INVESTMENT RISK APPRAISAL

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Abstract: Standard financial techniques neglect extreme situations and regards large market shifts as too unlikely to matter. This approach may account for what occurs most of the time in the market, but the picture it presents does not reflect the reality, as the major events happen in the rest of the time and investors are 'surprised' by 'unexpected' market movements. An alternative fuzzy approach permits fluctuations well beyond the probability type of uncertainty and allows one to make fewer assumptions about the data distribution and market behaviour. Fuzzifying the present value criteria, we suggest a measure of the risk associated with each investment opportunity and estimate the project's robustness towards market uncertainty. The procedure is applied to thirty-five UK companies and a neural network solution to the fuzzy criterion is provided to facilitate the decision-making process. Finally, we discuss the grounds for classical asset pricing model revision and argue that the demand for relaxed assumptions appeals for another approach to modelling the market environment.

Key words: fuzzy set theory, possibility theory, neural networks, investment project evaluation, capital asset pricing model

JEL Classification: C45 D81

1. Introduction

Investment projects are typically chosen on the basis of a restricted information set. Furthermore, the volatility literature claims that stock prices are too volatile to accord with simple present value models [37]. The technique applied here models the restricted information and incorporates the price uncertainty into the present value calculations. It is suggested that uncertain share prices and dividend yields, associated with a family of possible streams of future cash flows, as well as uncertain discount rates can be handled by the introduction of fuzzy variables. The value of a fuzzy variable is restricted by a possibility distribution [40,17]. Alternatively, one may use fuzzy numbers with corresponding membership functions [39,41,18].

As compared with the standard approach, the fuzzy present value is an effective method of project evaluation and investment risk appraisal under restricted market information. It allows one to make fewer assumptions about the underlying distribution and market behaviour. The solution procedure involves a multiple interval analysis, which enables investors to consider a project and take decisions based on varying levels of uncertainty. Further, increasing the range of uncertainty modelled in the fuzzy data, one can determine the robustness of the investment risk associated with each project. Finally, neural networks yield a mechanism to facilitate the solution of the fuzzy criterion. Once trained to evaluate a project, a neural net provides investors with a simple re-evaluation tool when the information available is subject to change.

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2. Net present value models

The present value criterion is familiar, widely adopted and simple to apply. It follows from the literature that NPV evaluations are increasingly being used in the UK, along with the readiness to implement more sophisticated techniques.

> 'When, ... economic uncertainty was most acutely experienced we saw a significant change in the use of certain investment practices in an attempt to handle the adverse effects of such economic factors.' [30]

Companies rely not only on one but on several methods and the greatest growth amongst all the evaluation practices is found in the adoption of the present value technique, it is concluded in Pike [30] and Sangster [34]. The classic form of the criterion may be represented as $NPV=\sum_{t=1}^{N}\frac{NCF_t}{(1-R)^t}-I_0$, where NCF_t stands for the net cash flow in period t, I_0 represents the initial cash outlay, R denotes the cost of capital, and N is the number of periods in the project.

Crisp NPV modifications: The standard formula has been continuously revised. Nickell [27] discusses the impact of costs of adjustment, while Precious [31] handles the effect of capital and labour market rationing in the context of rational expectation models. More recently, Dixit, Pindyck and Sodal [16] deal with investment irreversibility and the timing of the initial outlay. They show that the firm objective to maximise the expected present value of stochastically fluctuating net benefits has to be calculated at the time the project is evaluated. As a result, $NPV=D(V_0,V)(V-C)=E[e^{-RT}](V-C)$ depends on a random start time T. Here, the value function $\{V_t\}$ is a stationary Markov process representing the project benefit measured as the present value at the time the cost is incurred, V_0 is the project benefit at t=0, V stands for an arbitrary threshold, C indicates a constant sunk cost, R denotes a constant discount rate, and D is the expectation of the discount factor depending on V_0 and V. The start time T equals the time at which the benefit V_T reaches the threshold V. T is a random variable and its distribution can be determined from the known probability law of the evolution of V_t . The first order condition for the threshold V* maximising NPV is $\frac{V^*-C}{V^*}=\left[-\frac{V^*D_V(V_0,V^*)}{D(V_0,V^*)}\right]^{-1}=\frac{1}{\xi_D}$, where ξ_D denotes the elasticity of the discount factor D with respect to V*. The authors conclude that if the firm used a simple net-present-value rule, it would invest sooner and the discount factor would be larger. It is shown that the optimal rule implies a trade-off between a larger versus a later net benefit and can be interpreted as a markup formula based on the elasticity of the discount factor. Clearly, a method that permits more elasticity in the discount factor, will take some account of irreversibility.

Further, Stulz [38] argues that the mainstream approach to capital budgeting focuses on the special case where the volatility of the project's cash flows does not affect the value of the firm. The author proves that such risk matters, as each project has a cost equal to the impact of the project on the firm's total risk and consequently the net present value must be decreased by this cost. Total risk affects all the firm's decisions, including the choice of investment projects and the choice of capital structure. Consequently, capital budgeting and capital structure decisions must be consistent.

In conclusion, various modifications eliminate some of the problems associated with the standard criterion. It has been realised that the removal of any of the perfect market assumptions reduces the effectiveness of the method and typically destroys the foundations of generally accepted investment-selection techniques.

> ‘Having capital budgeting rules that lead to correct decisions in the presence of capital market imperfections would lead to a situation where modern finance theory can be applied consistently in firms and in a way that increases their value.’ [38]

The effort in this article is not to cope with a specific drawback of the present value technique but to simply permit in the model structure as much uncertainty as the market could possibly suffer. The outcome is an effective method under restricted information, uncertain data and market imperfections. Whatever reason one has for modifying the classic result, the allowances provided by the fuzzy criterion will cover these specific circumstances and will include the modified value, as well as a number of other possible values. The calculation based on the standard criterion may be adjusted, as in [38], because of the impact of a project on the investor’s total risk; or re-optimised due to irreversibility, as in [16]. The fuzzy present value method takes account of these and also lots of other possibilities, employing fuzzy cash flows and fuzzy discount rates and increasing the flexibility of the involved calculations. Our effort lies on the bridge towards a new paradigm of investment selection. Aluja [1] reasons that the perception of concepts inherent or surrounding the investment process, whose character is not principally measurable, is best handled by nonnumeric mathematics emerging from the theory of fuzzy sets. In response, we further develop here the technique presented in Hunter and Serguieva [20], introduce fuzzy dividend yields and a second type of calibration, suggest measures of the investment risk and its robustness, work out the neural network solution, and extend the number of companies from five to thirty-five. The classical method is broadly adopted in practice and managers are comfortable with it, which suggests why it might be sensible to accept the fuzzy version to take account of more general forms of uncertainty.

Fuzzy approaches to investment selection: Fuzzy logic enables research in areas involving a high level and diverse forms of uncertainty. Such problems are the norm in finance.

> ‘The initial development of the theory of fuzzy sets was motivated by the perception that traditional techniques of systems analysis are not effective in ... modelling the pervasive imprecision and uncertainty of the real world. ... More recently, the arrival of the information revolution has made the world of business, finance and management a magnet for methodologies ... such as neurocomputing, genetic computing and fuzzy logic.’ (L. Zadeh, Foreword to [5])

Thus, financial analysts and decision-makers are provided with effective modelling tools in the absence of complete or precise information and the presence of human involvement. Bojadziev and Bojadziev [5] demonstrate a number of potential applications of fuzzy techniques in finance. In particular, the authors consider, under close and under conflicting expert opinions, an investment-planning model that produces aggressive or conservative policies. Fuzzy zero-based budgeting is implemented to construct a conservative or an optimistic budget. Östermark [28] uses a fuzzy control model to manage a portfolio in a fuzzy dynamic environment in the presence of financial constraints. The author employs one riskless and n risky assets, models the factor imprecision using fuzzy sets and solves a fuzzy linear tracking problem using the Kalman filter. Next, Ng and Gan [26] construct a fuzzy dynamic hedging strategy under transaction costs. The goal is to insure investments in the Nikkei 225 Stock Index against market decline with Nikkei Sock Index Futures traded on SIMEX, while allowing full participation on the upside and maximising expected returns. The authors design a fuzzy logic trading system in order to resolve the trade-off between frequent rebalancing and increased transaction costs.

Proceeding closer towards the subject of this article, in one of the first publications on fuzzy set application to finance, Buckley [6] introduces the fuzzy analogous of the following financial calculations: future value, continuous interest, effective rate and regular annuities. He also provides two formulas for the fuzzy present value computation, depending on whether the cash amount is positive or negative. The author uses a fuzzy interest rate and fuzzy cash amounts and applies the standard arithmetic of fuzzy numbers: all the results are fuzzy numbers. Further, he derives fuzzy future and present values and fuzzy annuities under a fuzzy number of interest periods: the results need not be fuzzy numbers. Finally, two criteria for investment project evaluation are considered, net present value and internal rate of return. It is proved that the standard fuzzy arithmetic provides a complicated result for the former and no fuzzy extension for the latter method. In a subsequent article, LiCalzi [13] works in a broader framework and considers general accumulation and discount models characterised by their properties rather than specific functions. He derives the necessary and sufficient conditions for the future and present values to be compact fuzzy intervals (invertible fuzzy intervals) when the cash amount, the time and the interest/discount rate are compact fuzzy intervals (invertible fuzzy intervals). The author reasons that there should be only one present value definition and that the doubling in Bucley [6] is rather in the computational strategies required by the invertible fuzzy intervals employed. Further, [13] presents the natural extension of the accumulation and discount models to the cash flow case, where a different interest/discount rate can be associated to each cash amount. Finally, the author considers financial choice criteria and shows that the net present value is easily generalised, but it is not possible to prove the existence of the internal rate of return in broader terms. Later on, Buckley [7-10] develops a new technique for solving fuzzy equations, based on multiple interval analysis. The approach is quite effective and successfully applied to linear, nonlinear and differential equations and systems of equations. Particularly, [9] demonstrates how the technique enables a fuzzy solution to the internal rate of return criteria. Lastly, in a recent article, Kuchta [21] suggests fuzzy equivalents to a range of methods for investment project appraisal. She starts with the revenue per dollar and the payback procedures, continues through the net present and net future values and the net-present-value utility, and concludes with the standard and a modified internal rate of return. Likewise the previous authors, she considers fuzzy numbers as unequivocally determined by the set of their level cuts, but the main contribution consists in the alternative fuzzification of the project duration. While it is usually presented as a discrete fuzzy set with a membership function defined by a collection of positive integers, in [21] the duration is a real fuzzy number allowing for the project to finish at any moment, not only at the end of an interest/discount period.

Here, we apply the general technique for evaluating fuzzy expressions from [7-10] to the NPV problem and model uncertain market expectations using fuzzy dividend yields, fuzzy share prices and a fuzzy constant discount rate. Next, we take into account the gradually consolidating for the last two decades idea amongst financial economists that where stock markets are concerned, one should consider time-varying rather than constant rates [14]. A fuzzy time-varying discount rate and the fuzzy log present value are introduced to handle the problem. Still, if we compare the earlier fuzzy NPV studies and our work, the major difference is found in the emphasis we place on the empirical results. While the articles quoted in the preceding paragraph are strictly theoretical, here we evaluate thirty-five projects investing in shares of UK companies traded on the London Stock Exchange. However, it is the analysis of the empirical results that facilitates the consequent formulation of the investment risk and the project robustness towards market uncertainty. Also, the intuition that projects with a small and a highly robust investment risk are preferable further assists the definition of an alternative ranking technique. Thus the fuzzy criterion evolves into a considerably informative and advantageous to investors method. In conclusion, the theoretical and the empirical results are fairly balanced, with the driving force being the investors benefit.

