Oscillatory Behavior in a Model of Non-Markovian Mean Field Interacting Spins

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Abstract

We analyze a non-Markovian mean field interacting spin system, related to the Curie–Weiss model. We relax the Markovianity assumption by replacing the memoryless distribution of the waiting times of a classical spin-flip dynamics with a distribution with memory. The resulting stochastic evolution for a single particle is a spin-valued renewal process, an example of a two-state semi-Markov process. We associate to the individual dynamics an equivalent Markovian description, which is the subject of our analysis. We study a corresponding interacting particle system, where a mean field interaction-depending on the magnetization of the system-is introduced as a time scaling on the waiting times between two successive particle's jumps. Via linearization arguments on the Fokker–Planck mean field limit equation, we give evidence of emerging periodic behavior. Specifically, numerical analysis on the discrete spectrum of the linearized operator, characterized by the zeros of an explicit holomorphic function, suggests the presence of a Hopf bifurcation for a critical value of the temperature. The presence of a Hopf bifurcation in the limit equation matches the emergence of a periodic behavior obtained by simulating the N-particle system.

1 Introduction

Emerging periodic behavior in complex systems with a large number of interacting units is a commonly observed phenomenon in neuroscience [13], ecology [22], socioeconomics

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[4,23] and life sciences in general. From a mathematical standpoint, when modeling such a phenomenon it is natural to consider large families of microscopic identical units evolving through noisy interacting dynamics. Single units have no natural tendency to behave periodically and oscillations are rather an effect of self-organization as they emerge in the macroscopic limit when the number of particles tends to infinity. Within this modeling framework mean field models have received much attention due to their analytical tractability. For mean field models one can obtain in closed form the stochastic dynamics of a single unit in the limit of infinite many units; the time evolution of the associated distribution is a, possibly infinite-dimensional, dynamical system. Periodic trajectories of this dynamical system correspond to the emergence of self-organized periodic oscillations in the interacting system; this will be referred to with the term self-sustained periodic behavior. One of the goals of the mathematical theory in this field is to understand which types of microscopic interactions and mechanisms can lead to or enhance the above type self-organization. Among others, we cite noise [10,19,21], dissipation in the interaction potential [1,6,7,9], delay in the transmission of information and/or frustration in the interaction network [8,11,20]. In particular, in [11] the authors consider non-Markovian dynamics, studying systems of interacting nonlinear Hawkes processes for modeling neurons.

Although not proved in general, a strong belief in the literature is that, at least for Markovian dynamics, self-sustained periodic behavior cannot emerge if one does not introduce some time-irreversible phenomenon in the dynamics, as it is the case in all the above cited works (see e.g.[3,6,15]). The model treated here, in which the limit dynamics is still reversible with respect to the stationary distribution around which cycles emerge (see Remark 1 below), suggests that this paradigm could be false for the non-Markovian case.

Specifically, we give numerical and mathematical evidence of the emergence of self-sustained periodic behavior of the empirical magnetization in a mean field spin system related to the Curie–Weiss model, which happens to belong to the following universality class: it features the presence of a unique stable zero mean phase for values of the parameters corresponding to high temperatures, the emergence of periodic oscillations in an intermediate range of the parameter values, and a subsequent ferromagnetic ordered phase for increasingly lower temperatures. Our recipe consists in replacing the Poisson distribution of the spin-flip times with another renewal process, thus making the individual spin dynamics non-Markovian. In details, we consider the distribution of the interarrival times to have tails proportional to $e^{-t^{\gamma+1}}$, for $\gamma = 1, 2$. Then, we introduce an interaction among the spins via a time rescaling depending on the magnetization of the system. The specific choice of interarrival time distribution makes the computations developed in Sect. 4 as easiest as possible (to our knowledge), allowing for an explicit characterization of the discrete spectrum of the linearization of the limiting Fokker–Planck equation. A question which can arise naturally is whether similar results can be found for other classes of waiting times. Although we do not have a general answer to this, we want to remark that simulations with different distributions highlighted the same characteristics (e.g. Gamma distribution, tails proportional to $e^{-t^{\gamma+1}}$ with $\gamma \in \mathbb{R}$ such that $\gamma \geq 1$). All the working examples we considered feature exponentially or super-exponentially decaying tails. On the other hand, we have examples of polynomial tails (e.g. inverse Gamma distribution) where no oscillatory behavior was experienced.

The paper is organized as follows: in Sect. 2 we describe the model and the results obtained. In particular, before introducing the model (Sect. 2.2), we start by recalling basic facts about the Curie–Weiss model and its phase transition (Sect. 2.1); we then proceed with the results on the propagation of chaos (Sect. 2.3), and on the linearization of the Fokker–Planck equation

around a zero mean equilibrium, for two different choices of renewal dynamics (Sect. 2.4). Notably, we determine the discrete spectrum of the linearized operator in terms of the zeros of two holomorphic functions. Numerical investigations of the discrete spectrum, studied as a function of the interaction parameters, show the presence of a Hopf bifurcation: a pair of complex conjugate eigenvalues of the linearized Fokker–Planck equation around a stationary solution crosses the complex plane imaginary axis (see Sects. 2.4.1 and 2.4.2). Such a bifurcation of the mean field limit is reflected in the behavior of the finite particle system. Indeed, numerical simulations of the empirical magnetization in Sect. 3 confirm the transition from an incoherent state to self-organized rhythmic oscillations. Section 4 contains the proofs of the results of Sect. 2.

2 Model and Results

2.1 Motivation

As we mentioned above, the model we consider can be seen as a proper modification of the Curie–Weiss dynamics. When we refer to the latter, we mean a spin-flip type Markovian dynamics for a system of N interacting spins $\sigma_i \in \{-1, 1\}$, $i = 1, \ldots, N$. Such dynamics is *reversible* with respect to the equilibrium Gibbs probability measure on the space of configurations $\{-1, 1\}^N$,

$$P_{N,\beta}(\boldsymbol{\sigma}) := \frac{1}{Z_N(\beta)} \exp\left[-\beta H_N(\boldsymbol{\sigma})\right], \tag{1}$$

with $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$, $\beta > 0$ (*ferromagnetic* case), $Z_N(\beta)$ is a normalizing constant, and H_N is the *Hamiltonian*

$$H_N(\boldsymbol{\sigma}) := -\frac{1}{2N}\left(\sum_{i=1}^{N}\sigma_i\right)^2. \tag{2}$$

Denote also the empirical *magnetization* as $m^N := \frac{1}{N}\sum_{i=1}^{N}\sigma_i$. Note that the distribution (1) gives higher probability to the configurations with minimal energy, which by (2) are the ones where the individual spins are aligned in the same state. The equilibrium model undergoes a *phase transition* tuned by the interaction parameter $\beta > 0$, which can be recognized by proving a Law of Large Numbers for the equilibrium empirical magnetization

$$\text{Law}(m^N) \xrightarrow{N \to +\infty} \begin{cases} \delta_0, & \text{if } \beta \leq 1, \\ \frac{1}{2}\delta_{+m_\beta} + \frac{1}{2}\delta_{-m_\beta}, & \text{if } \beta > 1, \end{cases} \tag{3}$$

where $m_\beta > 0$ is the so-called *spontaneous magnetization* [2,12]. When we turn to the dynamics, different choices can be made in order to satisfy the above-mentioned reversibility with respect to (1). The prototype is a continuous-time spin-flip dynamics defined in terms of the infinitesimal generator L, applied to a function $f : \{-1, 1\}^N \to \mathbb{R}$,

$$Lf(\boldsymbol{\sigma}) = \sum_{i=1}^{N} e^{-\beta\sigma_i m^N}\left[f(\boldsymbol{\sigma}^i) - f(\boldsymbol{\sigma})\right], \tag{4}$$

where $\boldsymbol{\sigma}^i \in \{-1, 1\}^N$ is obtained from $\boldsymbol{\sigma}$ by *flipping* the i-th spin. Dynamics (4) induces a continuous-time Markovian evolution for the empirical magnetization process $m^N(t)$, which is given in terms of a generator $\mathcal{L}^N$ applied to a function $g : [-1, 1] \to \mathbb{R}$:

$$\mathcal{L}^N g(m) = N \frac{1+m}{2} e^{-\beta m} \left[g\left(m - \frac{2}{N}\right) - g(m) \right] + N \frac{1-m}{2} e^{\beta m} \left[g\left(m + \frac{2}{N}\right) - g(m) \right] \tag{5}$$

It is easy to obtain the weak limit of the sequence of processes $\left(m^N(t)\right)_{t \geq 0}$, by studying the uniform convergence of the generator (5) as $N \to +\infty$ (see e.g. [14]). The limit process $(m(t))_{t \geq 0}$ is deterministic and solves the Curie–Weiss ODE

$$\begin{cases} \dot{m}(t) = 2 \sinh(\beta m(t)) - 2m(t) \cosh(\beta m(t)), \\ m(0) = m_0 \in [-1, 1]. \end{cases} \tag{6}$$

The presence of the phase transition highlighted in (3) can be recognized as well in the out-of-equilibrium dynamical model (6). Indeed, studying the long-term behavior of (6), one finds that:

– for $\beta \leq 1$, (6) possesses a unique stationary solution, globally attractive, constantly equal to 0;
– for $\beta > 1$, 0 is still stationary but it is unstable; two other symmetric stationary locally attractive solutions, $\pm m_\beta$, appear: the two non-zero solutions to $m = \tanh(\beta m)$. The dynamics $m(t)$ gets attracted for $t \to +\infty$ to the polarized stationary state which has the same sign as the initial magnetization m_0.

Another concept which we refer to in what follows is that of *renewal* process, a generalization of the Poisson process. We identify a renewal process with the sequence of its *interarrival* times (also commonly referred to as sojourn times or waiting times in the literature) $\{T_n\}_{n=1}^\infty$, i.e. the holding times between the occurrences of two consecutive events. The Poisson process is characterized by having independent and identically distributed interarrival times, where each T_i is exponentially distributed. In particular, the *memoryless* property $\mathbb{P}(T_i > s + t | T_i > t) = \mathbb{P}(T_i > s)$, holds for any $s, t \geq 0$. The interarrival times of a renewal process are still independent and identically distributed, but their distribution is not required to be exponential. We recall that a continuous-time homogeneous Markov chain can be identified by a Poisson process, modeling the jump times, and a stochastic transition matrix, identifying the possible arrival states at each jump time. Due to the lack of the memoryless property, when one replaces the Poisson process in the definition of the spin-flip dynamics with a more general renewal process, the resulting evolution is thus non-Markovian. In the literature, the associated dynamics is referred to as *semi-Markov* process, first introduced by Lévy in [18].

