Fundamentalists heterogeneity and the role of the sentiment indicator

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Abstract

This paper is a contribution to the literature on the role of the sentiment indices in heterogeneous asset pricing models. We propose a new sentiment index in a financial market where we assume that transactions take place between two groups of fundamentalists that differentiate on the perception of the fundamental value. We assume that the fraction of fundamentalists in the two groups depends on the sentiment index. After studying the analytical properties of the deterministic discrete dynamical system we compare the new index with a previous index introduced in financial literature. For this purpose, by adding stochastic components to the fundamentalist' demands, we measure the performance of our model under different sentiment indices and we test its explanatory power to reproduce the stylized facts of financial data relying on the S&P500 index.

Keywords: Sentiment index; Market risk; Stochastic dynamic; Monte Carlo simulations

1   Introduction

Several scientific works have found that stock returns in financial markets exhibit some empirical regularities such as volatility clustering, asymmetry and excess of kurtosis (see Cont (2001) and Lux and Marchesi (2000) for example). In order to replicate these stylized facts many financial models have been developed, in particular, we refer to the models with heterogeneous agents(HAM) (see for example Brock and Hommes (1997), Chiarella *et al.* (2014), He and Li (2015)). The peculiarity of these kind of models is that the dynamic of a generic financial asset price comes from the interaction of agents with different attitudes to risk and expectation on the future value of the price. The interaction of agents is due to their adaptation of beliefs over time by choosing from different expectations functions, depending on past performance (realized profits). Recently, a number of empirical studies and financial experiments have shown that investor sentiment plays a significant role in the asset pricing. According to Corredor *et al.* (2013) investor sentiment can be defined as investor opinion, influenced by emotion, about future cash flows and investment risk. In particular, this field of research demonstrates that the stock excess returns

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are not easily explained by their fundamentals, and further demonstrates that the stock excess returns are affected by investor sentiment (Baker and Wurgler (2006), Baker *et al.* (2012), Chau *et al.* (2016), Yang and Zhou (2015)).

In line with the aforementioned literature on the role of the sentiment index, we aim to introduce a new sentiment index in a financial market model with two groups of fundamentalists traders which are homogeneous in their trading strategy, but they have heterogeneous beliefs about the fundamental value of the asset, and a market maker who mediates transactions out of equilibrium. Usually, beside fundamentalists, HAMs consider chartists or noisy traders showing that they are the main cause of complex dynamics in the model. Indeed, as suggested by Black (1986) and De Long *et al.* (1990), when investors trade unrelated to the fundamental, then asset prices will deviate from their intrinsic value. Differently, we consider only fundamentalists in our model and we assume that type-1 fundamentalists believe that the fundamental value (F_1) is greater than the fundamental value (F_2) assumed by type-2 fundamentalists. Therefore, when price is above the two fundamentals, both types of fundamentalists submit selling orders but type-2 fundamentalists sell more aggressively than type-1 fundamentalists. Conversely, when the price is below the two fundamental values, type-1 fundamentalists buy more aggressively than type-2 fundamentalists. Others few works consider models with only fundamentalists, it is the case for example of Naimzada and Ricchiuti (2009), De Grauwe and Kaltwasser (2012), Hommes (2013), and Cavalli *et al.* (2018). They suppose that one fundamentalist underestimates the true fundamental (pessimist) and the other overestimates it (optimist). Based on the above considerations, we assume that agents are bounded rational and do not know the true fundamental value of the asset price but they attempt to forecast it. However, differently from the previous contributes, we assume only that the fundamental value estimated by type-2 fundamentalists is always lower than the fundamental value estimated by type -1 fundamentalists. As we will see in Section 3, this assumption leads to different scenario (as described by Campisi and Muzzioli (2020), as well). Indeed, if fundamentalists heavily underestimate (overestimate) the true fundamental, it is possible that both the estimated fundamental values are below (above) the true fundamental value.

We explore the role of the investor sentiment in the dynamic of a generic asset price in a financial market with two fundamentalist traders. To this purpose, we conduct our analysis in three main steps.
First, we introduce the sentiment index which will be responsible of the switching mechanism of agents. Second, we study the deterministic skeleton of the resulting one dimensional discrete dynamical system highlighting the main characteristics of the model both analytical and numerical. Moreover, we stress the main differences with the model analysed by Campisi and Muzzioli (2020) focusing on the different role played by the two sentiment indices introduced. Finally, we consider a stochastic version of our model and that of Campisi and Muzzioli (2020) adding a stochastic component in the demand of both types of fundamentalists in order to reproduce the stylized facts and to capture what sentiment index fits better the real data. For this purpose, we consider the daily data of the S&P500 index as benchmark for our analysis. In detail, in the third part we follow He and Li (2007) and He and Zheng (2016), focusing mainly on the statistical properties of the returns, the analysis of the volatility clustering and the leverage effect.

The remainder of the paper is organized as follows. Section 2 reviews some more literature. Section 3 describes the model and introduces the new sentiment index. Section 4 details the analytical and

numerical results of the deterministic component of the model in relation to the work of Campisi and Muzzioli (2020). In Section 5 a statistical analysis of the stochastic model is given, moreover a comparison between our model with that of Campisi and Muzzioli (2020) is provided in order to establish the best sentiment index in replicating the stylized facts of S&P500 index. Section 6 concludes.

2   Relation to prior research

Our work is related to the literature on sentiment index and heterogeneous agent models. Some research has focused on how investor sentiment could affect the financial market. In general, the role of the sentiment index in determining the stock prices has been analysed by two points of view: a theoretical point of view and an empirical one. The theoretical strand of research found that stock price and the trading decisions of investors are affected by noisy traders relying on unpredictable changes in sentiment (see De Long *et al.* (1990), Barberis *et al.* (1998)). In particular, De Long *et al.* (1990) stress that the effects of investor sentiment change according to the investors who dominate the market. Barberis *et al.* (1998) study a Markov-switching model with homogeneous traders analysing the effect of investor sentiment on asset price.

The empirical literature has analysed what effects the sentiment indices have on the stock price formation too. For this purpose, Yang and Zhou (2015) find that both the investor trading behavior and investor sentiment have significant effects on the formation of excess returns. Yang and Yan (2011) study a sentiment asset pricing model with homogeneous sentiment investors. The work of Chau *et al.* (2016) takes into consideration the role sentiment play in the trading behavior of investor in the U.S. stock market finding that it is an important determinant of stock price variation. A review of investor sentiment measures is provided by Zhou (2018). Moreover, sentiment indices have been examined in relation to their ability to predict returns, this is explored by Brown and Cliff (2004) and Corredor *et al.* (2013) for example. However, both points of views have employed two different frameworks to study the role of sentiment indices; one a linear framework (Baker and Wurgler (2006), Verma and Soydemir (2009))and a non-linear one (Ding *et al.* (2004), Boswijk *et al.* (2007), Jawadi and Prat (2012)). Along this second line, some studies have analysed the role of investor sentiment in stock price formation suggesting that investor sentiment is one of the main determinants of asymmetry in stock returns (Jawadi *et al.* (2018), Tsai (2017)).

This paper is also closely related to the current HAMs with respect to heterogeneity and switching behavior of agents. From the seminal works of Brock and Hommes (1997, 1998) many other researchers successfully attempted to model the price dynamics of a generic asset using an heterogeneous agent models (HAMs) framework in order to replicate the stylized facts observed in financial markets. The main idea behind these models is that price dynamics is the result of the interaction of different investors with different trading rules. In the Adaptive Belief System introduced by Brock and Hommes (1997, 1998), investors adapt their beliefs over time by choosing from different predictors of future values of endogenous variables. Beyond heterogeneity, a further assumption of HAMs is that traders are bounded rational, that is they do not possess all the relevant information to forecast the true fundamental value of the market. Instead, they rely on simple rules of thumb (see for example Chiarella *et al.* (2009), Colasante *et al.* (2017), Brianzoni and Campisi (2020)). Heterogeneity and bounded rationality introduce non-linearity

in the model allowing the model to attain the main stylized facts of financial markets. The non-linear component in the model could be added in the excess demand function of traders or in the fraction of investors who adopt different decision rules. The one-dimensional map studied in Day and Huang (1990) for example, is non-linear since fundamentalists become increasingly aggressive as the price runs away from its fundamental value. The model of Tramontana *et al.* (2009) consider a non-linear model in which the stock markets of two countries are linked via and with the foreign exchange market. The authors observe random switches between bull and bear markets. A mechanism of switching between predictors is used in a variety of HAMs, such as Brock and Hommes (1998), Chiarella *et al.* (2006), Lux and Marchesi (2000). A common feature shared by all these works consists in building small scale models representing markets populated by heterogeneous and bounded rational agents. The small scale property permits their analytical study that supported by numerical simulations allow them to successfully explain various market behaviour such as asset bubbles and market crashes, volatility clustering, leverage effect and various power law behavior (Franke and Westerhoff (2016), Lux and Alfarano (2016), He and Li (2015), He and Li (2007), Lux and Marchesi (2000)).

Combining these two strands of research, we analyse the effects of a non-linear sentiment index in a HAMs when the fractions of traders vary according to this sentiment index. In particular, our model incorporates an endogenous switching mechanism generated by the sentiment index able to generate complex bull and bear scenarios. However, given that we are dealing with sentiment index, we prefer to refer to greed and fear scenarios respectively in line with the terminology used for risk indices (Whaley (2000)).

3   The model

Our work aims at providing an analysis of a market populated by traders with different beliefs about the fundamental value. In particular, the model includes a *market-maker* whose role is to announce an execution price and to execute transactions, two fundamentalists who believe in reversion to the mean (i.e. they expect prices return towards fundamental value). As a result, fundamentalists place orders to buy when the price is below their expected price and they place orders to sell when the price exceeds their expected price. They form their expected price based on the differences between fundamental value and current market price, and adjust their expected price each period.

Naimzada and Ricchiuti (2009), De Grauwe and Kaltwasser (2012), Hommes (2013) analyse financial models where agents adopt the same trading strategies, according to a fundamental rule based on different fundamental values. In particular, they assume that fundamentalists do not know the true value of the fundamental and they attempt to estimate it. Moreover, they assume that one fundamentalist always underestimates the fundamental value (pessimist) while the other always overestimates it (optimist). Our model incorporate two fundamentalists, but, differently from the previous authors, we assume that type-2 fundamentalists only underestimate the fundamental value (F_2) with respect to type-1 fundamentalists (F_1), i.e. $F_2 < F_1$. Thanks to this assumption, our model could entail the case where both fundamentalists are optimists or pessimists as well. This implies that when price (P_t) is lower than F_2, type-2 fundamentalists overestimate the price less than type-1 fundamentalists. On the other hand, when price P_t is higher than F_1, type-2 fundamentalists underestimate the price P_t more than type-1
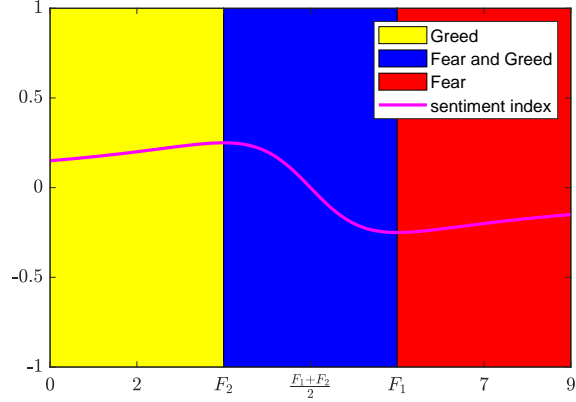
Figure 1: Fear and Greed predominance regions

fundamentalists. We assume that both types of fundamentalists have the same excess demand function:

$$D_t^{f_1} = \lambda(F_1 - P_t) \tag{1}$$

$$D_t^{f_2} = \lambda(F_2 - P_t) \tag{2}$$

where $D_t^{f_i}$ is the excess demand function for type-i fundamentalist; $i = 1, 2$, λ is a positive parameter and indicates how aggressively the fundamentalist reacts to the distance of the price to the corresponding fundamental value (F_1, F_2).

Within this framework, when the price is below F_2 type-2 fundamentalists buy less than type-1 fundamentalists. When price is above F_1, type-2 fundamentalists sell more than type-1 fundamentalists.

As in Campisi and Muzzioli (2020), depending on the price, we can distinguish the following fear or greed predominance regions (see Fig. 1):

a) when $P_t > F_1 > F_2$, in this case both fundamentalists sell, type-2 fundamentalists sell more than type-1 (Fear predominance region).

b) $F_2 < P_t < F_1$, type-2 fundamentalists sell whereas type-1 fundamentalists buy. This is similar to the bull and bear regime described in Day and Huang (1990) or Tramontana *et al.* (2009) (Fear and Greed mixed predominance region).

c) $P_t < F_2 < F_1$, both types of fundamentalists buy, but type-2 fundamentalists buy less than type-1 fundamentalists (Greed predominance region).

5

We assume that price adjustments are operated by a market-maker who knows the fundamental prices. The price-setting rule of the market-maker is given by:

$$P_{t+1} = P_t + \left(w_1 D_t^{f_1} + w_2 D_t^{f_2} \right) \tag{3}$$

where w_i is the proportion of fundamentalists of type i, $i = 1, 2$ and $w_1 + w_2 = 1$.

The number of traders is fixed and equal to $2N$, n_i is the number of fundamentalists of type $i = 1, 2$:

$$n_1 + n_2 = 2N \tag{4}$$

Therefore

$$w_1 = \frac{n_1}{2N} \qquad \text{and} \qquad w_2 = \frac{n_2}{2N} \tag{5}$$

The fractions n_1, n_2 of type-1 and type-2 fundamentalists, varies according to the following market sentiment index:

$$\eta_t = \frac{\left(\frac{F_1 + F_2}{2} \right) - P_t}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \tag{6}$$

The sentiment index measures in relative terms the distance between the price P_t and the mid value of the two fundamentals, F_1 and F_2. In particular the numerator measures how close the price is to the average fundamental value $\frac{F_1 + F_2}{2}$. This difference is normalized by the sum of the two distances (the denominator). Differently from Campisi and Muzzioli (2020), now the index is bounded between $[-0.25, 0.25]$ (see Fig.(1)). In particular, the index behaves similarly to that of Campisi and Muzzioli (2020) but now its extreme values are lesser than the fraction of agents, n_1, n_2. As a result, the switching mechanism is not only due to the information included in the index but it depends on the distance between the two fundamental values too. Indeed, it is possible that, for example, the index signals a buy strategy (greed scenario) but type 1 fundamentalists could not follow this strategy myopically. Moreover, the index at its extreme values, is characterized by a lower slope than the RAX index, as a consequence, it is more sensitive to small deviation of the price with respect to the fundamental values. After our evidence on the main characteristics of the new index, we now briefly recall its behavior which follows the RAX index of Campisi and Muzzioli (2020). In particular, the sentiment index is such that if the price is lower than F_2 it gradually returns to zero, since the buying behavior of both types of fundamentalists determines an overbought market. On the other hand, the closer price to the fundamental value F_2 the more the index is positive since for prices lower than F_2 both types of investors buy and for prices $P_t < \frac{F_1 + F_2}{2}$ and higher than F_2 type-1 investors buy with a higher intensity than the one that characterizes the selling

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behaviour of type-2 investors, yielding an overall buy signal. The sentiment index is such that if the price is higher than F_1 it gradually returns to zero, since the selling behavior of both types of fundamentalists determines an oversold market. Last, if $P_t = \frac{F_1 + F_2}{2}$ the sentiment index is equal to zero, indicating no buy or sell signal.