Finally, neural networks and fuzzy sets have been recently applied in a complementary and synergetic way and have coalesced to form the concept of soft computing. The effort of Kuo, L. Lee and C. Lee [22] is an illustration of such synergetic application to finance. They develop a stock-trading decision support system, integrating neural nets and fuzzy Delphi methods, and implement it to the Taiwan stock market. The result shows that without and under transaction costs the integrated model outperforms the single neural network. Another neuro-fuzzy example is the approach applied by Pan, Liu and Mejabi [29] to S&P500 forecasting. They develop a hybrid system where a fuzzy membership array is embedded in a neural network. Thus, inherent advantages of the resulting structure are both the ability of learning new patterns using the network training techniques and the interpretability of the weights and the transfer functions in the net. The out-of-sample forecasts of the hybrid system up to 24 months ahead without updating the model are compared with these of several regressions[2] and the neuro-fuzzy system is found to perform substantially better in terms of the root mean squared error. Our effort in this article to combine the advantages of the neural and

[2] OLS, AR(1), AR(2), EARCH(1), AR(1)+EARCH(1), AR(2)+ARCH(2), AR(2)+EARCH(2), and ARIMA(1,1,0).

fuzzy methodologies consists in building a three-layer feedforward network to solve the fuzzy investment criterion. We apply a modified Levenberg-Marquart backpropagation algorithm to impose certain sign restrictions on the weights in the net and to ensure that its output approximates the fuzzy solution. All the input vectors in the training set are presented to the neural net concurrently and its performance is evaluated over a separate test set. As a result, investors are provided with an effortless instrument for risk re-evaluation, any time they need to update and reconsider a project.

3. Investment project evaluation using a fuzzy criterion with a constant discount rate

For all intense and purposes, as the market evaluation of a company is based on its shares, the analysis is restricted to common stocks, as it is traditional in the finance literature. The stock's rate of return is defined by

$$R_{t_0+1} \equiv \frac{P_{t_0+1}\left(DY_{t_0+1}+1\right)}{P_{t_0}} - 1 \ , \qquad (1)$$

where P_{t_0} denotes the ex-dividend share price at the end of period t_0 and DY_{t_0+1} is the next period's dividend yield. The forward solution for P_{t_0} under constant returns is determined by

$$\hat{P}_{t_0} = \sum_{t=t_0+1}^{t_0+N} \frac{P_t DY_t}{(1+R)^{t-t_0}} + \frac{P_{t_0+N}}{(1+R)^N} = PV_{t_0} \ , \qquad (2)$$

which reveals that the estimated stream of future cash flows of projects investing in constant-return stocks at t_0 is (P_{t_0+1} DY_{t_0+1},..., P_{t_0+N} (DY_{t_0+N} +1)). The initial outlay of such projects is the share price at t_0, P_{t_0}. For simplicity, we will consider that the initial outlay is paid in the period in which the projects are evaluated, $t_0=0$, and P_0 is observed and known with certainty. The assumption of constant rates of return requires a project-evaluation technique using a constant discount rate R.

Notations: Before proceeding with the fuzzy criterion description, we will introduce some necessary notations. Let the positive real triangular fuzzy numbers to be substituted in (2) for P_t, DY_t and R, be respectively $\tilde{P}_t$, $\tilde{D}Y_t$ and $\tilde{R}$. The notations $\mu(x_{Pt}|\tilde{P}_t)=(P_{ta}/P_{tb}/P_{tc})$, $\mu(x_{DYt}|\tilde{D}Y_t)=(DY_{ta}/DY_{tb}/DY_{tc})$ and $\mu(x_R|\tilde{R})=(R_a/R_b/R_c)$ will stand for the triangular membership functions of $\tilde{P}_t$, $\tilde{D}Y_t$ and $\tilde{R}$. The graph of $\mu(x_{Pt}|\tilde{P}_t)$ is a triangle with base on the interval $[P_{ta},P_{tc}]$ and vertex at the point $x_{Pt}=P_{tb}$, where $0<P_{ta}<P_{tb}<P_{tc}$. The membership functions $\mu(x_{DYt}|\tilde{D}Y_t)$ and $\mu(x_R|\tilde{R})$ are described correspondingly. The weak α-cut $\tilde{P}_t(\alpha)$ of the fuzzy share price $\tilde{P}_t$ is defined by

$$\tilde{P}_i(\alpha) = \begin{cases} [P_{ta}, P_{tc}], & \alpha = 0 \\ \left\{x_{Pt} \mid \mu\left(x_{Pt} \mid \tilde{P}_t\right) \geq \alpha\right\}, & 0 < \alpha \leq 1 \end{cases}$$

and denoted as $\tilde{P}_t(\alpha)=[\underline{P_t(\alpha)}, \overline{P_t(\alpha)}]$, $0\leq\alpha\leq1$. Again, $\tilde{D}Y_t(\alpha)$ and $\tilde{R}(\alpha)$ are specified by analogy. When the time series of fuzzy share prices is considered, $\tilde{P}=\{\tilde{P}_t\}$, $1\leq t\leq N$, then $\tilde{P}(\alpha)$ is the Cartesian product of $\tilde{P}_t(\alpha)$, $\tilde{P}(\alpha)=\prod_{t=1}^{N}\tilde{P}_t(\alpha)$, $0\leq\alpha\leq1$. In a similar way, the Cartesian product $\tilde{D}Y(\alpha)=\prod_{t=1}^{N}\tilde{D}Y_t(\alpha)$,

$0\leq\alpha\leq1$, forms the α-cut of the dividend-yield time series, $\tilde{D}Y=\{\tilde{D}Y_t\}$, $1\leq t\leq N$. Further, we denote the stream of future cash flows by $\tilde{A}=(\tilde{A}_1,...,\tilde{A}_t,...,\tilde{A}_N)$, where $\tilde{A}_t=\tilde{P}_t\,\tilde{D}Y_t$, $1\leq t\leq N-1$, and $\tilde{A}_N=\tilde{P}_N(\tilde{D}Y_N+1)$. Then, the α-cut $\tilde{A}(\alpha)$ is introduced by $\tilde{A}(\alpha)=\prod_{t=1}^{N}\tilde{A}_t(\alpha)$, $0\leq\alpha\leq1$. Consequently, the condition $P\in\tilde{P}(\alpha)$ and $DY\in\tilde{D}Y(\alpha)$ is equivalent to $A\in\tilde{A}(\alpha)$,

$$\left.\begin{matrix} P\in\tilde{P}(\alpha) \\ DY\in\tilde{D}Y(\alpha)\end{matrix}\right\}\Leftrightarrow A\in\tilde{A}(\alpha)\ . \tag{3}$$

Next we present fuzzy-variable notations. The positive real fuzzy variables whose values are to be P_t, DY_t and R are denoted as $\breve{P}_t$, $\breve{D}Y_t$ and $\breve{R}$, respectively. Their values are restricted by the following triangular possibility distributions,

$$\mathrm{Poss}[\breve{P}_t=x_{Pt}]=\mu(x_{Pt}|\tilde{P}_t)\ ,\ \mathrm{Poss}[\breve{D}Y_t=x_{DYt}]=\mu(x_{DYt}|\tilde{D}Y_t)\ ,\ \mathrm{Poss}[\breve{R}=x_R]=\mu(x_R|\tilde{R})\ . \tag{4}$$

When interrelations (4) hold, then

$$\tilde{P}_t(\alpha)=[\,\underline{P_t(\alpha)}\,,\overline{P_t(\alpha)}\,]=\{x_{Pt}|\mathrm{Poss}[\breve{P}_t=x_{Pt}]\geq\alpha\},\ 0\leq\alpha\leq1\ , \tag{5a}$$

$$\tilde{D}Y_t(\alpha)=[\,\underline{DY_t(\alpha)}\,,\overline{DY_t(\alpha)}\,]=\{x_{DYt}|\mathrm{Poss}[\breve{D}Y_t=x_{DYt}]\geq\alpha\},\ 0\leq\alpha\leq1\ , \tag{5b}$$

$$\tilde{R}(\alpha)=[\,\underline{R(\alpha)}\,,\overline{R(\alpha)}\,]=\{x_R|\mathrm{Poss}[\breve{R}=x_R]\geq\alpha\},\ 0\leq\alpha\leq1\ . \tag{5c}$$

Again, if the share-price time series and the dividend-yield time series are considered, then $\pi_P=\min_{1\leq t\leq N}\{\mathrm{Poss}[\breve{P}_t=x_{Pt}]\}=\min_{1\leq t\leq N}\mu(x_{Pt}|\tilde{P}_t)\}$ is the joint possibility distribution of $\tilde{P}_t$, $1\leq t\leq N$, and $\pi_{DY}=\min_{1\leq t\leq N}\{\mathrm{Poss}[\breve{D}Y_t=x_{DYt}]\}=\min_{1\leq t\leq N}\mu(x_{DYt}|\tilde{D}Y_t)\}$ is the joint possibility distribution of $\tilde{D}Y_t$, $1\leq t\leq N$. The corresponding result for the stream of future cash flows presented as fuzzy variables, $\breve{A}=(\breve{A}_1,...,\breve{A}_t,...,\breve{A}_N)$, is

$$\pi_A=\min_{1\leq t\leq N}\{\mathrm{Poss}[\breve{A}_t=x_{A_t}]\}=\min\{\pi_P,\pi_{DY}\}, \tag{6}$$

assuming that $\breve{P}_t$ and $\breve{D}Y_t$ are non-interactive.

Fuzzy present value: We apply the two methods of evaluating fuzzy algebraic expressions presented in [8-10]. First, if positive real triangular fuzzy numbers $\tilde{P}_t$, $\tilde{D}Y_t$ and $\tilde{R}$ are substituted for P_t, DY_t and R in (2), then $\tilde{P}V_{FN}$ is the triangular-shaped fuzzy number providing a set of values that belong to the present value with various degrees of membership. The graph of the triangular-shaped membership function $\mu(x_{PV_{FN}}|\tilde{P}V_{FN})=(PV_{FVa}/PV_{FVb}/PV_{FVc})$ of the fuzzy present value is 0 outside $[PV_{FVa},PV_{FVc}]$ and 1 at $x_{PV_{FN}}=PV_{FVb}$, while $\mu(x_{PV_{FN}}|\tilde{P}V_{FN})$ is continuously increasing from 0 to 1 on the interval $[PV_{FVa},PV_{FVb}]$ and continuously decreasing from 1 to 0 on $[PV_{FVb},PV_{FVc}]$. For $t_0=0$, we form the Cartesian products $\tilde{P}(\alpha)$ and $\tilde{D}Y(\alpha)$, and find the α-cut $\Omega_{PV_{FN}}(\alpha)$,

$$\Omega_{PV_{FN}}(\alpha)=\left\{ \sum_{t=1}^{N}\frac{P_t DY_t}{(1+R)^t}+\frac{P_N}{(1+R)^N} \,|\, P_{1\times N}\in \tilde{P}(\alpha), DY_{1\times N}\in \tilde{D}Y(\alpha), R\in \tilde{R}(\alpha) \right\},\ 0\le\alpha\le 1 . \tag{7a}$$

Then, the first solution is defined by the following membership function:

$$\mu(x_{PV_{FN}}\,|\,\tilde{P}V_{FN})=\sup\{\alpha|\ x_{PV_{FN}}\in \Omega_{PV_{FN}}(\alpha)\} . \tag{7b}$$

