Effects of neutron stars magnetic dipole on the generation of gravitational waves

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Abstract

In this paper we shall consider the potential effect of the magnetic dipole moment of a neutron star (NS) in a binary NS-NS system.

We shall derive the Lagrangian of the binary system and show how to find a Multipolar Post Minkowskian (MPM) solution to the linearized Einstein-Maxwell system and the energy flux of the electromagnetic waves; we shall calculate at the higher order the equations of motion and precession. At the end, we will provide calculations proving that the effect of the magnetic moment on the binary system is barely observable.

Keywords: General Relativity; Binary neutron stars; Gravitational waves, Magnetic dipole

The hypothesis of the existence of gravitational waves was put forward in the early days of General Relativity. *Indirect* evidence of their existence came from the study of the decay of the orbit of the Hulse-Taylor binary PSR 1913+16 [1, 2, 3, 4]: for this work, the authors were awarded with the Nobel prize in 1993.

2016 started with the announcement of the *direct* observation of gravitational waves (GW) [5] by the advanced LIGO project [6]: the detector observed the merging of two black holes of masses $36^{+5}_{-4}\, M_{\odot}$ and $29^{+4}_{-4}\, M_{\odot}$ at a distance of 410^{+160}_{-180} Mpc and many other followed (see for example the catalogue of the observed merging binaries [7] and references therein). These observations marked the beginning of gravitational waves astronomy. The most important observation is, however, the neutron star coalescence event GW170817 and the associated gamma-ray-burst GRB170817, which confirmed with a photonic detection the non-photonic detection of the GWs. Other observatories are the ground based VIRGO [8, 9] and the upcoming cryogenic KAGRA [11] and the space-borne LISA [10].

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The discovery was preceded by a long experimental and theoretical work. The latter was aimed to the research of general solution to the Einstein equations that could describe compact inspiraling binaries. Nowadays the dynamics of this kind of systems and the gravitational waves energy flux are known up to 3.5 Post Newtonian order including Spin-Orbit (SO), Spin-Spin (SS) and Spin-Spin-Spin (SSS) effects [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 38, 28, 37, 39, 40] (quadratic and cubic-in-spin effect are usually neglected for a neutron stars binary system, since they are too small); see also the review [41] and references therein.

Neutron stars, however, are known to have large magnetic fields up to 10^{13} G (see [42] and references therein) (magnetars have even larger magnetic field [44] but, so far, have never been found in binary systems [61])[1] and magnetic moments $\approx 10^{25} \div 10^{30}$ $G\,cm^3$:[2] the aim of this work is to study the effect of the magnetic dipole of neutron stars in binary systems on the generation of gravitational waves.

Most neutron stars and pulsars in binary systems have a *small* magnetic field $10^8 \div 10^{10}$ G [42]: these are called *recycled pulsars* and are thought to be old stars that have been spun up by the accretion of material from a non relativistic companion. Younger neutron stars have much larger magnetic fields ($10^{11} \div 10^{13}$ G) but are usually found isolated [42, 71]; however, there are some known binary systems with an old, recycled star and a younger one such as the *double pulsar system* PSR J0737-3039 (the only binary known with two pulsars [67]) and the system of the pulsar PSR J1906+0746 [69].

In order to achieve our aim, we have to consider the linearized version of the Einstein-Maxwell system and extend the Multipolar Post Minkowskian (MPM) formalism developed in [48, 49, 41] to this case. We shall also calculate the Post-Newtonian expansion of the gravitational and electromagnetic potentials and study the effects of the electromagnetic field on the dynamics of the binary at the higher order. We shall also write the explicit form of the equations of motion and precession in the center of mass system and for circular orbits. At the end, we shall calculate the effects our newly calculated electromagnetic terms have on the accumulated number of gravitational waves cycles for a ground base detector.

The plan for this paper is as follows.

In the first section we write down the Lagrangian of the system mostly following [12] and we derive the equations of motion and precession, the stress-energy tensors for matter and electromagnetic field; we shall then write the expression of the 4-momentum in terms of the speed, spin and electromagnetic

[1] Even if magnetars have never been observed in a binary system, they might still be source of gravitational radiation, see [62, 63, 64, 65, 66] for example.

[2] This can be calculated assuming a star with radius of 10 km [43] and using the relation $\mathcal{M} = B\,R^3$, where B is the surface value of magnetic field of the neutron star.

potential and use it to find the conserved mass. In sections 4 and 5, we shall present the linearized Einstein-Maxwell systems and calculate the energy and angular momentum flux of electromagnetic waves. In section 6, we shall calculate the electromagnetic potential and the dynamics of the binary system at the Newtonian level. In section 7 we shall calculate the electromagnetic contributions to the gravitational waves flux, while in section 8 we shall calculate the electromagnetic waves flux. In section 9 we shall calculate the electromagnetic contributions to the accumulated number of gravitational waves cycles. In the conclusion we shall consider the case of the *double pulsar* system PSR J0737-3039 [67] and numerically estimate the electromagnetic contribution to the accumulated number of gravitational waves cycles.

As was pointed out in [41] and references therein, if we want to make all the c factors in our equations explicit, we should define:

$$S = cS_{physical} = Gm^2\chi \tag{1}$$

where χ is the dimensionless spin (≈ 0.1 for neutron stars, although the fastest known millisecond pulsar has $\chi \approx 0.4$ [39, 45]).

We use the Gauss-cgs unit system and the metric signature $(-,+,+,+)$.

1. The Lagrangian of the system and equations of motion

Following [12, 39], if $x(\tau)$ is the world-line of a particle and τ the proper time, we define the velocity:

$$u^\mu = \frac{dx^\mu}{d\tau}$$

Given the metric $g_{\mu\nu}$, we introduce the (body-fixed) orthonormal tetrad and cotetrad $e^{(a)}_\mu(\tau)$ and $e^\mu_{(a)}(\tau)$, defined in the usual way (η_{ab} is the Minkowski metric):

$$e^\mu_{(a)} e^\nu_{(b)} g_{\mu\nu} = \eta_{(a)(b)} \qquad e^{(a)}_\mu \, e^\mu_{(b)} = \delta^{(a)}_{(b)} \tag{2}$$

With these, we can define the antisymmetric rotation coefficients for the tetrad:[3]

$$\Omega_{\mu\nu} = e_{(a)\,[\mu}\, u^\rho \left(e^{(a)}_{\nu]}\right)_{;\rho} \tag{3}$$

[3]In this paper, we use $[\ldots]$ around indices to indicate anti symmetrization and $(\ldots)$ for symmetrization. Here we have:

$$e_{(a)\,[\mu}\, u^\rho \left(e^{(a)}_{\nu]}\right)_{;\rho} = \frac{1}{2}\left(e_{(a)\mu}\, a^{(a)}_{\nu;\rho} - e_{(a)\nu}\, a^{(a)}_{\mu;\rho}\right) u^\rho$$

and later in the stress-energy tensor:

$$p^{(\mu}u^{\nu)} = \frac{1}{2}\left(p^\mu u^\nu + p^\nu u^\mu\right)$$

Given the above definitions, we assume (see [12, 39]) that the Lagrangian depends only on the speed u^μ, the rotation coefficients $\Omega_{\mu\nu}$, the electromagnetic potential A_μ and its derivatives (in fact only its antisymmetric part, the Faraday tensor $F_{\mu\nu}$), on the metric $g_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\alpha\beta}$ and that it can be separated into three contributions:

$$\begin{aligned} L &= L\left(u^\mu, \Omega_{\mu\nu}, A_\mu, F_{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\alpha\beta}\right) = \\ &= L_{gr} + L_{elm}(A_\mu, F_{\mu\nu}, g_{\mu\nu}) + L_{matt}(g_{\mu\nu}, A_\mu, F_{\mu\nu}, \Omega_{\mu\nu}, u^\mu) \end{aligned} \tag{4a}$$

where L_{gr} is the usual lagrangian for the gravitational field (defined, for example, in [13, 14]), L_{matt} is the lagrangian for the matter while the one for the free electromagnetic field is given by

$$L_{elm} = -\frac{c}{16\pi} F_{\mu\nu}\, F^{\mu\nu} \tag{4b}$$

Following [12, 39], we can define the (free) current J^μ, the 4-momentum p_μ, the spin $S^{\mu\nu}$, and the quadrupole moment $J^{\mu\nu\rho\sigma}$:

$$J^\mu = \frac{\delta L_{matt}}{\delta A_\mu} \qquad\qquad p_\mu = \frac{\delta L_{matt}}{\delta u^\mu} \tag{4c}$$

$$S^{\mu\nu} = 2\,\frac{\delta L_{matt}}{\delta \Omega_{\mu\nu}} \qquad\qquad J^{\mu\nu\rho\sigma} = -6\,\frac{\delta L_{matt}}{\delta R_{\mu\nu\rho\sigma}} \tag{4d}$$

