# BASIC THEOREMS OF DISTRIBUTIONS AND FOURIER TRANSFORMS

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# Abstract

Distribution theory is an important tool in studying partial differential equations. Distributions are linear functionals that act on a space of smooth test functions. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense. In particular, any locally integrable function has a distributional derivative. There are different possible choices for the space of test functions, leading to different spaces of distributions. In this report, we take a look at some basic theory of distributions and their Fourier transforms. And we also solve some typical exercises at the end.

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# Chapter 1

# Introduction

Distribution theory was developed by the French mathematician Laurent Schwartz in the early fifties, after the work of P. Dirac, O. Heaviside, J. Leray and S. Sobolev. It is still an important tool in mathematical analysis today.

Distributions are linear functionals that map a set of test functions into the set of real numbers. The basic idea in distribution theory is to reinterpret functions as linear functionals acting on a space of test functions. We often define distribution by integrating standard functions against a test function. On distributions, we can define a (generalized) derivative so that many of the usual rules of calculus will hold. Moreover, distributional derivatives generalize classical derivatives: if f has a classical derivative, then its distributional derivative is the same as the classical one.

The Fourier transform is easily defined in $L^1$ as an integral, but image of the Fourier transformation. Hence the inverse can not always be defined as an integral. In $L^2$, the Fourier transform must be defined as limit, but turns out to be one-to-one onto $L^2$. The right space here is the space of Schwartz functions. The Fourier transform is an automorphism on the Schwartz space, which is contained in $L^1$. $\mathcal{S}$ being a topological vector space, $F$ induces an automorphism on its dual, the space of tempered distributions $\mathcal{S}'$.

Not all the proofs of theorems presented in this report are included. We selected a number of them, which we believe that the proof of those will illustrate the important techniques.

This work is mostly based on the notes of Josefina D. Alvarez Alonso[1]. We also consult the books from C. Zuily[2] and S. Kesavan[3]. For necessary real analysis background, we use the book from R. Wheeden and A. Zygmund[4].

# Chapter 2

# The Spaces $\mathcal{D}$ and $\mathcal{D}'$

**Definition 2.1.** *Given a function $\phi : \Omega \to \mathbb{C}$, $\phi \subset \mathbb{R}^n$ open, the* support *of $\phi$, denoted as* supp$(\phi)$, *is the closure of the set $\{x \in \Omega \mid \phi(x) \neq 0\}$. When a function is continuous and has continuous derivatives of all orders, we will say that it is infinitely differentiable.*
*The space of infinitely differentiable function with compact support is denoted by $C_0^\infty(\mathbb{R}^n)$ in $\mathbb{R}^n$.*

**Definition 2.2.** *Let $x \in \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$. A multi-index $\alpha$ is an n-tuple*

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

*The order of a multi-index $\alpha$ is defined as*

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

*Given $x \in \mathbb{R}, \alpha, \beta$ multi indices, we say $\alpha \leq \beta$ iff $\alpha_i \leq \beta_i, \forall 1 \leq i \leq n$. For $\alpha \leq \beta$, we define $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_n}{\beta_n}$. We set*

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

**Definition 2.3.** *Let* $\mathcal{D}$=$\{\phi : \mathbb{R}^n \to \mathbb{C} \mid \phi$ *is infinitely differentiable and the* $\operatorname{supp}(\phi)$ *is compact*$\}$. *And* $\mathcal{D}(\Omega) = \{\phi : \Omega \to \mathbb{C} \mid \phi$ *is infinitely differentiable and the* $\operatorname{supp}(\phi)$ *is compact* .$\}$ *We say that* $\phi_j \to \phi$ *in* $\mathcal{D}$ *iff* $\exists$ *compact set* $K \subset \mathbb{R}^n$ *such that* $\operatorname{supp}(\phi), \operatorname{supp}(\phi_j) \subset K$, *and* $\forall \alpha, \forall j \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}^n} |D^\alpha \phi_j - D^\alpha \phi| \to 0 \text{ as } j \to \infty$$

*Functions in* $\mathcal{D}$ *are called test functions.*

**Example 2.4.** *Let* $\rho(x) = \begin{cases} e^{\frac{-1}{1-|x|^2}}, & |x| < 1; \\ 0, & |x| \geq 1. \end{cases}$ *then* $\rho$ *in the space* $\mathcal{D}$ *and* $\operatorname{supp}(\rho) = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

**Example 2.5.** *If* $f \in L^1_{loc}(\mathbb{R}^n)$, *then* $(T_f, \phi) = \int f\phi dx$ *defines an element of* $\mathcal{D}$. *More examples are found in the exercises at the end of this report.*

**Definition 2.6.** *A linear funtional* $T : \mathcal{D} \to \mathbb{C}$ *is called a distribution if whenever* $\phi_m \to \phi$ *in* $\mathcal{D}$, *we have* $(T, \phi_m) \to (T, \phi)$. *The space of distributions, which is the dual of the space of test functions, is denoted by* $\mathcal{D}'$.

**Definition 2.7.** *A sequence* $\{T_j\}$ *in* $\mathcal{D}'$ *is said to converge to the distribution* $T \in \mathcal{D}'$ *if for each* $\phi \in \mathcal{D}, (T_j, \phi) \to (T, \phi)$.

**Definition 2.8.** *Given funtions* $f, g : \mathbb{R}^n \to \mathbb{C}$,*the convolution of* $f$ *with* $g$ *is defined by*

$$f * g(x) = \int_{\mathcal{R}^n} f(x-y)g(y)dy.$$

**Theorem 2.9.** *If* $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, *then the convolution* $f * g$ *is well-defined for almost everywhere* $x \in \mathbb{R}$ *and further,* $f * g \in L^p(\mathbb{R}^n)$ *with*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

*Proof.* Let $q$ be the conjugate exponent of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $h \in L^q(\mathbb{R}^n)$. Then

$(x, y) \to f(x-y)g(y)h(x)$ is measurable and,

$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)g(y)h(x)|dxdy &= \int_{\mathbb{R}} |h(x)| \int_{\mathbb{R}} |f(x-y)g(y)|dydx \\
&= \int_{\mathbb{R}} |h(x)| \int_{\mathbb{R}} |f(t)g(x-t)|dtdx \\
&= \int_{\mathbb{R}} |f(t)| \int_{\mathbb{R}} |h(x)||g(x-t)|dxdt \\
&\leq \|h\|_{L^q(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)} \\
&< +\infty
\end{aligned}$$

where we have used Holder's inequality and the fact that by the translation invariance of the Lebesgue measure $g(x)$ and $g(x-t)$ have the same $L^p$ norm. Thus by Fubini's theorem

$$\int_{\mathcal{R}} h(x)f(x-y)g(y)dy$$

exists for almost all x and we can choose $h(x) \neq 0$ for all $x$. Also

$$h \to \int (f * g)h$$

is a continuous linear functional on $L^q(\mathbb{R}^n)$ with norm bounded by $\|g\|_{L^p(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}$ which shows, by the Riesz Representation Theorem, that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}$ holds.

□

**Theorem 2.10.** *Given $K \subset \mathbb{R}^n$ compact, and $\epsilon > 0$, there exist $\phi \in \mathcal{D}$ such that*

$$0 \leq \phi(x) \leq 1, \forall x \in \mathbb{R}^n,$$

$$\phi(x) = 1, \forall x \in K,$$

$$\text{supp}(\phi) \subset K_\epsilon = \{x \in \mathbb{R}^n | d(x, K) < \epsilon\}.$$

**Theorem 2.11.** *$\mathcal{D}$ is dense in $L^p$ , $1 \leq p < \infty$.*

**Definition 2.12.** *A distribution $T \in \mathcal{D}$ is zero on an open set $\Omega \subset \mathbb{R}^n$, denoted as $T|_\Omega = 0$ if $\forall \phi \in \mathcal{D}$ with $\mathrm{supp}(\phi) \subset \Omega$,*

$$(T, \phi) = 0.$$

**Definition 2.13.** *Given $T in \mathcal{D}'$, the support of $T$ is denoted as $\mathrm{supp}(T)$,*

$$\mathrm{supp}(T) = \mathcal{R}^n \backslash \cup \{\Omega \subset \mathcal{R}^n, T|_\Omega = 0\}$$