Second, if one uses positive real fuzzy variables with triangular possibility distributions $\breve{P}_t$, $\breve{D}Y_t$ and $\breve{R}$, whose values are to be P_t, DY_t and R, correspondingly, then $\breve{P}V_{FV}$ is the fuzzy variable, whose values are possible solutions to (2). Form π_P and π_{DY}, and find the joint possibility distribution $\pi_{PV_{FV}}$,

$$\pi_{PV_{FV}}=\min\{\pi_P,\pi_{DY},\mathrm{Poss}[\breve{R}=x_R]\} , \tag{8a}$$

assuming that $\breve{P}_t$, $\breve{D}Y_t$ and $\breve{R}$ are non-interactive. Then, the triangular-shaped possibility distribution of the second solution is described by

$$\mathrm{Poss}[\breve{P}V_{FV}=x_{PV_{FV}}]=\sup\{\pi_{PV_{FV}}\,|\,x_{PV_{FV}}=\sum_{t=1}^{N}\frac{P_t DY_t}{(1+R)^t}+\frac{P_N}{(1+R)^N}\} . \tag{8b}$$

The comparison between $\tilde{P}V_{FN}$ and $\breve{P}V_{FV}$ produces the following result.

$$\mathrm{Poss}[\breve{P}V_{FV}=x_{PV_{FV}}]=\mu(x_{PV_{FN}}\,|\tilde{P}V_{FN}) , \tag{9a}$$

$$\tilde{P}V_{FN}(\alpha)=\Omega_{PV_{FN}}(\alpha)=\{x_{PV_{FV}}\,|\mathrm{Poss}[\breve{P}V_{FV}=x_{PV_{FV}}]\ge\alpha\} ,\ 0\le\alpha\le 1 . \tag{9b}$$

Consequently, the two solutions are identical. We have presented both of them, as the first solution $\tilde{P}V_{FN}$ provides the basis for the computational algorithm, and the second solution $\breve{P}V_{FV}$ gives us the opportunity to justify the uncertainty-modelling technique implemented. When one considers various levels of uncertainty, it is natural to find the present value of the investment project, using values of share prices, dividend yields and the discount rate, all at the same level of uncertainty. Thus, the project is evaluated at various levels of uncertainty. For each level of uncertainty in the data, the fuzzy method provides present values at the corresponding level of uncertainty. The values of P_t, DY_t and R at the same level of uncertainty are defined by their fuzzy-variable presentation and belong to the sets

$$\{x_{Pt}|\mathrm{Poss}[\breve{P}_t=x_{Pt}]\ge\alpha\},\ \{x_{DYt}|\mathrm{Poss}[\breve{D}Y_t=x_{DYt}]\ge\alpha\},\ \{x_R|\mathrm{Poss}[\breve{R}=x_R]\ge\alpha\},\ 0\le t\le N,\ 0\le\alpha\le 1 .$$

However, these sets are identical to $\tilde{P}_t(\alpha)$, $\tilde{D}Y_t(\alpha)$ and $\tilde{R}(\alpha)$ in the fuzzy-number presentation. Then, the set of present values at the corresponding level of uncertainty is delimited by the second solution, $\{x_{PV_{FV}}\,|\mathrm{Poss}[\breve{P}V_{FV}=x_{PV_{FV}}]\ge\alpha\}$, $0\le\alpha\le1$. This set can be determined by calculating $\sum_{t=1}^{N}\frac{P_t DY_t}{(1+R)^t}+\frac{P_N}{(1+R)^N}$ for $P_t\in\tilde{P}_t(\alpha)$, $DY_t\in\tilde{D}Y_t(\alpha)$, $R\in\tilde{R}(\alpha)$, $1\le t\le N$, $0\le\alpha\le1$, thus producing the α-cut $\Omega_{PV_{FN}}(\alpha)$ of the first solution $\tilde{P}V_{FN}$.

Data: Monthly (DataStream) data on share prices and dividend yields are employed for the period January 1975 – January 2000. The application covers thirty-five UK companies from various sectors and within diverse range of market capitalisation. They are selected for their positive and less than one price to book value at the beginning of the investment projects, which reveals undervaluation and potential investment attractiveness. Table 1 provides the company list and basic information. The discount rate employed is the average for the length of the projects 3-month UK treasury bill rate.

Table 1: ***Companies covered in the application***

Company	*sector*	*market capitalisation* (end-of-project), *£m*
BASS	Restaurants, Pubs & Breweries	7,616.94
BBA GROUP	Engineering – General	2,186.50
BENTALLS	Retailers - Multi Department	26.79
BLUE CIRCLE INDUSTRIES	Building & Construction Materials	2,912.45
BOC GROUP	Chemicals – Commodity	6,530.61
BOOTS CO.	Retailers - Multi Department	5,445.32
BP AMOCO	Oil – Integrated	121,241.85
BRITISH AMERICAN TOBACCO	Tobacco	7,654.29
BUNZL	Business Support Services	1,553.66
COATS VIYELLA	Other Textiles & Leather Goods	292.00
DIXONS GROUP	Retailers – Hardlines	7,103.14
GOODWIN	Engineering Fabricators	5.65
GREAT UNIVERSAL STORES	Retailers - Multi Department	3,641.28
HANSON	Building & Construction Materials	3,383.77
INCHCAPE	Vehicle Distribution	242.97
LEX SERVICE	Vehicle Distribution	436.58
MARKS & SPENCER	Retailers - Multi Department	8,464.95
NORTHERN FOODS	Food Processors	609.23
PILKINGTON	Building & Construction Materials	926.97
RANK GROUP	Leisure Facilities	1,516.28
RMC GROUP	Building & Construction Materials	2,218.21
SAINSBURY (J)	Food & Drug Retailers	6,707.54
SCOTTISH & NEWCASTLE	Restaurants, Pubs & Breweries	2,684.77
SMITH (WH) GROUP	Retailers - Soft Goods	1,200.80
SMITHS INDUSTRIES	Aerospace	2,912.40
TARMAC	Building & Construction Materials	1,126.40
TATE & LYLE	Food Processors	1,820.48
TAYLOR WOODROW	Other Construction	510.63
TI GROUP	Engineering – General	2,401.66
TRANSPORT DEVELOPMENT GROUP	Rail, Road & Freight	161.18
UNILEVER	Food Processors	13,261.69
UNITED BISCUITS HOLDINGS	Food Processors	1,255.64
WHITBREAD	Restaurants, Pubs & Breweries	3,102.43
WIMPEY (GEORGE)	House Building	415.45
WOLSELEY	Builders Merchants	2,725.25

Calculations: The support $\tilde{P}_t(0)=[\underline{P_t(0)}, \overline{P_t(0)}]$ of the triangular membership function of each $\tilde{P}_t$, $1 \le t \le N$, is 2.5% wider than the 99% normal-distribution confidence interval. This permits a broader range of possible values when modelling the uncertain price P_t. Further, $\tilde{P}_t(1)=P_t$ and the triangles are isosceles, except for the cases when the calibration produces $\underline{P_t(0)} \le 0$. Then, $\underline{P_t(0)}$ =eps is accepted. The procedure is repeated for $\tilde{DY}_t$, $1 \le t \le N$, and for $\tilde{R}$, where $\tilde{R}(1)=\mathrm{mean}(R_t)$. Thus, we assume no negative and zero share prices, dividend yields and discount rates. This is not a requirement of the method, but rather a feature of the investment projects considered. The calculations involve the following relation, where the same discount rate is applied to all companies.

$$\Omega_{PV_{FN}}(\alpha) = [\underline{PV_{FN}(\alpha)}, \overline{PV_{FN}(\alpha)}] = \left[\sum_{t=1}^{N} \frac{\underline{P_t(\alpha)DY_t(\alpha)}}{\left(1+\overline{R(\alpha)}\right)^t} + \frac{\underline{P_N(\alpha)}}{\left(1+\overline{R(\alpha)}\right)^N}, \sum_{t=1}^{N} \frac{\overline{P_t(\alpha)DY_t(\alpha)}}{\left(1+\underline{R(\alpha)}\right)^t} + \frac{\overline{P_N(\alpha)}}{\left(1+\underline{R(\alpha)}\right)^N} \right] \quad (10)$$

Consequently, the information that distinguishes the projects and affects their present values in Jan. 1975 comprises the streams of future cash flows from Feb. 1975 till Jan. 2000. Remember also relations (3) and (6). As an illustration, Figures 1 to 4 present the results[3] for four of the companies having different risk characteristics.[4]

Fig. 1a: BLUE CIRCLE INDUSTRIES

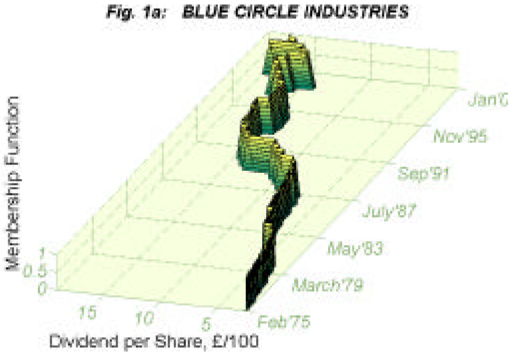


Fig. 1b: BLUE CIRCLE INDUSTRIES

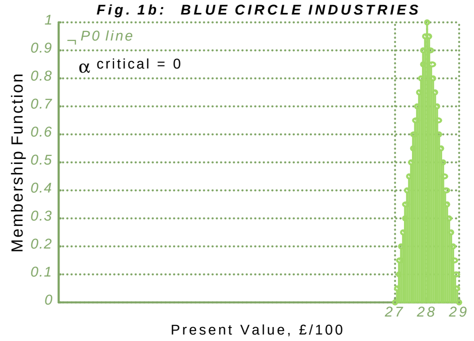


Fig. 2a: SCOTTISH & NEWCASTLE

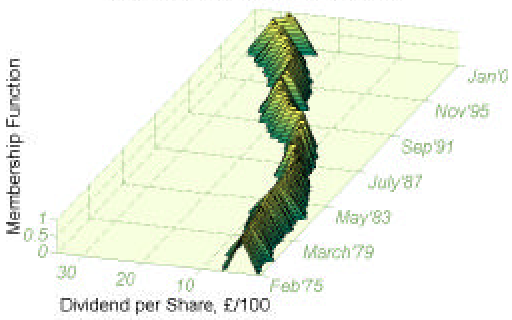


Fig. 2b: SCOTTISH & NEWCASTLE

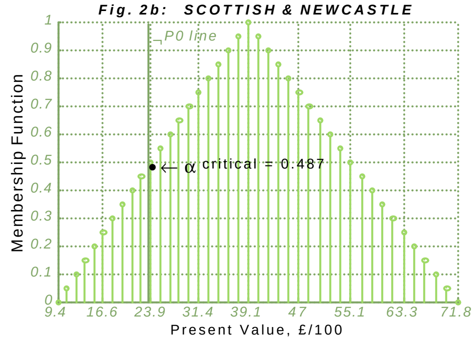


Fig. 3a: BBA GROUP

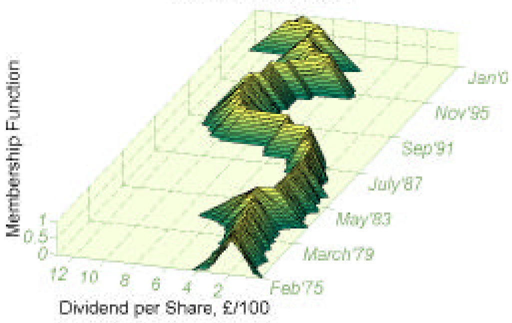


Fig. 3b: BBA GROUP

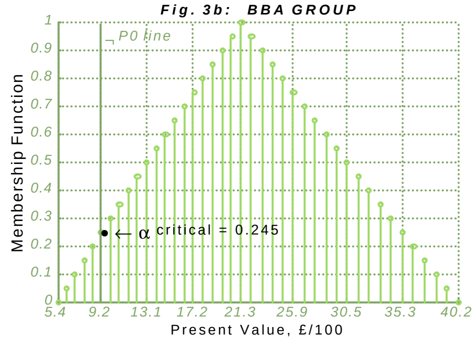


Fig. 4a: BP AMOCO

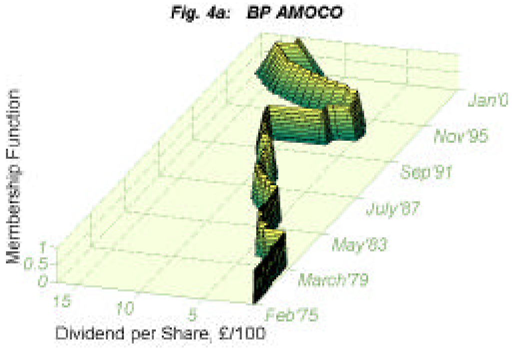
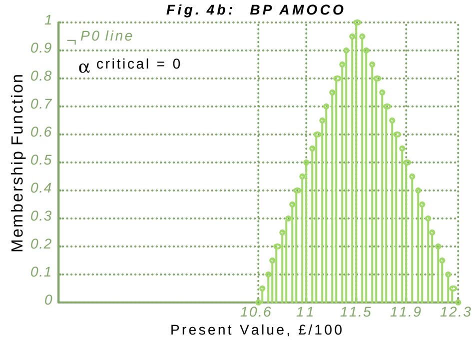


Fig. 4b: BP AMOCO



Results interpretation: The character of the fuzzy cash-flow stream affects the form of the fuzzy present value. Most importantly, there exists a unique for each company $\alpha_{critical}$, the level where the initial-outlay line crosses the present-value membership function. Since α is the degree of membership of a value to the set of the project's present

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[3] For better visualisation, only the dividends $D_t=P_tDY_t$, $1\leq t\leq N$, are included in the graphics of the cash-flow streams but not the final price P_N.