Since we want to study a binary system in which the components do not have electric charge, but do have a magnetic dipole moment (in their rest frame), a proper definition for the (free) 4-current in Gauss-cgs unit system is:

$$J^\mu = -c\,\nabla_\nu\, \left(\mathcal{M}^{\nu\mu}\delta^3(x - x_A(\tau))\right) \tag{5}$$

where $\mathcal{M}^{\nu\mu}$ is the antisymmetric magnetic dipole moment and x_A is the position of the body A; our definition of the current guarantees that the magnetic moment is conserved thanks to the antisymmetry of $\mathcal{M}^{\mu\nu}$. Moreover, we *do not* want our stars to have an electric dipole in their rest frame; we can achieve that, if we impose that (see for example [75, 51]):

$$\mathcal{M}^{\mu\nu}\, u_\nu \equiv 0 \tag{6}$$

Therefore, we have:

$$\delta L_{matt} = c\, p_\mu \delta u^\mu + \frac{1}{2\,c}\, S^{\mu\nu}\Omega_{\mu\nu} - \frac{c^2}{6}\, J^{\mu\nu\rho\sigma}\, \delta R_{\mu\nu\rho\sigma} - J^\mu \delta A_\mu + \frac{\delta L_{matt}}{\delta g_{\mu\nu}}\, \delta g_{\mu\nu}$$

Since coordinates have no physical meaning, we have to impose the invariance of L_{matt} with respect to transformations of coordinates such as $x^\mu \mapsto x^\mu + \xi^\mu$ (in this way L_{matt} will be a scalar under coordinates transformations) and scaling of the time $\tau \mapsto \lambda\tau$. We can achieve this if:

$$2\,\frac{\delta L_{matt}}{\delta g_{\mu\nu}} = c\, p^\mu\, u^\nu + \frac{1}{c}\, S^{\mu\rho}\Omega^\nu{}_\rho + c^2\, \frac{2}{3}\, R^\mu{}_{\lambda\rho\sigma} J^{\nu\lambda\rho\sigma} - J^\mu A^\nu \tag{7}$$

and if L_{matt} has degree one in u^μ and $\Omega^{\mu\nu}$ (see [12, 39]).

As a consequence, the Lagrangian of the system becomes:

$$L = L_{gr} + c\, p^\mu u_\mu + \frac{1}{2\,c}\, S^{\mu\nu}\Omega_{\mu\nu} - J^\mu\, A_\mu - \frac{c}{16\,\pi}\, F_{\mu\nu}F^{\mu\nu} \tag{8}$$

and the action is:

$$S = \int d^4x\, \sqrt{-g}\, L[u^\nu, \Omega_{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, A_\mu, F_{\mu\nu}] \tag{9}$$

Varying this action, we can write down the equations of motion and precession and the stress-energy tensors (see [12, 37]): the equation of precession is found by varying the action with respect to tetrads and using the equation (7), while the equations of motion arise by varying the action with respect to *world lines* (see [12]); as a result, the equations are given by (see also [12] and [40, 39, 73, 74] for the quadrupole part):

$$\frac{1}{c}\,\frac{Dp_\mu}{d\tau} = \frac{1}{2}\,\frac{1}{c}\, S^{\alpha\beta} R_{\alpha\beta\gamma\mu} u^\gamma - \frac{c}{3}\, J_\delta{}^{\alpha\beta\gamma} R^\delta{}_{\alpha\beta\gamma;\mu} + \frac{1}{2c}\, \mathcal{M}^{\rho\sigma} F_{\rho\sigma;\mu} \tag{10}$$

$$\frac{1}{c^2}\,\frac{1}{2}\,\frac{DS^{\mu\nu}}{d\tau} = p^{[\mu}u^{\nu]} - c^3\,\frac{2}{3}\, R^{[\mu}{}_{\lambda\rho\sigma} J^{\nu]\lambda\rho\sigma} - \frac{1}{c^2}\, F^{[\mu}{}_\rho \mathcal{M}^{\nu]\rho} \tag{11}$$

where we have defined $D/d\tau = u^\mu \nabla_\mu$ (∇_μ is the covariant derivative). In addition to these, varying the action with respect to A_μ and $\partial_\mu A_\nu$, we get the Maxwell's equations:

$$F^\mu{}_{\nu;\mu} = \frac{4\pi}{c}\, J_\nu = -\,4\pi\, \nabla_\nu\, \left(\mathcal{M}^{\nu\mu}\delta^3(x - x_A(\tau))\right) \tag{12}$$

$$F_{[\mu\nu,\rho]} = 0 \tag{13}$$

where in the first line we have used the definition of the current (5).

Finally, we have the evolution equation for the magnetic moment: two terms will contribute: one is due to the rotation of the star and the other is due to the precession and therefore has the same form of the spin precession equation; therefore the evolution equation for the magnetic moment is given by:

$$\frac{D\mathcal{M}^{\mu\nu}}{cd\tau} = \omega^{[\mu}{}_\alpha \mathcal{M}^{\alpha|\nu]} + p^{[\mu}u^{\nu]} - c^3\,\frac{2}{3}\, R^{[\mu}{}_{\lambda\rho\sigma} J^{\nu]\lambda\rho\sigma} - \frac{1}{c^2}\, F^{[\mu}{}_\rho \mathcal{M}^{\nu]\rho} \tag{14}$$

where ω is an antisymmetric tensor describing the rotation of the star. Given the moment of inertia $\mathcal{I}$ of the star (supposed spherical) and its spin S, we have that:

$$\omega = \frac{S}{c\mathcal{I}} = \frac{\omega'}{c} = \frac{5}{2}\,\frac{Gm}{cr^2}\,\chi. \tag{15}$$

where r is the radius of the star and χ is the dimensionless spin ($\chi \approx 0.1$ for NS).

The matter stress-energy tensor is given by:

$$T^{\mu\nu} = \sum_{A=1,2} \left[n_A \left(c\, p_A^{(\mu} u_A^{\nu)} + \frac{1}{3} R^{(\mu}{}_{\lambda\rho\sigma} J_A^{\nu)\lambda\rho\sigma} \right) + J_A^{(\mu} A^{\nu)} + \right.$$
$$\left. -\nabla_\rho \left(n_A S_A^{\rho(\mu} u_A^{\nu)} \right) - \frac{2}{3} \nabla_\rho \nabla_\sigma \left(n_A c^2 J_A^{\rho(\mu\nu)\sigma} \right) \right] \tag{16}$$

While the electromagnetic stress-energy tensor is:

$$T^{elm}_{\mu\nu} = -\frac{1}{4\pi} \left[\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - g^{\alpha\beta} F_{\mu\alpha} F_{\beta\nu} \right] \tag{17}$$

2. Contravariant 4-momentum and conserved mass

In this section we shall find the expression of the covariant 4-momentum in terms of u^μ, $S^{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and $F_{\mu\nu}$. This will allow us to find the conserved mass at quadratic order in spin.