2.2 The Dynamics

Glauber dynamics as the Curie–Weiss dynamics in (4) can be constructed in two stages. First one system of independent spin-flips: at the times of a Poisson process of intensity 1, the spin in a given site flips; different sites have independent Poisson processes. Then, the interaction can be introduced as a spin-dependent time scale in the waiting time for updates. In this section we illustrate this procedure, and generalize it by replacing the Poisson Process with a more general *renewal process*.

For the moment we focus on a single spin $\sigma(t) \in \{-1, 1\}$. If driven by a Poisson process of intensity 1, its dynamics has infinitesimal generator

$$\mathcal{L} f(\sigma) = f(-\sigma) - f(\sigma), \tag{7}$$

with $f : \{-1, 1\} \to \mathbb{R}$. If the Poisson process is replaced by a renewal process, the spin dynamics is not Markovian. In what follows, we refer to the resulting dynamics as a spin-valued renewal process, that is an example of q two-state semi-Markov process. We can associate a Markovian description to the latter: define $y(t)$ as the time elapsed since the last spin-flip occured up to time t. Suppose that the waiting times τ (interchangeably referred to as interarrival times) of the renewal satisfy

$$\mathbb{P}(\tau > t) = \varphi(t), \tag{8}$$

for some smooth function $\varphi : [0, +\infty) \to \mathbb{R}$. Then, the pair $(\sigma(t), y(t))_{t \geq 0}$ is Markovian with generator

$$\mathcal{L}f(\sigma, y) = \frac{\partial f}{\partial y}(\sigma, y) + F(y)[f(-\sigma, 0) - f(\sigma, y)], \tag{9}$$

for $f : \{-1, 1\} \times \mathbb{R}^+ \to \mathbb{R}$, with

$$F(y) := -\frac{\varphi'(y)}{\varphi(y)}. \tag{10}$$

This is equivalent to say that the couple $(\sigma(t), y(t))_{t \geq 0}$ evolves according to

$$\begin{cases} (\sigma(t), y(t)) \mapsto (-\sigma(t), 0), & \text{with rate } F(y(t)), \\ dy(t) = dt, & \text{otherwise.} \end{cases} \tag{11}$$

Note that the dynamics in (11) is well defined whenever F is continuous and nonnegative. Moreover, expression (10) for the jump rate follows by observing that, for an interarrival time τ of the jump process $\sigma(t)$, we have

$$\mathbb{P}(\sigma(t + h) = -\sigma | \sigma(t) = \sigma) = 1 - \mathbb{P}(\tau > t + h | \tau > t) = 1 - \frac{\varphi(t + h)}{\varphi(t)},$$

for any $h > 0$. Observe that when the τ's are exponentially distributed $F(y) \equiv 1$, so we get back to dynamics (7). Dynamics (9) can be perturbed by allowing the distribution of the waiting time for a spin-flip to depend on the current spin value σ; the simplest way is to model this dependence as a time scaling:

$$\mathbb{P}(\tau > t | \sigma) = \varphi(a(\sigma)t). \tag{12}$$

Under this distribution for the waiting times the generator of $(\sigma(t), y(t))_{t \geq 0}$ becomes:

$$\mathcal{L}f(\sigma, y) = \frac{\partial f}{\partial y}(\sigma, y) + a(\sigma)F(a(\sigma)y)[f(-\sigma, 0) - f(\sigma, y)].$$

On the basis of what seen above, it is rather simple to define a system of mean field interacting spins with non-exponential waiting times. For a collection of N pairs $(\sigma_i(t), y_i(t))_{i=1,...,N}$, we set $m^N(t) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t)$ to be the magnetization of the system at time t. The interacting dynamics is

$$\begin{cases} (\sigma_i(t), y_i(t)) \mapsto (-\sigma_i(t), 0), & \text{with rate } F\left(y_i(t)e^{-\beta\sigma_i(t)m^N(t)}\right)e^{-\beta\sigma_i(t)m^N(t)}, \\ dy_i(t) = dt, & \text{otherwise.} \end{cases} \tag{13}$$

where $\beta > 0$ is a parameter tuning the interaction between the particles.

Denoting $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$, $\boldsymbol{y} := (y_1, \ldots, y_N) \in (\mathbb{R}^+)^N$, $m^N := \frac{1}{N} \sum_{i=1}^N \sigma_i$, the associated infinitesimal generator is

$$\mathcal{L}^N f(\boldsymbol{\sigma}, \boldsymbol{y}) = \sum_{i=1}^N \frac{\partial f}{\partial y_i}(\boldsymbol{\sigma}, \boldsymbol{y}) + \sum_{i=1}^N F\left(y_i e^{-\beta \sigma_i m^N}\right) e^{-\beta \sigma_i m^N} \left[f(\boldsymbol{\sigma}^i, \boldsymbol{y}^i) - f(\boldsymbol{\sigma}, \boldsymbol{y}) \right],$$
(14)

where $\boldsymbol{\sigma}^i$ is obtained from $\boldsymbol{\sigma}$ by flipping the i-th spin, while $\boldsymbol{y}^i$ by setting to zero the i-th coordinate. The additional factor $e^{-\beta \sigma_i(t) m^N(t)}$ in the jump rate in (13) follows from the observation we made in (12) and the definition of $F(y) = -\frac{\varphi'(y)}{\varphi(y)}$. Note that, for $F \equiv 1$, we retrieve the Curie–Weiss dynamics (4) for the spins.

2.3 Propagation of Chaos

The macroscopic limit and propagation of chaos for the above class of models should be standard, although some difficulties may arise for general choices of F not globally Lipschitz. For computational reasons which will be made clear below, we focus on the case $F(y) = y^\gamma$, for $\gamma \in \mathbb{N}$, which corresponds to considering, in the single spin model (9), the tails of the distribution of the interarrival times to be $\varphi(t) \propto e^{-\frac{t^{\gamma+1}}{\gamma+1}}$.

When $F(y) = y^\gamma$, (13) becomes

$$\begin{cases} (\sigma_i(t), y_i(t)) \mapsto (-\sigma_i(t), 0), & \text{with rate } \quad y_i^\gamma(t) e^{-(\gamma+1)\beta \sigma_i(t) m^N(t)}, \\ dy_i(t) = dt, & \text{otherwise.} \end{cases}$$
(15)

As for the Curie–Weiss model, dynamics (15) is subject to a cooperative-type interaction: the spin-flip rate is larger for particles which are not aligned with the majority. Assuming propagation of chaos, at the macroscopic limit $N \to +\infty$ the representative particle $(\sigma(t), y(t))$ has a mean-field dynamics

$$\begin{cases} (\sigma(t), y(t)) \mapsto (-\sigma(t), 0), & \text{with rate } \quad y^\gamma(t) e^{-(\gamma+1)\beta \sigma(t) m(t)}, \\ dy(t) = dt, & \text{otherwise,} \end{cases}$$
(16)

with $m(t) = \mathbb{E}[\sigma(t)]$. To this dynamics we can associate (see [17]) the non-linear infinitesimal generator $\mathcal{L}(m(t))$ with depends on the value of $m(t)$ and acts on a function $f(\sigma, y)$ as follows:

$$\mathcal{L}(m(t)) f(\sigma, y) = \frac{\partial f}{\partial y}(\sigma, y) + y^\gamma e^{-(\gamma+1)\beta \sigma m(t)} [f(-\sigma, 0) - f(\sigma, y)],$$
(17)

where the non-linearity is due to the dependence of the generator on $m(t)$, a function of the joint law at time t of the processes $(\sigma(t), y(t))$. In Sect. 4 we study rigorously the well-posedness of the pre-limit and limit dynamics and the propagation of chaos. The main result is collected in the following

Theorem 1 (Propagation of chaos) *Fix $\gamma \in \mathbb{N}$, and let $T > 0$ be the final time in (15) and (16). Assume that $(\sigma_i(0), y_i(0))_{i=1,\ldots,N}$ are μ_0-chaotic for some probability distribution μ_0 on $\{-1, 1\} \times \mathbb{R}^+$. Then, the sequence of empirical measures $(\mu_t^N)_{t \in [0,T]}$ converges in distribution on the path space (in the sense of weak convergence of probability measures) to the deterministic law $(\mu_t)_{t \in [0,T]}$ of the unique solution to Eq. (16) with initial distribution μ_0.*

2.4 Local Analysis of the Fokker–Planck

In this section we illustrate the results on the local analysis of the Fokker–Planck equation for the mean field limit dynamics (16) with $\gamma = 1$ and $\gamma = 2$. We adopt the following approach: we find a stationary solution of interest, then we linearize formally the dynamics around that equilibrium. The specific choice of the values of γ allows for an explicit characterization of the discrete spectrum of the linearization of the Fokker–Planck equation. Indeed, we compute the discrete spectrum of the associated linearized operator, which we show to be given by the zeros of an explicit holomorphic function $H_{\beta,\gamma}(\lambda)$. We then study numerically the character of the eigenvalues when β varies: for both $\gamma = 1, 2$, we find that for all $\beta < \beta_c(\gamma)$ all eigenvalues have negative real part; at $\beta_c(\gamma)$ two eigenvalues are conjugate and purely imaginary, suggesting the possible presence of a Hopf bifurcation in the limit dynamics. These critical values of β are then compared to the ones estimated from the simulations of the finite particle system in Sect. 3.

The Fokker–Planck equation associated to (16) is a PDE describing the time evolution of the density function $f(t, \sigma, y)$ of the limit process $(\sigma(t), y(t))$. It is given by

$$\begin{cases} \frac{\partial}{\partial t} f(t, \sigma, y) + \frac{\partial}{\partial y} f(t, \sigma, y) + y^\gamma e^{-(\gamma+1)\beta\sigma m(t)} f(t, \sigma, y) = 0, \\ f(t, \sigma, 0) = \int_0^{+\infty} y^\gamma e^{(\gamma+1)\beta\sigma m(t)} f(t, -\sigma, y)dy, \\ m(t) = \int_0^\infty [f(t, 1, y) - f(t, -1, y)]dy, \\ 1 = \int_0^\infty [f(t, 1, y) + f(t, -1, y)]dy, \\ f(0, \sigma, y) = f_0(\sigma, y), \text{ for } \sigma \in \{-1, 1\}, \ y \in \mathbb{R}^+. \end{cases} \tag{18}$$

We postpone a derivation of the Fokker–Planck equation to Sect. 4.1. Here we just spend a few words on the boundary integral condition - second line of System (18) - which is specific to our model and may not be clear at first sight. It is a mass-balance condition: heuristically, at each time t, the mass of the spins at state $(\sigma, 0)$ (i.e. $f(t, \sigma, 0)$) equals the spins just jumped from $(-\sigma, y)$ weighted with their probabilities (i.e. $y^\gamma e^{(\gamma+1)\beta\sigma m(t)} f(t, -\sigma, y)dy$) integrated over all the possible jump times. While a general study of (18) is beyond the scope of this work, here we just observe that (18) can be seen as a system of two quasilinear PDEs (one for $\sigma = 1$ and another for $\sigma = -1$), where the non-linearity enters in an integral form through $m(t)$ in the exponent of the rate function. Moreover, the boundary integral condition in the second line poses additional challenges. Nevertheless, it is easy to exhibit a particular stationary solution to (18):

Proposition 1 *The function*

$$f^*(\sigma, y) = \frac{1}{2\Lambda} e^{-\frac{y^{\gamma+1}}{\gamma+1}}, \tag{19}$$

with $\Lambda := \int_0^{+\infty} e^{-\frac{y^{\gamma+1}}{\gamma+1}} dy$, is a stationary solution to Sys.(18) with $m = 0$.