**Theorem 2.14.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\{\Omega_i\}, i \in I$ constitute an open cover of $\Omega$. Let $T_i \in \mathcal{D}'(\Omega_i)$ such that whenever $\Omega_i \cap \Omega_j \neq \emptyset, i \neq j$, then*

$$T_i|_{\Omega_i \cap \Omega_j} = T_j|_{\Omega_i \cap \Omega_j}$$

*Then there exists a unique distribution $T \in \mathcal{D}'(\Omega)$ such that*

$$T|_{\Omega_i} = T_i, \forall i \in I.$$

*Proof.* Let $\{\phi_i\}, i \in I$ be a locally finite $C^\infty$ partition of unity subordinate to the cover $\{\Omega_i\}$. Let $\phi \in \mathcal{D}(\Omega)$, then the support of $\phi$ intersects only finitely many open sets $\Omega_i$ and $\phi\phi_i$ has support in $\Omega_i$. We define

$$T(\phi) = \sum_{i \in I} T_i(\phi\phi_i)$$

Which makes sense since the right-hand-side is a finite sum. Let $\tilde{\phi_m} \to 0$ in $\mathcal{D}(\Omega)$. Let K be a compact set containing $supp(\tilde{\phi_m})$ for all $m$. Let $i_1, i_2, ...i_l$ be the indices such that $K \bigcap \mathrm{supp}(\phi_{i_j})$ is non-empty for $1 \leq j \leq l$ and $K \bigcap \mathrm{supp}(\phi_i) = \emptyset$ for all other $i$.Thus,

$$T(\tilde{\phi_m}) = \sum_{j=1}^{l} T_{i_j}(\tilde{\phi_m}\phi_{i_j})$$

Note that $\tilde{\phi_m}\phi_{i_j} \to 0$ in $\mathcal{D}(\Omega_{i_j})$. Thus $T(\tilde{\phi_m}) \to 0$ and so $T \in \mathcal{D}'(\Omega)$

We now show that $T|_{\Omega_i} = T_i$, Let $\phi \in \mathcal{D}(\Omega_i)$. For any $j$,

$$\phi\phi_j \in \mathcal{D}(\Omega_i \cap \Omega_j)$$

then $$T_i(\phi\phi_j) = T_j(\phi\phi_j)$$

$$T(\phi) = \sum_j T_j(\phi\phi_j) = \sum_j T_i(\phi\phi_j) = T_i(\sum_j \phi\phi_j) = T_i(\phi).$$

□

# Chapter 3

# The Spaces $\mathcal{S}, \mathcal{S}', \mathcal{E}, \mathcal{E}'$

**Definition 3.1.** $\mathcal{E} = \{\varphi : \mathbb{R}^n \to \mathbb{C}, \varphi \in \mathbb{C}^\infty(\mathbb{R}^n)\}$

$\phi_j \to \phi$ *in* $\mathcal{E}$ *iff* $\forall \alpha \in \mathbb{N}_0^n$

$$\sup_{x\in\mathbb{R}^n} |D^\alpha \phi_j - D^\alpha \phi| \to 0 \text{ as } j \to \infty.$$

**Definition 3.2.** *The Schwartz Space, or the space of rapidly decreasing functions,* $\mathcal{S}$*, is given by*

$$\mathcal{S} = \{f \in \mathcal{E}(\mathbb{R}^n) | \lim_{|x|\to\infty} |x^\beta D^\alpha f(x)| = 0 \text{ for all multi-indices } \alpha \text{ and } \beta\}$$

*We say that* $\phi_j \to \phi$ *in* $\mathcal{S}$ *iff* $\forall$ *multi indices* $\alpha, \beta$,

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta \phi_j - D^\beta \phi)| \to 0 \text{ as } j \to \infty.$$

The following statements are easy to verify for $f \in \mathcal{E}(\mathbb{R}^n)$:

1. $f \in \mathcal{S}$ if, and only if, for every polynomial $P(x)$ and for every differential operator $L$ with constant coefficients, the function

$$P(x)Lf(x)$$

is bounded in $\mathbb{R}^n$.

2. $f \in \mathcal{S}$ if, and only if, for any integer $k \geq 0$, and any multi-index $\alpha$, the function

$$(1+|x|^2)^k D^\alpha f(x)$$

is bounded in $\mathbb{R}^n$.

**Theorem 3.3.** $\mathcal{S} \subset L^1(\mathbb{R}^n)$ *and the inclusion is continuous.*

*Proof.* Let $f \in \mathcal{S}$. Then for any integer $k \geq 0$, there exists a constant $M_k > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |f(x)|(1+|x|^2)^k \leq M_k$$

Now, for $k > n/2$, it is well known that $(1+|x|^2)^{-k} \in L^1(\mathbb{R}^n)$ (using polar coordinates). Hence

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|\, dx &= \int_{\mathbb{R}^n} |f(x)|\, (1+|x|^2)^k (1+|x|^2)^{-k} dx \\ &\leq M_k \int_{\mathbb{R}^n} (1+|x|^2)^{-k} dx \\ &< +\infty. \end{aligned}$$

Thus $f \in L^1(\mathbb{R}^n)$. Also if $C = \int_{\mathbb{R}^n} (1+|x|^2)^{-k} dx$, then

$$\|f\|_{L^1(\mathbb{R}^n)} \leq C \sup_{x \in \mathbb{R}^n} (|f(x)|(1+|x|^2)^k).$$

Hence if $f_m \to 0$ in $\mathcal{S}$ it follows that $\|f_m\|_{L^1(\mathbb{R}^n)} \to 0$ and the continuity of the inclusion follows. □

**Theorem 3.4.**

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E},$$

*with dense inclucions.*

**Theorem 3.5.** *$\mathcal{E}'$ can be identified with subset of $\mathcal{D}'$ of distributions of compact support.*

**Theorem 3.6.** *Let $T : \mathcal{D} \to \mathbb{C}$, the following are equivalent:*

*(i) $T$ is a distribution, i.e. $T \in \mathcal{D}'$*

*(ii) $T$ is linear, and given any compact set $K$, $\exists m = m(K) \in \mathbb{N}_0$ and $C_K > 0$ s.t. $\forall \phi \in \mathcal{D}(K)$*

$$|(T, \phi)| \leq C_K \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha \phi(x)|$$

*if $m$ does not depend on $K$, we say that $T$ is a distribution of order $m$.*

*Proof.* (ii)$\Rightarrow$ (i) is easy to show, $T$ is linear and well-defined by (ii). $\forall \phi_j \to 0$ in $\mathcal{D}$,

$$(T, \phi_j) \leq C_K \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha \phi_j)| \to 0$$

So T is a distribution.

(i) $\Rightarrow$ (ii) (proof by contradiction) Assume $\exists K \subset \mathbb{R}$, $K$ compact such that $\forall m \in \mathbb{N}$ and $C_K > 0$, $\exists$ $\phi$ with $supp(\phi) \subset K$ such that

$$|(T, \phi)| > C_K \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha \phi(x)|$$

Pick $C_K = m$ and let $\phi_m$ the function in $\mathcal{D}(\mathbb{R}^n)$ such that $supp(\phi_m) \subset K$, so

$$|(T, \phi_m)| > m \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha \phi_m(x)|$$

WLOG, may assue $(T, \phi_m) = 1$, then

$$\frac{1}{m} > \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq m}} |D^\alpha \phi_m(x)$$

So $\phi_m \to 0$ in $\mathcal{D}$, contradicts that $(T, \phi_m) = 1$. $\square$

**Definition 3.7.** *$\mathcal{D}'^{(m)}$ is the subspace of $\mathcal{D}'$ form by distributions of order $\leq m$*

**Definition 3.8.** *Given $m \in \mathbb{N}$,*

*$\mathcal{D}^{(m)} = \{\phi : \mathbb{R}^n \to \mathbb{C} | \phi$ has continuous derivatives for order $\leq m$ and $supp(\phi)$ is compact$\}$.*

*We say that $\phi_j \to \phi$ in $\mathcal{D}^{(m)}$ iff for any compact set $K$ such that $supp(\phi), supp(\phi_j) \subset K$ and $\forall \alpha, |\alpha| \leq m$,*

$$D^\alpha \phi_j(x) \to D^\alpha \phi(x) \text{ in } \mathbb{R}^n \quad .$$

**Theorem 3.9.** *(Leibniz's rule)*

*If $\phi, \psi \in \mathcal{E}$, given multi-index $\alpha \in \mathbb{N}^n$,*

$$D^\alpha(\phi\psi) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \phi D^{\alpha-\beta} \psi$$

# Chapter 4

# Differentiation of distributions

Let $f \in C^1, \phi \in \mathcal{D}$ , then

$$(T_{\frac{\partial f}{\partial x_1}}, \phi) = \int \frac{\partial f}{\partial x_1}(x)\phi(x)dx = \int_{-\infty}^{\infty} dx_n \dots \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_1}(x)\phi(x)dx$$

$$(T_{\frac{\partial f}{\partial x_j}}, \phi) = -(T_f, \frac{\partial \phi}{\partial x_j})$$

**Definition 4.1.**

$$(D^\alpha T, \phi) = (-1)^{|\alpha|}(T, D^\alpha \phi)$$

In any of these spaces $\mathcal{D}', \mathcal{S}', \mathcal{E}'$, we define $(D^\alpha T, \phi) = (-1)^{|\alpha|}(T, D^\alpha \phi)$ for $\phi$, respectively in $\mathcal{D}, \mathcal{S}, \mathcal{E}$.