First of all, the spin tensor is an antisymmteric tensor, and has therefore six independent components, but only three of them are physical (they are the spin vector components), since the other can be eliminated by fixing the center of body reference frame (see [28, 46]); following [12, 40, 28, 46, 47], we impose the Supplementary Spin Condition (SSC):

$$S^{\mu\nu} p_\mu = 0 \tag{18}$$

2.1. Contravariant 4-momentum

We start by differentiating the SSC equation (18) with respect to τ; substituting the equation of motion (10) and precession (11) and remembering that $p^\mu p_\mu = -m^2 c^2$, we find:[4]

$$\frac{DS^{\mu\nu} p_\mu}{d\tau} = \left(\frac{1}{c^2} \frac{DS^{\mu\nu}}{d\tau} \right) c^2 p_\mu + S^{\mu\nu} c \left(\frac{1}{c} \frac{Dp_\mu}{d\tau} \right)$$
$$0 = -(u^\mu p_\mu) p^\nu c^2 - u^\nu m^2 c^4 - \frac{4}{3} c^3 R^{[\mu}{}_{\lambda\rho\sigma} J^{\nu]\lambda\rho\sigma} - F^{[\mu}{}_{\alpha} \mathcal{M}^{\nu]\alpha} p_\mu + \frac{1}{2} S^{\mu\nu} S^{\alpha\beta} R_{\alpha\beta\gamma\mu} u^\gamma + O(S^3) \tag{19}$$

Since we already know that at the lowest order in spin $p_\mu = mc\, u_\mu + O(S^2)$, we can substitute this back in (19) remembering that the speed of the bodies u_μ has norm $u_\nu u^\nu = -1$; in this way, we find the following expression of the 4-momentum in terms of u, the spin and the Faraday tensor at $O(S^3)$ (see also [40] for the *gravitational* part of the equation):

$$p^\mu = mc\, u^\mu + \frac{c}{6} u^\mu R_{\rho\lambda\alpha\beta} J^{\rho\lambda\alpha\beta} + \frac{4c}{3} u_\beta R^{[\mu}{}_{\lambda\rho\sigma} J^{\nu]\lambda\rho\sigma} - \frac{S^{\alpha\beta} S^{\mu\nu}}{2mc^3} u^\lambda R_{\nu\lambda\alpha\beta} + \frac{1}{c^2} F^{[\mu}{}_{\alpha} \mathcal{M}^{\nu]\alpha} u_\nu + O(S^3) \tag{20}$$

[4]We remind that we only consider quadratic-in-spin terms.

2.2. Conserved mass

Now that we have the expression for the contravariant 4-momentum, we can calculate the conserved mass.

Following Bailey and Israel [12] (see also [40]), we contract (19) with (10) and we arrive at:

$$p^\mu \, \frac{Dp_\mu}{c\,d\tau} = \frac{1}{2}\,\frac{D(p_\mu p^\mu)}{c\,d\tau} = -m\,c^2\,\frac{Dm}{c\,d\tau} = \frac{1}{2}\,\frac{p_\mu p^\mu}{p_\mu u^\mu}\,\frac{D}{c\,d\tau}\,\left(\mathcal{M}^{\alpha\beta}F_{\alpha\beta}\right) + \frac{1}{6}\,\frac{p_\mu p^\mu}{p_\mu u^\mu}\,\frac{D}{c\,d\tau}\,\left(J_\alpha{}^{\beta\gamma\delta}R^\alpha{}_{\beta\gamma\delta}\right) + O(S^3) \tag{21}$$

If we retain only quadratic-in-spin terms and if we consider only Newtonian contributions, then the conserved mass $\widetilde{m}$ is given by (see [40]):

$$\widetilde{m} = m + \frac{1}{2}\,\mathcal{M}^{\alpha\beta}F_{\alpha\beta} - \frac{1}{6}\,J_\alpha{}^{\beta\gamma\delta}R^\alpha{}_{\beta\gamma\delta} + O(S^3) + O(c^{-2}) \tag{22}$$

3. Precession and magnetic moment evolution equations

We are now in a position to derive the evolution equation for the vector spin and magnetic moment in the presence of both gravitational and electromagnetic interactions.

First, we define the spin and magnetic dipole vectors (see [40, 39]):[5]

$$S_\mu = -\frac{1}{2}\,\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,\frac{p^\nu}{mc}\,S^{\rho\sigma} \tag{23a}$$

$$\mathcal{M}_\mu = -\frac{1}{2}\,\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,u^\nu\,\mathcal{M}^{\rho\sigma} \tag{23b}$$

the inverses are:

$$S^{\mu\nu} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\rho\sigma}\,\frac{p_\rho}{mc}\,S_\sigma \tag{23c}$$

$$\mathcal{M}^{\mu\nu} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\rho\sigma}\,u_\rho\,\mathcal{M}_\sigma \tag{23d}$$

where p^μ is given in equation (20). With these definitions, the spin vector automatically respects condition SSC given in equation (18) and our stars will not have an electric dipole in their rest frame as required by equation (6) because of the antisymmetry of the Levi Civita symbol.

To obtain the vector spin evolution equation, we differentiate (23a) with respect to τ and substitute equation (10) and (11) retaining only quadratic-in-

[5] The Levi Civita symbol is defined as usual: $\epsilon_{0123} = 1$.

spin terms; in this way, we obtain:[6]

$$\frac{DS_\mu}{dt} = -\frac{1}{2}\,\frac{1}{u^0}\,\epsilon_{\mu\nu\rho\sigma}\,\frac{1}{mc}\,\left[\frac{Dp^\nu}{d\tau}S^{\rho\sigma} + p^\nu\,\frac{DS^{\rho\sigma}}{d\tau}\right] =$$
$$= -\frac{1}{2}\,\frac{1}{u^0}\,\epsilon_{\mu\nu\rho\sigma}\,\left[-\frac{1}{2}\,S^{\alpha\beta}S^{\rho\sigma}R^\nu{}_{\gamma\alpha\beta}\,\frac{u^\gamma}{mc} - c^3\,\frac{4}{3}\,\frac{p^\nu}{m}\,R^{[\rho}{}_{\lambda\alpha\beta}J^{\sigma]\lambda\alpha\beta} - 2\,\frac{p^\nu}{mc}\,F^{[\rho|\alpha}\mathcal{M}^{\sigma]}{}_\alpha\right] + O(S^3) \tag{24}$$

For the gravitational sector of the equation (the first two terms on the right-hand side), we can follow [40] introducing the spin-precession frequency (antisymmetric) tensor

$$\Omega_{\mu\nu} = \frac{S_\lambda}{mc}\,\frac{1}{u^0}\,\left[u_{[\alpha}H_{\beta]}{}^\lambda - k\,\epsilon_{\alpha\beta\mu\nu}u^\mu G^{\nu\lambda}\right] \tag{25}$$

where $G_{\mu\nu}$ and $H_{\mu\nu}$ are respectively the mass-type quadrupole and the current-type quadrupole and are given by:

$$G_{\mu\nu} = -R_{\mu\lambda\nu\rho}u^\lambda u^\rho \tag{26}$$
$$H_{\mu\nu} = {}^*R_{\mu\kappa\nu\lambda}u^\kappa u^\lambda \tag{27}$$

For the electromagnetic contributions to the spin evolution, we first notice that:

$$F^{\mu\alpha}\mathcal{M}^\nu{}_\alpha = g_{\alpha\beta}\,F^{\mu\alpha}\,\mathcal{M}^{\nu\beta}$$

We now use the *Schouten identity* in the right hand side:

$$g_{\kappa\tau}\,\epsilon^{\mu\nu\rho\sigma} + \delta^\mu_\kappa\,g_{\lambda\tau}\,\epsilon_{\nu\sigma\rho\tau} + \delta^\nu_\kappa\,g_{\lambda\tau}\,\epsilon_{\sigma\rho\tau\mu} + \delta^\rho_\kappa\,g_{\lambda\tau}\,\epsilon_{\sigma\tau\mu\nu} + \delta^\sigma_\tau\,g_{\lambda\tau}\,\epsilon_{\tau\mu\nu\rho} \equiv 0$$

Remembering the condition (6) and the definition of the magnetic dipole vector, equation (23a), we find, after some lengthy algebra:

$$\left.\frac{DS_\mu}{dt}\right|^{elm} = \frac{1}{u^0}\,F_{\mu\nu}\,\mathcal{M}^\nu$$

Moreover, as was found in [40], $S_0 = O(S^3)$, so we can focus only on the space components of the spin vector: S_i.

Putting it all together, we finally find that the evolution equation for the spin vector is:

$$\frac{DS_i^A}{dt} = \Omega^A_{ij}\,S^j_A + \frac{1}{u^0}\,F^A_{ij}\,\mathcal{M}^j_A + O(S^3) \qquad A = \{1,2\}$$

[6]We remind that $\frac{dt}{d\tau} = u_0$.