[4] See [35] for the complete set of graphics.

values, then the level of uncertainty is calculated by $u=1-\alpha$, and correspondingly $u_{critical}=1-\alpha_{critical}$. A project is profitable at any $u<u_{critical}$, while there is a chance of being unprofitable when $u\geq u_{critical}$. For example, an investment in BBA GROUP at a level of uncertainty $u<0.755$ has a set of possible present values all bigger than the current shares price, consequently all possible net present values are positive. Any uncertainty equal or above $u_{critical}=0.755$ will force the corresponding set of possible NPVs to include zero and negative values. Thus, a critical level of uncertainty is associated with each project. (See Table 2.) For BOOTS CO., $u_{critical}=0$, the NPV set includes zero and negative values and the project is unprofitable at any level of uncertainty. On the opposite, BP AMOCO has $u_{critical}=1$ and the investment there is profitable at any u. Furthermore, if one selects the risk and decides in favour of a project at $u\geq u_{critical}$, when $0<u_{critical}<1$, there is a chance of getting an even more profitable project, as the possible NPV set at a higher level of uncertainty is wider and includes larger positive values. This can be considered as a kind of risk reward.

Table 2: ***Critical levels: constant discount rate and normal calibration***

company	$\alpha_{critical}$	$u_{critical}$
BASS	0.772	0.228
BBA GROUP	0.245	0.755
BENTALLS	0.528	0.472
BLUE CIRCLE INDUSTRIES	0.000	1.000
BOC GROUP	0.000	1.000
BOOTS CO.	1.000	0.000
BP AMOCO	0.000	1.000
BRITISH AMERICAN TOBACCO	0.901	0.099
BUNZL	0.839	0.161
COATS VIYELLA	0.000	1.000
DIXONS GROUP	0.656	0.344
GOODWIN	0.421	0.579
GREAT UNIVERSAL STORES	0.942	0.058
HANSON	0.507	0.493
INCHCAPE	1.000	0.000
LEX SERVICE	0.000	1.000
MARKS & SPENCER	0.944	0.056
NORTHERN FOODS	0.339	0.661
PILKINGTON	0.026	0.974
RANK GROUP	0.658	0.342
RMC GROUP	0.479	0.521
SAINSBURY (J)	1.000	0.000
SCOTTISH & NEWCASTLE	0.487	0.513
SMITH (WH) GROUP	0.986	0.014
SMITHS INDUSTRIES	0.584	0.416
TARMAC	0.000	1.000
TATE & LYLE	0.190	0.810
TAYLOR WOODROW	0.937	0.063
TI GROUP	0.000	1.000
TRANSPORT DEVELOPMENT GROUP	0.000	1.000
UNILEVER	1.000	0.000
UNITED BISCUITS HOLDINGS	0.000	1.000
WHITBREAD	0.765	0.235
WIMPEY (GEORGE)	1.000	0.000
WOLSELEY	0.000	1.000

The fuzzy present value provides investors with solutions on various levels of uncertainty. There are different agents on the market, individual investors and huge investment funds as the extremes, with diverse risk tolerance. They will consider the same project in divergent ways, as the risk or the level of uncertainty they are willing to accept is different. For example, $u_{acceptable}$ can be below or above $u_{critical}$, and while some agents will presumably profit from the project, others will regard it as too risky and more likely unprofitable.

4. Measuring the investment risk and its robustness

The results in the previous section show that a critical level of uncertainty is associated with each project. The solution procedure applied allows for finding the set of the project's present values that corresponds to all the share prices, dividend yields and discount rates possible at some level of uncertainty. This set is situated at the same level of uncertainty. Therefore, there is a critical level of uncertainty, $u_{critica}$, embodied in the market data we use to evaluate the project and this level delimits the project's investment risk. We suggest $1\text{-}u_{critical}=\alpha_{critical} \in [0,1]$ as a risk measure. The following reasons support such suggestion. First, the lower the critical level of uncertainty at which there is a chance for the project being unprofitable, the higher the investment risk. Second, $\alpha_{critical}$ is the membership level of the fuzzy net present value, below and at which it is certain that the solution includes negative and zero values, and above which the project is definitely profitable. Now, a new interpretation can be given to the results in Table 2, $u_{critical}\equiv$the critical level of uncertainty embodied in the market data, $\alpha_{critical}\equiv$the investment risk of the project. Moreover, evaluating the same projects under increased uncertainty of the market environment and comparing the resultant critical values one will derive an estimate of the investment risk robustness. To model increased market uncertainty, we apply a calibration procedure based on the 95% t6 confidence interval rather than the normal interval, thus producing fatter-tail possibility distributions. Table 3 below compares the critical values.

Table 3: ***Critical levels in the constant discount rate case: normal and t6 calibration***

company	*N-calibration*		t6-calibration	
	$\alpha_{Ncritical}$	$u_{Ncritical}$	$\alpha_{t6critical}$	$u_{t6critical}$
BASS	0.772	0.228	0.820	0.180
BBA GROUP	0.245	0.755	0.527	0.473
BENTALLS	0.528	0.472	0.822	0.178
BLUE CIRCLE INDUSTRIES	0.000	1.000	0.000	1.000
BOC GROUP	0.000	1.000	0.000	1.000
BOOTS CO.	1.000	0.000	1.000	0.000
BP AMOCO	0.000	1.000	0.529	0.471
BRITISH AMERICAN TOBACCO	0.901	0.099	0.920	0.080
BUNZL	0.839	0.161	0.880	0.120
COATS VIYELLA	0.000	1.000	0.000	1.000
DIXONS GROUP	0.656	0.344	0.713	0.287
GOODWIN	0.421	0.579	0.575	0.425
GREAT UNIVERSAL STORES	0.942	0.058	0.952	0.048
HANSON	0.507	0.493	0.573	0.427
INCHCAPE	1.000	0.000	1.000	0.000
LEX SERVICE	0.000	1.000	0.000	1.000
MARKS & SPENCER	0.944	0.056	0.955	0.045
NORTHERN FOODS	0.339	0.661	0.543	0.457
PILKINGTON	0.026	0.974	0.769	0.231
RANK GROUP	0.658	0.342	0.897	0.103
RMC GROUP	0.479	0.521	0.543	0.457
SAINSBURY (J)	1.000	0.000	1.000	0.000
SCOTTISH & NEWCASTLE	0.487	0.513	0.674	0.326
SMITH (WH) GROUP	0.986	0.014	0.991	0.009
SMITHS INDUSTRIES	0.584	0.416	0.639	0.361
TARMAC	0.000	1.000	0.000	1.000
TATE & LYLE	0.190	0.810	0.668	0.332
TAYLOR WOODROW	0.937	0.063	0.981	0.019
TI GROUP	0.000	1.000	0.563	0.437
TRANSPORT DEVELOPMENT GROUP	0.000	1.000	0.661	0.339
UNILEVER	1.000	0.000	1.000	0.000
UNITED BISCUITS HOLDINGS	0.000	1.000	0.000	1.000
WHITBREAD	0.765	0.235	0.806	0.194
WIMPEY (GEORGE)	1.000	0.000	1.000	0.000
WOLSELEY	0.000	1.000	0.000	1.000

Results interpretation: Raising the market uncertainty, we logically obtain decreased critical levels $u_{critical}$. For all the projects, the relation $u_{t6critical} \leq u_{Ncritical}$ holds, which indicates that the chance of a project being unprofitable now occurs at lower levels of uncertainty embodied in the data. If a project was profitable at any u or $u_{Ncritical}=1$, then it may still stay profitable and $u_{t6critical}=1$. It is the case with the following stocks: BLUE CIRCLE INDUSTRIES, BOC GROUP, COATS VIYELLA, LEX SERVICE, TARMAC, and UNITED BISCUITS HOLDINGS. These companies perform really well and the investment is believed to be rewarding despite the increased uncertainty in the environment. On the other hand, the critical level may decrease below 1, which is the case with BP AMOCO ($u_{t6critical}=0.471$), TI GROUP ($u_{t6critical}=0.437$) and TRANSPORT DEVELOPMENT GROUP ($u_{t6critical}=0.339$). Such companies perform relatively well but their results are not quite robust and this affects the profitability of the corresponding projects. For example, investing in BP AMOCO was gainful at any level of uncertainty and $u_{Ncritical}=1$; now there is some possibility for the project being unprofitable at levels equal or above $u_{t6critical}=0.471$. Further, if the critical level before increasing the uncertainty was strongly between zero and one, $0<u_{Ncritical}<1$, then the strong inequality $u_{t6critical}<u_{Ncritical}$ holds. Most of the companies fall in this category. Such projects are not the most favourable but still show positive results at levels below $u_{t6critical}$. The final outcome depends to some extent to the predictability of the market environment. To find out to what extent means to identify the project robustness towards the market uncertainty. Finally, if an investment was unrewarding at any level before and $u_{Ncritical}=0$, it logically stays unprofitable now and $u_{t6critical}=0$. Such examples are BOOTS CO., INCHCAPE, SAINSBURY (J), UNILEVER, and WIMPEY (GEORGE). The companies do not perform quite well and investment there is not believed to be gainful even in a less uncertain environment.

When the investment risk results, $\alpha_{critical}$, are considered, then the relation $\alpha_{t6critical} \geq \alpha_{Ncritical}$ is revealed, implying that the increased uncertainty causes the investment risk to rise. It will be helpful to find out the extent of this dependence or to identify the risk robustness towards market uncertainty. No investor will be interested in unprofitable projects, where $\alpha_{Ncritical}=\alpha_{t6critical}=1$. It is why Table 4 only includes companies with $\alpha_{Ncritical}<1$ and suggests

$$\Delta\alpha=\alpha_{t6critical}-\alpha_{Ncritical} \tag{11}$$

as a measure of the risk robustness. It is accepted that $0\leq\Delta\alpha<0.1$ represents highly robust projects, $0.1\leq\Delta\alpha<0.25$ indicates investments with medium risk robustness and $0.25\leq\Delta\alpha<0.5$ investments with low risk robustness, while $0.5\leq\Delta\alpha$ implies no robustness at all. The companies are first ordered in correspondence with their $\alpha_{t6critical}$, as the increased-uncertainty case better represents the reality. However, when choosing between projects with close investment risks, the more robust investments are preferable, leading to rearrangement according to the second criterion, $\Delta\alpha$. For example, RMC GROUP is preferable to NORTHERN FOODS, BBA GROUP and BP AMOCO, because the project is more robust, although with slightly higher investment risk. By analogy, HANSON is preferable to NORTHERN FOODS, BBA GROUP, BP AMOCO and TI GROUP. Also, SCOTTISH & NEWCASTLE is preferable to TATE & LYLE and TRANSPORT DEVELOPMENT GROUP. DIXONS GROUP is preferable to TRANSPORT DEVELOPMENT GROUP, etc. Consequently, the table suggests an order corresponding to the project's attractiveness to investors. It is important for investors to pick out projects having not only a small but also a highly robust investment risk. The fuzzy present value provides them with the necessary information and facilitates their decision.