Remark 1 Let $g^*(\sigma)$ be the marginal of $f^*(\sigma, y)$ with respect to the first coordinate. Then, $g^*(\sigma)$ is a stationary reversible distribution for the limit renewal process $(\sigma(t))_{t\geq 0}$. Indeed, by choosing $\sigma(0) \sim g^*$, $g^*(1) = g^*(-1) = \frac{1}{2}$, we have that $m(t) \equiv 0$ and $(\sigma(t))_{t\geq 0}$ is a renewal process with interarrival times τ such that $\mathbb{P}(\tau > t) \propto e^{-\frac{t^{\gamma+1}}{\gamma+1}}$ independently of the value of σ, so its law is invariant by time reversal.

The linearization of the operator associated to Sys. (18) around the equilibrium (19), yields the following eigen-system

$$
\begin{cases}
\frac{\partial}{\partial y} g(\sigma, y) + y^\gamma g(\sigma, y) - \frac{\beta \sigma k(\gamma+1)}{2\Lambda} y^\gamma e^{-\frac{y^{\gamma+1}}{\gamma+1}} = -\lambda g(\sigma, y), \\
g(\sigma, 0) = \frac{\beta \sigma k(\gamma+1)}{2\Lambda} + \int_0^\infty g(-\sigma, y) y^\gamma dy, \\
\int_0^\infty [g(\sigma, y) + g(-\sigma, y)] dy = 0, \quad (\sigma, y) \in \{-1, 1\} \times \mathbb{R}^+,
\end{cases}
\tag{20}
$$

where $k = 2 \int_0^\infty g(1, y) dy$, and $\Lambda = \int_0^\infty e^{-\frac{y^{\gamma+1}}{\gamma+1}} dy$. For a formal derivation see Sect. 4.3.1. We work out the computations of the discrete spectrum of the linearized operator for the two cases $\gamma = 1$, $\gamma = 2$.

2.4.1 Case $\gamma = 1$

In this case, $\Lambda = \sqrt{\frac{\pi}{2}}$, and the eigen-system (20) becomes

$$
\begin{cases}
\frac{\partial}{\partial y} g(\sigma, y) + y g(\sigma, y) + \lambda g(\sigma, y) = \beta \sigma k \left(\sqrt{\frac{\pi}{2}} \right)^{-1} y e^{-\frac{y^2}{2}} \\
g(\sigma, 0) = \beta \sigma k \left(\sqrt{\frac{\pi}{2}} \right)^{-1} + \int_0^\infty y g(-\sigma, y) dy, \\
\int_0^\infty [g(\sigma, y) + g(-\sigma, y)] dy = 0,
\end{cases}
\tag{21}
$$

where $k = 2 \int_0^\infty g(1, y) dy$. The eigenvalues of (21) are given by the zeros of an explicit holomorphic function.

Proposition 2 *The solutions in $\lambda \in \mathbb{C}$ to (21) are the zeros of the holomorphic function*

$$
H_{\beta,1}(\lambda) := H_1(\lambda) \left[-4\beta - \lambda^3 \sqrt{\frac{\pi}{2}} \right] + \sqrt{2\pi} \lambda^2 - 4\beta\lambda + 2\beta \sqrt{2\pi},
\tag{22}
$$

with

$$
H_1(\lambda) := \int_0^\infty e^{-\frac{y^2}{2}} e^{-\lambda y} dy.
\tag{23}
$$

Moreover, it holds

$$
H_1(\lambda) = \sqrt{\frac{\pi}{2}} \sum_{m=0}^\infty \frac{\lambda^{2m}}{(2m)!!} - \lambda \sum_{m=0}^\infty \frac{(2\lambda)^{2m} m!}{(2m+1)!} \frac{1}{2^m}.
\tag{24}
$$

Equation $H_{\beta,1}(\lambda) = 0$ can be numerically investigated. We used the power series expansion (24) and a numerical root finding built-in function of the software `Mathematica`, specifically `FindRoot`, starting the search from different initial points of the complex plane and from different values of β. Here we report the results:

(1.1) we find two conjugate purely imaginary solutions to $H_{\beta,1}(\lambda) = 0$, for $\lambda = \pm \lambda_c(1) := \pm i (1.171)$ and

$$
\beta = \beta_c(1) := 0.769;
\tag{25}
$$

(1.2) iterating the search around $(\beta_c(1), \lambda_c(1))$, the resulting complex eigenvalue goes from having a negative real part for $\beta < \beta_c(1)$ to a positive real part for $\beta > \beta_c(1)$;

(1.3) no other purely imaginary solution $\lambda = \pm ix$ is found for $0 \le x \le 500$ and $0 \le \beta \le 20$;

(1.4) for $\beta < \beta_c(1)$ all the eigenvalues $\lambda = ix + y$ are such that $y < 0$. This was verified for $-100 \le x \le 100$, $-100 \le y \le 100$.

2.4.2 Case $\gamma = 2$

In this case the eigen-system is given by

$$\begin{cases} \frac{\partial}{\partial y} g(\sigma, y) + y^2 g(\sigma, y) + \lambda g(\sigma, y) = \frac{3}{2\Lambda} \beta \sigma k y^2 e^{-\frac{y^3}{3}}, \\ g(\sigma, 0) = \frac{3}{2\Lambda} \beta \sigma k + \int_0^\infty y^2 g(-\sigma, y) dy, \\ \int_0^\infty [g(\sigma, y) + g(-\sigma, y)] dy = 0, \end{cases} \tag{26}$$

where $\Lambda = \int_0^\infty e^{-\frac{y^3}{3}} dy = \frac{\Gamma\left(\frac{1}{3}\right)}{3^{2/3}}$, $k = 2 \int_0^\infty g(1, y) dy$, and $\Gamma(\cdot)$ is the Gamma function. Analogously to the case $\gamma = 1$, the eigenvalues of (26) can be computed as the zeros of an explicit holomorphic function.

Proposition 3 *The solutions in $\lambda \in \mathbb{C}$ to (26) are the zeros of the holomorphic function*

$$H_{\beta,2}(\lambda) := H_2(\lambda)\bigg[12\beta - \lambda^4\Lambda + 6\beta\lambda\Lambda - 6\beta\lambda 3^{1/3}\Gamma(4/3) + 3\beta\lambda^2 3^{2/3}\Gamma(5/3) \tag{27}$$
$$- 6\beta\lambda^2 \frac{\Gamma(2/3)}{3^{1/3}} \bigg] + \bigg[2\Lambda\lambda^3 - 12\beta\Lambda + 12\beta\frac{\Gamma(2/3)}{3^{1/3}}\lambda - 6\beta\lambda^2 \bigg],$$

with

$$H_2(\lambda) := \int_0^\infty e^{-\lambda y} e^{-\frac{y^3}{3}} dy. \tag{28}$$

Moreover, it holds

$$H_2(\lambda) = \sum_{n=0}^\infty (-1)^n \frac{\lambda^n}{n!} 3^{\frac{1}{3}(n-2)} \Gamma\left(\frac{n+1}{3}\right). \tag{29}$$

We solved numerically $H_{\beta,2}(\lambda) = 0$ for different values of the parameters. Apart from being sensibly slower, it seems the numerical root finding for $\gamma = 2$ suffers from numerical instability issues. This is why we were able to check the results for much smaller intervals in this case. Our results are the following:

(2.1) we find two conjugate purely imaginary solutions $\lambda = \pm\lambda_c(2) := \pm i(1.978)$ for

$$\beta = \beta_c(2) := 0.362; \tag{30}$$

(2.2) analogous to (1.2);
(2.3) analogous to (1.3), verified for $0 \le x \le 10$ and $0 \le \beta \le 5$;
(2.4) analogous to (1.4), verified for $-25 \le x \le 25$, $-25 \le y \le 25$.

3 Finite Particle System Simulations

We ran several simulations of the particle system with $N = 1500$ spins for $\gamma = 1, 2$. Simulations are in accordance with the above numerical results on the eigenvalues. In particular, above the critical values of β (see (25) and (30)) periodic behavior of the finite interacting particle system appears. Description of the evidences is the following:

– For β small the magnetization goes to zero regardless of the initial datum (Fig. 1).
– There is a critical β (around 0.75 for $\gamma = 1$, around 0.35 for $\gamma = 2$) above which the magnetization starts oscillating. Close to the critical point oscillations do not look very regular (corrupted by noise?), but they soon become very regular if β is not too close to the critical value (see Fig. 2). We also made joint plots of the magnetization with the empirical mean of the y_i's (Fig. 3). Periodic oscillations seems to emerge.
– As β increases, the amplitude of the oscillation of the magnetization increases, while the period looks nearly constant (see Fig. 4). As β crosses another critical value (around 1.3 for $\gamma = 1$, around 1.65 for $\gamma = 2$) oscillations disappear, and the system magnetizes, i.e. the magnetization stabilizes to a non-zero value, actually close to ± 1 (Fig. 5).

