ON HANKEL OPERATORS ON HARDY AND BERGMAN SPACES AND RELATED QUESTIONS

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A Jean Poly, en témoignage de notre amitié

ABSTRACT. In this partly expository paper we analyze the (small) Hankel operator h_b on Hardy and Bergman spaces on a class of smoothly bounded domains of finite type in $\mathbb{C}^n$ which includes the strictly pseudoconvex domains and the convex domains.

We completely characterize the Hankel operators h_b that are bounded, compact, and belong to the Schatten ideal $\mathcal{S}_p$, for $0 < p < \infty$, for this class of domains, generalizing the results of [BPS2][1] where such results have been obtained when Ω is a convex domain of finite type. We describe the main ideas of the proofs which are basically the same as in [BPS2], and present some extensions and generalizations.

In order to characterize the bounded Hankel operators, we prove factorization theorems for functions in $H^1(\Omega)$ and $A^1(\Omega)$ respectively, results that are of independent interest.

1. INTRODUCTION

Let Ω be a smoothly bounded in $\mathbb{C}^n$. Let $0 < p < \infty$ and let $L^p(\Omega)$ denote the Lebesgue space with respect to the volume form. The Bergman space $A^p(\Omega)$ is the closed subspace of $L^p(\Omega)$ consisting of the holomorphic functions.

For $0 < p < \infty$ we denote the Lebesgue spaces on $\partial\Omega$ with respect to the induced surface measure $d\sigma$ by $L^p(\partial\Omega)$. We let $H^p(\Omega)$ denote the Hardy space of holomorphic functions on Ω, with norm given by

$$\|f\|^p_{H^p} := \sup_{0<\varepsilon<\varepsilon_0} \int_{\delta(w)=\varepsilon} |f(w)|^p \, d\sigma_\varepsilon(w),$$

where $d\sigma_\varepsilon$ denotes the surface measure on the manifold $\{\delta(w) = \varepsilon\}$, and $\delta(w) = \operatorname{dist}(w, \partial\Omega)$. To any $f \in H^p(\Omega)$ corresponds a unique boundary function in $L^p(\partial\Omega)$, that we still denote by f, obtained as normal almost everywhere limit, [St]. Thus, we may identify $H^p(\Omega)$ with a closed subspace of $L^p(\partial\Omega)$.

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[1]This paper can be retrieved at the URL http://calvino.polito.it/~peloso/research.html.

We denote by B_Ω and S_Ω the Bergman and Szegö kernel, respectively, and by P_B and P_S the Bergman and Szegö projection, respectively:

$$P_B f(z) := \int_\Omega B_\Omega(z,w) f(w) dV(w)$$
$$P_S g(z) := \int_{\partial\Omega} S_\Omega(z,\zeta) g(\zeta)\, d\sigma(\zeta)\,,$$

for $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$.

Let $b \in H^2(\Omega)$ ($b \in A^2(\Omega)$). The small Hankel operator h_b on the Hardy space $H^2(\Omega)$ (Bergman space $A^2(\Omega)$), with symbol b, is defined for $g \in H^2(\Omega)$ ($f \in A^2(\Omega)$) as

$$h_b(g) := P_S(b\overline{g}) \qquad \Big(h_b(f) := P_B(b\overline{f}) \Big).$$

We wish to describe the regularity properties of h_b in terms of the ones of the symbol b. More precisely, we will characterize the bounded, compact, and in the Schatten ideals (see Section 5) Hankel operators on Hardy and Bergman spaces, when Ω is in a class of pseudoconvex domains of finite type that now we introduce.

Definition 1.1. We say that Ω is an H-domain if it is a smoothly bounded pseudoconvex domain of finite type and if, moreover, for each $\zeta \in \partial\Omega$ there exist a neighborhood V_ζ and a biholomorphic map Φ_ζ defined on V_ζ such that $\Phi_\zeta(\Omega \cap V_\zeta)$ is geometrically convex.

We recall that a point $\zeta \in \partial\Omega$ is said to be of finite type if the (normalized) order of contact with $\partial\Omega$ of complex varieties at ζ is finite. By the result in [BoSt] and our assumption it suffices to consider the order of contact of $\partial\Omega$ at ζ with 1-dimensional complex manifolds, see [BoSt] and references therein. The domain Ω is said to be of finite type if every point on $\partial\Omega$ is of finite type. We denote by M_Ω the maximum of the types of points on $\partial\Omega$.

Notice that the class of H-domains contains both the convex domains of finite type and the strictly pseudoconvex domains.

The aim of this paper is two-fold. On one hand we wish to describe the results in [BPS2], presenting the main ideas and providing some details not explicitly included in that paper. In [BPS2] we concentrated our effort on the case of a convex domain of finite type Ω. Thus here, on the other hand, we present some extensions and generalization of those results, in particular in the context of H-domains.

2. Basic facts and notation

We begin by describing the geometry of an H-domain Ω. This is done locally, using a partition of unity. Moreover, in a neighborhood of a point $\zeta \in \partial\Omega$, using local coordinates and the assumption, we may in fact assume that Ω is geometrically convex. Thus, we do not lose generality if we assume that it is globally convex.

Then, there exist an $\varepsilon_0 > 0$ and a defining function ϱ for Ω such that for $-\varepsilon_0 < \varepsilon < \varepsilon_0$ the sets $\Omega_\varepsilon := \{z \in \mathbb{C}^n : \varrho(z) < \varepsilon\}$ are all convex. Moreover, denote by $U = U_{\varepsilon_0}$ the tubular neighborhood of $\partial\Omega$ given by $\{z \in \mathbb{C}^n : -\varepsilon_0 < \varrho(z) < \varepsilon_0\}$. By taking $\varepsilon_0 > 0$ sufficiently small, we may assume that on $\overline{U}$ the normal projection π of U onto $\partial\Omega$ is uniquely defined.

Let $z \in U$ and let v be a unit vector in $\mathbb{C}^n$. We denote by $\tau(z, v, r)$ the distance from z to the surface $\{z' : \varrho(z') = \varrho(z) + r\}$ along the complex line determined by v. One of the basic relations among the quantities defined above is the following. There exists a constant C depending only on the geometry of the domain such that given $z \in U$, any unit vector $v \in \mathbb{C}^n$ and $r \leq r_0$ and $\eta < 1$ we have

$$C^{-1}\eta^{1/2}\tau(z, v, r) \leq \tau(z, v, \eta r) \leq C\eta^{1/M_\Omega}\tau(z, v, r). \tag{1}$$

We next define the *r-extremal orthonormal basis* $\{v^{(1)}, \ldots, v^{(n)}\}$ at z. The first vector is given by the direction transversal direction to the level sets of ϱ, pointing outward. In the complex directions orthogonal to $v^{(1)}$ we choose $v^{(2)}$ in such a way that $\tau(z, v^{(2)}, r)$ is maximum. We repeat the same procedure to determine the remaining elements of the basis. We set

$$\tau_j(z, r) = \tau(z, v^{(j)}, r).$$

The polydisc $Q(z, r)$ is now given as

$$Q(z, r) = \{w : |w_k| \leq \tau_k(z, r),\ k = 1, \ldots, n\}.$$

Here $(w_1, \ldots, w_n)$ are the coordinates determined by r-extremal orthonormal basis $\{v^{(1)}, \ldots, v^{(n)}\}$ at z. Notice that these coordinates $(w_1, \ldots, w_n) = (w_1^{z,r}, \ldots, w_n^{z,r})$ depend on z and on r.

The quasi-distance is defined by setting

$$d_b(z, w) = \inf\{r : w \in Q(z, r)\}. \tag{2}$$

Notice that by the above properties the sets $Q(z, r)$ are in fact equivalent to the balls in the quasi-distance d_b. We also consider balls on the boundary $\partial\Omega$ defined in terms of d_b. For $\zeta \in \partial\Omega$ and $r > 0$ we set

$$B(\zeta, r) = \{z \in \partial\Omega : d_b(z, \zeta) < r\}.$$

These balls are equivalent to the sets $Q(\zeta, r) \cap \partial\Omega$.

Moreover, we define the function d on $\overline{\Omega} \times \overline{\Omega}$ by setting

$$d(z, w) = \delta(z) + \delta(w) + d_b\big(\pi(z), \pi(w)\big), \tag{3}$$

where π is the normal projection of a point z onto the boundary.

We now set

$$\tau(z, r) = \prod_{j=2}^{n} \tau_j(z, r).$$

Notice that, by construction, $\tau_1(z, r) \simeq r$.

When Ω is strictly pseudoconvex, we can simply set $\tau_j(z,r) = r^{\frac{1}{2}}$ when $j = 2,\ldots,n$, and consequently $\tau(z,r) = r^{\frac{n-1}{2}}$.

Let $\mathrm{Vol}\,(E)$ denote the Lebesgue measure of a set E, while σ denotes the induced surface measure. Then

$$\mathrm{Vol}\,\big(Q(w,r)\big) \simeq r^2\tau(w,r)^2, \qquad \sigma\big(B(w,r)\big) \simeq r\tau(w,r)^2.$$

As we said before, all these definitions are local, and may be given in the context of H-domains.