Table 4: ***Investment risk robustness: the constant discount rate case***

company	*investment risk*	*Dα*	*risk robustness*
BLUE CIRCLE INDUSTRIES	0.000	0.000	high
COATS VIYELLA	0.000	0.000	high
BOC GROUP	0.000	0.000	high
LEX SERVICE	0.000	0.000	high
TARMAC	0.000	0.000	high
UNITED BISCUITS HOLDINGS	0.000	0.000	high
WOLSELEY	0.000	0.000	high
RMC GROUP	0.543	0.064	high
HANSON	0.573	0.066	high
NORTHERN FOODS	0.543	0.204	medium
GOODWIN	0.575	0.154	medium
BBA GROUP	0.527	0.282	low
BP AMOCO	0.529	0.529	none
TI GROUP	0.563	0.563	none
SMITHS INDUSTRIES	0.639	0.055	high
SCOTTISH & NEWCASTLE	0.674	0.187	medium
TATE & LYLE	0.668	0.478	low
DIXONS GROUP	0.713	0.057	high
TRANSPORT DEVELOPMENT GROUP	0.661	0.661	none
WHITBREAD	0.806	0.041	high
BASS	0.820	0.048	high
BENTALLS	0.822	0.294	low
PILKINGTON	0.769	0.743	none
BUNZL	0.880	0.041	high
BRITISH AMERICAN TOBACCO	0.920	0.019	high
GREAT UNIVERSAL STORES	0.952	0.010	high
MARKS & SPENCER	0.955	0.011	high
RANK GROUP	0.897	0.239	medium
SMITH (WH) GROUP	0.991	0.005	high
TAYLOR WOODROW	0.981	0.044	high

Figures 5 to 8 illustrate how the increased uncertainty affects the cash-flow streams and shifts the present values. The four companies are correspondingly representative for the four zones of risk robustness: high, medium, low and none.

Fig. 5a: BLUE CIRCLE INDUSTRIES

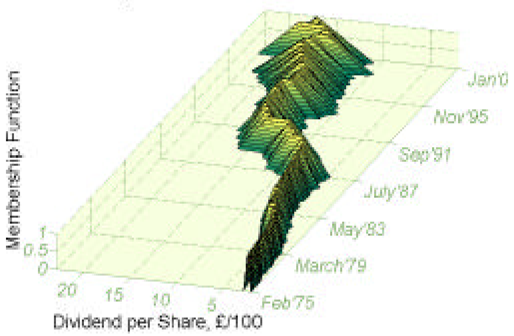


Fig. 5b: BLUE CIRCLE INDUSTRIES

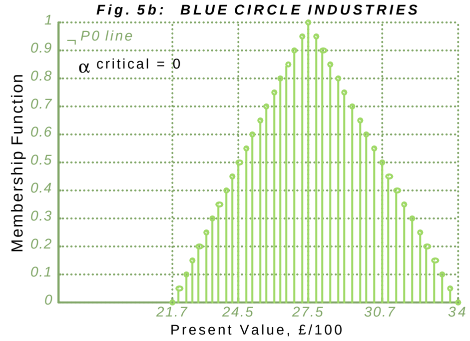


Fig. 6a: SCOTTISH & NEWCASTLE

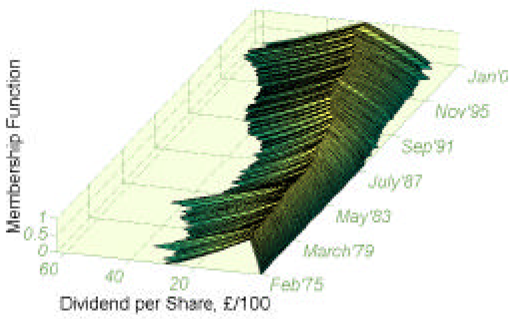


Fig. 6b: SCOTTISH & NEWCASTLE

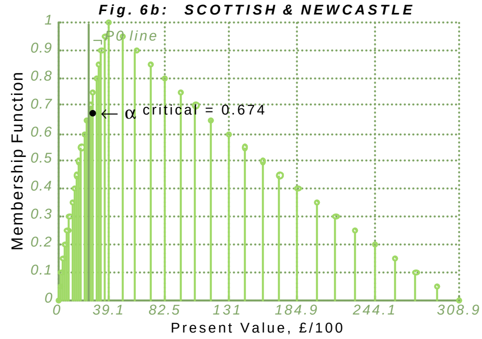


Fig. 7a: BBA GROUP

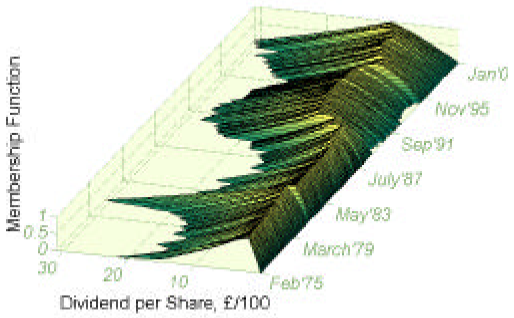


Fig. 7b: BBA GROUP

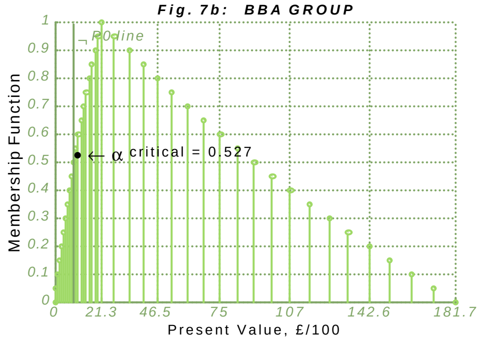


Fig. 8a: BP AMOCO

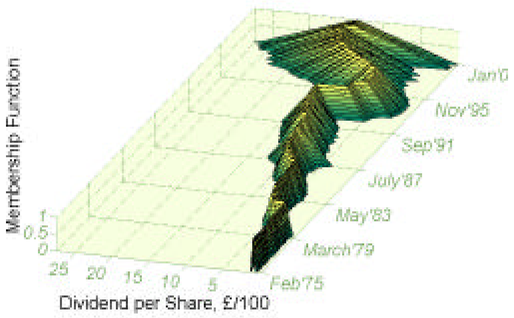
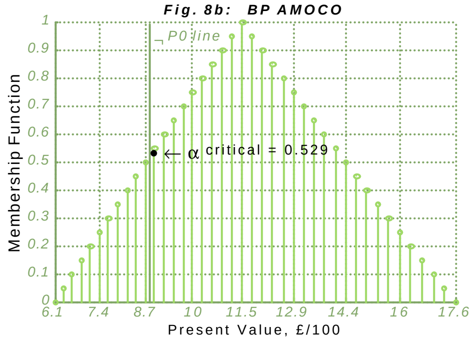


Fig. 8b: BP AMOCO



Remember that the standard method will not quite distinguish between the above projects, as they all have a positive crisp net present value. The generally accepted technique will not tell us how to choose between projects with close crisp net present values. It will not reveal whether projects with higher NPV are less robust and less preferable. In conclusion, the standard results are not quite informative and can be misleading.

5. Investment project evaluation using a fuzzy criterion with a time-varying discount rate

Since early 1980s, when several authors raised the topic that stock prices are too volatile to be rational forecasts of future dividends discounted at a constant rate, regression tests have convinced many financial economists that stock returns are time-varying rather than constant [14]. The assumption of time-varying returns transforms the price-dividend relation (2) into nonlinear and a loglinear approximation is required, where the analogous of the level return R_{t_0+1} in (1) will be the log return $r_{t_0+1} \equiv \ln(1+R_{t_0+1}) = \ln\left(\frac{P_{t_0+1}DY_{t_0+1}+P_{t_0+1}}{P_{t_0}}\right)$. Let p_{t_0+1} and dy_{t_0+1} stand for the log share price and the log dividend yield, correspondingly. Then, $r_{t_0+1} = p_{t_0+1} - p_{t_0} + \ln\left(1+e^{dy_{t_0+1}}\right)$ and using a first-order Taylor approximation of the last term, one obtains

$$r_{t_0+1} \approx k + p_{t_0+1} + (1-\rho)dy_{t_0+1} - p_{t_0} \quad . \tag{12}$$

Here, $k \equiv (1-\rho)\ln(1-\rho) - \rho\ln(\rho)$ and $\rho \equiv \frac{1}{1+e^{\overline{dy}}}$ are parameters of linearisation, and $\overline{dy}$ is the average log dividend yield. Solving (12) forward for p_{t_0} produces a linear equation for the log stock price under the assumption of time-varying stock returns,

$$\hat{p}_{t_0} = \sum_{t=t_0+1}^{t_0+N} \rho^{t-t_0-1}\left[(1-\rho)(dy_t + p_t) + k - r_t\right] + \rho^{t_0+N} p_{t_0+N} \equiv lpv_{t_0} \ . \qquad (13)^{5}$$

It is analogous to equation (2) for the level stock price in the constant-return case. To continue with the analogy, we now consider projects investing in time-varying-return stocks at t_0 and equate the log present value lpv_{t_0} of such projects with the log-current-price estimation in (13). The level-log transformation causes the following form of the stream of future cash flows, $\left(p_{t_0+1} + dy_{t_0+1}, \ldots, p_{t_0+N} + \ln\left(1 + e^{dy_{t_0+N}}\right)\right)$, and the log initial outlay is the log share price at t_0, p_{t_0}. We assume t_0=0 and p_0 known with certainty. The projects are evaluated under a time-varying discount rate r_t and they are profitable when $lpv_0 > p_0$.