At $O(S^3)$, in Ω_{ij} there are two contributions: Ω^{NS}, in which there are no spin terms, whose expression is given in [37], one with spin-orbit (SO) interaction, whose expression is given in [40]; we can separate these contributions and rewrite the previous equation in this way:

$$\frac{DS_i^A}{dt} = \left[\Omega_{ij}^{NS} + \Omega_{ij}^{SO}\right]_A S_A^j + \frac{1}{u^0} F_{ij}^A \mathcal{M}_A^j + O(S^3) \qquad A = \{1, 2\} \tag{28}$$

In a similar way, for the magnetic moment, we find:

$$\frac{D\mathcal{M}_A^i}{dt} = \omega'_{ij}\mathcal{M}_A^j + \Omega_{ij}^A \mathcal{M}_A^j + \frac{1}{c}\frac{1}{u^0} F_{ij}^A \mathcal{M}_A^j \qquad A = \{1, 2\} \tag{29}$$

4. The Einstein-Maxwell system

We can now study the Einstein-Maxwell system and look for a Multipolar Post Minkowskian (MPM) solution.

In section 5, we shall write the expression for the asymptotic wave form of the electromagnetic fields and calculate the energy and angular momentum flux carried away by the electromagnetic waves.

In this section and in the following, we deal with the general case, but in what follows we will only consider the higher order part of the equations.

4.1. The system

The Einstein-Maxwell system is (see [13], for example):

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T^m_{\mu\nu} + \frac{8\pi G}{c^4} T^{em}_{\mu\nu} \tag{30a}$$

$$F_{\mu}{}^{\nu}{}_{;\nu} = \frac{4\pi}{c} J_\mu \tag{30b}$$

where $T^{elm}_{\mu\nu}$ is the electromagnetic stress-energy tensor defined in (17) and $T^m_{\mu\nu}$ is the matter one defined in (16); J_μ is the (free) 4-current defined in (5). It is convenient to expand the covariant derivative in equation (5) as follows (see for example [13, 15]):

$$J_\mu = -\sum_A \partial_\nu \left(\mathcal{M}^\nu_{A\,\mu}\, \delta^3(\mathbf{x} - \mathbf{y}_A)\right) - \sum_A \left(\mathcal{M}^\nu_{A\,\mu}\, \delta^3(\mathbf{x} - \mathbf{y}_A)\right)\left(\partial_\nu \ln\left(\sqrt{-g}\right)\right) \tag{31}$$

where g is the determinant of the metric. Both the matter stress-energy tensor $T^m_{\mu\nu}$ and the current J_μ have compact support.

To solve the system, we have to add contour conditions: we assume (see [48, 41]) that in the far past the metric was asymptotically flat and stationary:

$$\partial_t\, g_{\mu\nu}(t, \mathbf{x}) = 0 \qquad \lim_{r\to+\infty} g_{\mu\nu}(t, \mathbf{x}) = \eta_{\mu\nu} \qquad \forall t \leq T \tag{32}$$

so that there is no incoming radiation at the source position.

4.2. The linerized system

In order to linearize our system, we introduce the *gothic* metric (see [41, 48, 13, 14]):

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{33}$$

and divide A_μ into a background potential $\overset{0}{A}_\mu$ and a perturbations potential $\widetilde{A}_\mu$:

$$A_\mu = \overset{0}{A}_\mu + \widetilde{A}_\mu. \tag{34}$$

We also divide the Faraday tensor $F_{\mu\nu}$ consistently:

$$F_{\mu\nu} = \overset{0}{F}_{\mu\nu} + \widetilde{F}_{\mu\nu} = 2\partial_{[\mu}\overset{0}{A}_{\nu]} + 2\partial_{[\mu}\widetilde{A}_{\nu]} \tag{35}$$

We assume that electromagnetic perturbations are *small*, in the following sense:

$$(\eta_{\mu\nu};\, \overset{0}{A}_\mu) = O(1) \qquad (h_{\mu\nu};\, \widetilde{A}_\mu) = O(G) \tag{36}$$

We also assume that the magnetic moment $\mathcal{M}_{\mu\nu}$ is $O(1)$. In this way we see that our current (31) naturally divides into an $O(1)$ part and an $O(G)$ part:

$$J_\mu = \overset{0}{J}_\mu + \widetilde{J}_\mu \tag{37a}$$

where:

$$\overset{0}{J}_\mu = -c \sum_{A=\{1,2\}} \partial_\nu \left(\mathcal{M}^{\nu}_{A\,\mu}\, \delta^3(\mathbf{x} - \mathbf{y}_A) \right) \tag{37b}$$

$$\widetilde{J}_\mu = -c \sum_{A=\{1,2\}} \left(\mathcal{M}^{\nu}_{A\,\mu}\, \delta^3(\mathbf{x} - \mathbf{y}_A) \right) \left(\partial_\nu \ln(\sqrt{-g}) \right) \tag{37c}$$

$\overset{0}{J}_\mu$ will be the source of the background field, while $\widetilde{J}_\mu$ will be a source for the perturbation fields.

All the fields we have introduced must be transverse:

$$\partial_\nu h_{\mu\nu} = 0 \qquad \partial_\mu \overset{0}{A}_\mu = 0 \qquad \partial_\mu \widetilde{A}_\mu = 0 \tag{38}$$

The first relation implies that we are using the *De Donder* or *harmonic* coordinate system, the other two, that the we use Lorenz gauge. The above relation do not, however, exploit the whole gauge freedom. If $f_{\mu\nu}$ respects the first of (38) another $f'_{\mu\nu}$ given by:

$$f'_{\mu\nu} = f_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\, \partial_\mu \xi^\mu \tag{39a}$$

will respect it too, if the vector ξ_μ respects the following (see [13, 41], for example):

$$\Box \xi_\mu = 0. \tag{39b}$$

Analogously, if b_μ respects the second (or the third) of (38), so will b'_μ given by:

$$b'_\mu = b_\mu + \partial_\mu \lambda \tag{40a}$$

if the scalar λ respects the following (see [59] for example):

$$\Box\lambda = 0 \tag{40b}$$

Substituting our definitions (33)-(37c) into the system (30) and using the gauge given in (38), we find (mind the position of the indices in the magnetic dipole):

$$\Box \overset{0}{A}_\alpha = -\frac{4\pi}{c}\,\overset{0}{J}_\alpha = -\,4\pi\left[\mathcal{M}_\alpha^{A\,\beta}\,\delta(x-x_A)\right]_{,\beta} \tag{41}$$

$$\Box h_{\mu\nu} = \frac{16\pi G}{c^4}\,|g|\,\left[\widetilde{T}^m_{\mu\nu}\right] + \left(\frac{16\pi G}{c^4}\,|g|\,\left[T^{em}_{\mu\nu}\right] + \Lambda_{\mu\nu}\right) \tag{42a}$$

$$\begin{aligned}\Box \widetilde{A}_\alpha &= -\,\frac{4\pi}{c}\,\widetilde{J}_\alpha + R_{\alpha\beta}\,\overset{0}{A}{}^\beta + R_{\alpha\beta}\,\widetilde{A}^\beta = \\ &= -\,4\pi\sum_A\left(\mathcal{M}_\alpha^{A\,\beta}\,\delta^3(\mathbf{x}-\mathbf{y}_A)\right)\left(\partial_\nu\,\ln\left(\sqrt{-g}\right)\right) + R_{\alpha\beta}\,\overset{0}{A}{}^\beta + R_{\alpha\beta}\,\widetilde{A}^\beta\end{aligned} \tag{42b}$$

where:

$$\widetilde{T}^{em}_{\mu\nu} = -\,\frac{\eta_{\mu\nu}}{4}\,\overset{0}{F}_{\mu\nu}\,\overset{0}{F}{}^{\mu\nu} + \eta^{\rho\sigma}\,\overset{0}{F}_{\mu\rho}\,\overset{0}{F}_{\sigma\nu} - \frac{h_{\mu\nu}}{4}\,\overset{0}{F}_{\mu\nu}\,\overset{0}{F}{}^{\mu\nu} + h^{\rho\sigma}\,\overset{0}{F}_{\mu\rho}\,\overset{0}{F}_{\sigma\nu} + \tag{43}$$

$$-\,\frac{\eta_{\mu\nu}}{2}\,\overset{0}{F}_{\mu\nu}\,\widetilde{F}^{\mu\nu} + 2\,\eta^{\rho\sigma}\,\overset{0}{F}_{\mu\rho}\,\widetilde{F}_{\sigma\nu} + T2 + T3 + \dots \tag{44}$$

$$\Gamma^\mu_{\alpha\mu} = -\frac{1}{2}\,h_{,\alpha} + \gamma 2 + \gamma 3 + \dots \tag{45}$$

