UNIT DISTANCE GRAPHS AND ALGEBRAIC INTEGERS

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Abstract. We answer a question of Brass about vertex degrees in unit distance graphs of finitely generated additive subgroups of $\mathbb{R}^2$.

The following problem was posed in [2, p. 186] (see also [1]).

Problem 1. *Does there exist a finitely generated additive subgroup $\mathcal{A} \subset \mathbb{R}^2$ such that there are infinitely many elements of $\mathcal{A}$ lying on the unit circle?*

If we look at the graph whose vertex set is a given finitely generated additive group $\mathcal{A} \subset \mathbb{R}^2$, and where $x, y \in \mathcal{A}$ are joined by an edge if and only if $|x - y| = 1$, then the question is whether such a graph can have infinite vertex degrees. This question was motivated by the problem of Erdős about maximal number of unit distances among n points in the plane [3]. In this short note we give a positive answer to Problem 1. Let us denote

$$\langle v_1, \dots, v_n \rangle_{\mathbb{Z}} := \{a_1 v_1 + \dots + a_n v_n \mid a_1, \dots, a_n \in \mathbb{Z}\}.$$

Theorem 1. *There exist four vectors $v_1, v_2, v_3, v_4 \in \mathbb{R}^2$ such that $\mathcal{A} = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{Z}}$ has infinitely many elements on the unit circle.*

Proof. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $p \in \mathbb{Z}[z]$ be the irreducible polynomial

$$p(z) = z^4 - z^3 - z^2 - z + 1.$$

Using Descartes' rule of sign and noting that $p(1) < 0$ we see that p has exactly two positive real roots. Since $z^4 - z^2 + 1 > 0$ for all $z \in \mathbb{R}$, it has no negative roots. Since p is reciprocal, the two remaining complex roots α and $\overline{\alpha}$ must be inverses of each other, so that $|\alpha| = 1$. Next, we have $\alpha^m \in \langle 1, \alpha, \alpha^2, \alpha^3 \rangle_{\mathbb{Z}}$ for all $m \in \mathbb{Z}$ because p is monic and has integer coefficients. Moreover, the numbers α^m, $m \in \mathbb{Z}$ satisfy $|\alpha^m| = 1$ and are all distinct since p is not divisible by any cyclotomic polynomial. Therefore, the set $\mathcal{A} = \langle 1, \alpha, \alpha^2, \alpha^3 \rangle_{\mathbb{Z}}$ has infinitely many elements on the unit circle. ☐

Of course, the above construction works for any algebraic integer $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. If we take α to be a primitive m-th root of unity, then the resulting unit distance graph has degree m or $2m$ depending on the parity of m. Taking $\alpha \in \mathbb{C}$ to be any other algebraic integer of absolute value 1 gives a unit distance graph of infinite degree. For example, one can take any non-real conjugate of a Salem number (a positive algebraic integer, all of whose conjugates have absolute value ≤ 1, see [4, Ch. 3]); this is exactly what was done in the proof above.

It is also natural to ask what is the minimal possible number of generators for such $\mathcal{A}$. The following result shows that in this sense Theorem 1 is optimal.

Theorem 2. *If an additive subgroup $\mathcal{A} \subset \mathbb{R}^2$ has rank ≤ 3, then it has only finitely many elements on the unit circle S^1.*

Proof. Suppose that $\mathcal{A} \cap S^1$ is infinite. We can restrict to the case when $\mathrm{rk}(\mathcal{A}) = 3$ and $\mathcal{A}$ spans $\mathbb{R}^2$ over reals, since otherwise $\mathcal{A}$ is contained in a lattice or in a line and thus intersects S^1 in a finite set. By the same argument, the subgroup generated by $\mathcal{A} \cap S^1$ must also have rank 3, and therefore we can choose three elements $v_1, v_2, v_3 \in \mathcal{A} \cap S^1$ and an integer $n > 0$ such that $\mathcal{A} \subseteq \langle v_1/n, v_2/n, v_3/n \rangle_{\mathbb{Z}}$.

Let us denote $(v_1, v_2) = \gamma$, $(v_2, v_3) = \alpha$, and $(v_3, v_1) = \beta$. Before we continue, let us record some elementary identities between v_i and the inner products α, β, γ. First, since the vectors v_i are in $\mathbb{R}^2$, the determinant of the Gram matrix of v_1, v_2, v_3 is 0, that is

$$1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 = 0. \tag{1}$$

Next, since v_i are linearly independent over $\mathbb{Q}$, we have $|\alpha|, |\beta|, |\gamma| < 1$, and thus, using (1), we see that $\alpha \neq \beta\gamma$, $\beta \neq \alpha\gamma$, and $\gamma \neq \alpha\beta$. Finally, we have the following identity

$$\frac{v_1}{\alpha - \beta\gamma} + \frac{v_2}{\beta - \alpha\gamma} + \frac{v_3}{\gamma - \alpha\beta} = 0. \tag{2}$$

Let $Q(x, y, z) = \|xv_1 + yv_2 + zv_3\|^2$ and let $\mathcal{S}$ be the set of solutions $(x, y, z) \in \mathbb{Z}^3$ of equation $Q(x, y, z) = n^2$; by our assumption this set is infinite. A triple (x, y, z) belongs to $\mathcal{S}$ if and only if

$$x^2 + y^2 + z^2 + 2\alpha yz + 2\beta zx + 2\gamma xy = n^2.$$

Since v_1, v_2, and v_3 are independent over $\mathbb{Q}$, the intersection of $\mathcal{S}$ with any plane in $\mathbb{Z}^3$ is finite. In particular, the set $\mathcal{S}_0 = \{(x, y, z) \in \mathcal{S} \mid xyz \neq 0\}$ is infinite, and the condition for $(x, y, z) \in \mathcal{S}_0$ can be rewritten as

$$\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = \frac{n^2 - x^2 - y^2 - z^2}{2xyz}. \tag{3}$$

Denote $\mathcal{V} = \{(1/x, 1/y, 1/z) \mid (x, y, z) \in \mathcal{S}_0\}$. We consider three cases.