Notations and fuzzy solutions: First, we will briefly introduce a few more notations. The linearisation parameters, k and ρ, are considered crisp. The real fuzzy numbers to be substituted for p_t, dy_t and r_t in (13) are correspondingly $\tilde{p}_t$, $\tilde{dy}_t$ and $\tilde{r}_t$. The level-log data transformation causes triangular-shaped rather then triangular membership functions,

$$\mu(x_{pt}|\tilde{p}_t)=(p_{ta}/p_{tb}/p_{tc})=\mu(x_{Pt}|\tilde{P}_t),\ \mu(x_{dyt}|\tilde{dy}_t)=(dy_{ta}/dy_{tb}/dy_{tc})=\mu(x_{DYt}|\tilde{DY}_t),\ \mu(x_{rt}|\tilde{r}_t)=(r_{ta}/r_{tb}/r_{tc})=\mu(x_{Rt}|\tilde{R}_t)\ . \quad (14)$$

Next, one can describe α-cuts, $\tilde{p}_t(\alpha) = \left[\underline{p_t(\alpha)}, \overline{p_t(\alpha)}\right] = \begin{cases} [p_{ta}, p_{tc}], & \alpha = 0 \\ \{x_{pt} \,|\, \mu(x_{pt}|\tilde{p}_t) \geq \alpha\}, & 0 < \alpha \leq 1 \end{cases}$, and Cartesian products, $\tilde{p}(\alpha) = \prod_{t=1}^{N} \tilde{p}_t(\alpha)$, 0≤α≤1. The fuzzy log stream of future cash flows is presented by $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_N) = \left(\tilde{p}_1 + \tilde{dy}_1, \ldots, \tilde{p}_N + \ln\left(1 + e^{dy_N}\right)\right)$ and its α-cut is defined by $\tilde{a}(\alpha) = \prod_{t=1}^{N} \tilde{a}_t(\alpha)$, 0≤α≤1. The condition $p \in \tilde{p}(\alpha)$ and $dy \in \tilde{dy}(\alpha)$ is equivalent to $a \in \tilde{a}(\alpha)$. Finally, the real fuzzy variables whose values are to be p_t, dy_t and r_t in (13) are denoted as $\breve{p}_t$, $\breve{dy}_t$ and $\breve{r}_t$, respectively, and their possibility distributions are described by Poss[$\breve{p}_t = x_{pt}$]=$\mu(x_{pt}|\tilde{p}_t)$, Poss[$\breve{dy}_t = x_{dyt}$]=$\mu(x_{dyt}|\tilde{dy}_t)$ and Poss[$\breve{r}_t = x_{rt}$]=$\mu(x_{rt}|\tilde{r}_t)$, where $\tilde{p}_t(\alpha) = \left[\underline{p_t(\alpha)}, \overline{p_t(\alpha)}\right] = \{x_{pt} \,|\, \mathrm{Poss}[\breve{p}_t = x_{pt}] \geq \alpha\}$, $\tilde{dy}_t(\alpha) = \left[\underline{dy_t(\alpha)}, \overline{dy_t(\alpha)}\right] = \{x_{dyt} \,|\, \mathrm{Poss}[\breve{dy}_t = x_{dyt}] \geq \alpha\}$, and $\tilde{r}_t(\alpha) = \left[\underline{r_t(\alpha)}, \overline{r_t(\alpha)}\right] = \{x_{rt} \,|\, \mathrm{Poss}[\breve{r}_t = x_{rt}] \geq \alpha\}$, 0≤α≤1.

Now, for t_0=0, form the Cartesian products $\tilde{p}(\alpha)$, $\tilde{dy}(\alpha)$ and $\tilde{r}(\alpha)$, and find the α-cut $\Omega_{lpv_{fn}}(\alpha)$,

$$\tilde{p}(\alpha) = \tilde{p}_1(\alpha) \times \ldots \times \tilde{p}(\alpha)_N,\ \tilde{dy}(\alpha) = \tilde{dy}_1(\alpha) \times \ldots \times \tilde{dy}_N(\alpha),\ \tilde{r}(\alpha) = \tilde{r}_1(\alpha) \times \ldots \times \tilde{r}_N(\alpha),\quad 0 \leq \alpha \leq 1,$$

$$\Omega_{lpv_{fn}}(\alpha) = \left\{ \sum_{t=1}^{N} \rho^{t-1}\left[(1-\rho)(dy_t + p_t) + k - r_t\right] + \rho^N p_N \,|\, p_{1\times N} \in \tilde{p}(\alpha), dy_{1\times N} \in \tilde{dy}(\alpha), r_{1\times N} \in \tilde{r}(\alpha) \right\} \ .$$

[5] $\lim_{N\to\infty}(p_{t_0}) \approx \frac{k}{1-\rho} + \sum_{t=t_0+1}^{\infty} \rho^{t-t_0-1}\left[(1-\rho)(dy_t + p_t) - r_t\right]$

Then the first solution for the fuzzy log present value is defined by its membership function,

$$\mu(x_{lpv_{fn}} \,|\, l\tilde{p}v_{fn})=\sup\{\alpha |\, x_{lpv_{fn}} \in \Omega_{lpv_{fn}}(\alpha)\} .$$

Next, form π_p , π_{dy} and π_r , and find the joint possibility distribution $\pi_{lpv_{fv}}$ =min{π_p,π_{dy},π_r}, assuming that $\breve{p}_t$, $\breve{d}y_t$ and $\breve{r}_t$ are non-interactive. Then, the triangular-shaped possibility distribution of the second solution is described by

$$\mathrm{Poss}[\, l\breve{p}v_{fv} = x_{lpv_{fv}}]=\sup\{ \pi_{lpv_{fv}} \,|\, x_{lpv_{fv}} = \sum_{t=1}^{N} \rho^{t-1}\left[(1-\rho)(dy_t + p_t) + k - r_t\right] + \rho^N p_N \} .$$

The solutions are identical and the relation $l\tilde{p}v_{fn}(\alpha)==\Omega_{lpv_{fn}}(\alpha)=\{ x_{lpv_{fv}} \,|\mathrm{Poss}[\, l\breve{p}v_{fv} = x_{lpv_{fv}}]\geq\alpha\}$, $0\leq\alpha\leq1$, holds, where

$$\Omega_{lpv_{fn}}(\alpha)=\left[\underline{lpv_{fn}(\alpha)},\overline{lpv_{fn}(\alpha)}\right]=\left[\sum_{t=1}^{N}\rho^{t-1}\left[(1-\rho)\left(\underline{dy_t(\alpha)}+\underline{p_t(\alpha)}\right)+k-\overline{r_t(\alpha)}\right]+\rho^N\underline{p_N(\alpha)},\sum_{t=1}^{N}\rho^{t-1}\left[(1-\rho)\left(\overline{dy_t(\alpha)}+\overline{p_t(\alpha)}\right)+k-\underline{r_t(\alpha)}\right]+\rho^N\overline{p_N(\alpha)}\right].$$

Calculations: The same 35 projects are evaluated, but the assumption of a time-varying discount rate enforces the employment of N fuzzy numbers $\tilde{r}_t$ with $\tilde{r}_t(1)=\ln(R_t)$, $1\leq t\leq N$, and not the average but all the monthly treasury-bill-rate data are involved. The calculations are performed for t6 calibration and the results are presented in Table 5.

Table 5: ***Critical levels for t6 calibration: constant and time-varying discount rate***

company	*constant discount rate*		*time-varying discount rate*	
	$a_{t6critical}$	$u_{t6ritical}$	$a_{logcritical}$	$u_{logcritical}$
BASS	0.820	0.180	1.000	0.000
BBA GROUP	0.527	0.473	0.696	0.304
BENTALLS	0.822	0.178	1.000	0.000
BLUE CIRCLE INDUSTRIES	0.000	1.000	0.000	1.000
BOC GROUP	0.000	1.000	0.000	1.000
BOOTS CO.	1.000	0.000	1.000	0.000
BP AMOCO	0.529	0.471	0.904	0.096
BRITISH AMERICAN TOBACCO	0.920	0.080	1.000	0.000
BUNZL	0.880	0.120	1.000	0.000
COATS VIYELLA	0.000	1.000	0.000	1.000
DIXONS GROUP	0.713	0.287	1.000	0.000
GOODWIN	0.575	0.425	0.925	0.075
GREAT UNIVERSAL STORES	0.952	0.048	1.000	0.000
HANSON	0.573	0.427	0.673	0.327
INCHCAPE	1.000	0.000	1.000	0.000
LEX SERVICE	0.000	1.000	0.000	1.000
MARKS & SPENCER	0.955	0.045	1.000	0.000
NORTHERN FOODS	0.543	0.457	0.656	0.344
PILKINGTON	0.769	0.231	0.919	0.081
RANK GROUP	0.897	0.103	1.000	0.000
RMC GROUP	0.543	0.457	0.739	0.261
SAINSBURY (J)	1.000	0.000	1.000	0.000
SCOTTISH & NEWCASTLE	0.674	0.326	0.775	0.225
SMITH (WH) GROUP	0.991	0.009	1.000	0.000
SMITHS INDUSTRIES	0.639	0.361	0.959	0.041
TARMAC	0.000	1.000	0.000	1.000
TATE & LYLE	0.668	0.332	0.778	0.222
TAYLOR WOODROW	0.981	0.019	1.000	0.000
TI GROUP	0.563	0.437	0.664	0.336
TRANSPORT DEVELOPMENT GROUP	0.661	0.339	0.736	0.264
UNILEVER	1.000	0.000	1.000	0.000
UNITED BISCUITS HOLDINGS	0.000	1.000	0.000	1.000
WHITBREAD	0.806	0.194	1.000	0.000
WIMPEY (GEORGE)	1.000	0.000	1.000	0.000
WOLSELEY	0.000	1.000	0.092	0.908

Results interpretation: Below the log critical level of uncertainty, $u<u_{logcritical}$, all the log present values are greater than the initial outlay p_0 . Consequently, $l\tilde{p}v(\alpha)>p_0$ for $\alpha_{logcritical}<\alpha\leq1$ and the project is profitable. At levels equal or above $u_{logcritical}$, the project may be unprofitable. We have not further increased the uncertainty modelled in the t6 data, only introduced a variable discount rate, but the comparison between the constant t6 and the time-varying t6 results reveals characteristics similar to an increased-market-uncertainty case: $u_{logcritical} \leq u_{t6critical} \leq u_{Ncritical}$ and $\alpha_{logcritical} \geq \alpha_{t6critical} \geq \alpha_{Ncritical}$. The chance of a project being unprofitable now occurs at lower levels of uncertainty embodied in the data and the investment risk is higher.

When one assess the projects under t6 calibration and a time-varying discount rate, he or she approaches the real market conditions. This allows an improved evaluation of the investment risk and its robustness. Table 6 includes companies with $\alpha_{logcritical}<1$ and repeats the procedure from Table 4 with the new results. The projects are first ordered according to their investment risk in the time-varying case, then the arrangement is refined corresponding to the robustness indicator. Under the new circumstances, we slightly change the qualifying conditions. Now, $0 \leq \Delta\alpha_{log}=\alpha_{logcritical}-\alpha_{Ncritical}<0.1$ represents highly robust projects, $0.1 \leq \Delta\alpha_{log}<0.3$ indicates investments with medium risk robustness, $0.3 \leq \Delta\alpha_{log}<0.6$ implies investments with low risk robustness, and $0.6 \leq \Delta\alpha_{log}$ reveals no robustness at all.

Table 6: ***Robustness of profitable projects***

company	*investment risk*	$\alpha_{logcritical}-\alpha_{Ncritical}$	*robustness*
BLUE CIRCLE INDUSTRIES	0.000	0.000	high
BOC GROUP	0.000	0.000	high
COATS VIYELLA	0.000	0.000	high
LEX SERVICE	0.000	0.000	high
TARMAC	0.000	0.000	high
UNITED BISCUITS HOLDINGS	0.000	0.000	high
WOLSELEY	0.092	0.092	high
HANSON	0.673	0.166	medium
NORTHERN FOODS	0.656	0.317	low
BBA GROUP	0.696	0.451	low
TI GROUP	0.664	0.664	none
RMC GROUP	0.739	0.260	medium
SCOTTISH & NEWCASTLE	0.775	0.288	medium
TATE & LYLE	0.778	0.588	low
TRANSPORT DEVELOPMENT GROUP	0.736	0.736	none
SMITHS INDUSTRIES	0.959	0.375	low
GOODWIN	0.925	0.504	low
BP AMOCO	0.904	0.904	none
PILKINGTON	0.919	0.893	none

A major aspect of the fuzzy criterion is its ability to identify projects with a small and highly robust investment risk. According to the above table, the best companies to invest in are BLUE CIRCLE INDUSTRIES, BOC GROUP, COATS VIYELLA, LEX SERVICE, TARMAC and UNITED BISCUITS HOLDINGS. The projects are profitable at any level of uncertainty and the investment risk is zero. Moving on, some risk is involved, and it is where the risk robustness meters. Some companies may look well at first glance, but investigating the risk robustness one reveals how the corresponding projects change from definitely or almost gainful to almost unprofitable. This is the case with BP AMOCO, PILKINGTON, TRANSPORT DEVELOPMENT GROUP and TI GROUP, and more or less with TATE & LYLE and BBA GROUP. On the other hand, HANSON, RMC GROUP and SCOTTISH & NEWCASTLE, although still risky, are quite robust and much preferable. Figures 9 to 12 illustrate each range of the robustness scale. Do not forget that the standard method will be in favour of all projects in Table 6, as they have a positive crisp net present value and may be a high one. So the standard procedure will not quite distinguish between eventual investments and will not inform investors about hidden risks.

