Equivariant coarse homotopy theory and coarse algebraic K-homology

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Abstract

We study equivariant coarse homology theories through an axiomatic framework. To this end we introduce the category of equivariant bornological coarse spaces and construct the universal equivariant coarse homology theory with values in the category of equivariant coarse motivic spectra.

As examples of equivariant coarse homology theories we discuss equivariant coarse ordinary homology and equivariant coarse algebraic K-homology.

Moreover, we discuss the cone functor, its relation with equivariant homology theories in equivariant topology, and assembly and forget-control maps. This is a preparation for applications in subsequent papers aiming at split-injectivity results for the Farrell–Jones assembly map.

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1. Introduction

In this paper we study equivariant coarse homology theories. We start with the equivariant generalization of the coarse homotopy theory developed by Bunke–Engel [BE16]. To this end we introduce the category of equivariant bornological coarse spaces and construct the universal equivariant coarse homology theory with values in the category of equivariant coarse motivic spectra.

As examples of equivariant coarse homology theories we discuss equivariant ordinary coarse homology and equivariant coarse algebraic K-homology of an additive category.

An important application of equivariant coarse homotopy theory is in the study of assembly maps which appear in isomorphism conjectures of Farrell–Jones or Baum–Connes type. The main tools for the transition between equivariant homology theories and equivariant coarse homology theories are the cone functor and the process of coarsification. In this paper we give a detailed account of the cone functor and the construction of equivariant homology theories from equivariant coarse homology theories. Then we introduce the coarsification functor and the forget-control map, and discuss its relation with the assembly maps.

The third part of the present paper provides the technical background for subsequent papers:

1. In [BEKW17] we show that a certain large scale geometric condition called finite decomposition complexity implies that a motivic version of the forget-control map is an equivalence.

2. In [BEKW] we study the descent principle. It states that under certain conditions the fact that the forget-control map becomes an equivalence after restriction of the action to all finite subgroups implies that it is split injective for the original (in general infinite) group.

   We combine this with the results of [BEKW17] and the technical results of the present paper to deduce split injectivity results for the original Farrell–Jones assembly map.

3. In [BE17] we study more formal aspects of the process of coarsification of homology theories and provide a general account for coarse assembly maps.

A bornological coarse space is a set equipped with a coarse and a bornological structure such that these structures are compatible with each other. The category of bornological coarse spaces **BornCoarse** was introduced in [BE16] as a general framework for coarse geometry and coarse topology. Interesting invariants of bornological coarse spaces up to coarse equivalence are coarse homology theories. In [BE16] the examples of coarse ordinary homology and various coarse versions of topological K-homology were discussed. In order to study general properties of coarse homology theories the category of motivic coarse spectra $\mathbf{Sp}\mathcal{X}$ was constructed as the target of the universal coarse homology theory

$$\mathrm{Yo}^s\colon \mathbf{BornCoarse} \to \mathbf{Sp}\mathcal{X}\ .$$

One of the motivations to consider equivariant coarse algebraic topology is that it appears as a building block of proofs that certain assembly maps are equivalences (Farrell–Jones conjecture [BL11], [BFJR04] and Baum–Connes conjecture [Yu95]).

In Section 2.1 we define for every group Γ the category Γ**BornCoarse** of Γ-bornological coarse spaces. We provide various constructions of Γ-bornological coarse spaces from Γ-sets, metric spaces with isometric Γ-action, or Γ-simplicial complexes. We further show that the category of Γ-bornological coarse spaces admits coproducts and fiber products. It also has an interesting symmetric monoidal structure $\otimes$.

In Section 3 we introduce the notion of an equivariant coarse homology theory, and the terminology necessary to state its defining properties:

1. coarse invariance,
2. coarse excision,
3. vanishing on flasques, and
4. u-continuity.

Strongness (introduced in Section 4.4) is an additional property which an equivariant coarse homology might have. It is important in order to interpret the forget-control map as a transformation between equivariant coarse homology theories.

In the present paper the most important additional property is continuity. We introduce this notion in Section 5. Continuity is crucial if one wants to relate the forget-control map with the assembly map for the family of finite subgroups.

Following the line of thought of [BE16], in Section 4 we construct the universal equivariant coarse homology theory

$$\mathrm{Yo}^s \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}$$

with values in the category of equivariant coarse motivic spectra. Similarly, in Section 5 we construct a universal continuous equivariant coarse homology theory

$$\mathrm{Yo}^s_c \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}_c\ .$$

As examples of equivariant coarse homology theories, in Part II we introduce equivariant coarse ordinary homology $H\mathcal{X}^\Gamma$ and equivariant coarse algebraic K-homology $K\mathbf{A}\mathcal{X}^\Gamma$ of an additive category $\mathbf{A}$. Our main results are the verification that the definitions indeed satisfy the four defining properties of an equivariant coarse homology theory. We also show that these examples have the additional properties of being strong, strongly additive and continuous. We calculate the evaluations of these equivariant homology theories on simple Γ-bornological coarse spaces. These calculations are important if one wants to understand which equivariant homology theories they induce after pull-back with the cone functor.

In most of the present paper we consider the theory for a fixed group Γ. But in Section 6 we consider a homomorphism of groups $H \to \Gamma$ and provide various transitions from H-bornological coarse spaces to Γ-bornological coarse spaces and back. The most important examples are restriction and induction. Our examples of equivariant coarse homology theories are defined for all groups, so in particular for H and Γ. The group-change construction on the level of bornological coarse spaces are accomplished by natural transformations relating the evaluations of the H- and Γ-equivariant versions of the equivariant homology theories.

In Section 9 we introduce the category of Γ-uniform bornological coarse spaces $\Gamma\mathbf{UBC}$ and the cone functor

$$\mathcal{O} \colon \Gamma\mathbf{UBC} \to \Gamma\mathbf{BornCoarse}\ .$$

The construction of the cone is motivated by Bartels–Farrell–Jones–Reich [BFJR04] and Mitchener [Mit01, Mit10], and its main ingredient in the construction is the hybrid coarse structure first introduced by Wright [Wri02] and studied in detail in [BE16]. Our main technical results are homotopy invariance and excisiveness of the cone. While the cone $\mathcal{O}(X)$ depends on the coarse structure on X, its germ at ∞, denoted by $\mathcal{O}^\infty(X)$, is essentially independent of the coarse structure. So $\mathcal{O}^\infty$ is very close to an equivariant homology theory. But it is still defined on $\Gamma\mathbf{UBC}$ and does not satisfy a wedge axiom.

In Section 10 we first review some general features of equivariant homotopy theory and then derive an equivariant homology theory

$$\mathcal{O}^\infty_{\mathrm{hlg}} \colon \Gamma\mathbf{Top} \to \Gamma\mathbf{Sp}\mathcal{X}$$

from the functor $\mathcal{O}^\infty$. We then introduce the classifying space $E_{\mathcal{F}}\Gamma$ for a family $\mathcal{F}$ of subgroups of Γ and define the motivic assembly map as the map

$$\alpha_{E_{\mathcal{F}}\Gamma} \colon \mathcal{O}^\infty_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma) \to \mathcal{O}^\infty_{\mathrm{hlg}}(*)$$

induced by the morphism $E_{\mathcal{F}}\Gamma \to *$. We discuss some conditions on a Γ-bornological coarse space Q implying that the twisted version $\alpha_{E_{\mathcal{F}}} \otimes \mathrm{Yo}^s(Q)$ of the assembly map becomes an equivalence.

In Section 11 we introduce the universal coarsification functor

$$F^\infty \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}$$

and the forget-control map

$$\beta \colon F^\infty \to \Sigma F^0 \ ,$$

a natural transformation of functors from $\Gamma\mathbf{BornCoarse}$ to $\Gamma\mathbf{Sp}\mathcal{X}$. The coarse geometry approach to the isomorphism conjectures provides conditions on a Γ-bornological coarse space X implying that the forget-control map $\beta_X \colon F^\infty(X) \to \Sigma F^0(X)$ is an equivalence, or becomes an equivalence after application of a suitable equivariant coarse homology theory. Since one is also interested in the assembly maps of equivariant homotopy theory like $\alpha_{E_{\mathcal{F}}\Gamma}$ we provide a comparison between this assembly map and the forget-control map.

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Part I.
General constructions

2. Equivariant bornological coarse spaces

2.1. Basic notions

In this section we introduce the equivariant version of the category of bornological coarse spaces introduced in [BE16]. We assume familiarity with [BE16, Section 2].

Let Γ be a group. If Γ acts on a bornological coarse space X by automorphisms, then it acts on the set of coarse entourages $\mathcal{C}$ of X. We let $\mathcal{C}^{\Gamma}$ denote the partially ordered subset of $\mathcal{C}$ of entourages of X which are fixed set-wise.

Definition 2.1. A Γ-*bornological coarse space* is a bornological coarse space X together with an action of Γ by automorphisms such that $\mathcal{C}^{\Gamma}$ is cofinal in $\mathcal{C}$.

A morphism between Γ-bornological coarse spaces is a morphism of bornological coarse spaces which is in addition Γ-equivariant. ♦

We let Γ**BornCoarse** denote the category of Γ-bornological coarse spaces and morphisms. By considering a bornological coarse space as a Γ-bornological coarse space with the trivial action we get a fully faithful functor

$$C\colon \mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse}\ . \tag{2.1}$$

Example 2.2. Let X be a set with an action of Γ and $A \subseteq \mathcal{P}(X \times X)^{\Gamma}$ be a family of Γ-invariant subsets. Then we can form the coarse structure $\mathcal{C} := \mathcal{C}\langle A\rangle$ generated by A. In this case $\mathcal{C}^{\Gamma}$ is cofinal in $\mathcal{C}$. Hence a Γ-coarse structure can be generated by a family of Γ-invariant entourages. ♦

Remark 2.3. Let **Coarse** denote the category of coarse spaces (i.e., sets with coarse structures) and controlled morphisms. We can consider Γ equipped with the minimal coarse structure $\mathcal{C}\langle \mathrm{diag}_{\Gamma}\rangle$ as a group object in **Coarse**.

Let X be a set with an action of a group Γ and $\mathcal{C}$ be a coarse structure on X. The following conditions are equivalent:

1. $\mathcal{C}^{\Gamma}$ is cofinal in $\mathcal{C}$.
2. For every entourage U in $\mathcal{C}$ the set $\bigcup_{\gamma\in\Gamma}(\gamma \times \gamma)(U)$ also belongs to $\mathcal{C}$.
3. The action is a morphism $\Gamma \times X \to X$ in **Coarse**.

The proof is straightforward.

We denote the category of Γ-coarse spaces consisting of coarse spaces with a Γ-action satisfying the above conditions and equivariant and controlled maps by Γ**Coarse**. ♦

Example 2.4. We consider Γ as a Γ-set with the left action. We furthermore let $\mathcal{B}_{min}$ be the minimal bornology on Γ consisting of the finite subsets. Finally, we let the coarse structure $\mathcal{C}_{can}$ on Γ be generated by the Γ-invariant sets $\Gamma(B \times B)$ for all B in $\mathcal{B}_{min}$.

Then $(\Gamma, \mathcal{C}_{can}, \mathcal{B}_{min})$ is a Γ-bornological coarse space called the canonical Γ-bornological coarse space associated to Γ. We will denote it by $\Gamma_{can,min}$. ♦

Example 2.5. Let X be a set with an action of Γ. It gives rise to the Γ-bornological coarse space $X_{min,min}$ with the minimal structures $\mathcal{B}_{min}$ consisting of the finite subsets of X and $\mathcal{C}_{min} := \mathcal{C}\langle \mathrm{diag}_X \rangle$. We will use the notation $X_{min,min}$ for this Γ-bornological coarse space.

For example, the identity of the underlying set of Γ is a morphism $\Gamma_{min,min} \to \Gamma_{can,min}$ of Γ-bornological coarse spaces. ♦

Example 2.6. Let X again be a set with an action of Γ. Then we can equip X with the maximal bornological and coarse structures. In this way we get a Γ-bornological coarse space $X_{max,max}$.

Our notation convention is such that the first subcript indicates the coarse structure, while the second subscript reflects the bornological structure.

For example, we also have a Γ-bornological coarse space $X_{min,max}$ and morphisms of Γ-bornological coarse spaces $X_{min,max} \to X_{max,max}$ and $X_{min,max} \to X_{min,min}$ given by the identity of the underlying set of X.

Note that in general $X_{max,min}$ does not make sense since the minimal bornology is not compatible with the maximal coarse structure. ♦

Example 2.7. Let (X, d) be a metric space with an isometric Γ-action. For r in $(0, \infty)$ we consider the invariant entourages

$$U_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\} \ .$$

The coarse structure associated to the metric is defined by these entourages, i.e., given by

$$\mathcal{C}_d := \mathcal{C}\langle \{U_r \mid r \in (0, \infty)\} \rangle \ .$$

Furthermore, the bornology associated to the metric is generated by the metrically bounded subsets, i.e., given by

$$\mathcal{B}_d := \mathcal{B}\langle \{B(x, r) \mid x \in X \ , r \in (0, \infty)\} \rangle \ ,$$

where $B(x, r)$ denotes the metric ball of radius r centered at x.

The associated bornological coarse space $X_d := (X, \mathcal{C}_d, \mathcal{B}_d)$ is a Γ-bornological coarse space.

The identity of the underlying set of X is a morphism $X_{min,max} \to X_d$ of Γ-bornological coarse spaces. ♦

Remark 2.8. Let Γ be a countable group equipped with a proper left invariant metric d. Then the Γ-bornological coarse spaces Γ_d and $\Gamma_{can,min}$ are equal. ♦

Example 2.9. Let X be a Γ-bornological coarse space with the coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$. For every invariant entourage U in $\mathcal{C}$ we consider the coarse structure $\mathcal{C}_U := \mathcal{C}\langle\{U\}\rangle$. The coarse structure $\mathcal{C}_U$ is compatible with $\mathcal{B}$. We let

$$X_U := (X, \mathcal{C}_U, \mathcal{B})$$

denote the resulting Γ-bornological coarse space. The identity of the underlying set is a morphism of Γ-bornological coarse spaces $X_U \to X$. If U' is a second invariant entourage such that $U \subseteq U'$, then we also have a morphism $X_U \to X_{U'}$. This construction is important for the formulation of the u-continuity condition in Definition 3.10. ◆

Example 2.10. Let X be a Γ-bornological coarse space with coarse structure $\mathcal{C}$ and bornology $\mathcal{B}$, and let Z be a Γ-invariant subset of X. Then we define the induced coarse structure and bornology on Z as follows:

1. $\mathcal{C}_Z := \{(Z \times Z) \cap U \mid U \in \mathcal{C}\}$
2. $\mathcal{B}_Z := \{Z \cap B \mid B \in \mathcal{B}\}$.

Then $Z_X := (Z, \mathcal{C}_Z, \mathcal{B}_Z)$ is a Γ-bornological coarse space. The inclusion $Z_X \to X$ is a morphism of Γ-bornological coarse spaces. ◆

Example 2.11. If Γ acts on the underlying set of a bornological coarse space $(X, \mathcal{C}, \mathcal{B})$, then we can define a Γ-bornological coarse space $\Gamma X := (X, \Gamma\mathcal{C}, \Gamma\mathcal{B})$, where

$$\Gamma\mathcal{C} := \mathcal{C}\left\langle \left\{ \bigcup_{\gamma \in \Gamma} (\gamma \times \gamma)(U) \mid U \in \mathcal{C} \right\} \right\rangle$$

and

$$\Gamma\mathcal{B} := \mathcal{B}\langle\{U[\gamma B] \mid U \in \Gamma\mathcal{C}, \gamma \in \Gamma \text{ and } B \in \mathcal{B}\}\rangle\ . \tag{2.2}$$

In general we must enlarge the bornology $\mathcal{B}$ to $\Gamma\mathcal{B}$ as described above in order to keep it compatible with the new coarse structure.

If X was a Γ-bornological coarse spaces, then $\Gamma X = X$. ◆

Definition 2.12. Let Γ denote a group and let X be a set with an action of Γ. If $\mathcal{B}$ is a bornology on X, then we let $\mathcal{B}_\Gamma$ denote the bornology on X which is generated by the sets ΓB for all B in $\mathcal{B}$.

Let $(X, \mathcal{C}, \mathcal{B})$ be a Γ-bornological coarse space. Then the new bornology $\mathcal{B}_\Gamma$ is compatible with the original coarse structure. The Γ-*completion* of $(X, \mathcal{C}, \mathcal{B})$ is defined to be the Γ-bornological coarse space $(X, \mathcal{C}, \mathcal{B}_\Gamma)$.

Let $(X, \mathcal{C}, \mathcal{B})$ be a Γ-bornological coarse space and Y be a subset of X. The subset Y is called Γ-*bounded* if it belongs to $\mathcal{B}_\Gamma$. ◆

Example 2.13. Let $(X, \mathcal{B})$ be a bornological space with an action of Γ by proper maps. We say that Γ acts *properly* if for every B in $\mathcal{B}$ the set $\{\gamma \in \Gamma \mid \gamma B \cap B \neq \emptyset\}$ is finite.

We define the coarse structure $\mathcal{C}_\mathcal{B}$ on X to be generated by the Γ-invariant entourages $U_B := \Gamma(B \times B)$ for all B in $\mathcal{B}$. If Γ acts properly, then the bornological structure is compatible with this coarse structure and we get a Γ-bornological coarse space $(X, \mathcal{C}_\mathcal{B}, \mathcal{B})$. ◆

Example 2.14. Let X be a Γ-complete Γ-bornological coarse space with coarse structure $\mathcal{C}$ and bornology $\mathcal{B}$. We equip the quotient set $\bar{X} := \Gamma\backslash X$ with the maximal bornology $\bar{\mathcal{B}}$ such that the projection $q\colon X \to \bar{X}$ is proper, i.e.,

$$\bar{\mathcal{B}} = \mathcal{B}\langle\{\bar{B} \subseteq \bar{X} \mid q^{-1}(\bar{B}) \in \mathcal{B}\}\rangle \ .$$

We furthermore equip $\bar{X}$ with the minimal coarse structure $\bar{\mathcal{C}}$ such that q is controlled, i.e.,

$$\bar{\mathcal{C}} := \mathcal{C}\langle\{(q \times q)(U) \mid U \in \mathcal{C}\}\rangle \ .$$

Then $\bar{\mathcal{C}}$ and $\bar{\mathcal{B}}$ are compatible and we obtain a bornological coarse space $(\bar{X}, \bar{\mathcal{C}}, \bar{\mathcal{B}})$.

This construction produces a functor

$$Q\colon \Gamma\mathbf{BornCoarse} \to \mathbf{BornCoarse} \ , \quad Q(X, \mathcal{C}, \mathcal{B}) := (\bar{X}, \bar{\mathcal{C}}, \bar{\mathcal{B}}) \ .$$

It is easy to check that the morphism of bornological coarse spaces $q\colon (X, \mathcal{C}, \mathcal{B}) \to (\bar{X}, \bar{\mathcal{C}}, \bar{\mathcal{B}})$ can be interpreted as the unit of an adjunction

$$Q\colon \Gamma\mathbf{BornCoarse} \leftrightarrows \mathbf{BornCoarse} : C \ ,$$

where C is the inclusion (2.1). ♦

Lemma 2.15. *The category* Γ**BornCoarse** *admits arbitrary coproducts and cartesian products.*

Proof. The coproduct of a family of Γ-bornological coarse spaces is represented by the coproduct of the underlying bornological coarse spaces with the induced Γ-action.

Similarly, the cartesian product of a family of Γ-bornological coarse spaces is represented by the cartesian product of the family of the underlying bornological coarse spaces with the induced Γ-action. □

For more information about limits and colimits in Γ**BornCoarse** we refer to Section 2.2.

Example 2.16. We consider a family $(X_i)_{i\in I}$ of Γ-bornological coarse spaces. We define the free union $\bigsqcup_{i\in I}^{\text{free}} X_i$ as follows:

1. The underlying Γ-set of the free union is the disjoint union of Γ-sets $\bigsqcup_{i\in I} X_i$.
2. The coarse structure of the free union is generated by entourages $\bigsqcup_{i\in I} U_i$ for all families $(U_i)_{i\in I}$, where U_i is an entourage of X_i for every i in I.
3. The bornology is generated by the set $\{B \mid B \in \mathcal{B}(X_i), i \in I\}$ of subsets of $\bigsqcup_{i\in I} X_i$.

If I is finite, then the free union is the coproduct of the family. In general, we have a morphism of Γ-bornological coarse spaces

$$\coprod_{i\in I} X_i \to \bigsqcup_{i\in I}^{\text{free}} X_i$$

induced by the identity of the underlying sets. ♦

Example 2.17. Let X and X' be two Γ-bornological coarse spaces. Then we can form the Γ-bornological coarse space $X' \otimes X$ whose coarse structure is the one of the cartesian product and the bornology is generated by the products $B' \times B$ for bounded subsets B' of X' and B of X. This construction defines a symmetric monoidal structure

$$- \otimes - \colon \Gamma\mathbf{BornCoarse} \times \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse}$$

with tensor unit given by the one-point space.

Let Y be a Γ-set. We can form the space $Y_{min,min} \otimes X$. For a second Γ-set Y' we have a canonical isomorphism

$$(Y' \times Y)_{min,min} \otimes X \cong Y'_{min,min} \otimes (Y_{min,min} \otimes X) \ . \qquad \blacklozenge$$

2.2. Limits and colimits in ΓBornCoarse

In this section we show that the category $\Gamma\mathbf{BornCoarse}$ admits all limits of non-empty diagrams and various colimits. We furthermore discuss some special cases.

Let Γ be a group. In the following arguments we let

$$\iota \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Set} \tag{2.3}$$

be the forgetful functor. For a Γ-bornological coarse space X we let $\mathcal{B}_X$ and $\mathcal{C}_X$ denote its bornology and coarse structure.

Proposition 2.18.

1. *The category* $\Gamma\mathbf{Coarse}$ *admits all small limits.*
2. *The category* $\Gamma\mathbf{BornCoarse}$ *admits all limits of diagrams indexed by non-empty small categories.*

Proof. We give the proof for $\Gamma\mathbf{BornCoarse}$. The statement for $\Gamma\mathbf{Coarse}$ can be obtained by ignoring all comments pertaining to bornologies and allowing in addition the index set I below to be empty.

Note that the non-emptiness assumption on I only enters into the part of the proof concerning bornologies. See also Remark 2.19.

Let I be a non-empty small category and

$$X \colon I \to \Gamma\mathbf{BornCoarse}$$

be a functor. We will show that $\lim_I X$ exists.

The category $\Gamma\mathbf{Set}$ is complete. In a first step we form the Γ-set

$$\widetilde{Y} := \lim_I \iota X \ .$$

For every i in I we have a map of Γ-sets $e_i\colon \widetilde{Y} \to \iota X(i)$.

On $\widetilde{Y}$ we define the bornology

$$\mathcal{B}_Y := \mathcal{B}\langle \{e_i^{-1}(B) \mid i \in I \text{ and } B \in \mathcal{B}_{X(i)}\}\rangle \ .$$

Since $\mathcal{B}_{X(i)}$ is Γ-invariant for every i in I, we see that $\mathcal{B}_Y$ is Γ-invariant.

We can view $\widetilde{Y}$ as a subset of $\prod_{i\in I} \iota X(i)$. On $\widetilde{Y}$ we define the coarse structure

$$\mathcal{C}_Y := \mathcal{C}\Big\langle \Big\{ \Big(\prod_{i\in I} U_i \Big) \cap (\widetilde{Y} \times \widetilde{Y}) \mid (U_i)_{i\in I} \in \prod_{i\in I} \mathcal{C}_{X(i)} \Big\} \Big\rangle \ .$$

Using that $\mathcal{C}_{X(i)}$ has a cofinal subset of invariant entourages for every i in I, we see that the coarse structure $\mathcal{C}_Y$ is generated by invariant entourages and is hence a Γ-coarse structure.

Finally, the relation

$$\Big(\prod_{i\in I} U_i \Big)[e_j^{-1}(B)] = e_j^{-1}(U_j[B])$$

for j in I and B a subset of X_j shows that the compatibility of $\mathcal{B}_{X(j)}$ with $\mathcal{C}_{X(j)}$ for all j in I implies that $\mathcal{B}_Y$ and $\mathcal{C}_Y$ are compatible.

We therefore have defined an object $Y := (\widetilde{Y}, \mathcal{C}_Y, \mathcal{B}_Y)$ in Γ**BornCoarse**. We now show that $e_j\colon Y \to X(j)$ is a morphism of Γ-bornological coarse spaces for all j in I. For every family $(U_i)_{i\in I}$ we have $e_j((\prod_{i\in I} U_i) \cap (\widetilde{Y} \times \widetilde{Y})) \subseteq U_j$. This implies that e_j is controlled. Furthermore, $\mathcal{B}_Y$ is defined such that e_j is proper for every j in I.

The morphisms $e_i\colon Y \to X(i)$ give a transformation $e\colon \underline{Y} \to X$ in **Fun**$(I, \Gamma$**BornCoarse**$)$, where $\underline{Y}$ is the constant functor with value Y. We now show that (Y, e) has the universal property of the limit .

We consider a pair (Z, f) with Z in Γ**BornCoarse** and with $f\colon \underline{Z} \to X$ a morphism in **Fun**$(I, \Gamma$**BornCoarse**$)$. Because the underlying Γ-set of Y is the limit of the diagram ιX there is a unique map $h\colon \iota Z \to \iota Y$ of Γ-sets such that $\iota f = \iota e \circ \underline{h}$.

It suffices to show that h is a morphism in Γ**BornCoarse**. Let U be an entourage of Z. Then $f_i(U)$ is an entourage of $X(i)$ for every i in I. We have

$$h(U) \subseteq \Big(\prod_{i\in I} (f_i \times f_i)(U) \Big) \cap (\widetilde{Y} \times \widetilde{Y}) \ ,$$

i.e., $h(U)$ is contained in one of the generating entourages of $\mathcal{C}_Y$. Hence h is a controlled map.

In order to show that h is proper, we consider a generating bounded subset $e_i^{-1}(B)$ for i in I and B in $\mathcal{B}_{X(i)}$. Then $h^{-1}(e_i^{-1}(B)) = f_i^{-1}(B)$ is bounded in Z. Since I is non-empty, the set of subsets $e_i^{-1}(B)$ for all i in I and B in $\mathcal{B}_{X(i)}$ cover Y. Therefore, every element of $\mathcal{B}_Y$ is contained in a finite union of such subsets. We conclude that h is proper. $\square$

Remark 2.19. Note that the last piece of the argument goes wrong if the index category of the diagram is empty. Then $Y = *$, and we need the subset $\{*\}$ (which is not of the form $e_i^{-1}(B)$) to generate the bornology. The empty limit does not exist since the category Γ**BornCoarse** does not have a final object. ♦

We now turn to colimits. Let I be a small category and

$$X\colon I \to \Gamma\mathbf{BornCoarse}$$

be a functor. The category Γ**Set** is cocomplete. We form the Γ-set

$$\widetilde{Y} := \operatorname*{colim}_I \iota X\ ,$$

where ι is the forgetful functor from (2.3). For every i in I we have a map $e(i)\colon \iota X(i) \to \widetilde{Y}$ of Γ-sets.

Definition 2.20. We say that the diagram X is colim-admissible if we have

$$e_i^{-1}((e_{j_1}(U_1) \circ \cdots \circ e_{j_k}(U_k))[\{y\}]) \in \mathcal{B}_{X(i)}$$

for every y in $\widetilde{Y}$, every i in I, every r in $\mathbb{N}$, every family $(j_1, \ldots, j_r)$ of objects of I, and every family of entourages U_k in $\mathcal{C}_{X(j_k)}$ for k in $\{1, \ldots, r\}$. ♦

Proposition 2.21.

1. *The category* Γ**Coarse** *admits all small colimits.*
2. *The category* Γ**BornCoarse** *admits colimits for all colim-admissible diagrams.*

Proof. We give the proof for Γ**BornCoarse**. The statement for Γ**Coarse** can be obtained by ignoring all comments pertaining to bornologies. Note that the assumption of colim-admissibility only enters to see that the bornology and the coarse structure on the colimit are compatible.

Assume that

$$X\colon I \to \Gamma\mathbf{BornCoarse}$$

is a colim-admissible diagram. We show that $\operatorname{colim}_I X$ exists.

In a first step we form the Γ-set

$$\widetilde{Y} := \operatorname*{colim}_I \iota X\ .$$

On $\widetilde{Y}$ we define the coarse structure

$$\mathcal{C}_Y := \mathcal{C}\langle\{(e_i \times e_i)(U) \mid i \in I \text{ and } U \in \mathcal{C}_{X(i)}\}\rangle\ .$$

Using the fact that $\mathcal{C}_{X(i)}$ has a cofinal subset of invariant entourages we see that the coarse structure $\mathcal{C}_Y$ is generated by invariant entourages and is hence a Γ-coarse structure.

We define the bornology $\mathcal{B}_Y$ to be the subset of $\mathcal{P}(\widetilde{Y})$ consisting of the sets B satisfying

$$e_i^{-1}((e_{j_1}(U_1) \circ \cdots \circ e_{j_k}(U_k))[B]) \in \mathcal{B}_{X(i)}$$

for every i in I, every r in $\mathbb{N}$, every family $(j_1, \dots, j_r)$ of objects of I, and every family of entourages U_k in $\mathcal{C}_{X(j_k)}$ for k in $\{1, \dots, r\}$. Since the diagram is colim-admissible, all one-point sets belong to $\mathcal{B}_Y$. Furthermore $\mathcal{B}_Y$ is obviously closed under forming finite unions and subsets. Consequently, $\mathcal{B}_Y$ is a bornology on Y. Since $\mathcal{B}_{X(i)}$ and $\mathcal{C}_{X(i)}$ are Γ-invariant for every i in I we see that $\mathcal{B}_Y$ is Γ-invariant. We finally observe that $\mathcal{C}_Y$ and $\mathcal{B}_Y$ are compatible by construction.

We define now the object $Y := (\widetilde{Y}, \mathcal{C}_Y, \mathcal{B}_Y)$ of Γ**BornCoarse**. By construction the maps $e_i \colon X(i) \to Y$ are morphisms in Γ**BornCoarse** for all i in I. The family of morphisms $(e_i)_{i \in I}$ provides a morphism $e \colon X \to \underline{Y}$ in **Fun**$(I, \Gamma$**BornCoarse**$)$, where $\underline{Y}$ is the constant functor with value Y.

We now show that (Y, e) has the universal property of a colimit.

Consider a pair (Z, f) with Z in Γ**BornCoarse** and $f \colon X \to \underline{Z}$. Since the underlying Γ-set of Y is the colimit of the diagram ιX there is a unique map $h \colon \iota Y \to \iota Z$ of Γ-sets such that $\iota f = \underline{h} \circ \iota e$.

It suffices to show that h is a morphism.

Let i in be I and U be in $\mathcal{C}_{X(i)}$. Then $h(e_i(U)) = f_i(U_i)$ is an entourage of Z. This implies that h is controlled.

Let now B be a bounded subset of Z. Since $\mathcal{C}_Z$ and $\mathcal{B}_Z$ are compatible and f_j is controlled, $f_j(U)[B]$ is bounded for every j in I and U in $\mathcal{C}_{X_j}$. We now have[1]

$$e_i^{-1}(e_j(U)[h^{-1}(B)]) \subseteq e_i^{-1}(h^{-1}(h(e_j(U))[B])) = f_i^{-1}(f_j(U)[B]) \ .$$

Since f_i is proper, we conclude that $e_i^{-1}(e_j(U)[h^{-1}(B)])$ is bounded. Since i, j in I and U in $\mathcal{C}_{X(j)}$ were arbitrary this shows that $h^{-1}(B)$ is bounded in Y. We conclude that h is a proper map. □

Example 2.22. If $(\mathcal{Y}, Z)$ is an equivariant complementary pair (see Definition 3.7) on X, then we have a push-out

$$\begin{array}{ccc} \operatorname{colim} \mathcal{Y} \cap Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ \operatorname{colim} \mathcal{Y} & \longrightarrow & X \end{array}$$

We first note that $\operatorname{colim} \mathcal{Y}$ is admissible since it is a filtered colimit of inclusions. It is straightforward to check that the diagram is colim-admissible and that the bornology on the space X is the one of the colimit. The only non-trivial fact to check is that the coarse structure on X given by the colimit is not too small. Let U be an entourage of X. Assume

[1]Here we use the general relation $U[f^{-1}(B)] \subseteq f^{-1}(f(U)[B])$ for a map $f \colon X \to Y$, entourage U of X and subset B of Y.

that i is in I such that $Y_i \cup Z = X$. Since the family is big, there exists j in I such that $U[Y_i] \subseteq Y_j$. Let $e\colon Y_j \to X$ and $f\colon Z \to X$ be the inclusions. Then

$$U \subseteq e(U \cap (Y_j \times Y_j)) \cup f(U \cap (Z \times Z)) \ .$$

♦

Example 2.23. By Proposition 2.18 the category Γ**BornCoarse** admits fiber products. Here we give an explicit description. We consider a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{b} & Z \end{array}$$

of Γ-bornological coarse spaces. We then form the cartesian product $X \times Y$ in the category Γ**BornCoarse**. We define the Γ-bornological coarse space $X \times_Z Y$ to be the subset of $X \times Y$ of pairs (x, y) with $a(x) = b(y)$ with the induced bornological and coarse structure. It is straightforward to check that the square

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{b} & Z \end{array}$$

is cartesian in Γ**BornCoarse**. ♦

Example 2.24. Assume that (Y, Z) is a coarsely excisive pair (see Definition 4.13) on X. Then we have a push-out

$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

It is straightforward to check that the diagram is colim-admissible and that the bornology on the space X is the one of the colimit. The only non-trivial fact to check is that the coarse structure on X given by the colimit is not too small. Let U be an entourage of X. Then there is an entourage W of X such that $U \subseteq W$ and $U[Z] \cap U[Y] \subseteq W[Z \cap Y]$. Let $e\colon Y \to X$ and $f\colon Z \to X$ be the inclusions. Then

$$U \subseteq e(W^2 \cap (Y \times Y)) \cup f(W^2 \cap (Z \times Z)) \ .$$

♦

Example 2.25. Let H be a group acting on a Γ-bornological coarse space X such that the set of H-invariant entourages of X is cofinal in all entourages.

We let $B_H(X)$ denote H-completion of X obtained from X by replacing the bornology $\mathcal{B}$ of X by the bornology $B_H(\mathcal{B})$ generated by the subsets HB for all B in $\mathcal{B}$. Then the coequalizer

$$(H_{min,min} \otimes X)_{max-\mathcal{B}} \rightrightarrows B_H(X) \xrightarrow{\pi} X/H$$

for the H-action exists. Here the index $-_{max-\mathcal{B}}$ indicates that we replaced the bornology by the maximal bornology. The two arrows are given by $(h, x) \mapsto x$ and $(h, x) \mapsto hx$. One checks easily that they are both morphisms of Γ-bornological coarse spaces: they are both proper since their domain has the maximal bornology, and if U is a H-invariant entourage of X then both maps send $\mathrm{diag}(H) \times U$ to U and hence the maps are controlled in view of our assumption on the coarse structure of X.

Finally, we check that the coequalizer diagram is colim-admissible. It suffices to check that for every H-invariant entourage U of X and point Hx in X/H the set $U[\pi^{-1}(Hx)]$ is bounded in $B_H(X)$. This is the case since $U[x]$ belongs to $\mathcal{B}$ and so $U[\pi^{-1}(Hx)] = HU[x]$ belongs to $B_H(\mathcal{B})$. ♦

Example 2.26. Let

$$M\colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Set}$$

be the functor which sends a Γ-bornological coarse space to its underlying Γ-set. In view of Proposition 2.18, it preserves all limits over non-empty small index categories. It is in fact the right-adjoint of an adjunction

$$(-)_{min,max}\colon \Gamma\mathbf{Set} \leftrightarrows \Gamma\mathbf{BornCoarse} : M\ ,$$

where $(-)_{min,max}$ sends a Γ-set S to the Γ-bornological coarse space obtained by equipping S with the minimal coarse structure and maximal bornology. Indeed, for a Γ-bornological coarse space X we have a natural identification

$$\mathrm{Hom}_{\Gamma\mathbf{BornCoarse}}(S_{min,max}, X) \cong \mathrm{Hom}_{\Gamma\mathbf{Set}}(S, M(X))\ .$$ ♦

3. Equivariant coarse homology theories

The following notions are the obvious generalizations from the non-equivariant situation considered in [BE16].

Two morphisms $f, f'\colon X \to X'$ between Γ-bornological coarse spaces are *close* to each other if the subset $\{(f(x), f'(x)) \mid x \in X\}$ of $X' \times X'$ is an entourage, i.e. if they are close as morphisms between the underlying bornological coarse spaces.

Definition 3.1. A morphism between Γ-bornological coarse spaces is an *equivalence* if it admits an inverse morphism up to closeness. ♦

Example 3.2. We consider a Γ-bornological coarse space X, a Γ-invariant subset A of X, and a Γ-invariant entourage U of X. Then we can form the U-thickening $U[A]$ which is again Γ-invariant. We now assume that U contains the diagonal. Then we have a natural inclusion $i\colon A \to U[A]$. This inclusion is in general not an equivalence of Γ-bornological coarse spaces.

For example, let the group $\mathbb{Z}$ act on $\mathbb{C}$ by $(n, z) \mapsto e^{2\pi i\theta n}z$, where θ is an irrational real number. Then the subset $\mathbb{C} \setminus \{0\}$ of $\mathbb{C}$ is $\mathbb{Z}$-invariant. Every non-trivial thickening of this subset contains the point 0. This point is fixed by the action, but $\mathbb{C} \setminus \{0\}$ does not contain any fixed point which could serve as the image of 0 under a potential inverse of the inclusion $\mathbb{C} \setminus \{0\} \to \mathbb{C}$. ♦

Let X be a Γ-bornological coarse space and A be a Γ-invariant subset of X.

Definition 3.3. The subset A is called *nice* if for every invariant entourage U of X containing the diagonal the inclusion $A \to U[A]$ is an equivalence. ♦

Example 3.4. Let X be a Γ-bornological coarse space and let Y be a bornological coarse space considered as a Γ-bornological coarse space with the trivial Γ-action.

For every subset A of Y the subset $A \times X$ of the Γ-bornological coarse space $Y \times X$ (or of $Y \otimes X$) is nice. ♦

A filtered family of subsets of a set X is a family $(Y_i)_{i\in I}$ of subsets indexed by a filtered partially ordered set I such that the map $I \to \mathcal{P}(X)$ given by $i \mapsto Y_i$ is order-preserving.

Let X be a Γ-bornological coarse space. Recall from [BE16] that a *big family* on X is a filtered family of subsets $(Y_i)_{i\in I}$ of X such that for every entourage U of X and i in I there exists j in I such that $U[Y_i] \subseteq Y_j$.

Definition 3.5. An *equivariant big family* on X is a big family consisting of Γ-invariant subsets. ♦

Example 3.6. Let X be a Γ-bornological coarse space and A be a Γ-invariant subset of X. Then the family

$$\{A\} := (U[A])_{U\in\mathcal{C}^\Gamma}$$

is an equivariant big family. ♦

Let X be a Γ-bornological coarse space. Recall from [BE16] that a complementary pair $(Z, \mathcal{Y})$ on X is a pair of a subset Z of X and a big family $\mathcal{Y} = (Y_i)_{i\in I}$ on X such that there exists i in I with $Z \cup Y_i = X$.

Definition 3.7. An *equivariant complementary pair* on X is a complementary pair $(Z, \mathcal{Y})$ such that Z is a Γ-invariant subset and $\mathcal{Y}$ is an equivariant big family. ♦

Definition 3.8. A Γ-bornological coarse space X is *flasque* if it admits a morphism $f\colon X \to X$ such that

1. f is close to id_X.
2. For every entourage U of X the subset $\bigcup_{n\in\mathbb{N}}(f^n \times f^n)(U)$ is an entourage of X.
3. For every bounded subset B of X there exists an integer n such that $\Gamma B \cap f^n(X) = \emptyset$.

We say that flasqueness of X is *implemented* by f. ♦

Remark 3.9. In Condition 3 above one could require the weaker condition $B \cap f^n(X) = \emptyset$ instead of $\Gamma B \cap f^n(X) = \emptyset$. Then much of the theory would go through, but we lose the possibility of descending the group change functors “H-completion”, “quotient” and “induction” to the motivic level. ♦

Let $\mathbf{C}$ be a cocomplete stable ∞-category. We consider a functor

$$E\colon \Gamma\mathbf{BornCoarse} \to \mathbf{C} \ .$$

If $\mathcal{Y} = (Y_i)_{i\in I}$ is a filtered family of Γ-invariant subsets of X, then we set

$$E(\mathcal{Y}) := \operatorname*{colim}_{i\in I} E(Y_i) \ . \tag{3.1}$$

In this formula we consider the subsets Y_i as Γ-bornological coarse spaces with the structures induced from X.