We now consider the behaviour of the Szegö and Bergman kernels. this is well known, and it described in terms of a pseudo-distance d on $\overline{\Omega}\times\overline{\Omega}$, both for strictly pseudoconvex domains [F] and for convex domains of finite type [Mc], [McS1], [McS2]. We now recall the fundamental estimates which describe this behaviour. There exist constants $C_{\gamma,\gamma'} > 0$ such that for all $(z,w) \in \overline{\Omega}\times\overline{\Omega}$ with $d(z,w) \neq 0$

$$|\partial_z^\gamma \overline{\partial}_w^{\gamma'} B_\Omega(z,w)| \leq C_{\gamma,\gamma'} d(z,w)^{-|\gamma|-|\gamma'|}\mathrm{Vol}\,\big(Q(z,d(z,w))\big)^{-1}, \tag{4}$$

[McS1], and

$$|\partial_z^\gamma \overline{\partial}_w^{\gamma'} S_\Omega(z,w)| \leq C_{\gamma,\gamma'} d(z,w)^{-|\gamma|-|\gamma'|}\sigma\big(B(\pi(z),d(z,w))\big)^{-1}, \tag{5}$$

[McS2]. In particular,

$$B_\Omega(w,w)^{-1} \simeq \mathrm{Vol}\,\big(Q(w,\delta(w))\big).$$

We point out that the above estimates do not reflect the full strength of the results in [McS2], since in (4) and (5) we do not distinguish between derivatives in different directions.

At this point we remark that the argument which has been used by Fefferman to extend estimates from strictly convex domains to strictly pseudo-convex domains may be used to extend the above estimates from convex domains of finite type to H-domains. Following [F] we prove this general, in fact well known, result.

Proposition 2.1. *Let D, W be smoothly bounded domains in $\mathbb{C}^n$, D of finite type and $W \subseteq D$. Let $\zeta \in \partial D$ and assume that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $Q(\zeta,\varepsilon_1)\cap\partial D = Q(\zeta,\varepsilon_1)\cap\partial W$ and that* $\mathrm{dist}\,\big(\mathrm{Q}(\zeta,\varepsilon_1),\partial\mathrm{W}\setminus\partial\mathrm{D}\big) > \varepsilon_2$. *Then, there exists $F \in \mathcal{C}^\infty(\mathbb{C}^n\times\mathbb{C}^n)$ such that*

$$B_D(z,w) = B_W(z,w) + F(z,w) \qquad \textit{on} \quad Q(\zeta,\varepsilon_1/2)\times Q(\zeta,\varepsilon_1/2).$$

Proof. The proof is the same as the one of Lemma 1 in [F]. It suffices to recall that the $\bar{\partial}$-Neumann problem is hypoelliptic up to the boundary, due to the subelliptic estimates [Ca]. However, we provide the details for sake of completeness.

First of all, we remark that instead of the polydiscs Q in the pseudo-distance, we could have used the classical Euclidean balls as well.

We wish to prove that

$$B_D(z,w) - B_W(z,w) \in \mathcal{C}^\infty\big(Q(\zeta,\varepsilon_1/2)\times Q(\zeta,\varepsilon_1/2)\big).$$

Let $w \in Q(\zeta, \varepsilon_1/2)$. If $|z-w| \geq \varepsilon_1/2$ then both B_D and B_W are smooth functions. So, we restrict to the case $|z-w| \leq \varepsilon_1/2$, in which case also $\operatorname{dist}(z, \partial W \setminus \partial D) > \varepsilon_2$. Now we set

$$u_w(z) = B_W(z,w)\chi_W(z) - B_D(z,w) \qquad\qquad z \in D.$$

We claim that u_w, which is in $L^2(D)$, is orthogonal to $A^2(D)$. In fact, for $h \in A^2(D)$,

$$\begin{aligned}\langle h, u_w\rangle &= \int_W h(z)B_W(w,z)dV(z) - \int_D h(z)B_D(w,z)dV(z)\\ &= h(w) - h(w) = 0.\end{aligned}$$

Moreover, $v_w = \bar\partial u_w$ has support in $\partial W \setminus \partial D$. Therefore, u_w is the canonical solution to the $\bar\partial$-equation with data having support contained in $\partial W \setminus \partial D$.

By the pseudo-local property of the solution of the above equation, it follows that u_w is smooth as a function of z away from $\partial W \setminus \partial D$, say for $z \in Q(\zeta, \varepsilon_1/2)$. Since u_w depends smoothly on $w \in Q(\zeta, \varepsilon_1/2)$ the conclusion follows. □

Corollary 2.2. *Let Ω be an H-domain. Then the Szegö and Bergman kernels S_Ω and B_Ω satisfy the estimates (4) and (5) respectively.*

Proof. Let $\zeta \in \partial\Omega$. By definition, there exists a neighborhood V_ζ of ζ in $\mathbb{C}^n$ and a biholomorphic map Φ_ζ defined on V_ζ such that $\Phi_\zeta(V_\zeta \cap \Omega) = W_\zeta$ is geometrically convex. By possibly shrinking V_ζ, we may assume that Φ_ζ is biholomorphic in a neighborhood of $\overline{V_\zeta}$. Then $\Phi_\zeta(V_\zeta \cap \partial\Omega)$ is a hypersurface of finite type and it is easy to see that there exists a smoothly bounded convex domain of finite type D such that $W_\zeta \subseteq D$ and $\partial D \cap \partial W_\zeta = \Phi_\zeta(V_\zeta \cap \partial\Omega)$.

By Proposition 2.1 and the transformation rule for the Bergman kernel, for some $\varepsilon_0 > 0$, on $Q(\zeta, \varepsilon_0) \times Q(\zeta, \varepsilon_0)$ we have

$$\begin{aligned}B_\Omega(z,w) &= B_{V_\zeta \cap \Omega}(z,w) + F_1(z,w)\\ &= J(z) B_{W_\zeta}\big(\Phi_\zeta(z), \Phi_\zeta(w)\big)\overline{J(w)} + F_1(z,w)\\ &= J(z)\big[B_D\big(\Phi_\zeta(z), \Phi_\zeta(w)\big) + F\big(\Phi_\zeta(z), \Phi_\zeta(w)\big)\big]\overline{J(w)} + F_1(z,w),\end{aligned}$$

where J denotes the determinant Jacobian of the mapping Φ.

Since the estimates (4) hold for the kernel B_D and F, F_1 are $\mathcal{C}^\infty$, the result for the Bergman kernel follows.

The result for the Szegö kernel now follows as well. By the estimates for the Bergman kernel, arguing as in [McS2] Propositions 3.1, 3.3–3.5 we obtain the estimates (5). □

Remark 2.3. Having the estimates (4) and (5) at our disposal on any H-domain, we can deduce mapping and duality theorems for the operators and function spaces under consideration. The results in [KL3] immediately apply since they were proved under the solely assumption on sizes of the kernels. The domains for which such theorems hold are called *strongly admissible* and satisfying a *homogeneity condition*, see [KL3], Definition 3.1, and the theorems that follow.

The next result is key in our analysis.

Theorem 2.4. *Let Ω be a smoothly bounded pseudoconvex H-domain of finite type in $\mathbb{C}^n$. Then there exist a neighborhood U of the boundary $\partial\Omega$ and a function $H \in \mathcal{C}^\infty(\mathbb{C}^n \times U)$ such that the following conditions hold:*

(i) $H(\cdot, w)$ is holomorphic on Ω for all $\zeta \in U$;

(ii) there exists a constant $c_1 > 1$ such that

$$\frac{1}{c_1} d(z,w) \le |H(z,w)| \le c_1 d(z,w).$$

Proof. For Ω a strictly pseudoconvex domain this is Henkin's classical result [H]. For Ω a convex domain of finite type is due to Diederich and Fornæss [DF]. In this case the support function H may be taken such that $\Re H(z,w) > 0$ for $(z,w) \in \Omega \times U$.

When Ω is an H-domain of finite type one can use the by now standard argument originated in [H] to construct a globally defined support function H patching together the support functions defined a finite covering of the neighborhood of the boundary U. □

The following lemma gives estimates for certain integrals that frequently occur. For a real number β we set $\beta^* = \beta/2$ if $\beta \ge 0$ and $\beta^* = \beta/M_\Omega$ if $\beta < 0$. Notice that if Ω is strictly pseudoconvex $\beta^* = \beta/2$ for all real β.

Lemma 2.5. *Let Ω be an H-domain and let $a, b, \alpha, \beta \in \mathbb{R}$ satisfy the following conditions:*

$$a + (n-1)b^* + 1 > 0, \qquad \text{and} \qquad a + \alpha + 2 - (n-1)\left(-(b+\beta+2)\right)^* < 0.$$

Then there exists a constant $c > 0$ such that

$$\int_\Omega d(z,w)^\alpha \tau(z, d(z,w))^\beta \delta(z)^a \tau(z,\delta(z))^b dV(z) \le c\delta(w)^{a+\alpha+2} \tau(w,\delta(w))^{b+\beta+2}.$$

If $\alpha + 1 - (n-1)\left(-(\beta+2)\right)^* < 0$, then there exists a constant $c > 0$ such that

$$\int_{\partial\Omega} d(z,w)^\alpha \tau(z, d(z,w))^\beta \, d\sigma(z) \le c\delta^{\alpha+1}(w) \tau^{\beta+2}(w, \delta(w)).$$

Proof. This result is in fact standard. See [PhS], [McS1] and [BPS2] for proofs. Again, since the results depend on local estimates, the extension of the proof in [BPS2] to H-domains is trivial. □

3. Factorization of Hardy and Bergman spaces

Our first results are the factorization of the spaces $H^1(\Omega)$ and $A^1(\Omega)$ when Ω is an H-domain.

Theorem 3.1. *Let Ω be an H-domain. Let $p>1$ and let p' be its conjugate exponent. Then there exists a constant $c>0$ such that every function $f\in H^1(\Omega)$ may be written as*

$$f=\sum_{i=1}^{\infty} F_i G_i\ ,$$

with

$$\sum_{i=1}^{\infty} \|F_i\|_{H^p}\|G_i\|_{H^{p'}} \le c\|f\|_{H^1}\ .$$

Proof. This theorem was first proved (when the dimension $n>1$) in the case of the unit ball in [CRW]. It was later extended to the case of a strictly pseudoconvex domain in [KL2]. Their proofs, which contain difficult technicalities, rely on the explicit expression of the Szegö kernel, or on Fefferman's asymptotic expansion.

