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On games without approximate equilibria

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Abstract

This note shows that the work by Simon and Tomkowicz (Israel J Math 227(1):215–231, 2018) answers another outstanding open question in game theory in addition to the non-existence of approximate Harsányi equilibrium in Bayesian games: it shows that strategic form games with bounded and separately continuous payoffs need not possess approximate equilibria.

Keywords Approximate equilibrium · Bayesian games · Discontinuous games

A strategic form game, with bounded and separately continuous payoffs, is a triple $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where N a finite collection of players, and for each $i \in N$, X_i is a compact convex set in some locally convex topological vector space, and $u_i : \prod_{i \in N} X_i \to \mathbb{R}$ is a bounded function which is affine in X_i and which is, for each $j \in N$, continuous in X_j when other components are held constant. A particular case of such games is when X_i is the set of probability measures $\Delta(A_i)$ over a compact set A_i in some metric space endowed with the topology of weak convergence, and u_i is the multi-linear extension of some bounded separately continuous payoff function on $\prod_i A_i$, known as the *mixed extension* (to this extent, $\prod_i A_i$ embeds naturally into $\prod_i X_i = \prod_i \Delta(A_i)$.)

In the two-player zero-sum case, Sion's minmax theorem, Sion (1958), shows that the separate continuity is sufficient to guarantee existence of the value; the existence of optimal strategies follows. (Indeed, if $u : X_1 \times X_2 \to \mathbb{R}$ is separately continuous, $x \to \min_{y \in Y} u(x, y)$ is upper-semicontinuous and hence obtains a maximum at an optimal strategy of the maximizer, i.e. of Player 1; and similarly for Player 2.) In the non-zero-sum case, however, an equilibrium need not exist; Vieille (1996) presents an example of a two-player non-zero-sum game 'on the square', i.e. in which each agent has action space [0, 1], the mixed extension of which possesses no equilibrium.

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It has remained open until now, however, the question of whether such games must possess ε-*equilibria* for each $\varepsilon > 0$; a profile $x = (x_i)_{i \in N}$ is a ε-equilibrium iff

$$\sup_{y_i \in X_i} u_i(y_i, x_{-i}) \leq u_i(x) + \varepsilon, \ \forall i \in N \tag{1}$$

(The case $\varepsilon = 0$ is simply an *equilibrium*.) This note shows that the example Simon and Tomkowicz (2018) gives a negative answer to this question.

The Bayesian game model of Simon and Tomkowicz (2018) fits into the following general model of Bayesian games: An underlying state space Ω with σ-algebra $\mathcal{F}$ and probability measure μ is given on $\mathcal{F}$. There are N players, each (for simplicity) with a finite action set A_i, a bounded measurable payoff function $r_i : \Omega \times \prod_{i \in N} A_i \to \mathbb{R}$, and a knowledge σ-algebra $\mathcal{F}_i \subseteq \mathcal{F}$. (This model is similar to the model in Stinchcombe (2011), except there the state space is a product of types spaces for each player, endowed with product σ-algebra; the general framework recalled here translates easily to this one.) A pure (resp. mixed) strategy σ_i of Player i is a mapping $\Omega \to A_i$ (resp. $\to \Delta(A_i)$) which is $\mathcal{F}_i$-measurable. (A pure strategy can be viewed a mixed strategy which assigns only Dirac measures.) For a profile $\sigma = (\sigma_i)_{i \in N}$ of strategies, the payoff to Player i is given by

$$u_i(\sigma) = \int_{\Omega} \left(\sum_{a=(a_1,\dots,a_n) \in \prod_{i \in N} A_i} r_i(\omega, a) \prod_{i \in N} \sigma_i(\omega)[a_i] \right) \mu(d\omega) \tag{2}$$

Clearly, two strategy profiles $\sigma = (\sigma_i)_{i \in N}$ and $\tau = (\tau_i)_{i \in N}$ which agree μ-a.e. yield the same payoffs, and hence one may treat the strategy space of Player i to be $X_i = L^{\infty}((\Omega, \mu), \Delta(A_i))$, the space of μ-measurable functions from Ω to $\Delta(A_i)$ modulo μ-a.e. equality, endowed with the weak-$*$ topology which makes it a compact metrizable space. (See, e.g., Rudin (2006) for the relevant background.) Furthermore, taking this collection of $(X_i)_{i \in N}$ of action spaces makes the payoff functons (u_i) defined in (2) clearly bounded but also separately continuous; indeed,

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{a_i \in A_i} \int_{\Omega} \sigma_i(\omega)[a_i] \cdot \left(\sum_{a_{-i} \in \prod_{j \neq i} A_j} r_i(\omega, a_i, a_{-i}) \prod_{j \neq i} \sigma_j(\omega)[a_j] \right) \mu(d\omega)$$

and for each $a_i \in A_i$, the function

$$\omega \to \left(\sum_{a_{-i} \in \prod_{j \neq i} A_j} r_i(\omega, a_i, a_{-i}) \prod_{j \neq i} \sigma_j(\omega)[a_j] \right)$$

is in $L^1((\Omega, \mu), \mathbb{R})$. Stinchcombe (2011) discusses, at length, how these functions in general are not jointly continuous in all strategies, and characterizes precisely when discontinuities arise.

The notions discussed in Simon and Tomkowicz (2018) of Harsányi equilibrium and ε-Harsányi equilibrium are precisely the usual notions of equilibrium and ε-equilibrium in the auxilliary strategic form game described above with actions spaces $(L^\infty((\Omega, \mu), \Delta(A_i)))_{i \in N}$ and payoffs defined by (2). Using standard measurable selection theorems, the definition given using (1) is equivalent to the following definition:

Definition 1 Fix a Bayesian game as above, and let $\varepsilon > 0$. A strategy profile $\sigma = (\sigma^i)_{i \in I}$, with $\sigma_i \in X_i := L^\infty((\Omega, \mu), \Delta(A_i))$, is an *Harsányi ε-equilibrium*, if for each $i \in I$,

$$\int_{\omega \in \Omega} \left[\max_{b^i \in A^i} r^i(\omega, b^i, \sigma^{-i}(\omega)) - r^i(\omega, \sigma(\omega)) \right] d\mu(\omega) \le \varepsilon. \tag{3}$$

where r^i has been extended multi-linearly to $\prod_{i=1}^N \Delta(A^i)$.

In particular, the example in Simon (2003) of a Bayesian game in which no Harsányi equilibrium exists, as well as the somewhat stronger example in Hellman (2014), gave yet more examples of a strategic form game with bounded separately continuous payoff not posessing equilibrium. However, both these examples can be shown to possess ε-Harsányi equilibria for each $\varepsilon > 0$; see Hellman and Levy (2020). The example in Simon and Tomkowicz (2018) of a Bayesian game possessing no ε-Harsányi equilibrium hence gives an example of a strategic form game possessing no ε-equilibrium for $\varepsilon > 0$ small enough.

It is to be noted that the example in Simon and Tomkowicz (2018) uses three players (see the discussion on Simon and Tomkowicz 2018, p. 230). It is still an open question as to whether two-player bounded separately continuous strategic form games must possess ε-equilibria.

Finally, it is noted:

Proposition 1 *The mixed extension of an N-player bounded separately continuous strategic form game, with compact metric action spaces, in which all but at most one player have jointly continuous payoff functions, possesses an ε-equilibrium for each $\varepsilon > 0$.*

Indeed, in the 2-player game presented in Vieille (1996), the payoff of Player 1 is a quadratic form in two variables, and hence ε-equilibria exist.

Proof Let $\varepsilon > 0$. W.l.o.g., $u_j(\cdot)$ is jointly continuous in all actions, i.e. continuous on $\prod_{i=1}^N X_i$, for each $j = 1, \ldots, N-1$. For each $j = 1, \ldots, N-1$, let $X'_j \subseteq X_j$ be finite, such that for each $x_j \in X_j$ there is $x'_j \in X'_j$ s.t. for all $x_{-j} \in \prod_{i \neq j} X_i$, $|u_j(x_j, x_{-j}) - u_j(x'_j, x_{-j})| < \varepsilon$, i.e., so that the functions $(u_j(x'_j, \cdot))_{x'_j \in X'_j}$ are ε-dense w.r.t. to supremum-norm in the set of functions $(u_j(x_j, \cdot))_{x_j \in X_j}$. Such $X'_1, \ldots, X'_{N-1}$ exist by the assumed joint continuity of $u_1, \ldots, u_{N-1}$ and the compactness of the action spaces.

For $j = 1, \ldots, N$, let v_j denote the restriction of u_j to $(\prod_{i=1}^{N-1} X'_i) \times X_N$. The $(v_i)_{i=1}^N$ are continuous. Let $x^* = (x_1^*, \ldots, x_N^*)$ be an equilibrium of the mixed extension of $(v_i)_{i=1}^N$. Clearly, in the original game $(u_j)_{j=1}^N$, Player N has no profitable

deviation against $(x_1^*, \ldots, x_{N-1}^*)$ in X_N, as $u_N(x_{-N}^*, \cdot) = v_N(x_{-N}^*, \cdot)$. Furtheremore, since x^* is an equilibrium under $(v_j)_{j=1}^N$, it holds for each $j = 1, \ldots, N-1$,

$$u_j(x^*) = v_j(x^*) = \max_{x \in X_j'} v_j(x, x_{-j}^*) = \max_{x \in X_j'} u_j(x, x_{-j}^*) \geq \max_{x \in X_j} u_j(x, x_{-j}^*) - \varepsilon$$

where the last inequality uses the approximation properties of X_j'. $\qquad\square$

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