The set $\{0,1\}_{max,max}$ is a Γ-bornological coarse space with the trivial Γ-action.

Definition 3.10. A *Γ-equivariant $\mathbf{C}$-valued coarse homology theory* is a functor

$$E\colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

with the following properties:

1. (Coarse invariance) For all $X \in \Gamma\mathbf{BornCoarse}$ the projection $\{0,1\}_{max,max}\otimes X \to X$ is sent by E to an equivalence.
2. (Excision) $E(\emptyset) \simeq 0$ and for every equivariant complementary pair $(Z,\mathcal{Y})$ on a Γ-bornological coarse space X the square

$$\begin{array}{ccc} E(Z\cap\mathcal{Y}) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(X) \end{array}$$

   is a push-out.
3. (Flasqueness) If a Γ-bornological coarse space X is flasque, then $E(X) \simeq 0$.
4. (u-Continuity) For every Γ-bornological coarse space X the natural map

$$\operatorname*{colim}_{U\in\mathcal{C}^\Gamma} E(X_U) \xrightarrow{\simeq} E(X)$$

   is an equivalence (see Example 2.9 for notation).

If the group Γ is clear from the context, then we will often just speak of an equivariant coarse homology theory. ♦

Remark 3.11. Condition 1 in the above definition is equivalent to the condition that E sends equivalences (Definition 3.1) of Γ-bornological coarse spaces to equivalences in $\mathbf{C}$:

The projection $\{0,1\}_{max,max}\otimes X \to X$ is an equivalence of Γ-bornological coarse spaces. If E preserves equivalences, then it sends this morphism to an equivalence.

Vice versa, if E satisfies Condition 1, then it sends pairs of close maps to equivalent maps. If f is an equivalence with inverse g up to closeness, then $E(f\circ g)$ and $E(g\circ f)$ are equivalent to $E(\mathrm{id})$ and therefore themselves equivalences. This implies by functoriality that $E(f)$ is an equivalence. ♦

Let the cocomplete stable ∞-category $\mathbf{C}$ have all small products. Let $(X_i)_{i\in I}$ be a family of Γ-bornological coarse spaces. If E is a $\mathbf{C}$-valued equivariant coarse homology theory, then by excision for every index i in I we have a projection $E(\bigsqcup_{i\in I}^{\mathrm{free}} X_i) \to E(X_i)$. The collection of these projections induces a morphism

$$E(\bigsqcup_{i\in I}^{\mathrm{free}} X_i) \to \prod_{i\in I} E(X_i) \tag{3.2}$$

Definition 3.12. E is called *strongly additive* if (3.2) is an equivalence for every family $(X_i)_{i\in I}$ of Γ-bornological coarse spaces. ♦

Let E be an equivariant coarse homology theory and let S be a Γ-set.

Lemma 3.13. *If E is strongly additive, then the twist $E(-\otimes S_{max,max})$ is strongly additive.*

Proof. This follows from the fact that for every family $(X_i)_{i\in I}$ of Γ-bornological coarse spaces we have an isomorphism

$$(\coprod_{i\in I}^{\mathrm{free}} X_i) \otimes S_{max,max} \cong \coprod_{i\in I}^{\mathrm{free}} (X_i \otimes S_{max,max})$$

of Γ-bornological coarse spaces. □

4. Equivariant coarse motivic spectra

4.1. Construction

In this section we define the stable ∞-category of coarse motives $\Gamma\mathbf{Sp}\mathcal{X}$. This is completely analogous to [BE16, Sec. 3 & 4]. The category $\Gamma\mathbf{Sp}\mathcal{X}$ is designed such that equivariant $\mathbf{C}$-valued coarse homology theories (Definition 3.10) are the same as colimit-preserving functors $\Gamma\mathbf{Sp}\mathcal{X} \to \mathbf{C}$. The precise formulation is Corollary 4.10.

Let $\mathbf{Spc}$ be the ∞-category of spaces, i.e., the universal presentable ∞-category generated by $*$. We start with the category

$$\mathbf{PSh}(\Gamma\mathbf{BornCoarse}) := \mathbf{Fun}(\Gamma\mathbf{BornCoarse}^{op}, \mathbf{Spc})$$

of $\mathbf{Spc}$-valued presheaves on $\Gamma\mathbf{BornCoarse}$. Let

$$\mathrm{yo}\colon \Gamma\mathbf{BornCoarse} \to \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \tag{4.1}$$

be the Yoneda embedding. Precomposition with it induces an equivalence

$$\mathbf{Fun}^{\lim}(\mathbf{PSh}(\Gamma\mathbf{BornCoarse})^{op}, \mathbf{Spc}) \simeq \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \ ,$$

where $\mathbf{Fun}^{\lim}$ denotes limit-preserving functors [Lur09, Thm. 5.1.5.6] (see also [BE16, Rem. 3.9]). We use this equivalence in order to evaluate presheaves on other presheaves.

For a filtered family $\mathcal{Y} = (Y_i)_{i\in I}$ of invariant subsets on some Γ-bornological coarse space X we write

$$\mathrm{yo}(\mathcal{Y}) := \operatorname*{colim}_{i\in I} \mathrm{yo}(Y_i) \in \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \ . \tag{4.2}$$

For E in $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ we set

$$E(\mathcal{Y}) := E(\mathrm{yo}(\mathcal{Y})) \ .$$

Remark 4.1. For a Γ-bornological coarse space X we have the equivalence

$$E(X) \simeq E(\mathrm{yo}(X)) \ .$$

Furthermore we have

$$E(\mathcal{Y}) \simeq \lim_{i\in I} E(Y_i) \tag{4.3}$$

for a filtered family $\mathcal{Y} = (Y_i)_{i\in I}$ of invariant subsets on X. ♦

Let E be an object of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$.

Definition 4.2. We say that E *satisfies descent* if

1. $E(\emptyset) \simeq *$, and
2. for every equivariant complementary pair $(Z, \mathcal{Y})$ on a Γ-bornological coarse space X the square
$$\begin{array}{ccc} E(X) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(Z\cap\mathcal{Y}) \end{array} \tag{4.4}$$
is cartesian.

Presheaves which satisfy descent are called *sheaves*. ♦

Remark 4.3. One can show that there is a subcanonical Grothendieck topology τ_χ on $\Gamma\mathbf{BornCoarse}$ such that the τ_χ-sheaves are exactly the presheaves which satisfy descent for equivariant complementary pairs. Since we will not be using this fact in this paper we will omit the arguments. ♦

We let $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ denote the full subcategory of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ of sheaves. We can characterize sheaves as presheaves which are local with respect to the morphisms

$$\begin{array}{rcl} \mathrm{yo}(\mathcal{Y}) \sqcup_{\mathrm{yo}(Z\cap\mathcal{Y})} \mathrm{yo}(Z) & \to & \mathrm{yo}(X) \\ \mathrm{yo}(\emptyset) & \to & * \ . \end{array} \tag{4.5}$$

Remark 4.4. In order to fix set-theoretic issues we assume that all Γ-bornological coarse spaces and the index sets I for the big families belong to some Grothendieck universe of small sets. The class of local objects is then generated by a small set of morphisms. The category $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ then belongs to a bigger universe. ♦

We have a sheafification adjunction

$$\mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \mathbf{Sh}(\Gamma\mathbf{BornCoarse}) : \mathit{inclusion} \ .$$

Remark 4.5. For a Γ-bornological coarse space X the presheaf $\mathrm{yo}(X)$ is a compact object of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$. If $\mathcal{Y}$ is a big family, then $\mathrm{yo}(\mathcal{Y})$ is an in general infinite colimit of compact objects and hence not compact anymore. Consequently, the morphisms (4.5) are not morphisms between compact objects. The localization $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ is therefore a presentable ∞-category, but it is not compactly generated. ♦

Let E be an object of $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$.

Definition 4.6.

1. E is *coarsely invariant* if it is local with respect to the morphisms

   $$\mathrm{yo}(\{0,1\}_{max,max} \otimes X) \to \mathrm{yo}(X)$$

   induced by the projection for every Γ-bornological coarse space X.

2. E *vanishes on flasques* if it is local with respect to the morphisms

   $$\mathrm{yo}(\emptyset) \to \mathrm{yo}(X)$$

   for every flasque Γ-bornological coarse space X.

3. E is *u-continuous* if it is local for the morphism

   $$\operatorname*{colim}_{U \in \mathcal{C}^\Gamma} \mathrm{yo}(X_U) \to \mathrm{yo}(X)$$

   for every Γ-bornological coarse space X (where $\mathcal{C}^\Gamma$ denotes the invariant entourages of the space X).

The above notions are just the equivariant analogues of the corresponding notions from [BE16, Sec. 3]. ♦

Definition 4.7. We define the ∞-category of Γ-*equivariant motivic coarse spaces* $\Gamma\mathbf{Spc}\mathcal{X}$ as the full localizing subcategory of $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ of coarsely invariant, u-continuous sheaves which vanish on flasques. ♦

The locality condition is generated by a small set of morphisms. Therefore we have a localization adjunction

$$\mathcal{L} \colon \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \Gamma\mathbf{Spc}\mathcal{X} : \mathit{inclusion} \ . \tag{4.6}$$

We define

$$\mathrm{Yo} := \mathcal{L} \circ \mathrm{yo} \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Spc}\mathcal{X} \ .$$

The ∞-category $\Gamma\mathbf{Spc}\mathcal{X}$ is a presentable ∞-category.

Definition 4.8. We define the category of *equivariant motivic coarse spectra* as the stabilization

$$\Gamma\mathbf{Sp}\mathcal{X} := \Gamma\mathbf{Spc}\mathcal{X}_*[\Sigma^{-1}]$$

in the realm of presentable ∞-categories. ♦

Then $\Gamma\mathbf{Sp}\mathcal{X}$ is a stable presentable ∞-category which fits into an adjunction

$$\Sigma^{mot}_+ \colon \Gamma\mathbf{Spc}\mathcal{X} \leftrightarrows \Gamma\mathbf{Sp}\mathcal{X} : \Omega^{mot} \ .$$

We further define the Yoneda functor

$$\mathrm{Yo}^s := \Sigma^{mot}_+ \circ \mathrm{Yo} \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X} \ .$$

Definition 4.9. We call $\mathrm{Yo}^s \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}$ the universal equivariant coarse homology theory. ♦

For a Γ-bornological coarse space X we consider the object $\mathrm{Yo}^s(X)$ of $\Gamma\mathbf{Sp}\mathcal{X}$ as the motive of X.

Let $\mathbf{C}$ be a cocomplete stable ∞-category. Let $\Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}}$ denote the full subcategory of $\mathbf{Fun}(\Gamma\mathbf{BornCoarse}, \mathbf{C})$ of functors which are Γ-equivariant $\mathbf{C}$-valued coarse homology theories in the sense of Definition 3.10. By $\mathbf{Fun}^{\mathrm{colim}}(\Gamma\mathbf{Sp}\mathcal{X}, \mathbf{C})$ we denote the full subcategory of $\mathbf{Fun}(\Gamma\mathbf{Sp}\mathcal{X}, \mathbf{C})$ of colimit preserving functors.

The construction of $\Gamma\mathbf{Sp}\mathcal{X}$ has the following consequence (see [BE16, Cor. 4.6]):

Corollary 4.10. *The functor Yo^s is a $\Gamma\mathbf{Sp}\mathcal{X}$-valued equivariant coarse homology theory. Furthermore, precomposition with Yo^s induces an equivalence of ∞-categories*

$$\mathbf{Fun}^{\mathrm{colim}}(\Gamma\mathbf{Sp}\mathcal{X}, \mathbf{C}) \to \Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}} \ .$$

4.2. Properties

If $\mathcal{Y} = (Y_i)_{i \in I}$ is an equivariant big family on a Γ-bornological coarse space X, then we define the equivariant motivic coarse spectrum

$$\mathrm{Yo}^s(\mathcal{Y}) := \Sigma^{mot}_+ \circ \mathcal{L} \circ \mathrm{yo}(\mathcal{Y}) \ . \tag{4.7}$$

Note that we have $\Sigma^{mot}_+ \circ \mathcal{L} \circ \mathrm{yo}(\mathcal{Y}) \simeq \mathrm{colim}_{i \in I} \mathrm{Yo}^s(Y_i)$.

We will use the notation

$$\mathrm{Yo}^s(X, \mathcal{Y}) := \mathrm{Cofib}(\mathrm{Yo}^s(\mathcal{Y}) \to \mathrm{Yo}^s(X)) \ . \tag{4.8}$$

By construction we have the following properties:

Corollary 4.11.

1. *We have a fiber sequence*

$$\mathrm{Yo}^s(\mathcal{Y}) \to \mathrm{Yo}^s(X) \to \mathrm{Yo}^s(X,\mathcal{Y}) \to \Sigma\mathrm{Yo}^s(\mathcal{Y})$$

2. *For an equivariant complementary pair* $(Z,\mathcal{Y})$ *on* X *the natural morphism*

$$\mathrm{Yo}^s(Z, Z\cap\mathcal{Y}) \to \mathrm{Yo}^s(X,\mathcal{Y})$$

   is an equivalence.

3. *If* $X \to X'$ *is an equivalence of* Γ*-bornological coarse spaces, then the induced morphism* $\mathrm{Yo}^s(X) \to \mathrm{Yo}^s(X')$ *is an equivalence in* $\Gamma\mathbf{Sp}\mathcal{X}$*.*

4. *If* X *is a flasque* Γ*-bornological coarse space, then* $\mathrm{Yo}^s(X) \simeq 0$*.*

5. *For every* Γ*-bornological coarse space* X *with coarse structure* $\mathcal{C}$ *the natural map*

$$\operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(X_U) \xrightarrow{\simeq} \mathrm{Yo}^s(X)$$

   is an equivalence.

Let X be a Γ-bornological coarse space and A be a Γ-invariant subset of X. Recall that $\{A\}$ denotes the equivariant big family generated by A (Example 3.6).

Corollary 4.12. *If* A *is nice, then we the natural map* $\mathrm{Yo}^s(A) \to \mathrm{Yo}^s(\{A\})$ *is an equivalence.*

Proof. Since A is nice, for every invariant entourage U of X the inclusion $A \to U[A]$ is an equivalence. The assertion now follows since Yo^s preserves equivalences. □

Let X be a Γ-bornological coarse space and Y, Z be invariant subsets such that $Y \cup Z = X$.

Definition 4.13. We say that (Y,Z) is a *coarsely excisive pair*, if:

1. For every entourage U of X there exists an entourage W of X such that

$$U[Y]\cap U[Z] \subseteq W[Y\cap Z]\ .$$

2. There exists a cofinal set of invariant entourages V of X such that $V[Y]\cap Z$ is nice.

Note that Condition 2 is a new aspect of the equivariant theory. ♦

Let X be a Γ-bornological coarse space and Y, Z be invariant subsets such that $Y \cup Z = X$.

Corollary 4.14. *If* (Y,Z) *is a coarsely excisive pair, then we have a cocartesian square*

$$\begin{array}{ccc} \mathrm{Yo}^s(Y\cap Z) & \longrightarrow & \mathrm{Yo}^s(Z) \\ \downarrow & & \downarrow \\ \mathrm{Yo}^s(Y) & \longrightarrow & \mathrm{Yo}^s(X) \end{array}$$

Proof. The proof of [BE16, Lem. 3.38] goes through literally. In the proof we need the equivalence

$$\mathrm{Yo}^s(V[Y] \cap Z) \simeq \mathrm{Yo}^s(\{V[Y] \cap Z\})$$

for sufficiently large invariant entourages V of X. This is ensured by Condition 2 in the Definition 4.13 of coarse excisiveness. □

Let X be a Γ-bornological coarse space. Given two bornological and Γ-invariant maps $p = (p_-, p_+)$ with $p_- \colon X \to (-\infty, 0]$ and $p_+ \colon X \to [0, \infty)$ we can form the coarse cylinder I_pX as in the non-equivariant case [BE16, Sec. 4.3]. With its natural Γ-action it is a Γ-bornological coarse space. The projection $I_pX \to X$ is a morphism. We will call it an equivariant cylinder in order to stress that the datum p was Γ-invariant.

Let X be a Γ-bornological coarse space and I_pX be an equivariant coarse cylinder.

Corollary 4.15. *The projection $I_pX \to X$ induces an equivalence $\mathrm{Yo}^s(I_pX) \to \mathrm{Yo}^s(X)$.*

Proof. We observe that the proof of [BE16, Prop. 4.16] goes through. At all places in the argument where Corollary 4.12 is used the corresponding subset is nice, see Example 3.4. □

We say that two morphisms $f_+, f_- \colon X \to X'$ between Γ-bornological coarse spaces are *homotopic* if there exists a cylinder I_pX such that $p_\pm$ are Γ-invariant, bornological and in addition controlled, and if there exists a morphism $h \colon I_pX \to X'$ such that $f_\pm = h \circ i_\pm$. This leads to an extension of the notion of coarse invariance.

Corollary 4.16. *If f_+ and f_- are homotopic, then $\mathrm{Yo}^s(f_+)$ and $\mathrm{Yo}^s(f_-)$ are equivalent.*

4.3. Symmetric monoidal structure

Recall that the category $\Gamma\mathbf{BornCoarse}$ has a symmetric monoidal structure

$$-\otimes- \colon \Gamma\mathbf{BornCoarse} \times \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse} \tag{4.9}$$

with tensor unit $*$.

Lemma 4.17. *$\Gamma\mathbf{Sp}\mathcal{X}$ has an induced closed symmetric monoidal structure $\otimes$ such that the functor $\mathrm{Yo}^s \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}$ is symmetric monoidal. The functor $\otimes$ commutes with colimits in each variable separately.*

Proof. We get an induced symmetric monoidal structure on $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ by the Day convolution product. The unit is given by $\mathrm{yo}(*)$, the Yoneda embedding is a strong symmetric monoidal functor, and $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ is closed symmetric monoidal.

For a Γ-bornological coarse space Q the functor

$$-\otimes Q \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse}$$

maps big families to big families and complementary pairs to complementary pairs. So the monoidal structure of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ restricts to one on $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ and the sheafification adjunction is a symmetric monoidal adjunction.

The functor $-\otimes Q$ furthermore respects closeness of morphisms and therefore coarse equivalences, and it respects flasqueness and u-continuity. So we get an induced symmetric monoidal structure on $\Gamma\mathbf{Spc}\mathcal{X}$ and $\mathrm{Yo}\colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Spc}\mathcal{X}$ is symmetric monoidal.

Since $\Gamma\mathbf{Spc}\mathcal{X}$ is presentable, we can equip its stabilization $\Gamma\mathbf{Sp}\mathcal{X}$ with a unique symmetric monoidal structure $\otimes$ such that stabilization $\Sigma^{mot}_{+}\colon \Gamma\mathbf{Spc}\mathcal{X} \to \Gamma\mathbf{Sp}\mathcal{X}$ is symmetric monoidal [GGN15, Thm. 5.1].

It follows from the construction that $\otimes$ commutes with colimits in each variable separately. □

4.4. Strong version

In this section we discuss an additional property (strongness) which an equivariant coarse homology theory can have. Another condition (continuity) will be discussed in Section 5.

By definition, a flasque Γ-bornological coarse space X admits a morphism $f\colon X \to X$ satisfying the conditions listed in Definition 3.8. The first condition is the condition that f is close to the identity. This fact is usually used in order to deduce that $\mathrm{Yo}^s(f) \simeq \mathrm{id}_{\mathrm{Yo}^s(X)}$. In the following we will use this weaker condition in order to define a more general notion of flasqueness.

Definition 4.18. X is called *weakly flasque* if it admits a morphism $f\colon X \to X$ satisfying

1. $\mathrm{Yo}^s(f) \simeq \mathrm{id}_{\mathrm{Yo}^s(X)}$.
2. For every entourage U of X the subset $\bigcup_{n\in\mathbb{N}}(f^n \times f^n)(U)$ is again an entourage of X.
3. For every bounded subset B of X there exists an integer n such that $\Gamma B \cap f^n(X) = \emptyset$.

We say that f *implements weak flasqueness* of X. ♦

Let $\mathbf{C}$ be a cocomplete stable ∞-category and consider a $\mathbf{C}$-valued equivariant coarse homology theory E.

Definition 4.19. E is called *strong* if $E(X) \simeq 0$ for all weakly flasque Γ-bornological coarse spaces X. ♦

Let us incorporate now the condition of strongness on the motivic level.

Definition 4.20. We define the version of equivariant motivic spectra $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$ as the localization of the category $\Gamma\mathbf{Sp}\mathcal{X}$ at the set of morphisms $0 \to \mathrm{Yo}^s(X)$ for all weakly flasque Γ-bornological coarse spaces X. ♦

The corresponding Yoneda functor is denoted by

$$\mathrm{Yo}^s_{\mathrm{wfl}} \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}} \ .$$

We consider now the ∞-category of strong Γ-equivariant coarse homology theories. The construction of $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$ has the following immediate consequence:

Corollary 4.21. *The functor $\mathrm{Yo}^s_{\mathrm{wfl}}$ is a $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$-valued equivariant coarse homology theory. Furthermore, precomposition with $\mathrm{Yo}^s_{\mathrm{wfl}}$ induces an equivalence of ∞-categories*

$$\mathbf{Fun}^{\mathrm{colim}}(\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}, \mathbf{C}) \to \mathbf{strong}\Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}} \ .$$

5. Continuity

The purpose of this section is to introduce the notion of continuity for equivariant coarse homology theories. This property will be crucially needed in Section 11.2. We will first introduce the notion of trapping exhaustions in Section 5.1. Section 5.2 contains the actual definition of continuous equivariant coarse homology theories, and Section 5.3 incorporates continuity motivically. In the last Section 5.4 we will show how one can force continuity for an equivariant coarse homology theory.

5.1. Trapping exhaustions

In this section we will introduce the notion of a trapping exhaustion of a Γ-bornological space and discuss some examples and basic properties of this notion. We will also introduce the stronger notion of a co-Γ-bounded exhaustion.

Let X be a bornological space and let F be a subset of X.

Definition 5.1. The subset F is called *locally finite* if $B \cap F$ is finite for every bounded subset B of X. ♦

Example 5.2. Every finite subset of X is locally finite. ♦

Example 5.3. If X has the minimal bornology on X, i.e., a subset is bounded if and only if it is finite, then every subset of X is locally finite. ♦

Example 5.4. If X has the maximal bornology on X, i.e., every subset of X is bounded, then the locally finite subsets of X are exactly the finite subsets. ♦

Let $f \colon X \to X'$ be a proper map between bornological spaces. Let F be a subset of X.

Lemma 5.5. *If F is locally finite, then $f(F)$ is locally finite.*

Proof. We use the relation $f(F) \cap B \subseteq f(F \cap f^{-1}(B))$. □

We consider in the following a Γ-bornological space X and a filtered family of invariant subsets $\mathcal{Y} = (Y_i)_{i\in I}$.

Definition 5.6. The family $\mathcal{Y}$ is called a *trapping exhaustion* if for every locally finite, invariant subset F of X there exists i in I such that $F \subseteq Y_i$. ♦

Example 5.7. The family consisting of all locally finite, invariant subsets is a trapping exhaustion.

It might happen that a Γ-bornological coarse space does note admit any non-empty invariant locally finite subset. Consider e.g. Γ with the maximal bornology. In this case the empty family is a trapping exhaustion. ♦

In the following we will introduce a particular kind of trapping exhaustions which we call co-Γ-bounded exhaustions.

We consider a Γ-bornological space X and a filtered family of invariant subsets $\mathcal{Y} = (Y_i)_{i\in I}$. We use the notation and terminology introduced in Definition 2.12.

Definition 5.8. The family $\mathcal{Y}$ is called a *co-Γ-bounded exhaustion* if

1. $\mathcal{Y}$ is an exhaustion, i.e., $\bigcup_{i\in I} Y_i = X$, and
2. $\mathcal{Y}$ is co-Γ-bounded, i.e., there exists i in I such that $X \setminus Y_i$ is Γ-bounded. ♦

We consider Γ-bornological spaces X and Z and a filtered family $\mathcal{Y} := (Y_i)_{i\in I}$ of invariant subsets of X. In the following we denote by $Z \otimes X$ the Γ-bornological space whose bornology is generated by products $A \times B$ for all bounded subsets A of Z and B of X.

Lemma 5.9. *If Z is bounded and $\mathcal{Y}$ is a co-Γ-bounded (resp., trapping) exhaustion of X, $(Z \times Y_i)_{i\in I}$ is a co-Γ-bounded (resp., trapping) exhaustion of $Z \otimes X$.*

Proof. The co-Γ-bounded case is straightforward.

For the trapping case assume that F is an invariant, locally finite subset of $Z \otimes X$. Since the projection $p\colon Z \otimes X \to X$ is proper, by Lemma 5.5 the subset $p(F)$ of X is locally finite. Hence there exists i in I such that $p(F) \subseteq Y_i$, and therefore $F \subseteq Z \times Y_i$. □

Lemma 5.10. *If $\mathcal{Y}$ is a co-Γ-bounded exhaustion of a Γ-bornological space X then it is a trapping exhaustion.*

Proof. Let F be an invariant, locally finite subset of X. Since $\mathcal{Y}$ is a co-Γ-bounded exhaustion of X there exists an index i in I and a bounded subset B of X such that $\Gamma B \cup Y_i = X$. Since the union of $\mathcal{Y}$ is X and $F \cap B$ is finite there exists an index j in I such that $i \leq j$ and $F \cap B \subseteq Y_j$. Since Y_j is invariant, then also $F \subseteq Y_j$. □

Example 5.11. Let Z be a Γ-bounded Γ-bornological space and let $\mathcal{Z} = (Z_i)_{i\in I}$ be an exhaustion by not necessarily Γ-invariant subsets. For every i in I we consider the subset

$$D_i := \Gamma(Z_i \times \{1\})$$

of $Z \times \Gamma$. We consider the Γ-bornological space $Z \otimes \Gamma$, where Γ has any Γ-invariant bornology. The family

$$\mathcal{D} := (D_i)_{i\in I}$$

is a co-Γ-bounded (and hence trapping) exhaustion of $Z \otimes \Gamma$. ♦

We consider $[0,\infty)$ as a Γ-bornological space with the trivial action and the bornology generated by the subsets $[0,n]$ for all integers n. In the following we will construct an interesting trapping exhaustion of the Γ-bornological space

$$[0,\infty) \otimes Z \otimes \Gamma$$

which will play an important role in Section 11.2.

Note that in general $([0,\infty) \times D_i)_{i\in I}$ is not trapping.

We consider the set of functions $I^{\mathbb{N}}$ with its partial order induced from I. Then the partially ordered set $I^{\mathbb{N}}$ is filtered. For a function κ in $I^{\mathbb{N}}$ we define the set

$$Y_\kappa := \bigcup_{n\in\mathbb{N}} [n-1,n] \times D_{\kappa(n)} \ .$$

Lemma 5.12. *If Z is Γ-bounded, then $\mathcal{Y} := (Y_\kappa)_{\kappa\in I^{\mathbb{N}}}$ is a trapping exhaustion of the space $[0,\infty) \otimes Z \otimes \Gamma$.*

Note that the exhaustion $\mathcal{Y}$ is not co-Γ-bounded.

Proof. The members of $\mathcal{Y}$ are Γ-invariant subsets. For every integer n the family given by $([n-1,n] \times D_i)_{i\in I}$ is a co-Γ-bounded exhaustion of $[n-1,n] \otimes Z \otimes \Gamma$, since $[n-1,n]$ is bounded and $(D_i)_{i\in I}$ is a co-Γ-bounded exhaustion of $Z \otimes \Gamma$. So it is trapping.

Let F be a Γ-invariant locally finite subset of $[0,\infty) \times Z \times \Gamma$. Then $F \cap [n-1,n] \times Z \times \Gamma$ is also locally finite and Γ-invariant. For every integer n we can choose $\kappa(n)$ in I such that $(F \cap [n-1,n] \times Z \times \Gamma) \subseteq [n-1,n] \times D_{\kappa(n)}$. This describes a function κ in $I^{\mathbb{N}}$ such that by construction $F \subseteq Y_\kappa$. □

Let $f\colon X' \to X$ be a proper map between Γ-bornological spaces.

Lemma 5.13. *If $\mathcal{Y}$ is a co-Γ-bounded (resp., trapping) exhaustion of X, then $f^{-1}\mathcal{Y}$ is a co-Γ-bounded (resp., trapping) exhaustion of X'.*

Proof. The co-Γ-bounded case is a direct consequence of properness and the fact that forming preimages commutes with forming complements.

For the trapping case one uses in addition Lemma 5.5. □

5.2. Continuous equivariant coarse homology theories

In this section we will introduce an additional continuity condition on an equivariant coarse homology theory. We then verify that a continuous equivariant coarse homology theory preserves coproducts.

Remark 5.14. In Section 7.3 we show continuity of equivariant coarse ordinary homology theory and in Proposition 8.17 continuity of equivariant coarse algebraic K-homology. In Example 8.31 we describe a version of the coarse algebraic K-homology theory which is not continuous. ♦

We can extend the notions of locally finite subsets and trapping / co-Γ-bounded exhaustions to Γ-bornological coarse spaces by just considering the underlying Γ-bornological spaces.

Let $\mathbf{C}$ be a stable cocomplete ∞-category and

$$E\colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

an equivariant coarse homology theory. We use the convention (3.1) for the evaluation $E(\mathcal{Y})$ on a filtered family $\mathcal{Y}$ of invariant subsets of a Γ-bornological coarse space. We have a natural morphism

$$E(\mathcal{Y}) \to E(X)\ . \tag{5.1}$$

Definition 5.15. E is called *continuous* if for every trapping exhaustion $\mathcal{Y}$ of some Γ-bornological coarse space X the morphism (5.1) is an equivalence. ♦

Remark 5.16. A continuous equivariant coarse homology theory E is determined by its values on locally finite, invariant spaces. More precisely, let $\mathcal{F}(X)$ be the filtered partially ordered set of locally finite, invariant subsets of X. In view of Example 5.7 we have a trapping exhaustion $\mathcal{Y} := (F)_{F\in\mathcal{F}(X)}$ of X. So we get

$$E(X) \simeq E(\mathcal{Y}) = \operatorname*{colim}_{F\in\mathcal{F}(X)} E(F)$$

showing the claim.

If $\mathcal{F}(X)$ is empty, i.e., X does not admit non-empty locally finite Γ-invariant subsets (see Example 5.7), and E is continuous, then $E(X) \simeq 0$. ♦

By excision an equivariant coarse homology theory preserves coproducts of finite families of Γ-bornological coarse spaces. By the following lemma, for a continuous equivariant coarse homology theory, we can drop the word *finite*.

Lemma 5.17. *A continuous equivariant coarse homology theory preserves coproducts.*

Proof. Let $(X_i)_{i\in I}$ be a family of Γ-bornological coarse spaces. We must show that the natural map

$$\bigoplus_{i\in I} E(X_i) \to E\Big(\coprod_{i\in I} X_i\Big)$$

is an equivalence. All invariant subsets of $\coprod_{i\in I} X_i$ have the induced bornological coarse structures.

Consider the following diagram, where the horizontal maps are equivalences by continuity and where $\mathcal{F}(-)$ is the trapping exhaustion consisting of all locally finite, invariant subsets.

$$\begin{array}{ccc} \bigoplus_{i\in I} \operatorname{colim}_{F_i\in\mathcal{F}(X_i)} E(F_i) & \xrightarrow{\simeq} & \bigoplus_{i\in I} E(X_i) \\ {\scriptstyle !}\downarrow & & \downarrow \\ \operatorname{colim}_{F\in\mathcal{F}(\coprod_{i\in I} X_i)} E(F) & \xrightarrow{\simeq} & E(\coprod_{i\in I} X_i) \end{array}$$

It remains to show that the map marked with ! is an equivalence. The bornology of the coproduct is described in [BE16, Lemma 2.24]. A subset of the coproduct is bounded if and only if its intersection with X_i for every i in I is bounded. This implies that for every invariant, locally finite subset F of $\coprod_{i\in I} X_i$ there exists a minimal finite subset $J(F)$ of I such that $F \subseteq \coprod_{i\in J(F)} X_i$. We write $F_i := F \cap X_i$. Then

$$F = \bigcup_{i\in J(F)} F_i$$

is a finite, coarsely disjoint decomposition. Hence

$$E(F) \simeq \bigoplus_{i\in J(F)} E(F_i) \simeq \bigoplus_{i\in I} E(F_i)\ .$$

That the map marked with ! is an equivalence now follows from the equivalence

$$\bigoplus_{i\in I} \operatorname*{colim}_{F_i\in\mathcal{F}(X_i)} E(F_i) \simeq \operatorname*{colim}_{(F_i)_i\in\mathcal{F}(\coprod_{i\in I} X_i)} \bigoplus_{i\in I} E(F_i)\ . \qquad \square$$

5.3. Continuous coarse motives

In this section we will explain how to incorporate continuity on the motivic level and we will discuss basic properties of this procedure. Recall the Yoneda embedding (4.1) and the notation (4.2). We have a natural morphism

$$\mathrm{yo}(\mathcal{Y}) \to \mathrm{yo}(X)\ . \tag{5.2}$$

Let E be an object of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$.

Definition 5.18. We call E *continuous* if it is local with respect to the morphisms (5.2) for all trapping exhaustions $\mathcal{Y}$ of Γ-bornological coarse spaces X. ♦

Remark 5.19. Let E be an object of $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ and recall (4.3). The collection of restriction morphisms $E(X) \to E(Y_i)$ for all i in I induce a natural morphism

$$E(X) \to E(\mathcal{Y}) \ . \tag{5.3}$$

Then E is continuous if and only if the morphism (5.3) is an equivalence for every trapping exhaustion $\mathcal{Y}$ of a Γ-bornological coarse space X. ♦

We now incorporate continuity on the motivic level by adding this relation to the list in Definition 4.7.

Definition 5.20. We define the ∞-category of continuous Γ-equivariant motivic coarse spaces $\Gamma\mathbf{Spc}\mathcal{X}_c$ to be the full localizing subcategory of $\mathbf{Sh}(\Gamma\mathbf{BornCoarse})$ of coarsely invariant, continuous and u-continuous sheaves which vanish on flasques. ♦

The locality condition is generated by a small set of morphisms. Therefore we have a localizing adjunction

$$\mathcal{L}_c \colon \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \Gamma\mathbf{Spc}\mathcal{X}_c : \mathit{inclusion} \ . \tag{5.4}$$

We define

$$\mathrm{Yo}_c := \mathcal{L}_c \circ \mathrm{yo} \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Spc}\mathcal{X}_c \ .$$

We furthermore have a localizing adjunction

$$C \colon \Gamma\mathbf{Spc}\mathcal{X} \leftrightarrows \Gamma\mathbf{Spc}\mathcal{X}_c : \mathit{inclusion}$$

and the relations

$$\mathcal{L}_c \simeq C \circ \mathcal{L} \ , \quad \mathrm{Yo}_c \simeq C \circ \mathrm{Yo} \ .$$

where $\mathcal{L}$ is as in (4.6).

The ∞-category $\Gamma\mathbf{Spc}\mathcal{X}$ is a presentable ∞-category.

Definition 5.21. We define the category of continuous equivariant motivic coarse spectra as the stabilization

$$\Gamma\mathbf{Sp}\mathcal{X}_c := \Gamma\mathbf{Spc}\mathcal{X}_{c,*}[\Sigma^{-1}]$$

in the realm of presentable ∞-categories. ♦

Then $\Gamma\mathbf{Sp}\mathcal{X}_c$ is a stable presentable ∞-category which fits into an adjunction

$$\Sigma^{mot}_{c,+} \colon \Gamma\mathbf{Spc}\mathcal{X}_c \leftrightarrows \Gamma\mathbf{Sp}\mathcal{X}_c : \Omega^{mot}_c \ .$$

We further define the following stable continuous version of the Yoneda functor:

$$\mathrm{Yo}^s_c := \Sigma^{mot}_{c,+} \circ \mathrm{Yo}_c \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}_c \ . \tag{5.5}$$

We obtain the functor C^s in the following commuting square from the universal property of the stabilization:

$$\begin{array}{ccc} \Gamma\mathbf{Spc}\mathcal{X} & \xrightarrow{\Sigma^{mot}_{+}} & \Gamma\mathbf{Sp}\mathcal{X} \\ \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle C^s} \\ \Gamma\mathbf{Spc}\mathcal{X}_c & \xrightarrow{\Sigma^{mot}_{c,+}} & \Gamma\mathbf{Sp}\mathcal{X}_c \end{array} \tag{5.6}$$

We furthermore have the relation

$$\mathrm{Yo}^s_c \simeq C^s \circ \mathrm{Yo}^s \ . \tag{5.7}$$

Let $\mathbf{C}$ be a cocomplete stable ∞-category. We let $\mathbf{Cont}\Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}}$ denote the full subcategory of $\mathbf{Fun}(\Gamma\mathbf{BornCoarse}, \mathbf{C})$ of functors which are continuous Γ-equivariant coarse homology theories in the sense of Definition 3.10.

The construction of $\Gamma\mathbf{Sp}\mathcal{X}_c$ has the following immediate consequence:

Corollary 5.22. *Precomposition with* Yo^s_c *induces an equivalence of* ∞*-catgeories*

$$\mathbf{Fun}^{\mathrm{colim}}(\Gamma\mathbf{Sp}\mathcal{X}_c, \mathbf{C}) \to \mathbf{Cont}\Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}} \ .$$

The Yoneda functor Yo^s_c has all the properties listed in Section 4.2. In addition it satisfies:

Corollary 5.23. *We have*

$$\mathrm{Yo}^s_c(\mathcal{Y}) \simeq \mathrm{Yo}^s_c(X)$$

for every trapping exhaustion $\mathcal{Y}$ *of a* Γ*-bornological coarse space* X*.*

So in particular, $\mathrm{Yo}^s_c(X)$ is determined by the collection of invariant, locally finite subsets of X:

$$\mathrm{Yo}^s_c(X) \simeq \operatorname*{colim}_{F \in \mathcal{F}(X)} \mathrm{Yo}^s_c(F) \ .$$

Note that Yo^s_c is a $\Gamma\mathbf{Sp}\mathcal{X}_c$-valued continuous Γ-equivariant coarse homology theory. Hence Lemma 5.17 implies:

Corollary 5.24. *The functor* $\mathrm{Yo}^s_c \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Spc}\mathcal{X}_c$ *preserves coproducts.*

If $\mathcal{Y} = (Y_i)_{i\in I}$ is a co-Γ-bounded (or trapping, respectively) exhaustion of a Γ-bornological space X and Z is a second Γ-bornological coarse space, then $\mathcal{Y} \times Z := (Y_i \times Z)_{i \in I}$ is not necessarily trapping in $X \otimes Z$. As a consequence the symmetric monoidal structure $\otimes$ does not descend to continuous motivic coarse spectra. But if Z is bounded, then $\mathcal{Y} \times Z$ is again a trapping exhaustion of $X \otimes Z$ by Lemma 5.9. We can conclude:

Corollary 5.25. *If* Z *is a bounded* Γ*-bornological coarse space, then the functor*

$$- \otimes Z \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse}$$

descends to a functor

$$- \otimes^{mot} Z \colon \Gamma\mathbf{Sp}\mathcal{X}_c \to \Gamma\mathbf{Sp}\mathcal{X}_c$$

such that

$$\mathrm{Yo}^s_c(X \otimes Z) \simeq \mathrm{Yo}^s_c(X) \otimes^{mot} Z \ . \tag{5.8}$$

5.4. Forcing continuity

To every $\mathbf{C}$-valued equivariant coarse homology theory E we can naturally associate a continuous version E_{cont}. This is actually the best approximation of E by some continuous $\mathbf{C}$-valued equivariant coarse homology theory. We will show that there is an adjunction

$$(-)_{cont}\colon \Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}} \leftrightarrows \mathbf{Cont}\Gamma\mathbf{CoarseHomologyTheories}_{\mathbf{C}} : \mathit{inclusion}\ .$$

Let us first construct the functor $(-)_{cont}$. For simplicity of the presentation we will only describe it on objects. The honest construction of the functor is similar and just involves more complicated diagrams.

