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CLASSICAL SYSTEM THEORY REVISITED FOR TURNPIKE IN STANDARD STATE SPACE SYSTEMS AND IMPULSE CONTROLLABLE DESCRIPTOR SYSTEMS*

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Abstract. The concept of turnpike connects the solution of long but finite time horizon optimal control problems with steady-state optimal controls. A key ingredient of the analysis of turnpike phenomena is the linear quadratic regulator problem and the convergence of the solution of the associated differential Riccati equation as the terminal time approaches infinity. This convergence has been investigated in linear systems theory in the 1980s. We extend classical system theoretic results for the investigation of turnpike properties of standard state space systems and descriptor systems. We present conditions for turnpike phenomena in the nondetectable case and for impulse controllable descriptor systems. For the latter, in line with the theory for standard linear systems, we establish existence and convergence of solutions to a generalized differential Riccati equation.

Key words. linear systems, descriptor systems, optimal control, long time behavior, Riccati equations

AMS subject classifications. 49N10, 49J15, 93B52

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1. Introduction.

The notion of *turnpike* has been used in economics and in control theory (see, e.g., the textbooks [39, 40]) for about 40 years with an increasing interest in the past decade. Turnpike denotes the property of the control and the solution to a finite time optimization problem to be close to the optimal values for the associated steady-state problem most of the time.

The backbone of most turnpike results for time autonomous systems is the turnpike property of a relevant linear quadratic regulator (LQR) optimization problem. And the turnpike property of the LQR problem is intimately linked to the decay of the solution to the associated generalized differential Riccati (gDRE) equation toward the stabilizing solution of an algebraic Riccati equation; see, e.g., [33, Lem. 2.6].

In the first part of this manuscript, we use classical mathematical systems theoretic results as presented by Callier, Winkin, and Willems [8] to show the turnpike property of the LQR problem. For that, we extend the results to the affine linear optimal control problem using an explicit formula of the state transition matrices of the closed-loop system. Having connected the system theoretic toolbox to the investi-

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gation of turnpike behaviors, we can immediately provide new general results for cases where the system is not detectable, which is a current research issue; see [13, 32].

In the second part of the paper, we derive turnpike properties of linear quadratic optimal control problems that are constrained by descriptor systems. Descriptor systems are also commonly referred to as DAEs. To our best knowledge, turnpike phenomena of DAEs have not been addressed so far.

In analogy with the standard LQR case, we will link the asymptotic behavior of an associated differential Riccati equation to the turnpike property.

Early considerations on Riccati equations for DAEs were made in [18] by examining LQ-regulators for singularly perturbed ODEs. An extensive investigation and fundamental results for the finite time LQR problem for DAEs have been provided by Bender and Laub [5], who expanded also on the work of Cobb [9] and Pandolfi [30].

Bender and Laub defined several equivalent relevant Riccati equations in standard state space form; see [5, sect. IV]. A generalized Riccati equation which, in particular, can be stated in the original system coefficients is not addressed in [5] apart from noting that the most obvious symmetric formulation is not well suited. We also mention the extension of the work by [5] to endpoint constraints [41]. The nonsymmetric differential Riccati equation that is formulated in the original coordinate system and that is also a main subject of this paper has been treated in [16]. There the relation of the LQR problem for descriptor systems has been discussed, and the existence of solutions under general conditions has been shown. This nonsymmetric differential generalized Riccati equation has been considered in [25], where necessary conditions for the existence of solutions in general and sufficient conditions for some special cases were derived. The related nonsymmetric generalized algebraic Riccati equation (gARE) has been investigated in [17] and applied in the context of model reduction for infinite time-horizon control systems in [29].

Apart from this branch, the literature on the DAE LQR optimization problem on finite time horizons has been enriched with results on suitable reformulations of the optimality conditions [20], on particularly structured cases [3, 14], and on the problem with time-varying coefficients [22, 24, 26]. In the course of the investigations, several formulations of gDREs have been proposed; see the discussion in [26].

More recently, based on a generalized *Kalman–Yakobovich–Popov* inequality [34] and the *Lur'e* equations, which provide a true generalization (cf. [35, sect. 6]) to the presented Riccati-based approach for DAEs, the linear quadratic regulation problem for DAEs has been considered in great generality; see [35].