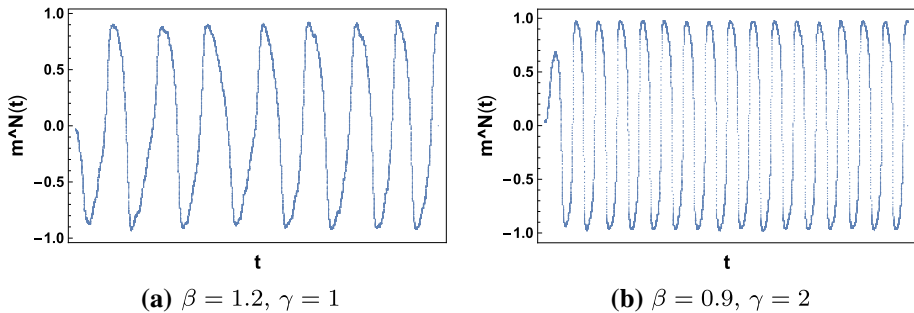**(a)** $\beta = 0.25, \gamma = 1$ **(b)** $\beta = 0.1, \gamma = 2$
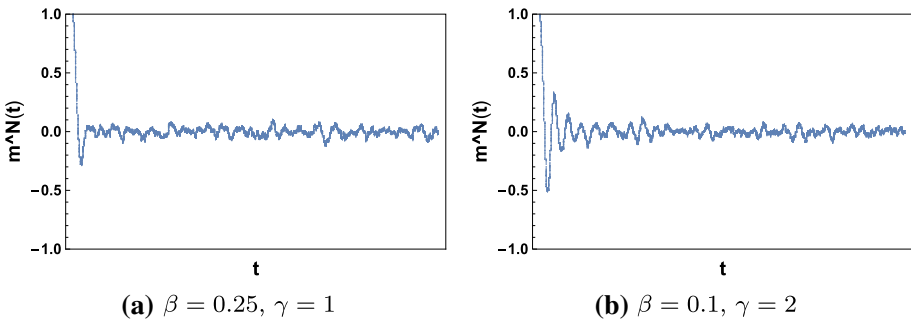
Fig. 1 Simulation of the finite particle system dynamics (15) for $\gamma = 1$ (left) and $\gamma = 2$ (right). The plot shows the time evolution of the empirical magnetization of $N = 1500$ spins. In the picture is depicted the evolution of magnetization, with initial data $\sigma_i(0) = 1$ for every $i = 1, \ldots, N$. Simulations suggest that the zero mean trajectory is attractive and the magnetization goes to zero regardless of the initial datum

**(a)** $\beta = 0.75, \gamma = 1$ **(b)** $\beta = 0.35, \gamma = 2$
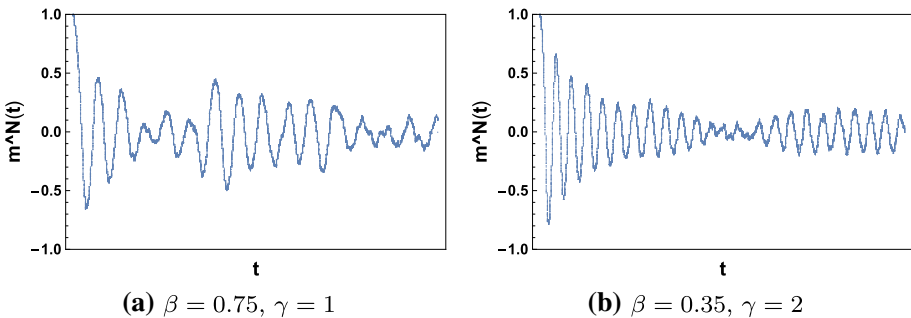
Fig. 2 Simulation of the finite particle system dynamics (15) for $\gamma = 1$ (left) and $\gamma = 2$ (right), with number of spins $N = 1500$. Increasing the value of β oscillations of the empirical magnetization appear. Initially not very regular, the rhythmic behavior becomes evident for larger β as shown in Fig. 3 and 4

(a) $\beta = 1.1$, $\gamma = 1$ **(b)** $\beta = 0.6$, $\gamma = 2$
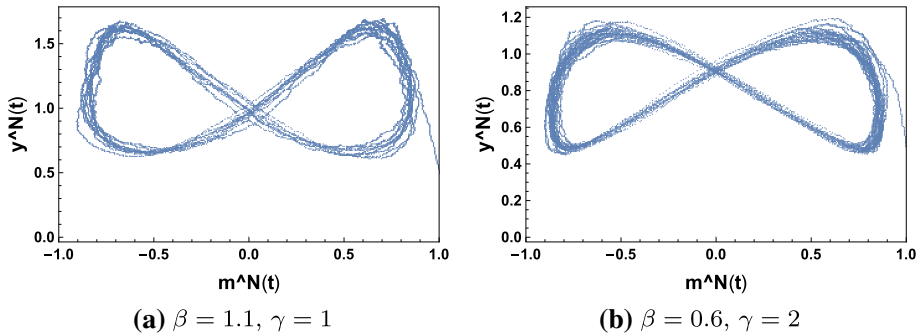
Fig. 3 Above a critical value of β the dynamics defined in (15) is periodic. In the figure the time evolution of the pair $(m^N(t), y^N(t))$ is plotted for $N = 1500$ particles. Here $m^N(t)$ is the empirical magnetization of the spins, whereas $y^N(t)$ indicates the empirical mean of the y_i's

(a) $\beta = 1.2$, $\gamma = 1$ **(b)** $\beta = 0.9$, $\gamma = 2$

Fig. 4 Simulation of the finite particle system dynamics (15) for $\gamma = 1$ (left) and $\gamma = 2$ (right), with number of spins $N = 1500$. For β above a critical value the empirical magnetization has a clear periodic behavior

(a) $\beta = 1.3$, $\gamma = 1$ **(b)** $\beta = 1.65$, $\gamma = 2$
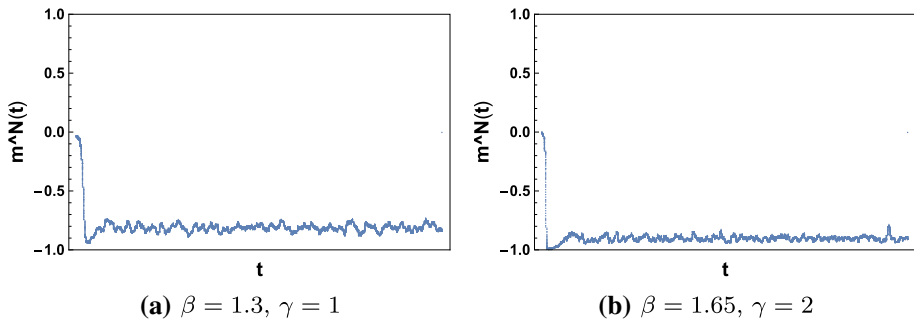
Fig. 5 Increasing β further above the first critical value, the system (15) crosses a second critical value above which oscillations disappear, and the system magnetizes, i.e. the magnetization stabilizes to a non-zero value. The simulation in the figure shows the evolution of the empirical magnetization for a system comprised of $N = 1500$ spins

– The oscillations are lasting for a wider interval of β's for $\gamma = 2$ (from $\beta \approx 0.35$ until $\beta \approx 1.65$) than $\gamma = 1$ (from $\beta \approx 0.75$ until $\beta \approx 1.3$). The period is instead smaller for $\gamma = 2$ than for $\gamma = 1$.

– For both $\gamma = 1, 2$, the appearance of oscillations in the particle system dynamics does not seem to depend on the initial data. This behavior of the finite volume system may reflect the presence of a *global* Hopf bifurcation in the mean field limit dynamics.

4 Proofs

4.1 Formal Derivation of the Fokker–Planck Equation

We give a sketch of the derivation of the Fokker–Planck equation (18). Let the process $(\sigma(t), y(t))_{t \geq 0}$ be distributed at time t according to some distribution $dm(t, \sigma, y) \in \mathcal{P}(\{-1, 1\} \times [0, +\infty))$, which we assume to be regular in y, i.e. absolutely continuous with respect to the Lebesgue measure on $[0, +\infty)$, with a smooth density $f(t, \sigma, y)dy$. Let $h : \{-1, 1\} \times [0, +\infty) \to \mathbb{R}$ be a smooth function, and compute

$$\frac{d}{dt} \mathbb{E}\left[h(\sigma(t), y(t))\right] = \mathbb{E}\left[\mathcal{L}(m(t))h(\sigma(t), y(t))\right],$$

with $m(t) := \int_{[0,+\infty)} [f(t, 1, y) - f(t, -1, y)]dy$. Under our regularity assumptions, the above expression is equivalent to

$$\int_{[0,+\infty)} h(1, y) \frac{\partial}{\partial t} f(t, 1, y)dy + \int_{[0,+\infty)} h(-1, y) \frac{\partial}{\partial t} f(t, -1, y)dy$$

$$= \int_{[0,+\infty)} \frac{\partial}{\partial y} h(1, y) f(t, 1, y)dy + \int_{[0,+\infty)} \frac{\partial}{\partial y} h(-1, y) f(t, -1, y)dy$$

$$+ \int_{[0,+\infty)} y^\gamma e^{-(\gamma+1)\beta m(t)} \left[h(-1, 0) - h1, y)\right] f(t, 1, y)dy$$

$$+ \int_{[0,+\infty)} y^\gamma e^{(\gamma+1)\beta m(t)} \left[h(1, 0) - h(-1, y)\right] f(t, -1, y)dy.$$

Integrating by parts in the first two integrals on the right hand side of the equality, and regrouping the terms, one finds

$$\int_{[0,+\infty)} h(1, y) \left[\frac{\partial}{\partial t} f(t, 1, y) + \frac{\partial}{\partial y} f(t, 1, y) + y^\gamma e^{-(\gamma+1)\beta m(t)} f(t, 1, y)\right] dy$$

$$+ \int_{[0,+\infty)} h(-1, y) \left[\frac{\partial}{\partial t} f(t, -1, y) + \frac{\partial}{\partial y} f(t, -1, y) + y^\gamma e^{(\gamma+1)\beta m(t)} f(t, -1, y)\right] dy$$

$$+ h(1, 0) \left[f(t, 1, 0) - \int_{[0,+\infty)} y^\gamma e^{(\gamma+1)\beta m(t)} f(t, -1, y)dy\right]$$

$$+ h(-1, 0) \left[f(t, -1, 0) - \int_{[0,+\infty)} y^\gamma e^{-(\gamma+1)\beta m(t)} f(t, 1, y)dy\right] = 0,$$

which, by the arbitrariness of h implies the first two equations in (18); the definition of $m(t)$ and the requirement for f to be a density function over $\{-1, 1\} \times [0, +\infty)$ complete the equations in (18).