Denote by $\Gamma\mathbf{BornCoarse}^{mb}$ the full subcategory of $\Gamma\mathbf{BornCoarse}$ spanned by the Γ-bornological coarse spaces which carry the minimal bornology. Given a functor

$$E\colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

with target a stable cocomplete ∞-category, we define the functor E_{cont} by left Kan extension:

$$\begin{array}{ccc} \Gamma\mathbf{BornCoarse}^{mb} & \xrightarrow{E} & \mathbf{C} \\ \downarrow & \nearrow_{E_{cont}} & \\ \Gamma\mathbf{BornCoarse} & & \end{array}$$

Since the image of every morphism originating in a Γ-bornological coarse space with minimal bornology is locally finite, the point-wise formula for the left Kan-extension implies that the canonical map

$$\operatorname*{colim}_{F\in\mathcal{F}(X)} E(F_X) \xrightarrow{\sim} E_{cont}(X) \tag{5.9}$$

is an equivalence.

Lemma 5.26. *If E is an equivariant coarse homology theory, then E_{cont} is a continuous equivariant coarse homology theory.*

Proof. We start with showing that E_{cont} is coarsely invariant. Let X be a Γ-bornological space. Since the subsets $\{0,1\}\times F$ of $\{0,1\}\times X$ for all locally finite invariant subsets F of X are cofinal in all locally finite invariant subsets of $\{0,1\}_{max,max}\otimes X$ we get the second equivalence in the chain

$$\begin{aligned} E_{cont}(\{0,1\}_{max,max}\otimes X) &\simeq \operatorname*{colim}_{F'\in\mathcal{F}(\{0,1\}\otimes X)} E(F'_{\{0,1\}_{max,max}\otimes X}) \\ &\simeq \operatorname*{colim}_{F\in\mathcal{F}(X)} E(\{0,1\}_{max,max}\otimes F_X) \\ &\simeq \operatorname*{colim}_{F\in\mathcal{F}(X)} E(F_X) \\ &\simeq E_{cont}(X)\ . \end{aligned}$$

See Example 2.10 for the notation F_X (the induced structures on the subset F). The third equivalence in the above chain of equivalences follows from the coarse invariance of E.

Next we show that E_{cont} satisfies excision. If $(Z, \mathcal{Y})$ is an invariant complementary pair on the Γ-bornological coarse space X, then $(F \cap Z, F \cap \mathcal{Y})$ is an invariant complementary pair on F_X. Hence

$$\begin{array}{ccc} E_{cont}(Z \cap \mathcal{Y}) & \longrightarrow & E_{cont}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ E_{cont}(Z) & \longrightarrow & E_{cont}(X) \end{array}$$

is the colimit of the push-out diagrams over F in $\mathcal{F}(X)$:

$$\begin{array}{ccc} E(F \cap Z \cap \mathcal{Y}) & \longrightarrow & E(F \cap \mathcal{Y}) \\ \downarrow & & \downarrow \\ E(F \cap Z) & \longrightarrow & E(F_X) \end{array}$$

(all subsets of F are equipped with the bornological coarse structures induced from F_X) and hence itself a push-out diagram.

We now show that E_{cont} vanishes on flasques. Assume that X is a flasque Γ-bornological coarse space and that flasqueness is implemented by the morphism $f\colon X \to X$. If F is an invariant, locally finite subset of X, then $\widetilde{F} := \bigcup_{n \in \mathbb{N}} f^n(F)$ is again an invariant, locally finite subset of X. Furthermore, $\widetilde{F}_X$ is flasque with flasqueness implemented by the restriction $f|_{\widetilde{F}}$. The inclusion $F \to \widetilde{F}$ belongs to the structure maps for the colimit over $\mathcal{F}(X)$. Since $E(\widetilde{F}) \simeq 0$ this finally implies that $E_{cont}(X) \simeq 0$.

Finally, we have u-continuity by

$$\begin{aligned} \operatorname*{colim}_{U \in \mathcal{C}} E_{cont}(X_U) &\simeq \operatorname*{colim}_{U \in \mathcal{C}} \operatorname*{colim}_{F \in \mathcal{F}(X)} E(F_{X_U}) \\ &\simeq \operatorname*{colim}_{U \in \mathcal{C}} \operatorname*{colim}_{F \in \mathcal{F}(X)} E(F_{(F \times F) \cap U}) \\ &\simeq \operatorname*{colim}_{F \in \mathcal{F}(X)} \operatorname*{colim}_{U \in \mathcal{C}} E(F_{(F \times F) \cap U}) \\ &\simeq \operatorname*{colim}_{F \in \mathcal{F}(X)} E(F_X) \\ &\simeq E_{cont}(X) \ . \end{aligned}$$

This finishes the proof that E_{cont} is an equivariant coarse homology theory.

We now argue that E_{cont} is continuous. Let X be a Γ-bornological coarse space and $\mathcal{Y} = (Y_i)_{i \in I}$ be a trapping exhaustion of X. For every invariant, locally finite subset F of X exists an index i in I such that $F \subseteq Y_i$. Furthermore, for every i in I we have an inclusion $\mathcal{F}(Y_i) \subseteq \mathcal{F}(X)$. This implies that

$$E_{cont}(\mathcal{Y}) \simeq \operatorname*{colim}_{i \in I} \operatorname*{colim}_{F \in \mathcal{F}(Y_i)} E(F_{Y_i}) \simeq \operatorname*{colim}_{F \in \mathcal{F}(X)} E(F_X) \simeq E_{cont}(X) \ .$$

This finishes the proof of Lemma 5.26. □

Proposition 5.27. *There exists an adjunction*

$$(-)_{cont}\colon \Gamma\mathbf{CoarseHomologyTheories_C} \leftrightarrows \mathbf{Cont}\Gamma\mathbf{CoarseHomologyTheories_C} : inclusion\ .$$

Proof. The collection of maps $F \to X$ for all locally finite invariant subsets F of X induces a transformation of functors

$$\eta\colon (-)_{cont} \to \mathrm{id}$$

on $\Gamma\mathbf{CoarseHomologyTheories_C}$. By Remark 5.16, if E is continuous, then the transformation $\eta_E\colon E_{cont} \to E$ is an equivalence. Moreover the transformations $\eta_{E_{cont}}$ and $(\eta_E)_{cont}$ (i.e., the functor $(-)_{cont}$ applied to η_E) are equivalences. By [Lur09, Prop. 5.2.7.4] we get the desired adjunction. □

Let the cocomplete stable ∞-category $\mathbf{C}$ admit all small products. Let E be a $\mathbf{C}$-valued equivariant coarse homology theory.

Lemma 5.28. *If E is strongly additive, then so is E_{cont}.*

Proof. Let $(X_i)_{i\in I}$ be a family of Γ-bornological coarse spaces and set $X := \bigsqcup^{\mathrm{free}}_{i\in I} X_i$ (Example 2.16). For a subset F of X and i in I we write $F_i := F \cap X_i$. Then F is locally finite in X if and only if F_i is locally finite in X_i for every i in I. Furthermore, we have an isomorphism of Γ-bornological coarse spaces $F_X \cong \bigsqcup^{\mathrm{free}}_{i\in I} F_{i,X_i}$. This implies

$$\operatorname*{colim}_{F\in\mathcal{F}(X)} \prod_{i\in I} E(F_{i,X_i}) \simeq \prod_{i\in I} \operatorname*{colim}_{F_i\in\mathcal{F}(X_i)} E(F_{i,X_i})\ .$$

We must show that the right vertical map in the diagram

$$\begin{array}{ccc}
\operatorname{colim}_{F\in\mathcal{F}(X)} E(F_X) & \xrightarrow{\simeq} & E_{cont}(X) \\
\downarrow{\simeq} & & \downarrow \\
\operatorname{colim}_{F\in\mathcal{F}(X)} \prod_{i\in I} E(F_{i,X_i}) & & \\
\downarrow{\simeq} & & \\
\prod_{i\in I} \operatorname{colim}_{F_i\in\mathcal{F}(X_i)} E(F_{i,X_i}) & \xrightarrow{\simeq} & \prod_{i\in I} E_{cont}(X_i)
\end{array}$$

is an equivalence. Indeed, the horizontal maps are equivalences by the definition of E_{cont} and the upper vertical map is an equivalence since E is strongly additive. □

6. Change of groups

In this section we describe various change of groups constructions. They induce adjunctions between the corresponding categories of equivariant motivic coarse spectra which are very similar to the base change functors in motivic homotopy theory.

Assume that we are given an equivariant coarse homology theory defined for every group (as it is the case for all our examples). The compatibility of the equivariant coarse homology theory with the change of groups functors is expressed by natural transformations. These transformations are additional data and will be discussed for every example of equivariant coarse homology theory separately.

What we describe here is the beginning of a story which should finally capture all group-change functors in a sort of spectral Mackey functor formalism (see, e.g., Barwick [Bar17] and Barwick–Glasman–Shah [BGS15]). The obvious task here is to capture the relations between these functors on the motivic level (like iterated restrictions or inductions and the Mackey relation, Lemma 6.8) together with all their higher coherences in a proper way.

All change of groups transformations are associated to a homomorphism of groups

$$\iota\colon H \to \Gamma \ .$$

6.1. Restriction

Every Γ-bornological coarse space gives rise to an H-bornological coarse space, where the action of H is induced from the action of Γ via ι. In this way we get a restriction functor

$$\mathrm{Res}^{\Gamma}_{H}\colon \Gamma\mathbf{BornCoarse} \to H\mathbf{BornCoarse} \ .$$

If the homomorphism ι is not clear from the context, then we add it to the notation and write $\mathrm{Res}^{\Gamma}_{H}(\iota)$.

The functor $\mathrm{Res}^{\Gamma}_{H}$ induces a pull-back functor $\mathrm{Res}^{\Gamma}_{H,pre}$ for presheaves, which preserves all limits and colimits. Since $\mathbf{PSh}(\Gamma\mathbf{BornCoarse})$ is a presentable ∞-category, by Lurie [Lur09, Cor. 5.5.2.9] the functor $\mathrm{Res}^{\Gamma}_{H,pre}$ is the right-adjoint of an adjunction

$$\mathrm{Res}^{\Gamma,pre}_{H}\colon \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \mathbf{PSh}(H\mathbf{BornCoarse}) : \mathrm{Res}^{\Gamma}_{H,pre} \ .$$

The functor $\mathrm{Res}^{\Gamma}_{H}$ sends equivariant complementary pairs on a Γ-bornological coarse space X to equivariant complementary pairs on the H-bornological coarse space $\mathrm{Res}^{\Gamma}_{H}(X)$. Consequently, the restriction functor $\mathrm{Res}^{\Gamma}_{H,pre}$ preserves sheaves. In the following, we decorate the Yoneda functors by the relevant group. Using $\mathrm{Res}^{\Gamma,pre}_{H} \circ \mathrm{yo}_{\Gamma} \simeq \mathrm{yo}_{H} \circ \mathrm{Res}^{\Gamma}_{H}$, we see that $\mathrm{Res}^{\Gamma,pre}_{H}$ sends the generators of the localization listed in Definition 4.6 for the group Γ to corresponding generators for H. We conclude that $\mathrm{Res}^{\Gamma}_{H,pre}$ preserves coarsely invariant sheaves, sheaves which vanish on flasque spaces, and u-continuous sheaves. Hence we get an adjunction

$$\mathrm{Res}^{\Gamma,*}_{H}\colon \Gamma\mathbf{Spc}\mathcal{X} \leftrightarrows H\mathbf{Spc}\mathcal{X} : \mathrm{Res}^{\Gamma}_{H,*} \ .$$

Here $\mathrm{Res}^{\Gamma}_{H,*}$ is given by the restriction of $\mathrm{Res}^{\Gamma}_{H,pre}$ to coarse motivic spaces, and its left adjoint satisfies

$$\mathrm{Res}^{\Gamma,*}_{H} \simeq \mathcal{L}_{H} \circ \mathrm{Res}^{\Gamma,pre}_{H} \ ,$$

where $\mathcal{L}_{H}$ is the localization as in (4.6) (we have added a subscript H in order to indicate the relevant group).

Hence by passing to the stabilizations we get an adjunction

$$\mathrm{Res}^{\Gamma,Mot}_H : \Gamma\mathbf{Sp}\mathcal{X} \leftrightarrows H\mathbf{Sp}\mathcal{X} : \mathrm{Res}^{\Gamma}_{H,Mot} \ ,$$

where $\mathrm{Res}^{\Gamma}_{H,Mot}$ is defined by the extension of the functor $\mathrm{Res}^{\Gamma}_{H,pre}$ to stable objects. The obvious equivalence

$$\mathrm{Res}^{\Gamma,pre}_H \circ \mathrm{yo}_\Gamma \simeq \mathrm{yo}_H \circ \mathrm{Res}^{\Gamma}_H$$

implies the equivalence

$$\mathrm{Res}^{\Gamma,Mot}_H \circ \mathrm{Yo}^s_\Gamma \simeq \mathrm{Yo}^s_H \circ \mathrm{Res}^{\Gamma}_H \ .$$

where we have decorated the Yoneda functors by the relevant group.

6.2. Completion

We consider a Γ-bornological coarse space X. Let $\mathcal{C}$ and $\mathcal{B}$ denote the coarse structure and the bornology of X. We define a new compatible bornology $\mathcal{B}_H$ on X generated by the $\iota(H)$-completions $\iota(H)B$ of the bounded subsets B of X. We observe that $\mathcal{C}$ and $\mathcal{B}_H$ are compatible and that $\mathcal{B}_H$ is $N_\Gamma(\iota(H))$-invariant (as a subset of $\mathcal{P}(X)$), where $N_\Gamma(\iota(H))$ denotes the normalizer of the subgroup $\iota(H)$ in Γ.

Definition 6.1. The H*-completion* of X is the $N_\Gamma(\iota(H))$-bornological coarse space defined by $B_H(X) := (X, \mathcal{C}, \mathcal{B}_H)$. ♦

In this way we define a functor

$$B_H : \Gamma\mathbf{BornCoarse} \to N_\Gamma(\iota(H))\mathbf{BornCoarse} \ .$$

The pull-back along B_H induces an adjunction

$$B^{pre}_H : \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \mathbf{PSh}(N_\Gamma(\iota(H))\mathbf{BornCoarse}) : B_{H,pre} \ .$$

It is easy to see that $B_{H,pre}$ preserves sheaves, u-continuity and coarse invariance, since its left-adjoint adjoint B^{pre}_H preserves the corresponding generating morphisms. See Section 6.1 for a similar argument.

If X is a flasque Γ-bornological coarse space with flasqueness implemented by $f : X \to X$, then $B_H(X)$ is flasque with flasqueness implemented by the same map. Here it is important to define flasqueness with Condition 3 in Definition 3.8 and not the weaker one discussed in Remark 3.9. Consequently, $B_{H,pre}$ preserves presheaves which vanish on flasques.

Similarly as in the case of the restriction, we get an adjunction

$$B^{Mot}_H : \Gamma\mathbf{Sp}\mathcal{X} \leftrightarrows N_\Gamma(\iota(H))\mathbf{Sp}\mathcal{X} : B_{H,Mot}$$

and an equivalence

$$B^{Mot}_H \circ \mathrm{Yo}^s_\Gamma \simeq \mathrm{Yo}^s_{N_\Gamma(\iota(H))} \circ B_H \ .$$

The identity of the underlying set induces a morphism $b_H\colon B_H(X) \to \mathrm{Res}^{\Gamma}_{N_\Gamma(\iota(H))}(X)$. The transformation b_H induces a natural transformation of functors

$$b_H\colon B_H \to \mathrm{Res}^{\Gamma}_{N_\Gamma(\iota(H))}\colon \Gamma\mathbf{BornCoarse} \to N_\Gamma(\iota(H))\mathbf{BornCoarse} \ .$$

By functoriality, we get a transformation

$$b_H\colon E \circ B_H \to E \circ \mathrm{Res}^{\Gamma}_{N_\Gamma(\iota(H))}$$

for any equivariant coarse homology theory E.

Remark 6.2. If H is a finite group, then $B_H \cong \mathrm{Res}^{\Gamma}_{N_\Gamma(\iota(H))}$. ♦

6.3. Quotients

We consider a Γ-bornological coarse space X. We form the quotient set $H\backslash X$ which carries an action of the group

$$W_\Gamma(H) := N_\Gamma(\iota(H))/\iota(H) \ ,$$

where $N_\Gamma(\iota(H))$ denotes the normalizer of the subgroup $\iota(H)$ in Γ.

Let $\pi\colon X \to H\backslash X$ denote the projection. We equip $H\backslash X$ with the maximal bornology such that the projection $\pi\colon B_H(X) \to H\backslash X$ is proper. Furthermore, we equip $H\backslash X$ with the minimal coarse structure such that $\pi\colon X \to H\backslash X$ is controlled. In this way we define a functor

$$Q_H\colon \Gamma\mathbf{BornCoarse} \to W_\Gamma(H)\mathbf{BornCoarse} \ .$$

The projection maps define a natural transformation $\pi_H\colon B_H \to \mathrm{Res}^{W_\Gamma(H)}_{N_\Gamma(\iota(H))}(Q_H)$ of $N_\Gamma(\iota(H))\mathbf{BornCoarse}$-valued functors.

For X in $\Gamma\mathbf{BornCoarse}$ one can interpret $\mathrm{Res}^{W_\Gamma(H)}_{N_\Gamma(\iota(H))}Q_H(X)$ as a coequalizer. For a bornological coarse space Z let $Z_{max-\mathcal{B}}$ denote the bornological coarse space obtained by replacing its bornology by the maximal bornology. The two maps $H \times X \to X$ in the lemma below are given by the projection $(h,x) \mapsto x$ and the action $(h,x) \mapsto hx$. The diagram and the colimit are considered in $N_\Gamma(\iota(H))\mathbf{BornCoarse}$, where the action of σ in $N_\Gamma(\iota(H))$ on $H \times X$ is given by $\sigma(h,x) := (\sigma h \sigma^{-1}, \sigma x)$.

Lemma 6.3. *We have an isomorphism*

$$\mathrm{Res}^{W_\Gamma(H)}_{N_\Gamma(\iota(H))}Q_H(X) \cong \operatorname{colim}\big((H_{min,max} \otimes X)_{max-\mathcal{B}} \rightrightarrows B_H(X)\big) \ .$$

Proof. One checks that the two morphisms in the coequalizer diagram are controlled. Since we replaced the bornology on the domain by the maximal one they are obviously proper. The coequalizer diagram is colim-admissible, see Example 2.25. One checks that the description of the structures on $Q_H(X)$ given above coincides with the explicit description of the structures of the colimit given in the proof of Proposition 2.21. □

The functor Q_H induces a pull-back in presheaves $Q_{H,pre}$. This functor preserves all limits and colimits. Again by Lurie [Lur09, Cor. 5.5.2.9] it fits into an adjunction

$$Q_H^{pre}\colon \mathbf{PSh}(\Gamma\mathbf{BornCoarse}) \leftrightarrows \mathbf{PSh}(W_\Gamma(H)\mathbf{BornCoarse}) : Q_{H,pre} \ .$$

Note that the functor Q_H induces a bijection between equivariant complementary pairs on a Γ-bornological coarse space X and on the $W_\Gamma(H)$-bornological coarse space $Q_H(X)$. Consequently, $Q_{H,pre}$ preserves sheaves.

It is furthermore clear that $Q_{H,pre}$ preserves coarsely invariant sheaves and u-continuous sheaves. If the Γ-bornological coarse space X is flasque with flasqueness implemented by $f\colon X \to X$, then the induced map $\bar{f}\colon Q_H(X) \to Q_H(X)$ implements flasqueness of $Q_H(X)$. It follows that $Q_{H,pre}$ preserves coarse motivic spaces.

Hence $Q_{H,pre}$ restricts to equivariant coarse motivic spaces. Similar as before we get an adjunction

$$Q_H^{Mot}\colon \Gamma\mathbf{Sp}\mathcal{X} \leftrightarrows W_\Gamma(H)\mathbf{Sp}\mathcal{X} : Q_{H,Mot} \ .$$

We have the relation

$$\mathrm{Yo}^s_{W_\Gamma(H)} \circ Q_H \simeq Q_H^{Mot} \circ \mathrm{Yo}^s_\Gamma \ .$$

6.4. Products

We consider two groups Γ and Γ'. For a Γ-bornological coarse space X and a Γ'-bornological coarse space X' we can form the product $X \otimes X'$ which is a $(\Gamma \times \Gamma')$ bornological coarse space. We fix the Γ-bornological coarse space X and consider the functor

$$P_X := X \otimes -\colon \Gamma'\mathbf{BornCoarse} \to (\Gamma \times \Gamma')\mathbf{BornCoarse} \ .$$

As in the preceding cases one can check that the restriction

$$P_{X,pre}\colon \mathbf{PSh}((\Gamma \times \Gamma')\mathbf{BornCoarse}) \to \mathbf{PSh}(\Gamma'\mathbf{BornCoarse})$$

along P_X preserves coarse motivic spaces and induces an adjunction

$$P_X^{Mot}\colon \Gamma'\mathbf{Sp}\mathcal{X} \leftrightarrows (\Gamma \times \Gamma')\mathbf{Sp}\mathcal{X} : P_{X,Mot}$$

such that

$$P_X^{Mot} \circ \mathrm{Yo}^s_{\Gamma'} \simeq \mathrm{Yo}^s_{(\Gamma\times\Gamma')} \circ P_X \ .$$

6.5. Induction

We now come back to our original situation and consider the homomorphism $\iota\colon H \to \Gamma$ of groups. We define a $(\Gamma \times H)$-bornological coarse space $\hat{\Gamma}$ as follows:

1. The underlying bornological coarse space of $\hat{\Gamma}$ is $\Gamma_{min,min}$.

2. The group $(\Gamma \times H)$ acts on the set $\hat{\Gamma}$ by
$$(\gamma, h)\gamma' := \gamma\gamma'\iota(h)^{-1} \ .$$

We define the functor
$$\hat{P}_\Gamma := \mathrm{Res}^{\Gamma\times H\times H}_{\Gamma\times H} \circ P_{\hat{\Gamma}} \colon H\mathbf{BornCoarse} \to (\Gamma \times H)\mathbf{BornCoarse} \ ,$$
where the restriction is along the homomorphism
$$\mathrm{id}_\Gamma \times \mathrm{diag}_H \colon \Gamma \times H \to \Gamma \times H \times H \ .$$

This functor extends to motives
$$\hat{P}^{Mot}_\Gamma := \mathrm{Res}^{\Gamma\times H\times H, Mot}_{\Gamma\times H} \circ P^{Mot}_{\hat{\Gamma}} \ .$$
By construction we have an adjunction
$$\hat{P}^{Mot}_\Gamma \colon H\mathbf{Sp}\mathcal{X} \leftrightarrows (\Gamma \times H)\mathbf{Sp}\mathcal{X} : \hat{P}_{\Gamma,Mot}$$
and the relation
$$\mathrm{Yo}^s_{\Gamma\times H} \circ \hat{P}_\Gamma \simeq \hat{P}^{Mot}_\Gamma \circ \mathrm{Yo}^s_H \ .$$

We consider the canonical embedding $\kappa \colon H \to \Gamma \times H$ into the second factor. Note that then we have $\Gamma \times H = N_{\Gamma\times H}(H)$ and hence $W_{\Gamma\times H}(H) \cong \Gamma$. We thus have the quotient functor
$$Q_H(\kappa) \colon (\Gamma \times H)\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse}$$
(it is useful to add the embedding κ as an argument since there is also the other obvious homomorphism $(\iota, \mathrm{id}) \colon H \to \Gamma \times H$). We define the induction functor as the composition
$$\mathrm{Ind}^\Gamma_H \colon H\mathbf{BornCoarse} \to \Gamma\mathbf{BornCoarse} \ , \quad \mathrm{Ind}^\Gamma_H := Q_H(\kappa) \circ \hat{P}_\Gamma \ .$$

The induction functor extends to motives
$$\mathrm{Ind}^{\Gamma,Mot}_H := Q^{Mot}_H(\kappa) \circ \hat{P}^{Mot}_\Gamma : H\mathbf{Sp}\mathcal{X} \to \Gamma\mathbf{Sp}\mathcal{X}$$
such that
$$\mathrm{Ind}^{\Gamma,Mot}_H \circ \mathrm{Yo}^s_H \simeq \mathrm{Yo}^s_\Gamma \circ \mathrm{Ind}^\Gamma_H \ .$$

Remark 6.4. The underlying Γ-set of $\mathrm{Ind}^\Gamma_H(X)$ is $\Gamma \times_H X$ with the left-action of Γ on the left factor. Here $\Gamma \times_H X$ stands for the quotient set $H\backslash(\Gamma \times X)$ with respect to the action given by
$$h(\gamma, x) = (\gamma\iota(h^{-1}), hx) \ .$$
The bornology on $\Gamma \times_H X$ is generated by the images of the subsets $\{\gamma\} \times B$ for all bounded subsets of X, and the coarse structure is generated by the images of $\mathrm{diag}_\Gamma \times U$ for all entourages U of X. ♦

Let $K := \ker(\iota\colon H \to \Gamma)$.

Lemma 6.5. *We have* $\mathrm{Ind}_H^\Gamma \cong \mathrm{Ind}_H^\Gamma \circ B_K$.

Proof. Straightforward. $\square$

Remark 6.6. Using Lemma 6.3 one can easily check that we have an isomorphism

$$\mathrm{Ind}_H^\Gamma(X) \cong \mathrm{colim}\,((H_{min,max} \otimes \Gamma_{min,min} \otimes X)_{max-\mathcal{B}} \rightrightarrows B_H(\Gamma_{min,min} \otimes X))$$

in $\Gamma\mathbf{BornCoarse}$. The two arrows are given by $(h,\gamma,x) \mapsto (\gamma,x)$ and $(h,\gamma,x) \mapsto (\gamma h^{-1}, hx)$. The group Γ acts on the factor $\Gamma_{min,min}$ by left multiplication. We used the more complicated description of the induction as a composition of various previously defined functors in order to deduce that induction descends to motives. ♦

Proposition 6.7. *If K is finite and the image of ι has finite index in Γ, then we have an adjunction*

$$\mathrm{Ind}_H^\Gamma\colon H\mathbf{BornCoarse} \leftrightarrows \Gamma\mathbf{BornCoarse} : \mathrm{Res}_H^\Gamma\ .$$

Proof. As we observed earlier, the underlying set of $\mathrm{Ind}_H^\Gamma(X)$ is given by $\Gamma \times_H X$. Recall that the functions [2]

$$X \to \Gamma \times_H X, \quad x \mapsto [e,x]$$

and

$$\Gamma \times_H \mathrm{Res}_H^\Gamma(Y) \to Y, \quad [\gamma, y] \mapsto \gamma y$$

define the unit and counit, respectively, of an adjunction $H\mathbf{Set} \rightleftarrows \Gamma\mathbf{Set}$. We must check that these functions define morphisms of equivariant bornological coarse spaces.
Since K is finite and normal in H, we have $B_K X \cong X$ for every H-bornological coarse space X. We claim that the function

$$X \to \Gamma \times X\ , \quad x \mapsto (e,x)$$

defines a natural morphism $B_K X \to B_H(\Gamma_{min,min} \otimes X)$. This function is obviously controlled. To see that it is proper, we note that the generating bounded subsets of $B_H(\Gamma_{min,min} \otimes X)$ are of the form $\iota(H)(\{\gamma\} \times B)$. The intersection of such a subset with $\{e\} \times X$ is equal to $\{e\} \times KB$, hence bounded because K is finite. Then the unit is given by the composition

$$X \cong B_K(X) \to \mathrm{Res}_H^\Gamma(B_H(\Gamma_{min,min} \otimes X)) \xrightarrow{\pi} \mathrm{Res}_H^\Gamma(Q_H(\Gamma_{min,min} \otimes X)) = \mathrm{Res}_H^\Gamma(\mathrm{Ind}_H^\Gamma(X)).$$

Consider now the composition

$$F\colon \Gamma \times \mathrm{Res}_H^\Gamma Y \to \Gamma \times_H \mathrm{Res}_H^\Gamma Y \to Y$$

of the projection map with the function which will provide the desired counit. It suffices to show that this composition is a morphism of bornological coarse spaces $B_H(\Gamma_{min,min} \otimes$

[2] We use the word "function" in order to denote maps between underlying sets

$\mathrm{Res}^{\Gamma}_{H}(Y)) \to Y$. If U is an entourage of Y, then $F(\mathrm{diag}_{\Gamma} \times U) = \bigcup_{\gamma\in\Gamma}(\gamma \times \gamma)(f \times f)(U)$ is an entourage of Y, see Remark 2.3. If B is a bounded subset of Y, then $F^{-1}(B) = \bigcup_{\gamma\in\Gamma}\{\gamma^{-1}\} \times \gamma B$ is bounded since $\bigcup_{h\in H}\{\gamma\iota(h)^{-1}\} \times \iota(h)B$ is bounded for every γ in Γ and the image of ι has finite index in Γ. This shows that F, and hence the counit, is a morphism. □

We now consider homomorphisms $H \to \Gamma$ and $H' \to \Gamma$, and set $K := \ker(H \to \Gamma)$ and $K' := \ker(H' \to \Gamma)$. Note that $W_H(K) \cong H/K =: \bar{H}$ and $W_{H'}(K') \cong H'/K' =: \bar{H}'$.

Lemma 6.8. *Assume that $H'\backslash\Gamma/H$ is finite.*

For an H-bornological coarse space X we have the relation

$$\mathrm{Res}^{\Gamma}_{H'} \circ \mathrm{Ind}^{\Gamma}_{H}(X) \;\cong\; \coprod_{[\gamma]\in\bar{H}'\backslash\Gamma/\bar{H}} \mathrm{Res}^{\bar{H}'}_{H'} \circ \mathrm{Ind}^{\bar{H}'}_{\bar{H}\cap\gamma^{-1}\bar{H}'\gamma} \circ \mathrm{Res}^{\bar{H}}_{\bar{H}\cap\gamma^{-1}\bar{H}'\gamma}(c_\gamma) \circ \mathrm{Ind}^{\bar{H}}_{H}(X) \ .$$

where $c_\gamma \colon \bar{H} \cap \gamma^{-1}\bar{H}'\gamma \to \bar{H}$ is given by $\bar{h} \mapsto \gamma^{-1}\bar{h}\gamma$.

Proof. One just makes all definitions explicit. □

The induction on the motivic level is given by

$$\mathrm{Ind}^{\Gamma,Mot}_{H} := Q^{Mot}_{H} \circ \hat{P}^{Mot}_{\Gamma} \ .$$

This functor fits into the adjunction

$$\mathrm{Ind}^{\Gamma,Mot}_{H} \colon H\mathbf{Sp}\mathcal{X} \leftrightarrows \Gamma\mathbf{Sp}\mathcal{X} : \mathrm{Ind}^{\Gamma}_{H,Mot}$$

and is compatible with the Yoneda functor:

$$\mathrm{Ind}^{\Gamma,Mot}_{H} \circ \mathrm{Yo}^{s}_{H} \simeq \mathrm{Yo}^{s}_{\Gamma} \circ \mathrm{Ind}^{\Gamma}_{H} \ .$$

Since the Yoneda functor preserves finite coproducts (since it is excisive), the Lemma 6.8 implies:

Corollary 6.9. *If $H'\backslash\Gamma/H$ is finite, then*

$$\mathrm{Res}^{\Gamma,Mot}_{H'} \circ \mathrm{Ind}^{\Gamma,Mot}_{H} \simeq \bigoplus_{[\gamma]\in\bar{H}'\backslash G/\bar{H}} \mathrm{Res}^{\bar{H}',Mot}_{H'} \circ \mathrm{Ind}^{\bar{H}',Mot}_{\bar{H}\cap\gamma^{-1}\bar{H}'\gamma} \circ \mathrm{Res}^{\bar{H},Mot}_{\bar{H}\cap\gamma^{-1}\bar{H}'\gamma}(c_\gamma) \circ Q^{Mot}_{K} \ .$$

Part II. Examples

7. Equivariant coarse ordinary homology

7.1. Construction

In this section we introduce equivariant coarse ordinary homology theory. Its construction is completely analogous to the non-equivariant case [BE16, Sec. 6.3].

Let X be a Γ-bornological coarse space and let n be a natural number. An *n-chain on X* is a function $c\colon X^{n+1} \to \mathbb{Z}$. Its *support* is defined by $\mathrm{supp}(c) := \{x \in X^{n+1} \mid c(x) \neq 0\}$. We typically think of n-chains as infinite linear combinations of points in X^{n+1}.

Let U be a coarse entourage of X and let B be a bounded subset. A point $(x_0, \dots, x_n)$ in X^{n+1} is *U-controlled* if $(x_i, x_j) \in U$ for every $0 \leq i, j \leq n$. The point $(x_0, \dots, x_n)$ in X^{n+1} *meets* B if there exists $0 \leq i \leq n$ such that x_i lies in B.

An n-chain c is *U-controlled* if every point of its support is U-controlled. The n-chain c is *controlled* if it is U-controlled for some coarse entourage U of X. Furthermore, c is *locally finite* if for every bounded subset B of X the set of points in $\mathrm{supp}(c)$ which meet B is finite. We let $C\mathcal{X}_n(X)$ denote the abelian group of controlled locally finite n-chains.

The boundary operator $\partial\colon C\mathcal{X}_n(X) \to C\mathcal{X}_{n-1}(X)$ (for all $n \geq 1$) is defined to be $\partial := \sum_{i=0}^{n} (-1)^i \partial_i$, where ∂_i is the linear extension of the operator $X^{n+1} \to X^n$ omitting the i'th entry. One checks that ∂ is well-defined and a differential of a chain complex. By $C\mathcal{X}(X)$ we denote the chain complex of locally finite and controlled chains on X.

Definition 7.1. For every natural number n we let $C\mathcal{X}_n^\Gamma(X)$ denote the subgroup of $C\mathcal{X}_n(X)$ of locally finite and controlled n-chains which are in addition Γ-invariant. ♦

For every natural number n the boundary operator $\partial\colon C\mathcal{X}_{n+1}(X) \to C\mathcal{X}_n(X)$ restricts to a boundary operator $\partial\colon C\mathcal{X}_{n+1}^\Gamma(X) \to C\mathcal{X}_n^\Gamma(X)$ between the subgroups of Γ-invariants. Hence we have defined a subcomplex $C\mathcal{X}^\Gamma(X)$ of $C\mathcal{X}(X)$. If $f\colon X \to X'$ is a morphism between Γ-bornological coarse spaces, then the induced map $C\mathcal{X}(f)\colon C\mathcal{X}(X) \to C\mathcal{X}(X')$ of chain complexes preserves the subcomplexes of Γ-invariants. Therefore, we obtain a functor

$$C\mathcal{X}^\Gamma\colon \Gamma\mathbf{BornCoarse} \to \mathbf{Ch}\ ,$$

where $\mathbf{Ch}$ denotes the category of chain complexes.

In order to go from chain complexes to spectra we use the Eilenberg–MacLane correspondence

$$\mathcal{EM}\colon \mathbf{Ch} \to \mathbf{Sp}\ . \tag{7.1}$$

One way to define this functor is as the composition

$$\mathcal{EM}\colon \mathbf{Ch} \to \mathbf{Ch}[W^{-1}] \to \mathbf{Sp}\ ,$$

where the first functor is the localization of the category of chain complexes at the quasi-isomorphisms, and the second functor is the mapping spectrum functor $\mathtt{map}(\mathbb{Z}[0], \dots)$ of the stable ∞-category $\mathbf{Ch}[W^{-1}]$, where $\mathbb{Z}[0]$ is the chain complex with $\mathbb{Z}$ placed in degree zero.

Definition 7.2. We define the functor $H\mathcal{X}^\Gamma\colon \mathbf{BornCoarse} \to \mathbf{Sp}$ by

$$H\mathcal{X}^\Gamma := \mathcal{EM} \circ C\mathcal{X}^\Gamma\ .$$

♦

Theorem 7.3. *$H\mathcal{X}^\Gamma$ is an equivariant coarse homology theory.*

Proof. We observe that the arguments given in the proof of the [BE16, Thm. 6.15] extend word-by-word to the equivariant case. □

7.2. Calculations for spaces of the form $\Gamma_{can,min} \otimes S_{min,max}$

In this section we will do some computations of equivariant coarse ordinary homology groups. In particular, we will relate it to ordinary group homology.

Example 7.4. If the Γ-bornological coarse space X has the trivial Γ-action, then we have an isomorphism $H\mathcal{X}^\Gamma(X) \cong H\mathcal{X}(X)$. ♦

In order to provide more examples we consider the group homology functor

$$H(\Gamma, -) \colon \mathbf{Mod}(\mathbb{Z}[\Gamma]) \to \mathbf{Sp}$$

which can be defined as the composition

$$\mathbf{Mod}(\mathbb{Z}[\Gamma]) \to \mathbf{Ch}_{\mathbb{Z}[\Gamma]} \to \mathbf{Ch}_{\mathbb{Z}[\Gamma]}[W^{-1}] \xleftarrow{\simeq} \mathbf{Ch}^{\mathit{free}}_{\mathbb{Z}[\Gamma]}[W^{-1}] \xrightarrow{\mathbb{Z}\otimes_{\mathbb{Z}[\Gamma]}-} \mathbf{Ch}_{\mathbb{Z}}[W^{-1}] \xrightarrow{\mathcal{EM}} \mathbf{Sp} \ .$$

The first functor sends a $\mathbb{Z}[\Gamma]$-module to a chain complex of $\mathbb{Z}[\Gamma]$-modules concentrated in degree 0. The second functor is the localization at quasi-isomorphisms. The equivalence in the third step is induced by the inclusion of the full subcategory of chain complexes of free $\mathbb{Z}[\Gamma]$-modules. It is essentially surjective by the existence of free resolutions. Finally, the functor $\mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} -$ is well-defined since it preserves quasi-isomorphisms between chain complexes of free $\mathbb{Z}[\Gamma]$-modules.

Since the above definition involves the inverse of an equivalence it does not directly provide an explicit formula. But for calculations it is useful to choose an explicit model for $H(\Gamma, -)$. The standard choice is as follows. We consider the chain complex of $\mathbb{Z}[\Gamma]$-modules $C(\Gamma)$ given by

$$\cdots \to \mathbb{Z}[\Gamma^{n+1}] \to \mathbb{Z}[\Gamma^n] \to \cdots \to \mathbb{Z}[\Gamma] \ .$$

The differential $C(\Gamma)_{n+1} \to C(\Gamma)_n$ is defined as the linear extension of the map

$$(\gamma_0, \dots, \gamma_{n+1}) \to \sum_{i=0}^{n+1} (-1)^i (\gamma_0, \dots, \widehat{\gamma}_i, \dots, \gamma_{n+1}) \ ,$$

where $\widehat{\gamma}_i$ indicates that this component gets omitted. The group Γ acts diagonally on the products Γ^n and this induces the $\mathbb{Z}[\Gamma]$-module structure on $C(\Gamma)$. We now consider the functor

$$C(\Gamma, -) \colon \mathbf{Mod}(\mathbb{Z}[\Gamma]) \to \mathbf{Ch}_{\mathbb{Z}} \ , \quad V \mapsto \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} (C(\Gamma) \otimes_{\mathbb{Z}} V) \ ,$$

which sends a $\mathbb{Z}[\Gamma]$-module V to its standard complex $C(\Gamma, V)$. Here Γ acts diagonally on $C(\Gamma) \otimes_{\mathbb{Z}} V$. Then we have an equivalence of functors

$$H(\Gamma, -) \simeq \mathcal{EM} \circ C(\Gamma, -) \ .$$

If S is a Γ-set, then we form the Γ-bornological coarse space $S_{min,max}$ given by the Γ-set S with the maximal bornological and the minimal coarse structure. In this way we get a functor

$$(-)_{min,max} \colon \Gamma\mathbf{Set} \to \Gamma\mathbf{BornCoarse} \ .$$

We have furthermore a functor

$$\Gamma\mathbf{Set} \to \mathbf{Mod}(\mathbb{Z}[\Gamma]) \ , \quad S \mapsto \mathbb{Z}[S] \ .$$

We now have two functors $\Gamma\mathbf{Set} \to \mathbf{Sp}$ given by

$$S \mapsto H\mathcal{X}^{\Gamma}(\Gamma_{can,min} \otimes S_{min,max}) \quad \text{and} \quad S \mapsto H(\Gamma, \mathbb{Z}[S]) \ .$$

Proposition 7.5 (cf. [Eng18, Prop. 3.8])**.** *There is a natural equivalence*

$$H\mathcal{X}^{\Gamma}(\Gamma_{can,min} \otimes S_{min,max}) \simeq H(\Gamma, \mathbb{Z}[S]) \ .$$

Proof. We claim that there is a natural isomorphism between $C\mathcal{X}^{\Gamma}(\Gamma_{can,min} \otimes S_{min,max})$ and the standard complex $C(\Gamma, \mathbb{Z}[S])$. To do so, we identify $\mathbb{Z}[\Gamma^{n+1}] \otimes_{\mathbb{Z}} \mathbb{Z}[S] \cong \mathbb{Z}[\Gamma^{n+1} \times S]$, where $\Gamma^{n+1} \times S$ carries the diagonal Γ-action. Then we define the homomorphism

$$\phi_n \colon C_n(\Gamma, \mathbb{Z}[S]) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma^{n+1} \times S] \to C\mathcal{X}^{\Gamma}(\Gamma_{can,min} \otimes S_{min,max}) \tag{7.2}$$

as the linear extension of

$$1 \otimes (\gamma_0, \gamma_1, \dots, \gamma_n, s) \mapsto \sum_{\gamma \in \Gamma} ((\gamma\gamma_0, \gamma s), \dots, (\gamma\gamma_n, \gamma s)) \ . \tag{7.3}$$

Note that all summands are different points on $(\Gamma \times S)^{n+1}$ so that the infinite sum makes sense, and it is Γ-invariant by construction. Every point $((\gamma\gamma_0, \gamma s), \dots, (\gamma\gamma_n, \gamma s))$ is controlled by the entourage $\Gamma\{(\gamma_i, \gamma_j) \mid 0 \le i, j \le n\} \times \mathrm{diag}_S$ of the Γ-bornological coarse space $\Gamma_{can,min} \otimes S_{min,max}$. To show that this chain is also locally finite, it suffices to check that there are only finitely many points in the support of the chain (7.3) which meet bounded sets of the form $B \times S$, where B is some finite subset of Γ. This is clear since Γ acts freely on Γ^{n+1}. This finishes the argument for the assertion that (7.2) is well-defined.