This work contributes to the theory on Riccati equations for descriptor systems in the following respects. We show that under the conditions used in [5, 16] and an additional definiteness condition on the optimization problem, the solution of the nonsymmetric gDREs has a distinguished structure and converges to the stabilizing solution of the associated algebraic Riccati equation. This structure also implies that the provided optimal feedback gains make the closed-loop system *impulse free* so that they are a *best choice* according to a conjecture stated in [5, sect. VII]. With the convergence of the gains and the closed loop being impulse free, we then can show that the DAE-constrained LQ optimization problem has the turnpike property.

The line of arguments and results in this paper are as follows. In section 2, we introduce the LQR problem for standard state space systems, the notion of turnpike, and classical results on the asymptotic behavior of the solutions and the controls that immediately imply well-known turnpike results. Next, in section 3, we derive explicit formulas for the solutions to the affine LQR problem, i.e., the LQR problem with nonzero target states. Then the arguments of the first section can be applied to conclude turnpike properties also in this case. In the second part of the paper, we consider the LQR problem with DAE constraints. Therefore, we introduce the relevant concepts in section 4 and prove the existence and asymptotic decay of solutions of the gDRE in section 5. Finally, we can prove turnpike properties of the affine LQR problem with DAE constraints in section 6. We conclude the paper with summarizing remarks and an overview of related open research questions.

2. Basic notations, notions, and results for the LQR problem. We consider the finite time horizon linear quadratic optimization problem.

Problem 2.1 (finite horizon optimal control problem). For coefficients $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$, and $F \in \mathbb{R}^{\ell\times n}$, for an initial value $x_0 \in \mathbb{R}^n$, for target outputs $y_c \in \mathbb{R}^k$ and $y_e \in \mathbb{R}^\ell$ and a terminal time $t_1 > 0$, consider the optimization of the cost functional

$$\frac{1}{2}\int_0^{t_1} \|Cx(s) - y_c\|^2 + \|u(s)\|^2 \,\mathrm{d}s \; + \frac{1}{2}\|Fx(t_1) - y_e\|^2 \to \min_u$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

We will investigate how solutions to Problem 2.1 will relate to solutions of the related state optimization problem in particular for large time horizons t_1.

Problem 2.2 (steady-state optimal control problem). Consider

$$\frac{1}{2}\|Cx - y_c\|^2 + \frac{1}{2}\|u\|^2 \,\mathrm{d}s \; \to \min_u$$

subject to

$$0 = Ax + Bu.$$

If $y_c = 0$ and $y_e = 0$, then we will refer to Problems 2.1 and 2.2 as *homogeneous* LQR problems, otherwise as *affine* LQR problems.

Definition 2.3. *The finite time optimal control problem has the* (exponential) turnpike *property if for some constant vectors x_s and u_s it holds that*

$$\|x(t) - x_s\| \leq \mathrm{const}(e^{\{\lambda t\}} + e^{\{\lambda(t_1 - t)\}})$$

and

$$\|u(t) - u_s\| \leq \mathrm{const}(e^{\{\lambda t\}} + e^{\{\lambda(t_1 - t)\}})$$

for $t \leq t_1$ and for constants $\mathrm{const} > 0$ *and* $\lambda < 0$ *independent of* t_1.

Remark 2.4. Throughout this manuscript, the notation const will be used to denote a generic constant value that is independent of t and t_1 but unspecified otherwise.

In general terms, the *turnpike property* means the existence of a trajectory—the *turnpike*—that is defined on the whole time line and only depends on the optimization criterion such that the solution to the optimal control problem on a finite time interval is close to the turnpike except at the beginning or the end of the interval; cf. the preface in [39]. In general considerations, the *closedness* is characterized through the measures of the time intervals in which the solution leaves an ϵ-neighborhood of the turnpike (cf. [39, p. xviii]) which can be time dependent; see, e.g., [11, 37].

In this work, we consider the exponential turnpike property (as in Definition 2.3) with a steady state as the turnpike, which is the solution to an associated steady-state optimal control problem; see also, e.g., [38]. When considering DAEs below, however, we will lay out that the notion of *an* associated steady-state problem might be not uniquely defined.

We start with the fundamental assumption for our considerations.

Assumption 2.5. We assume that A, B, and C in Problems 2.1 and 2.2 are such that the *algebraic Riccati equation*

$$A^*X + XA - XBB^*X + C^*C = 0 \tag{2.1}$$

has a stabilizing solution $P_+ \in \mathbb{R}^{n\times n}$, which means that the eigenvalues of

$$A_+ := A - BB^*P_+$$

all have negative real part.

We note that A_+ is invertible and that its *spectral abscissa* λ, i.e., the maximum real part among the eigenvalues of A_+, is strictly less than zero.

Remark 2.6. