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# Existence and uniqueness of the global solution to the Navier–Stokes equations

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A proof is given of the global existence and uniqueness of a weak solution to Navier–Stokes equations in unbounded exterior domains.


## 1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected $C^2$-smooth boundary $S$, and $D' := \mathbb{R}^3 \setminus D$ be the unbounded exterior domain.

Consider the Navier–Stokes equations:

$$u_t + (u, \nabla)u = -\nabla p + \nu \Delta u + f, \quad x \in D', \ t \geq 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u|_S = 0, \qquad u|_{t=0} = u_0(x). \tag{3}$$

Here $f$ is a given vector-function, $p$ is the pressure, $u = u(x, t)$ is the velocity vector-function, $\nu = const > 0$ is the viscosity coefficient, $u_0$ is the given initial velocity, $u_t := \partial_t u$, $(u, \nabla)u := u_a \partial_a u$, $\partial_a u := \frac{\partial u}{\partial x_a} := u_{;a}$, and $\nabla \cdot u_0 := u_{a;a} = 0$. Over the repeated indices $a$ and $b$ summation is understood, $1 \leq a, b \leq 3$. All functions are assumed real-valued.

We assume that $u \in W$,

$$W := \{u | L^2(0, T; H_0^1(D')) \cap L^\infty(0, T; L^2(D')) \cap u_t \in L^2(D' \times [0, T]); \nabla \cdot u = 0\},$$

where $T > 0$ is arbitrary.

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Let $(u,v) := \int_{D'} u_a v_a dx$ denote the inner product in $L^2(D')$, $\|u\| := (u,u)^{1/2}$. By $u_{ja}$ the $a$-th component of the vector-function $u_j$ is denoted, and $u_{ja;b}$ is the derivative $\frac{\partial u_{ja}}{\partial x_b}$. Eq. (2) can be written as $u_{a;a} = 0$ in these notations. We denote $\frac{\partial u^2}{\partial x_a} := (u^2)_{;a}$, $u^2 := u_b u_b$. By $c > 0$ various estimation constants are denoted.

*Let us define a weak solution to problem (1)–(3) as an element of $W$ which satisfies the identity:*

$$(u_t, v) + (u_a u_{b;a}, v_b) + \nu(\nabla u, \nabla v) = (f, v), \quad \forall v \in W. \tag{4}$$

Here we took into account that $-(\Delta u, v) = (\nabla u, \nabla v)$ and $(\nabla p, v) = -(p, v_{a;a}) = 0$ if $v \in H_0^1(D')$ and $\nabla \cdot v = 0$. Eq. (4) is equivalent to the integrated equation:

$$\int_0^t [(u_s, v) + (u_a u_{b;a}, v_b) + \nu(\nabla u, \nabla v)]ds = \int_0^t (f, v)ds, \quad \forall v \in W. \tag{$*$}$$

Eq. (4) implies Eq. $(*)$, and differentiating Eq. $(*)$ with respect to $t$ one gets Eq. (4) for almost all $t \geq 0$.

The aim of this paper is to prove the global existence and uniqueness of the weak solution to the Navier–Stokes boundary problem, that is, solution in $W$ existing for all $t \geq 0$. Let us assume that

$$\sup_{t\geq 0} \int_0^t \|f\| ds \leq c, \quad (u_0, u_0) \leq c. \tag{$A$}$$

**Theorem 1.** *If assumptions (A) hold and $u_0 \in H_0^1(D)$ satisfies Eq. (2), then there exists for all $t > 0$ a solution $u \in W$ to (4) and this solution is unique in $W$ provided that $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$.*

In Section 2 we prove Theorem 1. There is a large literature on Navier–Stokes equations, of which we mention only [1,2]. The global existence and uniqueness of the solution to Navier–Stokes boundary problems has not yet been proved without additional assumptions. Our additional assumption is $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$. The history of this problem see, for example, in [1]. In [2] the uniqueness of the global solution to Navier–Stokes equations is established under the assumption $\|u\|^8_{L^4(D')} \in L^1_{loc}(0, \infty)$.