Our proof instead relies on the existence of a support function H as in Theorem 2.4, and it turns out to be simpler even in the already known cases.

We present our proof in the case of a strictly pseudoconvex domain, situation in which the role played by the support function is already clear, while technicalities are simpler. We refer to [BPS2] for the general case. In general, we need to deal with atoms satisfying extra moment conditions (see definition below), since the relation between the Szegö kernel and the support function is not explicit. It is straightforward to extend the proof from convex domains to H-domains.

Let then Ω be a strictly pseudoconvex domain. In [KL2] it is proved that $H^1(\Omega)$ admits an atomic decomposition. We say that a function a defined on $\partial\Omega$ is an *atom* if its support is contained in a ball $B=B(\zeta,r)$ and the following two conditions are satisfied:

(*i*) $|a(z)|\le \sigma(B)^{-1}$;

(*ii*) $\displaystyle\int_{\partial\Omega} a(z)d\sigma(z)=0.$

We say that A is a holomorphic atom if $A=S_\Omega(a)$ for some atom a. It is important to remark that there exists a constant $c>0$ such that for all holomorphic atoms A we have $\|A\|_{H^1}\le c$, see [KL2] Theorem 2.6.

The atomic decomposition for $H^1(\Omega)$ implies that there exists a constant $c>0$ such that for each $f\in H^1(\Omega)$ there exist holomorphic atoms A_j and constants λ_j such that

$$f=\sum_{j=1}^{\infty}\lambda_j A_j, \qquad \text{and} \qquad \|f\|_{H^1}\le c\sum_j |\lambda_j|.$$

In order to prove the factorization theorem it suffices to factor each atom, that is, given any holomorphic atom A, to show that there exist holomorphic functions F and G such that $A=FG$ and

$$\|F\|_{H^p}\|G\|_{H^{p'}}\le c \tag{6}$$

for a constant $c>0$ independent of A.

Let $A = P_S(a)$, where $\operatorname{supp} a \subseteq B(w^{(0)}, r)$. We set $\tilde{w}^{(0)} = w^{(0)} - r\nu_{w^{(0)}}$, where $\nu_{w^{(0)}}$ is the outward unit vector at the point $w^{(0)}$ on $\partial\Omega$. Let $H(z,w)$ be the support function on Ω as in Theorem 2.4. We write

$$F(z) = A(z)\big(H(z,\tilde{w}^{(0)})\big)^{\alpha}, \qquad \text{and} \qquad G(z) = \big(H(z,\tilde{w}^{(0)})\big)^{-\alpha},$$

where $\alpha > 0$ is to be determined later.

Hence, we only need to prove (6). We have

$$\begin{aligned}
\|G\|_{H^p}^p &\le \int_{\partial\Omega} |H(z,\tilde{w}^{(0)})|^{-\alpha p} d\sigma(z) \\
&\le \int_{\partial\Omega} \big(r + d(z,w^{(0)})\big)^{-\alpha p} d\sigma(z) \\
&\le c r^{-\alpha p} \sigma(B(w^{(0)}, r)),
\end{aligned}$$

by Lemma 2.5, for $\alpha > n/p$, where the constant c is of course independent of A.

In order to estimate $\|F\|_{H^{p'}}$, let $C > 1$ be fixed. There exists a constant $c > 0$ such that

$$|A(z)| \le c\big(r/d(z,w^{(0)})\big)^{\beta} \sigma\big(B(w^{(0)}, d(z,w^{(0)}))\big)^{-1},$$

when $d(z,w^{(0)}) > Cr$ and $0 < \beta < 1/2$, see [KL2] or [BPS2] Lemma 4.7. Then,

$$\begin{aligned}
&\int_{\partial\Omega\setminus B(w^{(0)},Cr)} |F(z)|^{p'} d\sigma(z) \\
&\quad\le c \int_{\partial\Omega\setminus B(w^{(0)},Cr)} |A(z)|^{p'} \big(r + d(z,w^{(0)})\big)^{\alpha p'} d\sigma(z) \\
&\quad\le c r^{\beta p'} \int_{\partial\Omega\setminus B(w^{(0)},Cr)} d(z,w^{(0)})^{p'(\alpha-\beta)} \sigma\big(B(w^{(0)}, d(z,w^{(0)}))\big)^{-p'} d\sigma(z) \\
&\quad\le c r^{\beta p'} \int_{\partial\Omega} d(z,w^{(0)})^{p'(\alpha-\beta-n)} d\sigma(z) \\
&\quad\le c r^{\alpha p'} \sigma(B(w^{(0)}, r))^{1-p'},
\end{aligned}$$

by Lemma 2.5, provided that $p'(\alpha-\beta-n)+n < 0$, i.e. $\alpha < n/p+\beta$. Notice that in the estimate above we have used the assumption that Ω is a strictly pseudoconvex domain.

Next,

$$\begin{aligned}
\int_{B(w^{(0)},Cr)} |F(z)|^{p'} d\sigma(z) &\le \int_{B(w^{(0)},Cr)} |A(z)|^{p'} \big(r + d(z,w^{(0)})\big)^{\alpha p'} d\sigma(z) \\
&\le c r^{\alpha p'} \int_{B(w^{(0)},Cr)} |A(z)|^{p'} d\sigma(z) \\
&\le c r^{\alpha p'} \|A\|_{H^{p'}}^{p'} \\
&\le c r^{\alpha p'} \sigma(B(w^{(0)}, r))^{1-p'}.
\end{aligned}$$

Hence, if α has been chosen such that $n/p < \alpha < n/p + \beta$,

$$\|F\|_{H^{p'}}\|G\|_{H^p} \leq cr^\alpha \sigma(B)^{-1/p} r^{-\alpha}\sigma(B)^{1/p} \leq c,$$

where $B = B(w^{(0)}, r)$, as we wished to proved. $\square$

As a corollary we obtain the factorization theorem for the Bergman space $A^1(\Omega)$.

Corollary 3.2. *Let Ω be an H-domain. Let $p > 1$ and let p' be its conjugate exponent. Then there exists a constant $c > 0$ such that every function $f \in A^1(\Omega)$ may be written as*

$$f = \sum_{i=1}^{\infty} F_i G_i \ ,$$

with

$$\sum_{i=1}^{\infty} \|F_i\|_{A^p}\|G_i\|_{A^{p'}} \leq c\|f\|_{A^1} \ .$$

Proof. Let $\widetilde{\Omega}$ be the domain in $\mathbb{C}^{n+1}$ defined by

$$\widetilde{\Omega} = \{(z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} : \varrho(z) + |z_{n+1}|^2 < 0\}.$$

Notice that $\widetilde{\Omega}$ is smooth, bounded and pseudoconvex. Moreover, it is also an H-domain, as it is easy to check directly. Thus, Theorem 3.1 holds on $\widetilde{\Omega}$. By [KLR] Proposition 2.3 (see also [L]) we know that for $0 < p \leq \infty$,

$$A^p(\Omega) = \{g : g(z) = f(z, 0) \text{ for some } f \in H^p(\widetilde{\Omega})\}.$$

By Theorem 3.1 we have that

$$\begin{aligned} g(z) = f(z,0) &= \sum_{j=0}^{\infty} \tilde{F}_j(z,0)\tilde{G}_j(z,0) \\ &= \sum_{j=0}^{\infty} F_j(z)G_j(z), \end{aligned}$$

where $F_j \in A^p(\Omega)$ and $G_j \in A^{p'}(\Omega)$, by [KLR] Proposition 2.3 again. By part (b) in the same proposition we also have the required norm estimate, i.e. $\|F_j\|_{A^p(\Omega)} \simeq \|\tilde{F}_j\|_{H^p(\tilde{\Omega})}$ and analogously for G_j and $\tilde{G}_j$. This proves the corollary. $\square$

We remark that it is also possible to prove the factorization of $A^1(\Omega)$ directly, using the atomic decomposition and factorizing each atom, in the same fashion as in the Hardy case. This method would have the advantage of being more explicit. Here we prefer to follow the previous more direct route for sake of brevity.

4. Boundedness and compactness of Hankel operators

We now show how boundedness and compactness of Hankel operators follow from the factorization theorems.

We consider, in greater generality, the Hankel operator h_b defined on $H^p(\Omega)$ for $p > 1$, and on $A^p(\Omega)$ for the same range of p. The characterization of bounded and compact Hankel operators will not differ from the case $p = 2$.

In order to describe our results we need to introduce some more function spaces. Let

$$\|g\|_{BMO} := \sup_{\zeta,\varepsilon} \frac{1}{\sigma(B(\zeta,\varepsilon))} \int_{B(\zeta,\varepsilon)} |g(w) - g_{B(\zeta,\varepsilon)}| \, d\sigma(w),$$

where $g_{B(\zeta,\varepsilon)}$ is the average of g over the ball $B(\zeta,\varepsilon)$.

The space *BMO* is the space of functions modulo constants such that the above semi-norm is finite. Moreover, *VMO* is defined as the subspace of *BMO* which is the closure of the continuous functions in the *BMO* topology. We define *BMOA* and *VMOA* as the spaces of holomorphic functions in $H^1(\Omega)$ such that their boundary values, that we keep denoting by f, are in *BMO* and *VMO* respectively. It is well known that *BMOA* and *VMOA* are closed subspaces of *BMO* and *VMO*, respectively.