$$\Gamma^\alpha_{\mu\nu} = \frac{\eta^{\alpha\delta}}{2}\left[h_{\delta\mu,\nu} + h_{\delta\nu,\mu} - h_{\mu\nu,\delta} - \frac{1}{2}\left(h_{,\mu}\,\eta_{\delta\nu} + h_{,\nu}\,\eta_{\delta\mu} - h_{,\delta}\,\eta_{\mu\nu}\right)\right] + \Gamma 2 + \Gamma 3 + \dots \tag{46}$$

where Tn, γn and Γn are terms of order $O(G^n)$ and $\Lambda_{\mu\nu}$ is defined in [41, 18]. The determinant of the metric, on the other hand, can be calculated using the formula (valid for any matrix $\mathbf{M}$, see for example [15]):

$$det(\mathbf{1}-\mathbf{M}) = 1 - Tr\,(\mathbf{M}) + \frac{1}{2}\,(Tr\,(\mathbf{M}))^2 - \frac{1}{3}\,(Tr\,(\mathbf{M}))^3 + \dots$$

where $\mathbf{1}$ is the identity matrix; from the above formula, we find:

$$-g = 1 - h + \frac{1}{2}\,h^2 - \frac{1}{3}\,h^3 + O(h^4) \tag{47}$$

where $h = h^\mu{}_\mu = \eta^{\mu\nu}\,h_{\mu\nu}$.

The main difference between ours and the *usual* case is the presence of the electromagnetic stress energy tensor: unlike $T^m_{\mu\nu}$ it cannot be enclosed in a compact and its effects stretch throughout the whole spacetime, moreover $G\,\overset{0}{T}{}^{elm}_{\mu\nu} = O(G)$ (see equations (17) and (36)), therefore it must be taken into account already at linear order; this actually does not constitute a problem, in fact it has to be treated as $\Lambda_{\mu\nu}$ is treated in the usual case.

5. The energy and angular momentum flux of the electromagnetic field

In this section we will find the energy and angular momentum flux of the electromagnetic waves. We shall focus on the background field, the calculations for the perturbations being the same (one should just substitute $\overset{0}{}$ with a $\widetilde{}$).

In this section, we follow the usual notation and define (see [58, 48, 49, 50, 41]):

$$f^{(n)}(u) = \left(\frac{d}{du}\right)^n f(u)$$

where u is the retarded time $u = t - \frac{1}{c}\, r$.

In our case, the energy is carried away from the binary not only through gravitational waves (GW), but also through electromagnetic waves (EW) generated by the background potential $\overset{0}{A}_\mu$ and by the perturbations potential $\widetilde{A}_\mu$; the energy balance equation is, therefore (see [13]):

$$\frac{dE}{dt} = -\mathcal{F}^{GW} - \overset{0}{\mathcal{F}}{}^{EW} - \widetilde{\mathcal{F}}^{GW} \tag{48}$$

where we have indicated the fluxes with $\mathcal{F}$.

The energy flux per solid angle carried away by the electromagnetic waves (EW) is:

$$\frac{d\overset{0}{\mathcal{F}}{}^{EW}}{d\Omega} = \frac{d^2 \overset{0}{E}{}^{EW}}{dt d\Omega} = c\, r^2\, \overset{0}{T}{}^{em}_{00}. \tag{49}$$

We have found it easier to first introduce the transverse electric and magnetic fields (see for example [59]):

$$\overset{0}{\mathcal{B}}{}^T_i = \epsilon_{ilm}\, \partial_l\, \overset{0}{A}{}^T_m = \mathcal{P}_i^j \left(\epsilon_{jlm}\, \partial_l\, \overset{0}{A}_m\right) + O(r^{-2}) \tag{50}$$

$$\overset{0}{\mathcal{E}}{}^T_i = \epsilon_{ilm}\, N^l\, \overset{0}{\mathcal{B}}{}^T_m \tag{51}$$

where $\overset{0}{A}{}^T_i$ is the transverse asymptotic background electromagnetic potential given by (using *radiative coordinates* $(T, \mathbf{R})$ at null infinity):

$$\overset{0}{A}{}^T_i(u, \vec{x}) = \frac{1}{R}\, \mathcal{P}_i^j \sum_{l\geq 1} \frac{1}{c^l} \left[N_{L-1}\, \overset{0}{O}_{iL-1}(u) + \frac{1}{c}\, \frac{l}{l+1}\, \epsilon_{iab}\, N_{aL-1}\, \overset{0}{S}_{bL-1}(u)\right] + O(R^{-2}) \tag{52}$$

where $\mathcal{P}_i^j = \delta_i^j - N^i N_j$ projects the fields on the plane orthogonal to $\mathbf{N} = \frac{\mathbf{R}}{R}$, the direction of propagation of the fields. The multipoles $\overset{0}{O}_L$ and $\overset{0}{S}_L$ are resummations of the multipoles at every order.

Using the formulas in [48, 58] for symmetric trace free (STF) tensors, we obtain the following expressions for the fields:

$$\overset{0}{\mathcal{E}}{}_i^T = \frac{\mathcal{P}_i^j}{R\,c} \sum_{l\geq 1} \frac{1}{c^l}\,\frac{1}{l!}\left[N_{L-1}\,\overset{0}{O}{}^{(1)}_{L-1j} + \frac{1}{c}\,\frac{l}{l+1}\,\epsilon_{jab}\,N_{aL-1}\,\overset{0}{S}{}^{(1)}_{bL-1}\right] \tag{53}$$

$$\overset{0}{\mathcal{B}}{}_i^T = \frac{\mathcal{P}_i^j}{R\,c} \sum_{l\geq 1} \frac{1}{c^l}\,\frac{1}{l!}\left[\frac{1}{c}\,N_{L-1}\,\overset{0}{S}{}^{(1)}_{L-1j} + \frac{l}{l+1}\,\epsilon_{jab}\,N_{aL-1}\,\overset{0}{O}{}^{(1)}_{bL-1}\right] \tag{54}$$

With these definitions, we can rewrite the energy of the background field as:

$$\overset{0}{T}_{00} = \frac{1}{8\pi}\left(\overset{0}{\boldsymbol{\mathcal{E}}}{}^T\cdot\overset{0}{\boldsymbol{\mathcal{E}}}{}^T + \overset{0}{\boldsymbol{\mathcal{B}}}{}^T\cdot\overset{0}{\boldsymbol{\mathcal{B}}}{}^T\right) \tag{55}$$

In order to ease the calculations, we follow [58], and rewrite the electric field using vectorial spherical harmonics:

$$\overset{0}{\mathcal{E}}{}_i^T = \frac{1}{R\,c}\sum_{l\geq 1}\sum_{m=-l}^{l}\frac{1}{c^l}\left[E^{(1)}_{lm}\,Y_i^{E\,lm} + B^{(1)}_{lm}\,Y_i^{B\,lm}\right] \tag{56}$$

where $Y_i^{E\,lm}$ is the *electric* vectorial spherical harmonic and $Y_i^{B\,lm}$, the *magnetic* one (see section 2.D in [58]). The scalars E_{lm} and B_{lm} are linked to $\overset{0}{O}_L$ and $\overset{0}{S}_L$ by the relations:

$$\overset{0}{O}_L = -l!\,\sqrt{\frac{l}{l+1}}\,\sum_{m=-l}^{l} E_{lm}\,y_{Al}^{lm} \tag{57a}$$

$$\overset{0}{S}_L = -\,\frac{(l+1)!}{l}\,\sqrt{\frac{l}{l+1}}\,\sum_{m=-l}^{l} B_{lm}\,y_{Al}^{lm} \tag{57b}$$

The inverse relations are:

$$E_{lm} = -\frac{4\pi}{(2l+1)!!}\,\sqrt{\frac{l+1}{l}}\,\overset{0}{O}_L\,y_{Al}^{lm\,*} \tag{58a}$$

$$B_{lm} = -\frac{4\pi}{(2l+1)!!}\,\sqrt{\frac{l}{l+1}}\,\overset{0}{S}_L\,y_{Al}^{lm\,*} \tag{58b}$$

The same relations are valid also for the magnetic field, but from equation (50) it is easy to see that relations (57) are inverted and so are (58).

