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On the Hilbert vector of the Jacobian module of a plane curve

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Abstract. We identify several classes of complex projective plane curves $C : f = 0$, for which the Hilbert vector of the Jacobian module $N(f)$ can be completely determined, namely the 3-syzygy curves, the maximal Tjurina curves and the nodal curves, having only rational irreducible components. A result due to Hartshorne, on the cohomology of some rank 2 vector bundles on $\mathbb{P}^2$, is used to get a sharp lower bound for the initial degree of the Jacobian module $N(f)$, under a semistability condition.

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Introduction

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in three variables x, y, z with complex coefficients. Let $C : f = 0$ be a reduced curve of degree d in the complex projective plane $\mathbb{P}^2$. We denote by $J_f = (f_x, f_y, f_z)$ the Jacobian ideal, i.e. the homogeneous ideal in S spanned by the partial derivatives f_x, f_y, f_z of f. Since C is reduced, the singular subscheme of C, which is defined by the Jacobian ideal J_f, is 0-dimensional, and its degree is denoted by $\tau(C)$, and is called the global Tjurina number of C. Consider the graded S-module of Jacobian syzygies of f, namely

$$\mathrm{Syz}(J_f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$

Let $mdr(f) := \min\{k : \mathrm{Syz}(J_f)_k \neq (0)\}$ be the minimal degree of a Jacobian syzygy for f; in this paper we will assume $mdr(f) \geq 1$ unless otherwise specified. In fact, if $mdr(f) = 0$, the curve C is a pencil of lines.

We say that $C : f = 0$ is a m-syzygy curve if the S-module $\mathrm{Syz}(J_f)$ is minimally generated by m homogeneous syzygies, $r_1, r_2, \dots, r_m$, of degree $d_i = \deg r_i$, ordered such that

$$1 \le d_1 \le d_2 \le \cdots \le d_m.$$

The multiset $(d_1, d_2, \ldots, d_m)$ is called the exponents of the plane curve C and $\{r_1, r_2, \ldots, r_m\}$ is said to be a minimal set of generators for the S-module $\mathrm{Syz}(J_f)$. Some of the m-syzygy curves have been carefully studied. We recall that:

- a 2-syzygy curve C is said to be *free*, since then the S-module $\mathrm{Syz}(J_f)$ is a free module of the rank 2, see [3], [5], [10], [27], [28], [29];
- a 3-syzygy curve is said to be *nearly free* when $d_3 = d_2$ and $d_1 + d_2 = d$, see [2], [3], [5], [6], [11], [23];
- a 3-syzygy line arrangement is said to be a *plus-one generated line arrangement* of level d_3 when $d_1 + d_2 = d$ and $d_3 \ge d_2$, see [1]. By extension, a 3-syzygy curve C is said to be a *plus-one generated curve* of level d_3 when $d_1 + d_2 = d$ and $d_3 \ge d_2$, see [14].

The Jacobian module of f, or of the plane curve $C : f = 0$, is the quotient module $N(f) = \hat{J}_f / J_f$, with $\hat{J}_f$ the saturation of the ideal J_f with respect to the maximal ideal $\mathbf{m} = (x, y, z)$ in S. The Jacobian module $N(f)$ coincides with $H^0_{\mathbf{m}}(S/J_f)$, see [26]. Let $n(f)_j = \dim N(f)_j$, $T = 3(d-2)$ and recall that the Jacobian module $N(f)$ enjoys a weak Lefschetz type property, see [7] for this result, and [19], [20], [22] for Lefschetz properties of Artinian algebras in general. More precisely, we have

$$n(f)_0 \le n(f)_1 \le \cdots \le n(f)_{\lfloor T/2 \rfloor - 1} \le n(f)_{\lfloor T/2 \rfloor} \ge n(f)_{\lfloor T/2 \rfloor + 1} \ge \cdots \ge n(f)_T. \quad (1)$$

We consider the following two invariants for a curve $C : f = 0$

$$\sigma(C) := \min\{j : n(f)_j \ne 0\} = \mathrm{indeg}\big(N(f)\big), \qquad \nu(C) := \max\{n(f)_j\}_j.$$

The self duality of the graded S-module $N(f)$, see [21], [26], [30], implies that

$$n(f)_j = n(f)_{T-j}, \quad (2)$$

for any integer j, in particular $n(f)_k \ne 0$ exactly for $k = \sigma(C), \ldots, T - \sigma(C)$.