## 2. Proof of Theorem 1

**Proof of Theorem 1.** The steps of the proof are: (a) derivation of a priori estimates; (b) proof of the existence of the solution in $W$; (c) proof of the uniqueness of the solution in $W$.

(a) *Derivation of a priori estimates*

Take $v = u$ in (4). Then

$$(u_a u_{b;a}, u_b) = -(u_a u_b, u_{b;a}) = -\frac{1}{2}(u_a, (u^2)_{;a}) = \frac{1}{2}(u_{a;a}, u^2) = 0,$$

where the equation $u_{a;a} = 0$ was used. Thus, Eq. (4) with $v = u$ implies

$$\frac{1}{2}\partial_t(u, u) + \nu(\nabla u, \nabla u) = (f, u) \leq \|f\|\|u\|. \tag{5}$$

We will use the known inequality $\|u\|\|f\| \leq \epsilon\|u\|^2 + \frac{1}{4\epsilon}\|f\|^2$ with a small $\epsilon > 0$, and denote by $c > 0$ *various* estimation constants.

One gets from (5) the following estimate:

$$(u(t), u(t)) + 2\nu \int_0^t (\nabla u, \nabla u)ds \leq (u_0, u_0) + 2\int_0^t \|f\| ds \sup_{s\in[0,t]} \|u(s)\| \leq c + c \sup_{s\in[0,t]} \|u(s)\|. \tag{6}$$

Recall that assumptions (A) hold. Denote $\sup_{s\in[0,t]} \|u(s)\| := b(t)$. Then inequality (6) implies

$$b^2(t) \leq c + cb(t), \quad c = const > 0. \tag{7}$$

Since $b(t) \geq 0$, inequality (7) implies

$$\sup_{t\geq 0} b(t) \leq c. \tag{8}$$

Remember that $c > 0$ denotes various constants, and the constant in Eq. (8) differs from the constant in Eq. (7). From (6) and (8) one obtains

$$\sup_{t\geq 0}[(u(t), u(t)) + \nu \int_0^t (\nabla u, \nabla u) ds] \leq c. \tag{9}$$

A priori estimate (9) implies for every $T \in [0, \infty)$ the inclusions

$$u \in L^\infty(0, T; L^2(D')), \qquad u \in L^2(0, T; H_0^1(D')).$$

This and Eq. (4) imply that $u_t \in L^2(D' \times [0, T])$ because Eq. (4) shows that $(u_t, v)$ is bounded for every $v \in W$. Note that $L^\infty(0, T; L^2(D')) \subset L^2(0, T; L^2(D'))$, and that bounded sets in a Hilbert space are weakly compact. Weak convergence is denoted by the sign $\rightharpoonup$.

(b) *Proof of the existence of the solution $u \in W$ to (4) and (∗)*

The idea of the proof is to reduce the problem to the existence of the solution to a Cauchy problem for ordinary differential equations (ODE) of finite order, and then to use a priori estimates to establish convergence of these solutions of ODE to a solution of Eqs. (4) and (∗). This idea is used, for example, in [1]. Our argument differs from the arguments in the literature in treating the limit of the term $\int_0^t (u_s^n, v) ds$.

Let us look for a solution to Eq. (4) of the form $u^n := \sum_{j=1}^n c_j^n(t)\phi_j(x)$, where $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of the space $L^2(D')$ of divergence-free vector functions belonging to $H_0^1(D')$ and in the expression $u^n$ the upper index $n$ is not a power. If one substitutes $u^n$ into Eq. (4), takes $v = \phi_m$, and uses the orthonormality of the system $\{\phi_j\}_{j=1}^\infty$ and the relation $(\nabla\phi_j, \nabla\phi_m) = \lambda_m \delta_{jm}$, where $\lambda_m$ are the eigenvalues of the vector Dirichlet Laplacian in $D$ on the divergence-free vector fields, then one gets a system of ODE for the unknown coefficients $c_m^n$:

$$\partial_t c_m^n + \nu\lambda_m c_m^n + \sum_{i,j=1}^n (\phi_{ia}\phi_{jb;a}, \phi_{mb}) c_i^n c_j^n = f_m, \quad c_m^n(0) = (u_0, \phi_m). \tag{10}$$