Fig. 9: BLUE CIRCLE INDUSTRIES

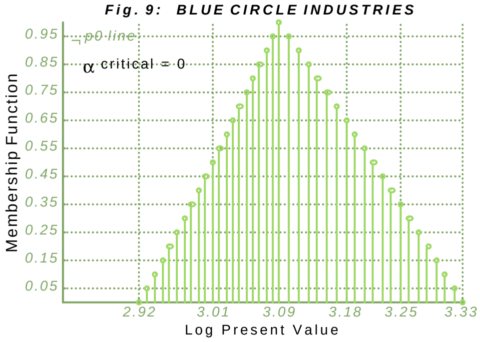


Fig. 10: SCOTTISH & NEWCASTLE

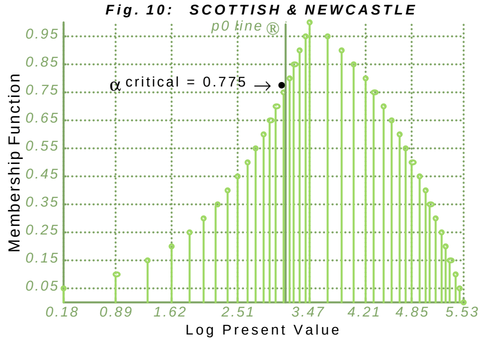


Fig. 11: BBA GROUP

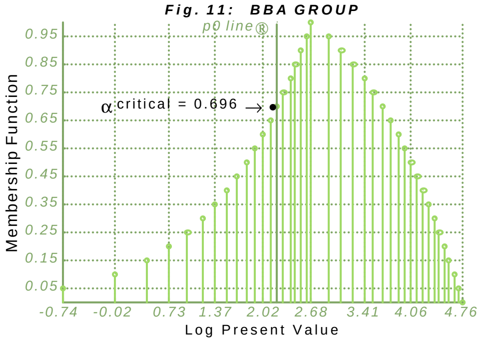
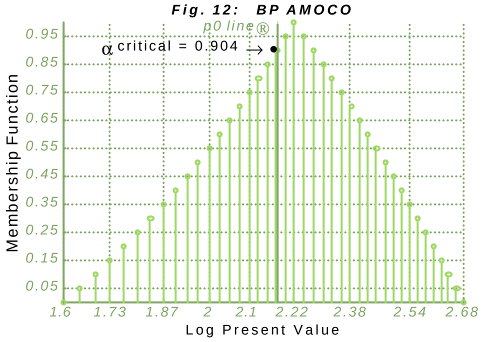


Fig. 12: BP AMOCO

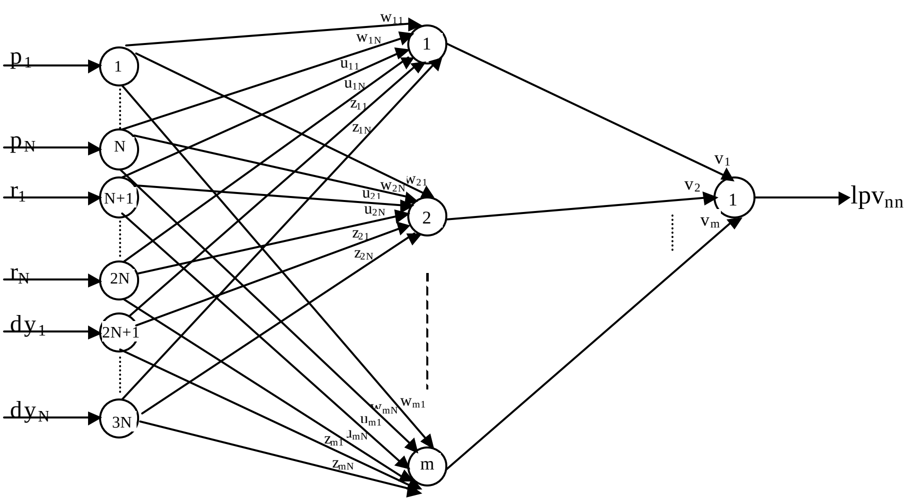


6. Using neural networks to evaluate the fuzzy criterion

The objective is to train a neural network to evaluate investment projects according to the fuzzy present value criterion. We apply a technique for solving fuzzy equations and evaluating fuzzy expressions using neural networks suggested by Buckley, Eslami and Hayashi [11] and further explained in Buckley and Feuring [12]. The time-varying-rate case is considered, as it is the most adequate one, and the three-layer feedforward neural net to be employed is presented in Figure 13. Its output for bias terms θ_j, sigmoidal transfer functions $g(x)=(1+e^{-x})^{-1}$ and weights w_{ji},u_{ji},z_{ji},v_j is

$$lpv_{nn}=\sum_{j=1}^{m} v_j g\left(\sum_{i=1}^{N}(w_{ji}p_i+u_{ji}r_i+z_{ji}dy_i)+\theta_j\right). \tag{15}$$

Fig. 13: ***Neural net architecture to solve the log present value problem***



The input neurons distribute the inputs - log share prices $p_1,\ldots,p_n$, log discount rates $r_1,\ldots,r_n$, and log dividend yields $dy_1,\ldots,dy_n$ - to the neurons in the second layer. N is the length of the projects. The input to node one in the second layer is $\sum_{i=1}^{N}(w_{1i}p_i+u_{1i}r_i+z_{1i}dy_i)+\theta_1$ with output $g\left(\sum_{i=1}^{N}(w_{1i}p_i+u_{1i}r_i+z_{1i}dy_i)+\theta_1\right)$. This is similar for nodes number 2 through m. Therefore, the input to the output node, which is the same as its output lpv_{nn}, is given in (15). For each project the net is to be trained, using values of p_i in the interval $[\underline{p_i(0)},\overline{p_i(0)}]$, r_i in $[\underline{r_i(0)},\overline{r_i(0)}]$ and dy_i in $[\underline{d_i(0)},\overline{d_i(0)}]$, so that

$$lpv_{nn}\approx lpv=\sum_{i=1}^{N}\rho^{i-1}\,[(1-\rho)(dy_i+p_i)+k-r_i]+\rho^n p_n\,. \qquad (16)$$

The training set consists of 362 vectors and the goal is a mean square error of 0.0001. We use the Neural Network Toolbox 3.0 for MATLAB 5.2[6] and choose the training function `trainlm` based on the Levenberg-Marquart technique, because it is the fastest backpropagation algorithm available. All the input vectors in the training set are presented to the network concurrently, as batching of concurrent inputs is computationally more efficient. The Levenberg-Marquart algorithm has higher storage requirements than the other training functions and to avoid running out of memory, we choose N=12. The same thirty-five companies are evaluated but now in one-year projects, from January 1999 till January 2000. After training, we will input the α-cuts of the triangular-shaped fuzzy numbers $\tilde{p}_i$, $\tilde{r}_i$ and $\tilde{dy}_i$, and perform interval arithmetic within the net. Thus, the output will be the corresponding α-cut of the triangular-shaped $l\tilde{p}v_{nn}$. In order the fuzzy output $l\tilde{p}v_{nn}$ to be an approximation to the solution $l\tilde{p}v_{fn}$ described earlier, certain sign restrictions have to be introduced on the weights in the net. As $\partial lpv/\partial p_i>0$, $\partial lpv/\partial dy_i>0$ and $\partial lpv/\partial r_i<0$, then the weight constraints should produce $\partial lpv_{nn}/\partial p_i>0$, $\partial lpv_{nn}/\partial dy_i>0$ and $\partial lpv_{nn}/\partial r_i<0$. (See [11,12].) Any of the following two sets of sign constraints will be satisfactory.

$$w_{ji}\geq 0,\ u_{ji}<0,\ z_{ji}\geq 0,\ v_j\geq 0,\qquad 1\leq i\leq N,\ 1\leq j\leq m, \qquad (17a)$$

$$w_{ji}<0,\ u_{ji}\geq 0,\ z_{ji}<0,\ v_j<0,\qquad 1\leq i\leq N,\ 1\leq j\leq m. \qquad (17b)$$

Figures 14 to 17 illustrate the $l\tilde{p}v_{fn}$ solution for four of the one-year projects and one may verify that the fuzzy log present values for all the companies are positive in January 1999. (See [35].) Consequently, we will apply (17a), because the constraint $v_j\geq 0$ provides for a nonnegative output lpv_{nn}.

Fig. 14: BLUE CIRCLE INDUSTRIES

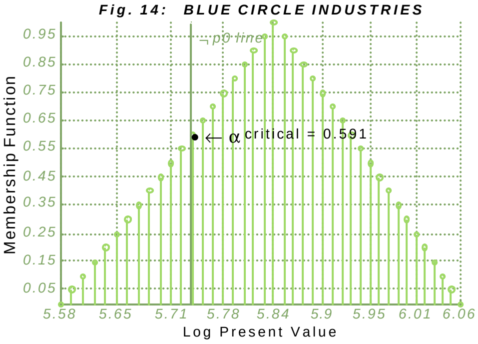


Fig. 15: SCOTTISH & NEWCASTLE

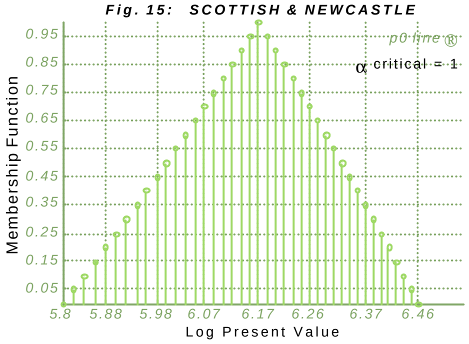


Fig. 16: BBA GROUP

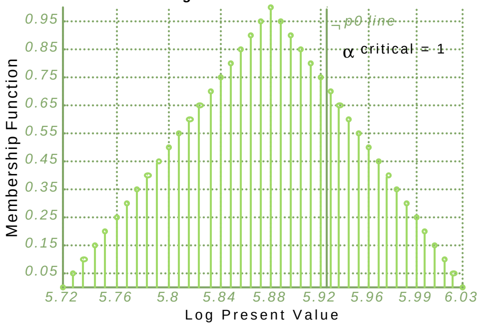
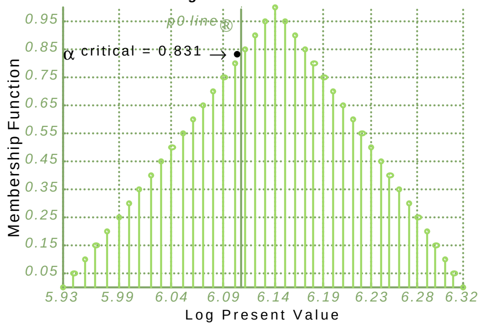