The number of type-1 and type-2 fundamentalists varies according to the sentiment index in the following way[1].

$$n_1 = n_2 - 2N\eta \tag{7}$$

therefore the number of type-1 fundamentalists is equal to the number of type-2 fundamentalists when $\eta = 0$. The number of type-1 fundamentalists is bigger than the number of type-2 fundamentalists when $\eta < 0$. The number of type-1 fundamentalists is lower than the number of fundamentalists of type 2 when $\eta > 0$. The closer the price to the fundamental value F_1 the more the proportion of type-1 fundamentalists increases since they performed better than the other group in forecasting the equilibrium price (and in the market we have an overall fear predominance since investors expect the price to decrease). On the other hand, the closer the price is to the fundamental value F_2 the more the proportion of type-2 fundamentalists increases since they performed better than the other group in forecasting the equilibrium price (and in the market we have an overall greed predominance since investors expect the price to increase).

The final map: Based on the above considerations, we obtain the first order nonlinear discrete dynamical equation which describes the price evolution over time. It takes the following form:

$$P_{t+1} = P_t + \left\{ \left[\frac{n_2}{2N} - \frac{(\frac{F_1 + F_2}{2}) - P_t}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \right] \lambda (F_1 - P_t) + \left[\frac{n_1}{2N} + \frac{(\frac{F_1 + F_2}{2}) - P_t}{(F_1 - P_t)^2 + (F_2 - P_t)^2} \right] \lambda (F_2 - P_t) \right\} \tag{8}$$

4   Dynamics of the deterministic model

In this section, we perform a stability analysis of map (8) in order to describe the local properties of the fixed points of the model under investigation. From straightforward computation, we can state the following.

Proposition 1 *Let $P_t > 0 \ \forall t$ and $F_1 > F_2$. The map(8) admits up to three fixed points*

$$P_1^* = \frac{F_1 + F_2}{2} - \frac{\sqrt{(F_1 - F_2)(F_2 - F_1 + 1)}}{2}$$
$$P_2^* = \frac{F_1 + F_2}{2} + \frac{\sqrt{(F_1 - F_2)(F_2 - F_1 + 1)}}{2}$$
$$P_3^* = \frac{F_1 + F_2}{2}$$
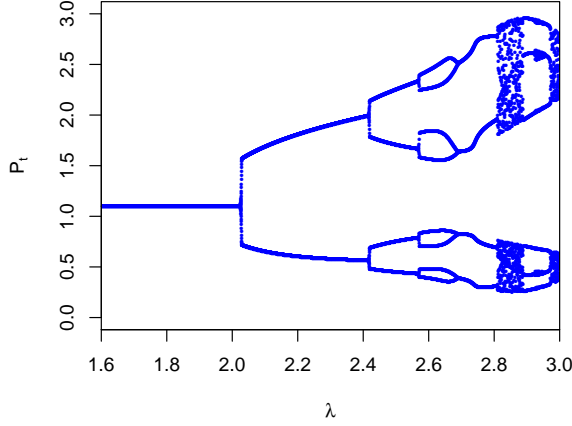
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[1]see Lux (1995)

Figure 2: Stability of fixed point P_3^* on varying the intensity of trading parameter λ in the interval $[1.6, 3]$ and other parameters fixed as follow: $n_2 = 0.2$, $F_1 = 3.2$, $F_2 = 1.2$, $N = 0.5$, i.c. $P_0 = 0.8$.

Moreover, the fixed points belong to the interval $[F_2, F_1]$. The fixed points (P_1^*, P_2^*) exist provided that

$$0 < F_1 - F_2 < 1$$

Note that, the fixed points (P_1^*, P_2^*) exist when the distance between the fundamental values of the two traders are not too far each other. In addition, we have three differences with respect to the model of Campisi and Muzzioli (2020). First, the fixed points of the two models are all different, moreover, in Campisi and Muzzioli (2020) there was always coexistence of attractors while in the present model coexistence comes only under some condition, i.e. lower degree of heterogeneity among fundamentalists. Third, under some condition on the reactivity parameter of fundamentalists (λ), the middle fixed point (P_3^*) can be stable. To establish the local stability of the fixed point P_3, we refer to the proposition below

Proposition 2 *Let $F_1 - F_2 > 1$ then the unique fundamental fixed point of map (8), P_3, is locally asymptotically stable provided that*

$$0 < \lambda < \frac{2}{F_1 - F_2 - 1}$$

For $\lambda = \frac{2}{F_1 - F_2 - 1}$, P_3 loses stability via flip bifurcation and two further stable fixed points arise.

Proof 1 *In order to analyse the local stability of the fixed point P_3^*, we need to calculate the derivative of price evaluated in the equilibrium. In our case, we find that the derivative in the fixed point P_3^* is equal to*

$$\frac{dP_{t+1}}{dP_t}(P_3^*) = 1 - \lambda + \frac{\lambda}{F_1 - F_2}$$

and applying the condition for the local stability

$$-1 < \frac{dP_{t+1}}{dP_t} < 1$$

8

we find the result of Proposition (2).

Fig. (2) shows qualitatively the results of Proposition (2). In particular, at the bifurcation point, $\lambda = \frac{2}{F_1 - F_2 - 1}$, the fixed point loses stability and a flip bifurcation occurs. Note that the case when $\lambda = 0$ is excluded by our analysis given that we assume $\lambda > 0$. As a further difference from the model of Campisi and Muzzioli (2020), we highlight that the two fixed points of the Proposition (2) are different from the fixed points (P_1^*, P_2^*) of Proposition (1). Instead, the assumptions made in Proposition (2) exclude the existence of (P_1^*, P_2^*). When there is lower degree of heterogeneity we lead to the same results of Campisi and Muzzioli (2020) that we brief recall for completeness of the analysis.

Property 1 *Assume $F_1 \neq F_2$. Let c_{min}, c_{max} be the local minimum and the local maximum of the Map (8); C_{min} and C_{max} are their iterates respectively, that is $C_{min} = P_{t+1}(c_{min})$ and $C_{max} = P_{t+1}(c_{max})$, then there exist two disjoint invariant[2] intervals $I = [P_{t+1}(c_{min}), P_{t+1}(C_{min})]$ and $J = [P_{t+1}(c_{max}), P_{t+1}(C_{max})]$ such that:*

1. ***Fear or Greed scenario:*** *if $P_{t+1}(C_{min}) < P_3^*$ and $P_{t+1}(C_{max}) > P_3^*$, then Map (8) has two coexistent attractors, P_1^* and P_2^*;*

2. ***Fear and Greed scenario:*** *for $P_{t+1}(C_{min}) = P_{t+1}(C_{max}) = P_3^*$ the two attractors merge and a contact bifurcation occurs.*

Our analysis enables us to summarize the more interesting cases with respect to the values of the parameters involved in our model. In particular, we resume all the possible cases in the following:

Proposition 3 *Consider map 8, moreover, let $F_1 > F_2$ and $\lambda > 0$, then:*

1. *Let $F_1 - F_2 > 1$, hence there exists one positive fixed point, $P_3^* = \frac{F_1 + F_2}{2}$ and it is globally asymptotically stable provided that $0 < \lambda < \dfrac{2}{F_1 - F_2 - 1}$,*

2. *Let $0 < F_1 - F_2 < 1$ then map 8 admits the three positive fixed points of Proposition 1. The fixed point P_3^* is always unstable and*

   (a) *If $F_1 - F_2$ is lower enough then we have coexistence between the fundamental prices P_1^*, P_2^* and the results of Property 1 hold,*

   (b) *Let F_1^{new}, F_2^{new} a new set of fundamental values then, if $F_1^{new} - F_2^{new} > F_1 - F_2$ the price converges to one of the two fixed points, P_1^*, P_2^*, and coexistence of prices is not possible.*

Concerning the local stability of the fixed points (P_1^*, P_2^*), we proceed via numerical simulations given that the analytical expression of the derivative in the two fixed points is hard to evaluate. For this purpose, we concentrate our attention in the cases of the main interest in order to make comparisons with the model of Campisi and Muzzioli (2020).

As we can see in Fig.(3), this case is new with respect to that of Campisi and Muzzioli (2020), it represents two scenarios where we do not assist to coexistence of attractors (this is the point 2(b) of Proposition 3). In Fig.(3 a-b) the trajectory of the price converges to the unique stable attractor in the fear region.

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[2]A set $I \subseteq \mathbb{R}_+$ is positively (negatively) invariant if $T^i(I) \subseteq I$ ($T^i(I) \supseteq I$) $\forall i \in \mathbb{Z}_+$. Moreover, I is invariant when it is both positively and negatively invariant. Finally, a closed and positively invariant region is called trapping.
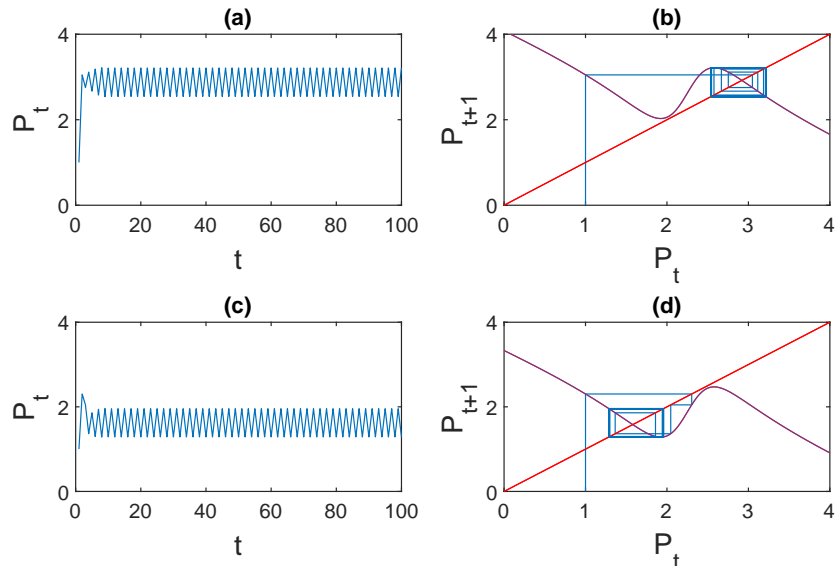
Figure 3: Symmetric convergence to Fear or Greed attractor for parameters $\lambda = 1.8$, $F_1 = 2.7$, $F_2 = 1.8$, $N = 0.5$, i.c. $P_0 = 1$. In (a) and (b) (fear scenario) for $n_2 = 0.73$ the price converges to the attractor P_2^*. While in (c) and (d) (greed scenario) for $n_2 = 0.27$ the price converges to the attractor P_1^*.

While in Fig. (3 c-d) the price converges to the stable attractor in the fear region. Note that the set of parameters is the same in both scenarios but the fraction of the two fundamentalists changes[3]. In addition, when a unique fixed point exists the dynamic of the map is symmetric. Figures (4) and (5) show two scenarios generated by our model when we assist to coexistence of attractors. In this case, the same dynamics of the model of Campisi and Muzzioli (2020) appear, that is, depending on the initial condition, P_0, the trajectory of price converges to the fear or greed attractor (Fig. (4(a-b) and (c-d), respectively). In Fig. (5) we find a homoclinic bifurcation. It is apparent the complex dynamics generated by our model when the two attractors merge giving rise to a unique complex attractor. An interesting situation is described in Fig. (6), where we report the different bifurcation scenario occurring. We choose the same set of colors used in Campisi and Muzzioli (2020) in order to facilitate comparisons between the two models. Our model differs with respect to that of Campisi and Muzzioli (2020) in all the three scenarios shown in Fig. (6). Indeed, both the fear and greed and the fear scenarios generate symmetric dynamics with respect the fraction of fundamentalists. The greater the proportion of one fundamentalist, the more complex the dynamic of the price (see Figure (6) fear and greed scenario and fear scenario). For what it concerns the greed scenario showed in Fig. (6), we note that it is very different with respect the scenario described in Campisi and Muzzioli (2020). In fact, in the greed scenario the model converges to a stable fixed point only if the fraction of one of the two fundamentalists is sufficiently large otherwise complex dynamics emerge. It is interesting to observe the dynamics in the fear scenario diagram of Fig.

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[3]For convenience, we represent all our simulation in terms of the number of type-2 fundamentalists without loss of generality. Indeed, a value of $n_2 = 0.27$ implies a value of $n_1 = 0.73$.
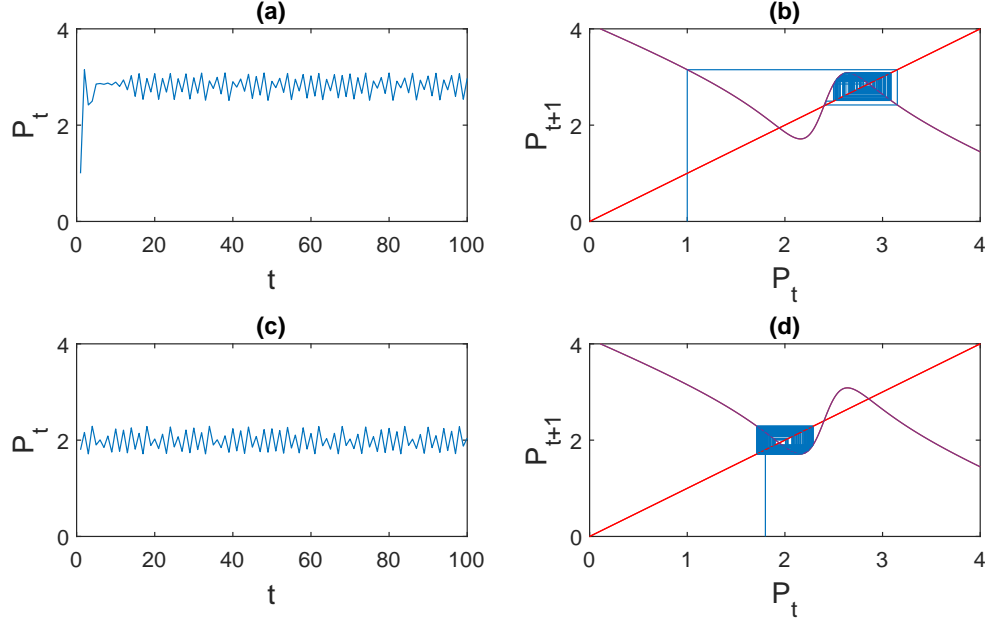
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Figure 4: Coexistence of attractors for parameters $\lambda = 1.8$, $F_1 = 2.7$, $F_2 = 2.1$, $N = 0.5$, $n_2 = 0.5$. In (a) and (b) (fear scenario) an i.c. $P_0 = 1.8$ generates a trajectory converging to the attractor P_2^* given by a stable two cycle. While in (c) and (d) (greed scenario) an i.c. $P_0 = 1$ generates a trajectory converging to the stable attractor P_1^* given by a stable two-cycle.