4.2 Proof of Theorem 1

Here we prove rigorously a propagation of chaos property for the N-particle dynamics to its mean-field limit, for any $\gamma \in \mathbb{N}$. Actually, we establish the proofs for $\gamma = 1$, where the rates enjoy globally Lipschitz properties, and then we generalize them to any $\gamma \in \mathbb{N}$ in Remark 2. The generalization to non-Lipschitz rates is possible because of the a-priori bound on the variables y_i's which, by definition, are such that $0 \leq y_i \leq T$, where $T < \infty$ is the final time horizon of the dynamics. For the convenience of the reader, we write again the dynamics

$$\begin{cases} (\sigma_i(t), y_i(t)) \mapsto (-\sigma_i(t), 0), & \text{with rate} \quad y_i^{\gamma}(t) e^{-(\gamma+1)\beta\sigma_i(t)m^N(t)}, \\ dy_i(t) = dt, & \text{otherwise}, \end{cases} \tag{31}$$

and the mean-field version

$$\begin{cases} (\sigma(t), y(t)) \mapsto (-\sigma(t), 0), & \text{with rate} \quad y^{\gamma}(t) e^{-(\gamma+1)\beta\sigma(t)m(t)}, \\ dy(t) = dt, & \text{otherwise}, \end{cases} \tag{32}$$

with $m(t) = \mathbb{E}[\sigma(t)]$. We represent both the microscopic and the macroscopic model as solutions of certain stochastic differential equations driven by Poisson random measures, in order to apply the results in [16]. As anticipated, in the proof we restrict to a finite interval of time $[0, T]$.

To begin with, let us fix a filtered probability space $\left((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{F}_t)_{t \in [0,T]}\right)$ satisfying the usual hypotheses, rich enough to carry an independent and identically distributed collection $\mathcal{N}, (\mathcal{N}_i)_{i \in \mathbb{N}}$ of stationary Poisson random measures on $[0, T] \times \mathcal{E}$, with intensity measure ν on $\mathcal{E} := [0, +\infty)$ equal to the restriction of the Lebesgue measure on $[0, +\infty)$. For any N, consider the system of Itô-Skorohod equations

$$\begin{cases} \sigma_i(t) = \sigma_i(0) + \int_0^t \int_{\mathcal{E}} f_1(\sigma_i(s^-), \xi, m^N(s^-), y_i(s^-)) \mathcal{N}_i(ds, d\xi), \\ y_i(t) = y_i(0) + t + \int_0^t \int_{\mathcal{E}} f_2(\sigma_i(s^-), \xi, m^N(s^-), y_i(s^-)) \mathcal{N}_i(ds, d\xi), \end{cases} \tag{33}$$

and the corresponding limit non-linear *reference particle*'s dynamics

$$\begin{cases} \sigma(t) = \sigma(0) + \int_0^t \int_{\mathcal{E}} f_1(\sigma(s^-), \xi, m(s^-), y(s^-)) \mathcal{N}(ds, d\xi), \\ y(t) = y(0) + t + \int_0^t \int_{\mathcal{E}} f_2(\sigma(s^-), \xi, m(s^-), y(s^-)) \mathcal{N}(ds, d\xi). \end{cases} \tag{34}$$

The functions $f_1, f_2 : \{-1, 1\} \times \mathbb{R}^+ \times [-1, 1] \times \mathbb{R}^+ \to \mathbb{R}$, modeling the jumps of the process, are given by

$$f_1(\sigma, \xi, m, y) := -2\sigma \mathbb{1}_{]0,\lambda[}(\xi), \quad f_2(\sigma, \xi, m, y) := -y \mathbb{1}_{]0,\lambda[}(\xi), \tag{35}$$

with $\lambda := \lambda(\sigma, m, y)$ being the rate function $\lambda(\sigma, m, y) = y^{\gamma} e^{-(\gamma+1)\beta\sigma m}$.

Proposition 4 *For $\gamma = 1$, Eqs. (33) and (34) possess a unique strong solution for $t \in [0, T]$.*

Proof With the choices in (35), the well-posedness of Eqs. (33) and (34) follows by Theorems 1.2 and 2.1 in [16]. Indeed, even though the function f_2 is not globally Lipschitz continuous in y, the L^1 Lipschitz assumption of the theorem still holds, by noting that

$$\int_{\Xi}\left|f_2(\sigma,\xi,m,y)-f_2(\tilde{\sigma},\xi,\tilde{m},\tilde{y})\right|d\xi=\int_{\Xi}\left|y\mathbb{1}_{]0,\lambda(\sigma,m,y)[}(\xi)-\tilde{y}\mathbb{1}_{]0,\lambda(\tilde{\sigma},\tilde{m},\tilde{y})[}(\xi)\right|d\xi$$

$$\leq|y|\left|\lambda(\sigma,m,y)-\lambda(\tilde{\sigma},\tilde{m},\tilde{y})\right|+\left|\lambda(\tilde{\sigma},\tilde{m},\tilde{y})\right|\left|y-\tilde{y}\right|$$

$$\leq|y|\left[\left|\lambda(\sigma,m,y)-\lambda(\tilde{\sigma},\tilde{m},\tilde{y})\right|+\left|\lambda(\tilde{\sigma},\tilde{m},\tilde{y})\right|\left|y-\tilde{y}\right|\right]$$

$$\leq CT\left[|m-\tilde{m}|+|y-\tilde{y}|+|\sigma-\tilde{\sigma}|\right],$$

where in the last step we have used that, by construction, the processes $y_i(t)\leq T$ for every $t\in[0,T]$, so that the rates are a priori bounded and the Lipschitz properties of $ye^{-2\beta\sigma m}$ for $(y,\sigma,m)\in\mathbb{R}^+\times\{-1,1\}\times[-1,1]$. □

Now, define the empirical measures $\mu^N:=\frac{1}{N}\sum_{i=1}^N\delta_{(\sigma_i,y_i)}$, and their evaluation along the paths of (33),

$$\mu_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{(\sigma_i(t),y_i(t))}.\tag{36}$$

The measures $(\mu_t^N)_{t\in[0,T]}$ can be viewed as random variables with values in $\mathcal{P}(D)$, the space of probability measures on D, where $D:=\mathcal{D}([0,T];\{-1,1\}\times\mathbb{R}^+)$ is the space of $\{-1,1\}\times\mathbb{R}^+$-valued càdlàg functions equipped with the Skorohod topology.

Proof of Theorem 1 Consider the i.i.d. processes $(\tilde{\sigma}_i(t),\tilde{y}_i(t))_{i=1,\ldots,N}$, coupled with the N-particle dynamics $(\sigma_i(t),y_i(t))_{i=1,\ldots,N}$,

$$\begin{cases}\tilde{\sigma}_i(t)=\tilde{\sigma}_i(0)+\int_0^t\int_{\Xi}f_1(\tilde{\sigma}_i(s^-),\xi,m(s^-),\tilde{y}_i(s^-))\mathcal{N}_i(ds,d\xi),\\\tilde{y}_i(t)=\tilde{y}_i(0)+t+\int_0^t\int_{\Xi}f_2(\tilde{\sigma}_i(s^-),\xi,m(s^-),\tilde{y}_i(s^-))\mathcal{N}_i(ds,d\xi),\end{cases}\tag{37}$$

with $m(t)=\mathbb{E}[\tilde{\sigma}_i(t)]$. Let $(\tilde{\mu}_t^N)_{t\in[0,T]}$ be the empirical measure associated to (37). Clearly, one has $(\tilde{\mu}_t^N)_{t\in[0,T]}\to(\mu_t)_{t\in[0,T]}$ in the weak convergence sense (by a functional LLN, see [16] for e.g.) where $(\mu_t)_{t\in[0,T]}$ is the deterministic law of the unique solution to Eq.(16) with initial distribution μ_0. We are thus left to show

$$d_1\Big(\mathrm{Law}\big((\mu_t^N)_{t\in[0,T]}\big),\mathrm{Law}\big((\tilde{\mu}_t^N)_{t\in[0,T]}\big)\Big)\xrightarrow{N\to+\infty}0,$$

with d_1 being the 1-Wasserstein distance (which metrizes the weak convergence of probability measures) on $\mathcal{P}(\mathcal{P}(D))$. Since (for instance see [9])

$$d_1\Big(\mathrm{Law}\big((\mu_t^N)_{t\in[0,T]}\big),\mathrm{Law}\big((\tilde{\mu}_t^N)_{t\in[0,T]}\big)\Big)\leq\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[d_{Sko}\big((\sigma_i,y_i),(\tilde{\sigma}_i,\tilde{y}_i)\big)\big],$$

with d_{Sko} the Skorohod metric on D, it is enough to show that

$$\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\sup_{t\in[0,T]}\Big(|\sigma_i(t)-\tilde{\sigma}_i(t)|+|y_i(t)-\tilde{y}_i(t)|\Big)\right]\xrightarrow{N\to+\infty}0.\tag{38}$$

The proof of (38) starts with the following inequalities which make use of the bounds for f_2 in Proposition 4. In the subsequent inequalities the value of the constant C may change from line to line. We have:

$$\mathbb{E}\left[\sup_{s\in[0,t]}|y_i(s)-\tilde{y}_i(s)|\right]\le\mathbb{E}\left[|y_i(0)-\tilde{y}_i(0)|\right]$$

$$+C\int_0^t\mathbb{E}\left[|m^N(s)-m(s)|+|y_i(s)-\tilde{y}_i(s)|+|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]ds\le C\int_0^t\mathbb{E}$$

$$\left[\sup_{r\in[0,s]}|m^N(r)-m(r)|+\sup_{r\in[0,s]}|y_i(r)-\tilde{y}_i(r)|+\sup_{r\in[0,s]}|\sigma_i(r)-\tilde{\sigma}_i(r)|\right]ds+C(N),$$

with $C(N)\xrightarrow{N\to+\infty}0$ because of the chaoticity assumption on the initial datum. We proceed similarly for the σ_i's, using the the Lipschitz continuity of f_1, we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]\le\mathbb{E}\left[|\sigma_i(0)-\tilde{\sigma}_i(0)|\right]$$

$$+C\int_0^t\mathbb{E}\left[|m^N(s)-m(s)|+|y_i(s)-\tilde{y}_i(s)|+|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]ds\le C\int_0^t\mathbb{E}$$

$$\left[\sup_{r\in[0,s]}|m^N(r)-m(r)|+\sup_{r\in[0,s]}|y_i(r)-\tilde{y}_i(r)|+\sup_{r\in[0,s]}|\sigma_i(r)-\tilde{\sigma}_i(r)|\right]ds+C(N).$$

