$

In order to prove (19) we show that the vector field $\tilde{D}_i^a := \sum_j B_i^j D_j$ solves equation (21) as well. We start by computing

$$\partial_a(A_i^j) = \partial_a(D_i(\Phi_a^*(x^j))) = D_i(\Phi_a^*(\tilde{V}_G(x^j))) = -D_i(\Phi_a^*(h^j)).$$

Since $B = A^{-1}$ the formula for derivative of the inverse matrix gives

$$\partial_a(B) = -B \cdot \partial_a(A) \cdot B.$$

This means that

$$\partial_a(B_j^i) = \sum_{k,r} B_j^k (D_k(\Phi_a^*(h^r)))B_r^i$$

and we get

$$\begin{aligned} \partial_a(\tilde{D}_j^a) &= \sum_i \partial_a(B_j^i) D_i \\ &= \sum_{i,k,r} B_j^k (D_k(\Phi_a^*(h^r)))B_r^i D_i \\ &= \sum_r \tilde{D}_j^a(\Phi_a^*(h^r))\tilde{D}_r^a. \end{aligned}$$

Hence both $\tilde{D}_i^a$ and D_i^a satisfy equation (21) and we have $\tilde{D}_i^a = D_i^a$. ■

Remark 7.2 *It is important to note that equation* (19) *holds in all $J^\infty(M, N)$ while equation* (20) *holds in $\mathcal{F}_k$ for k sufficiently large.*

Corollary 7.3 *Given an evolution vector field V_G with corresponding characteristic flow Φ_a, the expression $\Phi_a^*(u_\sigma^i)$ is a polynomial function with respect the variable $u_{\sigma'}^k$ if $|\sigma'|$ is sufficiently large.*

Proof. If we apply Theorem 7.1 to $f = u^i$ we get $\Phi_a^*(D_k(u^i)) = \sum_j B_k^j D_j(\Phi_a^*(u^i))$. Since $D_j(\Phi_a^*(u^i))$ is a linear function with respect the variable $u_{\sigma'}^i$ if $|\sigma'|$ is sufficiently large and $B_k^j \in \mathcal{F}_h$ for some $h \in \mathbb{N}$, applying iteratively Theorem 7.1 we obtain the thesis. ■

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