Theorem 4.1. *Let Ω be an H-domain, and let $1 < p < \infty$. Then the Hankel operator on the Hardy space*

$$h_b : H^p(\Omega) \to H^p(\Omega)$$

is bounded if and only if $b \in$ BMOA, with equivalence of norms. Moreover h_b is compact if and only if $b \in$ VMOA.

The Bloch space $\mathcal{B}$ is the space of holomorphic functions f in Ω such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\Omega} \delta(z) \left(|f(z)| + |\nabla f(z)|\right) < \infty.$$

It is well known that a holomorphic function f is in $\mathcal{B}$ if and only if there exists a positive integer k and a constant C_k such that, for $z \in \Omega$, one has $|\partial^\gamma f(z)| \leq C_k \delta(z)^{-k}$, for all γ with $|\gamma| \leq k$. The little Bloch space $\mathcal{B}_0$ is the subspace of $\mathcal{B}$ closure of the holomorphic functions continuous up the boundary. Equivalently, $f \in \mathcal{B}$ is in $\mathcal{B}_0$ if and only if $\lim_{\delta(z)\to 0} \delta(z)|\nabla f(z)| = 0$.

In the case of Bergman spaces we have the following result.

Theorem 4.2. *Let Ω be an H-domain, and let $1 < p < \infty$. Then the Hankel operator on the Bergman space*

$$h_b : A^p(\Omega) \to A^p(\Omega)$$

is bounded if and only if $b \in \mathcal{B}$, with equivalence of norms. Moreover h_b is compact if and only if $b \in \mathcal{B}_0$.

Proof of Theorems 4.1 and 4.2. We prove the theorem only in the Hardy space case, since the two proofs follow the same lines.

We begin by proving the necessity of the condition for boundedness.

By [KL3] we know that *BMOA* is the dual space of $H^1(\Omega)$, with equality of norms. Having the factorization of $H^1(\Omega)$ at our disposal, the proof is now the same as the one of Theorems VII and VIII in [CRW]. We indicate the argument for sake of completeness.

In order to prove that h_b bounded implies that $b \in BMOA$ it suffices to show that b is in the dual space of H^1. Let $g \in H^1$, and write $g = \sum_{i=1}^{\infty} F_i G_i$ with $F_i \in H^p(\Omega)$ and $G_i \in H^{p'}(\Omega)$, according to Theorem 3.1. Then,

$$\begin{aligned} |\langle b, g\rangle| &\le \sum_{i=1}^{\infty} |\langle b, F_i G_i\rangle| = \sum_{i=1}^{\infty} |\langle h_b(F_i), G_i\rangle| \\ &\le \sum_{i=1}^{\infty} \|h_b(F_i)\|_{H^p} \|G_i\|_{H^{p'}} \le c \sum_{i=1}^{\infty} \|F_i\|_{H^p} \|G_i\|_{H^{p'}} \\ &\le c\|g\|_{H^1} . \end{aligned}$$

Hence, $\|b\|_{BMO} \le c\|h_b\|$.

Conversely, if $b \in BMOA$, then b is the dual space of $H^1(\Omega)$, and

$$|\langle h_b(g), f\rangle| = |\langle b, gf\rangle| \le c\|gf\|_{H^1(\Omega)} \le c\|g\|_{H^p(\Omega)} \|f\|_{H^{p'}(\Omega)}.$$

This shows that h_b is bounded on $H^p(\Omega)$ and completes the proof in the bounded case.

We point out that the arguments given in [BPS2] for the factorization and the characterization of bounded Hankel operators are organized in a different way. We do not rely on the atomic decomposition of $H^1(\Omega)$. Such a decomposition is known, at least in the classical cases, to be equivalent to the duality between $H^1(\Omega)$ and *BMOA*, and this duality holds when Ω is a convex domain of finite type [KL3]. In [BPS2] we elected to follow a direct route, following [CRW], proving in fact the factorization theorem and the characterization of bounded Hankel operators essentially at the same time. The atomic decomposition for Hardy spaces $H^p(\Omega)$ on convex domains of finite type, $0 < p \le 1$, is proved in [GP].

In order to prove the statement about compactness, we rely again on duality theorems. Since some of these results seem not to appear explicitely in the literature, we present them here.

Theorem 4.3. *Let Ω be an H-domain. Then $P_S : L^\infty(\partial\Omega) \to BMOA$ is bounded. Moreover, the following conditions are equivalent:*

(*i*) *$b \in BMOA$;*

(*ii*) *$b = P_S(\phi)$ for some $\phi \in L^\infty(\partial\Omega)$.*

Furthermore, the following conditions are equivalent:

(*iii*) *$b = P_S(\phi)$ for some $\phi \in \mathcal{C}(\partial\Omega)$;*

(*iv*) *$b \in VMOA$;*

(*v*) *$\delta(z)^{2l-1}|\nabla^l b(z)|^2 dV(z)$ is a vanishing Carleson measure.*

Finally, $H^1(\Omega) = \big(VMOA\big)^*$*, with equivalence of norms.*

Proof. The proof of the equivalence of (i) and (ii) appears in [KL1] and [KL3], for Ω a strictly pseudoconvex domain, and a convex domain of finite type, respectively. These papers also contain the proof of the boundedness of P_S on the given spaces. The results in [KL3] are valid also in the case of H-domains, see Remark 2.3.

We now show that (iii) implies (iv), Let $\phi \in \mathcal{C}(\partial\Omega)$ and $\varepsilon > 0$ be given. Let $\psi \in \mathcal{C}^\infty(\partial\Omega)$ be such that $\|\phi - \psi\|_{L^\infty(\partial\Omega)} < \varepsilon$. It is easy to see that $P_S(\psi) \in VMOA$. By the boundedness of P_S it follows that $\|P_S(\phi) - P_S(\psi)\|_{BMOA} \leq c\varepsilon$. Since $VMOA$ is a closed subspace, $P_S(\phi) \in VMOA$.

The proof that (iv) is equivalent to (v) is slightly more technical, and appears in Propositions 4.6, 4.7 in [BPS2].

It remains to show that (iv) implies (iii), and duality. We shall prove both in the same time.

We first remark that $H^1(\Omega)$ is embedded into $(VMOA)^*$. Indeed, since $BMOA = \big(H^1(\Omega)\big)^*$, with equivalence of norms, each element f of H^1 gives rise to a continuous linear functional L_f on $VMOA$, with $\|L_f\| \leq c\|f\|_{H^1(\Omega)}$.

Next we set

$$E := \{b = P_S(\phi) \,:\, \phi \in \mathcal{C}(\partial\Omega)\}.$$

It is a Banach space when endowed with the norm

$$\|b\|_E = \inf\{\|\phi\|_\infty \,:\, b = P_S(\phi)\,, \phi \in \mathcal{C}(\partial\Omega)\}.$$

Moreover, by (iii), E is continuously embeds into $VMOA$, with dense image, since holomorphic functions which are $\mathcal{C}^\infty$ up to the boundary are dense.

We want to show that E coincides with $VMOA$. It suffices to prove that they have same dual. We actually prove also the statement about the duality, by showing that the dual of E is contained in $H^1(\Omega)$.

Let $L \in E^*$. Then,

$$\tilde{L} : \phi \in \mathcal{C}(\partial\Omega) \mapsto L(P_S(\phi))$$

is a continuous linear map. Hence, there exists a regular Borel measure μ on $\partial\Omega$ such that $\tilde{L}(\phi) = \int_{\partial\Omega} \phi d\mu$. Notice that $P_S\mu = \mu$ in the sense of distributions, that is

$$\langle \mu, P_S\phi \rangle = \langle \mu, \phi \rangle$$

for $\phi \in \mathcal{C}^\infty(\partial\Omega)$. Using the fact that $P_S^2 = P_S$, we now show that such a measure is absolutely continuous, and its density is a function in $H^1(\Omega)$.

We can approach μ weakly by a sequence of functions $f_n \in \mathcal{C}^\infty(\partial\Omega)$. Moreover, we can assume that $P_S f_n = f_n$, replacing if necessary f_n by $P_S f_n$. Thus, the functions f_n are boundary values of holomorphic functions F_n. For each $z \in \Omega$, $F_n(z)$ can be written as an integral on the boundary, using either the Euclidean harmonic Poisson kernel $\mathcal{P}_\Omega(z, \cdot)$, or the Szegö kernel. At the limit we can write

$$\int_{\partial\Omega} \mathcal{P}_\Omega(z, w) d\mu(w) = \int_{\partial\Omega} S_\Omega(z, w) d\mu(w).$$

The right hand side is clearly a holomorphic function in Ω, while the left hand side satisfies a uniform L^1 estimate on $\partial\Omega_\varepsilon$, and has μ as a boundary value. We conclude from these two facts that μ coincides with a H^1 function, which we wanted to prove.

We remark that from the proof it follows that one has equivalence of norms in (*iii*): There exists a constant c such that every function $b \in VMOA$ may be written as $P_S(\phi)$ for some $\phi \in \mathcal{C}(\partial\Omega)$ such that $\|\phi\|_{L^\infty(\partial\Omega)} \leq c\|b\|_{BMOA}$. $\square$

End of the proof of Theorems 4.1 and 4.2. If $b \in VMOA$ then by Theorem 4.3 above $b = P_S(\phi)$ for some $\phi \in \mathcal{C}(\partial\Omega)$. If $\psi_k \in \mathcal{C}^\infty$ and ψ_k tend to ϕ in $\mathcal{C}(\partial\Omega)$, then h_{b_k} tend to h_b, where $b_k = P_S(\psi_k) \in \mathcal{C}^\infty(\partial\Omega)$.