Denoting $\tilde{m}^N(t):=\frac{1}{N}\sum_{i=1}^N\tilde{\sigma}_i(t)$, we find

$$\mathbb{E}\left[\sup_{s\in[0,t]}|m^N(s)-m(s)|\right]\le\mathbb{E}\left[\sup_{s\in[0,t]}|m^N(s)-\tilde{m}^N(s)|\right]+\mathbb{E}\left[\sup_{s\in[0,t]}|\tilde{m}^N(s)-m(s)|\right]$$

$$\le\frac{1}{N}\sum_{j=1}^N\mathbb{E}\left[\sup_{s\in[0,t]}|\sigma_j(s)-\tilde{\sigma}_j(s)|\right]+\mathbb{E}\left[\sup_{s\in[0,t]}|\tilde{m}^N(s)-m(s)|\right]$$

$$=\mathbb{E}\left[\sup_{s\in[0,t]}|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]+C(N),$$

with $C(N)\xrightarrow{N\to+\infty}0$ because of the chaoticity of the i.i.d. processes $(\tilde{\sigma}_i(t),\tilde{y}_i(t))_{i=1,\dots,N}$, and where in the equality we have used the exchangeability of the processes $(\sigma_i,\tilde{\sigma}_i)_{i=1,\dots,N}$. Collecting the estimates, we have shown, for any $t\in[0,T]$,

$$\frac{1}{N}\sum_{i=1}^N\left\{\mathbb{E}\left[\sup_{s\in[0,t]}|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]+\mathbb{E}\left[\sup_{s\in[0,t]}|y_i(s)-\tilde{y}_i(s)|\right]\right\}$$

$$\le C(N)+C\int_0^t\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\sup_{r\in[0,s]}|\sigma_i(r)-\tilde{\sigma}_i(r)|+\sup_{r\in[0,s]}|y_i(r)-\tilde{y}_i(r)|\right]ds,$$

which by the Gronwall's lemma applied to

$$\phi(t):=\frac{1}{N}\sum_{i=1}^N\left\{\mathbb{E}\left[\sup_{s\in[0,t]}|\sigma_i(s)-\tilde{\sigma}_i(s)|\right]+\mathbb{E}\left[\sup_{s\in[0,t]}|y_i(s)-\tilde{y}_i(s)|\right]\right\},$$

implies (38), because $\phi(T)$ is an upper bound for the left hand side of (38). □

Remark 2 Proposition 4 and the proof of Theorem 1 can be generalized to any $\gamma \in \mathbb{N}$. Indeed, analogous Lipschitz L^1 estimates on the rates of Proposition 4 (used also in Theorem 1) hold by estimating

$$
\begin{aligned}
\left|\lambda(\sigma, m, y) - \lambda(\tilde{\sigma}, \tilde{m}, \tilde{y})\right| &= \left|y^\gamma e^{-(\gamma+1)\beta m \sigma} - \tilde{y}^\gamma e^{-(\gamma+1)\beta \tilde{m}\tilde{\sigma}}\right| \\
&\leq \left|y^\gamma e^{-(\gamma+1)\beta m \sigma} - \tilde{y}^\gamma e^{-(\gamma+1)\beta m \sigma}\right| + \left|\tilde{y}^\gamma e^{-(\gamma+1)\beta m \sigma} - \tilde{y}^\gamma e^{-(\gamma+1)\beta \tilde{m}\tilde{\sigma}}\right| \\
&\leq \left|e^{-(\gamma+1)\beta m \sigma}\right|\left|y^\gamma - \tilde{y}^\gamma\right| + \tilde{y}^\gamma \left|e^{-(\gamma+1)\beta m \sigma} - e^{-(\gamma+1)\beta \tilde{m}\tilde{\sigma}}\right| \\
&\leq C\left|y - \tilde{y}\right|\left|p(y, \tilde{y})\right| + \tilde{y}^\gamma \left[C|m - \tilde{m}| + C|\sigma - \tilde{\sigma}|\right] \leq C\left[|y - \tilde{y}| + |m - \tilde{m}| + |\sigma - \tilde{\sigma}|\right],
\end{aligned}
$$

with $p(y, \tilde{y})$ a polynomial of degree $\gamma - 1$. In the last step we have used the a priori bounds on $y \leq T$ to get $|p(y, \tilde{y})| \leq C(T)$ and the Lipschitz properties of $e^{-(\gamma+1)\beta m \sigma}$ for $(\sigma, m) \in \{-1, 1\} \times [-1, 1]$.

4.3 Proofs of the Local Analysis of Sect. 2.4

In this section we address the proofs of the results illustrated in Sect. 2.4.

Proof of Proposition 1 Setting $m = 0$ in Sys. (18), the stationary version of the first equation becomes

$$
\frac{\partial}{\partial y} f(\sigma, y) + y^\gamma f(\sigma, y) = 0, \tag{39}
$$

whose solution is of the form $f^*(\sigma, y) = c(\sigma) f(\sigma, 0) e^{-\frac{y^{\gamma+1}}{\gamma+1}}$. Denoting $\Lambda := \int_0^{+\infty} e^{-\frac{y^{\gamma+1}}{\gamma+1}} dy$, it is easy to see that the integral conditions imply $c(\sigma) = c(-\sigma) = \frac{1}{\Lambda}$ and $f(\sigma, 0) = f(-\sigma, 0) = \frac{1}{2}$. $\qquad\square$

4.3.1 Formal Derivation of Sys.(20)

We now compute formally the linearization of the operator associated to Sys. (18) around the solution (19) with $m = 0$. Namely, if we write the first equation in (18) in operator form

$$
\frac{\partial}{\partial t} f(t, \sigma, y) - \mathcal{L}_\gamma^{nl} f(t, \sigma, y) = 0,
$$

with $\mathcal{L}_\gamma^{nl} f(t, \sigma, y) := -\frac{\partial}{\partial y} f(t, \sigma, y) - y^\gamma e^{-(\gamma+1)\beta \sigma m(t)} f(t, \sigma, y)$, we want to find the linearized version of the operator $\mathcal{L}_\gamma^{nl}$.

For the purpose, we express a generic stationary solution to (18) as

$$
f(\sigma, y) = f^*(\sigma, y) + \varepsilon g(\sigma, y),
$$

where f^* is the stationary solution corresponding to $m = 0$ and g satisfies the condition

$$
\int_0^\infty [g(1, y) + g(-1, y)] dy = 0, \tag{40}
$$

so that $\int_0^\infty [f(1, y) + f(-1, y)] dy = 1$ holds. We also denote $m_f := \int_0^\infty [f(1, y) - f(-1, y)] dy$, which by the above consideration satisfies

$$
m_f = 2\varepsilon \int_0^\infty g(1, y) dy =: \varepsilon k. \tag{41}
$$

The stationary version of the first equation in (18) becomes

$$\frac{\partial}{\partial y} f^*(\sigma, y) + \varepsilon \frac{\partial}{\partial y} g(\sigma, y) + y^\gamma e^{-\beta \sigma \varepsilon k(\gamma+1)}[f^*(\sigma, y) + \varepsilon g(\sigma, y)] = 0.$$

By expanding at the first order in ε the term $e^{-\beta \sigma \varepsilon k(\gamma+1)} \approx 1 - (\gamma + 1)\beta \sigma \varepsilon k$, and by considering only the resulting linear terms in ε, we get

$$\frac{\partial}{\partial y} f^*(\sigma, y) + \varepsilon \frac{\partial}{\partial y} g(\sigma, y) + y^\gamma f^*(\sigma, y) + y^\gamma \varepsilon g(\sigma, y) - y^\gamma (\gamma + 1)\beta \sigma \varepsilon k f^*(\sigma, y) = 0.$$

Finally, using that f^* solves (39) and substituting its expression (19), we get

$$\frac{\partial}{\partial y} g(\sigma, y) + y^\gamma g(\sigma, y) - \frac{\beta \sigma k(\gamma + 1)}{2\Lambda} y^\gamma e^{-\frac{y^{\gamma+1}}{\gamma+1}} = 0.$$

We can define the linearized operator as

$$\mathcal{L}_\gamma^{\text{lin}} g(\sigma, y) := -\frac{\partial}{\partial y} g(\sigma, y) - y^\gamma g(\sigma, y) + \frac{\beta \sigma k(\gamma + 1)}{2\Lambda} y^\gamma e^{-\frac{y^{\gamma+1}}{\gamma+1}}. \tag{42}$$

We proceed with the linearization of the integral condition in the second line of Sys. (18):

$$f^*(\sigma, 0) + \varepsilon g(\sigma, 0) = \int_0^\infty [f^*(-\sigma, y) + \varepsilon g(-\sigma, y)] y^\gamma e^{\beta \sigma \varepsilon k(\gamma+1)} dy$$

$$\approx \int_0^\infty f^*(-\sigma, y) y^\gamma (1 + \beta \sigma \varepsilon k(\gamma + 1)) dy + \varepsilon \int_0^\infty g(-\sigma, y) y^\gamma (1 + \beta \sigma \varepsilon k(\gamma + 1)) dy$$

$$\approx \int_0^\infty f^*(-\sigma, y) y^\gamma dy + \beta \sigma \varepsilon k(\gamma + 1) \int_0^\infty f^*(-\sigma, y) y^\gamma dy + \varepsilon \int_0^\infty g(-\sigma, y) y^\gamma dy.$$

Using again that f^* solves (39) and its expression in (19), we get

$$g(\sigma, 0) = \frac{\beta \sigma k(\gamma + 1)}{2\Lambda} + \int_0^\infty g(-\sigma, y) y^\gamma dy. \tag{43}$$

In order to gain indications on the stability properties of the stationary solution to (18) with $m = 0$, we study the discrete spectrum of $\mathcal{L}_\gamma^{\text{lin}}$ defined in (42), i.e., we search for the eigenfunctions g and the eigenvalues $\lambda \in \mathbb{C}$, satisfying the linearized integral conditions (40) and (43) found above, and such that

$$\mathcal{L}_\gamma^{\text{lin}} g(\sigma, y) = \lambda g(\sigma, y), \tag{44}$$

which is equivalent to

$$\frac{\partial}{\partial y} g(\sigma, y) + y^\gamma g(\sigma, y) - \frac{\beta \sigma k(\gamma + 1)}{2\Lambda} y^\gamma e^{-\frac{y^{\gamma+1}}{\gamma+1}} = -\lambda g(\sigma, y). \tag{45}$$

The eigen-system around $m = 0$ is thus given by (20), where, recall by (41), $k = 2 \int_0^\infty g(1, y) dy$, and $\Lambda = \int_0^\infty e^{-\frac{y^{\gamma+1}}{\gamma+1}} dy$.

Remark 3 The derivation of the linearized operator (44) was formal. One could think to define it more rigorously, by indicating an Hilbert space where $\mathcal{L}_\gamma^{\text{lin}}$ acts on. The natural choice appears to be (a subspace of) $\left(L_{\mu_\gamma}^2(\mathbb{R}^+)\right)^2$ satisfying conditions (40) and (43), where

the outer square comes from the explicitation of the spin variable $\sigma = \pm 1$, and the measure μ_γ is defined as

$$\mu_\gamma(dy) := f^*(\sigma, y)dy = \frac{1}{2\Lambda}e^{-\frac{y^{\gamma+1}}{\gamma+1}}dy. \tag{46}$$

As in the computations we do not use the particular choice of domain of the operator or its properties, we do not investigate further on this.