It is straightforward to check that the collection $\{\phi_n\}_n$ is a chain map.

We now argue that the map (7.2) is an isomorphism. To this end we define an inverse

$$\psi \colon C\mathcal{X}_n(\Gamma_{can,min} \otimes S_{min,max}) \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma^{n+1} \times S] \cong C_n(\Gamma, \mathbb{Z}[S]) \ .$$

Let

$$c = \sum_{x \in (\Gamma \times S)^{n+1}} n_x x$$

be an invariant, controlled and locally finite n-chain on $\Gamma_{can,min} \otimes S_{min,max}$. We now define

$$\psi(c) := \sum_{(\gamma_1, \dots, \gamma_n, s) \in \Gamma^n \times S} n_{((1,s),(\gamma_1,s)\dots,(\gamma_n,s))} \otimes (1, \gamma_1, \dots, \gamma_n, s) \ .$$

Assume that c is U-controlled. Then only summands with $\{\gamma_1, \dots, \gamma_n\} \subseteq U[\{1\}]$ contribute to the sum. Since $U[\{1\}]$ is bounded and c is locally finite we see that the number of non-trivial summands is finite. This implies that $\psi(c)$ is well-defined.

It is straightforward to check that ϕ and ψ are inverse to each other: To see that $\psi\circ\phi = \mathrm{id}$, use that

$$1\otimes(\gamma_0,\gamma_1,\dots,\gamma_n,s) = 1\otimes(1,\gamma_0^{-1}\gamma_1,\dots,\gamma_0^{-1}\gamma_n,\gamma_0^{-1}s)\ .$$

The equality $\phi\circ\psi = \mathrm{id}$ follows from the Γ-invariance of an n-chain $c=\sum_{x\in(\Gamma\times S)^{n+1}} n_x x$ together with the observation that $n_{((\gamma_0,s_0),\dots,(\gamma_n,s_n))}=0$ unless $s_0=\dots=s_n$. The latter fact is due to S carrying the minimal coarse structure.

One easily checks that ϕ is natural for maps between Γ-sets. □

Remark 7.6. Assume that S is a transitive Γ-set. If we fix a point s in S and let Γ_s denote the stabilizer subgroup of s, then we have an isomorphism of $\mathbb{Z}[\Gamma]$-modules

$$\mathbb{Z}[S]\cong \mathrm{Ind}_{\Gamma_s}^{\Gamma}\mathbb{Z}\ .$$

The induction isomorphism in group homology now gives the chain of equivalences

$$H(\Gamma,\mathbb{Z}[S])\simeq H(\Gamma,\mathrm{Ind}_{\Gamma_s}^{\Gamma}\mathbb{Z})\simeq H(\Gamma_s,\mathbb{Z})\ .$$

Since this identification involves the choice of the base point s it is not reasonable to state any naturality (for morphisms of Γ-sets) of this equivalence. ♦

The following definition and proposition depend on definitions and results of the following sections. They are not needed later and can safely be skipped on first reading.

We can apply Proposition 7.5 in order to study the equivariant homology theory $H\mathbb{Z}^\Gamma$ on Γ-topological spaces induced by the equivariant coarse homology $H\mathcal{X}^\Gamma$ and the twist $\mathrm{Yo}^s(\Gamma_{can,min})$; see Definition 10.29 and Equation (10.17) for the notation.

Definition 7.7. We define the equivariant ordinary homology theory by

$$H\mathbb{Z}^\Gamma := H\mathcal{X}^\Gamma_{\mathrm{Yo}^s(\Gamma_{can,min})}\mathcal{O}^\infty_{\mathrm{hlg}} : \Gamma\mathbf{Top}\to\mathbf{Sp}\ .$$ ♦

By the results from Section 10.1 the associated equivariant homology theory is determined on Γ-CW complexes by the restriction of the functor $H\mathbb{Z}^\Gamma$ to transitive Γ-sets. The following result describes this functor explicitly.

We consider the following two functors $\mathbf{Orb}(\Gamma)\to\mathbf{Sp}$ given by

$$S\mapsto H\mathbb{Z}^\Gamma(S)\ ,\quad S\mapsto H(\Gamma,\mathbb{Z}[S])\ .$$

Proposition 7.8. *For every transitive Γ-set S we have a natural equivalence*

$$H\mathbb{Z}^\Gamma(S)\simeq \Sigma H(\Gamma,\mathbb{Z}[S])$$

of spectra.

Proof. By definition we have a natural equivalence

$$\begin{aligned} H\mathbb{Z}^\Gamma(S) &\simeq H\mathcal{X}^\Gamma(\mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(S))) \otimes \mathrm{Yo}^s(\Gamma_{can,min})) \\ &\simeq H\mathcal{X}^\Gamma(\mathcal{O}^\infty(S_{disc,max,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})) \ . \end{aligned}$$

By Proposition 9.35, $\mathcal{O}^\infty(S_{disc,max,max}) \simeq \Sigma\mathrm{Yo}^s(S_{min,max})$. This gives the equivalence

$$H\mathbb{Z}^\Gamma(S) \simeq \Sigma H\mathcal{X}^\Gamma(S_{min,max} \otimes \Gamma_{can,min}) \ .$$

By Proposition 7.5 we have a natural equivalence $H\mathcal{X}^\Gamma(S_{min,max} \otimes \Gamma_{can,min}) \simeq H(\Gamma, \mathbb{Z}[S])$ which finishes this proof. □

7.3. Additional properties

Recall Definition 5.15 of continuity of an equivariant coarse homology theory.

Lemma 7.9. *The equivariant coarse homology theory $H\mathcal{X}^\Gamma$ is continuous.*

Proof. Let X be a Γ-bornological coarse space and $\mathcal{Y} := (Y_i)_{i \in I}$ be a trapping exhaustion. If c is a chain in $C\mathcal{X}^\Gamma_n(X)$, then $\mathrm{supp}(c)$ is a Γ-invariant subset of X^{n+1} which meets every bounded subset of X in a finite set. For i in $\{0, \dots, n\}$ let $p_i \colon X^{n+1} \to X$ be the projection. Then we consider the Γ-invariant subset

$$F := \bigcup_{i=0}^n p_i(\mathrm{supp}(c)) \ .$$

Note that c belongs to the image of the map $C\mathcal{X}^\Gamma_n(F_X) \to C\mathcal{X}^\Gamma_n(X)$ induced by the inclusion of F into X.

Observe that F is locally finite. Hence there exists an index i in I such that $F \subseteq Y_i$. We conclude that

$$C\mathcal{X}^\Gamma_n(X) \cong \operatorname*{colim}_{i \in I} C\mathcal{X}^\Gamma_n(Y_i) \ .$$

The argument above implies that we have an isomorphism of chain complexes

$$C\mathcal{X}^\Gamma(X) \cong \operatorname*{colim}_{i \in I} C\mathcal{X}^\Gamma(Y_i) \ .$$

Since the Eilenberg–MacLane correspondence (7.1) preserves filtered colimits we conclude that

$$H\mathcal{X}^\Gamma(X) \simeq \operatorname*{colim}_{i \in I} H\mathcal{X}^\Gamma(Y_i)$$

which was to be shown. □

Recall Definition 4.19 of strongness of an equivariant coarse homology theory.

Lemma 7.10. *The equivariant coarse homology theory $H\mathcal{X}^\Gamma$ is strong.*

Proof. We can essentially repeat the proof of [BE16, Prop. 6.18].

Let $f\colon X \to X$ implement weak flasqueness of a Γ-bornological coarse space X. Then we can define the chain map

$$S := \sum_{n=0}^{\infty} C\mathcal{X}(f^n)\colon C\mathcal{X}(X) \to C\mathcal{X}(X)\ .$$

We refer to [BE16, Prop. 6.18] for the verification that this map is well-defined. We then have the identity of endomorphisms of $C\mathcal{X}^\Gamma(X)$

$$\mathrm{id}_{C\mathcal{X}(X)} + C\mathcal{X}(f) \circ S = S\ .$$

Applying the Eilenberg–MacLane correspondence $\mathcal{EM}$ this gives

$$\mathrm{id}_{H\mathcal{X}^\Gamma(X)} + H\mathcal{X}^\Gamma(f) \circ \mathcal{EM}(S) = \mathcal{EM}(S)\ .$$

Since we already know that $H\mathcal{X}^\Gamma$ is a coarse homology theory we have the equivalence $H\mathcal{X}^\Gamma(f) \simeq \mathrm{id}_{H\mathcal{X}^\Gamma(X)}$. Hence we get $\mathrm{id}_{H\mathcal{X}^\Gamma(X)} + \mathcal{EM}(S) \simeq \mathcal{EM}(S)$, and this implies that we must have $H\mathcal{X}^\Gamma(X) \simeq 0$. □

Recall the definition of the free union of a family of Γ-bornological coarse spaces which was given in Example 2.16 and the Definition 3.12 of the notion of strong additivity for an equivariant coarse homology theory.

Lemma 7.11. *The equivariant coarse homology theory $H\mathcal{X}^\Gamma$ is strongly additive.*

Proof. Let $(X_i)_{i\in I}$ be a family of Γ-bornological coarse spaces. By inspection of the definitions,

$$C\mathcal{X}^\Gamma\Big(\bigsqcup_{i\in I}^{\mathrm{free}} X_i\Big) \cong \prod_{i\in I} C\mathcal{X}^\Gamma(X_i)\ .$$

We then use that the Eilenberg–MacLane correspondence (7.1) preserves products. □

7.4. Change of groups

In this section we provide natural transformations which relate the equivariant coarse homology theory with the change of group functors considered in Section 6.

Let $\iota\colon H \to \Gamma$ be a homomorphism of groups. Let X be a Γ-bornological coarse space. Since every Γ-invariant chain is H-invariant we have an inclusion

$$C\mathcal{X}^\Gamma(X) \hookrightarrow C\mathcal{X}^H(\mathrm{Res}^\Gamma_H X)\ .$$

This gives a natural transformation between equivariant coarse homology theories

$$\mathrm{res}^\Gamma_H\colon H\mathcal{X}^\Gamma \to H\mathcal{X}^H \circ \mathrm{Res}^\Gamma_H\ . \tag{7.4}$$

We let the subset X' of X be a set of representatives of H-orbits in X. Then we have a restriction map

$$r\colon C\mathcal{X}_n(X) \to \mathbb{Z}^{X' \times X^n} \ , \quad c \mapsto c|_{X' \times X^n} \ .$$

We want to define a homomorphism

$$C\mathcal{X}_n^{\Gamma}(X) \to C\mathcal{X}_n^{W_\Gamma(H)}(Q_H(X)) \tag{7.5}$$

by linear extension of the projection map $X' \times X^n \to (H\backslash X)^{n+1}$ composed with the restriction r. We will argue now that this is well-defined, i.e., that the sums appearing in this extension are finite. For x in X we denote by $[x]$ its orbit in $H\backslash X$. We consider a point $([x_0], \dots, [x_n])$ in $(H\backslash X)^{n+1}$. Then $[x_0] \cap X'$ consists of a unique point x_0.

Let c be in $C\mathcal{X}_n(X)$ and assume that c is U-controlled for some entourage U of X. Then we have

$$\mathrm{supp}(c) \cap \{x_0\} \times X^n \subseteq (U[x_0])^{n+1} \ .$$

Since $U[x_0]$ is bounded the number of points of $\mathrm{supp}(c)$ which meet $U[x_0]$ is finite.

One checks that the homomorphism (7.5) does not depend on the choice of the set of representatives X' and has values in $W_\Gamma(H)$-invariant chains. Furthermore, it is compatible with the differential and takes values in controlled and locally finite chains.

We therefore get a natural transformation

$$q_H(\iota)\colon H\mathcal{X}^{\Gamma} \to H\mathcal{X}^{W_\Gamma(H)} \circ Q_H(\iota) \ . \tag{7.6}$$

Recall from Section 6.5 that $P_{\hat\Gamma}(X) = \Gamma_{min,min} \otimes X$ with the action of $\Gamma \times H$ given by $(\gamma, h)(\gamma', x) = (\gamma\gamma'\iota(h)^{-1}, hx)$. We define a morphism of chain complexes

$$C\mathcal{X}_n^{H} \to C\mathcal{X}_n^{\Gamma\times H}(P_{\hat\Gamma}(X))$$

by linear extension of the map

$$(x_0, \dots, x_n) \mapsto \sum_{\gamma' \in \Gamma} ((\gamma', x_0), (\gamma', x_1), \dots, (\gamma', x_n)) \ .$$

In this way we get a transformation

$$\hat p_\Gamma\colon H\mathcal{X}^{H} \to H\mathcal{X}^{\Gamma\times H} \circ \hat P_\Gamma \ . \tag{7.7}$$

We finally get a natural transformation

$$\mathrm{ind}_H^\Gamma \simeq q_H(\kappa) \circ \hat p_\Gamma\colon H\mathcal{X}^{H} \to H\mathcal{X}^{\Gamma} \circ \mathrm{Ind}_H^\Gamma \ , \tag{7.8}$$

where $\kappa\colon H \to \Gamma \times H$ is the inclusion of the second factor.

Proposition 7.12. *If $\iota\colon H \to \Gamma$ is injective, then the transformation* (7.8) *is an equivalence of H-equivariant coarse homology theories.*

Proof. By Remark 6.4 the map $X \to \Gamma \times_H X = \mathrm{Ind}_H^\Gamma(X)$ given by $x \mapsto [1, x]$ is an embedding of an H-invariant coarse component. Restriction along this map gives a map of chain complexes $C\mathcal{X}^\Gamma(\mathrm{Ind}_H^\Gamma(X)) \to C\mathcal{X}^H(X)$ which induces the inverse to (7.8). □

8. Equivariant coarse algebraic K-homology

In this section, we define for every additive category $\mathbf{A}$ with Γ-action its Γ-equivariant coarse algebraic K-homology

$$K\mathbf{A}\mathcal{X}^{\Gamma} \colon \Gamma\mathbf{BornCoarse} \to \mathbf{Sp} \ .$$

The construction associates to a Γ-bornological coarse space X an additive category of equivariant X-controlled $\mathbf{A}$-objects $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$, and defines $K\mathbf{A}\mathcal{X}^{\Gamma}$ to be the (non-connective) algebraic K-theory spectrum of this category.

8.1. The algebraic K-theory functor

We describe the properties of the K-theory functor that we will use subsequently. See [BFJR04, Sec. 2.1] for similar statements. Let **Add** denote the 1-category of small additive categories and exact functors. In the following, all additive categories will be small so that we can omit this adjective safely.

The K-theory functor is a functor

$$K \colon \mathbf{Add} \to \mathbf{Sp}$$

which has the following properties (we will recall the occurring notions further below):

1. (Normalization) It sends (a skeleton of) the additive category of finitely generated free modules over a ring R to the non-connective K-theory (see e.g. [Sch04]) of that ring.
2. (Invariance) It sends isomorphic exact functors to equivalent maps.
3. (Colimits) If $\mathbf{A} = \operatorname{colim}_i \mathbf{A}_i$ is a filtered colimit of additive subcategories, then the natural map $\operatorname{colim}_i K(\mathbf{A}_i) \xrightarrow{\simeq} K(\mathbf{A})$ is an equivalence.
4. (Additivity) If $\Phi, \Psi \colon \mathbf{A} \to \mathbf{A}'$ are exact functors between additive categories, then we have an equivalence $K(\Phi) + K(\Psi) \simeq K(\Phi \oplus \Psi)$ of morphisms between K-theory spectra.
5. (Exactness) If $\mathbf{A}$ is a Karoubi filtration of $\mathbf{C}$, then we have a fiber sequence
$$K(\mathbf{A}) \to K(\mathbf{C}) \to K(\mathbf{C}/\mathbf{A}) \xrightarrow{\partial} K(\mathbf{A}) \ .$$
6. (Products) If we have a family $(\mathbf{A}_i)_{i\in I}$ of additive categories, then the natural map $K(\prod_{i\in I} \mathbf{A}_i) \to \prod_{i\in I} K(\mathbf{A}_i)$ is an equivalence.
7. (Flasqueness) It sends flasque additive categories to zero.

Let us recall the definition of some notions appearing above.

A category is additive if admits a zero object and biproducts such that the operation $\mathrm{Hom}(A,B) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B)$ which sends f, g to

$$f + g \colon A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\mathrm{id} + \mathrm{id}} B$$

defines an abelian group structure on morphism sets. In applications, it is useful to consider the equivalent characterization of additive categories as categories which are enriched over abelian groups, have a zero object and admit finite coproducts Lurie [Lur14, Sec. 1.1.2].

A morphism between additive categories is a functor between the underlying categories which preserves the zero object and finite coproducts. Equivalently, one can require that the functor is compatible with the enrichement in abelian groups, see Mac Lane [ML98, Prop. VIII.2.4].

Given morphisms $\Phi, \Psi \colon \mathbf{A} \to \mathbf{A}'$ between additive categories, we define a new morphism $\Phi \oplus \Psi \colon \mathbf{A} \to \mathbf{A}'$ by choosing for every object A of $\mathbf{A}$ an object of $\mathbf{A}'$ representing the sum $\Phi(A) \oplus \Psi(A)$. Since the sum of two functors is unique up to unique isomorphism, there is an essentially unique map $K(\Phi \oplus \Psi)$ by virtue of Property (2).

Definition 8.1. An additive category $\mathbf{A}$ is called *flasque* if there exists a functor $\Sigma \colon \mathbf{A} \to \mathbf{A}$ such that $\mathrm{id}_{\mathbf{A}} \oplus \Sigma \cong \Sigma$. ♦

Note that Property (4) implies Property (7): Assume that $\mathbf{A}$ is flasque and that $\Sigma \colon \mathbf{A} \to \mathbf{A}$ is a functor satisfying $\mathrm{id}_{\mathbf{A}} \oplus \Sigma \cong \Sigma$. By Property 4 we then have an equivalence $K(\Sigma) + \mathrm{id}_{K(\mathbf{A})} \simeq K(\Sigma)$, which implies that $K(\mathbf{A}) \simeq 0$.

Let $\mathbf{A} \subseteq \mathbf{C}$ be a full additive subcategory. For C, D in $\mathbf{C}$, let $\mathrm{Hom}_{\mathbf{C}}(C, \mathbf{A}, D)$ denote the set of all morphisms in $\mathrm{Hom}_{\mathbf{C}}(C, D)$ which factor through some object in $\mathbf{A}$. Then $\mathrm{Hom}_{\mathbf{C}}(C, \mathbf{A}, D)$ is a subgroup of $\mathrm{Hom}_{\mathbf{C}}(C, D)$. We let $\mathbf{C}/\mathbf{A}$ be the category with the same objects as $\mathbf{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{C}/\mathbf{A}}(C, D) := \mathrm{Hom}_{\mathbf{C}}(C, D) / \mathrm{Hom}_{\mathbf{C}}(C, \mathbf{A}, D) \ .$$

Note that $\mathbf{C}/\mathbf{A}$ is again an additive category. It has the universal property that exact functors on $\mathbf{C}/\mathbf{A}$ correspond bijectively to exact functors on $\mathbf{C}$ which vanish on $\mathbf{A}$.

Let $\mathbf{C}$ be an additive category.

Definition 8.2. The inclusion $\mathbf{A} \subseteq \mathbf{C}$ of a full additive subcategory is a *Karoubi filtration* if every diagram

$$A \xrightarrow{f} C \xrightarrow{g} B$$

in $\mathbf{C}$, where A, B are objects of $\mathbf{A}$, admits an extension to a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xrightarrow{g} & B \\ \downarrow & & \downarrow{\cong} & & \uparrow \\ D & \overset{inc}{\rightarrowtail} & D \oplus D^{\perp} & \overset{\mathrm{pr}}{\twoheadrightarrow} & D \end{array}$$

for some object D of $\mathbf{A}$. ♦

In [Kas15, Lem. 5.6] it is shown that Definition 8.2 is equivalent to the standard definition of a Karoubi filtration as considered in [CP97].

An algebraic K-theory functor as described here can be furnished by restricting the K-theory functor constructed by Schlichting [Sch06] to additive categories. For Property (1), see [Sch06, Thms. 5 & 8]. Property (2) is so elementary that it is rarely stated explicitly, but can be easily read off from the construction in [Sch06]. Property (3) is a combination of [Qui73, Eq. (9)] and [Sch06, Cor. 5]. Property (4) follows from Property (5) (cf. also [Wal85, Prop. 1.3.2]); the latter is proved in [Sch04, Thm. 2.10]. Property (6) is shown in [KW17, Thm 1.2]; for connective K-theory, this is originally due to Carlsson [Car95].

8.2. X-controlled A-objects

Let X be a bornological coarse space with the bornology $\mathcal{B}$ and the coarse structure $\mathcal{C}$, and let $\mathbf{A}$ be an additive category with a (strict) Γ-action. The poset $\mathcal{B}$ is ordered by the subset inclusion and will be regarded as a category. For a functor $A\colon \mathcal{B} \to \mathbf{A}$ we define $\gamma A\colon \mathcal{B} \to \mathbf{A}$ to be the functor sending a bounded set B to $\gamma(A(\gamma^{-1}(B)))$.

Definition 8.3. An *equivariant X-controlled* $\mathbf{A}$*-object* is a pair (A, ρ) consisting of a functor $A\colon \mathcal{B} \to \mathbf{A}$ and a family $\rho = (\rho(\gamma))_{\gamma\in\Gamma}$ of natural isomorphisms $\rho(\gamma)\colon A \to \gamma A$ satisfying the following conditions:

1. $A(\emptyset) \cong 0$.
2. For all B, B' in $\mathcal{B}$, the commutative square

$$\begin{array}{ccc} A(B \cap B') & \longrightarrow & A(B) \\ \downarrow & & \downarrow \\ A(B') & \longrightarrow & A(B \cup B') \end{array}$$

   is a push-out.
3. For all B in $\mathcal{B}$, there exists a finite subset F of B such that the inclusion $F \subseteq B$ induces an isomorphism $A(F) \xrightarrow{\cong} A(B)$.
4. For all pairs of elements γ, γ' of Γ we have $\rho(\gamma\gamma') = \gamma\rho(\gamma') \circ \rho(\gamma)$, where $\gamma\rho(\gamma')$ is the natural transformation from γA to $\gamma\gamma' A$ induced from $\rho(\gamma')$. ♦

Let (A, ρ) be an equivariant X-controlled $\mathbf{A}$-object.

Lemma 8.4. *1. a) The canonical morphism*

$$\bigoplus_{x\in F} A(\{x\}) \xrightarrow{\sum_{x\in F} A(\{x\}\subseteq F)} A(F)$$

is an isomorphism for every finite subset F of X.

b) *For two finite subsets F, F' of X with $F \subseteq F'$ we have a commuting square*

$$\begin{array}{ccc} \bigoplus_{x\in F} A(\{x\}) & \xrightarrow{\sum_{x\in F} A(\{x\}\subseteq F)} & A(F) \\ \downarrow & & \downarrow \\ \bigoplus_{x\in F'} A(\{x\}) & \xrightarrow{\sum_{x\in F'} A(\{x\}\subseteq F')} & A(F') \end{array} .$$

2. a) *For a bounded subset B of X there exists a unique minimal finite subset F_B of B such that $A(F_B) \to A(B)$ is an isomorphism.*

   b) *If B and F_B are as in 2a, then for any subset B' of X with $F_B \subseteq B' \subseteq B$, the morphisms $A(F_B) \to A(B')$ and $A(B') \to A(B)$ are isomorphisms.*

Proof. Item 1 is a direct consequence of Definition 8.3(1) and (2).

Suppose now that F, F' are two finite subsets of B as in Definition 8.3(3). Then we obtain a push-out square

$$\begin{array}{ccc} A(F\cap F') & \longrightarrow & A(F) \\ \downarrow & & \downarrow \cong \\ A(F') & \xrightarrow{\cong} & A(B) \end{array}$$

in which all arrows are inclusions of direct summands. It follows that $A(F \cap F') \to A(B)$ is also an isomorphism. This implies the existence of F_B.

For a bounded subset B' with $F_B \subseteq B' \subseteq B$, inspection of a similar push-out square implies that $A(F_B) \to A(B')$ is an isomorphism, and the claim follows. □

Let (A, ρ) be an equivariant X-controlled **A**-object.

Definition 8.5. The function σ which sends a bounded subset B of X to the finite subset F_B from Lemma 8.4 is called the *support function* of (A, ρ). ♦

The support function is an order preserving, equivariant function from $\mathcal{B}$ to the set of finite subsets of X with the property that $\sigma(\sigma(B)) = \sigma(B)$ for every bounded subset B.

Let $(A, \rho), (A', \rho')$ be equivariant X-controlled **A**-objects and let U be an invariant entourage of X.

Definition 8.6. An *equivariant U-controlled morphism* $f\colon (A, \rho) \to (A', \rho')$ is a natural transformation

$$f\colon A(-) \to A'(U[-]) ,$$

such that $\rho'(\gamma) \circ f = (\gamma f) \circ \rho(\gamma)$ for all elements γ of Γ. ♦

We let $\mathrm{Mor}_U((A,\rho),(A',\rho'))$ denote the set of equivariant U-controlled morphisms. Furthermore, we define the set of controlled morphisms from A to A' by

$$\mathrm{Hom}_{\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)}((A,\rho),(A',\rho')) := \operatorname*{colim}_{U\in\mathcal{C}^{\Gamma}} \mathrm{Mor}_U((A,\rho),(A',\rho')) \ .$$

We denote the resulting category of equivariant X-controlled $\mathbf{A}$-objects and equivariant controlled morphisms by $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$.

We observe that the composition of a U-controlled and a U'-controlled morphism is a $U\circ U'$-controlled morphism. We conclude that composition in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ is well-defined.

Lemma 8.7. *The category $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ is additive.*

Proof. Let $(A,\rho),(A',\rho')$ be equivariant X-controlled $\mathbf{A}$-objects. Denote by $A\oplus A'$ their direct sum in $\mathbf{Fun}(\mathcal{B},\mathbf{A})$. Note that $(A\oplus A',\rho\oplus\rho')$ is an X-controlled $\mathbf{A}$-object because finite unions of bounded sets are bounded. Since finite unions of coarse entourage are coarse entourages and there are canonical isomorphisms of Γ-sets

$$\mathrm{Nat}(A\oplus A', C\circ U[-]) \cong \mathrm{Nat}(A, C\circ U[-]) \times \mathrm{Nat}(A', C\circ U[-])$$

and

$$\mathrm{Nat}(C, (A\oplus A')\circ U[-]) \cong \mathrm{Nat}(C, A\circ U[-]) \times \mathrm{Nat}(C, A'\circ U[-])$$

(we use the symbol Nat to denote the morphism sets in $\mathbf{Fun}(\mathcal{B},\mathbf{A})$) for all U in $\mathcal{C}^{\Gamma}$ and equivariant X-controlled $\mathbf{A}$-objects C, it follows that $(A\oplus A',\rho\oplus\rho')$ is also a direct sum in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$.

Similarly, addition of morphisms in $\mathbf{Fun}(\mathcal{B},\mathbf{A})$ induces the addition operation of morphisms in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$. □

Next we discuss the functoriality of $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ in the variables X and $\mathbf{A}$. For a Γ-equivariant exact functor $\Phi\colon \mathbf{A}\to\mathbf{A}'$ of additive categories, there exists an induced exact functor $\mathbf{V}^{\Gamma}_{\Phi}(X)\colon \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)\to\mathbf{V}^{\Gamma}_{\mathbf{A}'}(X)$ which sends an object (A,ρ) to $(\Phi\circ A,\Phi(\rho))$. Therefore, we have a functor

$$\mathbf{V}^{\Gamma}_{-}(X)\colon \mathbf{Fun}(B\Gamma,\mathbf{Add})\to\mathbf{Add} \ .$$

Let $\phi\colon (X,\mathcal{B},\mathcal{C})\to(X',\mathcal{B}',\mathcal{C}')$ be a morphism of Γ-bornological coarse spaces, and let (A,ρ) be an equivariant X-controlled $\mathbf{A}$-object. Since ϕ is proper, we can define a functor $\phi_*A\colon \mathcal{B}'\to\mathbf{A}$ by

$$\phi_*A(B) := A(\phi^{-1}(B)) \ ,$$

and we define

$$\phi_*\rho(\gamma)(B) = \rho(\gamma)(\phi^{-1}(B)) \ .$$

All properties of Definition 8.3 except (3) are immediate. To see that (3) also holds, we note that $\sigma(\phi^{-1}(B))\subseteq\phi^{-1}(\phi(\sigma(\phi^{-1}(B)))\subseteq\phi^{-1}(B)$ and apply Lemma 8.4 to see that $\phi_*A(\phi(\sigma(\phi^{-1}(B))))\to\phi_*A(B)$ is an isomorphism.

Let $f\colon (A,\rho) \to (A',\rho')$ be an equivariant U-controlled morphism. Then there exists some V in $\mathcal{C}'^{,\Gamma}$ such that $(\phi \times \phi)(U) \subseteq V$. Then $U[\phi^{-1}(B)] \subseteq \phi^{-1}(V[B])$ for all bounded subsets B of X, so we obtain an induced V-controlled morphism

$$\phi_* f = \{f_{\phi^{-1}(B)}\colon \phi_* A(B) \to \phi_* A(V[B])\}_{B \in \mathcal{B}'} \ .$$

This defines a functor

$$\phi_*\colon \mathbf{V}^\Gamma_{\mathbf{A}}(X) \to \mathbf{V}^\Gamma_{\mathbf{A}}(X') \ .$$

We thus have constructed a functor

$$\mathbf{V}^\Gamma_{\mathbf{A}}\colon \Gamma\mathbf{BornCoarse} \to \mathbf{Add} \ .$$

8.3. Coarse algebraic K-homology

Let Γ be a group and $\mathbf{A}$ be an additive category with a Γ-action.

Definition 8.8. We define the *coarse algebraic K-homology* $K\mathbf{A}\mathcal{X}^\Gamma$ associated to $\mathbf{A}$ as

$$K\mathbf{A}\mathcal{X}^\Gamma := K \circ \mathbf{V}^\Gamma_{\mathbf{A}}\colon \Gamma\mathbf{BornCoarse} \to \mathbf{Sp} \ . \qquad \blacklozenge$$

This section discusses the homological properties of $K\mathbf{A}\mathcal{X}^\Gamma$. Our first goal is to prove the following theorem.

Theorem 8.9. *The functor $K\mathbf{A}\mathcal{X}^\Gamma$ is an equivariant coarse homology theory.*

We divide the proof of Theorem 8.9 into a sequence of lemmas.

Lemma 8.10. *The functor $K\mathbf{A}\mathcal{X}^\Gamma$ is u-continuous.*

Proof. Let X be a Γ-bornological coarse space, and let U be an invariant entourage of X. The natural map $X_U \to X$ induces a functor $\Phi_U\colon \mathbf{V}^\Gamma_{\mathbf{A}}(X_U) \to \mathbf{V}^\Gamma_{\mathbf{A}}(X)$. Since the definition of equivariant X-controlled $\mathbf{A}$-objects is independent of the coarse structure, Φ_U is the identity on objects. Additionally, since inclusions of direct summands are monomorphisms, Φ_U is faithful.

This allows us to view $\mathbf{V}^\Gamma_{\mathbf{A}}(X_U)$ as a subcategory of $\mathbf{V}^\Gamma_{\mathbf{A}}(X)$, and we have

$$\mathbf{V}^\Gamma_{\mathbf{A}}(X) = \bigcup_{U \in \mathcal{C}^\Gamma} \mathbf{V}^\Gamma_{\mathbf{A}}(X_U)$$

since every morphism in $\mathbf{V}^\Gamma_{\mathbf{A}}(X)$ is U-controlled for some U in $\mathcal{C}^\Gamma$.

Since the algebraic K-theory functor is compatible with filtered colimits (Property (3)), the claim of the lemma follows. $\square$

Let $\phi, \psi\colon X \to X'$ be morphisms of Γ-bornological coarse spaces.

Lemma 8.11. *If ϕ and ψ are close, then ϕ_* and ψ_* are isomorphic.*

Proof. Let U' be a symmetric entourage of X' containing the diagonal such that $(\phi(x), \psi(x))$ lies in U' for all x in X. Note that this implies $\phi^{-1}(B') \subseteq \psi^{-1}(U'[B'])$ and $\psi^{-1}(B') \subseteq \phi^{-1}(U'[B'])$ for all bounded subsets B' of X'.

Let (A, ρ) be an equivariant X-controlled $\mathbf{A}$-object. The maps

$$A(\phi^{-1}(B')) \to A(\psi^{-1}(U'[B']))$$

define a natural morphism $f\colon \phi_*A \to \psi_*A$, and similarly we have a natural morphism $g\colon \psi_*A \to \phi_*A$. Since the composition $g \circ f$ is given by the natural transformation

$$\{A(\phi^{-1}(B') \subseteq \phi^{-1}((U')^2[B']))\colon \phi_*A \to \phi_*A \circ (U')^2[-]\}_{B' \in \mathcal{B}'} \ ,$$

we have $g \circ f = \mathrm{id}_{\phi_*A}$. Similarly, $f \circ g = \mathrm{id}_{\psi_*A}$. It follows that $\phi_* \cong \psi_*$. □

Corollary 8.12. *The functor $K\mathbf{A}\mathcal{X}^\Gamma$ is coarsely invariant.*

Proof. This is a direct consequence of Lemma 8.11 together with the Property (2) of the algebraic K-theory functor. □

Lemma 8.13. *The functor $K\mathbf{A}\mathcal{X}^\Gamma$ vanishes on flasque Γ-bornological coarse spaces.*

Proof. Let X be a Γ-bornological coarse space with flasqueness implemented by $\phi\colon X \to X$. We claim that the functor

$$\Sigma := \bigoplus_{n \in \mathbb{N}} (\phi^n)_*\colon \mathbf{V}^\Gamma_{\mathbf{A}}(X) \to \mathbf{V}^\Gamma_{\mathbf{A}}(X)$$

is well-defined (up to canonical isomorphism).

For every bounded subset B of X, there exists some n in $\mathbb{N}$ such that $(\phi^n)^{-1}(B) = \emptyset$, so the direct sum $\bigoplus_{n \in \mathbb{N}} (\phi^n)_*A$ exists for every equivariant X-controlled object (A, ρ). Let $f\colon (A, \rho) \to (A', \rho')$ be a U-controlled morphism. Then $\bigoplus_{n \in \mathbb{N}} (\phi^n)_*f$ is V-controlled, where $V := \bigcup_{n \in \mathbb{N}} (\phi \times \phi)^n(U)$ is again a coarse entourage of X by assumption on ϕ. So Σ is an exact functor.

Since ϕ is close to id_X, we conclude from Lemma 8.11 that $\phi_* \circ \Sigma$ and Σ are isomorphic. Hence,

$$\mathrm{id}_{\mathbf{V}^\Gamma_{\mathbf{A}}(X)} \oplus \Sigma \cong \mathrm{id}_{\mathbf{V}^\Gamma_{\mathbf{A}}(X)} \oplus (\phi_* \circ \Sigma) \cong \Sigma \ ,$$

and we deduce the lemma from Property (7) of the algebraic K-theory functor. □

Let X be a Γ-bornological coarse space and Z a Γ-invariant subset of X. For an equivariant X-controlled object (A, ρ), we denote by $(A|_Z, \rho|_Z)$ the restriction to Z, that is $A|_Z$ is the restriction of A to $\mathcal{B} \cap Z$ and $\rho|_Z$ the appropriate restriction of ρ.

Let X be a Γ-bornological coarse space and let $\mathcal{Y} = (Y_i)_{i \in I}$ be an equivariant big family in X. For each i in I, the canonical exact functor $\mathbf{V}^{\Gamma}_{\mathbf{A}}(Y_i) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ is injective on objects and fully faithful, so we can regard $\mathbf{V}^{\Gamma}_{\mathbf{A}}(Y_i)$ as a full subcategory of $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$. Define

$$\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y}) := \bigcup_{i \in I} \mathbf{V}^{\Gamma}_{\mathbf{A}}(Y_i) \ ,$$

considered as a full subcategory of $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$.

Lemma 8.14. *The inclusion $\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ is a Karoubi filtration.*

Proof. Let $(A, \rho), (A', \rho')$ be objects in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})$, (C, ρ_C) be an object in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$, and let $f \colon A \to C$ and $g \colon C \to A'$ be morphisms. Choose i in I such that both A and A' are objects in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(Y_i)$, and pick an invariant and symmetric entourage U which contains the diagonal such that f and g are U-controlled. Let j in I be such that $U[Y_i] \subseteq Y_j$. Since f is a natural transformation $A \to C \circ U[-]$ and $g|_{C|_{X \setminus Y_j}} = 0$, the following diagram commutes:

$$\begin{array}{ccccc}
A & \xrightarrow{\ f\ } & C & \xrightarrow{\ g\ } & A' \\
\downarrow & & \downarrow \cong & & \uparrow \\
C|_{Y_j} & \xrightarrow{\ inc\ } & C|_{Y_j} \oplus C|_{X \setminus Y_j} & \xrightarrow{\ \mathrm{pr}\ } & C|_{Y_j}
\end{array}$$

Hence, the inclusion $\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ is a Karoubi filtration. □

Proposition 8.15. *The functor $K\mathbf{A}\mathcal{X}^{\Gamma}$ is excisive.*

Proof. The category of $\emptyset$-controlled $\mathbf{A}$-objects is the zero category, which has trivial K-theory by Property (1).

Let X be a Γ-bornological coarse space, and let $(Z, \mathcal{Y})$ be an equivariant complementary pair on X. Both inclusions $\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z \cap \mathcal{Y}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(Z)$ and $\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ are Karoubi filtrations by Lemma 8.14. Therefore, we obtain by Property (5) of the algebraic K-theory functor a map of fiber sequences

$$\begin{array}{ccccccc}
K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z \cap \mathcal{Y})) & \longrightarrow & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z)) & \longrightarrow & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z \cap \mathcal{Y})) & \xrightarrow{\ \partial\ } & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z \cap \mathcal{Y})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})) & \longrightarrow & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)) & \longrightarrow & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})) & \xrightarrow{\ \partial\ } & K(\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y}))
\end{array}$$

Consider the induced exact functor $\Phi \colon \mathbf{V}^{\Gamma}_{\mathbf{A}}(Z)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(Z \cap \mathcal{Y}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}(X)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})$.

Let (A, ρ) in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ and consider the natural morphisms $f \colon (A|_Z, \rho|_Z) \to (A, \rho)$ and $p \colon (A, \rho) \to (A|_Z, \rho|_Z)$. Clearly, $pf = \mathrm{id}_{(A|_Z, \rho|_Z)}$. Pick i in I such that $X \setminus Z \subseteq Y_i$. Then $\mathrm{id}_{(A,\rho)} - fp$ factors through $(A|_{Y_i}, \rho|_{Y_i})$, so f and p define mutually inverse isomorphisms in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})$. We conclude that $\Phi \circ \Psi \cong \mathrm{id}_{\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)/\mathbf{V}^{\Gamma}_{\mathbf{A}}(\mathcal{Y})}$, so Φ is an equivalence of categories.