Now, h_{b_k} is of trace class, since by integration by parts (see [BPS1] Lemma 6.5)

$$b_k(z) = \int_\Omega S_\Omega(z,w) D^{m+1} b_k(w) \delta^m(w) dV(w),$$

for some differential operator D^{m+1} of order $m+1$ with smooth coefficients so that

$$\|h_{b_k}\|_{\mathcal{S}_1} \leq \int_\Omega S_\Omega(w,w) |D^{m+1} b_k(w)| \delta^m(w) dV(w) < \infty,$$

for m sufficiently large. (Recall that the Hankel operator with symbol $z \mapsto S(\cdot, w)$ is of rank one, and norm $S_\Omega(w,w)$.)

On the other hand, assume that h_b is compact. Then, in order to prove that $b \in VMOA$, it suffices to show that

$$\Big| \int_{\partial\Omega} b(z) \overline{a_m(z)} dV(z) \Big| \to 0,$$

for any sequence of atoms a_m with support in $B(\zeta^{(m)}, r_m)$ with $r_m \to 0$. From now on, we shall restrict to strictly pseudoconvex domains, and argue as in the proof of the factorization theorem. If $A_m = P_S(a_m)$,

$$\begin{aligned} \Big| \int_{\partial\Omega} b(z) \overline{a_m(z)} d\sigma(z) \Big| &= \Big| \int_{\partial\Omega} b(z) \overline{A_m(z)} d\sigma(z) \Big| \\ &= \Big| \int_{\partial\Omega} b(z) \overline{A_m(z) H_m(z)^\alpha H_m(z)^{-\alpha}} d\sigma(z) \Big| \\ &= |\langle h_b(\beta_m A_m H_m^\alpha), \beta_m^{-1} H_m^{-\alpha} \rangle|, \end{aligned} \tag{7}$$

where $H_m(z) = r_m + H(z, \zeta^{(m)})$, as in the proof of the factorization theorem. We have chosen α such that $n/p < \alpha < n/p + 1/2$ and set $\beta_m = r_m^{-\alpha} \sigma(B(\zeta^{(m)}, r_m))^{1/p'}$. With this choice, $\beta_m A_m H_m^\alpha$ and $\beta_m^{-1} H_m^{-\alpha}$ are uniformly bounded in norm in H^p and $H^{p'}$ respectively.

Recall that if $T : X \to Y$ is a compact operator between Banach spaces, and $x_n \to x$ weakly, then $Tx_n \to Tx$ in norm.

Hence, if we show that $\{\beta_m^{-1} H_m^{-\alpha}\}$ weakly tends to 0, the result will follow from the compactness of h_b and (7) above.

Thus, let $g \in L^p(\partial\Omega)$. We want to show that $\langle \beta_m^{-1} H_m^{-\alpha}, g\rangle$ tends to 0. By density we may assume that g is a continuous function. Then it is sufficient to prove that $\beta_m^{-1} H_m^{-\alpha}$ tends to 0 in the L^1 norm. This follows from Lemma 2.5. □

5. Schatten ideal Hankel operators

Let T be a compact operator on a Hilbert space and let $0 < p < \infty$. We say that T belongs to the Schatten class $\mathcal{S}_p$ if $\|T\|_{\mathcal{S}_p}^p = \sum_j s_j^p < \infty$, where

$$s_j := \{\inf \|T - E\| : \ \operatorname{rank} E \leq j\}\,.$$

Notice that $\mathcal{S}_1$ are the trace class operators, $\mathcal{S}_1$ consists of the Hilbert-Schmidt operators, and when $p = \infty$, $\mathcal{S}_\infty$ becomes the space of the compact operators. It is well known that for all p, $\mathcal{S}_p$ is an ideal in the space of bounded operators.

In this part we characterize the compact Hankel operators h_b on Hardy and Bergman spaces that are in the Schatten ideals, in terms of smoothness of the symbol b.

Let $p > 0$. The Besov spaces of holomorphic functions $\mathcal{B}_p(\Omega)$ are defined by

$$\mathcal{B}_p(\Omega) := \big\{ g \in H^2(\Omega) : \sum_{|\gamma|\leq l} \int_\Omega |\delta(z)^l \partial^\gamma g(z)|^p B_\Omega(z,z) dV(z) < \infty \big\},$$

where l is some integer such that $lp > n$, and γ denotes a multi-index of derivation. We set

$$\|g\|_{\mathcal{B}_p} := \sum_{|\gamma|\leq l} \left[\int_\Omega |\delta(z)^l \partial^\gamma g(z)|^p B_\Omega(z,z) dV(z) \right]^{1/p} .$$

Theorem 5.1. *Let Ω be an H-domain. Let h_b denote either the Hankel operator on the Hardy space $H^2(\Omega)$ or the Hankel operator on the Bergman space $A^2(\Omega)$. Let $0 < p < \infty$. Then, h_b is in the Schatten class $\mathcal{S}_p$ if and only if $b \in \mathcal{B}_p$, with equivalence of norms.*

This theorem was first proved in the Hardy space case, by Peller for $p = 1$ when Ω is the unit disc in $\mathbb{C}$. It was later extended to the case $p > 1$ by Peller [Pe1] and Rochberg [R], and to the case $0 < p < 1$ by Peller [Pe2] and Semmes [Se]. This result in the Bergman space case, for $1 < p < \infty$, on the unit disc, follows from the results in [AFP].

For higher dimensional cases, this result is known to hold for $p > 1$ when Ω is the unit ball [FR], and more generally, a strictly pseudoconvex domain [BPS1], for both Hankel operators on Hardy and Bergman spaces.

In [BPS2] we prove this result in the context of a convex domain of finite type Ω, for the full range $0 < p < \infty$.

This result holds also in the case of a strictly pseudoconvex domain Ω. For the range $1 \leq p < \infty$ the proof appears in [BPS1]. Here we show how to extend the proof of Theorem 7.1 in [BPS2] to the case of a strictly pseudoconvex domain Ω. The same argument, with all the technicalities of [BPS2], is valid for all H-domains.

A key technical tool in the proof of Theorem 5.1 is the atomic decomposition of the Besov spaces.

Definition 5.2. Given $\eta > 0$ we say that a sequence of points $\{w^{(j)}\}$ is an η-lattice if:

(*i*) $\cup_j Q(w^{(j)}, \eta\delta(w^{(j)})) = \Omega$,

(*ii*) the sets $Q(w^{(j)}, \eta\delta(w^{(j)}))$ are almost disjoint, in the sense that any given point in Ω belongs to at most N_Ω of the polydiscs, and the integer N_Ω depends only on the geometry,

(*iii*) $Q(w^{(j)}, \eta\delta(w^{(j)})/C_\Omega) \cap Q(w^{(k)}, \eta\delta(w^{(k)})/C_\Omega) = \emptyset \quad \text{if} \quad j \neq k.$

The following kind of decomposition was first studied by Coifman and Rochberg [CR]. For a proof in this context see [BPS2].

Theorem 5.3. *Let Ω be an H-domain and $0 < p < +\infty$. There exists $\eta_0 > 0$ such that for each η-lattice $\{w^{(j)}\}$ with $0 < \eta < \eta_0$, the following conditions hold.*

(*i*) *There exists $c > 0$ such that for any f in $\mathcal{B}_p(\Omega)$ there exists $\{\nu_j\} \in \ell^p$ such that*

$$f(z) = \sum_j \nu_j \sigma_j S_\Omega(z, w^{(j)})$$

where $\sigma_j := \delta(w^{(j)})^{-1} B_\Omega(w^{(j)}, w^{(j)})^{-1}$, and

$$\sum_j |\nu_j|^p \leq c\|f\|^p_{\mathcal{B}_p}.$$

(*ii*) *Conversely, given any sequence $\{\nu_j\} \in \ell^p$ the function f defined as in (i) is in $\mathcal{B}_p(\Omega)$ and*

$$\|f\|^p_{\mathcal{B}_p} \leq c \sum_j |\nu_j|^p.$$

Proof. The proof of this fact is by now standard. It relies on the size estimates for the Szegö and Bergman kernels. See [BPS2] Theorem 5.1 for the significant case of a convex domain of finite type. □

A simpler result, which will be crucial later and whose proof follows the same lines, is the following.

Proposition 5.4. *Let $\{w^{(j)}\}$ be an η-lattice. Let $p > 0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha + (n-1)\beta^* > -1$ be given. Then there exists an $\eta_0 > 0$ such that if $\eta < \eta_0$, there exists a constant $c = c_p$ independent of η such that for all holomorphic functions f which are in the weighted space $L^p(\delta(z)^\alpha \tau(z, \delta(z))^\beta dV(z))$, one has*

$$\int_\Omega |f(z)|^p \delta(z)^\alpha \tau(z, \delta(z))^\beta dV(z) \leq c \sum_j \delta(w^{(j)})^{\alpha+2} \tau(w^{(j)}, \delta(w^{(j)}))^{\beta+2} |f(w^{(j)})|^p. \tag{8}$$

In particular, for $f \in \mathcal{B}_p$, one has

$$\|f\|^p_{\mathcal{B}_p} \leq c \sum_j \delta(w^{(j)})^{pk} \sum_{|\gamma|\leq k} |\partial^\gamma f(w^{(j)})|^p.$$

Proof. We only sketch the proof, refering to Corollary 5.7 in [BPS2] for full details.

We first majorize the integral over Ω with the sum of the integrals over $Q_j = Q(w^{(j)}, \eta\delta(w^{(j)}))$. On each Q_j we majorize the quantity $|f(z)|$ by $|f(w^{(j)})| + |f(z) - f(w^{(j)})|$. Now the first terms give rise to the right hand side. We now prove that the second terms give rise to an error term which can be substracted to the left-hand side in (8).