Case 1. The $\mathbb{Q}$-span of $\mathcal{V}$ is $\mathbb{Q}^3$. By choosing any three linearly independent vectors in $\mathcal{V}$ and solving (3) for α, β, γ, we get that α, β, and γ must be rational. But then (2) gives a linear relation over $\mathbb{Q}$ contradicting the fact that $\mathrm{rk}(\mathcal{A}) = 3$.

Case 2. The $\mathbb{Q}$-span of $\mathcal{V}$ is 1-dimensional. Then $\mathcal{S}_0$ is contained in a line through the origin, and by homogeneity of Q we get that $|\mathcal{S}_0| \leq 2$, a contradiction.

Case 3. The $\mathbb{Q}$-span of $\mathcal{V}$ is 2-dimensional. Then there is a unique nonzero triple $(a, b, c) \in \mathbb{Z}^3$ such that for all $(x, y, z) \in \mathcal{S}_0$ we have $a/x + b/y + c/z = 0$. In this case, by solving (3) for α, β, γ, we get

$$\alpha = \lambda a + \alpha_0, \quad \beta = \lambda b + \beta_0, \quad \gamma = \lambda c + \gamma_0 \tag{4}$$

for some $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{Q}^3$ and $\lambda \in \mathbb{R}$. We may assume that $abc \neq 0$, since otherwise $\mathcal{S}_0$ would be contained in one of the three planes $ay + bx = 0$, $az + cx = 0$, or $bz + cy = 0$, and hence would be finite.

Next, recall that all integer solutions of $1/x + 1/y + 1/z = 0$ are given by $x = dst$, $y = drt$, $z = drs$ for some nonzero integers d, r, s, t with $r + s + t = 0$ (simply note that the equation is equivalent to $z^2 = (z + x)(z + y)$ that is in turn equivalent to $z^2 = x'y'$). By applying this observation to the equation $a/x + b/y + c/z = 0$ we deduce that any solution $(x, y, z) \in \mathcal{S}_0$ can be written as

$$x = \frac{dst}{bc}, \quad y = \frac{drt}{ac}, \quad z = \frac{drs}{ab}$$

for some nonzero integers d, r, s, t with $r + s + t = 0$. Plugging these expressions back into Q and setting $v_3 = \kappa_1 v_1 + \kappa_2 v_2$ (where κ_i can be computed from (2)) we get

$$n^2 = \|xv_1 + yv_2 + zv_3\|^2 = \frac{d^2}{(abc)^2}\|s((\kappa_1 c - a)r - as)v_1 + r((\kappa_2 c - b)s - br)v_2\|^2,$$

and thus, setting $\varphi_1(r, s) = (\kappa_1 c - a)r - as$ and $\varphi_2(r, s) = (\kappa_2 c - b)s - br$, we get

$$\|s\varphi_1(r, s)v_1 + r\varphi_2(r, s)v_2\|^2 = \frac{(abcn)^2}{d^2} \leq (abcn)^2.$$

Since v_1 and v_2 form a basis in $\mathbb{R}^2$ and $|s|, |r| \geq 1$ we get that $|\varphi_i(r,s)| \leq C$ for some constant C that depends only on a, b, c, n, and γ. If φ_1 and φ_2 are not proportional, then these two inequalities define a bounded region in the (r,s)-plane, and hence there are only finitely many solutions in this case, contrary to our assumption. Therefore, φ_1 and φ_2 must be proportional, that is

$$(\kappa_1 c - a)(\kappa_2 c - b) - ab = 0$$

or equivalently

$$a(\alpha - \beta\gamma) + b(\beta - \alpha\gamma) + c(\gamma - \alpha\beta) = 0. \tag{5}$$

If we now substitute (4) into (1) and into (5), we obtain two different polynomial equations on λ, $p_1(\lambda) = 0$ and $p_2(\lambda) = 0$, where

$$\begin{aligned} p_1(\lambda) \;&=\; 1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 = 2abc\lambda^3 + (2bc\alpha_0 + 2ac\beta_0 + 2ab\gamma_0 - a^2 - b^2 - c^2)\lambda^2 \\ &+ 2(a\beta_0\gamma_0 + b\alpha_0\gamma_0 + c\alpha_0\beta_0 - a\alpha_0 - b\beta_0 - c\gamma_0)\lambda + (2 + \alpha_0\beta_0\gamma_0 - \alpha_0^2 - \beta_0^2 - \gamma_0^2) \end{aligned}$$

and

$$\begin{aligned} p_2(\lambda) \;&=\; a(\alpha - \beta\gamma) + b(\beta - \alpha\gamma) + c(\gamma - \alpha\beta) = -3abc\lambda^2 \\ &- (2bc\alpha_0 + 2ac\beta_0 + 2ab\gamma_0 - a^2 - b^2 - c^2)\lambda - (a\beta_0\gamma_0 + b\alpha_0\gamma_0 + c\alpha_0\beta_0 - a\alpha_0 - b\beta_0 - c\gamma_0). \end{aligned}$$

We observe that $2p_2(\lambda) = -p_1'(\lambda)$, so that λ is a double root of a cubic polynomial p_1 with rational coefficients. It is well-known that this implies $\lambda \in \mathbb{Q}$, but then α, β, and γ are also rational, and once again we arrive at a contradiction. □

References

[1] P. Brass, *Erdős's distance problems in normed spaces*, Computational Geometry 6 (4), pp. 195–214 (1996).

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