We can substitute (56) into equation (55) and use (49) (we suppress for the moment the magnetic field since the calculations are similar; its contribution will be added at the end):

$$\frac{d^2E}{dtd\Omega} = \frac{1}{8\pi}\sum_{lm}\frac{1}{c^{2l+1}}\Big[E^{(1)}_{lm}\,E^{(1)}_{l'm'}\,Y_i^{E\,lm}\,Y_i^{E\,l'm'} + B^{(1)}_{lm}\,B^{(1)}_{l'm'}\,Y_i^{B\,lm}\,Y_i^{B\,l'm'} + \\ + B^{(1)}_{lm}\,E^{(1)}_{l'm'}\,Y_i^{B\,lm}\,Y_i^{E\,l'm'} + E^{(1)}_{m}\,B^{(1)}_{l'm'}\,Y_i^{E\,lm}\,Y_i^{B\,l'm'}\Big] \tag{59}$$

thanks to the orthonormality of the vectorial spherical harmonics (see [58]), after an integration over the solid angle, we find:

$$\frac{dE}{dt} = \frac{1}{8\pi} \sum_{lm} \frac{1}{c^{2l+1}} \left[|E_{lm}|^2 + |B_{lm}|^2 \right]; \tag{60}$$

One can see that this formula correctly reproduces the one given by Jackson [59].

Substituting (58) into the previous equation, we find:

$$\frac{dE}{dt} = \frac{1}{2} \sum_{l\geq 1} \frac{1}{c^{2l+1}} \left[\frac{l+1}{l} \frac{1}{l!\,(2l+1)!!} \overset{0}{O}{}^{(1)}_L \overset{0}{O}{}^{(1)}_L + \frac{1}{c^2} \frac{l}{(l+1)!(2l+1)!!} \overset{0}{S}{}^{(1)}_L \overset{0}{S}{}^{(1)}_L \right]$$

Adding the magnetic field contributions, we find the formula for the flux of the electromagnetic waves we were looking for:

$$\begin{aligned} \frac{d\overset{0}{E}{}^{EW}}{dt} &= \frac{1}{2} \sum_{l\geq 1} \left[\frac{1}{c^{2l+1}} \frac{l+1}{l} \frac{1}{l!\,(2l+1)!!} \left(\overset{0}{O}{}^{(1)}_L \overset{0}{O}{}^{(1)}_L + \frac{1}{c^2} \overset{0}{S}{}^{(1)}_L \overset{0}{S}{}^{(1)}_L \right) \right] + \\ &+ \frac{1}{2} \sum_{l\geq 1} \left[\frac{1}{c^{2l+1}} \frac{l}{(l+1)!(2l+1)!!} \left(\overset{0}{O}{}^{(1)}_L \overset{0}{O}{}^{(1)}_L + \frac{1}{c^2} \overset{0}{S}{}^{(1)}_L \overset{0}{S}{}^{(1)}_L \right) \right] \end{aligned} \tag{61}$$

As we said at the beginning of this section, the energy flux for the perturbation field has the same form as (61), but with the *tilde* multipole moments, $\widetilde{O}_L$ and $\widetilde{S}_L$.

Since electromagnetic interactions are linear, there are *no tails* contributions for the background electromagnetic field; therefore, we have:

$$\overset{0}{O}_L \equiv \overset{0}{Q}_L \qquad \overset{0}{S}_L \equiv \overset{0}{M}_L. \tag{62}$$

where $\overset{0}{Q}$ and $\overset{0}{M}$ are the instantaneous contribution to the energy flux given by (see [50]):

$$\overset{0}{Q}_L = \int d^3x \int_{-1}^{+1} dz \left[\delta_l(z)\, \hat{x}_L\, \overset{0}{J}_0 - \frac{1}{c^2} \frac{2l+1}{(l+1)(2l+3)}\, \delta_{l+1}\, \hat{x}_{aL}\, \frac{\partial}{\partial u}\, \overset{0}{J}_i \right] \quad l \geq 1 \tag{63a}$$

$$\overset{0}{M}_L = \int d^3x \int_{-1}^{+1} dz \delta_l(z)\, \epsilon_{iab}\, \hat{x}_{aL-1}\, \overset{0}{J}_{bL} \qquad l \geq 1 \tag{63b}$$

where $\overset{0}{J}_\mu$ is given in (37b) and (see [41, 49, 50]):

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}\, l!}\, (1-z^2)^l$$

On the other hand, tails effects for the electromagnetic perturbations *do* exist as a consequence of the interaction with the gravitational field (see terms $R_{\alpha\beta}\overset{0}{A}{}^{\beta} + R_{\alpha\beta}\tilde{A}^{\beta}$ in the right hand side of (42b)).

The above calculations can be repeated for the angular momentum flux, but one should consider term proportional to $\dfrac{1}{R^2}$, since other terms vanish upon integration over the solid angle (see also [58]). Considering only the electric field (the contributions of the magnetic field being analogous), we find:

$$\begin{aligned}\frac{dJ_j}{dt} = \frac{i}{4\pi}\sum_{l\geq 1}\sum_{m=-l}^{m=l}\Bigg[&\frac{\xi_j^{-1}}{\sqrt{2}}\sqrt{(l-m+1)(l-m)}\left(E_{l\,m}\,E^*_{l\,m-1} + B_{l\,m}\,B^*_{l\,m-1}\right)+\\ &-\frac{\xi_j^{+1}}{\sqrt{2}}\sqrt{(l-m+1)(l+m)}\left(E_{l\,m}\,E^*_{l\,m+1} + B_{l\,m}\,B^*_{l\,m+1}\right)+\\ &+\xi_j^0\, m\left(E_{lm}\,E^*_{l\,m} + B_{l\,m}\,B^*_{l\,m}\right)\Bigg]+\end{aligned} \tag{64}$$

Notice that the third component in the last line is proportional to the energy, as it should (see [59, 60], for example).

If we add the magnetic field contributions and use equations (57) together with equation (2.26) in [58], we find the angular momentum flux for the electromagnetic waves:

$$\begin{aligned}\frac{dJ_j}{dt} &= \epsilon_{jab}\sum_{l\geq 1}\frac{1}{c^{2l+1}}\left[\frac{l+1}{l!\,(2l+1)!!}\left(Q_{aL-1}\,Q^{(1)}_{aL-1} + \frac{1}{c^2}\,M_{aL-1}\,M^{(1)}_{aL-1}\right)\right]+\\ &+\epsilon_{jab}\sum_{l\geq 1}\frac{1}{c^{2l+1}}\left[\frac{l^2}{(l+1)!\,(2l+1)!!}\left(Q_{aL-1}\,Q^{(1)}_{aL-1} + \frac{1}{c^2}\,M_{aL-1}\,M^{(1)}_{aL-1}\right)\right]\end{aligned} \tag{65}$$

6. Metric, electromagnetic potentials and dynamics at Newtonian level

Now that we have the linearized version of the Einstein-Maxwell system and we know (from the appendix) how to match the PN interior solution to the MPM exterior, we can calculate the potentials and of the equations of motion. From now on we shall work only at the higher order.

In what follows, we shall deal with functions of the type:

$$F(\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2) \sim \frac{1}{|\mathbf{x}-\mathbf{y}_A|^n} \qquad A = \{1, 2\} \quad n \geq 1$$

which are divergent when evaluated at the position of the particle A. To solve this problem, we use Hadamard regularization procedure as described in [17, 41, 54, 55, 56, 57]. In fact we don't need the whole machinery here, what we actually need is to assume that the function $F(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)$ admits the series expansion:

$$F(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)=\sum_{k=-k0}^{k=0} r_1^k\, f_k(\mathbf{n}_1,\mathbf{y}_1,\mathbf{y}_2)+O(r_1)$$

then, we can define the regularized function at point 1 to be the *Hadamard partie finie* (see [17]):

$$\left(F\right)_1 := F(\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)=\int\frac{d\Omega(\mathbf{n}_1)}{4\pi}f_0(\mathbf{n}_1,\mathbf{y}_1,\mathbf{y}_2) \tag{66}$$

6.1. The electromagnetic potential

The equation (41) for the background potential is a normal wave equation. We can easily find the Post Newtonian expansion of the solution at every order using the formula [17]:

$$\Box^{-1}f(t,x)=-\frac{1}{4\pi}\left[\int d^3x'\,\frac{f(t,x')}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{c}\int d^3x'\,f(t,x')+\frac{1}{2c^2}\int d^3x'\,|\mathbf{x}-\mathbf{x}'|\,f(t,x')\right]+\dots \tag{67}$$