**Theorem 4.2.** *Let $K$ be any the of spaces $\mathcal{D}', \mathcal{S}', \mathcal{E}'$, then $D^\alpha$ is a linear and continuous operator from $K$ into itself. In addition, if $T \in \mathcal{D}'$,*

$$supp(D^\alpha T) \subset supp(T)$$

*Proof.* We will show this for $K = \mathcal{S}'$, linearity of $D^\alpha$ is trivial. Assume $T_j \to T$ in $\mathcal{S}'$, then

$\forall \phi \in \mathcal{S}, (T_j, \phi) \to (T, \phi)$, so

$$\begin{aligned}(D^\alpha T_j, \phi) &= (-1)^{|\alpha|}(T_j, D^\alpha \phi) \\ &\to (-1)^{|\alpha|}(T, D^\alpha \phi) \\ &= (D^\alpha T, \phi)\end{aligned}$$

Next to show $supp(D^\alpha T) \subset supp(T)$, assume $supp(\phi) \subset \mathbb{R}^n \backslash supp(T)$, then $(T, \phi) = 0$, since $supp(D^\alpha \phi) \subset supp(\phi)$, so $(T, D^\alpha \phi) = 0$,

$$(D^\alpha T, \phi) = (-1)^{|\alpha|}(T, D^\alpha \phi) = 0$$

so $supp(D^\alpha T) \subset supp(T)$. □

**Definition 4.3.** *The translation operator $\tau_{-h}$ is defined by*

$$\tau_{-h} f(x) = f(x+h)$$

*Where $h = (0, \ldots 0, h_j, 0, \ldots, 0)$*

**Theorem 4.4.** *There exist*

$$\lim_{h \to 0} \frac{\tau_{-h} T - T}{h_j} = \frac{\partial T}{\partial x_j}$$

*That is , $\forall \phi \in \mathcal{D}$,*

$$\lim_{h \to 0} \frac{(\tau_{-h} T, \phi) - (T, \phi)}{h_j} = (\frac{\partial T}{\partial x_j}, \phi).$$

*Proof.* $\forall \phi \in \mathcal{D}$, left hand side is

$$\frac{(\tau_{-h} T, \phi) - (T, \phi)}{h_j} = \frac{(T, \tau_h \phi) - (T, \phi)}{h_j} = (T, \frac{\tau_h \phi - \phi}{h_j})$$

Each function $\frac{\tau_h\phi-\phi}{h_j}$ has compact support independent of $h$, say $|h|\leq 1$,

$$D^\alpha[\frac{\phi(x-h)-\phi(x)}{h_j}] = \frac{D^\alpha\phi(x-h)-D^\alpha\phi(x)}{h_j}$$

So it is sufficient to prove that $\frac{\tau_h\phi-\phi}{h_j} \to \frac{-\partial\phi}{\partial x_i}$ in $\mathcal{D}$ as $h\to 0$. We will show this by using Mean Value Theorem twice,

$$\begin{aligned} |\frac{\phi(x-h)-\phi(x)}{h_j} + \frac{\partial\phi}{\partial x_j}(x)| &= |-\frac{\partial\phi}{\partial x_j}(\xi) + \frac{\partial\phi}{\partial x_j}(x)| \\ &= |\frac{\partial^2\phi}{\partial x_j^2}(\eta)||x-\xi| \end{aligned}$$

Where $\xi$ is between $x$ and $x-h$, $\eta$ is between $x$ and $\xi$. Since $|x-\xi|\to 0$ as $h\to 0$ and $\phi$ has compact support, so $|\frac{\partial^2\phi}{\partial x_j^2}(\eta)|$ is bounded. So

$$\lim_{h\to 0}\frac{(\tau_{-h}T,\phi)-(T,\phi)}{h_j} = (\frac{\partial T}{\partial x_j},\phi).$$

□

**Theorem 4.5.** *If $T\in\mathcal{S}'$, there exists a rapid decreasing function $f$ and multi-index $\alpha\in N^n$ such that*

$$T = D^\alpha f$$

**Theorem 4.6.** *Let $T\in\mathcal{D}'$, and $\Omega\in\mathbb{R}^n$, $\Omega$ is open such that $\bar{\Omega}$ is compact. Then there is a continuous function $f=f(\Omega): R^n\to C$ and $m=m(\Omega)\in N$ such that*

$$T = \frac{\partial^{mn}}{\partial x_1^m\ldots\partial x_n^m}f$$

.

# Chapter 5

# Tensor Product, Convolution and Multiplication of Distributions

**Definition 5.1.** *Given* $f = f(x) : \mathcal{R}^n \to \mathcal{C}$, $g = g(y) : \mathcal{R}^m \to \mathcal{C}$, *then the tensor product is defined by* $f(x) \times g(y) : \mathcal{R}^{n+m} \to \mathcal{C}$.
*If* $f \in L^1_{loc,x}$ *and* $g \in L^1_{loc,y}$, *then* $f \times g \in L^1_{loc}$,

$$\begin{aligned}
(f \times g, \phi) &= \int f(x)g(y)\phi(x,y)dxdy \\
&= \int f(x)\left[\int g(y)\phi(x,y)dy\right]dx \\
&= \int g(y)\left[\int f(x)\phi(x,y)dx\right]dy
\end{aligned}$$

*Notation:* $T_f$ *is on* $\mathcal{D}'_x$, $T_g$ *on* $\mathcal{D}'_y$, $T_{fg}$ *on* $\mathcal{D}'_{xy}$.

**Definition 5.2.** *A function is* $\phi(x,y)$ *is called a function of separated variables if* $\phi(x,y) = \alpha(x)\beta(y)$ *for functions* $\alpha, \beta$. *If we define* $W = T_x \times S_y$, *then*

$$(W, \phi(x,y)) = (T_x, \alpha)(S_y, \beta).$$

**Theorem 5.3.** *Let* $T \in \mathcal{D}'_x, S \in \mathcal{D}'_y$ *, then*

*i) Given* $\phi \in \mathcal{D}_{xy}$ *, the function* $\psi(x) = (S_y, \phi(x,y)) \in \mathcal{D}_x$

*ii)The application*

$$\mathcal{D}_{xy} \to \mathbb{C}$$

$$\phi \to (T_x, (S_y, \phi(x,y))) \in \mathcal{D}'_{xy}.$$

**Theorem 5.4.** *Functions of separate variables are dense in* $\mathcal{D}_{xy}$.

**Theorem 5.5.** *There exists a unique extension of* $T \times S$ *to* $\mathcal{D}_{xy}$ *such that*

$$(T_x \times S_y, \alpha(x)\beta(y)) = (T, \alpha)(S, \beta).$$

**Theorem 5.6.** *Let* $T \in \mathcal{D}'_x, S \in \mathcal{D}'_y$*, then*

*i)* $$supp(T \times S) = supp(T) \times supp(S)$$

*ii) Given* $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m$,

$$\mathcal{D}_x^\alpha \mathcal{D}_y^\beta (T \times S) = \mathcal{D}_x^\alpha T \times \mathcal{D}_y^\beta S.$$

**Definition 5.7.** *Given* $\phi \in \mathcal{D}$,

$$\begin{aligned}(f * g, \phi) &= \int \left[ \int f(x-y)g(y)dy \right] \phi(x)dx \\ &= \int \int f(x-y)g(y)\phi(x)dxdy \\ &= \int \int f(x)g(y)\phi(x+y)dxdy\end{aligned}$$

*Define*

$$(T * S, \phi) = (T_x \times S_y, \phi(x+y)).$$

**Theorem 5.8.** *If $T \in \mathcal{D}', S \in \mathcal{E}'$, then $(T * S, \phi)$ defines a distribution on $\mathcal{D}(\mathbb{R}^{2n})$ and*

$$supp(f * g) \subset supp(T) + supp(S)$$

.