The main aim of this paper is to identify classes of curves $C : f = 0$ for which the Hilbert vector $\big(n(f)_j\big)$ of the Jacobian module $N(f)$ can be completely determined. In [13], Theorem 3.1, Theorem 3.2, recalled below in Theorem 1.1, there is a description of the dimensions $n(f)_j$ for a certain range of j. Moreover, in [14], Theorem 3.9, Corollary 3.10, recalled below in Theorem 1.3 and Corollary 1.4, there are descriptions of the minimal resolution of $N(f)$, when $C : f = 0$ is a 3-syzygy curve, and respectively a plus-one generated curve of degree $d \ge 3$.

Using these results, we first give a general formula for the Hilbert vector $(n(f)_j)$ of the Jacobian module of a 3-syzygy curve in Theorem 2.1, as well as a graphic representation of its behavior. Then we determine the Hilbert vector $(n(f)_j)$ of the Jacobian module $N(f)$ when $C : f = 0$ is a maximal Tjurina curve, see Proposition 3.1. Next we get some information on the Hilbert vector $(n(f)_j)$ when $C : f = 0$ is a nodal curve, which is complete if in addition all the irreducible components of C are rational, see Theorem 3.2.

In the final section we use a result due to Hartshorne, see [18], Theorem 7.4, to relate the cohomology of some rank 2 vector bundles on $\mathbb{P}^2$ to the Hilbert vector $(n(f)_j)$ of the Jacobian module $N(f)$. More precisely, we get in this way a sharp lower bound for the initial degree $\sigma(C)$ of the Jacobian module $N(f)$, under the condition $mdr(f) \geq (d-1)/2$, see Theorem 4.2.

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1. Preliminaries

We recall some notations and results. Let $C : f = 0$ be a reduced complex plane curve of $\mathbb{P}^2$, assumed not free, and consider the Milnor algebra $M(f) = S/J_f$, where $J_f = (f_x, f_y, f_z)$ is the Jacobian ideal. The general form of the minimal resolution for the Milnor algebra $M(f)$ of such a curve $C : f = 0$ is

$$0 \to \bigoplus_{i=1}^{m-2} S(-e_i) \to \bigoplus_{i=1}^{m} S(1-d-d_i) \to S^3(1-d) \to S, \tag{3}$$

with $e_1 \leq e_2 \leq \cdots \leq e_{m-2}$ and $1 \leq d_1 \leq d_2 \leq \cdots \leq d_m$. It follows from [21], Lemma 1.1 that one has

$$e_j = d + d_{j+2} - 1 + \varepsilon_j,$$

for $j = 1, \dots, m-2$ and some integers $\varepsilon_j \geq 1$. The minimal resolution of $N(f)$ obtained from (3), by [21], Proposition 1.3, is

$$0 \to \bigoplus_{i=1}^{m-2} S(-e_i) \to \bigoplus_{i=1}^{m} S(-\ell_i) \to \bigoplus_{i=1}^{m} S\big(d_i - 2(d-1)\big) \to \bigoplus_{i=1}^{m-2} S\big(e_i - 3(d-1)\big),$$

where $\ell_i = d + d_i - 1$. It follows that

$$\sigma(C) = 3(d-1) - e_{m-2} = 2(d-1) - d_m - \varepsilon_{m-2}. \tag{4}$$

The following result describes the central part of the Hilbert vector of $N(f)$.

Theorem 1.1. *Let $C : f = 0$ be a reduced, non free curve of degree d and set $r = mdr(f)$. Then one has the following.*

(i) *if $r \geq \frac{d}{2}$ and $2d - 4 - r \leq j \leq d - 2 + r$, then*

$$n(f)_j = \begin{cases} 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) & \text{for } d = 2d' + 1 \\ 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) & \text{for } d = 2d' \end{cases} \tag{5}$$

(ii) *if $r < \frac{d}{2}$ and $d + r - 3 \leq j \leq 2d - r - 3$, then $n(f)_j = \nu(C)$. Moreover $n(f)_{d+r-4} = n(f)_{2d-r-2} = \nu(C) - 1$.*

Proof. See [13], Theorem 3.1 and Theorem 3.2. □

By Theorem 1.1, in case (i), the points $(j, n(f)_j)$ lie on an upward pointing parabola. Moreover, using the formulas (9) and (10) and Remark 4.1, the claim (5) can be written:

$$n(f)_j = \begin{cases} \nu(C) - \left(j - \lfloor \frac{T}{2} \rfloor\right)\left(j - \lceil \frac{T}{2} \rceil\right) & \text{for } d = 2d' + 1 \\ \nu(C) - \left(j - \frac{T}{2}\right)^2 & \text{for } d = 2d' \end{cases}$$

with $T = 3(d-2)$ as above. On the other hand, in the case (ii), the points $(j, n(f)_j)$ lie on a horizontal line segment, with a one-unit drop at the extremities, as represented in Figure 1 below.

Recall the following definition, see [5], [10].

Definition 1.2. For a plane curve $C : f = 0$, the *coincidence threshold* of f is the integer

$$ct(f) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of the same degree d as f and such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

It is known that $ct(f) \geq d - 2 + mdr(f)$.

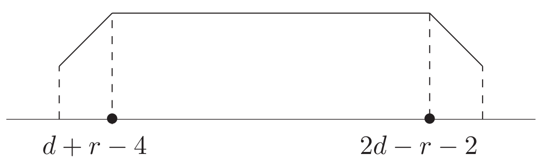

Figure 1. The case $r < \frac{d}{2}$.

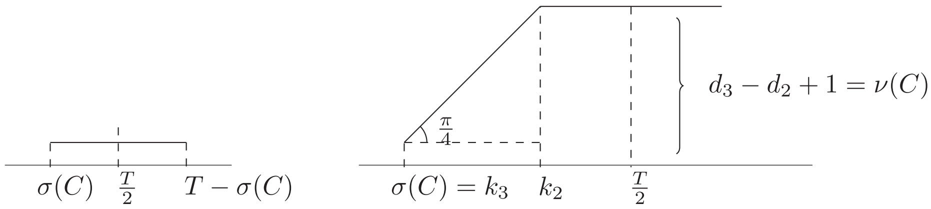

Figure 2. The case of plus-one generated curves

Note that $C: f=0$ is a free curve if and only if $\nu(C)=0$, hence $N(f)=0$. In other words, the Jacobian ideal is satured in this case, i.e. $\hat{J}_f=J_f$, see [8], [29]. The first nontrivial case is that of nearly free curves; indeed, by [11], Corollary 2.17, for a nearly free curve $C: f=0$, one has $N(f)\neq 0$ and $\nu(C)=1$. Moreover $\sigma(C)=d+d_1-3$ and this describes completely the Hilbert vector of the Jacobian module of a nearly free curve. In particular it has the shape described on the left side of Figure 2. Recall that a nearly free curve is exactly a plus-one generated curve with exponents satisfying $d_2=d_3$. For the more general case of the 3-syzygy curves, we recall the following result.

Theorem 1.3 ([14], Theorem 3.9). *Let $C: f=0$ be a 3-syzygy curve with exponents (d_1,d_2,d_3) and set $e=d_1+d_2+d_3$. Then the minimal free resolution of $N(f)$ as a graded S-module has the form*

$$0\to S(-e)\to\bigoplus_{i=1}^{3}S(1-d-d_i)\to\bigoplus_{i=1}^{3}S(d_i+2-2d)\to S(e+3-3d), \tag{6}$$

where the leftmost map is the same as in the resolution (3), when $m=3$. In particular,

$$\sigma(C)=3(d-1)-(d_1+d_2+d_3).$$

This implies the following.

Corollary 1.4 ([14], Corollary 3.10). *Let $C: f=0$ be a plus-one generated curve of degree $d\geq 3$ with (d_1,d_2,d_3), which is not nearly free, i.e. $d_2<d_3$. Set $k_j=2d-d_j-3$ for $j=1,2,3$. Then one has the following minimal free resolution of $N(f)$ as a graded S-module:*

$$\begin{aligned}0&\to S(-d-d_3)\to S(-d-d_3+1)\oplus S(-k_1-2)\oplus S(-k_2-2)\\&\to S(-k_1-1)\oplus S(-k_2-1)\oplus S(-k_3-1)\to S(-k_3).\end{aligned}$$

In particular $\sigma(C) = k_3 < k_2 \leq \frac{T}{2}$ *and the Hilbert vector of* $N(f)$ *is given by following formulas:*

(1) $n(f)_j = 0$ *for* $j < k_3$;

(2) $n(f)_j = j - k_3 + 1$ *for* $k_3 \leq j \leq k_2$;

(3) $n(f)_j = d_3 - d_2 + 1 = \nu(C)$ *for* $k_2 \leq j \leq \frac{T}{2}$.

By above corollary, the Hilbert vector of the Jacobian module of a plus-one generated curve of degree d and level d_3 has the shape given on the right hand side of Figure 2, *where we have drawn only the part corresponding to* $j \leq \frac{T}{2}$, *due to the symmetry* (2).

2. Results on the Hilbert vector of $N(f)$ for 3-syzygy curves

As a simple example of a 3-syzygy curve which is not a plus-one generated curve, let $C : f = 0$ be a smooth curve of degree $d \geq 3$, where $d_1 = d_2 = d_3 = d - 1$. It is known that the Hilbert function of the Milnor algebra $M(f)$ is in this case $\left(\frac{1-t^{d-1}}{1-t}\right)^3$. For a smooth curve we have $N(f) = M(f)$, hence $n(f)_j = \dim M(f)_j$ and the Hilbert vector of the Jacobian module $N(f)$ has the shape described in Figure 3. It is interesting to notice the change in convexity when we pass through the value $j = d - 1$.

For a general 3-syzygy curve, we have the following result.

Theorem 2.1. *Let* $C : f = 0$ *be a* 3*-syzygy curve of degree* d*, not plus-one generated, with exponents* $d_1 \leq d_2 \leq d_3$. *Set* $e = d_1 + d_2 + d_3$ *and* $k_i = 2(d - 1) - d_i$ *for* $i = 1, 2, 3$. *Then the following hold.*

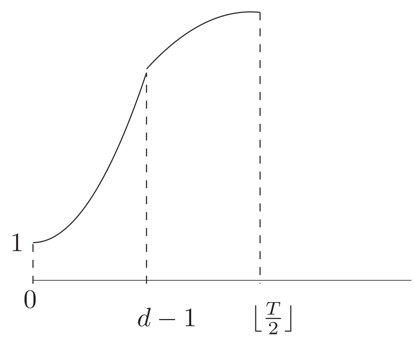

Figure 3. The case of smooth curves

$$n(f)_k = \begin{cases} 0 \quad \textit{for } k < \sigma \\ \binom{k-\sigma+2}{2} \quad \textit{for } \sigma \le k < k_3 \\ \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} \quad \textit{for } k_3 \le k < k_2 \\ \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2} - \binom{k-k_2+2}{2} \quad \textit{for } k_2 \le k < T_0, \end{cases}$$

where $\sigma = \sigma(C) = 3(d-1) - e$ *and*

$$T_0 = \begin{cases} k_1 - 1 \quad \textit{if } d_1 \ge \frac{d}{2} \\ d + d_1 - 2 \quad \textit{if } d_1 < \frac{d}{2}. \end{cases}$$

Note that $n(f)_k$ is known for $T_0 \le k \le \frac{T}{2}$ in view of Theorem 1.1, hence the information on the Hilbert vector of $N(f)$ is complete in this situation.

Proof. Note that $\sigma \ge 0$, since, by [14], Theorem 2.4, we have $d_j \le d-1$ for $j = 1, 2, 3$. Then, by [14], Theorem 2.3, we have $d_1 + d_2 > d > d-1$ and hence

$$\sigma = 3(d-1) - (d_1 + d_2 + d_3) < 3(d-1) - (d-1) - d_3 = 2(d-1) - d_3 = k_3.$$

By Theorem 1.3, the minimal resolution of $N(f)$ is

$$0 \to S(-e) \to \bigoplus_{j=1}^{3} S(-\ell_j) \to \bigoplus_{i=1}^{3} S(-k_i) \to S\big(e - 3(d-1)\big),$$

where $\ell_j = d - 1 + d_j$. We note that $k_3 \le k_2 \le k_1$ and also $k_2 \le T_0$. If we fix k with $\sigma \le k < k_3$, the minimal resolution of $N(f)$ above yields

$$n(f)_k = \dim S_{k-\sigma} = \binom{k-\sigma+2}{2}.$$

Now we consider the case $k_3 \le k < k_2$. We have $\ell_1 > k_2$, since $d_1 + d_2 > d$ as we have seen above. It follows that

$$n(f)_k = \dim S_{k-\sigma} - \dim S_{k-k_3} = \binom{k-\sigma+2}{2} - \binom{k-k_3+2}{2}.$$

This difference is a linear form in k, and the coefficient of k is given by $(k_3 - \sigma)$. Note that $k_3 - \sigma = 2(d-1) - d_3 - 3(d-1) + e = d_1 + d_2 - (d-1) \ge 2$. To continue, we need to discuss the position of ℓ_1 with respect to k_1. Note that $\ell_1 > k_1$ if and only if $d_1 \ge d/2$. Hence we have to consider two cases.

Case 1: $d_1 \geq d/2$. In this case, we can compute the value $n(f)_k$ for $k \leq k_1 - 1$ exactly as above, and we get

$$n(f)_k = \binom{k - \sigma + 2}{2} - \binom{k - k_3 + 2}{2} - \binom{k - k_2 + 2}{2}.$$

Note that in this case $T_0 = k_1 - 1 = 2(d - 1) - d_1 - 1 > 2d - 4 + d_1$ and hence all the Hilbert vector $(n(f)_j)$ is known by using Theorem 1.1 (i).

Case 2: $d_1 < d/2$. In this case $\ell_1 \leq k_1$, and we can compute the value $n(f)_k$ for $k \leq \ell_1 - 1$ exactly as above, obtaining the same formula. Note that in this case $T_0 = \ell_1 - 1 = d - 2 + d_1 > d - 3 + d_1$, and hence again all the Hilbert vector $(n(f)_j)$ is known by using Theorem 1.1 (ii). □

Example 2.2. Let $C : f = (x^9 + y^4z^5)^7 + xz^{62} = 0$, a singular curve of degree $d = 63$. It is a 3-syzygy curve, not plus-one generated, with $d_1 = 9$, $d_2 = 56$ and $d_3 = 62$. We have

$$e = \sum_{i=1}^{3} d_i = 127, \qquad \sigma = 59, \qquad k_3 = 62, \qquad k_2 = 68.$$

Since $d_1 < \frac{d}{2}$, $T_0 = d + d_1 - 2 = 70$. The first quadratic part is for $k \in [59, 61]$, the middle linear part is for $k \in [62, 67]$, and the second quadratic part is for $k \in [68, 70]$. This second quadratic part is too short, containing only 3 points $(j, n(f)_j)$, to be seen in a graphical representation of the corresponding Hilbert vector. Note also that one has $n(f)_k = \nu(C) = 27$ for $k \in [69, 91]$, where $91 = \lfloor T/2 \rfloor$. In particular, the last two points on the second quadratic part are in fact situated on this horizontal line segment.

Example 2.3. Let $C : f = (x + y)^2(x - y)^2(x + 2y)^2(x - 2y)^2(x + 3y)^2(x - 3y)^2 \cdot (x + 4y)^2(x - 4y)^2(x + 5y)^2(x - 5y)^2 + z^{20} = 0$, a singular curve of degree $d = 20$. It is a 3-syzygy curve not plus-one generated, with $d_1 = 9$ and $d_2 = d_3 = 19$. We have

$$e = \sum_{i=1}^{3} d_i = 47, \qquad \sigma = 10, \qquad k_3 = k_2 = 19.$$

Since $d_1 < \frac{d}{2}$, $T_0 = d + d_1 - 2 = 27 = T/2$. The first quadratic part is for $k \in [10, 18]$, the middle linear part is missing since $k_2 = k_3$, the second quadratic part is for $k \in [19, 27]$ and one has $n(f)_k = \nu(C) = 81$ for $k \in [26, 27]$, where $27 = T/2 = T_0$.

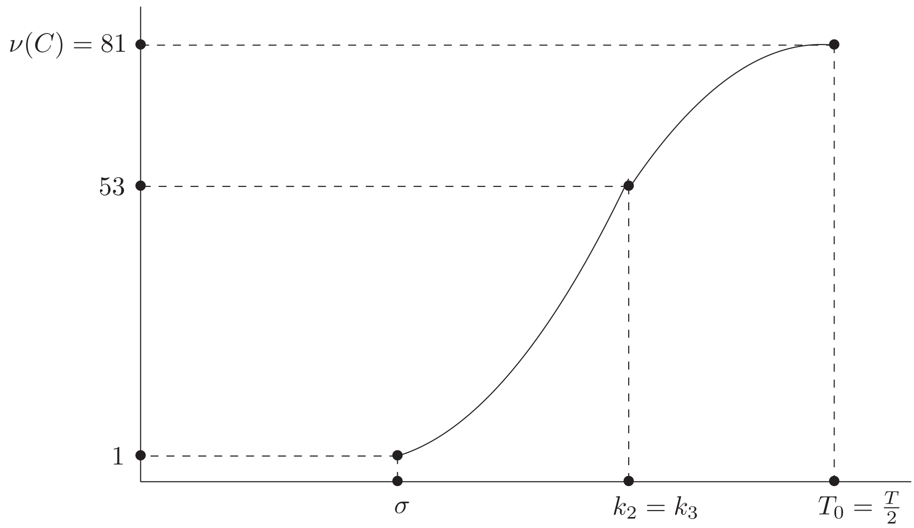

Figure 4. The Hilbert vector for Example 2.3

3. Maximal Tjurina curves and nodal curves

We assume in this section that $r = d_1 \geq d/2$.

A reduced plane curve $C : f = 0$ of degree d is called a maximal Tjurina curve if the global Tjurina number $\tau(C)$ equals the du Plessis-Wall upper bound, namely if

$$\tau(C) = (d-1)(d-r-1) + r^2 - \binom{2r-d+2}{2}, \tag{7}$$

see [15], [16], [17]. We know that a reduced plane curve $C : f = 0$ of degree d is a maximal Tjurina curve if and only if one has $d_1 = d_2 = \cdots = d_m = r$, $e_1 = e_2 = \cdots = e_{m-2} = d + r$ and $m = 2r - d + 3$, see [15], Theorem 3.1. Using now the equality (4), it follows that in this case

$$\sigma(C) = 2d - r - 3. \tag{8}$$

Theorem 1.1 yields then the following result.

Proposition 3.1. *Let $C : f = 0$ be a maximal Tjurina curve of degree d with $r = d_1 \geq d/2$. Then the Hilbert vector of the Jacobian module $N(f)$ is given by the following*

$$n(f)_j = \begin{cases} 3(d')^2 - (j - 3d' + 2)(j - 3d' + 1) - \tau(C) & \textit{for } d = 2d' + 1 \\ 3(d')^2 - 3d' + 1 - (j - 3d' + 3)^2 - \tau(C) & \textit{for } d = 2d' \end{cases}$$

for $2d - 3 - r \leq j \leq d - 3 + r$ *and* $n(f)_j = 0$ *otherwise.*

Consider now an arbitrary nodal curve $C : f = 0$ of degree d in $\mathbb{P}^2$. Let $\mathcal{N}$ denote the set of nodes of the curve C and $n(C)$ the number of irreducible components of C. For such curves we have the following result.

Theorem 3.2. *Let* $C : f = 0$ *be a nodal curve in* $\mathbb{P}^2$ *of degree* $d \geq 4$. *Then one has the following, with* f_s *as in Definition* 1.2.

$$n(f)_k = \begin{cases} m(f_s)_k - |\mathcal{N}| & \textit{for } d - 3 < k \leq T/2 \\ m(f_s)_k - |\mathcal{N}| + n(C) - 1 & \textit{for } k = d - 3. \end{cases}$$

Moreover, when all the irreducible components of C are rational, one has in addition $n(f)_k = 0$ *for* $k \leq d - 3$.

Proof. For any reduced plane curve $C : f = 0$, one clearly has

$$n(f)_k = m(f)_k - d(f)_k,$$

where $m(f)_k = \dim M(f)_k$ and $d(f)_k = \dim S_k/(\hat{J}_f)_k$. Since we have to determine $n(f)_k$ only for $k \leq T/2$ by symmetry, and since $ct(f) \geq d - 2 + r > T/2$ when $r = d_1 \geq d/2$, it follows that

$$n(f)_k = m(f_s)_k - d(f)_k,$$

with f_s as in Definition 1.2 and $k \leq T/2$. In particular, for such curves, we have to determine only the values $d(f)_k$ for $k \leq T/2$. On the other hand, we know that

$$d(f)_k = \tau(C),$$

for $k \geq T - ct(C)$, see [4], Proposition 2. In particular, this equality holds for $k \geq 3(d-2) - (d - 2 + r) = 2d - 4 - r$, see also the proof of [13], Theorem 3.1. Assume now that $C : f = 0$ is a nodal curve in $\mathbb{P}^2$. Then $r = d_1 \geq d - 2 \geq d/2$ for $d \geq 4$, see [8], Example 2.2 (i). Let $\operatorname{def} S_k(\mathcal{N})$ denote the defect of the set of nodes $\mathcal{N}$ with respect to the linear system S_k. Then it is known that

$$d(f)_k = |\mathcal{N}| - \operatorname{def} S_k(\mathcal{N}),$$

see [4]. On the other hand, ([9], Corollary 1.6) implies that $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k > d - 3$ and $\operatorname{def} S_k(\mathcal{N}) = n(C) - 1$ for $k = d - 3$. If all the irreducible components of C are rational, then ([12], Theorem 2.7) shows that $n(f)_k = 0$ for $k \leq d-3$. These facts imply our claims. □

4. Relation to a result by Hartshorne

Let $C : f = 0$ be a curve of degree d in $\mathbb{P}^2$, and let $r = mdr(f)$ be the minimal degree of a Jacobian syzygy for f. In this section we give some informations about the invariant $\sigma(C)$, using a result by Hartshorne, namely [18], Theorem 7.4. We recall that the sheafification of $\mathrm{Syz}(J_f)$, denoted by $E_C := \widetilde{\mathrm{Syz}(J_f)}$, is a rank two vector bundle on $\mathbb{P}^2$, see [2], [25], [26]. We set

$$e(f)_m = \dim \mathrm{Syz}(J_f)_m = \dim H^0\big(\mathbb{P}^2, E_C(m)\big),$$

for any integer m. Associated to the vector bundle E_C there is the normalized vector bundle $\mathscr{E}_C$, which is the twist of E_C such that $c_1(\mathscr{E}_C) \in \{-1, 0\}$. More precisely,

when $d = 2d' + 1$ is odd:

$$\mathscr{E}_C = E_C(d'), \qquad c_1(\mathscr{E}_C) = 0, \qquad c_2(\mathscr{E}_C) = 3(d')^2 - \tau(C), \tag{9}$$

and

when $d = 2d'$ is even:

$$\mathscr{E}_C = E_C(d'-1), \qquad c_1(\mathscr{E}_C) = -1, \qquad c_2(\mathscr{E}_C) = 3(d')^2 - 3d' + 1 - \tau(C), \tag{10}$$

see [13], Section 2.

Remark 4.1. The vector bundle E_C is stable if and only if $\mathscr{E}_C$ has no sections, see [24], Lemma 1.2.5. This is equivalent to $r = mdr(f) \geq \frac{d}{2}$, see [26], Proposition 2.4. Moreover by [13], Theorem 2.2 and using the formulas (9) and (10), we have that for a stable vector E_C, $c_2(\mathscr{E}_C) = \nu(C)$. Moreover, the vector bundle E_C is semistable if and only if $r = mdr(f) \geq (d-1)/2$, see again [24], Lemma 1.2.5, a condition that occurs in our Theorem 4.2 below.

The important key point is the identification

$$H^1\big(C, E_C(k)\big) = N(f)_{k+d-1}$$

for any integer k, see [26], Proposition 2.1. Hence the study of the Hilbert vector of the Jacobian module $N(f)$ is equivalent to the study of the dimension of $H^1\big(C, E_C(k)\big)$.

Theorem 4.2. *Let $C : f = 0$ be a curve of degree d, and let $r = mdr(f)$ be the minimal degree of a Jacobian syzygy for f. Assume that $r \geq (d-1)/2$, in other words that the rank 2 vector bundle E_C is semistable. Then we have the following.*

(1) *If $d = 2d' + 1$ is odd, then*

$$\sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1.$$

(2) *If $d = 2d'$ is even, then*

$$\sigma(C) \geq \tau(C) - 2(d')^2 - 2rd' + r^2 + 5d' + r - 3.$$

The above inequalities are sharp, in particular they are equalities when C is a maximal Tjurina curve with $r \geq d/2$.

Proof. We discuss only the case $d = 2d' + 1$, the other case being completely similar. One has

$$n(f)_k = h^1(\mathbb{P}^2, \mathscr{E}_C(k - 3d')).$$

Moreover $h^0(\mathbb{P}^2, \mathscr{E}_C(t)) = h^0(\mathbb{P}^2, E_C(t + d')) \neq 0$ if and only if $t + d' \geq r$. Hence the minimal t satisfying this condition is $t_m = r - d' \geq 0$. Then [18], Theorem 7.4 implies that $n(f)_k = 0$ when

$$k - 3d' \leq -c_2(\mathscr{E}) + t_m^2 - 2.$$

Using the formula for t_m above, and the formula for $c_2(\mathscr{E})$ given in the equations (9), we get that $n(f)_k = 0$ when

$$k \leq \tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 2,$$

which clearly implies our claim (1). The fact that the inequality in (1) is in fact an equality when C is a maximal Tjurina curve with $r \geq d/2$ follows by a direct computation. Indeed, using the above definition of a maximal Tjurina curve of degree $d = 2d' + 1$, namely the equality (7), we see that

$$\tau(C) = 2(d')^2 + 2rd' - r^2 - r + d'.$$

Hence

$$\tau(C) - 2(d')^2 - 2rd' + r^2 + 3d' - 1 = 2d - r - 3 = \sigma(C),$$

where the last equality follows from (8). □

Example 4.3. Let $C : f = 0$ be a curve of degree $d = 2d' + 1$, having a unique node as singularities. Then it is known that $r = d - 1 = 2d'$, and $\tau(C) =$

$\sigma(C) = 1$. The inequality in Theorem 4.2 (1) is in this case

$$1 \geq d'(3 - 2d'),$$

hence the two terms in this inequality can be far apart in some cases.

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References

[1] T. Abe, *Plus-one generated and next to free arrangements of hyperplanes*, International Mathematics Research Notices, rnz099, https://doi.org/10.1093/imrn/rnz099.

[2] T. Abe, A. Dimca, *On the splitting types of bundles of logarithmic vector fields along plane curves*, Internat. J. Math. **29** (2018), no. 8.

[3] E. A. Bartolo, L. Gorrochategui, I. Luengo, A. Melle-Hernándes, *On some conjecture about free and nearly free divisors*, Singularities and Computer Algebra, (2017), Springer 1–19.

[4] A. Dimca, *Syzygies of Jacobian ideals and defects of linear systems*, Bull. Math. Soc. Sci. Math. Roumanie Tome **56** (104) No. 2, 2013, 191–203.

[5] A. Dimca, *Freeness versus maximal global Tjurina number for plane curves*, Math Proc. Cambridge Phil. Soc. **163** (2017), 161–172.

[6] A. Dimca, *On rational cuspidal plane curves and the local cohomology of Jacobian rings*, Comment. Math. Helv. **94** (2019), 689–700.

[7] A. Dimca, D. Popescu, *Hilbert series and Lefschetz properties of dimension one almost complete intersections*, Comm. Algebra **44** (2016), 4467–4482.

[8] A. Dimca, E. Sernesi, *Syzygies and logarithmic vector fields along plane curves*, Journal de l'Ecole polytechnique-Mathématiques **1** (2014), 247–267.

[9] A. Dimca, G. Sticlaru, *Koszul complexes and pole order filtrations*, Proc. Edinburg. Math. Soc. **58** (2015), 333–354.

[10] A. Dimca, G. Sticlaru, *Free divisors and rational cuspidal plane curves*, Math. Res. Lett. **24** (2017), 1023–1042.

[11] A. Dimca, G. Sticlaru, *Free and nearly free curves vs rational cuspidal plane curves*, Publ. RIMS Kyoto Univ. **54** (2018), 163–179.

[12] A. Dimca, G. Sticlaru, *Line and rational curve arrangements, and Walther's inequality*, Rend. Lincei Mat. Appl. **30** (2019), 615–633.

[13] A. Dimca, G. Sticlaru, *On the jumping lines of bundles of logarithmic vector fields along plane curves*, Publicacions Matemàtiques **64** (2020), 513–542.

[14] A. Dimca, G. Sticlaru, *Plane curves with three syzygies, minimal Tjurina curves and nearly free cuspidal curves*, Geometriae Dedicata 10.1007/s10711-019-00485-7.

[15] A. Dimca, G. Sticlaru, *Jacobian syzygies and plane curves with maximal global Tjurina numbers*, arXiv: 1901.05915.

[16] A. A. du Plessis, C. T. C. Wall, *Application of the theory of the discriminant to highly singular plane curves*, Math. Proc. Cambridge Phil. Soc. **126** (1999), 259–266.

[17] Ph. Ellia, *Quasi complete intersections and global Tjurina number of plane curves*, J. Pure Appl. Algebra **224** (2020), 423–431.

[18] R. Hartshorne, *Stable vector bundles of rank* 2 *on* $\mathbb{P}^3$, Math Ann. **238** (1978), 229–280.

[19] T. Harima, J. Migliore, U. Nagel, J. Watanabe, *The weak and strong Lefschetz properties for artinian* $\mathbb{K}$*-algebras*, J. Algebra **262** (2003), 99–126.

[20] T. Harima, T. Maeno, Y. Numada, A. Wachi, J. Watanabe, *The Lefschetz properties*, Lecture Notes in Mathematics **2080** (2013), Springer.

[21] S. H. Hassanzadeh, A. Simis, *Plane Cremona maps: Saturation and regularity of the base ideal*, J. Algebra **371** (2012), 620–652.

[22] G. Ilardi, *Jacobian ideals, arrangements and Lefschetz properties*, J. Algebra **508** (2018), 418–430.

[23] S. Marchesi, J. Vallès, *Nearly free curves and arrangements: a vector bundle point of view*, Math. Proc. Camb. Phil. Soc. (2019). https://doi.org/10.1017/S0305004119000318.

[24] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math. **3** (1980).

[25] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. univ. Tokyo, **27** (1980), 265–291.

[26] E. Sernesi, *The local cohomology of the Jacobian ring*, Documenta Mathematica **19** (2014), 541–565.

[27] A. Simis, *Differential idealizers and algebraic free divisors*, Commutative Algebra: Geometric, Homological, Combinatorial and Computational Aspects, **244** (2005), Chapman & Hall, 211–226.

[28] A. Simis, *The depth of the Jacobian ring of a homogeneous polynomial in three variables*, Proc. Amer. Math. Soc. **134** (2006), 1591–1598.

[29] A. Simis, S. O. Tohaneanu, *Homology of homogeneous divisors*, J. Math. **200** (2014), 449–487.

[30] D. Van Straten, T. Warmt, *Gorenstein duality for one-dimensional almost complete intersections with an application to non-isolated real singularities*, Math. Proc. Cambridge Phil. Soc. **158** (2015), 249–268.

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