We have

$$\begin{aligned}
&\sum_j \int_{Q_j} |f(z) - f(w^{(j)})|^p \delta(z)^\alpha \tau(z, \delta(z))^\beta dV(z) \\
&\qquad \leq c_p \eta^{p/M_\Omega} \sum_j \delta(w^{(j)})^\alpha \tau(w^{(j)}, \delta(w^{(j)}))^\beta \frac{\mathrm{Vol}\,(Q_j)}{\mathrm{Vol}\,(Q_j^*)} \int_{Q_j^*} |f(z)|^p dV(z) \\
&\qquad \leq c_p \eta^{p/M_\Omega} \int_\Omega \delta(z)^\alpha \tau(z, \delta(z))^\beta |f(z)|^p dV(z),
\end{aligned}$$

where $Q_j^* = Q(w^{(j)}, \theta\delta(w^{(j)}))$, for some other constant θ larger than η for which this new family is assumed to be also almost disjoint. If η has been chosen small enough, this quantity is indeed smaller than half of the left hand side (8). $\square$

Remark 5.5. The last proposition gives an *a priori* estimate. In fact, a modification of the proof allows to replace the assumption that f belongs to the weighted space $L^p(\delta(z)^\alpha \tau(z, \delta(z))^\beta dV(z))$ by the much weaker assumption that f belongs to some $L^p(\delta(z)^N dV(z))$ for some large N. We will show how to modify this kind of proof in such a direction later on. Then, the proposition may be seen as a tool to prove that some function is in a precise weighted space. We will use it for this purpose.

We also point out that the inequality converse to (8) follows from the mean value inequalities.

Before discussing the proof of Theorem 5.1 we give one more definition. Let a be a positive real number, m be a positive integer and γ be a multi-index with $|\gamma| \leq m$. Let $\{w^{(j)}\}$ be an η-lattice and

$$\frac{n}{2} < a < m.$$

We set

$$e_{j,\gamma} = e_j := \delta^{m-a}(w^{(j)}) {\sigma_j}^{\frac{1}{2}} H^a(\cdot, w^{(j)}) \overline{\partial}_w^\gamma S_\Omega(\cdot, w^{(j)}); \tag{9}$$

$$f_j := \delta^a(w^{(j)}) {\sigma_j}^{-\frac{1}{2}} H^{-a}(\cdot, w^{(j)})\,. \tag{10}$$

It is worth mentioning that, by Lemma 2.5, the norms of e_j and f_j in $H^2(\Omega)$ are bounded uniformly in j. In fact, one can prove more (see [BPS2] Section 2): There exist linear operators $\mathcal{K}_\gamma, \mathcal{K}'$ bounded from $L^2(\partial\Omega)$ into $H^2(\Omega)$ and orthonormal sequences $\{\psi_j\}$ and $\{\psi'_j\}$ such that $e_{j,\gamma} = \mathcal{K}_\gamma(\psi_j)$ and $f_j = \mathcal{K}'(\psi'_j)$.

Proof of Theorem 5.1. We begin discussing the proof of the sufficient condition. This is by now classical, and the argument relies on the atomic decomposition of the space of symbols. If $b \in \mathcal{B}_p$ then there exist an η-lattice $\{w^{(j)}\}$ and a sequence $\{\lambda_j\} \in \ell^p$ such that

$$b(z) = \sum_j \lambda_j \sigma_j S_\Omega(z, w^{(j)}).$$

We present the proof of the sufficient condition in the case $0 < p \le 1$. The proof of the other case is much more difficult, see [R] [Sy1] and [BPS2] for instance.

Let $g \in H^2(\Omega)$ and $z \in \Omega$. Then

$$\begin{aligned} h_b g(z) &= P_S(b\overline{g})(z) \\ &= \sum_j \lambda_j \sigma_j \int_{\partial\Omega} S_\Omega(z,\zeta)\overline{g(\zeta)} S_\Omega(\zeta, w^{(j)})\, d\sigma(\zeta) \\ &= \sum_j \lambda_j \sigma_j S_\Omega(z, w^{(j)}) \overline{g(w^{(j)})} \\ &= \sum_j \lambda_j \sigma_j S_\Omega(z, w^{(j)}) \langle S_\Omega(\cdot, w^{(j)})\,,\, g\rangle\,. \end{aligned}$$

Hence, we can write h_b as an infinite sum of operator of rank 1, namely $g \mapsto \sigma_j S_\Omega(z, w^{(j)}) \langle S_\Omega(\cdot, w^{(j)})\,,\, g\rangle$. Their $\mathcal{S}_p$ norm equals their operator norm, which is uniformly bounded in j. Then,

$$\|h_b\|^p_{\mathcal{S}_p} \le c \sum_j |\lambda_j|^p \le c\|b\|^p_{\mathcal{B}_p}\,.$$

We now turn to the necessary condition.

Let $1 \le p < \infty$ first. By Proposition 5.4 we have

$$\begin{aligned} \|b\|^p_{\mathcal{B}_p} &\le c \sum_{|\gamma|\le m} \sum_j \delta^{mp}(w^{(j)}) \big|\partial^\gamma b(w^{(j)})\big|^p \\ &= c \sum_{|\gamma|\le m} \sum_j \delta^{mp}(w^{(j)}) \Big| \int_{\partial\Omega} \partial^\gamma_w S_\Omega(w^{(j)}, \zeta) b(\zeta)\, d\sigma(\zeta)\Big|^p \\ &= c \sum_{|\gamma|\le m} \sum_j \Big| \int_{\partial\Omega} b(\zeta) \overline{e_{j,\gamma}(\zeta) f_j(\zeta)}\, d\sigma(\zeta)\Big|^p \\ &= c \sum_{|\gamma|\le m} \sum_j |\langle h_b(e_{j,\gamma}), f_j\rangle|^p, \end{aligned} \tag{11}$$

since the function $e_{j,\gamma}, f_j$ are holomorphic.

Now, for each γ we obtain that

$$\sum_j |\langle h_b(e_{j,\gamma}), f_j\rangle|^p = \sum_j |\langle \mathcal{K}'^* h_b \mathcal{K}_\gamma(\psi_j), \psi'_j\rangle|^p.$$

Since the operators $\mathcal{K}_\gamma, \mathcal{K}'$ are bounded, $\{\psi_j\}$ and $\{\psi'_j\}$ are orthonormal sequences, and $1 \le p < \infty$, this last quantity is bounded by $c\|h_b\|^p_{\mathcal{S}_p}$, which is what we wanted to prove.

We now turn to the case $0 < p < 1$.

Having fixed an η-lattice $\{w^{(j)}\}$, we decompose it into a finite number of subsequences whose elements satisfy a separation condition.

We say that a sequence of points $\{z^{(j)}\}$ form an M*-sequence* if

$$Q_M(z^{(j)}) \cap Q_M(z^{(k)}) = \emptyset \quad \text{if} \quad j \ne k,$$

where $M > 0$ and

$$Q_M(z) = Q(z, M\delta(z)) \cap \{w : \, \delta(w) \ge \delta(z)/M\}.$$

In order to better understand the set $Q_M(z)$, take Ω to be the upper-half plane in $\mathbb{C}$. Then $Q_M(x+iy)$ is equivalent to the set $\{w = u+iv : |u-x| < M, y/M < v < My\}$, which in turn is equivalent to the disc of center z and radius M in the hyperbolic metric.

We have the following result, for whose proof we refer to Proposition 7.3 in [BPS2]. However, we remark that this idea originated in [Se], where Semmes first proved the characterization of Schatten class Hankel operators on the unit disc, when $0 < p < 1$.

Proposition 5.6. *Let $0 < \eta < \eta_0$ and let $\{w^{(j)}\}$ be an η-lattice in Ω. Let $M > 0$. Then there exists an integer $N = N(M, \eta)$ such that $\{w^{(j)}\}$ can be decomposed into a finite number of M-sequences $\{z^{(j,l)}\}$, $j \in \mathbb{N}$, $l \in \{1, 2, \ldots, N\}$.*

Having fixed the η-lattice $\{w^{(j)}\}$ and its decomposition into M-sequences $\{z^{(j,l)}\}$, $l = 1, \ldots, N(M)$, $j = 1, 2, \ldots$, we decompose the sequences $\{e_j\}$ and $\{f_j\}$ accordingly by setting

$$e^l_j = e^l_{j,\gamma} := \delta^{m-a}(z^{(j,l)}) \sigma_{(j,l)}{}^{\frac{1}{2}} H^a(\cdot, z^{(j,l)}) \overline{\partial}^\gamma_w S(\cdot, z^{(j,l)}); \tag{12}$$

$$f^l_j := \delta^a(z^{(j,l)}) \sigma_{(j,l)}{}^{-\frac{1}{2}} H^{-a}(\cdot, z^{(j,l)}), \tag{13}$$

where $\sigma_{(j,l)} = \delta(z^{(j,l)})^{-1} B(z^{(j,l)}, z^{(j,l)})^{-1}$ as before. We remark that these two sequences coincide with the previous ones. They only have been relabelled.

Proposition 5.7. *Let $\{w^{(j)}\}$ be an η-lattice and let $\{z^{(j,l)}\}$, $l = 1, \ldots, N(M)$ be a decomposition into M-sequences. It is possible to choose $a \in \mathbb{R}$ and $m \in \mathbb{N}$ in the definition of e^l_j and f^l_j large enough so that the following condition holds. There exist $\varepsilon_0 > 0$ and a constant $c > 0$ independent of M such that for all $b \in \mathcal{B}_p$ we have*

$$\sum_{l \le N} \sum_{j \ne k} |\langle h_b(e^l_{j,\gamma}), f^l_k\rangle|^p \le \frac{c}{M^{\varepsilon_0}} \|b\|^p_{\mathcal{B}_p}.$$

Again, we refer to [BPS2] Proposition 7.4 for a proof.