Proof of Proposition 2 Recall that in this case we have $\gamma = 1$. In order to solve the first equation in (21), we set $h(\sigma, y) := g(\sigma, y)e^{\frac{y^2}{2}}$. It holds

$$\frac{\partial}{\partial y}h(\sigma, y) = -\lambda h(\sigma, y) + \frac{y\beta\sigma k}{\sqrt{\frac{\pi}{2}}},$$

whose solution is

$$h(\sigma, y) = e^{-\lambda y}\left[h(\sigma, 0) + \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}}\int_0^y ue^{\lambda u}du\right].$$

Noting that $\int_0^y ue^{\lambda u}du = \frac{1}{\lambda^2} - \frac{e^{\lambda y}}{\lambda^2} + \frac{e^{\lambda y}}{\lambda}y$, we obtain

$$g(\sigma, y) = e^{-\frac{y^2}{2}}e^{-\lambda y}\left[g(\sigma, 0) + \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}}\left(\frac{1}{\lambda^2} - \frac{e^{\lambda y}}{\lambda^2} + \frac{e^{\lambda y}}{\lambda}y\right)\right]. \tag{47}$$

We now impose the integral conditions. First, we note that $\int_0^\infty[g(\sigma, y) + g(-\sigma, y)]dy = 0$ is equivalent to $g(\sigma, y) + g(-\sigma, y) = 0$ for every $y \in \mathbb{R}^+$ because of expression (47). For the computation of k, recalling notation (23), we find

$$k = 2\int_0^\infty g(1, y)dy = 2g(1,0)H_1(\lambda) + 2\frac{\beta k}{\sqrt{\frac{\pi}{2}}}\frac{1}{\lambda^2}H_1(\lambda) - 2\frac{\beta k}{\lambda^2}\frac{1}{\sqrt{\frac{\pi}{2}}}\sqrt{\frac{\pi}{2}} + 2\frac{\beta k}{\sqrt{\frac{\pi}{2}}}\frac{1}{\lambda},$$

so that

$$k = \frac{2g(1,0)H_1(\lambda)}{1 - 2\frac{\beta}{\lambda\sqrt{\frac{\pi}{2}}} - 2\frac{\beta H_1(\lambda)}{\lambda^2\sqrt{\frac{\pi}{2}}} + 2\frac{\beta}{\lambda^2}}. \tag{48}$$

The integral condition in the second line of (21) gives

$$g(\sigma, 0) = \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} + \int_0^\infty y\left[e^{-\frac{y^2}{2}}e^{-\lambda y}\left(g(-\sigma, 0) - \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}}\left(\frac{1}{\lambda^2} - \frac{e^{\lambda y}}{\lambda^2} + \frac{e^{\lambda y}}{\lambda}y\right)\right)\right]dy$$

$$= \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} - g(\sigma, 0)(1 - \lambda H_1(\lambda)) - \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}}\frac{(1 - \lambda H_1(\lambda))}{\lambda^2}$$

$$+ \frac{1}{\lambda^2}\frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} - \frac{1}{\lambda}\frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}}\int_0^\infty y^2 e^{-\frac{y^2}{2}}dy$$

$$= \frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} - g(\sigma, 0)(1 - \lambda H_1(\lambda)) - \frac{(1 - \lambda H_1(\lambda))}{\lambda^2}\frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} + \frac{1}{\lambda^2}\frac{\beta\sigma k}{\sqrt{\frac{\pi}{2}}} - \frac{1}{\lambda}\beta\sigma k.$$

In the second equality we have used that $\int_0^\infty y e^{-\frac{y^2}{2}} e^{-\lambda y} dy = 1 - \lambda H_1(\lambda)$ which can be obtained by an integration by parts using the property $g(\sigma, 0) = -g(-\sigma, 0)$. Solving for $g(1, 0)$ in the above

$$g(1,0)[2 - \lambda H_1(\lambda)] = \beta k \left[\frac{1}{\sqrt{\frac{\pi}{2}}} - \frac{(1 - \lambda H_1(\lambda))}{\lambda^2} \frac{1}{\sqrt{\frac{\pi}{2}}} + \frac{1}{\lambda^2} \frac{1}{\sqrt{\frac{\pi}{2}}} - \frac{1}{\lambda} \right].$$

Substituting the value of k we found in (48), we get

$$g(1,0)[2 - \lambda H_1(\lambda)] = \frac{2\beta g(1,0) H_1(\lambda)}{1 - 2\frac{\beta}{\lambda \sqrt{\frac{\pi}{2}}} - 2\frac{\beta H_1(\lambda)}{\lambda^2 \sqrt{\frac{\pi}{2}}} + 2\frac{\beta}{\lambda^2}} \left[\frac{1}{\sqrt{\frac{\pi}{2}}} - \frac{(1 - \lambda H_1(\lambda))}{\lambda^2} \frac{1}{\sqrt{\frac{\pi}{2}}} + \frac{1}{\lambda^2} \frac{1}{\sqrt{\frac{\pi}{2}}} - \frac{1}{\lambda} \right],$$

which is equivalent to

$$2 - \lambda H_1(\lambda) = \frac{2\beta H_1(\lambda) \left[\lambda^2 + \lambda H_1(\lambda) - \lambda \sqrt{\frac{\pi}{2}} \right]}{\lambda^2 \sqrt{\frac{\pi}{2}} - 2\beta\lambda - 2\beta H_1(\lambda) + 2\beta \sqrt{\frac{\pi}{2}}}. \tag{49}$$

As a polynomial in λ, (49) can be written as

$$-\lambda^3 H_1(\lambda) \sqrt{\frac{\pi}{2}} + \lambda^2 \sqrt{2\pi} - 4\beta\lambda - 4\beta H_1(\lambda) + 2\sqrt{2\pi}\beta = 0,$$

or, grouping for $H_1(\lambda)$,

$$H_1(\lambda) \left[-4\beta - \lambda^3 \sqrt{\frac{\pi}{2}} \right] + \sqrt{2\pi}\lambda^2 - 4\beta\lambda + 2\beta\sqrt{2\pi} = 0,$$

i.e. the zeros of $H_{\beta,1}(\lambda)$. In fact, as defined in (23), $H_1(\lambda)$ is a holomorphic function on $\mathbb{C}$, whose expression in series is

$$H_1(\lambda) = \int_0^\infty e^{-\frac{y^2}{2}} e^{-\lambda y} dy = \sum_{n=0}^\infty (-1)^n \frac{\lambda^n}{n!} \int_0^\infty y^n e^{-\frac{y^2}{2}} dy.$$

The latter integral is known

$$\int_0^\infty y^n e^{-\frac{y^2}{2}} dy = 2^{\frac{1}{2}(n-1)} \Gamma\left(\frac{n+1}{2}\right), \tag{50}$$

where $n \in \mathbb{N}$ and $\Gamma(\cdot)$ is the Gamma function. When $n = 2m + 1$ with $m \in \mathbb{N}$, for the properties of the Gamma function on $\mathbb{N}$, (50) reduces to

$$\int_0^\infty y^n e^{-\frac{y^2}{2}} dy = 2^{\frac{1}{2}(n-1)} \Gamma\left(\frac{n+1}{2}\right) = 2^m m!.$$

For $n = 2m$ with $m \in \mathbb{N}$ instead we have, by the property $\Gamma\left(l + \frac{1}{2}\right) = \frac{(2l-1)!!}{2^l}\sqrt{\pi}$ for any $l \in \mathbb{N}$,

$$\int_0^\infty y^n e^{-\frac{y^2}{2}} dy = 2^{\frac{1}{2}(n-1)} \Gamma\left(\frac{n+1}{2}\right) = \sqrt{\frac{\pi}{2}}(2m - 1)!!.$$

We use these equalities, and reorder the terms of the absolutely convergent series of $H_1(\lambda)$ to finally get

$$H_1(\lambda) = \sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{\lambda^{2m}}{(2m)!!} - \lambda \sum_{m=0}^{\infty} \frac{(2\lambda)^{2m} m!}{(2m+1)!} \frac{1}{2^m}.$$

$\square$

Proof of Proposition 3 Now we set $\gamma = 2$ and proceed as for $\gamma = 1$, by setting $h(\sigma, y) := g(\sigma, y) e^{\frac{y^3}{3}}$, so that

$$\frac{\partial}{\partial y} h(\sigma, y) = -\lambda h(\sigma, y) + \frac{3}{2\Lambda} \beta \sigma k y^2.$$

Thus,

$$h(\sigma, y) = e^{-\lambda y} \left[h(\sigma, 0) + \frac{3\beta\sigma k}{2\Lambda} \int_0^y u^2 e^{\lambda u} du \right].$$

Since $\int_0^y u^2 e^{\lambda u} du = \frac{1}{\lambda^3} [-2 + e^{\lambda y}(2 + \lambda y(-2 + \lambda y))]$, we can write

$$g(\sigma, y) = e^{-\frac{y^3}{3}} e^{-\lambda y} \left[g(\sigma, 0) + \frac{3\beta\sigma k}{2\Lambda} \frac{1}{\lambda^3} \left(-2 + 2e^{\lambda y} - 2\lambda y e^{\lambda y} + \lambda^2 y^2 e^{\lambda y} \right) \right]. \quad (51)$$

Recalling notation (28), we compute

$$k = 2 \int_0^{\infty} g(1, y) dy = 2H_2(\lambda) g(1, 0) - 2H_2(\lambda) \frac{3\beta k}{\Lambda \lambda^3} + 2 \frac{3\beta k}{\Lambda \lambda^3} \int_0^{\infty} e^{-\frac{y^3}{3}} dy$$

$$- 2 \frac{3\beta k}{\Lambda \lambda^2} \int_0^{\infty} y e^{-\frac{y^3}{3}} dy + \frac{3\beta k}{\Lambda \lambda} \int_0^{\infty} y^2 e^{-\frac{y^3}{3}} dy$$

$$= 2H_2(\lambda) g(1, 0) - 2H_2(\lambda) \frac{3\beta k}{\Lambda \lambda^3} + 2 \frac{3\beta k}{\lambda^3} - 2 \frac{3\beta k}{\Lambda \lambda^2} \frac{\Gamma(2/3)}{3^{1/3}} + \frac{3\beta k}{\Lambda \lambda},$$

which gives

$$k = \frac{2g(1, 0) H_2(\lambda)}{1 + 2H_2(\lambda) \frac{3\beta}{\Lambda \lambda^3} - 2 \frac{3\beta}{\lambda^3} + 2 \frac{3\beta}{\Lambda \lambda^2} \frac{\Gamma(2/3)}{3^{1/3}} - \frac{3\beta}{\Lambda \lambda}}. \quad (52)$$