**Theorem 5.9.** *The inclusions*

$$\mathcal{E} \subset \mathcal{D}'$$

$$\mathcal{D} \subset \mathcal{E}'$$

*holds, and they are continuous and dense.*

# Chapter 6

# Fourier transform

**Definition 6.1.** *Let $f \in L^1(\mathbb{R}^n)$. The Fourier Transform of $f$, denoted by $\widehat{f}$ or $F[f]$, is a function defined on $\mathbb{R}^n$ by the formula*

$$\widehat{f} = \int_{\mathbb{R}^n} e^{2\pi i x\cdot(x)dx,} \quad ,$$

*where $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$ is the usual Euclidean inner-product in $\mathbb{R}^n$.*

**Theorem 6.2.** *If $f \in L^1(\mathbb{R}^n)$, then $F[f]$ is bounded, continuous and*

$$\lim_{|\xi|\to\infty} F[f](\xi) = 0.$$

**Theorem 6.3.** *i) If $\forall \beta$, such that $|\alpha| \leq k, k \geq 1, D^\beta f$ is continuous and integrable, then*

$$F[D^\beta f](\xi) = (-2\pi i\xi)^\beta F[f](\xi), |\beta| \leq k.$$

*ii)If $f$ and $|x|^k f$ are integrable for some $k \geq 1$, then $\hat{f}$ has continuous derivatives up to and including order $k$, and*

$$D^\beta \hat{f}(\xi) = F[(2\pi i x)^\beta f](\xi), |\beta| \leq k.$$

**Theorem 6.4.** *If* $f \in L^1, h \in \mathbb{R}^n, k \in \mathbb{R}$ *and* $k \neq 0$, *then*

$$F[\tau_h f](\xi) = e^{2\pi i\xi\cdot h}\hat{f}(\xi)$$

$$F[f(kx)](\xi) = \frac{1}{|k|^n}\hat{f}(\frac{\xi}{k}).$$

**Definition 6.5.** *Conjugate Fourier transform of* $f \in L^1$, *denoted as* $\bar{F}[f]$, *is defined by*

$$\bar{F}[f](\xi) = \int e^{-2\pi i\xi\cdot x}f(x)dx.$$

If we define $\check{f}(x) = f(-x)$, then

$$\bar{F}[f] = \check{\hat{f}}$$

$$\bar{F}[f] = \overline{F[\bar{f}]}.$$

**Theorem 6.6.** *If* $f$ *and* $F[f] \in L^1$, *then*

$$\bar{F}F[f] = f \ a.e.$$

**Theorem 6.7.** *If* $f \in L^1 \bigcap L^2$, *then* $\hat{f} \in L^2$ *and*

$$\|F[f]\|_{L^2} = \|f\|_{L^2}.$$

**Theorem 6.8.** $F$ *and* $\bar{F}$ *are isomorphism from* $\mathcal{S}$ *to itself.*

*Given* $\phi \in \mathcal{S}$,

$$F \circ \bar{F}[\phi] = \bar{F} \circ F[\phi] = \phi.$$

*Proof.* If $F : \mathcal{S} \to \mathcal{S}$ is continuous, then so is $\bar{F}$ since $\bar{F}(\phi) = \overline{F(\bar{\phi})}$

If $\forall\phi \in \mathcal{S}, \bar{F}F[\phi] = \phi$, then

$$\overline{F\bar{F}[\phi]} = \overline{F\overline{F[\bar{\phi}]}} = \bar{F}[F[\bar{\phi}]] = \bar{\phi}$$

Next, want to prove that $F$ continuous from $\mathcal{S}$ to $\mathcal{S}$. Take $\phi \in \mathcal{S}$,

$$\begin{aligned}
\sup_{\xi \in \mathbb{R}^n} |\xi^\alpha D^\beta \hat{\phi}(\xi)| &\leq c_\alpha \int |D^\alpha[(2\pi i x)^\beta \phi(x)]| dx \\
&\leq c_\alpha \int \frac{(1+|x|^2)^{n+1}}{(1+|x|^2)^{n+1}} |D^\alpha[(2\pi i x)^\beta \phi(x)]| dx \\
&\leq c_\alpha \sup_{x \in \mathbb{R}^n} |(1+|x|^2)^{n+1} D^\alpha[(2\pi i x)^\beta \phi(x)]| \int (1+|x|^2)^{-n-1} dx
\end{aligned}$$

Since $\phi \in \mathcal{S}$, so $(2\pi i x)^\beta \phi(x) \in \mathcal{S}$, by alternate definition, $\sup_{x \in \mathbb{R}^n} |(1+|x|^2)^{n+1} D^\alpha[(2\pi i x)^\beta \phi(x)]|$ is bounded, $\int (1+|x|^2)^{-n-1} dx$ is also bounded, then $\hat{\phi}(\xi) \in \mathcal{S}$.

Next, need to show that $\bar{F}F(\phi) = \phi, \forall \phi \in \mathcal{S}$. $\forall y \in \mathbb{R}^n$,

$$\phi(y) = \int e^{-2\pi i \xi y} \hat{\phi}(\xi) d\xi = \int [\int e^{-2\pi i \xi (x-y} \phi(\xi) dx] d\xi$$

We could not reverse the order of the integral, however, given $\psi \in \mathcal{S}$, for each $j = 1, 2, \ldots,$ the following double integral exists:

$$\int \int e^{-2\pi i \xi (x-y)} \phi(\xi) \psi(\frac{\xi}{j}) dx d\xi = \int e^{-2\pi i \xi y} \hat{\phi}(\xi) \psi(\frac{\xi}{j}) d\xi$$

Changing variables by letting u=$\frac{\xi}{j}, v = j(x-y)$, so

$$\int e^{2\pi i u v} \phi(\frac{v}{j} + y) \psi(u) du dv = \int \phi(\frac{v}{j} + y) \hat{\psi}(v) dv$$

Hence, $\int e^{-2\pi i \xi y} \hat{\phi}(\xi) \psi(\frac{\xi}{j}) d\xi = \int \phi(\frac{v}{j} + y) \hat{\psi}(v) dv$

Then we take the limit of both sides, we could do this because functions are in $\mathcal{S}$. So

$$\psi(0) \bar{F}F[\phi](y) = \phi(y) \int \hat{\psi}(v) dv$$

Then we prove this theorem if we can find a $\psi \in \mathcal{S}$ satisfying

$\psi(0) = 1$ and $\int \hat{\psi}(v) dv = 1$

Function $\psi(x) = e^{-\pi|x|^2}$ would do this job. Since $\hat{\psi} = \psi$ and $\int \psi dx = 1$. $\square$

**Definition 6.9.** *Given $T \in \mathcal{S}', \forall \phi \in \mathcal{S}$, define:*

$$(F[T], \phi) = (T, F[\phi])$$

**Theorem 6.10.** *$F$ and $\bar{F}$ are isomorphism from $\mathcal{S}'$ to itself.*

$$F\bar{F}[T] = \bar{F}F[T] = T.$$

**Theorem 6.11.** *Let $f \in L^1(\mathbb{R}^n)$, then*

$$F[T_f] = T_{\hat{f}} \text{ in } \mathcal{S}'.$$

**Theorem 6.12.** *If $T$ has compact support, then*

$$F[T]_\xi = (T_x, e^{2\pi i \xi \cdot x}), \text{ in the sense of } \mathcal{D}'_\xi.$$

*Proof.* Since $T \in \mathcal{S}'$, so there exists a function $f$ slowing increasing at $\infty$ and continuous. Let $\beta \in \mathbb{N}_0^n$ such that $T = D^\beta f$. In addition, since $supp(T)$ is compact, if $\chi \in C_0^\infty$ such that $\chi \equiv 1$ on a neighborhood of $supp(T)$, then

$$\begin{aligned}
(\chi T, \phi) &= (T, \chi\phi) \\
&= (D^\alpha f, \chi\phi) \\
&= (-1)^{|\alpha|}(f, D^\alpha(\chi\phi)) \\
&= (-1)^{|\alpha|} \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} (f, D^\gamma \chi D^{\alpha-\gamma}\phi) \\
&= \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} (-1)^{|\gamma|} (D^{\alpha-\gamma} f D^\gamma \chi, \phi) \\
&= (\sum_{\beta \le \alpha} D^\beta f_\beta, \phi).
\end{aligned}$$

$D^\beta f_\beta$ is compact supported and slowly increasing at $\infty$,

$$\begin{aligned} F[T] &= \sum_{\beta\le\alpha} F[D^\beta f_\beta] \\ &= \sum_{\beta\le\alpha} (-2\pi i\xi)^\beta F[f_\beta] \\ &= \sum_{\beta\le\alpha} (-2\pi i\xi)^\beta \int e^{2\pi i\xi x} f_\beta(x) \\ &= \sum_{\beta\le\alpha} (-2\pi i\xi)^\beta (f_\beta, e^{2\pi i\xi x}) \\ &= \sum_{\beta\le\alpha} (-1)^\beta (f_\beta, D_x^\beta e^{2\pi i\xi x}) \\ &= \sum_{\beta\le\alpha} (D^\beta f_\beta, e^{2\pi i\xi x}) \\ &= (T_x, e^{2\pi i\xi x}). \end{aligned}$$

□

**Theorem 6.13.** *(Paley-Wiener) Given $T \in \mathcal{S}'$, the following are equivalent:*

*i)* $$supp(T) \subset x \in \mathbb{R}^n : |x_1| \le c, |x_2| \le c, \dots |x_n| \le c$$

*ii) $\hat{T}$ is a continuous function that extends as an extire function to $\mathbb{C}^n$, satisfying:*