(6). It reproduces the same dynamics of the greed scenario until $n_2 \in [0.2, 0.8]$ while for values of n_2 out of this interval more complicated dynamics arise. Finally, concerning the fear and greed scenario, we note that it emerges when the distance between the two fundamental values is greater than the distance of the fundamental values both in the greed and the fear scenario.

To sum up, our model is able to generate different intricate dynamics of price, moreover in comparison with respect to the model of Campisi and Muzzioli (2020) there are substantially two main differences. First, the new map can exhibit coexistence of attractors and in this case the same dynamics of the map of Campisi and Muzzioli (2020) emerge. In addition, this model is able to reproduce scenario where there is existence of one attractor that can be globally asymptotically stable providing that conditions on parameters established in Proposition (2) hold. Second, the trajectory of the price is quite different in the three scenarios that the model can generate, that is, fear scenario, fear and greed scenario and greed scenario. In particular, both in the fear and greed scenario that the fear scenario exhibit symmetric dynamics of the price. Moreover for extreme values of n_2 (with slightly change in the value of the parameter of reaction of fundamentalists, λ) more intricate dynamics emerge. On the other hand, the greed scenario admits stability only when the proportion of one of the two fundamentalists is larger enough.

In the next section we consider a stochastic version of both models in order to compare their capability in reproducing the stylized facts of the $S\&P500$ index.
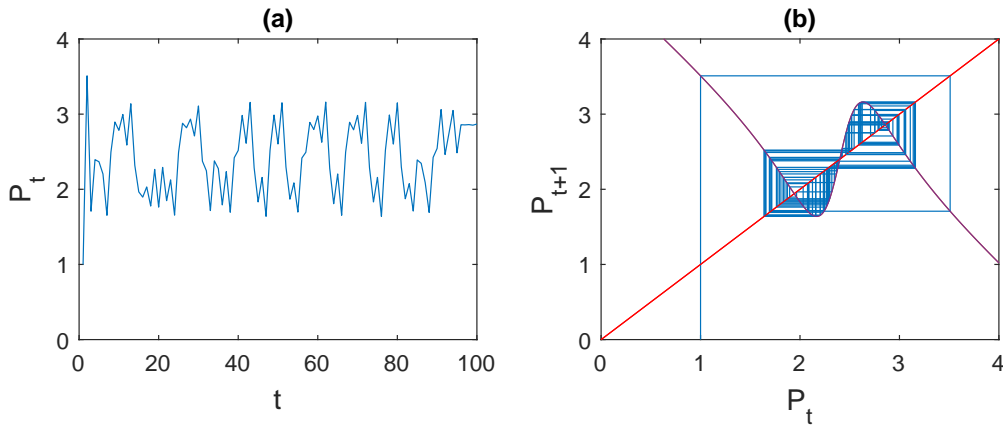
Figure 5: Homoclinic bifurcation scenario. In (a) and (b) two complex attractors merge for $n_2 = 0.5$, $N = 0.5$, $F_1 = 2.7$, $F_2 = 2.1$, $\lambda = 2.1$, i.c. $P_0 = 1$.
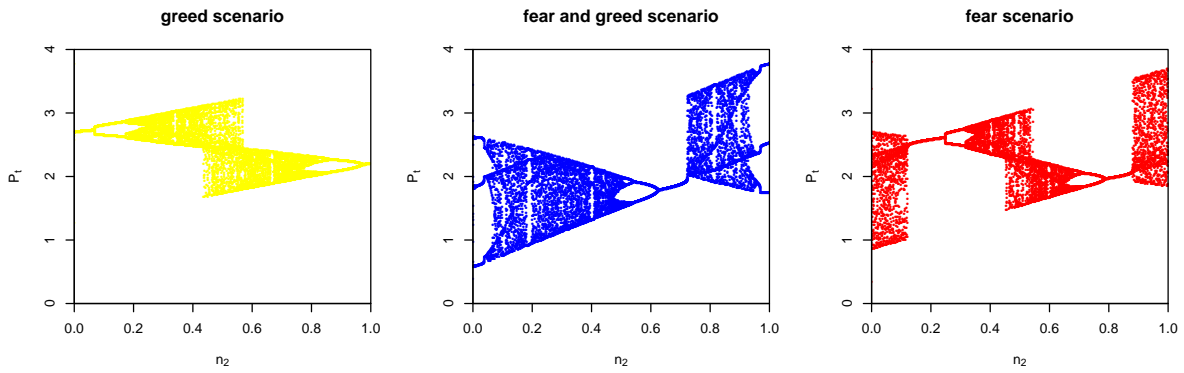

Figure 6: The three scenarios. Greed scenario (yellow diagram) with parameters $\lambda = 1.8$, $F_1 = 2.7$, $F_2 = 2.2$, $N = 0.5$, i.c. $P_0 = 1.2$. Fear and greed scenario (blue diagram) with parameters $\lambda = 2.1$, $F_1 = 2.6$, $F_2 = 1.75$, $N = 0.5$, i.c. $P_0 = 1.2$. Fear scenario (red diagram) with parameters $\lambda = 2.2$, $F_1 = 2.6$, $F_2 = 1.95$, $N = 0.5$, i.c. $P_0 = 1.2$.

5 Statistical properties of the stochastic model

After introduced the new sentiment index we test its performance in reproducing stylized facts in comparison with the RAX sentiment index of Campisi and Muzzioli (2020). The analysis performed in this section is inspired to the works of He and Zheng (2016), Huang *et al.* (2013). To this purpose, we add noise to each of the demand components and examine the dynamics of the stochastic model introduced in this work ($SM1$) and the stochastic model of Campisi and Muzzioli (2020) ($SM2$). To see how well the simulated data matches with the real data in terms of statistical and qualitative properties, we use the $S\&P500$ index ($S\&P500$) as empirical evidence and benchmark for our comparison. For our analysis we use 5464 daily observations which start from 1 June 1998 to 14 February 2020. Each simulation is run for 5464 time periods and the first 1000 are dropped to wash out the initial effect of the results. For comparison, the same set of parameters and noisy demand are used (see Table(1)).

5.1 Properties of the returns

Using P_t to denote the price index , we study the times series of price returns, defined as

$r_t = ln(P_t) - ln(P_{t-1})$

focusing on the deviation from normality and volatility of their distribution. In Table2 we provide the main summary statistic of the returns for all the models under investigation. As we can note, both $SM1$ and $SM2$ share some empirical facts of $S\&P500$ index. In particular, negative skewness and high kurtosis. Regarding skewness, empirical evidence shows an asymmetric relation between stock index returns and changes in the volatility, that is, negative returns have a greater impact on the volatility index than positive returns. High kurtosis implies that the distribution of the returns displays fat tails comparing with that of the normal distribution (which presents kurtosis equal to 3) and it causes that the returns are frequently far from their average. All these facts are captured by our model and that of Campisi and Muzzioli (2020) as shown in Table2.

In the next paragraph we analyse phenomenon of volatility clustering in more detail, but a simple inspection of Fig.7(a-c) reveals the presence of this stylized fact. Indeed, with volatility clustering returns display consecutive large volatility periods alternated with several consecutive periods characterized by small volatility. In Fig.7(d-f) we report the Q-Q plots of the three models and they describe that the returns are not normally distributed. In the Q-Q plot, indeed, we display the quantile of the sample data (returns) versus the theoretical quantiles of the Normal distribution. If the distribution of returns is Normal, then the plot appears to be linear, which is not the case in this instance. Moreover, the tails lay below and above the 45 degree line implying that their distributions are fat-tailed, as well (see Fig.7(g-i)). As a further complement, to confirm the non-normality of the returns we conduct the Jarque-Bera test (see Table2). In detail, we test the null hypothesis that returns follow a normal distribution at the 1% significant level. The returned value of $J - B = 1$ indicates that the Jarque-Bera test rejects the null hypothesis at the 1% significance level, additionally, the test statistic, J-B statistic, is greater than the critical value, which is 5.8461, indicating a rejection of the null hypothesis.