Problem (10) has a unique global solution because of the a priori estimate that follows from (9) and from Parseval's relations:

$$\sup_{t\geq 0}(u^n(t), u^n(t)) = \sup_{t\geq 0} \sum_{j=1}^n [c_j^n(t)]^2 \leq c. \tag{11}$$

Consider the set $\{u^n = u^n(t)\}_{n=1}^\infty$. Inequalities (9) and (11) for $u = u^n$ imply the existence of the weak limits $u^n \rightharpoonup u$ in $L^2(0, T; H_0^1(D'))$ and in $L^\infty(0, T; L^2(D'))$. This allows one to pass to the limit in Eq. (∗) in all the terms except the first, namely, in the term $\int_0^t (u_s^n, v(s)) ds$. The weak limit of the term $(u_a^n u_{b;a}^n, v_b)$ exists and is equal to $(u_a u_{b;a}, v_b)$ because

$$(u_a^n u_{b;a}^n, v_b) = -(u_a^n u_b^n, v_{b;a}) \to -(u_a u_b, v_{b;a}) = (u_a u_{b;a}, v_b).$$

Note that $v_{b;a} \in L^2(D')$ and $u_a^n u_b^n \in L^4(D')$. The relation $(u_a^n u_{b;a}^n, v_b) = -(u_a^n u_b^n, v_{b;a})$ follows from an integration by parts and from the equation $u_{a;a}^n = 0$.

The following inequality is essentially known:

$$\|u\|_{L^4(D')} \leq 2^{1/2}\|u\|^{1/4}\|\nabla u\|^{3/4}, \qquad \|u\| := \|u\|_{L^2(D')}, \quad u \in H_0^1(D'). \tag{12}$$

In [1] this inequality is proved for $D' = \mathbb{R}^3$, but a function $u \in H_0^1(D')$ can be extended by zero to $D = \mathbb{R}^3 \setminus D'$ and becomes an element of $H^1(\mathbb{R}^3)$ to which inequality (12) is applicable.

It follows from (12) and Young's inequality ($ab \le \frac{a^p}{p} + \frac{b^q}{q}$, $p^{-1} + q^{-1} = 1$) that

$$\|u\|^2_{L^4(D')} \le \epsilon \|\nabla u\|^2 + \frac{27}{16\epsilon^3}\|u\|^2, \quad u \in H_0^1(D'), \tag{13}$$

where $\epsilon > 0$ is an arbitrary small number, $p = \frac{4}{3}$ and $q = 4$. One has $u_a^n u_b^n \rightharpoonup u_a u_b$ in $L^2(D')$ as $n \to \infty$, because bounded sets in a reflexive Banach space $L^4(D')$ are weakly compact. Consequently, $(u_a^n u_{b;a}^n, v_b) \to (u_a u_{b;a}, v_b)$ when $n \to \infty$, as claimed. Therefore, $\int_0^t (u_a^n u_{b;a}^n, v_b) ds \to \int_0^t (u_a u_{b;a}, v_b) ds$. The weak limit of the term $\nu \int_0^t (\nabla u^n, \nabla v) ds$ exists because of the a priori estimate (9) and the weak compactness of the bounded sets in a Hilbert space. Since Eq. (∗) holds, and the limits of all its terms, except $\int_0^t (u_s^n, v) ds$, do exist, then there exists the limit $\int_0^t (u_s^n, v(s)) ds \to \int_0^t (u_s, v(s)) ds$ for all $v \in W$. By passing to the limit $n \to \infty$ one proves that the limit $u$ satisfies Eq. (∗). Differentiating Eq. (∗) with respect to $t$ yields Eq. (4) almost everywhere.