*$\forall \epsilon > 0$ , $\exists A(\epsilon)$ such that if $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, then*

$$|\hat{T}(z)| \le A(\epsilon) e^{2\pi(c+\epsilon)(|z_1|+\dots+|z_n|)}.$$

**Theorem 6.14.** *If $\phi, \psi \in \mathcal{S}$, then*

$$F[\phi \bar{F}[\psi]] = F[\phi] * \psi$$

*Similarly, we can change $\bar{F}$ and $F$.*

*Proof.*

$$\begin{aligned} F[\phi \bar{F}[\psi]] &= \int e^{2\pi i z\cdot\xi}\phi(\xi)[\int e^{-2\pi i \xi\cdot x}\psi(x)dx]d\xi \\ &= \int[\int e^{2\pi i(z-x)\cdot\xi}\phi(\xi)\psi(x)dx]d\xi \\ &= \int \psi(x)[\int e^{2\pi i(z-x)\cdot\xi}\phi(\xi)d\xi]dx \\ &= F[\phi] * \psi \end{aligned}$$

Since the double integral exists so we could change the order of integration. □

**Theorem 6.15.** *If* $f, g \in L^1$, *then* $\forall \xi \in \mathbb{R}^n$,

$$F[f * g](\xi) = F[f](\xi)F[g](\xi).$$

**Theorem 6.16.** *If* $T \in \mathcal{E}', S \in \mathcal{S}'$,

$$F[T * S] = F[T]F[S], \text{ in the sense of } \mathcal{S}$$

# Chapter 7

# Exercises

**Exercise 7.1.** *Demonstrate the following inclusions are strict and continuous:*

$$L^p_{loc} \subset L^q_{loc} \subset \mathcal{D}', 1 \leq q < p \leq \infty.$$

*Proof.* Let $f \in L^p_{loc}$, then $\forall K$compact set, $\int\limits_K |f|^p dx < \infty$, let $m = \frac{p}{q}$, since $p > q$, then $m > 1$, choose $n$ such that $\frac{1}{m} + \frac{1}{n} = 1$. Then

$$\begin{aligned} \int |f|^q dx &= \int |f|^q * 1 dx \\ &\leq \left( \int (|f|^q)^m \, dx \right)^{\frac{1}{m}} \left( \int (1)^n dx \right)^{\frac{1}{n}} \\ &\leq \left( \int |f|^p \right)^{\frac{1}{m}} dx |K|^{\frac{1}{n}} \\ &< \infty \end{aligned}$$

So $f \in L^q_{loc}$. Next, need to show that the inclusion is continuous. Given $f, f_n \in L^p_{loc}$ such that $f_n \to f$ in $L^p_{loc}$, i.e $(\int |f - f_n|^p dx)^{\frac{1}{p}} \to 0$, need to show that $(\int |f - f_n|^q dx)^{\frac{1}{q}} \to 0$. If we define function $g = f - f_n$, then the proof will be similar as above. Let $f = \frac{1}{|x|^{\frac{1}{q}}}$, then $f \in L^q_{loc}$, but $f \notin L^p_{loc}$, so the inclusion is strict.

Let $f \in L^q_{loc}$, then $\forall \varphi \in \mathcal{D}, (T_f, \varphi) = \int f\varphi dx$,

$$\begin{aligned} |(T_f, \varphi)| &\leq \int |f||\varphi| dx \\ &\leq (\int |f|^q)^{\frac{1}{q}} (\int \varphi^p)^{\frac{1}{p}} \\ &\leq (\int |f|^q)^{\frac{1}{q}} |supp\varphi|^{\frac{1}{p}} \\ &< +\infty. \end{aligned}$$

If $f_n \to f \in L^q_{loc}$, then $(T, f_n) \to (T, f)$. □

**Exercise 7.2.** *Given any $\phi \in \mathcal{D}(\mathbb{R})$, consider*

$$\lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx$$

*Prove that the limit exists and it defines a distribution, we will call it principal value of $\frac{1}{x}$, denoted as p.v.$\frac{1}{x}$.*

*Proof.* Linearity is ok since both integral and limit are linear.

$$\begin{aligned} \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx &= \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \frac{\phi(x) + \phi(-x)}{2x} + \frac{\phi(x) - \phi(-x)}{2x} dx \\ &= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} \frac{\phi(x) - \phi(-x)}{x} dx \\ &= \int_0^{\infty} \frac{\phi(x) - \phi(-x)}{x} dx \end{aligned}$$

By FTC, $|\phi(x) - \phi(-x)| = |\int_{-x}^{x} \phi'(s) ds| \leq \int_{-x}^{x} |\phi'(s)| ds \leq 2|x| \sup |\phi'|$

$$\begin{aligned} \int_0^{\infty} \frac{\phi(x) - \phi(-x)}{x} dx &\leq \int_0^{\infty} \frac{2x \sup |\phi'|}{x} dx \\ &\leq 2 \sup |\phi'| \int_{supp(\phi) \cap [0,\infty]} 1 dx \\ &= 2 \sup |\phi'| |supp(\phi) \cap [0, \infty]| \end{aligned}$$

So $p.v.\frac{1}{x}$ is a distribution. $\square$

**Exercise 7.3.** *Given $p \in \mathbb{R}, p < n+1$, and a function $\alpha \in \mathcal{D}$ such that $\alpha \equiv 1$ in a neighborhood of 0, show that*

$$\int |x|^{-p}[\phi(x) - \alpha(x)\phi(0)]dx, \phi \in \mathcal{D}(\mathbb{R})$$

*defines a distribution of order at most 1.*

*Proof.* Let $r \in \mathbb{R}$ be a small number, then $\alpha \equiv 1$ in $B(0,r), let K = supp(\phi)$

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx &= \int_{B(0,r)} \frac{\phi(x) - \alpha(x)}{|x|} \times \frac{1}{|x|^{p-1}}dx + \int_{\mathbb{R}\backslash B(0,r)} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx \\
&\leq \int_{B(0,r)} \sup_{\substack{c \in B(0,r) \\ x \in \mathbb{R}^n}} |\phi'(c)| \frac{1}{|x|^{p-1}}dx + \int_{\mathbb{R}\backslash B(0,r)} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx \\
&\leq \sup_{\substack{c \in K \\ x \in \mathbb{R}^n}} \phi'(c)\| |x|^{1-p}\|_{L^1_{B(0,r)}} + \int_{\mathbb{R}\backslash B(0,r)} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx \\
&< C(K)\|D\phi\| + \int_{K\backslash B(0,r)} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx
\end{aligned}$$

$$\begin{aligned}
\left|\int_{K\backslash B(0,r)} \frac{\phi(x) - \alpha(x)\phi(0)}{|x|^p}dx\right| &\leq \int_{K\backslash B(0,r)} \frac{|\phi(x) - \alpha(x)\phi(0)|}{|x|^p}dx \\
&\leq \int_{K\backslash B(0,r)} \frac{|\phi(x)| + |\alpha(x)||\phi(0)|}{|x|^p}dx \\
&\leq \|\phi\|_{L^\infty}(\|\alpha\|_{L^\infty} + 1) \int_{K\backslash B(0,r)} \frac{1}{|x|^p}dx \\
&< C(\tilde{K})\|\phi\|_{L^\infty}(\|\alpha\|_{L^\infty} + 1)
\end{aligned}$$

Since $\phi, \alpha \in \mathcal{D}$, then they are bounded. $|\phi(x)| \leq \|\phi\|_{L^\infty}, |\alpha(x)| \leq \|\alpha\|_{L^\infty}$ and $\int_{K\backslash B(0,r)} \frac{1}{|x|^p}dx$ is a constant depending on K.

$\square$

**Exercise 7.4.** *Given $T \in \mathcal{D}'$, show that there exists an infinite number of distributions $S$ $\in \mathcal{D}'$ such that S=T.*

*Proof.* Let $H = \{\psi \in \mathcal{D} : \int_{-\infty}^{\infty} \psi = 0\}$,define $\phi(x) = \int_{-\infty}^{x} \psi(t)dt$
Fix $\gamma_0 \in \mathcal{D}$ to be some function so that $\int_{-\infty}^{\infty} \gamma_0 \equiv 1$, then for $\forall \gamma \in \mathcal{D}, \gamma = \lambda\gamma_0 + \psi$ when $\lambda = \int_{-\infty}^{\infty} \gamma$ and $\psi \in H$. Define $\langle S, \psi \rangle = -\langle T, \phi \rangle, \forall \psi \in H$. Let $\gamma \in \mathcal{D}$, then

$$\begin{aligned} \langle S, \gamma \rangle &= \langle S, \lambda\gamma_0 \rangle + \langle S, \psi \rangle \\ &= \lambda\langle S, \gamma_0 \rangle + \langle S, \psi \rangle \end{aligned}$$

Make a choice $c \in \mathbb{C}$ and set $\langle S, \gamma_0 \rangle = c$, then $\langle S, \gamma_0 \rangle = \lambda c + \langle S, \psi \rangle$, Since $S \in \mathcal{D}'$

$$\begin{aligned} \langle S', \gamma \rangle &= -\langle S, \lambda\gamma' \rangle \\ &= -\lambda\langle S, \lambda\gamma_0' \rangle - \langle S, \psi' \rangle \\ &= -\lambda\langle S, \gamma_0' \rangle - \langle S, \psi' \rangle \end{aligned}$$

Next, claim that $\gamma_0', \psi' \in H$, recall that $\int_{-\infty}^{\infty} \gamma_0 = 1, \int_{-\infty}^{\infty} \psi = 0$,$\gamma_0(x) \to 0$ as $x \to \pm\infty$, so $\int_{-\infty}^{\infty} \gamma_0' = 0, \int_{-\infty}^{\infty} \psi' = 0$, hence

$$\begin{aligned} -\lambda\langle S, \gamma_0' \rangle - \langle S, \psi' \rangle &= = \lambda\langle T, \int_{-\infty}^{x} \gamma_0'(t)dt \rangle + \langle T, \int_{-\infty}^{x} \psi'(t)dt \rangle \\ &= \lambda\langle T, \gamma_0' \rangle - \langle T, \psi \rangle \\ &= \langle T, \lambda\gamma_0 + \psi \rangle \\ &= \langle T, \gamma \rangle \end{aligned}$$

So $S' = T$ in $\mathcal{D}'$. Since c is arbitrary chosen, so we will have infinite number of distributions $S \in \mathcal{D}'$ such that $S' = T$. □

**Exercise 7.5.** *If $f \in L^1$, prove that*

$$f = \lim_{R \to \infty} \int_{|\xi| \le R} e^{-2\pi i x \cdot \xi} \hat{f}(\xi) d\xi, \textit{ in the sense of } \mathcal{S}'.$$

*Proof.* Let $f_R = \int e^{-2\pi i x\cdot\xi}\hat{f}(\xi)\chi_{|\xi|\leq R}d\xi$, need to show that $\forall\varphi \in \mathcal{S}, (T_{f_R},\varphi) \to (T_f,\varphi)$

$$\begin{aligned}
\lim_{R\to\infty}(T_{f_R},\varphi) &= \lim_{R\to\infty}\int \varphi(x)\int_{|\xi|\leq R} e^{-2\pi i x\cdot\xi}\hat{f}(\xi)d\xi dx \\
&= \lim_{R\to\infty}\int\int \varphi(x)e^{-2\pi i x\cdot\xi}\chi_{|\xi|\leq R}\hat{f}(\xi)d\xi dx \\
&= \lim_{R\to\infty}\int \hat{f}(\xi)\chi_{|\xi|\leq R}\int \varphi(x)e^{2\pi i x\cdot(-\xi)}dxd\xi \\
&= \int_R \hat{f}(\xi)\check{\varphi}(\xi)d\xi \\
&= (T_{\hat{f}(\xi)},\check{\varphi}(\xi)) \\
&= (\hat{T}_f,\check{\varphi}_x) \\
&= (T_f,\hat{\check{\varphi}}(x)) \\
&= (T_f,\varphi).
\end{aligned}$$

Using the Fubini's theorem and Dominated Convergence theorem. □

**Exercise 7.6.** *Prove that the derivative $D^\alpha$ is continuous from $\mathcal{D}'^{(m)}$ to $\mathcal{D}'^{(m+|\alpha|)}$.*

*Proof.* First need to show that if $T \in \mathcal{D}'^{(m)}$, then $D^\alpha T \in \mathcal{D}'^{(m+|\alpha|)}$. Let $K$ be any compact set such that $\forall\varphi \in C_0^\infty(\mathbb{R}^n), supp(\varphi) \subset K$, since $T \in \mathcal{D}'^{(m)}$, so

$$|(T,\varphi)| \leq C_K \sup_{|\beta|\leq m}\|D^\beta\varphi\|_\infty$$

C is any constant depends on K, then we have

$$\begin{aligned}
|(D^\alpha T,\varphi)| &= |(-1)^\alpha(T,D^\alpha\varphi| \\
&\leq C_K \sup_{|\beta|\leq m}\|D^\beta D^\alpha\varphi\|_\infty \\
&\leq C_K \sup_{|\gamma|\leq m+|\alpha|}\|D^\gamma\varphi\|_\infty
\end{aligned}$$

So $D^\alpha T \in \mathcal{D}'^{(m+|\alpha|)}$. Next need to show that if $T_j \to T$ in $\mathcal{D}'^{(m)}$, then $D^\alpha T_j \to D^\alpha T$ in

$\mathcal{D}'^{(m+|\alpha|)}$, so if $\exists T_j \in \mathcal{D}$ such that $\forall \varphi \in \mathcal{D}$, the $\lim\limits_{N\to\infty}(\sum_{j=1}^{N} T_j, \varphi)$ exists. Then

$$\lim_{N\to\infty}(\sum_{j=1}^{N} D^\alpha T_j, \varphi) \quad = \quad (-1)^{|\alpha|} \lim_{N\to\infty}(\sum_{j=1}^{N} T_j, D^\alpha \varphi)$$

exists because if $\varphi \in \mathcal{D}$, then $\psi = D^\alpha \varphi \in \mathcal{D}$, so $D^\alpha T_j \to D^\alpha T$ in $\mathcal{D}'^{(m+|\alpha|)}$.

□

**Exercise 7.7.** *Calculate* $x^k \delta^{(l)}$ *for* $k, l \in \mathbb{N}$

*Proof.* $\forall \varphi \in C^\infty(\mathbb{R})$

$$\begin{aligned}
(x^k \delta^{(l)}, \varphi) &= (\delta^{(l)}, x^k \varphi) \\
&= (-1)^l (\delta, D^l x^k \varphi) \\
&= (-1)^l (\delta, \sum_{j=0}^{l} \binom{l}{j} D^j x^k D^{l-j} \varphi)
\end{aligned}$$

If $l \geq k$ ,$D^j x^k = k(k-1)\dots(k-j-1)x^{k-j} = 0$, when $j - k > 0$

$$\begin{aligned}
(-1)^l (\delta, \sum_{j=0}^{l} \binom{l}{j} D^j x^k D^{l-j} \varphi) &= (-1)^l \sum_{j=0}^{l} \binom{l}{j} D^j (\delta, k(k-1)\dots(k-j-1)x^{k-j} D^{l-j} \varphi) \\
&= (-1)^l (\delta, \sum_{j=0}^{l} \binom{l}{j} D^j x^k D^{l-j} \varphi) \\
&= 0, if k - j > 0
\end{aligned}$$

aslo $= 0$ if $k - j > 0$, so it only has value when $j = k$ and $l > k$, therefore,

$$(x^k \delta^{(l)}, \varphi) = (-1)^l \binom{l}{k} D^{l-k} \varphi(0)$$

□

**Exercise 7.8.** *If $T \in \mathcal{D}'^{(m)}$ and $\alpha \in \mathcal{D}^{(p)}, p \geq m$, show that*

$$(\alpha T, \phi) = (T, \alpha\phi), \phi \in \mathcal{D}^{(m)}$$

*defines a distribution of order $\leq m$.*

*Proof.* Let $K \subset \mathbb{R}^n$ be compact, take $\phi, \varphi \in C^\infty$, such that $supp(\varphi) \subset K$. Since $T \in \mathcal{D}'^{(m)}$, so $|(T,\phi)| \leq C_K \sup_{|\beta|\leq m} |D^\beta \phi|$. Want to show $|(\alpha T, \varphi)| \leq C_K \sup_{|\beta|\leq m} |D^\beta \varphi|$

$$\begin{aligned}
|(\alpha T, \varphi)| &= |(T, \alpha\varphi)| \\
&\leq C_K \sup_{|\beta|\leq m} |D^\beta \alpha \phi| \\
&= C_K \sup_{|\beta|\leq m} |\sum_{0\leq j\leq \beta} \binom{\beta}{j} D^j \alpha D^{\beta-j}\varphi| \\
&\leq C_K \sum_{0\leq j\leq \beta} \binom{\beta}{j} |D^j \alpha||D^{\beta-j}\varphi| \\
&\leq C_K M \sum_{0\leq j\leq \beta} \|D^{\beta-j}\varphi\|_{L^\infty(K)} \\
&\leq C_K \tilde{M} \sup_{\beta-j} |D^{\beta-j}\varphi(x)|
\end{aligned}$$

□

**Exercise 7.9.** *Calculate the following the derivative in $\mathcal{D}'$*

*a)* $\frac{\partial^n}{\partial x_1 \ldots \partial x_n} Y(x)$, *where*

$$Y(x) = \begin{cases} 1 & \text{if } x_1 \geq 0, \ldots x_n \geq 0; \\ 0 & \text{else} \end{cases}$$

*Proof.* When $n = 2$, let $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
(\frac{\partial^2}{\partial x_1 \partial x_2} Y, \varphi) &= (Y, \frac{\partial^2}{\partial x_1 \partial x_2}\varphi) \\
&= \int_0^\infty \int_0^\infty \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\varphi(x_1, x_2)\right) dx_1 dx_2 \\
&= \int_0^\infty \lim_{R\to\infty} \int_0^R \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\varphi(x_1, x_2)\right) dx_1 dx_2 \\
&= \int_0^\infty \lim_{R\to\infty} \frac{\partial}{\partial x_2}\varphi(x_1, x_2)\Big|_0^R dx_2 \\
&= \int_0^\infty -\frac{\partial \varphi}{\partial x_2}(0, x_2) dx_2 \quad (\textit{since} \lim_{R\to\infty} \varphi(R, x_2) = 0) \\
&= -\int_0^\infty \frac{\partial \varphi}{\partial x_2}(0, x_2) dx_2 \\
&= -\lim_{R\to\infty} \varphi(0, x_2)\Big|_0^R \\
&= \varphi(0, 0)
\end{aligned}$$

Using the same way, we can get the $(\frac{\partial^n}{\partial x_1 \ldots \partial x_n} Y, \varphi) = (-1)^n \varphi(\underbrace{0, \ldots, 0}_{n})$ □

**Exercise 7.10.** *We set with* $r = (\sum_{i=1}^n x_i^2)^{1/2} \neq 0$,

$$E_n = \begin{cases} \log r & \textit{if } n = 2; \\ r^{2-n} & \textit{if } n \geq 3 \end{cases}$$

*a) Prove that* $E_n$ *belongs to* $\mathcal{D}'(\mathbb{R}^n)$.

*b) Let* $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. *Compute* $\Delta E_n$ *in the sense of distributions.*

*Proof.* a) It is easy to see that the function $E_n$ is locally integrable in $\mathbb{R}^n$. Indeed if we use polar coordinates we get

$$\int_{|x|\leq 1} |E_n(x)|dx = \begin{cases} -2\pi \int_0^1 r\log r dr = \frac{\pi}{2} & \text{if } n = 2; \\ \\ 2\pi \int_0^1 r dr & \text{if } n \geq 3 \end{cases}$$

b) We have $< \Delta E_n, \varphi >=< E_n, \Delta\varphi >$. $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$, so

$$< \Delta E_n, \varphi >= \int_{\mathbb{R}} E_n(x)\Delta\varphi(x)dx$$

Since not all the derivatives of $E_n$ are locally integrable we cannot integrate by parts in the above integral. We shall overcome this difficulty in the following way. Since $E_n$ is locally integrable, by Lebesgue's theorem, we can write

$$< \Delta E_n, \varphi >= \lim_{\epsilon\to 0} \int_{|x|\geq\epsilon} E_n(x)\Delta\varphi(x)dx = \lim_{\epsilon\to 0} I_\epsilon$$

Now $E_n \in C^\infty$ for $|x| = r \geq \epsilon$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, so we have

$$I_\epsilon = \int_{|x|\geq\epsilon} \Delta E_n(x)\varphi(x)dx - \int_{|x|=\epsilon} (E_n \frac{\partial\varphi}{\partial r} - \varphi\frac{\partial E_n}{\partial r})d\sigma_\epsilon$$

Let us compute $\Delta E_n$ in $\{x : |x| \geq \epsilon\}$.

1. $n = 2$:

$$\frac{\partial}{\partial x}\log(x^2 + y^2) = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2}{\partial x^2}\log(x^2 + y^2) = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2}{\partial y^2}\log(x^2 + y^2) = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

So $\Delta E_2 = 0$

2. If $n \geq 3$:

$$\frac{\partial}{\partial x_i} r^{2-n} = \frac{2-n}{2} \cdot 2x \cdot r^{-n} = (2-n)x_i \cdot r^{-n}$$

$$\frac{\partial^2}{\partial x_i^2} = (2-n)r^{-n} + (2-n)x_i \cdot \frac{-n}{2} \cdot 2x_i \cdot r^{-n-2}$$

So $\Delta E_n = (2-n) \cdot nr^{-n} - n(2-n)(\sum_{i=1}^{n} x_i^2)r^{-n-2} = 0$ since $\sum_{i=1}^{n} x_i^2 = r^2$, i.e. $\Delta E_n = 0$.

Therefore

$$-I_\epsilon = \int_{|x|=\epsilon} (E_n \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial E_n}{\partial r}) d\sigma_\epsilon$$

To compute $I_\epsilon$ we use polar coordinates

$$x_i = r \cdot f_i(\theta_1, \ldots, \theta_{n-1}) \quad i = 1, \ldots, n$$

so we get

$$dx = F(\theta_1, \ldots, \theta_{n-1}) r^{n-1} d\theta_1 \ldots d\theta_{n-1}$$

and the measure on the sphere of radius $\epsilon$ is equal to

$$d\sigma_\epsilon = \epsilon^{n-1} F(\theta_1, \ldots, \theta_{n-1}) d\theta_1 \ldots d\theta_{n-1} = \epsilon^{n-1} d\sigma_1$$

where $d\sigma_1 = F(\theta_1, \ldots, \theta_{n-1}) d\theta_1 \ldots d\theta_{n-1}$ is the measure on the unit sphere.

On the other hand:

$$\frac{\partial}{\partial r} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial x_i}{\partial r} = \sum_{i=1}^{n} \frac{x_i}{r} \frac{\partial}{\partial x_i}$$

since $\frac{\partial x_i}{\partial r} = f_i(\theta_1, \ldots, \theta_{n-1}) = \frac{x_i}{r}$.

Let us compute now the limit of $I_\epsilon$ when $\epsilon$ goes to 0.

1. $n = 2$

$$-I_\epsilon = \int_{|x|=\epsilon} (\log \epsilon \frac{\partial \varphi}{\partial r} - \varphi \cdot \frac{1}{\epsilon}) \epsilon d\sigma_1 = \underbrace{\int_{|x|=\epsilon} \epsilon \log \epsilon \frac{\partial \varphi}{\partial r} d\sigma_1}_{<1>} - \underbrace{\int_{|x|=\epsilon} \varphi d\sigma_1}_{<2>}$$

We have $|\frac{\partial \varphi}{\partial r}| \leq \sum_{i=1}^{n} |\frac{x_i}{r}||\frac{\partial \varphi}{\partial x_i}| \leq \sum_{i=1}^{n} \sup_{\mathbb{R}} |\frac{\partial \varphi}{\partial x_i}|$ since $|\frac{x_i}{r}| \leq 1$, so we get

$$< 1 >= \Big| \int_{|x|=\epsilon} \epsilon \log \epsilon \frac{\partial \varphi}{\partial r} d\sigma_1 \Big| \leq C |\epsilon \log \epsilon \frac{\partial \varphi}{\partial r}| \cdot \int d\sigma_1$$

So this term tends to 0 when $\epsilon \to 0$. For the second term we write

$$< 2 >= - \int \widetilde{\varphi}(\epsilon, \theta) d\sigma_1 \text{ where } \widetilde{\varphi}(\epsilon, \theta) = \varphi(r \cos \theta, r \sin \theta)$$

When $\epsilon \to 0$, by Lebesgue's theorem $< 2 > \to -\widetilde{\varphi}(0, \theta) \cdot \int d\sigma_1$ and since $\widetilde{\varphi}(0, \theta) = \widetilde{\varphi}(0, 0)$ we get

$$\lim_{\epsilon \to 0} I_\epsilon = 2\pi \varphi(0, 0) = 2\pi < \delta, \varphi >$$

2. $n \geq 3$

$$\begin{aligned} -I_\epsilon &= \int_{r=\epsilon} \frac{1}{\epsilon^{n-2}} \frac{\partial \varphi}{\partial r} \epsilon^{n-1} d\sigma_1 - \int_{r=\epsilon} \widetilde{\varphi}(\epsilon, \theta_1, \ldots, \theta_{n-1})(2-n) \cdot \frac{1}{\epsilon^{n-1}} \epsilon^{n-1} d\sigma_1 \\ &= \int_{r=\epsilon} \epsilon \frac{\partial \varphi}{\partial r} d\sigma_1 + (n-2) \int_{r=\epsilon} \widetilde{\varphi}(\epsilon, \theta_1, \ldots, \theta_{n-1}) d\sigma_1 \end{aligned}$$

The first term tends to 0 since $|\frac{\partial \varphi}{\partial r}| \leq \sup_{\mathbb{R}} |\frac{\partial \varphi}{\partial x_i}| \leq C$.
By Lebesgue's theorem the second term tends to

$$(n-2)\varphi(0) \Big\{ \int d\sigma_1 \Big\}$$

so

$$\lim_{\epsilon \to 0} I_\epsilon = C_n(2-n)\varphi(0) = (2-n)C_n < \delta, \varphi >$$

where $C_n$ is the measure of the unit sphere in $\mathbb{R}^n$.