At the higher order and in terms of the magnetic dipole vector, we have:

$$\overset{0}{A}_i=\epsilon_{ijk}\mathcal{M}_1^k\frac{n_1^j}{r_1^2}+1\leftrightarrow 2 \qquad \overset{0}{A}_0=\frac{1}{c}\epsilon_{klm}v_1^l\mathcal{M}_1^m\frac{n_1^k}{r_1^2}+1\leftrightarrow 2 \tag{68}$$

When evaluated at the position of the particle 1, these equations become:

$$\left(\overset{0}{A}_i\right)_1=\epsilon_{ijk}\mathcal{M}_1^k\frac{n_{12}^j}{r_{12}^2} \qquad \left(\overset{0}{A}_0\right)_1=\frac{1}{c}\epsilon_{klm}v_1^l\mathcal{M}_1^m\frac{n_{12}^k}{r_{12}^2} \tag{69}$$

At the higher order, the Faraday tensor at the position of particle 1 is therefore given by:

$$\left(\overset{0}{F}_{ij}\right)_1=2\frac{\epsilon_{ijk}\mathcal{M}_2^k}{r_{12}^3}-3\frac{\epsilon_{jsk}\mathcal{M}_2^k n_{12}^s-\epsilon_{isk}\mathcal{M}_2^k n_{12}^j}{r_{12}^3}n_{12}^s+O(c^{-1}) \tag{70}$$

$$\left(\overset{0}{F}_{0i}\right)_1=\frac{1}{c}\frac{\epsilon_{slm}v_2^l\mathcal{M}_2^m}{r_{12}^3}\left(\delta_{ks}-3n_{12}^k n_{12}^s\right)-\frac{1}{c}\frac{\epsilon_{ksl}\mathcal{M}_2^l}{r_{12}^3}\left(v_2^s-(n_{12}v_2)n_{12}^s\right)+O(c^{-2}). \tag{71}$$

6.2. The metric and electromagnetic perturbation potentials

In analogy to [17, 41], we define the sources of the gravitational potentials from the matter stress-energy tensor:

$$\sigma=\frac{T_{00}+T_{ii}}{c^2} \qquad \sigma_i=\frac{T_{0i}}{c} \qquad \sigma_{ij}=T_{ij} \tag{72}$$

On the other hand, from the electromagnetic stress-energy tensor, we define

$$\sigma^{elm} = T_{00}^{elm} + T_{ii}^{elm} \qquad \sigma_i^{elm} = T_{0i}^{elm} \qquad \sigma_{ij}^{elm} = T_{ij}^{elm} \tag{73}$$

Using the potentials defined in [17, 41, 18, 57], we see that the direct contributions from the electromagnetic stress-energy tensor start at the order $O(c^{-2})$, and the indirect contributions coming from the equations of motion start at the order $O(c^{-2})$: there are therefore no electromagnetic contributions at the Newtonian level.

6.3. Equations of motion

From equation (10), and using equation (70), we find at Newtonian level that the electromagnetic contributions to the acceleration of particle 1 is given by:

$$\begin{aligned}(a_i)_1^{elm} &= \frac{1}{2}\left(\overset{0}{F}_{ij}\right)_1 \mathcal{M}_1^j = \\ &= -\frac{3}{2m_1}\frac{\mathcal{M}_1^k\mathcal{M}_2^s}{r_{12}^4}\left[\left(\delta_{ks} - 5n_{12}^k n_{12}^s\right) - \delta_{ik}n_{12}^s + \delta_{is}n_{12}^k\right]\end{aligned} \tag{74}$$

In order to write down the equations of motion in the center of mass frame, we define (see for example [41]):

$$m = m_1 + m_2 \qquad \frac{1}{\nu} = m\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad \Delta = \frac{m_1 - m_2}{m}$$

with this, we can rewrite (74) as:

$$(a_1)_1^{elm} - (a_i)_2^{elm} = a_i^{elm} = -\frac{3}{2m\nu}\frac{\mathcal{M}_1^k\mathcal{M}_2^s}{r_{12}^4}\left[\left(\delta_{ks} - 5n_{12}^k n_{12}^s\right) - \delta_{ik}n_{12}^s + \delta_{is}n_{12}^k\right] \tag{75}$$

We can rewrite the previous equation introducing the orthonormal triad $\vec{n}, \vec{\ell}, \vec{\lambda}$ (see [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41]); (75) becomes $\vec{a}^{elm} = a_n\vec{n} + a_\ell\vec{\ell} + a_\lambda\vec{\lambda}$, where:

$$a_n = \vec{a}\cdot\vec{n} = -\omega^2 r = -\frac{3}{2}\frac{\mathcal{M}_1^k\mathcal{M}_2^s}{r_{12}^4}\left(\delta_{ks} - 3n_{12}^k n_{12}^s\right) \tag{76}$$

including the gravitational terms at Newtonian level, we read $\omega^2 = \frac{Gm}{r_{12}} + \frac{3}{2m\nu}\frac{(\mathcal{M}_1\cdot\mathcal{M}_2) - 3(\mathcal{M}_1 n)(\mathcal{M}_2 n)}{r_{12}^3}$. We also have:

$$a_\ell = \vec{a}\cdot\vec{\ell} = -\frac{3}{2m\nu}\frac{(\mathcal{M}_2 n)(\mathcal{M}_1\ell) + (\mathcal{M}_1 n)(\mathcal{M}_2\ell)}{r_{12}^4}. \tag{77}$$

Finally, we can discard the third term, since $a_\lambda \approx c^{-5}$ (see [41]).

At Newtonian level, the electromagnetic energy of the particle 1 is given by:

$$\begin{aligned}(E)_1^{elm} &= -\frac{1}{2}\mathcal{M}_1^{ik}\left(\overset{0}{F}_{ik}\right)_1 = \\ &= -\frac{1}{2}\frac{\mathcal{M}_1^k\mathcal{M}_2^s}{r_{12}^3}\left(\delta_{ks} - 3n_{12}^k n_{12}^s\right)\end{aligned} \tag{78}$$

In the center of mass frame, we have:

$$E^{elm} = \frac{\mathcal{M}_1^k\mathcal{M}_2^s}{r_{12}^3}\left(\delta_{ks} - 3n_{12}^k n_{12}^s\right). \tag{79}$$

6.4. Spin precession

From equation (11), we see that at Newtonian order only the electromagnetic torque will act on the spin vector, therefore we have:

$$\begin{aligned}\left.\frac{dS_1^i}{dt}\right|^{elm} &= \overset{0}{F}{}^{ij}\mathcal{M}_{1j} + O(c^{-2}) = \\ &= 2\frac{\epsilon_{ijk}\mathcal{M}_1^j\mathcal{M}_2^k}{r_{12}^3} - 3\frac{\epsilon_{jsk}\mathcal{M}_2^k\mathcal{M}_1^j n_{12}^i - \epsilon_{isk}\mathcal{M}_2^k(\mathcal{M}_1 n)}{r_{12}^3}n_{12}^s\end{aligned} \tag{80}$$

6.5. The magnetic moment evolution equation

At the higher order, only the star rotation is relevant so we have:

$$\frac{d\mathcal{M}_1^i}{dt} = \omega_1^{ij}\mathcal{M}_{1j} + O(c) \tag{81}$$

.

7. Electromagnetic contribution to the energy flux of gravitational waves

The general expressions for the gravitational waves flux and for the multipoles were given in [41, 58] and references therein. Since we are interested to the expression at the higher order, we only need the mass quadrupole:

$$\mathcal{F}^{GW} = \frac{G}{c^5}\left[\frac{1}{5}\, I_{ij}^{(3)} I_{ij}^{(3)} + O(c^{-2})\right] \tag{82}$$

where at the relevant order, the quadrupole moment is given by:

$$I_{ij} = \sigma\, x_{<ij>} + O(c^{-2}) \tag{83}$$

At the higher order, we have $\sigma = m_1\delta(x - x_1(t)) + m_2\delta(x - x_2(t))$, where $x_i(t)$ is the position of the body i at time t.