Fig. 17: BP AMOCO



The input of $[\underline{\tilde{p}_i}(\alpha), \overline{\tilde{p}_i(\alpha)}]$, $[\underline{\tilde{r}_i}(\alpha), \overline{\tilde{r}_i(\alpha)}]$ and $[\underline{\tilde{dy}_i}(\alpha), \overline{\tilde{dy}_i(\alpha)}]$ produces the output

$$[\underline{l\tilde{p}v_{nn}}(\alpha), \overline{l\tilde{p}v_{nn}(\alpha)}] = \sum_{j=1}^{m} v_j g\left(\sum_{i=1}^{N}\left(w_{ji}[\underline{\tilde{p}_i}(\alpha), \overline{\tilde{p}_i(\alpha)}] + u_{ji}[\underline{\tilde{r}_i}(\alpha), \overline{\tilde{r}_i(\alpha)}] + z_{ji}[\underline{\tilde{dy}_i}(\alpha), \overline{\tilde{dy}_i(\alpha)}]\right) + \theta_j\right) =$$

$$= \sum_{j=1}^{m} v_j g\left(\sum_{i=1}^{N}\left([w_{ji}\underline{\tilde{p}_i}(\alpha), w_{ji}\overline{\tilde{p}_i(\alpha)}] + [u_{ji}\overline{\tilde{r}_i(\alpha)}, u_{ji}\underline{\tilde{r}_i}(\alpha)] + z_{ji}[z_{ji}\underline{\tilde{dy}_i}(\alpha), z_{ji}\overline{\tilde{dy}_i(\alpha)}]\right) + \theta_j\right) =$$

$$= \sum_{j=1}^{m} v_j g\left(\left[\sum_{i=1}^{N}\left(w_{ji}\underline{\tilde{p}_i}(\alpha) + u_{ji}\overline{\tilde{r}_i(\alpha)} + z_{ji}\underline{\tilde{dy}_i}(\alpha) + \theta_j\right), \sum_{i=1}^{n}\left(w_{ji}\overline{\tilde{p}_i(\alpha)} + u_{ji}\underline{\tilde{r}_i}(\alpha) + z_{ji}\overline{\tilde{dy}_i(\alpha)} + \theta_j\right)\right]\right) =$$

$$= \left[\sum_{j=1}^{m} v_j g\left(\sum_{i=1}^{N}\left(w_{ji}\underline{\tilde{p}_i}(\alpha) + u_{ji}\overline{\tilde{r}_i(\alpha)} + z_{ji}\underline{\tilde{dy}_i}(\alpha) + \theta_j\right)\right), \sum_{j=1}^{m} v_j g\left(\sum_{i=1}^{N}\left(w_{ji}\overline{\tilde{p}_i(\alpha)} + u_{ji}\underline{\tilde{r}_i}(\alpha) + z_{ji}\overline{\tilde{dy}_i(\alpha)} + \theta_j\right)\right)\right], \tag{18}$$

since (17a) holds and g is monotonically increasing and positive. The Neural Network Toolbox 3.0 allows customisation of many of the functions, thus giving the user control over the initialising, simulating and training algorithms. We have modified `trainlm` to provide the satisfaction of the sign constraints. Next, the net is trained to approximate (16) and one obtains the following result, having in mind (16) and (18).

$$[\underline{l\tilde{p}v_{nn}}(\alpha), \overline{l\tilde{p}v_{nn}(\alpha)}] \approx \left[\sum_{i=1}^{N}\rho^{i-1}\{(1-\rho)(\underline{\tilde{dy}_i}(\alpha) + \underline{\tilde{p}_i}(\alpha)) + k - \overline{\tilde{r}_i(\alpha)}\} + \rho^N\underline{\tilde{p}_N}(\alpha), \sum_{i=1}^{N}\rho^{i-1}\{(1-\rho)(\overline{\tilde{dy}_i(\alpha)} + \overline{\tilde{p}_i(\alpha)}) + k - \underline{\tilde{r}_i}(\alpha)\} + \rho^N\overline{\tilde{p}_N(\alpha)}\right]$$

After training, we simulate the net for each project using 39 test vectors, while no element of the training set is included in the test set. For all companies, $\max_j |net_j - target_j| = \max\left(\max_\alpha|\underline{l\tilde{p}v_{fn}}(\alpha) - \underline{l\tilde{p}v_{nn}}(\alpha)|, \max_\alpha|\overline{l\tilde{p}v_{fn}(\alpha)} - \overline{l\tilde{p}v_{nn}(\alpha)}|\right) \le 0.021$, $1 \le j \le 39$, where $\alpha \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1\}$. It is a good approximation, and we may conclude that $l\tilde{p}v_{nn} \approx l\tilde{p}v_{fn}$. In fact, in most cases $\max_j |net_j - target_j| \le 0.01$. (See Table 7.)

Results interpretation: Let us consider the situation when an investment decision has to be taken within certain period of time. First, fuzzy data are modelled using the information available at the beginning of the period and a neural network is trained to approximate the fuzzy log present value of the project. Next, the decision-maker is provided with the trained network and at any moment he or she acquires new information, the net is simulated with modified inputs. For example, he or she has just confirmed the news that a company is willing to approve a merge considered recently

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[6] The programmes we use in Sections 2, 3 and 4 are also realised in MATLAB and presented in [35].

Table 7: ***Neural net performance***

company	*mse (training),* 10^{-5}	$\max_j \lvert net_j - target_j \rvert$ *(test)*
BASS	9.87969	0.0129
BBA GROUP	8.76903	0.0099
BENTALLS	8.62142	0.0096
BLUE CIRCLE INDUSTRIES	1.40298	0.0086
BOC GROUP	7.44909	0.0095
BOOTS CO.	0.09898	0.0018
BP AMOCO	0.35005	0.0019
BRITISH AMERICAN TOBACCO	3.49424	0.0062
BUNZL	3.29825	0.0098
COATS VIYELLA	9.66064	0.0210
DIXONS GROUP	2.35558	0.0055
GOODWIN	9.58095	0.0210
GREAT UNIVERSAL STORES	6.08490	0.0137
HANSON	0.53559	0.0036
INCHCAPE	9.38748	0.0139
LEX SERVICE	0.12500	0.0014
MARKS & SPENCER	2.59878	0.0115
NORTHERN FOODS	0.19096	0.0021
PILKINGTON	1.61479	0.0067
RANK GROUP	6.51450	0.0123
RMC GROUP	1.48070	0.0068
SAINSBURY (J)	3.15770	0.0063
SCOTTISH & NEWCASTLE	9.55638	0.0141
SMITH (WH) GROUP	1.91102	0.0089
SMITHS INDUSTRIES	4.23372	0.0186
TARMAC	9.80010	0.0141
TATE & LYLE	0.69747	0.0047
TAYLOR WOODROW	4.53601	0.0140
TI GROUP	3.84633	0.0125
TRANSPORT DEVELOPMENT GROUP	0.19929	0.0039
UNILEVER	1.20821	0.0054
UNITED BISCUITS HOLDINGS	9.67377	0.0110
WHITBREAD	9.25426	0.0208
WIMPEY (GEORGE)	2.33830	0.0152
WOLSELEY	0.58552	0.0048

and this will shift its share price from month x onwards of the investment project. If the fuzzy data initially modelled only moderate possibility of a merge sometime during the project, now they should present high merge possibility and a price shift in month x. In result, the data are less uncertain, the new neural-net output gives a tolerable investment risk and the project is accepted.

7. Future research

The fuzzy NPV criterion involves evaluation of the present value of a stream of uncertain future cash flows using a constant or time-varying uncertain discount rate. Then the resultant fuzzy value is compared with the known crisp initial outlay, thus producing an estimate of the investment risk. If one considers investing in the stock market now, then the initial outlay is equal to the current share price and the uncertain future cash flows comprise future dividends and the final price. Apparently, the present value of the future dividends and the final price produces a current-price estimate. Consequently, the method includes fuzzification of an asset pricing technique. We focused earlier on the present value calculations and used a fuzzy discount rate constructed from the prevailing risk-free rate by making allowances for a risk premium and shifts in response to shocks in some economic indicators. However, the general procedure can be applied to any asset pricing model.[7] In this context, it is relevant to explore briefly the grounds for and the influence of the fuzzification of the capital asset pricing model (CAPM).

Conventionally, the CAPM is formulated as $E[R_i]=R_f+\beta_i(E[R_M]-R_f)$, where $E[R_i]$ is the expected rate of return on asset i, R_f is the risk-free rate of return, β_i stands for the asset's beta, and $E[R_M]$ denotes the expected rate of return on the market portfolio. One will instantly notice that the model only holds under certain assumptions. In his seminal article [36], Sharpe admits that the assumptions are 'highly restrictive and undoubtedly unrealistic'. They include the following: (i) all investors have homogenous expectations for the returns on the available assets and there exists a common joint normal probability distribution, (ii) all investors are risk averse and maximise the expected utility of their end-of-period wealth, (iii) any investor can take a long or short position of any size in any asset and can borrow or lend any amount at the same risk-free interest rate. Each presumption is prone to criticism and a number of CAPM extensions have been developed in order to relax some of the restrictions. Starting with (i), Lintner [23] shows that under heterogeneous investor expectations the expected returns and covariances in the standard CAPM can be expressed as complex weighted averages of investor expectations. Fama [19] investigates the empirical distribution of daily returns on NYSE and finds it symmetrical but fat-tailed and with no finite variance. The conclusion is that investors can use some measures of dispersion rather then the variance. Moving to the second assumption, Merton deals in [25] with the static single-period nature of the model and suggests that under continuous trading the returns and the changes in the opportunity set can be described by continuous-time stochastic processes. He derives a three-fund separation theorem, where the return on the third fund is perfectly negatively correlated with changes in the interest rate and allows investors to hedge against unfavourable intertemporal shifts in the risk-free rate and correspondingly in the opportunity set. Finally, the third assumption is often considered most restrictive. Black [4] derives a new structure of the model, first with no risk-free asset, and then under restricted risk-free borrowing. In the former case, he uses the expected return on the minimum-variance zero-beta portfolio $E[R_z]$ instead of the return on the risk-free asset, while in the latter case the derived model includes both $E[R_z]$ and R_f. In conclusion, the CAPM defenders argue that the simplifications are acceptable for the cause of quantifying and pricing the risk, they claim that the model is 'approximately true' and predicts 'relatively well'. However, one should not forget that the CAPM is derived in a hypothetical world and a number of empirical studies have already rejected it [2,3,24,32]. Finally, in [33] Roll concludes that the model is non-testable, as the market portfolio is unobservable.

The above review reveals the demand for relaxed assumptions and the need for bringing the standard CAPM closer to reality. We find that in general such a need appeals for another approach to modelling the market environment. Instead of simplifying the way the real market works and getting results perfectly logical in theory but questionable in

[7] See [35] for an extended discussion on the grounds for fuzzification of the capital asset pricing model and the arbitrage theory pricing approximation.

practice, one can find a representation of market data incorporating as much uncertainty as the real environment possibly embodies and work out a solution based on real not abstract data. Thus, the data will not be bound by normal probability distributions and the calculations will involve all forms of potential uncertainty (compare the mean-variance case). As a result, the necessity for market-behaviour assumptions will significantly diminish. Fuzzy sets have proved to provide exceptionally adequate description of real-world data. For example, a fuzzy risk-free rate will reflect the fact that varied investors can borrow at varied risk-free rates and will indicate that the borrowing and the lending risk-free rates may differ for the same investor. It will communicate to some extent that the risk-free resource is not fully divisible or unrestrictedly available and will also help in coping with the single-period nature of the model by making allowances for intertemporal shifts in the risk-free rate. Further, a fuzzy beta and a fuzzy market return will reveal the heterogeneous investor expectations and will critically weaken the requirement for a common joint normal probability distribution. They will also express to some degree the reality that the market portfolio is unobservable and not all the assets are fully divisible or available. Finally, they will make allowances for discrepancies between ex ante and ex post returns.

The need for classical model revision and the grounds for new types of models are well recognised. The mathematics underlying the standard financial techniques neglects extreme situations and regards large market shifts as too unlikely to matter. Such techniques may account for what occurs most of the time in the market, but the picture they presents does not reflect the reality, as the major events happen in the rest of the time and investors are 'surprised' by 'unexpected' market movements. The fuzzy approach allows for market fluctuations well beyond the probability type of uncertainty permitted by the standard financial methods. It does not impose predefined data or market behaviour, there is only an attempt to model as much uncertainty as the environment can possibly embody, thus producing better estimates of the investment risk.

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