Table 1: Parameter setting and initial values

F_1	F_2	n_2	N	λ	ϵ_1	ϵ_2	P_0
2700	2000	0.5	0.5	0.1	0.1	0.1	1091

Table 2: Summary statistics of returns. The table reports the summary statistics including mean, standard deviation (sd), skewness, kurtosis, minimum and maximum value, Jarque-Bera test and statistic of $S\&P500$, $SM1$ and $SM2$.

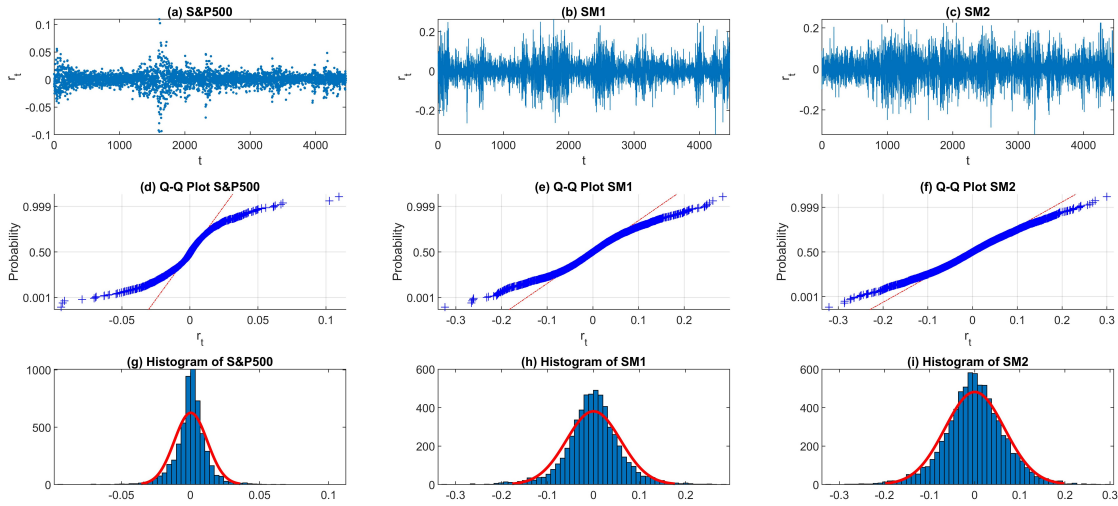|	Mean	sd	Min	Max	Skewness	Kurtosis	J-B	J-B statistic
S&P500	0.000207	0.0120	-0.0947	0.1096	-0.2374	11.0846	1	14932
SM1	0.000036	0.0589	-0.3240	0.2846	-0.0764	5.0633	1	974.53
SM2	0.000089	0.0661	-0.3209	0.2999	-0.0896	4.053	1	361.32

Figure 7: Returns series for $S\&P500$ (left panel), $SM1$ (middle panel) and $SM2$ (right panel)

5.2 Volatility clustering

In this section we intend to demonstrate that both models $SM1$ and $SM2$ exhibit volatility clustering and long-range dependence, in line with the evidence on $S\&P500$ index. In detail, in the presence of volatility clustering periods of quiescence and turbulence tend to cluster together (Lux and Marchesi (2000)). In order to show the existence of volatility clustering we first plot the autocorrelation functions (ACFs) of returns, absolute returns and squared returns for each model, that is for $S\&P500$ (Fig.8(a)), SM1 (Fig.8(b)) and SM2 (Fig.8(c)). As stated by Cont (2001), while returns themselves are uncorrelated, absolute returns $|r_t|$ or their squares display a positive, significant and slowly decaying autocorrelation function: $corr(|r_t|, |r_{t+q}|) > 0$ for q ranging from a few minutes to a several weeks, where q denotes the lags. In Panels (a)-(c) of Fig.(8) we can see that both $SM1$ and $SM2$, as the $S\&P500$, exhibit volatility clustering by plotting ACFs as a function of the number of lags, and $SM1$ fits better the ACFs of $S\&P500$ than $SM2$. In particular, there are no significant and decaying ACFs in returns but the ACFs of the absolute returns and squared returns are large and persistent even after 50 lags. Moreover, to confirm this stylized fact, we provide an estimate of the power component in the ACFs of absolute returns. Indeed, the pertinent literature converged on the insight that the empirical autocorrelation are similar to a power law, then, in addition we measure how fast the ACF of absolute returns decays following He and Zheng (2016), Huang *et al.* (2013) and Cont (2001):

$$corr(|r_t|, |r_{t+q}|) \simeq \frac{\zeta}{q^d} \tag{9}$$

where q is the number of lags, ζ is a parameter capturing the ACF of absolute returns with lag one, and d is the power exponent capturing the decay of the ACFs. We have estimated Eq.(9) with non-linear least square and the results of each model are shown in Table(3). As we can note, our estimates are in line with the empirical evidence which postulates a value of d in the interval $[0.2\,0.4]$.

As stressed by He and Zheng (2016), volatility clustering is an indicator of long memory but it does not necessarily lead to long memory. Long-range dependence or long memory, may be defined for stationary processes as the significant correlation of far distant observations (Hauser (1997)). To account for this other evidence, we approach the current analysis testing the hypothesis of long-range dependence in the volatility measured by the time series of $|r_t|$. To this end, we compute the Lo-modified range over standard deviation or R/S statistic (also called re-scaled range) (see Lo (1991)) given by:

$$\mathcal{Q}_n \equiv \frac{1}{\hat{\sigma}_n(q)} \left[\max_{1 \leq k \leq n} \sum_{j=1}^{j=k} (r_j - \bar{r}_n) - \min_{1 \leq k \leq n} \sum_{j=1}^{j=k} (r_j - \bar{r}_n) \right] \tag{10}$$

where $\hat{\sigma}_n(q)$ is the sample variance:

$$\hat{\sigma}_n(q) = \frac{1}{n} \sum_{j=1}^{j=n} (r_j - \bar{r}_n) + \frac{2}{n} \sum_{j=1}^{j=q} \left(1 - \frac{j}{q+1} \right) \sum_{i=j+1}^{i=n} (r_i - \bar{r}_n)(r_{i-j} - \bar{r}_n)$$

15

Table 3: Persistence of ACFs of absolute returns. For each return series in S&P500, SM1 and SM2, we estimate $corr(|r_{t+q}|,|r_t| \simeq \zeta/q^d$ with nonlinear least squares and report ζ and d.

	S&P500 $corr(|r_{t+q}|,|r_t|)$	SM1 $corr(|r_{t+q}|,|r_t|)$	SM2 $corr(|r_{t+q}|,|r_t|)$
d	0.3073***	0.3868***	0.3274***
	(0.0414)	(0.0412)	(0.0229)
ζ	0.4700***	0.3426***	0.1714***
N	200	200	200
R^2	0.7578	0.7009	0.6845
Root MSE	0.032	0.0281	0.0139

*** significant at 1%.

The first and second terms in the bracket of Eq.(10) are the maximum of the partial sums of the first k deviations of r_t from the sample mean $\bar{r}_t$ and the minimum of the same sequence of partial sums, respectively. Under the null hypothesis it is confirmed the long-range dependence otherwise it is rejected. In Fig.8(d), we report the R/S statistic for lags ranging from 1 to 100, and it is possible to see that we reject the null hypothesis of no long-range dependence when the number is not too large, i.e. $q \leq 50$.