(c) *Proof of the uniqueness of the solution $u \in W$*

Suppose there are two solutions to Eq. (4), $u$ and $w$, $u, w \in W$, and let $z := u - w$. Then

$$(z_t, v) + \nu(\nabla z, \nabla v) + (u_a u_{b;a} - w_a w_{b;a}, v_b) = 0. \tag{14}$$

Since $z \in W$, one may set $v = z$ in (14) and get

$$(z_t, z) + \nu(\nabla z, \nabla z) + (u_a u_{b;a} - w_a w_{b;a}, z_b) = 0, \qquad z = u - w. \tag{15}$$

Note that $(u_a u_{b;a} - w_a w_{b;a}, z_b) = (z_a u_{b;a}, z_b) + (w_a z_{b;a}, z_b)$, and $(w_a z_{b;a}, z_b) = 0$ due to the equation $w_{a;a} = 0$. Thus, Eq. (15) implies

$$\partial_t (z, z) + 2\nu(\nabla z, \nabla z) \le 2|(z_a u_{b;a}, z_b)|. \tag{16}$$

Since $|z_a u_{b;a} z_b| \le |z|^2 |\nabla u|$, one has the following estimate:

$$|(z_a u_{b;a}, z_b)| \le \int_{D'} |z|^2 |\nabla u| dx \le \|z\|^2_{L^4(D')} \|\nabla u\| \le \|\nabla u\| \Big(\epsilon \|\nabla z\|^2 + \frac{27}{16\epsilon^3}\|z\|^2\Big). \tag{17}$$

Denote $\phi := (z, z)$, take into account that $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$, choose $\epsilon = \frac{\nu}{\|\nabla u\|}$ in the inequality (13), in which $u$ is replaced by $z$, use inequality (17) and get

$$\partial_t \phi + \nu(\nabla z, \nabla z) \le \frac{27}{16\nu^3}\|\nabla u\|^4 \phi, \quad \phi|_{t=0} = 0. \tag{18}$$

In the derivation of inequality (18) the idea is to compensate the term $\nu\|\nabla z\|^2$ on the left side of inequality (16) by the term $\epsilon\|\nabla u\|\|\nabla z\|^2$ on the right side of inequality (17). To do this, choose $\|\nabla u\|\epsilon = \nu$ and obtain inequality (18). It follows from inequality (18) that

$$\partial_t \phi \le \frac{27\|\nabla u\|^4}{16\nu^3}\phi, \quad \phi|_{t=0} = 0.$$

Since we have assumed that $\|\nabla u\|^4 \in L^1_{loc}(0, \infty)$ this implies that $\phi = 0$ for all $t \ge 0$.

Theorem 1 is proved. □

**Remark 1.** One has (summation is understood over the repeated indices):

$$2|(z_a u_{b;a}, z_b)| = 2|(z_a u_b, z_{b;a})| \le 18\|\nabla z\|\||z||u|\| \le \nu\|\nabla z\|^2 + \frac{81}{\nu}\||z||u|\|^2.$$

Thus,

$$\partial_t \phi + \nu(\nabla z, \nabla z) \le \frac{81}{\nu}\| \mid z \mid\mid u \mid \|^2.$$

If one assumes that $|u(\cdot,t)| \leq c(T)$ for every $t \in [0,T]$, then $\partial_t \phi \leq c\phi, \phi(0) = 0$, on any interval $[0,T]$, $c = c(T,\nu) > 0$ is a constant. This implies $\phi = 0$ for all $t \geq 0$. The same conclusion holds under a weaker assumption $\|u(\cdot,t)\|_{L^4(D')} \leq c(T)$ for every $t \in [0,T]$, or under even weaker assumption $\|u(\cdot,t)\|^8_{L^4(D')} \in L^1_{loc}(0,\infty)$.

In [1] it is shown that the smoothness properties of the solution $u$ are improved when the smoothness properties of $f$, $u_0$ and $S$ are improved.

## References

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[2] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, North Holland, Amsterdam, 1984.