It follows from Property (2) of the algebraic K-theory functor that

$$\begin{array}{ccc} K(\mathbf{V}^\Gamma_{\mathbf{A}}(Z\cap\mathcal{Y})) & \longrightarrow & K(\mathbf{V}^\Gamma_{\mathbf{A}}(Z)) \\ \downarrow & & \downarrow \\ K(\mathbf{V}^\Gamma_{\mathbf{A}}(\mathcal{Y})) & \longrightarrow & K(\mathbf{V}^\Gamma_{\mathbf{A}}(X)) \end{array}$$

is a push-out. By Property (3) of algebraic K-theory we have $K(\mathbf{V}^\Gamma_{\mathbf{A}}(\mathcal{Y})) \simeq K\mathbf{A}\mathcal{X}^\Gamma(\mathcal{Y})$. This proves excision. □

Remark 8.16. Let $(X,\mathcal{B},\mathcal{C})$ be a Γ-bornological coarse space and let Y be a Γ-invariant subspace of X with the property that $U[Y]=Y$ for every U in $\mathcal{C}$. Then $(Y, X\setminus Y)$ is a coarsely excisive pair. Inspecting the proof of Proposition 8.15, we obtain the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & K(\mathbf{V}^\Gamma_{\mathbf{A}}(Y)) & \xrightarrow{\simeq} & K(\mathbf{V}^\Gamma_{\mathbf{A}}(Y)) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ K(\mathbf{V}^\Gamma_{\mathbf{A}}(X\setminus Y)) & \longrightarrow & K(\mathbf{V}^\Gamma_{\mathbf{A}}(X)) & \longrightarrow & K(\mathbf{V}^\Gamma_{\mathbf{A}}(X)/\mathbf{V}^\Gamma_{\mathbf{A}}(X\setminus Y)) \end{array}$$

In addition, we observe that an inverse to the right vertical equivalence is induced by the functor Ψ which is given by $\Psi(A,\rho)=(A|_Y,\rho|_Y)$. Since this functor is already well-defined as a functor $\Psi\colon \mathbf{V}^\Gamma_{\mathbf{A}}(X)\to\mathbf{V}^\Gamma_{\mathbf{A}}(Y)$, we see that the projection map

$$K(\mathbf{V}^\Gamma_{\mathbf{A}}(X)) \simeq K(\mathbf{V}^\Gamma_{\mathbf{A}}(X\setminus Y)) \oplus K(\mathbf{V}^\Gamma_{\mathbf{A}}(Y)) \to K(\mathbf{V}^\Gamma_{\mathbf{A}}(Y))$$

arising from excision coincides with $K(\Psi)$. ♦

Theorem 8.9 follows now by combining Lemma 8.10, Corollary 8.12, Lemma 8.13 and Proposition 8.15. In the remainder of this section, we establish some additional properties of the equivariant coarse homology theory $K\mathbf{A}\mathcal{X}^\Gamma$. For the next two propositions recall the notions of continuity (Definition 5.15) and strongness (Definition 4.19).

Proposition 8.17. *The equivariant coarse homology theory $K\mathbf{A}\mathcal{X}^\Gamma$ is continuous.*

Proof. Let (A,ρ) be an equivariant X-controlled object. Set $S := \{x\in X \mid A(\{x\})\not\cong 0\}$. By definition, we have $S\cap B=\sigma(B)$, where σ is the support function of A. Hence, S is a locally finite subset of X, so (A,ρ) lies in the full subcategory $\mathbf{V}^\Gamma_{\mathbf{A}}(S)$ of $\mathbf{V}^\Gamma_{\mathbf{A}}(X)$. This shows that $\mathbf{V}^\Gamma_{\mathbf{A}}(X)=\bigcup_{S\subseteq X \text{ locally finite}}\mathbf{V}^\Gamma_{\mathbf{A}}(S)$. By Property (3) of the algebraic K-theory functor, it follows that $K\mathbf{A}\mathcal{X}^\Gamma$ is continuous. □

Proposition 8.18. *The equivariant coarse homology theory $K\mathbf{A}\mathcal{X}^\Gamma$ is strong.*

Proof. Let X be a Γ-bornological coarse space with weak flasqueness implemented by $\phi\colon X\to X$. As in the proof of Lemma 8.13, the functor $\Sigma\colon \mathbf{V}^\Gamma_{\mathbf{A}}(X)\to\mathbf{V}^\Gamma_{\mathbf{A}}(X)$ given by

$$\Sigma := \bigoplus_{n\in\mathbb{N}}(\phi^n)_*$$

is well-defined. By assumption, we have $\mathrm{id}_{K\mathbf{A}\mathcal{X}^\Gamma(X)} = K\mathbf{A}\mathcal{X}^\Gamma(\phi)$. Now apply Property (4) of algebraic K-theory to deduce that

$$\mathrm{id}_{K\mathbf{A}\mathcal{X}^\Gamma(X)} + K(\Sigma) \simeq \mathrm{id}_{K\mathbf{A}\mathcal{X}^\Gamma(X)} + K\mathbf{A}\mathcal{X}^\Gamma(\phi) \circ K(\Sigma) \simeq K(\mathrm{id}_{\mathbf{V}^\Gamma_{\mathbf{A}}(X)} \oplus \phi_* \circ \Sigma) \simeq K(\Sigma) ,$$

so $\mathrm{id}_{K\mathbf{A}\mathcal{X}^\Gamma(X)} \simeq 0$. □

Recall the definition of the free union of a family of Γ-bornological coarse spaces which was given in Example 2.16 and Definition 3.12 of the notion of strong additivity for an equivariant coarse homology theory.

Proposition 8.19. *The equivariant coarse homology theory $K\mathbf{A}\mathcal{X}^\Gamma$ is strongly additive.*

Proof. Let $(X_i)_{i\in I}$ be a family of Γ-bornological coarse spaces. The functors

$$\Phi_j \colon \mathbf{V}^\Gamma_{\mathbf{A}}(\bigsqcup_{i\in I}^{\mathrm{free}} X_i) \to \mathbf{V}^\Gamma_{\mathbf{A}}(X_j)$$

sending a $\bigsqcup_{i\in I}^{\mathrm{free}} X_i$-controlled $\mathbf{A}$-object (A,ρ) to $(A|_{X_j}, \rho|_{X_j})$ for j in I assemble to a functor

$$\Phi \colon \mathbf{V}^\Gamma_{\mathbf{A}}(\bigsqcup_{i\in I}^{\mathrm{free}} X_i) \to \prod_{i\in I} \mathbf{V}^\Gamma_{\mathbf{A}}(X_i) \ .$$

For j in I, let $\iota_j \colon X_j \to \bigsqcup_{i\in I}^{\mathrm{free}} X_i$ denote the inclusion. We claim that the functor

$$\Psi \colon \prod_{i\in I} \mathbf{V}^\Gamma_{\mathbf{A}}(X_i) \to \mathbf{V}^\Gamma_{\mathbf{A}}(\bigsqcup_{i\in I}^{\mathrm{free}} X_i)$$

which sends a sequence $(A_i,\rho_i)_i$ to $\bigoplus_{i\in I}(\iota_i)_*(A_i,\rho_i)$ is well-defined (up to canonical isomorphism). We only have to check that the direct sum exists. This follows from the fact that for every bounded subset B of $\bigsqcup_{i\in I}^{\mathrm{free}} X_i$ the subset $\{i \in I \mid B \cap X_i \neq \emptyset\}$ is finite, and that $B \cap X_i$ is bounded for all i in I.

Clearly, $\Psi \circ \Phi$ is isomorphic to the identity. The composition $\Psi \circ \Phi$ is also isomorphic to the identity since $(A,\rho) \cong \bigoplus_{i\in I}(A|_{X_i}, \rho|_{X_i})$ for all objects (A,ρ).

Using Properties (2) and (6), we conclude that

$$K(\mathbf{V}^\Gamma_{\mathbf{A}}(\bigsqcup_{i\in I}^{\mathrm{free}} X_i)) \xrightarrow{K(\Phi)} K(\prod_{i\in I} \mathbf{V}^\Gamma_{\mathbf{A}}(X_i)) \xrightarrow{\simeq} \prod_{i\in I} K(\mathbf{V}^\Gamma_{\mathbf{A}}(X_i))$$

is an equivalence. Note that the j-th component of this equivalence is given by the map $K(\Phi_j)$. Now apply Remark 8.16 to see that $K(\Phi_j)$ agrees with the projection map coming from excision. □

8.4. Calculations

8.4.1. Examples of the form $(\Gamma/H)_{min,min}$

Let H be a subgroup of Γ. Let $\mathbf{A}$ be an additive category with trivial Γ-action.

Lemma 8.20. *We have an equivalence*

$$K\mathbf{A}\mathcal{X}^{\Gamma}((\Gamma/H)_{min,min}) \simeq K(\mathbf{Fun}(BH,\mathbf{A})) \ .$$

Proof. In view of the Property 2 of the K-theory functor it suffices to construct an equivalence of additive categories

$$\Phi\colon \mathbf{V}^{\Gamma}_{\mathbf{A}}((\Gamma/H)_{min,min}) \to \mathbf{Fun}(BH,\mathbf{A}) \ .$$

This functor sends an object (A,ρ) of $\mathbf{V}^{\Gamma}_{\mathbf{A}}((\Gamma/H)_{min,min})$ to the functor sending γ to $\rho(\gamma)(\{eH\})\colon A(\{eH\}) \to A(\{eH\})$.

Furthermore, the functor Φ sends a morphism

$$f\colon (A,\rho) \to (A',\rho')$$

in $\mathbf{V}^{\Gamma}_{\mathbf{A}}((\Gamma/H)_{min,min})$ to the transformation $f(\{eH\})\colon A(\{eH\}) \to A'(\{eH\})$.

In order to define an inverse functor we choose a section $s\colon G/H \to G$ of the projection $G \to G/H$. Then

$$\Psi\colon \mathbf{Fun}(BH,\mathbf{A}) \to \mathbf{V}^{\Gamma}_{\mathbf{A}}((\Gamma/H)_{min,min})$$

sends a functor $F\colon BH \to \mathbf{A}$ to the following object (A,ρ) of $\mathbf{V}^{\Gamma}_{\mathbf{A}}((\Gamma/H)_{min,min})$: we choose $A(B) = \bigoplus_{b\in B} F(*)$ and ρ is defined on an element γ of Γ such that $\rho(\gamma)(B)$ is the morphism $\bigoplus_{b\in B} F(*) \to \bigoplus_{b\in\gamma^{-1}(B)} F(*)$ sending the summand with index $b = gH$ to the summand with index $\gamma^{-1}gH$ via $F(h)$, where h is the element of H which is uniquely determined by the equation $\gamma^{-1}s(gH) = s(\gamma^{-1}gH)h$.

It is an easy exercise to construct the isomorphisms from the compositions $\Psi\circ\Phi$ and $\Phi\circ\Psi$ to the respective identity functors. □

8.4.2. Examples of the form $X_{min,max} \otimes \Gamma_{can,min}$

We consider the group Γ as a Γ-bornological coarse space $\Gamma_{can,min}$. In applications of coarse homotopy theory to proofs of the Farrell–Jones conjecture the coarse algebraic K-homology $K\mathbf{A}\mathcal{X}^{\Gamma}$ twisted by $\mathrm{Yo}^s(\Gamma_{can,min})$ plays an important role. Therefore it is relevant to calculate the spectra

$$K\mathbf{A}\mathcal{X}^{\Gamma}_{\mathrm{Yo}^s(\Gamma_{can,min})}((\Gamma/H)_{min,max}) \simeq K\mathbf{A}\mathcal{X}^{\Gamma}((\Gamma/H)_{min,max} \otimes \Gamma_{can,min}) \ .$$

More generally, we will replace Γ/H by any Γ-set X.

Definition 8.21 ([BR07, Def. 2.1]). Let $\mathbf{A}$ be an additive category with a Γ-action and let X be a Γ-set. We define a new additive category denoted $\mathbf{A} *_\Gamma X$ as follows. An object A in $\mathbf{A} *_\Gamma X$ is a family $A = (A_x)_{x \in X}$ of objects in $\mathbf{A}$ where we require that $\{x \in X \mid A_x \neq 0\}$ is a finite set. A morphism $\phi \colon A \to B$ is a collection of morphisms $\phi = (\phi_{x,g})_{(x,g) \in X \times \Gamma}$, where $\phi_{x,g} \colon A_x \to g(B_{g^{-1}x})$ is a morphism in $\mathbf{A}$. We require that the set of pairs (x,g) in $X \times \Gamma$ with $\phi_{x,g} \neq 0$ is finite. Addition of morphisms is defined componentwise. Composition of morphisms is defined as the convolution product. ♦

Remark 8.22. In [BR07, Def. 2.1] additive categories with right Γ-action are used. For us it is more convenient to consider left Γ-actions. ♦

Let H be a subgroup of Γ.

Definition 8.23. We will denote $\mathbf{A} *_\Gamma (\Gamma/H)$ by $\mathbf{A}[H]$. ♦

If $\mathbf{A}$ is the category of finitely generated, free R-modules for some ring R, then $\mathbf{A}[H]$ is equivalent to the category of finitely generated, free $R[H]$-modules.

The following calculation closely follows Bartels–Farrell–Jones–Reich [BFJR04, Sec. 6.1 and Proof of Prop. 6.2].

Proposition 8.24. *For every Γ-set X we have an equivalence*

$$\mathbf{V}^\Gamma_{\mathbf{A}}(X_{min,max} \otimes \Gamma_{can,min}) \simeq \mathbf{A} *_\Gamma X \ .$$

Proof. The desired equivalence is given by an exact functor

$$\Phi \colon \mathbf{V}^\Gamma_{\mathbf{A}}(X_{min,max} \otimes \Gamma_{can,min}) \to \mathbf{A} *_\Gamma X \ .$$

We define Φ as follows:

1. For an object (A,ρ) in $\mathbf{V}^\Gamma_{\mathbf{A}}(X_{min,max} \otimes \Gamma_{can,min})$, we define $\Phi(A,\rho)_x$ as $A(\{(x,1)\})$.
2. For a morphism $f \colon (A,\rho) \to (A',\rho')$ we define $\Phi(f)_{x,g}$ as the composition

$$A(\{x,1\}) \xrightarrow{f} A'(\{x\} \times F) \xrightarrow{p_g} A'(\{x,g\}) \xrightarrow{\rho'(g)} gA'(\{g^{-1}x,1\}) \ ,$$

   where F is a finite subset of Γ containing g and p_g is the projection arising from the identification $\bigoplus_{f \in F} A(\{x,f\}) \xrightarrow{\cong} A(\{x\} \times F)$.

Note that $\Phi(f)_{x,g}$ is independent of the choice of F.

We will first show that Φ is fully faithful. A morphism $f \colon (A,\rho) \to (A',\rho')$ is determined by its values on $A(\{x,1\})$ by equivariance. Since $X_{min,max}$ has the minimal coarse structure, the family $(\Phi(f)_{x,g})_{(x,g) \in X \times \Gamma}$ determines f. Hence Φ is faithful.

Since the Γ-action on $X_{min,max} \otimes \Gamma_{can,min}$ is free, for every finite subset F of Γ and every family of morphisms $A(\{x,1\}) \to A'(\{x\} \times F)$ indexed by points x in X there exists a unique equivariant extension to a morphism $f \colon (A,\rho) \to (A,\rho')$. Let $\phi \colon \Phi(A,\rho) \to \Phi(A',\rho')$

be any morphism in $A *_\Gamma X$. Let $F := \{g \in \Gamma \mid \exists x \in X : \phi_{g,x} \neq 0\}$, then F is a finite subset of Γ. The family of morphisms

$$\left((A(\{x,1\}) \xrightarrow{\bigoplus_{g\in F} \rho'_{g^{-1}} \circ \phi_{x,g}} \bigoplus_{g \in F} A'(\{x,g\}) \xrightarrow{\cong} A'(\{x\} \times F) \right)_{x \in X}$$

extends to a morphism $f\colon (A,\rho) \to (A',\rho')$ with $\Phi(f) = \phi$. This shows that Φ is fully faithful.

We now show that Φ is essentially surjective. Every finitely supported family $(A_x)_{x\in X}$ of objects of A extends essentially uniquely to an equivariant object (A,ρ) with $A(\{x,1\}) = A_x$ for all x in X. This uses the choice of finite sums of the objects A_x. □

Let X be a Γ-set.

Corollary 8.25. *We have an equivalence*

$$K\mathbf{A}\mathcal{X}^\Gamma(X_{min,max} \otimes \Gamma_{can,min}) \simeq K(\mathbf{A} *_\Gamma X) \ .$$

8.4.3. Examples of the form $(\Gamma/H)_{min,min} \otimes \Gamma_{?,max}$.

Let H be a subgroup of Γ and let $\mathbf{A}$ be an additive category with Γ-action.

Lemma 8.26. *If H is finite, then*

$$K\mathbf{A}\mathcal{X}^\Gamma((\Gamma/H)_{min,min} \otimes \Gamma_{max,max}) \simeq K\mathbf{A}\mathcal{X}^\Gamma((\Gamma/H)_{min,min} \otimes \Gamma_{can,max}) \simeq K(\mathbf{A}[H]) \ .$$

Otherwise

$$K\mathbf{A}\mathcal{X}^\Gamma((\Gamma/H)_{min,min} \otimes \Gamma_{max,max}) \simeq K\mathbf{A}\mathcal{X}^\Gamma((\Gamma/H)_{min,min} \otimes \Gamma_{can,max}) \simeq 0 \ .$$

Proof. We argue similarly as in the proof of Proposition 8.24 for $X = (\Gamma/H)_{min,min}$. If H is finite, then the set F appearing in the proof is still finite, but for a different reason.

If H is infinite, then there are no non-trivial Γ-invariant $(\Gamma/H)_{min,min} \otimes \Gamma_{?,max}$ controlled modules (this does not depend on the coarse structures). □

8.5. Change of groups

Let H be a subgroup of Γ.

Theorem 8.27. *There is an equivalence of H-equivariant coarse homology theories*

$$\mathrm{ind}_H^\Gamma\colon K\mathbf{A}\mathcal{X}^H \xrightarrow{\simeq} K\mathbf{A}\mathcal{X}^\Gamma \circ \mathrm{Ind}_H^\Gamma \ .$$

In the proof we will construct a equivalence in the other direction. We state the theorem in this form since this is the more common direction.

Proof. Let $(X, \mathcal{B}, \mathcal{C})$ be a bornological coarse space. Recall from the Remark 6.4 that the bornological coarse space $\operatorname{Ind}_H^\Gamma X$ is given by the set $\Gamma \times_H X$ with bornology generated by the subsets $\{g\} \times B$ for B in $\mathcal{B}$ and coarse structure generated by the entourages $\operatorname{diag}_\Gamma \times U$ for U in $\mathcal{C}$.

Note that $H \times_H X \simeq X$ is an H-invariant coarse component of $\operatorname{Ind}_H^\Gamma X$. Hence, restricting an object (A, ρ) of $\mathbf{V}^\Gamma_{\mathbf{A}}(\operatorname{Ind}_H^\Gamma X)$ to $(A|_X, \rho_X)$ yields a functor

$$\mathbf{V}^\Gamma_{\mathbf{A}}(\operatorname{Ind}_H^\Gamma X) \to \mathbf{V}^H_{\mathbf{A}}(X) \ .$$

Similarly to the proof of Lemma 8.20, one checks that this functor is an equivalence. □

Let H be a subgroup of Γ. Sending a Γ-equivariant X-controlled object (A, ρ) to the object $(A, \{\rho(h)\}_{h \in H})$ yields a natural transformation

$$\operatorname{res}_H^\Gamma \colon K\mathbf{A}\mathcal{X}^\Gamma \to K\mathbf{A}\mathcal{X}^H \circ \operatorname{Res}_H^\Gamma \ . \qquad (8.1)$$

8.6. Variations on the definition

Let again $\mathbf{A}$ be an additive category with a Γ-action and $(X, \mathcal{B}, \mathcal{C})$ be a Γ-bornological coarse space. Intuitively, an equivariant X-controlled $\mathbf{A}$-object (A, ρ) is some (infinite) sum of objects in $\mathbf{A}$ parametrized by points in X together with an action of Γ. One may want to keep track of the "global" object associated to an X-controlled object explicitly. The purpose of this section is to give an alternative definition of $\mathbf{V}^\Gamma_{\mathbf{A}}(X)$ which accomplishes precisely this, and discuss a variation of this definition which leads to an example of a non-continuous coarse homology theory.

Since an equivariant X-controlled $\mathbf{A}$-object usually involves an infinite number of objects in $\mathbf{A}$, we need to enlarge our coefficient category appropriately. Therefore, let $\mathbf{A} \to \widehat{\mathbf{A}}$ be a fully faithful and Γ-equivariant embedding of $\mathbf{A}$ into an additive category $\widehat{\mathbf{A}}$ with Γ-action which admits sufficiently large direct sums. The sum completion of $\mathbf{A}$ is a canonical choice for $\widehat{\mathbf{A}}$.

For an object A in $\widehat{\mathbf{A}}$, let $\Pi(A)$ denote the set of idempotents on A whose image splits off as a direct sum and is isomorphic to an object in $\mathbf{A}$.

Definition 8.28. A *global equivariant X-controlled $\mathbf{A}$-object* is a triple (A, ϕ, ρ) consisting of

1. an object A in $\widehat{\mathbf{A}}$;
2. a function $\phi \colon \mathcal{B} \to \Pi(A)$;
3. a morphism $\rho(\gamma) \colon A \to \gamma A$ for every element γ in Γ;

such that the following conditions are satisfied:

1. The function ϕ satisfies the following relations:
   a) $\phi(\emptyset) = 0$;
   b) $\phi(B_1 \cup B_2) = \phi(B_1) + \phi(B_2) - \phi(B_1 \cap B_2)$;
   c) $\phi(B_1 \cap B_2) = \phi(B_1) \circ \phi(B_2)$.
2. For every bounded subset B of X, there exists some finite subset F of B such that $\phi(B) = \phi(F)$.
3. For all pairs of elements γ, γ' in Γ, we have $\rho(\gamma'\gamma) = \gamma\rho(\gamma') \circ \rho(\gamma)$.
4. For all elements γ of Γ and bounded subsets B of X, we have the equality
$$\phi(\gamma^{-1}B) = \rho(\gamma)^{-1} \circ \gamma\phi(B) \circ \rho(\gamma)\ . \qquad \blacklozenge$$

In contrast to the data specifying an equivariant X-controlled object, this definition does not fix a chosen image for each of the idempotents $\phi(B)$.

Definition 8.29. A morphism $f\colon (A, \phi, \rho) \to (A', \phi', \rho')$ of global equivariant X-controlled $\mathbf{A}$-objects is a morphism $f\colon A \to A'$ in $\widehat{\mathbf{A}}$ satisfying the following conditions:

1. f is equivariant in the sense that $(\gamma f) \circ \rho(\gamma) = \rho'(\gamma) \circ f$;
2. f is controlled in the sense that the set
$$\bigcap\{U \subseteq X \times X \mid \forall\ B, B' \in \mathcal{B}\colon U[B] \cap B' = \emptyset \Rightarrow \phi'(B')f\phi(B) = 0\}$$
is an entourage of X. $\blacklozenge$

Morphisms of global equivariant X-controlled $\mathbf{A}$-objects can be composed. We denote the resulting category by $\mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}(X)$. Similar to the discussion in Section 8.2, one shows that $\mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}(X)$ is additive. If $f\colon X \to X'$ is a morphism of Γ-bornological coarse spaces, then we define a functor
$$f_*\colon \mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}(X) \to \mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}(X')$$
which sends (A, ϕ, ρ) to $(A, f_*\phi, \rho)$, were $f_*\phi(B') := \phi(f^{-1}(B'))$ for all B' in $\mathcal{B}'$. Furthermore, the construction is functorial with respect to commutative squares of additive functors
$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \widehat{\mathbf{A}} \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\widehat{\Phi}} \\ \mathbf{A}' & \longrightarrow & \widehat{\mathbf{A}}' \end{array}$$
in which the horizontal arrows are fully faithful and Γ-equivariant embeddings.

We have already indicated how objects in $\mathbf{V}^{\Gamma}_{\mathbf{A}}(X)$ correspond to objects in $\mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}(X)$ and vice versa, namely by summing up all values of an equivariant X-controlled object and choosing images of idempotents, respectively. In fact, it is not difficult to show the following.

Proposition 8.30. *There is a zig-zag of equivalence between the functors* $\mathbf{V}^{\Gamma}_{\mathbf{A}}$ *and* $\mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}$ *from* Γ**BornCoarse** *to* **Add***.*

Example 8.31. We now provide a modification of the definition of $\mathbf{V}^{\Gamma}_{\mathbf{A}\subseteq\widehat{\mathbf{A}}}$ which leads to a non-continuous coarse homology theory.

Let (A, ϕ, ρ) be a global equivariant X-controlled $\mathbf{A}$-object. Condition (2) implies, in the presence of Condition (1), that the object (A, ϕ) is essentially determined by its restriction to the poset of finite subsets of X, and that the support of (A, ϕ), i.e. the set $\{x \in X \mid \phi(\{x\} \neq 0\}$, is a locally finite subset of X.

Dropping Condition (2), we obtain another additive category $\mathbf{V}^{\Gamma}_{\mathbf{A},\Psi}(X)$ which is also functorial in Γ-bornological coarse spaces. Taking non-connective algebraic K-theory of this category also gives rise to an equivariant coarse homology theory $K\mathbf{A}\mathcal{X}^{\Gamma}_{\Psi}$ since the proofs of Lemma 8.10, Lemma 8.11, Lemma 8.13 and Proposition 8.15 go through without change. However, the proof of continuity (Proposition 8.17) does not apply to $K\mathbf{A}\mathcal{X}^{\Gamma}_{\Psi}$ since the condition ensuring that the support of an object is locally finite has been omitted.

In fact, the following example shows that the coarse homology theory $K\mathbf{A}\mathcal{X}_{\Psi}$ is not continuous: Suppose that there exists an object A in $\mathbf{A}$ whose class in K_0 is non-trivial. Consider the bornological coarse space $\mathbb{N}_{min,max}$. Choose an ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and define the function $\phi\colon \mathcal{P}(\mathbb{N}) \to \Pi(A)$ by

$$\phi(B) := \begin{cases} \mathrm{id}_A & B \in \mathcal{F}, \\ 0 & B \notin \mathcal{F}. \end{cases}$$

Then (A, ϕ) is a global $\mathbb{N}_{min,max}$-controlled $\mathbf{A}$-object since $\mathcal{F}$ is an ultrafilter. Moreover, the morphism of bornological coarse spaces $\mathbb{N}_{min,max} \to *$ induces a homomorphism

$$\pi_0 K\mathbf{A}\mathcal{X}_{\Psi}(\mathbb{N}_{min,max}) \to \pi_0 K\mathbf{A}\mathcal{X}_{\Psi}(*) \cong K_0(\mathbf{A})$$

which maps the class $[(A, \phi)]$ to $[A] \neq 0$.

On the other hand, locally finite subsets of $\mathbb{N}_{min,max}$ are precisely the finite subsets of $\mathbb{N}$. Since each finite subset F is a union of coarse components of $\mathbb{N}_{min,max}$, the pair of subsets $(F, \mathbb{N} \setminus F)$ is a complementary pair. From excision, we obtain a direct sum decomposition

$$K\mathbf{A}\mathcal{X}_{\Psi}(\mathbb{N}_{min,max}) \simeq K\mathbf{A}\mathcal{X}_{\Psi}(F_{min,max}) \oplus K\mathbf{A}\mathcal{X}_{\Psi}((\mathbb{N} \setminus F)_{min,max}) \ ,$$

in which the projection $K\mathbf{A}\mathcal{X}_{\Psi}(\mathbb{N}_{min,max}) \to K\mathbf{A}\mathcal{X}_{\Psi}(F_{min,max})$ is induced by the functor restricting $\mathbb{N}_{min,max}$-controlled objects to F (cf. Remark 8.16). Since the restriction of (A, ϕ) to any finite subset of $\mathbb{N}$ is the zero object, we conclude that the class $[(A, \phi)]$ does not lie in the image of the comparison map

$$\operatorname*{colim}_{F \subseteq \mathbb{N} \text{ finite}} \pi_0 K\mathbf{A}\mathcal{X}_{\Psi}(F_{min,max}) \to \pi_0 K\mathbf{A}\mathcal{X}_{\Psi}(\mathbb{N}_{min,max}) \ .$$

In particular, $K\mathbf{A}\mathcal{X}_{\Psi}$ is not continuous. ♦

Part III. Cones and assembly maps

9. Cones

In this section we will introduce the cone and the 'cone at infinity' functors, and discuss their properties. The cone at infinity is later used in Section 10.3 to define the universal assembly map, and in Section 10.4 to transform coarse homology theories into topological homology theories. In conjunction with the Rips complex construction the cone at infinity will be used in Section 11.3 to construct the universal coarse assembly map.

The first three Sections 9.1–9.3 are technical preparations. The cone functor is then defined and discussed in Section 9.4 and Section 9.5.

9.1. Γ-uniform bornological coarse spaces

In this section we introduce the category $\Gamma\mathbf{UBC}$ of Γ-uniform bornological coarse spaces. Its objects are Γ-bornological coarse spaces with an additional Γ-uniform structure. The additional datum of a Γ-uniform structure is needed in order to define hybrid structures, see Section 9.2.

We start with recalling some basics on uniform spaces. Let X be a set and $\mathcal{T}$ be a subset of $\mathcal{P}(X \times X)$, the power set of $X \times X$.

Definition 9.1. The set $\mathcal{T}$ is a uniform structure if it is non-empty, closed under composition, inversion, supersets, finite intersection, every element of $\mathcal{T}$ contains the diagonal of X, and if for every U in $\mathcal{T}$ there exists V in $\mathcal{T}$ such that $V \circ V \subseteq U$. ⧫

Remark 9.2. Note that any subset S of $\mathcal{P}(X \times X)$ with the property that for every U in S there exists V in X such that $V \circ V \subseteq U$ generates a uniform structure on X by taking the closure of S under composition, inversion, supersets and finite intersection. ⧫

The elements of $\mathcal{T}$ are called *uniform entourages.* We will consider $\mathcal{T}$ as a filtered partially ordered set whose order relation is the opposite of the inclusion relation.

Definition 9.3. A *uniform space* is a pair $(X, \mathcal{T})$ of a set X together with a uniform structure $\mathcal{T}$. ⧫

Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be uniform spaces and $f \colon X \to X'$ be a map of sets.

Definition 9.4. The map f is called a *uniform map* if for every uniform entourage U' of X' we have $(f \times f)^{-1}(U') \in \mathcal{T}$. ⧫

If a group Γ acts on a uniform space $(X, \mathcal{T})$, then it acts on the set of uniform entourages $\mathcal{T}$. We let $\mathcal{T}^\Gamma$ denote the subset of $\mathcal{T}$ of Γ-invariant uniform entourages.

Definition 9.5. A Γ-*uniform space* is a uniform space $(X, \mathcal{T})$ with an action of Γ by automorphisms such that $\mathcal{T}^\Gamma$ is cofinal in $\mathcal{T}$. ♦

A uniform structure $\mathcal{T}$ on a Γ-set X such that X is a Γ-uniform space will be called a Γ-*uniform structure.*

Example 9.6. The uniform structure $\mathcal{T}_d$ of a metric space (X, d) is generated by the uniform entourages

$$U_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\} \tag{9.1}$$

for all r in $(0, \infty)$.

If Γ acts isometrically on a metric space (X, d), then the associated uniform space $X_{u,d} := (X, \mathcal{T}_d)$ is a Γ-uniform space.

If the metric is implicitly clear, then we will also write X_u instead of $X_{u,d}$.

A uniformly continuous map between metric spaces induces a uniform map between the associated uniform spaces.

The standard metric turns $\mathbb{R}$ into a uniform space $\mathbb{R}_u$. The action of $\mathbb{Z}$ on $\mathbb{R}$ by dilatations $(n, x) \mapsto 2^n x$ is an action on $\mathbb{R}_u$ by automorphisms of uniform spaces, but $\mathbb{R}_u$ is not a $\mathbb{Z}$-uniform space. ♦

Let $\Gamma\mathbf{U}$ be the category of Γ-uniform spaces and uniform equivariant maps.

Example 9.7. Let K be a simplicial complex with a simplicial Γ-action. On K we consider the path quasi-metric induced by the spherical metric on the simplices. This quasi-metric is preserved by Γ and the associated uniform structure is a Γ-uniform structure. Therefore a simplicial complex with a simplicial Γ-action gives rise to a Γ-uniform space K_u.

If $K \to K'$ is an equivariant simplicial map, then it is a uniform map $K_u \to K'_u$ between the Γ-uniform spaces. ♦

We now consider the combination of uniform and bornological coarse structures. We consider a Γ-set X with a Γ-coarse structure $\mathcal{C}$ and a Γ-uniform structure $\mathcal{T}$.

Definition 9.8. We say that $\mathcal{C}$ and $\mathcal{T}$ are *compatible* if $\mathcal{C}^\Gamma \cap \mathcal{T}^\Gamma \neq \emptyset$. ♦

In words, the coarse and the uniform structures are compatible if there exists an invariant entourage which is both a coarse entourage and a uniform entourage.

Definition 9.9. We define the category $\Gamma\mathbf{UBC}$ of Γ-uniform bornological coarse spaces as follows:

1. The objects of $\Gamma\mathbf{UBC}$ are Γ-bornological coarse spaces with an additional compatible Γ-uniform structure.

2. The morphisms of Γ**UBC** are morphisms of Γ-bornological coarse spaces which are in addition uniform. ♦

Example 9.10. Let (X, d) be a quasi-metric space with an action of Γ by isometries and a Γ-invariant bornology $\mathcal{B}$. Assume that the metric and the bornology are compatible in the sense that for every r in $(0, \infty)$ and B in $\mathcal{B}$ we have $U_r[B] \in \mathcal{B}$, where U_r is as in (9.1). Then we get a Γ-uniform bornological coarse space X_{du} with the following structures:

1. The coarse structure is generated by the coarse entourages U_r for all r in $(0, \infty)$.
2. The uniform structure is generated by the uniform entourages U_r for all r in $(0, \infty)$.
3. The bornology is $\mathcal{B}$.

If (X', d') is a second quasi-metric space with isometric Γ-action and $f\colon X \to X'$ is a proper (this refers to the bornologies), Γ-equivariant contraction, then $f\colon X_{du} \to X'_{du}$ is a morphism of Γ-uniform bornological coarse spaces. ♦

Example 9.11. Let K be a Γ-simplicial complex equipped with the spherical quasi-metric. Then we can equip K with the bornology of metrically bounded subsets and obtain a Γ-uniform bornological coarse space K_{du}. Alternatively we can equip it with the maximal bornology and get the Γ-uniform bornological coarse space $K_{du,max}$.

A morphism of Γ-simplicial complexes $f\colon K \to K'$ always induces the two morphisms $f\colon K_{du,max} \to K'_{du,max}$ and $K_{du,max} \to K'_{du}$ of Γ-uniform bornological coarse spaces. If f is proper (in the sense that preimages of simplices are finite complexes), then it also induces a morphism $f\colon K_{du} \to K'_{du}$. ♦

Example 9.12. For a Γ-uniform bornological coarse space X let $F_{\mathcal{T}}(X)$ denote the underlying Γ-bornological coarse space obtained by forgetting the datum of the uniform structure. Let X and Y be Γ-uniform bornological coarse spaces. Then we define the Γ-uniform bornological coarse space $X \otimes Y$ such that $\mathcal{F}_{\mathcal{T}}(X \otimes Y) = F_{\mathcal{T}}(X) \otimes F_{\mathcal{T}}(Y)$ and the Γ-uniform structure on $X \otimes Y$ is generated by the products $U \times V$ for all pairs of uniform entourages U of X and V of Y. ♦

9.2. Hybrid structures

In this section we will define hybrid coarse structures, which will feature in the definition of cones in Section 9.4.

We consider a Γ-uniform bornological coarse space X with coarse structure $\mathcal{C}$, bornology $\mathcal{B}$, and uniform structure $\mathcal{T}$. Let furthermore a Γ-invariant big family $\mathcal{Y} = (Y_i)_{i \in I}$ be given.

Definition 9.13. The pair $(X, \mathcal{Y})$ is called *hybrid data.* ♦

In this situation we can define the hybrid coarse structure $\mathcal{C}_h$ as follows.

Note that $\mathcal{P}(X \times X)^{\Gamma}$ is a filtered poset with the opposite of the inclusion relation. We consider a function $\phi\colon I \to \mathcal{P}(X \times X)^{\Gamma}$, i.e., an order-preserving map.

Definition 9.14. The function ϕ is called $\mathcal{T}^\Gamma$-admissible, if for every U in $\mathcal{T}^\Gamma$ there exists i in I such that $\phi(i) \subseteq U$. ♦

Given a $\mathcal{T}^\Gamma$-admissible function $\phi\colon I \to \mathcal{P}(X \times X)^\Gamma$ we define the entourage

$$U_\phi := \big\{(x,y) \in X \times X \mid \big(\forall i \in I \mid (x,y) \in Y_i \times Y_i \text{ or } (x,y) \in \phi(i)\big)\big\} \ .$$

Note that U_ϕ is Γ-invariant.

Definition 9.15. The *hybrid coarse structure* $\mathcal{C}_h$ is the coarse structure generated by the entourages $U \cap U_\phi$ for all U in $\mathcal{C}^\Gamma$ and $\mathcal{T}^\Gamma$-admissible functions $\phi\colon I \to \mathcal{P}(X \times X)^\Gamma$. ♦

Definition 9.16. The *hybrid space* X_h is defined to be the Γ-bornological coarse space with underlying set X, the hybrid coarse structure $\mathcal{C}_h$ and the bornological structure $\mathcal{B}$. ♦

A morphism of hybrid data $f\colon (X,\mathcal{Y}) \to (X',\mathcal{Y}')$ is a morphism of Γ-uniform bornological coarse spaces which is compatible with the big families $\mathcal{Y} = (Y_i)_{i\in I}$ and $\mathcal{Y}' = (Y'_{i'})_{i'\in I'}$ in the sense that for every i in I there exists i' in I' such that $f(Y_i) \subseteq Y'_{i'}$.

Lemma 9.17. *If f is a morphism of hybrid data, then the underlying map of sets is a morphism $f\colon X_h \to X'_h$ of Γ-bornological coarse spaces.*

Proof. [BE16, Lem. 5.15]. □

Remark 9.18. One could set up a category of hybrid data and understand the construction of the hybrid structure as a functor from hybrid data to Γ-bornological coarse spaces. ♦

9.3. Decomposition Theorem and Homotopy Theorem

In this section we discuss the Decomposition Theorem and the Homotopy Theorem for hybrid coarse structures. These two theorems constitute important technical results which are needed to prove crucial properties of the cone functor in Section 9.5.

Let A, B be Γ-invariant subsets of a Γ-uniform space Y with uniform structure $\mathcal{T}$. For an entourage U of Y we set

$$\mathcal{T}^\Gamma_{\subseteq U} := \{V \in \mathcal{T}^\Gamma \mid V \subseteq U\} \ .$$

Definition 9.19. The pair (A, B) is an *equivariant uniform decomposition* if

1. $Y = A \cup B$, and
2. there is an invariant uniform entourage U of Y and a function

   $$s\colon \mathcal{P}(Y \times Y)^\Gamma_{\subseteq U} \to \mathcal{P}(Y \times Y)^\Gamma$$

   such that for every W in $\mathcal{P}(Y \times Y)^\Gamma_{\subseteq U}$ we have the inclusion

   $$W[A] \cap W[B] \subseteq s(W)[A \cap B]$$

   and the restriction $s_{|\mathcal{T}^\Gamma_{\subseteq U}}$ is $\mathcal{T}^\Gamma$-admissible. ♦

According to Definition 9.14, the function $s_{|\mathcal{T}^\Gamma_{\subseteq U}}$ is $\mathcal{T}^\Gamma$-admissible if for every entourage V in $\mathcal{T}^\Gamma$ there is an entourage W in $\mathcal{T}^\Gamma_{\subseteq U}$ such that $s(W) \subseteq V$.

Example 9.20. Assume that K is a Γ-simplicial complex, and A and B are Γ-invariant subcomplexes such that $K = A \cup B$. Then (A, B) is an equivariant uniform decomposition of K_u. This follows from [BE16, Ex. 5.19]. ♦

Let Y be a Γ-uniform bornological coarse space with an invariant big family $\mathcal{Y} = (Y_i)_{i\in I}$. We further assume that (A, B) is an equivariant uniform decomposition of Y. We let Y_h denote the associated bornological coarse space with the hybrid structure.

We write A_h for the Γ-bornological coarse space obtained from the Γ-uniform bornological coarse structure on A induced from Y by first restricting the hybrid data to A and then forming the hybrid structure. By A_{Y_h} we denote the Γ-bornological coarse space obtained by restricting the structures of Y_h to the subset A. It was shown in [BE16, Lem. 5.17] that $A_h = A_{Y_h}$.

Definition 9.21. A Γ-uniform space $(Y, \mathcal{T})$ is called Hausdorff, if $\bigcap_{U\in\mathcal{T}} U = \mathrm{diag}_Y$. ♦

Theorem 9.22. *(Decomposition Theorem) If $I = \mathbb{N}$ and Y is Hausdorff, then the following square in $\Gamma\mathbf{Sp}\mathcal{X}$ is cocartesian:*

$$\begin{array}{ccc} \mathrm{Yo}^s((A\cap B)_h, A\cap B\cap\mathcal{Y}) & \longrightarrow & \mathrm{Yo}^s(A_h, A\cap\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathrm{Yo}^s(B_h, B\cap\mathcal{Y}) & \longrightarrow & \mathrm{Yo}^s(Y_h, \mathcal{Y}) \end{array} \tag{9.2}$$

Proof. The proof of [BE16, Thm. 5.20] goes through word-for-word. One just works with invariant entourages or invariant uniform neighbourhoods everywhere. □

We consider a Γ-uniform bornological coarse space Y with an invariant big family $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$. We consider the unit interval $[0, 1]$ with the trivial Γ-action as a Γ-uniform bornological coarse space $[0, 1]_{du}$ with the structures induced from the metric. On the tensor product $[0, 1]_{du} \otimes Y$ (Example 9.12) we consider the big family $([0, 1] \times Y_n)_{n\in\mathbb{N}}$. Let $\mathcal{B}$ denote the bornology of Y.

Theorem 9.23 (Homotopy Theorem)**.** *Assume that for every B in $\mathcal{B}$ there exists n in $\mathbb{N}$ such that $B \subseteq Y_n$. Then the projection induces an equivalence*

$$\mathrm{Yo}^s(([0, 1]_{du} \otimes Y)_h) \to \mathrm{Yo}^s(Y_h)\ .$$

Proof. The proof of [BE16, Thm. 5.25] goes through in the present equivariant case. □

9.4. The cone functor

In this section we define the cone functor

$$\mathcal{O}\colon \Gamma\mathbf{UBC} \to \Gamma\mathbf{BornCoarse} \tag{9.3}$$

and prove that decompositions of the spaces which are simultaneously uniformly and coarsely excisive lead to corresponding coarsely excisive decompositions of the cones.

We consider the metric space $[0,\infty)$ with the metric induced from the inclusion into $\mathbb{R}$ and the trivial Γ-action. We get a Γ-uniform bornological coarse space $[0,\infty)_{du}$. For a Γ-uniform bornological coarse space Y we form the Γ-uniform bornological coarse space

$$[0,\infty)_{du} \otimes Y$$

(Example 9.12). This Γ-uniform bornological coarse space has a canonical big family given by

$$\mathcal{Y}(Y) := ([0,n] \times Y)_{n\in\mathbb{N}} \ . \tag{9.4}$$

A morphism $f\colon Y \to Y'$ of Γ-uniform bornological coarse spaces induces a morphism of hybrid data

$$([0,\infty)_{du} \otimes Y, \mathcal{Y}(Y)) \to ([0,\infty)_{du} \otimes Y', \mathcal{Y}(Y'))$$

given by the map $\mathrm{id}_{[0,\infty)} \times f$ on the underlying sets.

Definition 9.24. The *cone functor* (9.3) is defined such that it sends the Γ-uniform bornological coarse space Y to the Γ-bornological coarse space

$$\mathcal{O}(Y) := ([0,\infty)_{du} \otimes Y)_h$$

and a morphism $f\colon Y \to Y'$ of Γ-uniform bornological coarse spaces to the morphism

$$\mathcal{O}(f)\colon \mathcal{O}(Y) \to \mathcal{O}(Y')$$

of Γ-bornological coarse spaces given by the map $\mathrm{id}_{[0,\infty)} \times f$ of the underlying sets. ♦

Example 9.25. If the Γ-uniform bornological coarse space Y is discrete as a uniform and as a coarse space, then $\mathcal{O}(Y)$ is flasque: Flasqueness of $\mathcal{O}(Y)$ can be implemented by the map $(t,y) \mapsto (t + \frac{1}{t+1}, y)$.

If the uniform structure of Y is discrete, but the coarse structure of Y is strictly larger than the discrete one, then the above morphism f does not implement flasqueness of $\mathcal{O}(Y)$, because Condition 2 in Definition 3.8 is violated. ♦

Let X and Y be Γ-uniform bornological coarse spaces. Recall that $F_{\mathcal{T}}$ is the functor from Γ-uniform bornological coarse spaces to Γ-bornological coarse spaces which forgets the uniform structure.

Lemma 9.26. *If Y is discrete as a coarse space, then*

$$\mathcal{O}(X) \otimes F_{\mathcal{T}}(Y) \cong \mathcal{O}(X \otimes Y) \ .$$

Proof. Immediate from the definitions. □

Let Y be a Γ-uniform bornological coarse space with coarse and uniform structures $\mathcal{C}$, $\mathcal{T}$, and let A, B be Γ-invariant subsets of Y.

Lemma 9.27. *If (A, B) is an equivariant uniform (Definition 9.19) and coarsely excisive decomposition of Y, then*

$$([0,\infty)\times A, [0,\infty)\times B)$$

is a coarsely excisive pair on $\mathcal{O}(Y)$.

Proof. We have $([0,\infty)\times A)\cup([0,\infty)\times B) = [0,\infty)\times Y$.

Let s and U be as in Definition 9.19, let $\phi\colon \mathbb{N}\to\mathcal{P}(Y\times Y)^\Gamma$ be a $\mathcal{T}^\Gamma$-admissible function and let $\kappa\colon [0,\infty)\to[0,\infty)$ be monotoneously decreasing such that $\lim_{u\to\infty}\kappa(u)=0$. The pair $\psi := (\phi,\kappa)$ determines the invariant entourage

$$U_\psi := \{((a,x),(b,y))\in([0,\infty)\times Y)^{\times 2} \mid |a-b|\le\kappa(\max\{a,b\})\;\&\;(x,y)\in\phi(\lceil a\rceil)\cap\phi(\lceil b\rceil)\}\ .$$

For W in $\mathcal{C}^\Gamma$ and r in $(0,\infty)$ we consider the entourage $W_r := U_r\times W$ of $[0,\infty)_d\otimes Y$. The entourages of the form $U_\psi\cap W_r$ for all ψ as above, r in $(0,\infty)$ and W in $\mathcal{C}^\Gamma$ are cofinal in the hybrid coarse structure of $\mathcal{O}(Y)$.

We now fix ψ, W and r as above. We must show that there exist r' in $(0,\infty)$, W' in $\mathcal{C}^\Gamma$, and ψ' such that

$$(U_\psi\cap W_r)[[0,\infty)\times A]\cap(U_\psi\cap W_r)[[0,\infty)\times B]\subseteq(U_{\psi'}\cap W'_{r'})[[0,\infty)\times(A\cap B)]\ . \quad (9.5)$$

Using coarse excisiveness of (A,B) we can choose an invariant entourage W' of X such that $W[A]\cap W[B]\subseteq W'[A\cap B]$. We further set $r' := r$. Then

$$W_r[[0,\infty)\times A]\cap W_r[[0,\infty)\times B]\subseteq W'_r[[0,\infty)\times(A\cap B)]. \quad (9.6)$$

By $\mathcal{T}^\Gamma$-admissibility of ϕ there is u_0 in $\mathbb{N}$ such that $\phi(u)\subseteq U$ for all u in $\mathbb{N}$ with $u\ge u_0$. We define

$$\phi'\colon \mathbb{N}\to\mathcal{P}(Y\times Y)^\Gamma\ ,\quad \phi'(u) := \begin{cases} W' & u<u_0 \\ s(\phi(u)) & u\ge u_0\end{cases}$$

Then ϕ' is $\mathcal{T}^\Gamma$-admissible. We further set $\psi' := (\phi',\kappa)$. We claim that

$$(U_\psi\cap W_r)[[0,\infty)\times A]\cap(U_\psi\cap W_r)[[0,\infty)\times B]\subseteq U_{\psi'}[[0,\infty)\times(A\cap B)]\ . \quad (9.7)$$

Consider a point (u,z) in $(U_\psi\cap W_r)[[0,\infty)\times A]\cap(U_\psi\cap W_r)[[0,\infty)\times B]$. Then there exists (a,x) in $[0,\infty)\times A$ such that we have

$$|a-u|\le\kappa(\max\{a,u\})\quad\text{and}\quad(z,x)\in\phi(\lceil u\rceil)\cap\phi(\lceil a\rceil)\ ,$$

and there exist (b,y) in $[0,\infty)\times B$ such that

$$|b-u|\le\kappa(\max\{b,u\})\quad\text{and}\quad(z,y)\in\phi(\lceil u\rceil)\cap\phi(\lceil b\rceil)\ .$$

In particular, we have $z \in (\phi(\lceil u \rceil) \cap W)[A] \cap (\phi(\lceil u \rceil) \cap W)[B]$. If $u \geq u_0$, then we have $z \in \phi'(\lceil u \rceil)[A \cap B]$ by the corresponding property of s (see Definition 9.19). Let w in $A \cap B$ be such that we have $(z, w) \in \phi'(\lceil u \rceil)$. Then $((u, z), (u, w)) \in U_{\psi'}[[0, \infty) \times (A \cap B)]$. If $u < u_0$, then again $((u, z), (u, w)) \in U_{\psi'}[[0, \infty) \times (A \cap B)]$ because of the choice of W'.

The relations (9.6) and (9.7) together imply (9.5). □

Remark 9.28. In the above proof, in contrast to the general Decomposition Theorem 9.22 for hybrid structures, we do not use that Y is Hausdorff. ♦

9.5. The cone at infinity

In this section we will define the "cone at infinity" functor. It fits into the cone fiber sequence. We discuss invariance under coarsenings and calculate it for discrete spaces. Furthermore, we show that the "cone at infinity" is excisive and homotopy invariant.

If Y is a Γ-uniform bornological coarse space Y, then $\mathcal{O}(Y)$ has a canonical big family $\mathcal{Y}(Y)$ given by (9.4). Recall the notation (4.8).

Definition 9.29. We define the functor

$$\mathcal{O}^\infty \colon \Gamma\mathbf{UBC} \to \Gamma\mathbf{Sp}\mathcal{X}$$

by

$$\mathcal{O}^\infty(Y) := \mathrm{Yo}^s(\mathcal{O}(Y), \mathcal{Y}(Y)) \ . \quad ♦$$

Recall that $F_{\mathcal{T}}$ is the functor from Γ-uniform bornological coarse spaces to Γ-bornological coarse spaces which forgets the uniform structure. For n in $\mathbb{N}$ let $([0, n] \times Y)_{\mathcal{O}(Y)}$ denote the Γ-bornological coarse space given by the subset $[0, n] \times Y$ of $\mathcal{O}(Y)$ with the induced structures. The inclusion

$$F_{\mathcal{T}}(Y) \to ([0, n] \times Y)_{\mathcal{O}(Y)} \ , \quad y \mapsto (0, y)$$

is an equivalence of Γ-bornological coarse spaces for every integer n. Hence we have an equivalence

$$\mathrm{Yo}^s(Y) \simeq \mathrm{Yo}^s(\mathcal{Y}(Y)_{\mathcal{O}(Y)}) \ .$$

Corollary 9.30. *For every Γ-uniform bornological coarse space Y we have a natural fiber sequence*

$$\mathrm{Yo}^s(F_{\mathcal{T}}(Y)) \to \mathrm{Yo}^s(\mathcal{O}(Y)) \to \mathcal{O}^\infty(Y) \xrightarrow{\partial} \Sigma \mathrm{Yo}^s(F_{\mathcal{T}}(Y)) \qquad (9.8)$$

in $\Gamma\mathbf{Sp}\mathcal{X}$.

Proof. The fiber sequence is associated to the pair $(\mathcal{O}(Y), \mathcal{Y}(Y))$, see Corollary 4.11.1. □

Let Y be a Γ-uniform bornological coarse space. Then we consider the Γ-bornological coarse space $\mathcal{O}(Y)_-$ obtained from the Γ-uniform bornological coarse space $\mathbb{R}\otimes Y$ by taking the hybrid coarse structure 9.15 associated to the big family $((-\infty,n]\times Y)_{n\in\mathbb{N}}$. Note that the subset $[0,\infty)\times Y$ of $\mathcal{O}(Y)_-$ with the induced structures is the cone $\mathcal{O}(Y)$. We then have maps of Γ-bornological coarse spaces

$$F_{\mathcal{T}}(Y)\xrightarrow{i}\mathcal{O}(Y)\xrightarrow{j}\mathcal{O}(Y)_-\xrightarrow{d}F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y)\ .$$

The first two maps i and j are the inclusions, and the last map d is given by the identity of the underlying sets.

Proposition 9.31. *We have a commutative diagram in* $\Gamma\mathbf{Sp}\mathcal{X}$

$$\begin{array}{ccccc}\mathrm{Yo}^s(\mathcal{O}(Y)) & \xrightarrow{j} & \mathrm{Yo}^s(\mathcal{O}(Y)_-) & \xrightarrow{d} & \mathrm{Yo}^s(F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y)) \\ \| & & \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Yo}^s(\mathcal{O}(Y)) & \longrightarrow & \mathcal{O}^\infty(Y) & \xrightarrow{\partial} & \Sigma\mathrm{Yo}^s(F_{\mathcal{T}}(Y))\end{array}\qquad(9.9)$$

Remark 9.32. This proposition identifies a segment of the cone sequence (9.8) with a sequence represented by maps between Γ-bornological coarse spaces. It in particular shows that the cone $\mathcal{O}^\infty(Y)$ is represented by the Γ-bornological coarse space $\mathcal{O}(Y)_-$. ♦

Proof of Proposition 9.31. We consider the diagram of motivic coarse spectra

$$\begin{array}{ccccc}\mathrm{Yo}^s(F_{\mathcal{T}}(Y)) & \xrightarrow{i} & \mathrm{Yo}^s(\mathcal{O}(Y)) & \longrightarrow & \mathrm{Yo}^s(F_{\mathcal{T}}([0,\infty)\otimes Y)) \\ \downarrow & & \downarrow j & & \downarrow \\ \mathrm{Yo}^s(F_{\mathcal{T}}((-\infty,0]\otimes Y)) & \longrightarrow & \mathrm{Yo}^s(\mathcal{O}(Y)_-) & \xrightarrow{d} & \mathrm{Yo}^s(F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y))\end{array}\qquad(9.10)$$

The left and right vertical and the lower left horizontal map are given by the canonical inclusions. The upper right horizontal map is the identity map of the underlying sets. This diagram commutes since it is obtained by applying Yo^s to a commuting diagram of bornological coarse spaces.

The left square in (9.10) is cocartesian since the pair $((-\infty,0]\times Y,\mathcal{O}(Y))$ in $\mathcal{O}(Y)_-$ is coarsely excisive. Furthermore, since $((-\infty,0]\times Y,[0,\infty)\times Y)$ is coarsely excisive in $F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y)$ the outer square is cocartesian. It follows that the right square is cocartesian.

Since the upper right and the lower left corners in (9.10) are trivial by flasqueness of the rays the diagram is equivalent to the composition

$$\begin{array}{ccccc}\mathrm{Yo}^s(F_{\mathcal{T}}(Y)) & \xrightarrow{i} & \mathrm{Yo}^s(\mathcal{O}(Y)) & \longrightarrow & 0 \\ \downarrow & & \downarrow j & & \downarrow \\ 0 & \longrightarrow & \mathrm{Yo}^s(\mathcal{O}(Y)_-) & \xrightarrow{d} & \mathrm{Yo}^s(F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y))\end{array}$$

of cocartesian squares. Note that $\mathcal{O}^\infty(Y)$ is defined as the cofiber of the left upper horizontal map i. Hence the left square yields the middle vertical equivalence in (9.9).

The outer square yields the equivalence $\mathrm{Yo}^s(F_{\mathcal{T}}(\mathbb{R}_{du}\otimes Y))\simeq\Sigma\mathrm{Yo}^s(Y)$. The right square then identifies d with the boundary map ∂ of the cone sequence. □

Next we will observe that $\mathcal{O}^\infty(Y)$ is essentially independent of the coarse structure on Y. Let Y be a Γ-uniform bornological coarse space with coarse structure $\mathcal{C}$, bornology $\mathcal{B}$ and uniform structure $\mathcal{T}$. Let $\mathcal{C}'$ be a Γ-coarse structure on Y such that $\mathcal{C} \subseteq \mathcal{C}'$ and $\mathcal{C}'$ is still compatible with the bornology. We write Y' for the Γ-uniform bornological coarse space obtained from Y by replacing the coarse structure $\mathcal{C}$ by $\mathcal{C}'$. Then the identity map of the underlying sets is a morphism $Y \to Y'$ of Γ-uniform bornological coarse spaces. We will call such a morphism a *coarsening.*

Let Y be a Γ-uniform bornological coarse space.

Proposition 9.33. *If $Y \to Y'$ is a coarsening, then the induced map*

$$\mathcal{O}^\infty(Y) \to \mathcal{O}^\infty(Y')$$

is an equivalence.

Proof. By definition of $\mathcal{O}^\infty(Y)$ we have

$$\mathcal{O}^\infty(Y) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{Yo}^s(([0,\infty)_{du} \otimes Y)_h, [0,n] \times Y) \ , \tag{9.11}$$

where the subsets $[0,n] \times Y$ of $([0,\infty)_{du} \otimes Y)_h$ have the induced bornological coarse structure. By u-continuity of Yo^s we have

$$\mathcal{O}^\infty(Y) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \operatorname*{colim}_{U} \mathrm{Yo}^s(([0,\infty)_{du} \otimes Y)_U, ([0,n] \times Y)_U) \ , \tag{9.12}$$

where U runs over the Γ-invariant entourages of $([0,\infty)_{du} \otimes Y)_h$. Here for a subset X' of X the notation X'_U denotes the set X' with the structures induced from X_U, i.e., X'_U is a short-hand notation for X'_{X_U}. For every integer n, there is a cofinal set of entourages U such that the pair

$$([0,n] \times Y, [n,\infty) \times Y)$$

is coarsely excisive on $([0,\infty) \times Y)_U$. In fact, this is a coarsely excisive pair for any entourage U that allows propagation from $\{n\} \times Y$ in the direction of the ray. Since the Yoneda functor Yo^s is excisive we get the equivalence

$$\mathcal{O}^\infty(Y) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \operatorname*{colim}_{U} \mathrm{Yo}^s(([n,\infty) \times Y)_U, (\{n\} \times Y)_U) \ . \tag{9.13}$$

In general, for a Γ-bornological coarse space X with coarse structure $\mathcal{C}$ and an invariant subset Z we have an equivalence

$$\operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(Z_U) \simeq \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(Z_{(Z\times Z)\cap U}) \tag{9.14}$$

(here we must not omit the colimit). We insert this into (9.13) and get

$$\mathcal{O}^\infty(Y) \simeq \operatorname*{colim}_{n\in\mathbb{N}} \operatorname*{colim}_{U} \mathrm{Yo}^s(([n,\infty) \times Y)_{U_n}, (\{n\} \times Y)_{U_n}) \ , \tag{9.15}$$

where we use the abbreviation $U_n := (([n,\infty)\times Y)\times([n,\infty)\times Y))\cap U$. We can now interchange the order of the colimits and get

$$\mathcal{O}^\infty(Y)\simeq \operatorname*{colim}_{U}\operatorname*{colim}_{n\in\mathbb{N}}\mathrm{Yo}^s(([n,\infty)\times Y)_{U_n},(\{n\}\times Y)_{U_n})\ . \tag{9.16}$$

We argue now that in this formula we can replace the colimit over the invariant entourages U of $([0,\infty)_{du}\otimes Y)_h$ by the colimit over all invariant entourages U' of $([0,\infty)_{du}\otimes Y')_h$. To this end we consider the generating entourages $U_\psi\cap W'$ of $\mathcal{O}(Y')$ (see the proof of Lemma 9.27 for notation). Since $\mathcal{T}$ and $\mathcal{C}$ are compatible, there exists an integer n_0 sufficiently large such that $\phi(n_0)\in\mathcal{C}$. But then, since ϕ is monotoneous, we have $\phi(x)\in\mathcal{C}$ for every x in $[n_0,\infty)$. We conclude that for every integer n with $n\geq n_0$ we have

$$(U_\psi\cap W')\cap\big(([n,\infty)\times Y)\times([n,\infty)\times Y)\big)\subseteq U_\psi\cap(W'\cap\phi(n))$$

and $W'\cap\phi(n)\in\mathcal{C}$.

This gives

$$\mathcal{O}^\infty(Y)\simeq \operatorname*{colim}_{U'}\operatorname*{colim}_{n\in\mathbb{N}}\mathrm{Yo}^s(([n,\infty)\times Y)_{U'_n},(\{n\}\times Y)_{U'_n})\ , \tag{9.17}$$

where now U' runs over the invariant entourages of $([0,\infty)_{du}\otimes Y')_h$. Going the argument above backwards with Y replaced by Y' we end up with

$$\mathcal{O}^\infty(Y)\simeq \operatorname*{colim}_{n\in\mathbb{N}}\mathrm{Yo}^s(([0,\infty)_{du}\otimes Y')_h,[0,n]\times Y)\simeq\mathcal{O}^\infty(Y') \tag{9.18}$$

and this completes the proof. □

In the following proposition we use the invariance under coarsening in order to calculate the value of the $\mathcal{O}^\infty$-functor on Γ-uniform bornological coarse spaces whose underlying uniform structure is discrete.

For a Γ-uniform bornological coarse space X which is discrete as a uniform space let X_{disc} denote the Γ-uniform bornological coarse space obtained by replacing the coarse structure by the discrete coarse structure.

Remark 9.34. If X is not discrete as a uniform space, then the discrete coarse structure is not compatible with the uniform structure. ♦

Let X be a Γ-uniform bornological coarse space.

Proposition 9.35. *If X is discrete as a uniform space, then we have an equivalence*

$$\mathcal{O}^\infty(X)\simeq\Sigma\mathrm{Yo}^s(F_{\mathcal{T}}(X_{disc}))$$

in $\Gamma\mathbf{Sp}\mathcal{X}$.

Proof. Since $X_{disc} \to X$ is a coarsening, by Proposition 9.33 we have an equivalence

$$\mathcal{O}^\infty(X_{disc}) \xrightarrow{\simeq} \mathcal{O}^\infty(X) \ .$$

By Example 9.25 we know that $\mathcal{O}(X_{disc})$ is flasque and hence $\mathrm{Yo}^s(\mathcal{O}(X_{disc})) \simeq 0$. The fiber sequence obtained in Corollary 9.30 yields an equivalence $\mathcal{O}^\infty(X_{disc}) \simeq \Sigma\mathrm{Yo}^s(F_{\mathcal{T}}(X_{disc}))$ as desired. □

Next we discuss excision and homotopy invariance for $\mathcal{O}^\infty$.

Let Y be a Γ-uniform bornological coarse space and A, B be Γ-invariant subsets of Y.

Corollary 9.36. *If (A, B) is an equivariant uniformly and coarsely excisive decomposition, then the following square in $\Gamma\mathbf{Sp}\mathcal{X}$ is cocartesian:*

$$\begin{array}{ccc} \mathcal{O}^\infty(A\cap B) & \longrightarrow & \mathcal{O}^\infty(B) \\ \downarrow & & \downarrow \\ \mathcal{O}^\infty(A) & \longrightarrow & \mathcal{O}^\infty(Y) \end{array} \tag{9.19}$$

Proof. Since (A, B) is coarsely excisive the square

$$\begin{array}{ccc} \mathrm{Yo}^s(\mathcal{F}_{\mathcal{T}}(A)\cap \mathcal{F}_{\mathcal{T}}(B)) & \longrightarrow & \mathrm{Yo}^s(\mathcal{F}_{\mathcal{T}}(B)) \\ \downarrow & & \downarrow \\ \mathrm{Yo}^s(\mathcal{F}_{\mathcal{T}}(A)) & \longrightarrow & \mathrm{Yo}^s(\mathcal{F}_{\mathcal{T}}(Y)) \end{array}$$

is cocartesian. Furthermore, by Lemma 9.27 the square

$$\begin{array}{ccc} \mathcal{O}(A\cap B) & \longrightarrow & \mathcal{O}(B) \\ \downarrow & & \downarrow \\ \mathcal{O}(A) & \longrightarrow & \mathcal{O}(Y) \end{array}$$

is cocartesian. Now it just remains to use the cone sequence (9.8) in order to conclude that the square (9.19) is cocartesian. □

Remark 9.37. If we assume that the underlying uniform space of Y is Hausdorff, then we could drop the assumption that (A, B) is coarsely excisive. In this case it will follow from the Decomposition Theorem 9.22 applied to the equivariant uniform decomposition $([0,\infty)\times A, [0,\infty)\times B)$ of $[0,\infty)_{du}\otimes Y$ that (9.19) is cocartesian. ♦

Corollary 9.38. *The functor $\mathcal{O}^\infty\colon \Gamma\mathbf{UBC} \to \Gamma\mathbf{Sp}\mathcal{X}$ is homotopy invariant.*

Proof. Let Y be a Γ-uniform bornological coarse space. Let $h\colon [0,1]_{du}\otimes Y \to Y$ be the projection. By functoriality of the cone we get the morphism

$$\mathcal{O}(h)\colon \mathcal{O}([0,1]_{du}\otimes Y) \to \mathcal{O}(Y) \ .$$

We now observe that

$$\mathcal{O}([0,1]_{du}\otimes Y)\cong([0,\infty)_{du}\otimes[0,1]_{du}\otimes Y)_h\cong([0,1]_{du}\otimes[0,\infty)_{du}\otimes Y)_h\ .$$

By the Homotopy Theorem 9.23 we get an equivalence

$$\mathrm{Yo}^s(\mathcal{O}([0,1]_{du}\otimes Y))\simeq\mathrm{Yo}^s(([0,1]_{du}\otimes[0,\infty)_d\otimes Y)_h)\simeq\mathrm{Yo}^s(([0,\infty)_{du}\otimes Y)_h)\simeq\mathrm{Yo}^s(\mathcal{O}(Y))\ .$$

Since the projections $[0,1]_{du}\otimes[0,n]_{du}\otimes Y\to[0,n]_{du}\otimes Y$ induce equivalences of underlying Γ-bornological coarse spaces we conclude that the projection h induces an equivalence

$$\mathcal{O}^\infty([0,1]_{du}\otimes Y)\to\mathcal{O}^\infty(Y)$$

in $\Gamma\mathbf{Sp}\mathcal{X}$. □

10. Topological assembly maps

The overall theme of this section is the interplay between equariant coarse homology theories and equivariant homology theories.

We start in Section 10.1 with the general discussion of equivariant homology theories, and then in Section 10.2 we modify the "cone at infinity" functor $\mathcal{O}^\infty$ to get an equivariant $\Gamma\mathbf{Sp}\mathcal{X}$-valued homology theory

$$\mathcal{O}^\infty_{\mathrm{hlg}}\colon\Gamma\mathbf{Top}\to\Gamma\mathbf{Sp}\mathcal{X}\ .$$

We introduce classifying spaces $E_{\mathcal{F}}\Gamma$ for families $\mathcal{F}$ of subgroups and define the motivic assembly map

$$\alpha_{\mathcal{F}}\colon\mathcal{O}^\infty_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma)\to\mathcal{O}^\infty_{\mathrm{hlg}}(*)$$

in Section 10.3. Using the cone sequence we define further versions $\alpha_{X,Q}$ of the motivic assembly map with twist Q and discuss some instances where it is an equivalence. Finally, in Section 10.4 we use the functor $\mathcal{O}^\infty_{\mathrm{hlg}}$ in order to derive equivariant homology theories from coarse homology theories.

10.1. Equivariant homology theories

In this section we will recall the notion of a (strong) equivariant homology theory on Γ-topological spaces.

Our basic category of topological spaces is the convenient category **Top** of compactly generated weakly Hausdorff spaces. A map between topological spaces is a weak equivalence if it induces an isomorphism between the sets of connected components and isomorphisms of homotopy groups in all positive degrees and for all choices of base points. The ∞-category obtained from (the nerve of) **Top** by inverting these weak equivalences is a model for the presentable ∞-category **Spc** of spaces. In particular, we have the localization functor

$$\kappa\colon \mathbf{Top} \to \mathbf{Spc}\ . \tag{10.1}$$

A Γ-topological space is a topological space with an action of the group Γ by automorphisms. We denote the category of Γ-topological spaces and equivariant continuous maps by $\Gamma\mathbf{Top}$. A weak equivalence between Γ-topological spaces is a Γ-equivariant map which induces weak equivalences on fixed-point spaces for all subgroups of Γ. We will model this homotopy theory by presheaves on the orbit category of Γ.

The orbit category $\mathbf{Orb}(\Gamma)$ of Γ is the category of transitive Γ-sets and equivariant maps. A Γ-set can naturally be considered as a discrete Γ-topological space. In this way we get a fully faithful functor $\mathbf{Orb}(\Gamma) \to \Gamma\mathbf{Top}$. For a transitive Γ-set S and Γ-topological space X we consider the topological space $\mathrm{Map}_{\Gamma\mathbf{Top}}(S, X)$ of equivariant maps from S to X.

Remark 10.1. We consider a transitive Γ-set S. If we fix a base point s in S and denote the stabilizer of s by Γ_s, then we get an identification $\mathrm{Map}(S, X) \simeq X^{\Gamma_s}$, where X^{Γ_s} is the subspace of Γ_s-fixed points. ♦

We define a functor

$$\ell\colon \Gamma\mathbf{Top} \to \mathbf{PSh}(\mathbf{Orb}(\Gamma)) \ \text{ by } \ \ell(X)(S) := \kappa(\mathrm{Map}_{\Gamma\mathbf{Top}}(S, X)) \ \text{ for } \ S \in \mathbf{Orb}(\Gamma)\ . \tag{10.2}$$

Remark 10.2. A map between topological spaces is a weak equivalence if and only if its image under κ is an equivalence. Consequently, a map between Γ-topological spaces is a weak equivalence if and only if its image under ℓ is an equivalence. By Elmendorf's theorem [May96, Thm. VI.6.3] (which boils down to the assertion that ℓ is essentially surjective) the functor ℓ induces an equivalence

$$\Gamma\mathbf{Top}[W^{-1}] \xrightarrow{\simeq} \mathbf{PSh}(\mathbf{Orb}(\Gamma))\ , \tag{10.3}$$

where $\Gamma\mathbf{Top}[W^{-1}]$ denotes the ∞-category obtained from $\Gamma\mathbf{Top}$ by inverting the weak equivalences. Occasionally we will use the fact that a weak equivalence in $\Gamma\mathbf{Top}$ between Γ-CW-complexes is actually a homotopy equivalence in $\Gamma\mathbf{Top}$. ♦

Let $\mathbf{C}$ be a cocomplete ∞-category. By the universal property of the presheaf category we have an equivalence of ∞-categories

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\mathbf{Orb}(\Gamma)), \mathbf{C}) \simeq \mathbf{Fun}(\mathbf{Orb}(\Gamma), \mathbf{C})\ , \tag{10.4}$$

where the superscript colim stands for colimit-preserving. The localization functor (10.2) induces a faithful restriction functor

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\mathbf{Orb}(\Gamma), \mathbf{C}) \to \mathbf{Fun}(\Gamma\mathbf{Top}, \mathbf{C})\ . \tag{10.5}$$

From now on we assume that $\mathbf{C}$ is cocomplete and stable.

Our preferred definition of the notion of an equivariant $\mathbf{C}$-valued homology theory would be the following.

Let $E\colon \Gamma\mathbf{Top} \to \mathbf{C}$ be a functor.

Definition 10.3. E is called a *strong equivariant* $\mathbf{C}$*-valued homology theory* if it is in the essential image of (10.5). ♦

Assume that we are given a functor E as above. If it sends weak equivalences to equivalences, then, using the equivalence (10.3), it extends essentially uniquely to a functor $\mathbf{PSh}(\mathbf{Orb}(\Gamma)) \to \mathbf{C}$. The functor E is an equivariant $\mathbf{C}$-valued homology theory if this extension preserves colimits. In general it seems to be complicated to check these conditions if E is given by some geometric construction. For this reason we add the adjective *strong* in order to distinguish this notion from the Definition 10.4 of an equivariant $\mathbf{C}$-valued homology theory that we actually work with.

Let $E\colon \Gamma\mathbf{Top} \to \mathbf{C}$ be a functor. We extend E to pairs (X, A) of Γ-topological spaces and subspaces by setting

$$E(X, A) := \mathrm{Cofib}\big(E(*) \to E(X \cup_A \mathrm{Cone}(A))\big) ,$$

where $\mathrm{Cone}(A)$ denotes the cone over A and $*$ is the base point of the cone.

Definition 10.4. The functor E is called an *equivariant* $\mathbf{C}$*-valued homology theory* if it has the following properties:

1. (Homotopy invariance) For every Γ-topological space X the projection induces an equivalence
$$E([0, 1] \times X) \to E(X) .$$
2. (Excision) If (X, A) is a pair of Γ-topological spaces and U is an invariant open subset of A such that $\overline{U}$ is contained in the interior of A, then the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces an equivalence
$$E(X \setminus U, A \setminus U) \to E(X, A) .$$
3. (Wedge axiom) For every family $(X_i)_{i\in I}$ of Γ-topological spaces the canonical map
$$\bigoplus_{i\in I} E(X_i) \xrightarrow{\simeq} E\Big(\coprod_{i\in I} X_i\Big)$$
is an equivalence. ♦

Remark 10.5. In order to verify that E satisfies excision one must show that E sends the square

$$\begin{array}{ccc} \mathrm{Cone}(A \setminus U) & \longrightarrow & (X \setminus U) \cup_{A\setminus U} \mathrm{Cone}(A \setminus U) \\ \downarrow & & \downarrow \\ \mathrm{Cone}(A) & \longrightarrow & X \cup_A \mathrm{Cone}(A) \end{array}$$

to a push-out square. For homotopy invariant functors E this is equivalent to the property that E sends the right vertical map in the square above to an equivalence.

For homotopy invariant functors E excision follows from the stronger condition of closed excision, i.e., that for every decomposition (A, B) of X into closed invariant subsets the diagram

$$\begin{array}{ccc} E(A \cap B) & \longrightarrow & E(A) \\ \downarrow & & \downarrow \\ E(B) & \longrightarrow & E(X) \end{array}$$

is a push-out square. This can be seen as follows. Assume that E is homotopy invariant and satisfies closed excision. Let X, A, and U be as above. Then we have a closed decomposition

$$(X \setminus U \cup_{A \setminus U} \mathrm{Cone}(A \setminus U)\ ,\ \mathrm{Cone}(\overline{U}))$$

of $X \cup_A \mathrm{Cone}(A)$ with intersection $\mathrm{Cone}(\overline{U} \setminus U)$. By closed excision we get the cocartesian square

$$\begin{array}{ccc} E(\mathrm{Cone}(\overline{U} \setminus U)) & \longrightarrow & E(X \setminus U \cup_{A \setminus U} \mathrm{Cone}(A \setminus U)) \\ \downarrow & & \downarrow \\ E(\mathrm{Cone}(\overline{U})) & \longrightarrow & E(X \cup_A \mathrm{Cone}(A)) \end{array}$$

Since E is homotopy invariant it sends the left vertical map to an equivalence since cones are contractible. Consequently, the right vertical map is an equivalence, too. ♦

Remark 10.6. Let E be an equivariant $\mathbf{C}$-valued homology theory. In general we can not expect that it factorizes over the localization (10.2).

Using the equivalence (10.4) the restriction of E to the orbit category gives rise to a strong equivariant $\mathbf{C}$-valued homology theory $E^{\%}$ which comes with a natural transformation $E^{\%} \to E$. Using the theory developed by Davis–Lück [DL98, Sec. 3] one can check that

$$E^{\%}(X) \xrightarrow{\simeq} E(X)$$

for all Γ-CW-complexes X. ♦

10.2. The cone as an equivariant homology theory

In Section 9.5 we have seen that the "cone at infinity" functor $\mathcal{O}^\infty$ is a homotopy invariant and excisive functor from $\Gamma\mathbf{UBC}$ to $\mathbf{Sp}\mathcal{X}$. In this section we modify this functor in order to get an equivariant homology theory $\mathcal{O}^\infty_{\mathrm{hlg}} \colon \Gamma\mathbf{Top} \to \Gamma\mathbf{Sp}\mathcal{X}$.

If X is a Γ-uniform space, then we can consider X as a Γ-uniform bornological coarse space $X_{max,max}$ by equipping the uniform space X in addition with the maximal coarse structure and the maximal bornology. In this way we get a functor

$$\mathcal{M} \colon \Gamma\mathbf{U} \to \Gamma\mathbf{UBC}\ , \quad X \mapsto \mathcal{M}(X) := X_{max,max}\ . \tag{10.6}$$

Let Y be a Γ-set and Q be a Γ-bornological coarse space. The projection $Y_{max,max} \to *$ induces a morphism

$$\mathrm{Yo}^s(Y_{max,max}) \otimes \mathrm{Yo}^s(Q) \to \mathrm{Yo}^s(Q) \tag{10.7}$$

in $\Gamma\mathbf{Sp}\mathcal{X}$.

Lemma 10.7. *If the underlying set of Q is a free Γ-set, then* (10.7) *is an equivalence.*

Proof. Since Q is a free Γ-set we can choose a Γ-equivariant map of sets $\kappa\colon Q \to Y$. The map $(\kappa, \mathrm{id})\colon Q \to Y_{max,max} \otimes Q$ is then a morphism of Γ-bornological coarse spaces. It is an inverse to the projection up to equivalence. Hence the projection is an equivalence of Γ-bornological coarse spaces and therefore $\mathrm{Yo}^s(Y_{max,max} \otimes Q) \to \mathrm{Yo}^s(Q)$ is an equivalence. We now use that Yo^s is a symmetric monoidal functor (see Section 4.3) in order to rewrite the domain of the morphism as in (10.7). □

The functor

$$\mathcal{O}^\infty \circ \mathcal{M}\colon \Gamma\mathbf{U} \to \Gamma\mathbf{Sp}\mathcal{X} \tag{10.8}$$

behaves very much like an equivariant homology theory. Of course, it is not defined on $\Gamma\mathbf{Top}$, but on $\Gamma\mathbf{U}$. On the other hand it is homotopy invariant by Corollary 9.38, and sends equivariant uniform decompositions to push-outs by Corollary 9.36. The drawback is that it in general only preserves finite coproducts.

In order to improve these points we define a new functor

$$\mathcal{O}^\infty_{\mathrm{hlg}}\colon \Gamma\mathbf{Top} \to \Gamma\mathbf{Sp}\mathcal{X}$$

by first restricting $\mathcal{O}^\infty \circ \mathcal{M}$ to the subcategory of Γ-compact Γ-metrizable spaces and then left-Kan extending the result to $\Gamma\mathbf{Top}$. In the following we describe the details.

Let X be a Γ-toplogical space. Recall that X is Γ-*compact* if there exists a compact subset K of X such that $\Gamma K = X$, and that X is Γ-*metrizable* if there exists a Γ-invariant metric on X which induces the topology of X.

We denote by $\Gamma\mathbf{Top}^{cm}$ the full sub-category of $\Gamma\mathbf{Top}$ spanned by all the Γ-compact and Γ-metrizable Γ-topological spaces. Associated to any X in $\Gamma\mathbf{Top}^{cm}$ we define

$$\mathcal{N}(X) := \{N \subseteq X \times X \mid N \text{ contains a } \Gamma\text{-invariant neighborhood of the diagonal}\}\ .$$

Furthermore we set $\mathcal{U}(X) := (X, \mathcal{N}(X))$. For a Γ-invariant metric on X which is compatible with the topology we let $\mathcal{T}_d$ denote the associated metric uniform structure on X.

Lemma 10.8. *Assume that X is in $\Gamma\mathbf{Top}^{cm}$.*

1. *$\mathcal{N}(X)$ is a Γ-uniform structure.*
2. *If d is a Γ-invariant metric compatible with the topology, then $\mathcal{N}(X) = \mathcal{T}_d$.*
3. *The assignment $X \mapsto \mathcal{U}(X)$ defines a functor $\mathcal{U}\colon \Gamma\mathbf{Top}^{cm} \to \Gamma\mathbf{U}$.*

Proof. By assumption we can choose a Γ-invariant metric d which is compatible with the topology. We claim that for every Γ-invariant neighborhood N of the diagonal there exists some T in $\mathcal{T}_d$ such that $T \subseteq N$. The case $N = X \times X$ is trivial and we assume that N is a proper subset. We define a function

$$b\colon X \to (0,\infty), \quad x \mapsto \sup\{\epsilon \in (0,\infty) \mid B_\epsilon(x) \times B_\epsilon(x) \subseteq N\} ,$$

where $B_\epsilon(x)$ denotes the open ball of radius ϵ around x (with respect to d). Since d is compatible with the topology the argument of the sup is non-empty for every x in X and the value $b(x)$ is indeed positive. Furthermore, the supremum is attained. Moreover, the supremun can not be infinite since we assume that N is a proper subset. Finally, we observe that b is Γ-equivariant with respect to the trivial Γ-action on $(0,\infty)$.

We now show that the function b is 1-Lipschitz. Let x, y be two points in X and let δ be their distance. If both $b(x)$ and $b(y)$ are less than δ, then so is their distance. Therefore, we can assume that $b(x) \geq b(y)$ and $b(x) \geq \delta$. By the triangle inequality we have

$$B_{b(x)-\delta}(y) \times B_{b(x)-\delta}(y) \subseteq B_{b(x)}(x) \times B_{b(x)}(x) \subseteq N$$

This implies $b(x) \geq b(y) \geq b(x) - \delta$. In particular, $|b(x) - b(y)| \leq \delta$ and b is indeed 1-Lipschitz and thus continuous.

By assumption we can choose a compact subset K of X such that $\Gamma K = X$. Since b is continuous, the restriction $b|_K\colon K \to (0,\infty)$ attains a positive minimal value, which we denote by ϵ_0. By Γ-equivariance of b, we then have $b(x) \geq \epsilon_0$ for all x in X. We conclude that the metric uniform entourage $\{(x,y) \in X \times X \mid d(x,y) < \epsilon_0\}$ is contained in N.

Note that every metric uniform entourage is a neighbourhood of the diagonal since d is compatible with the topology. Since both $\mathcal{N}(X)$ and $\mathcal{T}_d$ are closed under taking supersets, we have shown that $\mathcal{N}(X) = \mathcal{T}_d$. This shows the first two assertions of the Lemma.

We now show the third assertion. Let $f\colon X \to X'$ be an equivariant continuous map between two Γ-compact and Γ-metrizable Γ-topological spaces. We must show that $f\colon (X,\mathcal{N}(X)) \to (X',\mathcal{N}(X'))$ is uniformly continuous. Let V' belong to $\mathcal{N}(X')$. Then V' contains a Γ-invariant neighbourhood of the diagonal U'. Since f is equivariant and continuous, $(f^{-1} \times f^{-1})(U)$ is a Γ-invariant neighbourhood of the diagonal of X contained in $(f^{-1} \times f^{-1})(V')$. Consequently, $(f^{-1} \times f^{-1})(V') \in \mathcal{N}(X)$. This implies that $f\colon (X,\mathcal{N}(X)) \to (X',\mathcal{N}(X'))$ is uniformly continuous. □

Let X be a Γ-topological space, and let A and B be closed Γ-invariant subsets of X such that $A \cup B = X$.

Lemma 10.9. *If X is Γ-compact and Γ-metrizable, then (A, B) is an equivariant uniform decomposition of $\mathcal{U}(X)$ (Definition 9.19).*

Proof. We choose a Γ-invariant metric d on X which is compatible with the topology. By Lemma 10.8, $(X, \mathcal{T}_d) = \mathcal{U}(X)$.

For a subset Z of Y and e in $(0,\infty)$ we consider the e-thickening

$$U_e[Z] := \{y \in Y \mid d(y, Z) \leq e\}$$

(note that U_e is defined in (9.1) with a $\leq$-relation, too) of Z. If Z is invariant, then $U_e[Z]$ is again invariant.

If $A \cap B = \emptyset$, then by Γ-compactness of Y the subsets A and B are uniformly separated and (A, B) is an equivariant uniform decomposition.

Assume now that $A \cap B \neq \emptyset$. It suffices to define a monotoneous function $s\colon (0,\infty) \to (0,\infty)$ such that:

1. $\lim_{e \to 0} s(e) = 0$.
2. For all e in $(0,\infty)$ we have
$$U_e[A] \cap U_e[B] \subseteq U_{s(e)}[A \cap B] \ .$$

We define a function $s\colon (0,\infty) \to [0,\infty]$ by

$$s(e) := \inf\{e' \in (0,\infty) \mid U_e[A] \cap U_e[B] \subseteq U_{e'}[A \cap B]\} \ .$$

By construction, this function is monotoneous. Since $A \cap B \neq 0$, and since by Γ-compactness of Y there exists R in $(0,\infty)$ such that $U_R[A \cap B] = Y$, the function s is finite. Condition 2 follows from this observation.

We claim that

$$A \cap B = \bigcap_{e>0} U_e[A] \cap U_e[B] \ . \tag{10.9}$$

It is clear that

$$A \cap B \subseteq \bigcap_{e>0} U_e[A] \cap U_e[B] \ .$$

On the other hand, assume that $y \in Y \setminus (A \cap B)$. Without loss of generality (interchange the roles of A and B, if necessary) we can assume that $y \in Y \setminus A$. Since A is closed there exists e in $(0,\infty)$ such that $y \notin U_e[A]$. Hence $y \in Y \setminus \bigcap_{e>0} U_e[A] \cap U_e[B]$. This shows the opposite inclusion

$$\bigcap_{e>0} U_e[A] \cap U_e[B] \subseteq A \cap B.$$

We now show Condition 1. Assume the contrary. Then there exists ϵ in $(0,\infty)$ such that $s(e) \geq \epsilon > 0$ for all e in $(0,\infty)$. For every integer n there exists a point y_n in $U_{1/n}[A] \cap U_{1/n}[B]$ such that $y_n \notin U_\epsilon[A \cap B]$. We can assume (after replacing y_n by $\gamma_n y_n$ for suitable elements γ_n of Γ) by Γ-compactness that $y_n \to y$ for $n \to \infty$. Then $y \notin U_{\epsilon/2}[A \cap B]$ but $y \in A \cap B$ by (10.9). This is a contradiction, and so we have verified Condition 1. $\square$

Definition 10.10. We define the functor

$$\mathcal{O}^\infty_{\mathrm{hlg}} \colon \Gamma\mathbf{Top} \to \Gamma\mathbf{Sp}\mathcal{X}$$

as the left Kan extension

$$\begin{array}{ccc} \Gamma\mathbf{Top}^{cm} & \xrightarrow{\mathcal{O}^\infty \circ \mathcal{M} \circ \mathcal{U}} & \Gamma\mathbf{Sp}\mathcal{X} \\ \big\downarrow & \overset{\mathcal{O}^\infty_{\mathrm{hlg}}}{\nearrow} & \\ \Gamma\mathbf{Top} & & \end{array}$$

of $\mathcal{O}^\infty \circ \mathcal{M} \circ \mathcal{U}$ along the fully faithful functor $\Gamma\mathbf{Top}^{cm} \hookrightarrow \Gamma\mathbf{Top}$. ♦

Proposition 10.11. $\mathcal{O}^\infty_{\mathrm{hlg}}$ *is an equivariant* $\Gamma\mathbf{Sp}\mathcal{X}$*-valued homology theory.*

Proof. We first note that the objectwise formula for the left Kan-extension gives

$$\mathcal{O}^\infty_{\mathrm{hlg}}(X) \simeq \operatorname*{colim}_{(Y \to X) \in \Gamma\mathbf{Top}^{cm}/X} \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(Y))) \ . \tag{10.10}$$

Let Y be a Γ-compact and Γ-metrizable Γ-topological space and $(\phi, f) \colon Y \to [0,1] \times X$ be an equivariant map. Then we have the factorization

$$(\phi, f) \colon Y \xrightarrow{(\phi, \mathrm{id}_Y)} [0,1] \times Y \xrightarrow{(\mathrm{id}_{[0,1]}, f)} [0,1] \times X \ .$$

Note that $[0,1] \times Y$ is again a Γ-compact and Γ-metrizable Γ-topological space. This shows that the category of maps of the form $(\mathrm{id}_{[0,1]}, f) \colon [0,1] \times Y \to [0,1] \times X$ for morphisms of Γ-topological spaces $f \colon Y \to X$ with Y in $\Gamma\mathbf{Top}^{cm}$ is cofinal in $\Gamma\mathbf{Top}^{cm}/([0,1] \times X)$. By Lemma 10.8, we have an isomorphism of Γ-uniform spaces

$$\mathcal{U}([0,1] \times Y) \cong [0,1]_u \otimes \mathcal{U}(Y) \ .$$

By the homotopy invariance of $\mathcal{O}^\infty$ (Corollary 9.38) the projection $[0,1]_u \otimes \mathcal{U}(Y) \to \mathcal{U}(Y)$ induces an equivalence $\mathcal{O}^\infty(\mathcal{M}(\mathcal{U}([0,1] \times Y))) \simeq \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(Y)))$. Hence we conclude that $\mathcal{O}^\infty_{\mathrm{hlg}}([0,1] \times X) \simeq \mathcal{O}^\infty_{\mathrm{hlg}}(X)$, i.e., that $\mathcal{O}^\infty_{\mathrm{hlg}}$ is homotopy invariant.

Assume now that $(X_i)_{i \in I}$ is a filtered family of Γ-topological spaces and set $X := \operatorname{colim}_{i \in I} X_i$. Then every morphism $Y \to X$ for a Y in $\Gamma\mathbf{Top}^{cm}$ factorizes as $Y \to X_i \to X$ for some i in I since Y is Γ-compact. It follows that

$$\begin{aligned} \mathcal{O}^\infty_{\mathrm{hlg}}(X) &\simeq \operatorname*{colim}_{(Y \to X) \in \Gamma\mathbf{Top}^{cm}/X} \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(Y))) \\ &\simeq \operatorname*{colim}_{i \in I} \operatorname*{colim}_{(Y \to X_i) \in \Gamma\mathbf{Top}^{cm}/X_i} \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(Y))) \\ &\simeq \operatorname*{colim}_{i \in I} \mathcal{O}^\infty_{\mathrm{hlg}}(X_i) \ . \end{aligned}$$

This in particular implies the wedge axiom for $\mathcal{O}^\infty_{\mathrm{hlg}}$.

In order to show that $\mathcal{O}^\infty_{\mathrm{hlg}}$ satisfies excision, by Remark 10.5 it suffices to show the stronger result that for every decomposition (A, B) of a Γ-topological space X into two closed invariant subsets we have a push-out square

$$\begin{array}{ccc} \mathcal{O}^\infty_{\mathrm{hlg}}(A \cap B) & \longrightarrow & \mathcal{O}^\infty_{\mathrm{hlg}}(A) \\ \downarrow & & \downarrow \\ \mathcal{O}^\infty_{\mathrm{hlg}}(B) & \longrightarrow & \mathcal{O}^\infty_{\mathrm{hlg}}(X) \end{array} \quad . \tag{10.11}$$

Let $r\colon Y \to X$ be an object of $\Gamma\mathbf{Top}^{cm}/X$. Then $(r^{-1}(A), r^{-1}(B))$ is a closed decomposition of Y into Γ-compact and Γ-metrizable subspaces. Note that the objects of the form $r^{-1}(A) \to A$ of $\Gamma\mathbf{Top}^{cm}/A$ for all $r\colon Y \to X$ are cofinal in $\Gamma\mathbf{Top}^{cm}/A$.

We conclude that (10.11) is a colimit of commuting squares

$$\begin{array}{ccc} \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(r^{-1}(A) \cap r^{-1}(B)))) & \longrightarrow & \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(r^{-1}(A)))) \\ \downarrow & & \downarrow \\ \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(r^{-1}(B)))) & \longrightarrow & \mathcal{O}^\infty(\mathcal{M}(\mathcal{U}(Y))) \end{array} \tag{10.12}$$

Since Y is a Γ-compact and Γ-metrizable Γ-topological space, by Lemma 10.9 the decomposition $(r^{-1}(A), r^{-1}(B))$ is an equivariant uniform decomposition of $\mathcal{U}(Y)$. Therefore the square (10.12) is a push-out square by Corollary 9.36. Being a colimit of push-out squares the square (10.11) is therefore also a push-out square. □

Remark 10.12. Instead of $\mathcal{O}^\infty \circ \mathcal{M}$ we could consider any functor

$$A\colon \Gamma\mathbf{U} \to \mathbf{C}$$

for some cocomplete stable ∞-category $\mathbf{C}$, which is excisive for closed decompositions and homotopy invariant. The construction above produces an equivariant $\mathbf{C}$-valued homology theory $A_{\mathrm{hlg}}\colon \Gamma\mathbf{Top} \to \mathbf{C}$ as a left Kan-extension

$$\begin{array}{ccc} \Gamma\mathbf{Top}^{cm} & \xrightarrow{A\circ\mathcal{U}} & \mathbf{C} \\ \downarrow & \nearrow_{A_{\mathrm{hlg}}} & \\ \Gamma\mathbf{Top} & & \end{array}$$

The proof that this is indeed an equivariant $\mathbf{C}$-valued homology theory is word-for-word the same as the proof of Proposition 10.11.

Note that the proof also shows that for a filtered family of Γ-topological spaces $(X_i)_{i\in I}$ the following natural morphism is an equivalence

$$\operatorname*{colim}_{i\in I} A_{\mathrm{hlg}}(X_i) \xrightarrow{\simeq} A_{\mathrm{hlg}}(\operatorname*{colim}_{i\in I} X_i) \ . \tag{10.13}$$

We will use this equivalence later in this paper. ♦

A Γ-uniform space X has an underlying Γ-topological space which we will denote by $\tau(X)$. The topology of $\tau(X)$ is generated by the sets $V[x]$ for all points x and uniform entourages V of X. We actually get a functor

$$\tau\colon \Gamma\mathbf{U} \to \Gamma\mathbf{Top}\ . \tag{10.14}$$

Remark 10.13. Let Y be a Γ-compact and Γ-metrizable Γ-topological space, and let X be a Γ-uniform space. Then every continuous map $Y \to \tau(X)$ induces a uniform map $\mathcal{U}(Y) \to X$. Therefore, we obtain a natural morphism

$$A_{\mathrm{hlg}}(\tau(X)) \to A(X)\ . \tag{10.15}$$

Since the inclusion functor $\Gamma\mathbf{Top}^{cm} \hookrightarrow \Gamma\mathbf{Top}$ is fully faithful, the morphism (10.15) is an equivalence for $X := \mathcal{U}(Y)$ if Y is a Γ-compact and Γ-metrizable Γ-topological space. ♦

10.3. Families of subgroups and the universal assembly map

In this section we will define the classifying space $E_{\mathcal{F}}\Gamma$ for a family $\mathcal{F}$ of subgroups of Γ, and we will define the corresponding universal assembly map

$$\alpha_{\mathcal{F}}\colon \mathcal{O}^{\infty}_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma) \to \mathcal{O}^{\infty}_{\mathrm{hlg}}(*)\ .$$

We will also introduce the motivic assembly map

$$\alpha_{X,Q}\colon \mathcal{O}^{\infty}_{\mathrm{hlg}}(X) \otimes \mathrm{Yo}^{s}(Q) \to \Sigma\mathrm{Yo}^{s}(Q)$$

for a Γ-topological space X twisted by a Γ-bornological coarse space Q.

Definition 10.14. A *family of subgroups* $\mathcal{F}$ of Γ is a non-empty subset of the set of subgroups of Γ which is closed under conjugation and taking subgroups. ♦

Example 10.15. Examples of families of subgroups are:

1. $\{\mathbf{1}\}$ – the family containing only the trivial subgroup
2. **Fin** – the family of all finite subgroups
3. **Vcyc** – the family of all virtually cyclic subgroups
4. **All** – the family of all subgroups

In Section 11 we will mostly work with the family **Fin** since one can model the corresponding classifying space (see Definition 10.17) by the Rips complex. ♦

For a family of subgroups $\mathcal{F}$ we let $\mathbf{Orb}_{\mathcal{F}}(\Gamma) \subseteq \mathbf{Orb}(\Gamma)$ be the full subcategory of transitive Γ-sets whose stabilizers belong to $\mathcal{F}$.

Example 10.16. Note that $\mathbf{Orb}_{\mathbf{All}}(\Gamma) = \mathbf{Orb}(\Gamma)$.

There is furthermore an equivalence $B\Gamma \to \mathbf{Orb}_{\{1\}}(\Gamma)^{op}$ which sends the unique object $*$ of $B\Gamma$ to the Γ-set Γ (with the left action). On morphisms this equivalence sends the morphism γ in $\mathrm{Hom}_{B\Gamma}(*, *) = \Gamma$ to the automorphism of the Γ-set Γ given by right-multiplication with γ. ♦

For two families of subgroups $\mathcal{F}, \mathcal{F}'$ satisfying $\mathcal{F} \subseteq \mathcal{F}'$ we have an inclusion

$$\mathbf{Orb}_{\mathcal{F}}(\Gamma) \hookrightarrow \mathbf{Orb}_{\mathcal{F}'}(\Gamma)$$

of orbit categories. On presheaves this inclusion of orbit categories defines a restriction functor which is the right-adjoint of an adjunction

$$\mathrm{Ind}_{\mathcal{F}}^{\mathcal{F}'} : \mathbf{PSh}(\mathbf{Orb}_{\mathcal{F}}(\Gamma)) \leftrightarrows \mathbf{PSh}(\mathbf{Orb}_{\mathcal{F}'}(\Gamma)) : \mathrm{Res}_{\mathcal{F}}^{\mathcal{F}'} \ . \tag{10.16}$$

We let $*_{\mathcal{F}}$ denote the final object in $\mathbf{Orb}_{\mathcal{F}}(\Gamma)$.

Definition 10.17. The object

$$E_{\mathcal{F}}\Gamma := \mathrm{Ind}_{\mathcal{F}}^{\mathbf{All}}(*_{\mathcal{F}})$$

of $\mathbf{PSh}(\mathbf{Orb}(\Gamma))$ is called the classifying space of Γ for the family $\mathcal{F}$. ♦

Remark 10.18. Assume that X is a Γ-CW-complex with an equivalence $\ell(X) \simeq E_{\mathcal{F}}\Gamma$. Then the following Lemma 10.19 shows that for every subgroup H of Γ the fixed point set X^H is contractible or empty depending on whether H belongs to $\mathcal{F}$ or not. This property is the usual characterization of a classifying space of Γ for the family $\mathcal{F}$. We say that X is a model for $E_{\mathcal{F}}\Gamma$. A model for $E_{\mathcal{F}}\Gamma$ is unique up to contractible choice. ♦

Let $y \colon \mathbf{Orb}(\Gamma) \to \mathbf{PSh}(\mathbf{Orb}(\Gamma))$ denote the Yoneda embedding.

Lemma 10.19. *For $T \in \mathbf{Orb}(\Gamma)$ we have*

$$\mathrm{Map}(y(T), E_{\mathcal{F}}\Gamma) \simeq \begin{cases} \emptyset & \text{if } T \notin \mathbf{Orb}_{\mathcal{F}}(\Gamma) \ , \\ * & \text{if } T \in \mathbf{Orb}_{\mathcal{F}}(\Gamma) \ . \end{cases}$$

Proof. For a presheaf E on $\mathbf{Orb}_{\mathcal{F}}(\Gamma)$ and object T in $\mathbf{Orb}(\Gamma)$ we have (we interpret the Hom-sets as discrete spaces)

$$\mathrm{Ind}_{\mathcal{F}}^{\mathbf{All}}(E)(T) \simeq \operatorname*{colim}_{(S \to T) \in \mathbf{Orb}_{\mathcal{F}}(\Gamma)^{op}/T} E(S) \ .$$

Consequently, we have

$$\mathrm{Map}(y(T), E_{\mathcal{F}}\Gamma) \simeq \operatorname*{colim}_{(S \to T) \in \mathbf{Orb}_{\mathcal{F}}(\Gamma)^{op}/T} *_{\mathcal{F}}(S) \ .$$

Note that $*_{\mathcal{F}}(S) \simeq *$ and $\mathbf{Orb}_{\mathcal{F}}(\Gamma)^{op}/T$ is empty if T is not in $\mathbf{Orb}_{\mathcal{F}}(\Gamma)$ and otherwise has the identity on T as final object. □

Remark 10.20. We must consider models for $E_{\mathcal{F}}\Gamma$ since $\mathcal{O}^\infty_{\mathrm{hlg}}$ or the equivariant homology theory A_{hlg} constructed in Remark 10.12 are not expected to be strong equivariant homology theories (Definition 10.3) and therefore can not be applied to the object $E_{\mathcal{F}}\Gamma$ of the ∞-category $\mathbf{PSh}(\mathbf{Orb}(\Gamma))$. But on the other hand, since these functors are homotopy invariant, the evaluations $\mathcal{O}^\infty_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma^{cw})$ or $A_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma^{cw})$ on a model $E_{\mathcal{F}}\Gamma^{cw}$ for $E_{\mathcal{F}}\Gamma$ are well-defined up to equivalence.

From now on $E_{\mathcal{F}}\Gamma^{cw}$ will denote some choice of a model. ♦

The projection $E_{\mathcal{F}}\Gamma^{cw} \to *$ is a morphism in $\Gamma\mathbf{Top}$.

If $F\colon \Gamma\mathbf{Top} \to \mathbf{C}$ is an equivariant $\mathbf{C}$-valued homology theory, then the Farrell–Jones and Baum–Connes type question is for which family $\mathcal{F}$ this projection induces an equivalence

$$F(E_{\mathcal{F}}\Gamma^{cw}) \to F(*) \ .$$

The coarse geometric approach to the question uses that this projection induces a morphism in $\Gamma\mathbf{Sp}\mathcal{X}$

$$\alpha_{E_{\mathcal{F}}\Gamma^{cw}}\colon \mathcal{O}^\infty_{\mathrm{hlg}}(E_{\mathcal{F}}\Gamma^{cw}) \to \mathcal{O}^\infty_{\mathrm{hlg}}(*) \ .$$

Definition 10.21. The morphism $\alpha_{E_{\mathcal{F}}\Gamma^{cw}}$ is called the *universal assembly map* for the family of subgroups $\mathcal{F}$. ♦

One could ask if there is an interesting family $\mathcal{F}$ (i.e., not the family of all subgroups of Γ) for which the universal assembly map is an equivalence.

Most of the study of the assembly map is based on an identification of this map with a forget-control map, or equivalently, a boundary operator of a cone sequence. We develop this point of view below.

The following diagram is one of the starting points of [BLR08]. We consider a Γ-topological space X and the projection $X \to *$. It induces the vertical maps in the diagram in $\Gamma\mathbf{Sp}\mathcal{X}$ whose horizontal parts are segments of the cone sequence Corollary 9.30:

$$\begin{array}{ccccc}
\operatorname*{colim}_{(Y\to X)\in\Gamma\mathbf{Top}^{cm}/X}\mathrm{Yo}^s(Y_{max,max}) & \longrightarrow & \operatorname*{colim}_{(Y\to X)\in\Gamma\mathbf{Top}^{cm}/X}\mathrm{Yo}^s(\mathcal{O}(\mathcal{M}(\mathcal{U}(Y)))) & \longrightarrow & \mathcal{O}^\infty_{\mathrm{hlg}}(X) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_X} \\
\mathrm{Yo}^s(*) & \longrightarrow & \mathrm{Yo}^s(\mathcal{O}(*)) & \longrightarrow & \mathcal{O}^\infty_{\mathrm{hlg}}(*)
\end{array}$$

Note that $\mathrm{Yo}^s(\mathcal{O}(*)) \simeq 0$ since $\mathcal{O}(*)$ is flasque. Furthermore, if we tensor-multiply (with respect to $-\otimes-$) the diagram with $\mathrm{Yo}^s(Q)$ for a Γ-bornological coarse space Q whose underlying Γ-set is free, then by Lemma 10.7 the left vertical map becomes an equivalence.

Definition 10.22. We define the *obstruction motive* to be the object

$$M(X) := \operatorname*{colim}_{(Y\to X)\in\Gamma\mathbf{Top}^{cm}/X}\mathrm{Yo}^s(\mathcal{O}(\mathcal{M}(\mathcal{U}(Y))))$$

of the category $\Gamma\mathbf{Sp}\mathcal{X}$. ♦

The discussion above has the following corollary. Let Q be a Γ-bornological coarse space.

Corollary 10.23. *If the underlying* Γ*-set of* Q *is free, then we have a fiber sequence*

$$M(X)\otimes \mathrm{Yo}^s(Q)\to \mathcal{O}^\infty_{\mathrm{hlg}}(X)\otimes \mathrm{Yo}^s(Q)\xrightarrow{\alpha_{X,Q}}\Sigma\mathrm{Yo}^s(Q)\to \Sigma(M(X)\otimes \mathrm{Yo}^s(Q))\ .$$

Definition 10.24. The morphism $\alpha_{X,Q}:=\alpha_X\otimes \mathrm{id}_{\mathrm{Yo}^s(Q)}$ is called the *motivic assembly map* with twist Q. ♦

Let Q be a Γ-bornological coarse space.

Corollary 10.25. *Let the underlying* Γ*-set of* Q *be free. Then the motivic assembly map with* Q*-twist* $\alpha_{X,Q}$ *is an equivalence if and only if* $M(X)\otimes \mathrm{Yo}^s(Q)\simeq 0$*.*

A typical example for Q is $\Gamma_{can,min}$, and variants like $\Gamma_{can,max}$, $\Gamma_{max,max}$, $\Gamma_{min,min}$, etc.

In general we expect the assembly map $\alpha_{E_{\mathcal{F}}\Gamma,Q}$ to become an equivalence only after applying suitable equivariant coarse homology theories. But the following is an example where the assembly map is an equivalence already motivically.

Let $\mathcal{F}$ be a family of subgroups of Γ and consider a Γ-bornological coarse space Q.

Lemma 10.26. *Assume that:*

1. $E_{\mathcal{F}}\Gamma^{cw}$ *is* Γ*-compact and* Γ*-metrizable.*
2. Q *is discrete as a coarse space.*
3. Q *has stabilizers in* $\mathcal{F}$*.*
4. Q *is* Γ*-finite.*

Then we have

$$M(E_{\mathcal{F}}\Gamma^{cw})\otimes \mathrm{Yo}^s(Q)\simeq 0\ .$$

Remark 10.27. Note that if the family $\mathcal{F}$ is **VCyc**, i.e., the family of all virtually cyclic subgroups, then the condition of $E_{\mathcal{F}}\Gamma^{cw}$ being Γ-compact is very restrictive: by a conjecture of Juan-Pineda–Leary [JPL06] this implies that Γ is virtually cyclic itself.

This conjecture is proven for hyperbolic groups (Juan-Pineda–Leary [JPL06]), elementary amenable groups (Kochloukova–Martinez-Perez–Nucinkis [KMPN09] and Groves–Wilson [GW13]) and for one-relator groups, acylindrically hyperbolic groups, 3-manifold groups, CAT(0) cube groups, linear groups (von Puttkamer–Wu [vPW16, vPW17]). ♦

Proof. Since $E_{\mathcal{F}}\Gamma^{cw}$ is Γ-compact and Γ-metrizable the colimit in the Definition 10.22 of $M(E_{\mathcal{F}}\Gamma)^{cw}$ stabilizes. We consider Q as a Γ-uniform bornological coarse space Q_{disc} with the discrete uniform structure. Using Lemma 9.26, we get the equivalence

$$M(E_{\mathcal{F}}\Gamma^{cw})\otimes \mathrm{Yo}^s(Q)\simeq \mathrm{Yo}^s(\mathcal{O}(\mathcal{M}(\mathcal{U}(E_{\mathcal{F}}\Gamma^{cw})))\otimes Q_{disc})\ .$$

The space $E_{\mathcal{F}}\Gamma^{cw}$ has the following universal property: for every Γ-space X with stabilizers in $\mathcal{F}$ which is homotopy equivalent to a CW-complex the space $\mathrm{Map}_{\Gamma}(X, E_{\mathcal{F}}\Gamma^{cw})$ is contractible. It follows from the universal property of $E_{\mathcal{F}}\Gamma^{cw}$ that there is a unique homotopy class of maps $\iota\colon Q \to E_{\mathcal{F}}\Gamma^{cw}$. Furthermore, the maps $\iota\circ \mathrm{pr}_Q, \mathrm{pr}_{E_{\mathcal{F}}\Gamma^{cw}}\colon E_{\mathcal{F}}\Gamma^{cw}\times Q \to E_{\mathcal{F}}\Gamma^{cw}$ are homotopic. Since Q is Γ-finite and $E_{\mathcal{F}}\Gamma^{cw}$ is Γ-compact Γ-metrizable these maps and homotopies are automatically uniform maps when we consider $E_{\mathcal{F}}\Gamma^{cw}$ as a uniform space by applying $\mathcal{U}$. Hence $\mathrm{id}_{\mathcal{M}(\mathcal{U}(E_{\mathcal{F}}\Gamma^{cw}))\otimes Q_{disc}}$ is homotopic to the composition

$$\mathcal{M}(\mathcal{U}(E_{\mathcal{F}}\Gamma^{cw}))\otimes Q_{disc} \xrightarrow{\mathrm{pr}_Q} Q_{disc} \xrightarrow{(\iota,\mathrm{id}_Q)} \mathcal{M}(\mathcal{U}(E_{\mathcal{F}}\Gamma^{cw}))\otimes Q_{disc}$$

of morphisms between Γ-uniform bornological coarse spaces. Using that $\mathcal{O}$ is homotopy invariant we conclude that $\mathrm{id}_{M(E_{\mathcal{F}}\Gamma^{cw})\otimes \mathrm{Yo}^s(Q)}$ factorizes over $\mathrm{Yo}^s(\mathcal{O}(Q_{disc}))$. We now use Example 9.25 in order to conclude that $\mathrm{Yo}^s(\mathcal{O}(Q_{disc}))\simeq 0$. This finishes this proof. □

For the following we just combine Lemma 10.26 and Corollary 10.25.

Corollary 10.28. *If $E_{\mathcal{F}}\Gamma^{cw}$ is Γ-compact and Γ-metrizable, then the motivic assembly map $\alpha_{E_{\mathcal{F}}\Gamma^{cw},\Gamma_{min,?}}$ is an equivalence for $?\in\{\min,\max\}$.*

10.4. Homology theories from coarse homology theories

Let E be an equivariant $\mathbf{C}$-valued coarse homology theory, i.e., a colimit preserving functor $E\colon \Gamma\mathbf{Sp}\mathcal{X}\to\mathbf{C}$. By Proposition 10.11 we can use the cone $\mathcal{O}^{\infty}_{\mathrm{hlg}}$ in order to pull-back E to an equivariant homology theory:

Definition 10.29. Let us define

$$E\mathcal{O}^{\infty}_{\mathrm{hlg}} := E\circ\mathcal{O}^{\infty}_{\mathrm{hlg}}\colon \Gamma\mathbf{Top}\to\mathbf{C}\ ,$$

which is an equivariant $\mathbf{C}$-valued homology theory. ♦

Recall that for an equivariant coarse motivic spectrum L the functor

$$-\otimes L\colon \Gamma\mathbf{Sp}\mathcal{X}\to\Gamma\mathbf{Sp}\mathcal{X}$$

preserves colimits. If E is a $\mathbf{C}$-valued equivariant coarse homology theory, then we can define a new equivariant coarse homology theory

$$E_L(-) := E(-\otimes L)\colon \Gamma\mathbf{Sp}\mathcal{X}\to\mathbf{C}\ . \tag{10.17}$$

We will refer to L as a *twist*.

The Farrell–Jones/Baum–Connes type question for $E_{\mathrm{Yo}^s(Q)}\mathcal{O}^{\infty}_{\mathrm{hlg}}$ is now the question for which family $\mathcal{F}$ of subgroups the morphism $E_{\mathrm{Yo}^s(Q)}(\alpha_{E_{\mathcal{F}}\Gamma})$ is an equivalence of spectra. The next corollary is a consequence of Corollary 10.25.

Let Q be a Γ-bornological coarse space.

Corollary 10.30. *Assume that the underlying* Γ*-set of* Q *is free. Then the assembly map* $E_{\mathrm{Yo}^s(Q)}(\alpha_{E_{\mathcal{F}}\Gamma})$ *is an equivalence if and only if*

$$E_{\mathrm{Yo}^s(Q)}(M(E_{\mathcal{F}}\Gamma)) \simeq 0 \ .$$

Given the coarse homology theory E and a twist L it is of particular interest to characterize the homology theory $E_L\mathcal{O}^\infty_{\mathrm{hlg}}$. As explained in Section 10.1, the restriction of an equivariant homology theory from $\Gamma\mathbf{Top}$ to the full subcategory $\Gamma\mathbf{CW}$ of Γ-CW complexes is determined by its restriction to the orbit category $\mathbf{Orb}(\Gamma)$. So we must understand the evaluations $E_L\mathcal{O}^\infty_{\mathrm{hlg}}(S)$ for all transitive Γ-sets S. The main tool is Proposition 9.35. It gives

$$E_L\mathcal{O}^\infty_{\mathrm{hlg}}(S) \simeq \Sigma E_L(\mathrm{Yo}^s(S_{min,max})) \ .$$

We have calculated these evaluations for equivariant coarse ordinary homology and for equivariant coarse algebraic K-homology explicitly:

1. see Section 7.2 for $H\mathcal{X}^\Gamma_{\mathrm{Yo}^s(\Gamma_{can,min})}$, and
2. see Section 8.4 for $K\mathbf{A}\mathcal{X}^\Gamma_{\mathrm{Yo}^s(\Gamma_{?,?})}$.

11. Forget-control and assembly maps

11.1. The forget-control map

Let X be a coarse space and U an entourage of X.

Let $\mu\colon \mathcal{P}(X) \to [0,1]$ be a probability measure on the measurable space $(X, \mathcal{P}(X))$.

Definition 11.1. The measure μ is called finite U-bounded if there is a finite U-bounded subset F of X with $\mu(F) = 1$. We then define the support $\mathrm{supp}(\mu)$ to be the smallest subset of X with measure one.

We let $P_U(X)$ denote the topological space of finite U-bounded probability measures on X equipped with the topology induced by the evaluation against finitely supported functions. ♦

Every point x in X gives rise to a Dirac measure δ_x at x. If U contains the diagonal of X, then we get a map $X \to P_U(X)$, $x \mapsto \delta_x$, of sets. A probability measure μ which is finite U-bounded can be written as a finite convex combination of Dirac measures. More concretely,

$$\mu = \sum_{x \in \mathrm{supp}(\mu)} \mu(\{x\})\delta_x \ .$$

The set $P_U(X)$ has a natural structure of a simplicial complex. In other words, $P_U(X)$ is the simplicial complex with vertex set X such that a subset $\{x_1, \dots, x_m\}$ spans a simplex if and only if $(x_i, x_j) \in U$ for all i, j in $\{1, \dots, m\}$. Let $P_U(X)_u$ denote the uniform space with the uniform structure induced by the canonical path quasi-metric on the simplicial complex, see Example 9.7. Note that the topology induced by this uniform structure is the topology on $P_U(X)$ induced by the evaluation against finitely supported functions, i.e., we have

$$\tau(P_U(X)_u) = P_U(X) \ ,$$

where τ is the functor associating the underlying topological space to a uniform space, cf. (10.14). Eilenberg–Steenrod [ES52, p. 75] called this the metric topology. In general this topology differs from the weak topology on the simplicial complex in the sense of Eilenberg–Steenrod [ES52, p. 75].

If X' is a second coarse space with entourage U' and $f\colon X \to X'$ is a map such that $(f \times f)(U) \subseteq U'$, then we get a map of simplicial complexes

$$f_*\colon P_U(X) \to P_{U'}(X') \ , \quad \mu \mapsto f_*\mu \ ,$$

where the measure $f_*\mu$ is the push-forward of the measure μ.

If X is a Γ-coarse space and the entourage U is Γ-invariant, then Γ acts on the simplicial complex $P_U(X)$ such that the latter becomes a Γ-simplicial complex, so in particular it becomes a Γ-topological space. Furthermore we obtain a Γ-uniform space $P_U(X)_u$. If X' is a second Γ-coarse space, U' is a Γ-invariant entourage, and $f\colon X \to X'$ is equivariant with $(f \times f)(U) \subseteq U'$, then $f_*\colon P_U(X) \to P_{U'}(X')$ is equivariant.

Let X be a Γ-coarse space.

Definition 11.2. The Γ-topological space

$$\mathrm{Rips}(X) := \operatorname*{colim}_{U \in \mathcal{C}^\Gamma} P_U(X)$$

is called the *Rips complex* of X. Note that the colimit is taken in $\Gamma\mathbf{Top}$. ♦

Remark 11.3. Note that in general the simplicial complex $P_U(X)$ is not locally finite. In this case its topology does not exhibit it as a CW-complex (this happens if and only if the simplicial complex is locally finite).

But by a result of Dowker [Dow52, Thm. 1 on P. 575] $P_U(X)$ has the homotopy type of a CW-complex (see also Milnor [Mil59, Thm. 2] for a short proof of this). Concretely, the underlying identity map of the set induces a homotopy equivalence $\mathrm{CW}(P_U(X)) \to P_U(X)$, where $\mathrm{CW}(P_U(X))$ denotes $P_U(X)$ retopologized as a CW-complex. ♦

Let Γ_{can} be the group Γ considered as a Γ-coarse space with its canonical coarse structure. Recall the localization (10.2).

Lemma 11.4. *We have an equivalence* $\ell(\mathrm{Rips}(\Gamma_{can})) \simeq E_{\mathbf{Fin}}\Gamma$.

Proof. By definition, $\ell(\mathrm{Rips}(\Gamma))$ is the presheaf on $\mathbf{Orb}(\Gamma)$ given by

$$S \mapsto \kappa(\mathrm{Map}_{\Gamma\mathbf{Top}}(S, \mathrm{Rips}(\Gamma_{can}))) \ ,$$

where κ is as in (10.1). We now observe that the stabilizers of the points of $\mathrm{Rips}(\Gamma_{can})$ are finite subgroups of Γ. Consequently, the presheaf belongs to the subcategory of presheaves supported on the orbits with finite stabilizers. Since, by definition, $E_{\mathbf{Fin}}\Gamma$ corresponds to the final object in this subcategory there exists a morphism $\ell(\mathrm{Rips}(\Gamma_{can})) \to E_{\mathbf{Fin}}\Gamma$. In order to show that it is an equivalence it suffices to show that the spaces $\mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, \mathrm{Rips}(X))$ for all finite subgroups H of Γ are equivalent to $*$, i.e., that these spaces are connected and that their homotopy groups are trivial.

We now study these homotopy groups. Since the spheres S^n are compact for all n in $\mathbb{N}$ and the structure maps

$$\mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, P_U(\Gamma_{can})) \to \mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, P_{U'}(\Gamma_{can}))$$

for $U \subseteq U'$ are inclusions of CW-complexes, we have

$$\pi_*(\mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, \mathrm{Rips}(\Gamma_{can}))) \cong \operatorname*{colim}_{U \in \mathcal{C}^\Gamma} \pi_*(\mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, P_U(\Gamma_{can}))) \ .$$

We have a homeomorphism

$$\mathrm{Map}_{\Gamma\mathbf{Top}}(\Gamma/H, P_U(\Gamma_{can})) \cong P_U(\Gamma_{can})^H \ .$$

We fix an integer n and consider a map

$$f \colon S^n \to P_U(\Gamma_{can})^H \ .$$

The image $f(S^n)$ is a compact subset of $P_U(\Gamma_{can})^H$. In the case $n = 0$, since Γ_{can} is coarsely connected, we can increase the entourage U such that the image $f(S^0)$ belongs to a connected component of $P_U(\Gamma_{can})$. Let now n be arbitrary. We can now assume that $f(S^n)$ belongs to a connected component of $P_U(\Gamma_{can})$. It is hence bounded in diameter by some integer N. We conclude that under the map $P_U(\Gamma_{can}) \to P_{U^N}(\Gamma_{can})$ the image $f(S^n)$ is mapped to a subset of a single simplex of $P_{U^N}(\Gamma_{can})$. The intersection of the H-fixed points with this simplex is a convex subset and hence itself contractible. We conclude that f is homotopic to a constant map. □

Definition 11.5. If X is a Γ-bornological coarse space and U an invariant entourage of X, then we equip the Γ-simplicial complex $P_U(X)$ with the bornology generated by the subsets $P_U(B)$ for all bounded subsets B of X. Equipped with this bornology and the coarse structure induced by the metric we obtain a Γ-bornological coarse space which we denote by $P_U(X)_{bd}$. We furthermore write $P_U(X)_{bdu}$ for the corresponding Γ-uniform bornological coarse space. Note in contrast that the notation $P_U(X)_d$ (or $P_U(X)_{du}$, respectively) would mean the Γ-bornological coarse space (or Γ-uniform bornological coarse space, respectively) whose bornological and coarse (and uniform, respectively) structures are induced from the metric, see Example 9.11. ♦

Let X be a Γ-bornological coarse space and U be an invariant entourage of X. The Dirac measures provide a morphism of Γ-bornological coarse spaces

$$\delta \colon X_U \to P_U(X)_{bd} \ . \tag{11.1}$$

Lemma 11.6. *Assume:*

1. Γ *is torsion-free.*
2. *The underlying* Γ*-set of* X *is free.*

Then the morphism (11.1) *is an equivalence of* Γ*-bornological coarse spaces.*

Proof. We first observe that Γ acts freely on $P_U(X)$. Indeed, for γ in Γ and μ in $P_U(X)$ satisfying $\gamma\mu = \mu$ the subgroup of Γ generated by γ has a finite orbit contained in $\mathrm{supp}(\mu)$. Since Γ is torsion-free and X is a free Γ-set this can only happen if $\gamma = 1$.

To define an inverse morphism $g\colon P_U(X)_{bd} \to X_U$ we first choose representatives for the orbits $P_U(X)/\Gamma$. Then we define $g(\mu)$ for every chosen representative μ to be a point in $\mathrm{supp}(\mu)$, and extend equivariantly. Then $g \circ \delta = \mathrm{id}_{X_U}$ and $\delta \circ g$ is close to $\mathrm{id}_{P_U(X)}$. □

The following corollary follows immediately from the above lemma.

Corollary 11.7. *If* Γ *is a finitely generated torsion-free group, then* $\Gamma_{can,min} \to P_U(\Gamma)_{bd}$ *is an equivalence for every invariant generating entourage* U *of* Γ.

Assume that Q is a Γ-bornological coarse space. Let X be a Γ-bornological coarse space and U be an invariant entourage of X.

Lemma 11.8. *If the underlying* Γ*-set of* Q *is free, then*

$$\delta \times \mathrm{id}_Q\colon X_U \otimes Q \to P_U(X)_{bd} \otimes Q$$

is an equivalence of Γ*-bornological coarse spaces.*

Note that we do not assume that X is a free Γ-set.

Proof. First we note that Γ acts freely on the set $P_U(X) \times Q$. We choose representatives for the orbits $(P_U(X) \times Q)/\Gamma$. Then we choose for every representative $(\mu, q) \in P_U(X) \times Q$ a point $x \in \mathrm{supp}(\mu)$ and set $g(\mu, q) := (x, q)$. Then we extend this to an equivariant map $g\colon P_U(X) \times Q \to X \times Q$. This map of sets is a morphism

$$g\colon P_U(X)_{bd} \otimes Q \to X_U \otimes Q\ .$$

By construction $g \circ (\delta \times \mathrm{id}_Q) = \mathrm{id}_{X \times Q}$ and $g \circ (\delta \times \mathrm{id}_Q)$ is close to the identity. □

For the following recall the cone functor $\mathcal{O}$ from Section 9.4 and the "cone at infinity" functor $\mathcal{O}^\infty$ from Section 9.5.

Let X be a Γ-bornological coarse space.

Definition 11.9. We define the following equivariant coarse motivic spectra:

$$\begin{aligned} F(X) &:= \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(\mathcal{O}(P_U(X)_{bdu}))\ , \\ F^\infty(X) &:= \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathcal{O}^\infty(P_U(X)_{bdu})\ , \\ F^0(X) &:= \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(P_U(X)_{bd})\ . \end{aligned}$$

♦

Using standard Kan-extension techniques one can refine the above description to functors

$$F, F^\infty, F^0 \colon \Gamma\mathbf{BornCoarse} \to \Gamma\mathbf{Sp}\mathcal{X}\ ,$$

see Remark 11.11 for details. The fiber sequence from Corollary 9.30 provides a natural fiber sequence of functors

$$F^0(X) \to F(X) \to F^\infty(X) \xrightarrow{\beta_X} \Sigma F^0(X)\ . \tag{11.2}$$

Definition 11.10. We call β_X the *forget-control map.* ♦

Remark 11.11. We let $\Gamma\mathbf{BornCoarse}^{\mathcal{C}}$ denote the category of pairs (X, U), where X is a Γ-bornological coarse space and U is an invariant entourage of X containing the diagonal. A morphism $(X, U) \to (X', U')$ is a morphism $f\colon X \to X'$ in $\Gamma\mathbf{BornCoarse}$ such that $(f \times f)(U) \subseteq U'$. We have a forgetful functor

$$\Gamma\mathbf{BornCoarse}^{\mathcal{C}} \to \Gamma\mathbf{BornCoarse}\ , \quad (X, U) \mapsto X\ . \tag{11.3}$$

Let

$$\widetilde{E}\colon \Gamma\mathbf{BornCoarse}^{\mathcal{C}} \to \mathbf{C}$$

be a functor to some cocomplete target $\mathbf{C}$ and let E be the left Kan extension of $\widetilde{E}$ along (11.3). The evaluation of E on a Γ-bornological coarse space X is then given as follows:

Lemma 11.12. *We have an equivalence*

$$E(X) \simeq \operatorname*{colim}_{U\in\mathcal{C}^\Gamma(X)} \widetilde{E}(X, U)\ .$$

Proof. By the pointwise formula for the left Kan extension we have an equivalence

$$E(X) \simeq \operatorname*{colim}_{((X',U'),f\colon X'\to X)\in\Gamma\mathbf{BornCoarse}^{\mathcal{C}}/X} \widetilde{E}(X', U')\ .$$

If $((X', U'), f\colon X' \to X)$ belongs to $\Gamma\mathbf{BornCoarse}^{\mathcal{C}}/X$, then we have a morphism

$$(X', U') \to (X, f(U') \cup \mathrm{diag}(X))$$

in $\Gamma\mathbf{BornCoarse}^{\mathcal{C}}/X$. This easily implies that the full subcategory of objects of the form $((X, U), \mathrm{id}_X)$ of $\Gamma\mathbf{BornCoarse}^{\mathcal{C}}/X$ with U in $\mathcal{C}^\Gamma(X)$ is cofinal in $\Gamma\mathbf{BornCoarse}^{\mathcal{C}}/X$. ♦

We have a functor

$$P\colon \Gamma\mathbf{BornCoarse}^{\mathcal{C}} \to \Gamma\mathbf{UBC}\ , \quad (X,U) \mapsto P_U(X)_{bdu}\ .$$

We construct the fibre sequence (11.2) by applying the left Kan extension to the fibre sequence of functors $\Gamma\mathbf{BornCoarse}^{\mathcal{C}} \to \Gamma\mathbf{Sp}\mathcal{X}$

$$\mathrm{Yo}^s \circ \mathcal{F}_{\mathcal{T}} \circ P \to \mathrm{Yo}^s \circ \mathcal{O} \circ P \to \mathcal{O}^\infty \circ P \to \Sigma\mathrm{Yo}^s \circ \mathcal{F}_{\mathcal{T}} \circ P$$

obtained by precomposing the sequence from Corollary 9.30 with P. ♦

Let X be a Γ-bornological space. In the following two corollaries we identify the Γ-coarse motivic spectrum $F^0(X)$.

Corollary 11.13. *If Γ is torsion-free and the underlying Γ-set of X is free, then*

$$F^0(X) \simeq \mathrm{Yo}^s(X)\ .$$

Proof. We have equivalences

$$F^0(X) = \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(P_U(X)_{bd}) \stackrel{Lemma\ 11.6}{\simeq} \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathrm{Yo}^s(X_U) \stackrel{Corollary\ 4.11.5}{\simeq} \mathrm{Yo}^s(X)\ ,$$

which proves the claim. □

Let Q and X be Γ-bornological coarse spaces.

Corollary 11.14. *If the underlying Γ-set of Q is free, then*

$$F^0(X) \otimes \mathrm{Yo}^s(Q) \simeq \mathrm{Yo}^s(X) \otimes \mathrm{Yo}^s(Q)\ .$$

Proof. Here we use Lemma 11.8, that Yo^s is symmetric monoidal, and that the functor

$$-\otimes \mathrm{Yo}^s(Q)\colon \Gamma\mathbf{Sp}\mathcal{X} \to \Gamma\mathbf{Sp}\mathcal{X}$$

preserves colimits. Therefore we can write down an analogous sequence of equivalences as in the above proof of Corollary 11.13. □

11.2. Comparison of the assembly and the forget-control map

In this section we will compare the assembly map for the family of finite subgroups with the forget-control map.

For every two Γ-bornological coarse spaces X and L we have the forget-control morphism (Definition 11.10)

$$\beta_{X,L}\colon \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \mathcal{O}^\infty(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(L) \to \operatorname*{colim}_{U\in\mathcal{C}^\Gamma} \Sigma\mathrm{Yo}^s(P_U(X)_{bd} \otimes L)\ .$$

We have furthermore the assembly map (see Definition 10.24)

$$\alpha_{\mathrm{Rips}(X),L}\colon \mathcal{O}^\infty_{\mathrm{hlg}}(\mathrm{Rips}(X)) \otimes \mathrm{Yo}^s(L) \to \mathcal{O}^\infty_{\mathrm{hlg}}(*) \otimes \mathrm{Yo}^s(L) \simeq \Sigma\mathrm{Yo}^s(L)$$

induced by the morphism $\mathrm{Rips}(X) \to *$ of Γ-topological spaces.

Recall [BE16, Def. 2.28] that a coarse space $(X, \mathcal{C})$ is called *coarsely connected* if for any two points x, y in X there exists an entourage U in $\mathcal{C}$ such that $(x, y) \in U$.

Definition 11.15. A Γ-bornological coarse space X is *eventually coarsely connected* if there exists a coarse entourage U such that X_U is coarsely connected.

Note that an eventually coarsely connected space is in particular coarsely connected. ♦

While one is interested in $\alpha_{\mathrm{Rips}(X),\Gamma_{can,min}}$, using descent methods (like in [BEKW]) we will only be able to derive split-injectivity of $\beta_{X,\Gamma_{max,max}}$ in several cases. The following theorem allows us to compare both maps. This comparison does not hold directly but only after forcing continuity.

Let X be a Γ-bornological coarse space. Recall [BE16, Def. 6.100] that X has *strongly bounded geometry* if it is equipped with the minimal compatible bornology and if for every entourage U of X there exists a uniform finite upper bound on the cardinalities of U-bounded subsets of X. Furthermore recall the functor C^s from (5.6).

Theorem 11.16. *Assume:*

1. *X has strongly bounded geometry.*
2. *X is Γ-finite.*
3. *The action of Γ on X is proper (Example 2.13).*
4. *X is eventually coarsely connected.*

Then the morphisms $C^s(\alpha_{\mathrm{Rips}(X),\Gamma_{can,min}})$ and $C^s(\beta_{X,\Gamma_{max,max}})$ are equivalent.

Remark 11.17. Note that X being Γ-finite and the action of Γ on X being proper implies that X has the minimal bornology.

Furthermore, if X is Γ-finite and U is a Γ-invariant entourage of X, then the assumption that every U-bounded subset is finite, already implies a uniform upper bound on the cardinality of U-bounded subsets. Hence Assumption 1 in Theorem 11.16 above could be equivalently replaced by the seemingly weaker assumption that every U-bounded subset is finite. ♦

The rest of this section is devoted to the proof of Theorem 11.16.

We consider a Γ-simplicial complex K. Recall the notation introduced in Example 9.11:

1. K_u denotes the Γ-uniform space associated to K.

2. $K_{u,max,max}$ denotes the Γ-uniform bornological coarse space which has the uniform structure of K_u, but the maximal coarse and bornological structures.

3. $K_{u,d,max}$ denotes the Γ-uniform bornological coarse space with the uniform structure of K_u, the metric coarse structure and the maximal bornological structure.

4. $K_{d,max}$ denotes the Γ-bornological coarse space underlying $K_{u,d,max}$.

We denote by

$$\beta^{max,max}_{X,L}\colon \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\mathcal{O}^\infty(P_U(X)_{u,max,max})\otimes \mathrm{Yo}^s(L)\to \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\Sigma\mathrm{Yo}^s(P_U(X)_{max,max}\otimes L)$$

and

$$\beta^{d,max}_{X,L}\colon \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\mathcal{O}^\infty(P_U(X)_{u,d,max})\otimes \mathrm{Yo}^s(L)\to \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\Sigma\mathrm{Yo}^s(P_U(X)_{d,max}\otimes L)$$

the forget-control maps.

The proof of Theorem 11.16 consists of a sequence of lemmas.

Let X and L be Γ-bornological coarse spaces.

Lemma 11.18. *Assume:*

1. *X has strongly bounded geometry.*
2. *X is Γ-finite.*
3. *The underlying set of L is a free Γ-set.*

Then $\alpha_{\mathrm{Rips}(X),L}$ and $\beta^{max,max}_{X,L}$ are equivalent.

Proof. The assumptions on X imply that $P_U(X)$ is Γ-compact and locally finite, hence Γ-metrizable, for every invariant coarse entourage U of X. Therefore, by (10.15), we have an equivalence

$$\mathcal{O}^\infty_{\mathrm{hlg}}(P_U(X))\simeq \mathcal{O}^\infty(P_U(X)_{u,max,max})\ .$$

Using Definition 11.2 of the Rips complex as a filtered colimit of Γ-topological spaces and the relation (10.13) we get

$$\begin{aligned}\mathcal{O}^\infty_{\mathrm{hlg}}(\mathrm{Rips}(X)) &\simeq \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\mathcal{O}^\infty_{\mathrm{hlg}}(P_U(X))\\ &\simeq \operatorname*{colim}_{U\in\mathcal{C}^\Gamma}\mathcal{O}^\infty(P_U(X)_{u,max,max})\ .\end{aligned}$$

Hence we get the following commutative diagram:

$$\begin{array}{ccc}\operatorname{colim}_{U\in\mathcal{C}^\Gamma}\mathcal{O}^\infty(P_U(X)_{u,max,max})\otimes\mathrm{Yo}^s(L) & \xrightarrow{\beta^{max,max}_{X,L}} & \operatorname{colim}_{U\in\mathcal{C}^\Gamma}\Sigma\mathrm{Yo}^s(P_U(X)_{max,max}\otimes L)\\ \downarrow\simeq & & \\ \mathcal{O}^\infty_{\mathrm{hlg}}(\mathrm{Rips}(X))\otimes\mathrm{Yo}^s(L) & & \downarrow\simeq\\ \downarrow\alpha_{\mathrm{Rips}(X),L} & & \\ \mathcal{O}^\infty_{\mathrm{hlg}}(*)\otimes\mathrm{Yo}^s(L) & \xrightarrow{\simeq} & \Sigma\mathrm{Yo}^s(L)\end{array}$$

The vertical arrow on the right is an equivalence by Lemma 10.7. □

Recall the functor C^s from (5.6). Let X be a Γ-bornological coarse space.

Lemma 11.19. *Assume:*

1. *X has strongly bounded geometry.*
2. *X is eventually coarsely connected.*

Then $C^s(\beta^{max,max}_{X,\Gamma_{can,min}})$ and $C^s(\beta^{d,max}_{X,\Gamma_{can,min}})$ are equivalent.

Proof. The morphism $P_U(X)_{u,d,max} \to P_U(X)_{u,max,max}$ of Γ-uniform bornological coarse spaces induces a diagram

$$\begin{array}{ccc} C^s(\mathcal{O}^\infty(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})) & \xrightarrow{\beta^{d,max}_{X,\Gamma_{can,min}}} & C^s(\Sigma\mathrm{Yo}^s(P_U(X)_{d,max} \otimes \Gamma_{can,min})) \\ \downarrow & & \downarrow \\ C^s(\mathcal{O}^\infty(P_U(X)_{u,max,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})) & \xrightarrow{\beta^{max,max}_{X,\Gamma_{can,min}}} & C^s(\Sigma\mathrm{Yo}^s(P_U(X)_{max,max} \otimes \Gamma_{can,min})) \end{array}$$

Since $P_U(X)_{u,d,max} \to P_U(X)_{u,max,max}$ is a coarsening, the left vertical arrow is an equivalence by Proposition 9.33. To show that the right vertical arrow is an equivalence for large entourages U we will need continuity.

For an invariant entourage U of X we denote the set of finite subcomplexes of $P_U(X)$ by $\mathcal{F}(P_U(X))$. It is a filtered partially ordered set with respect to the inclusion relation. For every finite subcomplex F we define a Γ-invariant subcomplex $D_F := \Gamma(F \times \{1\})$ of $P_U(X) \times \Gamma$. We consider the family of Γ-invariant subsets

$$\mathcal{D} := (D_F)_{F \in \mathcal{F}(P_U(X))}$$

of $P_U(X) \times \Gamma$. By Example 5.11 the family $\mathcal{D}$ is a co-Γ-bounded exhaustion of both spaces $P_U(X)_{max,max} \otimes \Gamma_{can,min}$ and $P_U(X)_{d,max} \otimes \Gamma_{can,min}$. By continuity, it suffices to show that the bornological coarse structures on $\mathcal{D}$ induced from $P_U(X)_{max,max} \otimes \Gamma_{can,min}$ and from $P_U(X)_{d,max} \otimes \Gamma_{can,min}$, respectively, agree.

Since the bornologies of $P_U(X)_{max,max} \otimes \Gamma_{can,min}$ and $P_U(X)_{d,max} \otimes \Gamma_{can,min}$ agree, we only have to care about the coarse structures. Let U be large enough, such that $P_U(X)$ is connected.

The coarse structure on D_F induced by $P_U(X)_{d,max} \otimes \Gamma_{can,min}$ is generated by the entourages

$$(D_F \times D_F) \cap (U_r \times V_B) \ ,$$

where U_r is a metric entourage of $P_U(X)$ of size r in $(0, \infty)$ and $V_B := \Gamma(B \times B)$ for a finite subset B is one of the generating entourages of the canonical structure of Γ.

The coarse structure on D_F induced by $P_U(X)_{max,max} \otimes \Gamma_{can,min}$ is generated by the entourages

$$(D_F \times D_F) \cap ((P_U(X) \times P_U(X)) \times V_B)$$

for finite subsets B of Γ. It is clear that the coarse structure of the latter is larger than the one of the first, and it remains to show to other inclusion.

We have

$$(D_F \times D_F) \cap ((P_U(X) \times P_U(X)) \times V_B) \cong \bigcup_{(\gamma,\gamma') \in V_B} (\gamma F \times \gamma' F) \times \{(\gamma, \gamma')\} \ .$$

Since F is a finite subcomplex and $P_U(X)$ is connected, there exists an r in $(0, \infty)$ such that $BF \times BF \subseteq U_r$. This implies that

$$(\gamma F \times \gamma' F) \times \{(\gamma, \gamma')\} \subseteq U_r \times V_B$$

for all pairs (γ, γ') in V_B. We conclude that

$$(D_F \times D_F) \cap ((P_U(X) \times P_U(X)) \times V_B) \subseteq (D_F \times D_F) \cap (U_r \times V_B) \ .$$

This finishes the proof. □

Let X be a Γ-bornological coarse space.

Lemma 11.20. *Assume:*

1. *X is Γ-finite.*
2. *X has strongly bounded geometry.*
3. *the Γ-action on X is proper.*

Then the maps $C^s(\beta^{d,max}_{X,\Gamma_{can,min}})$ and $C^s(\beta_{X,\Gamma_{can,max}})$ are equivalent.

Proof. Recall from Definition 11.5 that $P_U(X)_{bd}$ denotes the Γ-bornological coarse space whose coarse structure is induced from the metric and whose bornology is generated by the subsets $P_U(B)$ for all bounded subsets B of X. Recall furthermore that we write $P_U(X)_{bdu}$ for the corresponding Γ-uniform bornological coarse space.

Recall from (5.7) that $\mathrm{Yo}^s_c \simeq C^s \circ \mathrm{Yo}^s$. It suffices to produce diagrams

$$\begin{array}{ccc} \mathrm{Yo}^s_c([0,k] \otimes P_U(X)_{bd} \otimes \Gamma_{can,max}) & \longrightarrow & C^s(\mathcal{O}(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max})) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Yo}^s_c([0,k] \otimes P_U(X)_{d,max} \otimes \Gamma_{can,min}) & \longrightarrow & C^s(\mathcal{O}(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})) \end{array}$$

for every natural number k which are compatible with increasing k and U.

To produce the diagram we will use continuity with the exhaustion $\mathcal{Y} = (Y_\kappa)_{\kappa \in \mathcal{F}(P_U(X))^{\mathbb{N}}}$ from Lemma 5.12, where $\mathcal{F}(P_U(X))$ denotes the set of all finite subcomplexes of $P_U(X)$. Recall that for κ in $\mathcal{F}(P_U(X))^{\mathbb{N}}$ we set

$$Y_\kappa := \bigcup_{n \in \mathbb{N}} [n-1, n] \times D_{\kappa(n)} \ . \qquad (11.4)$$

Since $P_U(X)_{bd}$ and $P_U(X)_{d,max}$ are Γ-bounded the exhaustion is trapping by Lemma 5.12 for both spaces

$$\mathcal{O}(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min}) \quad \text{and} \quad \mathcal{O}(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max}) \ .$$

Note that the hybrid coarse structure does not play a role here since trapping exhaustions are a bornological concept.

Since the definition of the exhaustion is independent of k and compatible with increasing U, it remains for us to show that the bornological coarse structures on Y_κ induced from $\mathcal{O}(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max})$ and $\mathcal{O}(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})$, respectively, agree in order to obtain (by continuity) the equivalences in the above diagram. Since the coarse structures of $\mathcal{O}(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max})$ and $\mathcal{O}(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})$ agree, we only have to consider the bornologies.

Every bounded subset of $\mathcal{O}(P_U(X)_{u,d,max}) \otimes \mathrm{Yo}^s(\Gamma_{can,min})$ or $\mathcal{O}(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max})$ is contained in $[0,n] \times P_U(X) \times \Gamma$ for some n. It therefore suffices to see that the induced bornologies on $([0,n] \times P_U(X) \times \Gamma) \cap Y_\kappa$ coincide. We can now further finitely decompose

$$([0,n] \times P_U(X) \times \Gamma) \cap Y_\kappa \subseteq \bigcup_{i=1}^{n+1} [i-1,i] \times D_{\kappa(i)} \ .$$

It suffices to show that the induced bornologies on $[i-1,i] \times D_{\kappa(i)}$ coincide. For this we have to show that for every $F \in \mathcal{F}(P_U(X))$ the bornologies on D_F induced from $P_U(X)_{bd}$ and $P_U(X)_{d,max}$, respectively, agree.

Since X is Γ-finite and the Γ-action on X is proper, X carries the minimal bornology. Consequently, every bounded subset of $P_U(X)_{bd}$ is contained in a finite subcomplex. Hence, the bornology on D_F induced by $P_U(X)_{bd}$ is generated by the sets $D_F \cap (F' \times \Gamma)$ for all F' in $\mathcal{F}(P_U(X))$. This set is equal to

$$\Big(\bigcup_{\gamma\in\Gamma} \gamma F \times \{\gamma\}\Big) \cap (F' \times \Gamma) = \bigcup_{\{\gamma\in\Gamma \,|\, \gamma F \cap F' \neq \emptyset\}} (\gamma F \cap F') \times \{\gamma\} \ .$$

Note that the index set of the union on the right hand side is finite since the Γ-action is proper.

The bornology induced by $P_U(X)_{d,max}$ is generated by the sets $D_F \cap (P_U(X) \times B)$ for all finite subsets B of Γ. This set can be written in the form

$$\bigcup_{\gamma\in B} \gamma F \times \{\gamma\} \ .$$

The families of subsets

$$\Big(\bigcup_{\{\gamma\in\Gamma|\gamma F\cap F'\neq\emptyset\}} (\gamma F \cap F') \times \{\gamma\}\Big)_{F'\in\mathcal{F}(P_U(X))} \quad \text{and} \quad \Big(\bigcup_{\gamma\in B} \gamma F \times \{\gamma\}\Big)_{B\subseteq\Gamma, |B|<\infty}$$

generate the same bornologies. This finishes the proof of the lemma. □

Let X be a Γ-bornological coarse space.

Lemma 11.21. *Assume:*

1. *X has strongly bounded geometry.*
2. *X is Γ-finite.*

Then the morphisms $C^s(\beta_{X,\Gamma_{can,max}})$ and $C^s(\beta_{X,\Gamma_{max,max}})$ are equivalent.

Proof. The morphism of Γ-bornological coarse spaces $\Gamma_{can,max} \to \Gamma_{max,max}$ induces the commutative diagram

$$\begin{array}{ccc} C^s(\mathcal{O}^\infty(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{can,max})) & \longrightarrow & \Sigma \mathrm{Yo}^s_c(P_U(X)_{bd} \otimes \Gamma_{can,max}) \\ \downarrow & & \downarrow \\ C^s(\mathcal{O}^\infty(P_U(X)_{bdu}) \otimes \mathrm{Yo}^s(\Gamma_{max,max})) & \longrightarrow & \Sigma \mathrm{Yo}^s_c(P_U(X)_{bd} \otimes \Gamma_{max,max}) \end{array} \tag{11.5}$$

The functor $\mathcal{O}^\infty$ from $\Gamma\mathbf{UBC}$ to $\mathbf{Sp}\mathcal{X}$ is homotopy invariant and excisive for equivariant uniform decompositions. Since X is Γ-finite and has strongly bounded geometry, for every invariant entourage U the complex $P_U(X)$ is a Γ-finite simplicial complex. Using excision and homotopy invariance we conclude that the left vertical map in the above diagram is an equivalence if

$$C^s(\mathcal{O}^\infty(S) \otimes \mathrm{Yo}^s(\Gamma_{can,max})) \to C^s(\mathcal{O}^\infty(S) \otimes \mathrm{Yo}^s(\Gamma_{max,max}))$$

is an equivalence for every Γ-uniform bornological coarse space S which is a transitive Γ-set that has the minimal bornology and the discrete uniform structure. Note that in this case

$$\mathcal{O}^\infty(S) \simeq \mathcal{O}^\infty(S_{disc,min,min}) \simeq \Sigma \mathrm{Yo}^s(S_{min,min}) \ ,$$

since $S_{disc,min,min} \to S$ is a coarsening and $\mathcal{O}(S_{disc,min,min})$ is flasque.

If we can show that for every Γ-bounded Γ-bornological coarse space X the map

$$\mathrm{Yo}^s_c(X \otimes \Gamma_{can,max}) \to \mathrm{Yo}^s_c(X \otimes \Gamma_{max,max})$$

is an equivalence, then we can conclude that the right vertical map in (11.5) is an equivalence, and by the above argument the left vertical map is an equivalence, too. The lemma then follows by taking the colimit over all invariant entourages U of X.

By continuity, it suffices to show that if F is a locally finite, invariant subset of $X \times \Gamma_{?,max}$ (the coarse structure on Γ does not matter since local finiteness is a bornological concept), then the bornological coarse structures on F induced by $X \times \Gamma_{can,max}$ and $X \times \Gamma_{max,max}$, respectively, agree. Since the bornologies are the same, we only have to care about the coarse structures.

Every entourage of $F_{\Gamma_{can,max}}$ is an entourage of $F_{\Gamma_{max,max}}$. So it remains to show the other inclusion. We choose a bounded subset A of X such that $\Gamma A = X$. Let U be an invariant entourage of X containing the diagonal. Then $U[A]$ is bounded. Furthermore, we have

$U \subseteq \Gamma(A \times U[A])$. The set $W' := F \cap (U[A] \times \Gamma)$ is finite since F is locally finite and $U[A] \times \Gamma$ is bounded. We let W denote the projection of W' to Γ. Then we have

$$(U \times \Gamma \times \Gamma) \cap (F \times F) \subseteq (U \times \Gamma(W \times W)) \cap (F \times F) .$$

Now note that $\Gamma(W \times W)$ is an entourage of $\Gamma_{can,max}$. This shows that every entourage of $F_{\Gamma_{max,max}}$ is an entourage of $F_{\Gamma_{can,max}}$. □

Theorem 11.16 follows from combining Lemma 11.18 (with $L = \Gamma_{can,min}$), Lemma 11.19, Lemma 11.20 and Lemma 11.21.

11.3. Homological properties of pull-backs by the cone

Recall Definition 11.9 of the three functors F, F^0, and F^∞. In this section we analyze the homological properties of the functor F^∞. It turns out that this functor is almost a coarse homology theory. The only problematic axiom is vanishing on flasques. In order to improve on this point recall the definition of $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$ from Definition 4.20 and consider the composition

$$F^\infty_{\mathrm{wfl}} \colon \Gamma\mathbf{BornCoarse} \xrightarrow{F^\infty} \Gamma\mathbf{Sp}\mathcal{X} \to \Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}} .$$

In a similar manner, we derive functors F^0_{wfl} and F_{wfl} from F^0 and F, respectively. For every Γ-bornological coarse space X we have a fiber sequence in $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$

$$F^0_{\mathrm{wfl}}(X) \to F_{\mathrm{wfl}}(X) \to F^\infty_{\mathrm{wfl}}(X) \xrightarrow{\beta_{X,\mathrm{wfl}}} \Sigma F^0_{\mathrm{wfl}}(X) . \tag{11.6}$$

The morphism $\beta_{X,\mathrm{wfl}}$ is a version of the forget-control morphism from Definition 11.10.

Proposition 11.22. *The functor F^∞_{wfl} is an equivariant $\Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}}$-valued coarse homology theory.*

Proof. We verify the axioms.

1. (Coarse invariance) We consider a Γ-bornological coarse space X. For i in $\{0,1\}$ let

   $$\iota_i \colon X \to \{0,1\}_{max,max} \otimes X$$

   denote the corresponding inclusions. It suffices to show that $F^\infty_{\mathrm{wfl}}(\iota_0)$ and $F^\infty_{\mathrm{wfl}}(\iota_1)$ are equivalent. For every invariant entourage U of X we consider the invariant entourage $\widetilde{U} := \{0,1\}^2 \times U$ of $\{0,1\}_{max,max} \otimes X$. Then the map

   $$[0,1]_{du} \otimes P_U(X)_{bdu} \to P_{\widetilde{U}}(\{0,1\}_{max,max} \otimes X)_{bdu}$$

   given by

   $$(t,\mu) \mapsto (1-t)\iota_{0,*}\mu + t\iota_{1,*}\mu$$

   is a homotopy between the morphisms of Γ-uniform bornological coarse spaces

   $$P_U(X)_{bdu} \to P_{\widetilde{U}}(\{0,1\}_{max,max} \otimes X)_{bdu}$$

induced by ι_0 and ι_1.

By the homotopy invariance of the functor $\mathcal{O}^\infty$ (Corollary 9.38) we conclude that the morphisms

$$\mathcal{O}^\infty(P_U(X)_{bdu}) \to \mathcal{O}^\infty(P_{\widetilde{U}}(\{0,1\}_{max,max} \otimes X)_{bdu})$$

induced by ι_0 and ι_1 are equivalent. Since the entourages of the form $\widetilde{U}$ for all U in $\mathcal{C}^\Gamma$ are cofinal in the entourages of $\{0,1\} \otimes X$, we get the equivalence of $F^\infty_{\mathrm{wfl}}(\iota_0)$ and $F^\infty_{\mathrm{wfl}}(\iota_1)$ as desired.

2. (Excision) Let X be a Γ-bornological coarse space and Z an invariant subset. For an invariant entourage U of X the subset $P_U(Z)$ of $P_U(X)_{bdu}$ is invariant and closed.

   Let $(\mathcal{Y}, Z)$ be an equivariant complementary pair on X with $\mathcal{Y} = (Y_i)_{i\in I}$. Let i_0 in I be such that $Y_{i_0} \cup Z = X$. Let i_1 in I be such that $U[Y_{i_0}] \subseteq Y_{i_1}$. Then for every i in I with $i \ge \max\{i_0, i_1\}$ we have $P_U(Y_i) \cup P_U(Z) = P_U(X)$. The pair of invariant subsets $(P_U(Y_i), P_U(Z))$ is then an equivariant uniform decomposition of $P_U(X)_{bdu}$. By Corollary 9.36 and Remark 9.37 the functor $\mathcal{O}^\infty$ sends equivariant uniform decompositions to push-outs. We conclude that for i in I with $i \ge \max\{i_0, i_1\}$ we have a push-out

$$\begin{array}{ccc} \mathcal{O}^\infty(P_U(Z \cap Y_i)_{bdu}) & \longrightarrow & \mathcal{O}^\infty(P_U(Y_i)_{bdu}) \\ \downarrow & & \downarrow \\ \mathcal{O}^\infty(P_U(Z)_{bdu}) & \longrightarrow & \mathcal{O}^\infty(P_U(X)_{bdu}) \end{array}$$

   Since colimits of push-out squares are push-out squares, we now take the colimits over the invariant entourages U in $\mathcal{C}^\Gamma$ and over i in I to get the push-out square

$$\begin{array}{ccc} F^\infty(Z \cap \mathcal{Y}) & \longrightarrow & F^\infty(\mathcal{Y}) \\ \downarrow & & \downarrow \\ F^\infty(Z) & \longrightarrow & F^\infty(X) \end{array}$$

   We get the desired push-out

$$\begin{array}{ccc} F^\infty_{\mathrm{wfl}}(Z \cap \mathcal{Y}) & \longrightarrow & F^\infty_{\mathrm{wfl}}(\mathcal{Y}) \\ \downarrow & & \downarrow \\ F^\infty_{\mathrm{wfl}}(Z) & \longrightarrow & F^\infty_{\mathrm{wfl}}(X) \end{array}$$

3. (Flasqueness) We assume that X is flasque with the flasqueness implemented by the equivariant map $f\colon X \to X$. For an invariant entourage U of X with the property $(\mathrm{id}, f)(\mathrm{diag}_X) \subseteq U$ we form the entourage of X

$$\widetilde{U} := \bigcup_{n\in\mathbb{N}} (f^n \times f^n)(U) \ .$$

Note that $(f \times f)(\widetilde{U}) \subseteq \widetilde{U}$. Therefore we have a morphism

$$P_{\widetilde{U}}(f) \colon P_{\widetilde{U}}(X)_{bdu} \to P_{\widetilde{U}}(X)_{bdu} \ .$$

Like every simplicial map it is distance decreasing. Moreover, for every $\mu \in P_{\widetilde{U}}(X)$ we have

$$d(\mu, P_{\widetilde{U}}(f)(\mu)) \le 2 \ .$$

Finally, if B is a bounded subset of X and n is an integer such that $\widetilde{U}[B] \cap f^n(X) = \emptyset$, then

$$P_{\widetilde{U}}(f^n)(P_{\widetilde{U}}(X)) \cap P_{\widetilde{U}}(B) = \emptyset \ .$$

We conclude that $P_{\widetilde{U}}(f)$ implements flasqueness of the bornological coarse space $P_{\widetilde{U}}(X)_{bd}$. The set of invariant entourages of the form $\widetilde{U}$ as above is cofinal in all invariant entourages of X. Therefore, we get $F^0(X) \simeq 0$ and hence $F^0_{\mathrm{wfl}}(X) \simeq 0$ by taking the colimit over these entourages.

We now claim that $\mathcal{O}(P_{\widetilde{U}}(f))$ implements weak flasqueness of $\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})$. In the following we verify the conditions stated in Definition 4.18.

Since f is U-close to id_X, as in 1 we can conclude that the map $P_{\widetilde{U}}(f)_{bdu}$ is uniformly homotopic to $\mathrm{id}_{P_{\widetilde{U}}(X)_{bdu}}$. By the homotopy invariance of $\mathcal{O}$ we conclude that

$$\mathrm{Yo}^s(\mathcal{O}(P_{\widetilde{U}}(f))) \simeq \mathrm{id}_{\mathrm{Yo}^s(\mathcal{O}(P_{\widetilde{U}}(f)_{bdu}))}$$

as required in Definition 4.18.1.

In order to save notation we define the map

$$Q \colon \mathcal{P}([0,\infty) \times P_{\widetilde{U}}(X) \times [0,\infty) \times P_{\widetilde{U}}(X)) \to \mathcal{P}([0,\infty) \times P_{\widetilde{U}}(X) \times [0,\infty) \times P_{\widetilde{U}}(X))$$

by

$$Q(V) := \bigcup_{n \in \mathbb{N}} \big(([0,\infty) \times P_{\widetilde{U}}(f))^n \times ([0,\infty) \times P_{\widetilde{U}}(f))^n\big)(V) \ .$$

Let now V be an entourage of $\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})$. We must show that $Q(V)$ is again an entourage of $\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})$. After enlarging V we can assume that it is of the form $V = U_\psi \cap W_r$ as in the proof of Lemma 9.27, where the function ϕ (which is the first component of ψ) is such that $\phi(i)$ is a uniform entourage of the form $U_{r(i)}$ for every i in $\mathbb{N}$, see (9.1). Since $P_{\widetilde{U}}(f)$ is distance decreasing we see that $Q(W_r) \subseteq W_r$. Since $P_{\widetilde{U}}(f)$ preserves the first coordinate of the cone and is distance decreasing we also see that $Q(U_\psi) \subseteq U_\psi$. Hence we actually get $Q(V) \subseteq V$.

Finally, for every bounded subset A of $\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})$ there exists r in $(0,\infty)$ and a bounded subset B of X such that $A \subseteq [0,r] \times P_{\widetilde{U}}(B)$. We can choose an integer n such that $f^n(X) \cap B = \emptyset$. Then $\mathcal{O}(P_{\widetilde{U}}(f))^n(\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})) \cap A = \emptyset$.

We conclude that

$$\mathrm{Yo}^s_{\mathrm{wfl}}(\mathcal{O}(P_{\widetilde{U}}(X)_{bdu})) \simeq 0 \ .$$

Taking the colimit over the invariant entourages entourages U and again using the cofinality of the resulting family of entourages $\widetilde{U}$ we get $F_{\mathrm{wfl}}(X) \simeq 0$.

From the fiber sequence (11.6) we now conclude that

$$F^{\infty}_{\mathrm{wfl}}(X) \simeq 0 \ .$$

4. (u-continuity) This is just a cofinality check:

$$\operatorname*{colim}_{U\in\mathcal{C}^{\Gamma}} F^{\infty}_{\mathrm{wfl}}(X_U) \simeq \operatorname*{colim}_{U\in\mathcal{C}^{\Gamma}} \operatorname*{colim}_{V\in\mathcal{C}\langle U\rangle^{\Gamma}} \mathcal{O}^{\infty}_{\mathrm{wfl}}(P_V(X)_{bdu}) \simeq \operatorname*{colim}_{V\in\mathcal{C}^{\Gamma}} \mathcal{O}^{\infty}_{\mathrm{wfl}}(P_V(X)_{bdu}) \simeq F^{\infty}_{\mathrm{wfl}}(X) \ .$$

This finishes the proof of Proposition 11.22. □

Remark 11.23. Let X be flasque. In the above proof we have shown that $F_{\mathrm{wfl}}(X) \simeq 0$. Note that we do not expect that $F(X) \simeq 0$. ♦

Let E be a strong Γ-equivariant $\mathbf{C}$-valued coarse homology theory. Then we have an essentially unique factorization $E_{\mathrm{wfl}} \colon \Gamma\mathbf{Sp}\mathcal{X}_{\mathrm{wfl}} \to \mathbf{C}$. The composition

$$E_{\mathrm{wfl}} \circ F^{\infty}_{\mathrm{wfl}} \colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

is then a Γ-equivariant $\mathbf{C}$-valued coarse homology theory. We have an equivalence

$$E \circ F^{\infty} \simeq E_{\mathrm{wfl}} \circ F^{\infty}_{\mathrm{wfl}} \ .$$

Corollary 11.24. *If E is a strong equivariant $\mathbf{C}$-valued coarse homology theory, then*

$$E \circ F^{\infty} \colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

is a Γ-equivariant coarse homology theory.

Let E be an equivariant coarse homology theory and Q be a Γ-bornological coarse space.

Lemma 11.25. *If E is strong, then E_Q is also strong.*

Proof. If X is a weakly flasque Γ-bornological coarse space with weak flasqueness implemented by $f \colon X \to X$, then $f \otimes \mathrm{id}_Q$ implements weak flasqueness of $X \otimes Q$. This implies the lemma. □

If the underlying Γ-set of Q is free, then by Corollary 11.14 we have

$$E_Q(F^0(X)) \simeq E(F^0(X) \otimes \mathrm{Yo}^s(Q)) \simeq E(X \otimes Q) \simeq E_Q(X) \ .$$

In particular, the functor

$$E_Q \circ F^0 \colon \Gamma\mathbf{BornCoarse} \to \mathbf{C}$$

is an equivariant coarse homology theory.

Let Q be a Γ-bornological coarse space and E be an equivariant coarse homology theory.

Corollary 11.26. *If E is strong and the underlying Γ-set of Q is free, then the forget-control map*

$$\beta\colon E_Q \circ F^\infty \to \Sigma E_Q \circ F^0$$

is a transformation between equivariant coarse homology theories.

This aspect of the theory (in the case of a trivial group Γ) is further studied in [BE17].

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