5.3 Leverage effect

Empirical evidence converged on the insight of an asymmetric relation between stock index returns and changes in the volatility, that is, negative returns have a greater impact on the volatility index than positive returns. In particular, there are two main hypotheses that may explain this asymmetric relation: the leverage effect (Black (1976) and Christie (1982)) and feedback effect hypotheses (French *et al.* (1987), Campbell and Hentschel (1992), Bekaert and Wu (2000)). According to the leverage effect hypothesis, when returns are negative, the leverage ratio of the firm increases, and the firm's debt exceed total equity. As the equity of a firm is more exposed to the firm's total risk, the volatility of the equity should increase in turn. On the other hand, the feedback hypothesis states that the asymmetric relation is based on the change in conditional volatility, which implies a change in the stock market price (for further details see Badshah (2013) and Bekiros *et al.* (2017)). For this purpose, we assess the asymmetric returns-volatility relation following Cont (2001), Huang *et al.* (2013) and He and Zheng (2016) graphing the $corr(|r_{t+q}|^2, r_t)$ as a function of lag q in Fig.(9). In this case, we can see that volatility is negatively correlated with past return of the $S\&P500$ (Fig.9(a)), SM1 (Fig.9(b))and SM2 (Fig.9(c)), indeed the $corr(|r_{t+q}|^2, r_t)$ starts from negative values in all the cases showing that negative returns lead to a rise in volatility.

6 Conclusions

In this work we have analysed both analytically and numerically a financial market model with two types of fundamentalists. In particular, we have extended the model of Campisi and Muzzioli (2020) introducing a new sentiment index which is considered by investors for their trading decisions. Although the model takes into account rational traders, we assume that investors are not able to forecast the
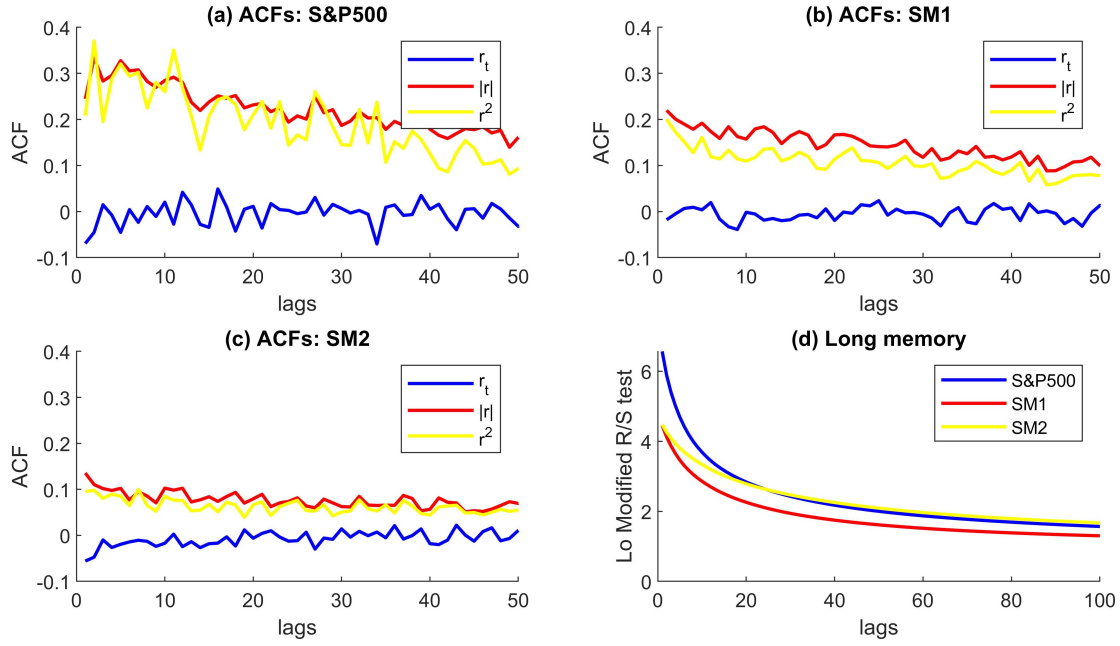
Figure 8: Volatility clustering and long-range dependence. Panels (a)-(c) plots the ACFs of returns (blue line), the absolute returns (red line) and the squared returns (yellow line) for S&P500, SM1 and SM2 respectively. Panel (d) plots the Lo modified R/S statistic of the absolute returns.
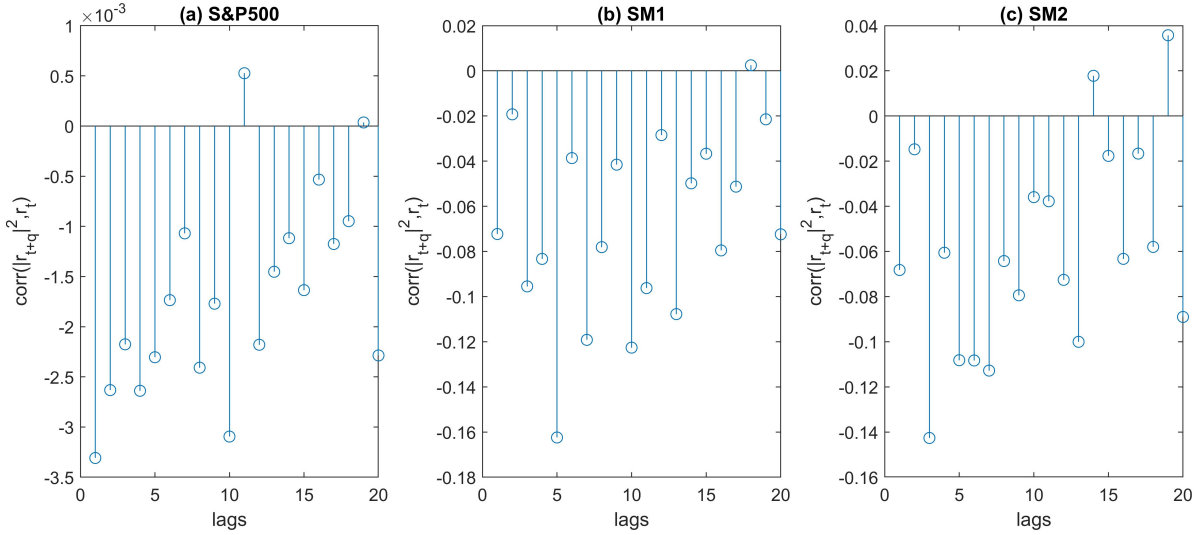

Figure 9: The leverage effect. This figure plots the degree of leverage effect measured by $corr(|r_{t+q}|^2, r_t)$.

fundamental value exactly, that is, we are assuming uncertainty about the true fundamental value. We have seen that the model is able to generate further dynamics with respect to that of Campisi and Muzzioli (2020), moreover when we introduce stochastic shocks, both models match the stylized fact observed on the $S\&P500$ index. In particular, after analysed the analytical part with the support of the numerical simulations, we have added a stochastic component in both the demand of fundamentalists. Then we focused to test some empirical evidence documented by financial literature, i.e. volatility clustering, kurtosis, asymmetry of returns, long-range dependence and leverage effect. We can conclude that the two sentiment indices considered in the two models under investigation are able to attained the global sentiment of the market and to address trading decisions of investors.

References

Badshah, I. U. (2013). Quantile regression analysis of the asymmetric return-volatility relation. *Journal of Futures Markets*, **33**(3), 235–265.

Baker, M. and Wurgler, J. (2006). Investor sentiment and the cross-section of stock returns. *The Journal of Finance*, **61**(4), 1645–1680.

Baker, M., Wurgler, J., and Yuan, Y. (2012). Global, local, and contagious investor sentiment. *Journal of Financial Economics*, **104**(2), 272–287.

Barberis, N., Shleifer, A., and Vishny, R. (1998). A model of investor sentiment. *Journal of Financial Economics*, **49**(3), 307–343.