As before, the condition $\int_0^{\infty} [g(\sigma, y) + g(-\sigma, y)] dy = 0$ in (26) is equivalent to $g(\sigma, y) + g(-\sigma, y) = 0$ for every $y \in \mathbb{R}^+$ because of (51). Using this observation for $y = 0$ in the other integral condition - see second line of system (26) - we compute

$$g(\sigma, 0) = \frac{3\beta\sigma k}{2\Lambda} + \int_0^{\infty} y^2 \left[e^{-\frac{y^3}{3}} e^{-\lambda y} \left(-g(\sigma, 0) \right. \right.$$

$$\left. \left. - \frac{3\beta\sigma k}{2\Lambda} \frac{1}{\lambda^3} (-2 + 2e^{\lambda y} - 2\lambda y e^{\lambda y} + \lambda^2 y^2 e^{\lambda y}) \right) \right] dy.$$

Observing that, by integration by parts, $\int_0^{\infty} y^2 e^{-\frac{y^3}{3}} e^{-\lambda y} dy = 1 - \lambda H_2(\lambda)$, we find

$$g(\sigma, 0) = \frac{3\beta\sigma k}{2\Lambda} - (1 - \lambda H_2(\lambda)) g(\sigma, 0) + \frac{3\beta\sigma k}{\Lambda \lambda^3} (1 - \lambda H_2(\lambda)) - \frac{3\beta\sigma k}{\Lambda \lambda^3}$$

$$+ \frac{3\beta\sigma k}{\Lambda \lambda^2} 3^{1/3} \Gamma(4/3) - \frac{3\beta\sigma k}{2\Lambda \lambda} 3^{2/3} \Gamma(5/3).$$

Computing in $\sigma = 1$ and grouping for $g(1,0)$,

$$g(1,0)[2 - \lambda H_2(\lambda)] = k\left[\frac{3\beta}{2\Lambda} + \frac{3\beta}{\Lambda\lambda^3}(1 - \lambda H_2(\lambda)) - \frac{3\beta}{\Lambda\lambda^3} + \frac{3\beta}{\Lambda\lambda^2}3^{1/3}\Gamma(4/3) - \frac{3\beta}{2\Lambda\lambda}3^{2/3}\Gamma(5/3)\right]$$

$$= k\left[\frac{3\beta}{2\Lambda} - \frac{3\beta}{\Lambda\lambda^2}H_2(\lambda) + \frac{3\beta}{\Lambda\lambda^2}3^{1/3}\Gamma(4/3) - \frac{3\beta}{2\Lambda\lambda}3^{2/3}\Gamma(5/3)\right].$$

Plugging expression (52) for k,

$$2 - \lambda H_2(\lambda) = \frac{2H_2(\lambda)}{1 + 2H_2(\lambda)\frac{3\beta}{\Lambda\lambda^3} - 2\frac{3\beta}{\lambda^3} + 2\frac{3\beta}{\Lambda\lambda^2}\frac{\Gamma(2/3)}{3^{1/3}} - \frac{3\beta}{\Lambda\lambda}}\left[\frac{3}{2\Lambda}\beta - \frac{3\beta}{\Lambda\lambda^2}H_2(\lambda)\right.$$

$$\left. + \frac{3\beta}{\Lambda\lambda^2}3^{1/3}\Gamma(4/3) - \frac{3\beta}{2\Lambda\lambda}3^{2/3}\Gamma(5/3)\right].$$

This gives

$$2 - \lambda H_2(\lambda) = \frac{2\lambda H_2(\lambda)\left[\frac{3}{2}\beta\lambda^2 - 3\beta H_2(\lambda) + 3\beta 3^{1/3}\Gamma(4/3) - \frac{3}{2}\beta\lambda 3^{2/3}\Gamma(5/3)\right]}{\Lambda\lambda^3 + 6\beta H_2(\lambda) - 6\beta\Lambda + 6\beta\lambda\frac{\Gamma(2/3)}{3^{1/3}} - 3\beta\lambda^2},$$

which is equivalent to

$$2\Lambda\lambda^3 + 12\beta H_2(\lambda) - 12\beta\Lambda + 12\beta\lambda\frac{\Gamma(2/3)}{3^{1/3}}$$

$$- 6\beta\lambda^2 - \lambda^4\Lambda H_2(\lambda) + 6\beta\lambda H_2(\lambda)\Lambda - 6\beta\lambda^2\frac{\Gamma(2/3)}{3^{1/3}}H_2(\lambda)$$

$$= 6\beta\lambda H_2(\lambda)3^{1/3}\Gamma(4/3) - 3\beta\lambda^2 H_2(\lambda)3^{2/3}\Gamma(5/3).$$

As a polynomial in λ, this is

$$- \Lambda H_2(\lambda)\lambda^4 + 2\Lambda\lambda^3$$

$$+ \lambda^2\left[-6\beta - 6\beta\frac{\Gamma(2/3)}{3^{1/3}}H_2(\lambda) + 3\beta H_2(\lambda)3^{2/3}\Gamma(5/3)\right]$$

$$+ \lambda\left[6\beta\Lambda H_2(\lambda) - 6\beta H_2(\lambda)3^{1/3}\Gamma(4/3) + 12\beta\frac{\Gamma(2/3)}{3^{1/3}}\right] + 12\beta H_2(\lambda) - 12\beta\Lambda = 0.$$

Equivalently, in terms of $H_2(\lambda)$ we have

$$H_2(\lambda)[12\beta - \lambda^4\Lambda + 6\beta\lambda\Lambda - 6\beta\lambda 3^{1/3}\Gamma(4/3)$$

$$+ 3\beta\lambda^2 3^{2/3}\Gamma(5/3) - 6\beta\lambda^2\frac{\Gamma(2/3)}{3^{1/3}}]$$

$$+ \left[2\Lambda\lambda^3 - 12\beta\Lambda + 12\beta\frac{\Gamma(2/3)}{3^{1/3}}\lambda - 6\beta\lambda^2\right] = 0,$$

i.e. the zeros of $H_{\beta,2}(\lambda)$ in (27). As defined in (28), $H_2(\lambda)$ is a holomorphic function on $\mathbb{C}$, which can be expressed in series as

$$H_2(\lambda) = \int_0^\infty e^{-\lambda y}e^{-\frac{y^3}{3}}dy = \sum_{n=0}^\infty (-1)^n\frac{\lambda^n}{n!}\int_0^\infty y^n e^{-\frac{y^3}{3}}dy$$

$$= \sum_{n=0}^\infty (-1)^n\frac{\lambda^n}{n!}3^{\frac{1}{3}(n-2)}\Gamma\left(\frac{n+1}{3}\right),$$

which is expression (29), where we have used the formula for $\int_0^\infty y^n e^{-\frac{y^3}{3}} dy =$
$3^{\frac{1}{3}(n-2)} \Gamma \left(\frac{n+1}{3} \right)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

References

1. Andreis, L., Tovazzi, D.: Coexistence of stable limit cycles in a generalized Curie-Weiss model with dissipation. J. Stat. Phys. **173**(1), 163–181 (2018)
2. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. Elsevier, Amsterdam (2016)
3. Bertini, L., Giacomin, G., Pakdaman, K.: Dynamical aspects of mean field plane rotators and the Kuramoto model. J. Stat. Phys. **138**(1–3), 270–290 (2010)
4. Chen, W.-C.: Nonlinear dynamics and chaos in a fractional-order financial system. Chaos Solitons Fractals **36**(5), 1305–1314 (2008)
5. Collet, F., Dai Pra, P.: The role of disorder in the dynamics of critical fluctuations of mean field models. Electron. J. Probab. **17**(26), 1–40 (2012)
6. Collet, F., Dai Pra, P., Formentin, M.: Collective periodicity in mean-field models of cooperative behavior. Nonlinear Differ. Equ. Appl. NoDEA **22**(5), 1461–1482 (2015)
7. Collet, F., Formentin, M.: Effects of local fields in a dissipative Curie–Weiss model: Bautin Bifurcation and large self-sustained Oscillations. J. Stat. Phys. **176**, 478 (2019)
8. Collet, F., Formentin, M., Tovazzi, D.: Rhythmic behavior in a two-population mean-field Ising model. Phys. Rev. E **94**(4), 042139 (2016)
9. Dai Pra, P., Fischer, M., Regoli, D.: A Curie–Weiss model with dissipation. J. Stat. Phys. **152**(1), 37–53 (2013)
10. Dai Pra, P., Giacomin, G., Regoli, D.: Noise-induced periodicity: some stochastic models for complex biological systems. In: Mathematical Models and Methods for Planet Earth, pp 25-35. Springer (2014)
11. Ditlevsen, S., Löcherbach, E.: Multi-class oscillating systems of interacting neurons. Stoch Process. Appl. **127**, 1840–1869 (2017)
12. Ellis, R.S., Newman, C.M.: Limit theorems for sums of dependent random variables occurring in statistical mechanics. Z. Wahrscheinlichkeitstheorie verw Gebiete **44**, 117–139 (1978)
13. Ermentrout, G.B., Terman, D.H.: Mathematical Foundations of Neuroscience. Springer Science & Business Media, New York (2010)
14. Ethier, S.N., Kurtz, T.G.: Markov Processes. Characterization and Convergence. Wiley, Hoboken (1986)
15. Giacomin, G., Poquet, C.: Noise, interaction, nonlinear dynamics and the origin of rhythmic behaviors. Braz. J. Probab. Stat. **29**(2), 460–493 (2015)
16. Graham, C.: McKean–Vlasov Itô–Skorokhod equations, and nonlinear diffusions with discrete jump sets. Stoch. Process. Appl. **40**(1), 69–82 (1992)
17. Kolokoltsov, V.N.: Nonlinear Markov Processes and Kinetic Equations, vol. 182. Cambridge University Press, Cambridge (2010)
18. Lévy, P.: Processus semi-markoviens. In: Proceedings of the International Congress of Mathematicians, Amsterdam pp. 416–426 (1954)
19. Scheutzow, M.: Noise can create periodic behavior and stabilize nonlinear diffusions. Stoch. Process. Appl. **20**(2), 323–331 (1985)
20. Touboul, J.D.: The hipster effect: when anti-conformists all look the same. Discret. Contin. Dyn. Syst. B **24**(8), 4379–4415 (2019)
21. Touboul, J.D., Hermann, G., Faugeras, O.: Noise-induced behaviors in neural mean field dynamics. SIAM J. Appl. Dyn. Syst. **11**(1), 49–81 (2012)
22. Turchin, P., Taylor, A.D.: Complex dynamics in ecological time series. Ecology **73**(1), 289–305 (1992)

23. Weidlich, W., Haag, G.: Concepts and Models of a Quantitative Sociology: The Dynamics of Interacting Populations, vol. 14. Springer Science & Business Media, New York (2012)

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