End of the proof of Theorem 5.1. We begin by assuming *a priori* that $b \in \mathcal{B}_p$. We are going to remove this assumption later on.

Now, arguing as in (11) we have

$$\begin{aligned}\|b\|^p_{\mathcal{B}_p} &\le c \sum_{|\gamma|\le m} \sum_j |\langle h_b(e_{j,\gamma}), f_j\rangle|^p \\ &= c \sum_{|\gamma|\le m} \sum_{l\le N} \sum_j |\langle h_b(e^l_{j,\gamma}), f^l_j\rangle|^p \\ &\le c \sum_{|\gamma|\le m} \sum_{l\le N} \Big(\|\mathcal{K}_\gamma\|^p \|\mathcal{K}'\|^p \|h_b\|^p_{\mathcal{S}_p} + \sum_{j\ne k} |\langle h_b(e^l_{j,\gamma}), f^l_k\rangle|^p \Big).\end{aligned}$$

This last inequality follows from elementary Hilbert space arguments, and we refer to [BPS2] Proposition 7.1 for a proof. Notice however, that this last step is key in our analysis. In fact, by decomposing the sequences $\{e_{j,\gamma}\}$ and $\{f_k\}$ into finitely many subsequences $\{e^l_{j,\gamma}\}$ and $\{f^l_k\}$, $l = 1, \dots, N(M)$, we may estimate the left-hand side by a large constant times $\|h_b\|_{\mathcal{S}_p}$ and an error term. This error term is estimated by using Proposition 5.7 above. Hence, we obtain

$$\|b\|^p_{\mathcal{B}_p} \le C_M \|h_b\|^p_{\mathcal{S}_p} + \frac{c}{M^{\varepsilon_0}} \|b\|^p_{\mathcal{B}_p},$$

where C_M is a large constant that depends on M, the parameter that measures the amount of separation between elements in each subsequence $\{z^{(j,l)}\}$.

From the above estimate it follows that

$$\|b\|_{\mathcal{B}_p} \le c\|h_b\|_{\mathcal{S}_p},$$

which proves the theorem under the assumption that we know *a priori* that $b \in \mathcal{B}_p$.

Our next task is to remove this assumption when it is possible, and to propose a refinement of the proof when it is not.

When Ω is a convex domain of finite type in $\mathbb{C}^n$ this is done in [BPS2], and in fact it only requires a simple approximation argument. Without loss of generality we may assume that $0 \in \Omega$. For $0 < r < 1$ we define $T_r f(z) = f(rz)$. Then T is an operator on $H^2(\Omega)$, uniformly bounded in r. For each b holomorphic in Ω, $T_r b$ in holomorphic across the boundary and hence in $\mathcal{B}_p$ for all p. Then, we may apply the *a priori* estimate to $h_{T_r b}$ and pass to the limit.

When Ω is strictly pseudoconvex domain (non star-like), or more generally an H-domain, the argument outlined above does not work anymore. We now show how to refine the arguments when we cannot rely on an *a priori* assumption.

For simplicity, we assume that Ω is strictly pseudoconvex, the proof in the general case being completely analogous. Let $0 < p < 1$ and $h_b \in \mathcal{S}_p$. We set

$$A_j = \sum_{|\gamma|\le m} \int_{2^{-j}<\delta<2^{-j+1}} |\delta(z)^m \partial^\gamma b(z)|^p \frac{dV}{\delta^{n+1}}.$$

We want to show that $\sum_{j=1}^{\infty} A_j < \infty$.

Notice that we may assume that $b \in \mathcal{B}_1$ so that

$$\sum_{|\gamma| \leq m} \int_{\Omega} |\delta(z)^m \partial^\gamma b(z)| \frac{dV}{\delta^{n+1}} < \infty.$$

By Hölder's inequality it follows that

$$A_j \leq c \|b\|^p_{\mathcal{B}_1} 2^{jn(1-p)}.$$

This weak inequality will play the role of the *a priori* assumption.

Using the same argument as in the proof of Proposition 5.4 (see the proof of Corollary 5.7 [BPS2]) we get

$$\sum_{j \leq J} A_j \leq C_\eta \sum_{|\gamma| \leq m} \sum_{(j,l) \in \mathcal{N}_J} |\delta(z^{(j,l)})^m \partial^\gamma b(z^{(j,l)})|^p$$
$$+ C\eta^{\frac{1}{2}} \sum_{|\gamma| \leq m} \int_{\delta(z) \geq 2^{-J-1}} |\delta^m(z) \partial^\gamma b(z)|^p \frac{dV}{\delta^{n+1}},$$

where $\{z^{(j,l)}\}$ is as before with $\eta < \eta_0$ small to be determined later, and we have set $\mathcal{N}_J = \{(j,l) : \delta(z^{(j,l)}) \geq s^{-J}\}$. The last inequality may be written as

$$\sum_{j \leq J} A_j \leq C_\eta \sum_{|\gamma| \leq m} \sum_{(j,l) \in \mathcal{N}_J} |\delta(z^{(j,l)})^m \partial^\gamma b(z^{(j,l)})|^p + C\eta^{\frac{1}{2}} A_{J+1}.$$

The same argument as in (11) gives

$$\sum_{(j,l) \in \mathcal{N}_J} |\delta(z^{(j,l)})^m \partial^\gamma b(z^{(j,l)})|^p \leq C_M \|h_b\|^p_{\mathcal{S}_p} + c \sum_l \sum_{(s,l),(t,l) \in \mathcal{N}_J} |\langle b, e^l_s f^l_t \rangle|^p.$$

By the usual integration by parts argument (see Lemma 6.5 in [BPS1])

$$\begin{aligned} |\langle b, e^l_s f^l_t \rangle|^p &= \left| \int D^m b(z) e^l_s(z) f^l_t(z) \delta(z)^{m-1} dV(z) \right| \\ &\leq \int |D^m b(z) e^l_s(z) f^l_t(z)| \delta(z)^{m-1} dV(z) \\ &\leq c \sum_k |D^m b(w^{(k)}) e^l_s(w^{(k)}) f^l_t(w^{(k)})| \delta(w^{(k)})^{m+n}, \end{aligned}$$

for some η'-lattice $\{w^{(k)}\}$, where D^m denotes a differential operator of order m with smooth coefficients. We now consider

$$E_J(z) = \sum_{(s,l),(t,l) \in \mathcal{N}_J} |e^l_s(z)|^p |f^l_t(z)|^p,$$

where e^l_s, f^l_t are defined as in (12) and (13). Then,

$$\sum_{l \leq N} \sum_{(s,l),(t,l) \in \mathcal{N}_J} |\langle b, e^l_s f^l_t \rangle|^p \leq c \sum_{|\gamma| \leq m} \sum_k |\partial^\gamma b(w^{(k)})|^p E_J(w^{(k)})^p \delta(w^{(k)})^{p(m+n)}. \tag{14}$$

We claim that

$$|E_J(z)|^p \le \frac{c}{M^\alpha}\big(\delta(z)+2^{-J}\big)^{-np} \tag{15}$$

Let us assume this inequality for the moment, and finish the proof of the theorem. The left-hand side in (14) is bounded by

$$\frac{C_\eta}{M^\alpha}\sum_{|\gamma|\le m}\int_\Omega \delta(z)^{mp}|\partial^\gamma b(z)|^p\left(\frac{\delta(z)}{\delta(z)+2^{-J}}\right)^{np}\frac{dV}{\delta^{n+1}},$$

or, equivalently, by

$$\frac{C_\eta}{M^\alpha}\left(\sum_{j=1}^J A_j + 2^{npJ}\sum_{j=J+1}^\infty 2^{-npj}A_j\right).$$

Therefore, for every η, M and J, we have the inequality

$$\sum_{j=1}^J A_j \le C_\eta C_M\|h_b\|_{\mathcal{S}_p} + C\eta^{\frac12}A_{J+1} + \frac{C_\eta}{M^\alpha}\left(\sum_{j=1}^J A_j + 2^{npJ}\sum_{j=J+1}^\infty 2^{-npj}A_j\right).$$

Recall now that $A_j \le c_0 2^{jn(1-p)}$. For $p > \frac12$, we can find β such that $n(1-p) < \beta < np$. Then $2^{-\beta j}A_j \to 0$, and there exists an infinite number of values of J such that $2^{-\beta j}A_j \ge 2^{-\beta J}A_J$ for $j > J$. For such J, the above inequality becomes

$$\sum_{j=1}^J A_j \le C_\eta C_M\|h_b\|_{\mathcal{S}_p} + C\eta^{\frac12}2^\beta A_J + \frac{C_\eta}{M^\alpha}\left(\sum_{j=1}^J A_j + C_\beta A_J\right).$$

Let us choose η so that $C2^\beta\eta^{\frac12} < \frac14$, then M so that $\frac{C_\eta}{M^\alpha}(1+C_\beta) < \frac14$. With this choice,

$$\sum_1^J A_j \le 2C_\eta C_M\|h_b\|_{\mathcal{S}_p},$$

which gives the desired estimate when J tends to infinity and $p > \frac12$. If p is smaller, we take $\frac12 < \tilde p < 1$ and use the fact that we already know that $b \in B_{\tilde p}$, from which we get $A_j \le c2^{jn(1-p/\tilde p)}$. The proof given above will work when $1 - p/\tilde p < p$, that is $p(1+1/\tilde p) > 1$, i.e. $p > \frac13$. For smaller values of p we proceed recursively, using at each step the estimate coming from the previous step. We then obtain that the estimate is valid for $p > p_n$, with

$$p_{n+1}(1+p_n) = 1, \qquad \text{that is} \quad p_n = 1/n \to 0.$$

This completes the proof of the theorem.