Bekaert, G. and Wu, G. (2000). Asymmetric volatility and risk in equity markets. *The Review of Financial Studies*, **13**(1), 1–42.

Bekiros, S., Jlassi, M., Naoui, K., and Uddin, G. S. (2017). The asymmetric relationship between returns and implied volatility: Evidence from global stock markets. *Journal of Financial Stability*, **30**, 156–174.

Black, F. (1976). Studies of stock market volatility changes. *1976 Proceedings of the American Statistical Association Bisiness and Economic Statistics Section*.

Black, F. (1986). Noise. *The Journal of Finance*, **41**(3), 528–543.

Boswijk, H. P., Hommes, C. H., and Manzan, S. (2007). Behavioral heterogeneity in stock prices. *Journal of Economic Dynamics and Control*, **31**(6), 1938–1970.

Brianzoni, S. and Campisi, G. (2020). Dynamical analysis of a financial market with fundamentalists, chartists, and imitators. *Chaos, Solitons & Fractals*, **130**, 109434.

Brock, W. A. and Hommes, C. H. (1997). A rational route to randomness. *Econometrica: Journal of the Econometric Society*, pages 1059–1095.

Brock, W. A. and Hommes, C. H. (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, **22**(8-9), 1235–1274.

Brown, G. W. and Cliff, M. T. (2004). Investor sentiment and the near-term stock market. *Journal of Empirical Finance*, **11**(1), 1–27.

Campbell, J. Y. and Hentschel, L. (1992). No news is good news: An asymmetric model of changing volatility in stock returns. *Journal of Financial Economics*, **31**(3), 281–318.

Campisi, G. and Muzzioli, S. (2020). Investor sentiment and trading behavior. DEMB Working Paper series 163, University of Modena and Reggio Emilia, Modena, Italy. ISSN: 2281-440X.

Cavalli, F., Naimzada, A. K., Pecora, N., and Pireddu, M. (2018). Agents' beliefs and economic regimes polarization in interacting markets. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **28**(5), 055911.

Chau, F., Deesomsak, R., and Koutmos, D. (2016). Does investor sentiment really matter? *International Review of Financial Analysis*, **48**, 221–232.

Chiarella, C., He, X.-Z., and Wang, D. (2006). Statistical properties of a heterogeneous asset pricing model with time-varying second moment. In *The Complex Networks of Economic Interactions*, pages 109–123. Springer.

Chiarella, C., Dieci, R., and He, X. (2009). Heterogeneity, market mechanisms and asset price dynamics. *Handbook of Financial Markets: Dynamics and Evolution*, pages 277–344.

Chiarella, C., He, X.-Z., and Zwinkels, R. C. (2014). Heterogeneous expectations in asset pricing: Empirical evidence from the s&p500. *Journal of Economic Behavior & Organization*, **105**, 1–16.

Christie, A. A. (1982). The stochastic behavior of common stock variances: Value, leverage and interest rate effects. *Journal of Financial Economics*, **10**(4), 407–432.

Colasante, A., Palestrini, A., Russo, A., and Gallegati, M. (2017). Adaptive expectations versus rational expectations: Evidence from the lab. *International Journal of Forecasting*, **33**(4), 988–1006.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *QUANTITATIVE FINANCE*, **1**, 223–236.

Corredor, P., Ferrer, E., and Santamaria, R. (2013). Investor sentiment effect in stock markets: Stock characteristics or country-specific factors? *International Review of Economics & Finance*, **27**, 572–591.

Day, R. H. and Huang, W. (1990). Bulls, bears and market sheep. *Journal of Economic Behavior & Organization*, **14**(3), 299–329.

De Grauwe, P. and Kaltwasser, P. R. (2012). Animal spirits in the foreign exchange market. *Journal of Economic Dynamics and Control*, **36**(8), 1176–1192.

De Long, J. B., Shleifer, A., Summers, L. H., and Waldmann, R. J. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, **98**(4), 703–738.

Ding, D. K., Charoenwong, C., and Seetoh, R. (2004). Prospect theory, analyst forecasts, and stock returns. *Journal of Multinational Financial Management*, **14**(4-5), 425–442.

Franke, R. and Westerhoff, F. (2016). Why a simple herding model may generate the stylized facts of daily returns: explanation and estimation. *Journal of Economic Interaction and Coordination*, **11**(1), 1–34.

French, K. R., Schwert, G. W., and Stambaugh, R. F. (1987). Expected stock returns and volatility. *Journal of Financial Economics*, **19**(1), 3.

Hauser, M. A. (1997). Semiparametric and nonparametric testing for long memory: A monte carlo study. *Empirical Economics*, **22**(2), 247–271.

He, X.-Z. and Li, Y. (2007). Power-law behaviour, heterogeneity, and trend chasing. *Journal of Economic Dynamics and Control*, **31**(10), 3396–3426.

He, X.-Z. and Li, Y. (2015). Testing of a market fraction model and power-law behaviour in the dax 30. *Journal of Empirical Finance*, **31**, 1–17.

He, X.-Z. and Zheng, H. (2016). Trading heterogeneity under information uncertainty. *Journal of Economic Behavior & Organization*, **130**, 64–80.

Hommes, C. (2013). *Behavioral rationality and heterogeneous expectations in complex economic systems.* Cambridge University Press.

Huang, W., Zheng, H., and Chia, W.-M. (2013). Asymmetric returns, gradual bubbles and sudden crashes. *The European Journal of Finance*, **19**(5), 420–437.

Jawadi, F. and Prat, G. (2012). Arbitrage costs and nonlinear adjustment in the g7 stock markets. *Applied Economics*, **44**(12), 1561–1582.

Jawadi, F., Namouri, H., and Ftiti, Z. (2018). An analysis of the effect of investor sentiment in a heterogeneous switching transition model for g7 stock markets. *Journal of Economic Dynamics and Control*, **91**, 469–484.

Lo, Andrew, W. (1991). Long-term memory in stock market prices. *Econometrica*, **59**(5), 1279–1313.

Lux, T. (1995). Herd behaviour, bubbles and crashes. *The Economic Journal*, **105**(431), 881–896.

Lux, T. and Alfarano, S. (2016). Financial power laws: Empirical evidence, models, and mechanisms. *Chaos, Solitons & Fractals*, **88**, 3–18.

Lux, T. and Marchesi, M. (2000). Volatility clustering in financial markets: a microsimulation of interacting agents. *International journal of Theoretical and Applied Finance*, **3**(04), 675–702.

Naimzada, A. K. and Ricchiuti, G. (2009). Dynamic effects of increasing heterogeneity in financial markets. *Chaos, Solitons & Fractals*, **41**(4), 1764–1772.

Tramontana, F., Gardini, L., Dieci, R., and Westerhoff, F. (2009). The emergence of bull and bear dynamics in a nonlinear model of interacting markets. *Discrete Dynamics in Nature and Society*, **2009**.

Tsai, I.-C. (2017). Diffusion of optimistic and pessimistic investor sentiment: An empirical study of an emerging market. *International Review of Economics & Finance*, **47**, 22–34.

Verma, R. and Soydemir, G. (2009). The impact of individual and institutional investor sentiment on the market price of risk. *The Quarterly Review of Economics and Finance*, **49**(3), 1129–1145.

Whaley, R. E. (2000). The investor fear gauge. *The Journal of Portfolio Management*, **26**(3), 12–17.

Yang, C. and Yan, W. (2011). Does high sentiment cause negative excess return. *International Journal of Digital Content Technology and Its Application*, **5**(12), 211–217.

Yang, C. and Zhou, L. (2015). Investor trading behavior, investor sentiment and asset prices. *The North American Journal of Economics and Finance*, **34**, 42–62.

Zhou, G. (2018). Measuring investor sentiment. *Annual Review of Financial Economics*, **10**, 239–259.