We need the third time derivative of (83), which is given by:

$$I_{ij}^{(3)} = \left(6\, v_1^{<i} a_1^{j>} + 2\, y_1^{<i} \dot{a}_1^{j>}\right) m_1 + \left(6\, v_2^{<i} a_2^{j>} + 2\, y_2^{<i} \dot{a}_2^{j>}\right) m_2 \qquad (84)$$

One has now to substitute the equation (74) and the analogous expression for the particle 2; it is also necessary to use the evolution equation for the magnetic moment (81). In this way, using the dimensionless parameter $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$ (see for example [41]), at the higher order one gets:

$$\mathcal{F}^{GW} = \frac{8c^5x^5}{5G^3m^3}\left(4G^2m^3 + 3(\mathcal{M}_2 n)(v\omega_1\mathcal{M}_1)\nu + 3(\mathcal{M}_1 n)(v\omega_2\mathcal{M}_2)\nu + 3(\mathcal{M}_2 v)(n\omega_1\mathcal{M}_1)\nu + 3(\mathcal{M}_1 v)(n\omega_2\mathcal{M}_2)\nu\right) \qquad (85)$$

where, for example, $(v\omega_1\mathcal{M}_1) = v_i\omega_1^{ij}\mathcal{M}_{1j}$.

Now, imposing $\vec{v} = \hat{w}\sqrt{\frac{Gm}{r_{12}} + \frac{3}{2m\nu}\frac{(\mathcal{M}_1 \cdot \mathcal{M}_2) - 3(\mathcal{M}_1 n)(\mathcal{M}_2 n)}{r_{12}^3}}$, where $\hat{w}$ is the direction of the relative velocity vector, and Taylor expanding the square root, we get, at the higher order:

$$\mathcal{F}^{GW} = \frac{32}{5}\frac{c^5\nu^2x^5}{G} + \frac{48c^6\nu}{5G^3m^3}\left((\mathcal{M}_2 w)(n\omega_1\mathcal{M}_1) + (\mathcal{M}_1 w)(n\omega_2\mathcal{M}_2) + (\mathcal{M}_2 n)(w\omega_1\mathcal{M}_1) + (\mathcal{M}_1 n)(w\omega_2\mathcal{M}_2)\right) x^{11/2} \qquad (86)$$

8. Electromagnetic waves flux

In our case, part of the energy is radiated also through electromagnetic waves. We have calculated the flux in section 5 equation (61); since we stop at the higher order order, the only contributions come from $\overset{0}{Q}_L$, in particular from the dipole moment $\overset{0}{Q}_i$; using the magnetic moment vector, we have:

$$\overset{0}{Q}_i = \epsilon_{ikl}\frac{v_1^K}{c}\mathcal{M}_1^l + \epsilon_{ikl}\frac{v_2^K}{c}\mathcal{M}_2^l \qquad (87)$$

With this, we find that the electromagnetic waves flux for circular orbit is given by:

$$\begin{aligned}\mathcal{F}^{EW} = &\frac{5}{24}\frac{x^7c^7}{G^4m^4}\left[(\Delta - 1)\left(\mathcal{M}_1^2 + (\mathcal{M}_1 w)^2\right) + (\Delta + 1)\Big((\mathcal{M}_2 w)^2 + \mathcal{M}_2^2\Big)\right] + \\ &+ \frac{15}{16}\frac{x^9c^{11}}{G^7m^7}\left[(\Delta - 1)\left(\mathcal{M}_1^2 + (\mathcal{M}_1 w)^2\right) + (\Delta + 1)\Big((\mathcal{M}_2 w)^2 + \mathcal{M}_2^2\Big)\right]\Big((\mathcal{M}_1 \cdot \mathcal{M}_2) - 3(\mathcal{M}_1 n)(\mathcal{M}_2 n)\Big)\end{aligned} \qquad (88)$$

9. Orbital phase evolution and number of gravitational waves cycles

To illustrate the quantitative importance of the newly calculated electromagnetic terms in the flux of GW one can calculate the accumulated number of gravitational waves cycles for a ground based detector.

As usual (see [41, 40, 37, 53, 39]), we start from the energy balance:

$$\frac{dE}{dt} = -\mathcal{F} \tag{89}$$

where $\mathcal{F}$ is the flux. Defining $\dot{\phi} = \omega$ (where ϕ is the orbital phase), after some manipulations, we can rewrite the previous equation as:

$$\frac{d\phi}{dx} = -\frac{\mathcal{F}}{\omega}\left(\frac{dE}{dx}\right)^{-1} \tag{90}$$

where ω must be expressed in terms of x. The resulting right hand side must be expanded in series and eventually integrated term by term.

In our case, on the right hand side of the energy balance (89) we have to include both electromagnetic and gravitational waves energy flux; therefore, we have two contributions to the phase evolution:

$$\frac{d\phi}{dt} = \frac{d\phi^{GW}}{dt} + \frac{d\phi^{EW}}{dt} = -\frac{\mathcal{F}^{GW} + \mathcal{F}^{EW}}{\omega}\left(\frac{dE}{dx}\right)^{-1} \tag{91}$$

where $\mathcal{F}^{GW}$ is the *total* gravitational waves flux calculated in [41, 40, 37, 53, 39] and in this paper and $\mathcal{F}^{EW}$ is given in (88). ϕ^{GW} and ϕ^{EW} are those terms of the phase evolution ϕ only contain contributions coming from the gravitational and electromagnetic waves flux respectively. Detectors, however, can measure directly only the former; the contributions due to the emission of electromagnetic waves can only be inferred indirectly from the decay of the orbit.

After the integration of the first of (91), for the gravitational waves contributions to the phase evolution, we find that electromagnetic corrections start at relative order x^2:

$$\phi^{GW}_{elm} = -\frac{c^5 x^{-5/2}}{32\nu}\left[-\frac{15}{16\nu}\frac{c}{G^3 m^3}\left((\mathcal{M}_2 w)(n\omega_1\mathcal{M}_1)+(\mathcal{M}_1 w)(n\omega_2\mathcal{M}_2)+(\mathcal{M}_2 n)(w\omega_1\mathcal{M}_1)+(\mathcal{M}_1 n)(w\omega_2\mathcal{M}_2)\right)x^{1/} \right. \tag{92}$$

For the electromagnetic wave contribution, we find:

$$\phi^{EW} = -\frac{c^5 x^{-5/2}}{32\nu}\left[\frac{125}{384}\frac{c^2}{G^3 m^4}\left((\Delta-1)\left(\mathcal{M}_1^2-\mathcal{M}_1 w)^2\right)+(\Delta+1)\left(\mathcal{M}_2^2-\mathcal{M}_2 w)^2\right)\right)x^2\right] \tag{93}$$

10. Conclusion

Motivated by the intense magnetic field of NS, we have studied the effect of the star's magnetic moment on the production of gravitational waves in a NS-NS binary system.

We have found the expression of the equations of motion and precession for a system of point-like, uncharged, magnetized NS; with these equations, we have found the covariant 4-momentum and the conserved mass. We have described the Einstein-Maxwell system of equations, found the MPM solution and written the energy and angular momentum for the electromagnetic field.

We have calculated the equations of motion and precession at Newtonian order and with these we have calculated the higher order term for the gravitational and electromagnetic waves flux and the electromagnetic contribution to the accumulated number of cycles for a ground based detector.

Considering a system similar to the *double pulsar* system PSR J0737-3039 [67, 68] in which both stars have radius $r \approx 10$ km and mass $m_1 = m_2 = 1.4\mathrm{M}_\odot$ and in which one NS has a magnetic field of about 10^{12}G (magnetic dipole $\mathcal{M} \approx 10^{30}$ G cm^{-3}) and the other has magnetic field of about 10^{10}G (magnetic dipole 10^{28} G cm^{-3}), remembering equation (15), we can see that the electromagnetic contributions to gravitational waves cycle amounts to

$$-1.6\Big((\mathcal{M}_2 w)(n\omega_1\mathcal{M}_1)\chi_1+(\mathcal{M}_1 w)(n\omega_2\mathcal{M}_2)\chi_2+(\mathcal{M}_2 n)(w\omega_1\mathcal{M}_1)\chi_1+(\mathcal{M}1 n)(w\omega_2\mathcal{M}_2)\chi_2\Big) \tag{94}$$

where we have defined $(n\omega_1\mathcal{M}_1) = n^i_{12}\omega^1_{ij}\mathcal{M}^j_1$, $(\mathcal{M}_1 n) = \mathcal{M}^i_1 n^i_{12}$ etc. For comparison, the purely gravitational contribution to the number of cycles is 15952.6 for a neutron satr binary system [41], therefore we can say that the effect of the intense NS magnetic field on the production of gravitational waves is barely observable. The electromagnetic waves flux contribution amounts to about 10^{-32} and is, therefore, negligible.

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