It remains to show that (15) is valid. One has to slightly modify the corresponding part of the proof of Proposition 7.4 in [BPS2]. We give some details for completness.

Let

$$\phi_1(u) = \frac{\delta(u)^{(m-a+n/2)p-n-1}}{d(z,u)^{p(n+m-a)}},$$
$$\phi_2(v) = \frac{\delta(v)^{(a-n/2)p-n-1}}{d(z,v)^{pa}}.$$

Then,

$$E_J(z) \le \iint_{\bigcup_{l,\,s\neq t} Q_s^l\times Q_t^l, \delta(u)>c2^{-J}, \delta(v)>c2^{-J}} \phi_1(u)\phi_2(v)dV(u)dV(v), \tag{16}$$

where $Q_s^l = Q\big(z^{(s,l)}, \eta\delta(z^{(s,l)})/C_\Omega\big)$.

Recall that the Q_s^l are pairwise disjoint, that $(s,l),(t,l)\in\mathcal{N}_J$, and that

$$Q_M(z) = \{w:\, d_b(z,w) < M\delta(z),\, \delta(w) > \delta(z)/M\}.$$

Then, $Q_M(z^{(s,l)})\cap Q_M(z^{(t,l)}) = \emptyset$ when $s\neq t$. So, if $(u,v)\in \cup_{l,\,s\neq t} Q_s^l\times Q_t^l$, then $u\in Q_s^l$ for some s and $v\notin Q_M(z^{(s,l)})$, which means that either $d(u,v) > M\delta(u)$ (where M has been changed into a smaller constant), or $\delta(u)/M > \delta(v)$.

In the first case, since $d(u,v) < d(u,z) + d(z,w)$, we must have either $d(z,v) > M\delta(u)/2$ or $d(z,u) > M\delta(v)/2$, that is, one of the two following conditions holds:

(i) $d(z,v)^{-1} < \big(M\delta(u)\big)^{-1}$;
(ii) $d(z,u)^{-1} < \big(M\delta(u)\big)^{-1}$.

Moreover, $\delta(u),\delta(v) > 2^{-J}$, so that the two conditions above become

(i') $d(z,v)^{-1} \simeq \big(d(z,v)+2^{-J}\big)^{-1}$;
(ii') $d(z,u)^{-1} \simeq \big(d(z,u)+2^{-J}\big)^{-1}$.

Let E', F' denote the subsets of $\Omega\times\Omega$ on which the two conditions (i') or (ii') hold. Then, the integral in (16) is bounded by a constant times

$$\iint_{E'} \phi_1(u)\phi_2(v)dV(u)dV(v) + \iint_{F'} \phi_1(u)\phi_2(v)dV(u)dV(v).$$

By symmetry it obviously suffices to estimate one of the two integrals.

We choose a so that $ap > n+1$ and $(a-n/2)p-n-1 > -1$. Hence, for $0<\alpha<ap-n-1$,

$$\begin{aligned}\int_{\substack{d(z,v)^{-1}<(M\delta(u))^{-1},\\ \delta(v)>2^{-J}}} \phi_2(v)dV(v) &\le \frac{1}{M^\alpha}\frac{\delta(z)^{(a-n/2)p-n-1}}{\delta(u)^\alpha[d(z,v)+2^{-J}]^{ap-\alpha}}\\ &\le \frac{c}{M^\alpha}\big(\delta(u)+2^{-J}\big)^{\alpha-np/2}\delta(u)^{-\alpha}.\end{aligned}$$

Now,

$$\int_{\delta(u)>2^{-J}} \frac{\delta(u)^{(m-a+n/2)p-n-1-\alpha}}{d(z,u)^{p(n+m-a)}}dV(u) \le \frac{c}{(\delta(z)+2^{-J})^{\alpha+np/2}},$$

for α such that $(m-a+n/2)p-n-1-\alpha > -1$.

Hence,

$$\iint_{E'} \phi_1(u)\phi_2(v)dV(u)dV(v) \le \frac{c}{M^\alpha}\big(\delta(z) + 2^{-J}\big)^{-np},$$

for α small enough. This gives (15), and finishes the proof of the theorem. □

References

[AFP] J. Arazy, S. Fisher, J. Peetre, Hankel operators on weighted bergman spaces, *Amer. J. Math.* **110** (1998), 989-1053.

[Be] F. Beatrous, Estimates for derivatives of holomorphic functions in pseudoconvex domains, *Math. Z.* **191** (1986), 91-116.

[BeLi] F. Beatrous, S-Y. Li, On the boundedness and compactness of operators of Hankel type, *J. Funct. Anal.* **111** (1993), 350-379.

[BoSt] H. P. Boas, E. J. Straube, On equality of line type and variety type of real hypersurfaces in $\mathbb{C}^n$, *J. Geom. Anal.* **2** (1992), 95-98.

[BPS1] A. Bonami, M. M. Peloso, F. Symesak, Powers of the Szegö kernel and Hankel operators on Hardy spaces, *Mich. Math. J.* **46** (1999), 225-250.

[BPS2] ———, Factorization of Hardy spaces and Hankel operators on convex domains in $\mathbb{C}^n$, (1999) preprint.

[Ca] D. Catlin, Subelliptic estimates for the $\bar\partial$-Neumann problem, *Ann. Math.* **126** (1987), 131-192.

[CR] R. Coifman, R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, *Astérisque* **77** (1980), 11-66.

[CRW] R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.* **103** (1976), 611-635.

[Co] B. Coupet, Décomposition Atomique des Espaces de Bergman, *Ind. J. Math.* **38** (1989), 917-941.

[DFiF] K. D. Diederich, B. Fisher, J. E. Fornæss, Hölder estimates on convex domains of finite type, *Math. Z.* **232**, 43-61 (1999), .

[DF] K. D. Diederich, J. E. Fornæss, Support functions for convex domains of finite type, *Math. Z.* **230** (1999), 145-164.

[F] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Inven. Math.* **26** (1974), 1-65.

[FR] M. Feldman, R. Rochberg, Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group, *Analysis and partial differential equations*, Lecture notes in Pure and Appl. Math. 122, Marcel Dekker, New York 1990.

[GL] J. B. Garnett, R. H. Latter, The atomic decomposition for Hardy spaces in several complex variables, *Duke J. Math.* **45** (1978), 815-845.

[H] G. M. Henkin, Integral representation of functions in striclty pseudoconvex domains and some applications, *Math. U.S.S.R. Sb.* **7** (1969), 597-616.

[GP] S. Grellier, M. M. Peloso, Atomic decomposition for Hardy spaces on convex domains of finite type, in preparation.

[KL1] S. G. Krantz, S-Y. Li, A note on Hardy spaces and functions of bounded mean oscillation on domains in $\mathbb{C}^n$, *Mich. Math. J.* **41** (1994), 71-97.

[KL2] ———, On Decomposition Theorems for Hardy Spaces on Domains in $\mathbb{C}^n$ and Applications, *Journ. of Fourier Anal. and Appl.* **2** (1995), 65-107.

[KL3] ———, Duality theorems for Hardy and Bergman spaces on convex domains of finite type in $\mathbb{C}^n$, *Ann. Inst. Fourier, Grenoble* **45** (1995), 1305-1327.

[KLR] S. G. Krantz, S-Y. Li, R. Rochberg, Analysis of certain function spaces associated with Hankel operators, *Ill. J. Math.* **41** (1997), 398-411.

[L] E. Ligocka, On the Forelli-Rudin construction and weighted Bergman projections, *Studia Math.* **94** (1989), 257-272.

[Mc] J. D. McNeal, Estimates on the Bergman kernels of convex domains, *Adv. in Math* **109** (1994), 108-139.

[McS1] J. D. McNeal, E. M. Stein, Mapping properties of the Begrman projection on convex domains of finite type, *Duke J. Math.* **73** (1994), 177-199.

[McS2] ———, The Szegö projection on convex domains, *Math. Z.* **224** (1997), 519-553.

[Pe1] V. V. Peller, Hankel operators of class $\mathcal{S}_p$ and their application (rational approximation, Gaussian processes, the problem of majorizing operators), *Math. of the USSR-Sbornik* **113** (1982), 538-581.

[Pe2] ———, A description of Hankel operators of class $\mathcal{S}_p$ for $p > 0$, and investigation on the rate of rational approximation, and other applications, *Math. of the USSR-Sbornik* **50** (1985), 465-494.

[PhS] D. H. Phong, E. M. Stein, Estimates for the Bergman and Szegö projections on strongly pseudoconvex domains, *Duke Math. J.* **44**(1977), 695-704.

[R] R. Rochberg, Trace ideal criteria for Hankel operators and commutators, *Indiana J. Math.* **31** (1982), 913-925.

[Se] S. Semmes, Trace ideal criteria for Hankel operators and applications to Besov spaces, *Equations and Op. Theory* **7** (1984), 241-281.

[St] E. M. Stein, *Boundary behaviour of holomorphic functions of several complex variables*, Math. Notes, Princeton University Press, Princeton 1972.

[Sy1] F. Symesak, Hankel operators on complex ellipsoids, *Ill. J. Math.* **40** (1996), 632-647.

[Sy2] ———, Décomposition atomique des espaces de Bergman, *Publ. Matemàtiques* **39** (1995), 285-299.

[Sy3] ———, Hankel operators on pseudoconvex domains of finite type in $\mathbb{C}^2$, *Can. J. Math.* **50** (1998), 658-672.

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