Therefore in all cases we have $\Delta E_n = a_n \delta$ where $a_n$ is a constant. □

**Exercise 7.11.** *Let* $f(x) = e^{-|x|^2}$, $x \in \mathbb{R}^n$. *Find the Fourier Transfrom of f.*

*Proof.* It is easy to see that $f \in L^1(\mathbb{R}^n)$. We first compute the Fourier transform when $n = 1$. Thus for $\xi \in \mathbb{R}$,

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{(-2\pi i x \xi)} f(x) dx \\ &= \int_{-\infty}^{\infty} e^{(-\pi^2 \xi^2)} e^{-(x+\pi i \xi)^2} dx \end{aligned}$$

We can evaluate this integral using Cauchy's theorem in the complex plane since the function is holomorphic. Consider the contour $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$ shown in the following figure:

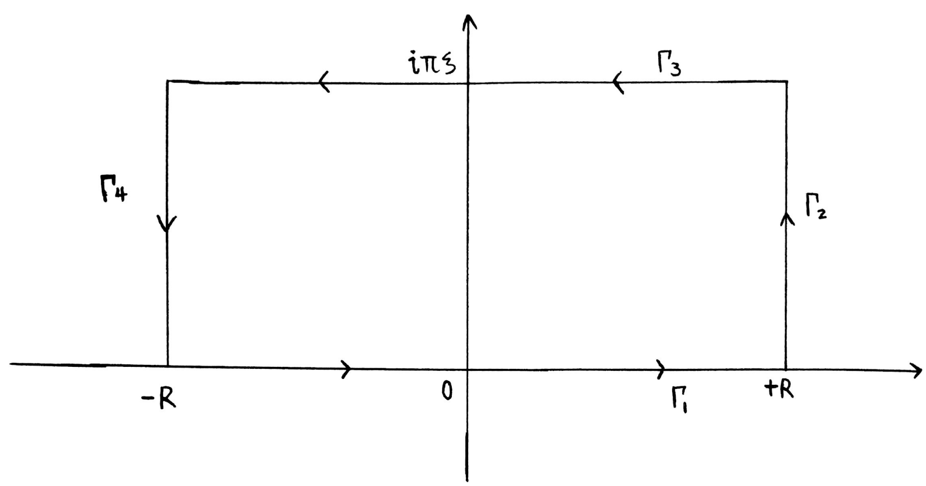


**Figure 7.1**: *Contour* Γ

By Cauchy's theorem, $\int_\Gamma e^{(-z^2)} dz = 0$. Further

$$\begin{aligned}
\Big|\int_{\Gamma_2} e^{-z^2}dz\Big| &= \Big|\int_0^{\pi\xi} e^{-(R+iy)^2}dy\Big| \\
&= \Big|\int_0^{\pi\xi} e^{-R^2}e^{-2iRy}\exp(y^2)dy\Big| \\
&\leq Ce^{-R^2}
\end{aligned}$$

and so this integral tends to 0 as $R \to +\infty$.

Similarly

$$\lim_{R\to+\infty}\int_{\Gamma_4} e^{-z^2}dz = 0.$$

Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(x+i\pi\xi)^2}dx &= -\lim_{R\to\infty}\int_{\Gamma_1} e^{-z^2}dz \\
&= \int_{-\infty}^{\infty} e^{-x^2}dx \\
&= \sqrt{\pi}
\end{aligned}$$

as is well-known. Hence,

$$\widehat{f}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2}.$$

Now for any general $n$,

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}e^{-|x|^2}dx \\
&= \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n}(x_j^2+2\pi i x_j\xi_j)}dx \\
&= (\sqrt{\pi})^n e^{-\sum_{j=1}^{n}\pi^2\xi_j^2} \\
&= (\pi)^{n/2}e^{-\pi^2|\xi|^2}
\end{aligned}$$

**Exercise 7.12.** *Let $S \in \mathcal{E}'(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$. Show that for $k \in \mathbb{N}$:*

$$x^k(S * T) = \sum_{j=0}^{k} \binom{k}{j} (x^j S) * (x^{k-j} T)$$

*Proof.* By definition, for $\varphi \in \mathcal{D}(\mathbb{R})$,

$$< x^k(S * T), \varphi >=< S * T, x^k\varphi >=< S_x, < T_y, (x+y)^k \varphi(x+y) >>$$

Now

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

It follows that

$$< x^k(S * T), \varphi >=< S_x, < T_y, \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} \varphi(x+y) >>$$

Now

$$< T_y, \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} \varphi(x+y) >= \sum_{j=0}^{k} \binom{k}{j} x^j < T_y, y^{k-j}\varphi(x+y) >$$

and

$$< T_y, y^{k-j}\varphi(x+y) >=< y^{k-j} T_y, \varphi(x+y) >$$

$$< S_x, x^j \Psi >=< x^j S_x, \Psi > \text{ for all } \Psi \in C^\infty(\mathbb{R})$$

It follows that

$$\begin{aligned} < x^k(S * T), \varphi > &= \sum_{j=0}^{k} \binom{k}{j} < x^j S_x, < y^{k-j} T_y, \varphi(x+y) >> \\ &= \sum_{j=0}^{k} \binom{k}{j} < (x^j S) * (x^{k-j} T), \varphi > \end{aligned}$$

□

**Exercise 7.13.** *Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be such that $\rho \geq 0$ and $\int_{\mathbb{R}^n} \rho(x)dx = 1$. For $\epsilon > 0$ we set $\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho(\frac{x}{\epsilon})$, and for $u \in \mathcal{D}'(\mathbb{R}^n)$, $u_\epsilon = u * \rho_\epsilon$. Show that when $\epsilon \to 0$:*

- *If $u \in L^p(\mathbb{R}^n)$, $1 \leq \rho \leq +\infty$, $u_\epsilon \to u$ in $L^p(\mathbb{R}^n)$, and prove the inequality $\|v * \rho_\epsilon\|_{L^p} \leq \|v\|_{L^p}$, $\forall v \in L^p(\mathbb{R}^n)$.*

*Proof.* First of all $\rho_\epsilon \to \delta$ in $\mathcal{E}'$ when $\epsilon \to 0$. Indeed $supp\rho \subset \{|x| \leq M\}$ and

$$\int \rho_\epsilon(x)\varphi(x)dx = \int_{|x|\leq M} \rho(x)\varphi(\epsilon x)dx \qquad \forall \varphi \in C^\infty(\mathbb{R}^n)$$

Then:

- $\rho(x)\varphi(\epsilon x) \to \rho(x)\varphi(0)$ a.e. if $\epsilon \to 0$
- $|l_{(|x|\leq M)}\rho(x)\varphi(\epsilon x)| \leq \sup_{|y|\leq M} |\varphi(y)|\rho(x) \in L^1(\mathbb{R}^n)$

The result follows from the Lebesgue theorem and from the fact that $\int \rho x dx = 1$.
Let $u \in L^p(\mathbb{R}^n)$. Since $C^0_\epsilon(\mathbb{R}^n)$ is dense in $L^p$, there exists a sequence $(u_j)$ in $C^0_\epsilon$ such that

$$(1) \forall \alpha > 0, \exists J : j \geq J \Rightarrow \|u_j - u\|_{L^p} < \frac{\alpha}{3}$$

Let $j_0$ be fixed, $j_0 \geq J$. Then

$$(2) \|u * \rho_\epsilon - u\|_{L^p} \leq \|u * \rho_\epsilon - u_{j_0} * \rho_\epsilon\|_{L^p} + \|u_{j_0} * \rho_\epsilon - u_{j_0}\|_{L^p} + \|u_{j_0} - u\|_{L^p}$$

It follows from (1)

$$(3) \|u_{j_0} - u\|_{L^p} < \frac{\alpha}{3}$$

Moreover we have

$$\forall \alpha > 0, \exists \epsilon_0 : \epsilon < \epsilon_0 \Rightarrow \sup |u_{j_0} * \rho_\epsilon(x) - u_{j_0}(x)| < \delta$$

Now

$$
\begin{aligned}
\|u_{j_0} * \rho_\epsilon - u_{j_0}\|_{L^p} &= (\int_K |u_{j_0} * \rho_\epsilon(x) - u_{j_0}(x)|^\rho)^{1/\rho} \\
&\leq C \sup_K |u_{j_0} * \rho_\epsilon(x) - u_{j_0}(x)| \\
&< C\delta
\end{aligned}
$$

So if $\epsilon < \epsilon_1$

$$(4) \|u_{j_0} * \rho_\epsilon - u_{j_0}\|_{L^p} < \frac{\alpha}{3}$$

Let us assume the following inequality has been proved

$$(5) \|v * \rho\|_{L^p} \leq \|v\|_{L^p} \qquad \forall v \in L^p$$

Then we shall have

$$(6) \|u * \rho_\epsilon - u_{j_0} * \rho_\epsilon\|_{L^p} \leq \|u_{j_0} - u\| < \frac{\alpha}{3}$$

Using (2), (3), (4) and (6) we shall get

$$\forall \alpha > 0, \exists \epsilon_1 : \epsilon < \epsilon_1 \Rightarrow \|u * \rho_\epsilon - u\|_{L^p} < \alpha$$

□

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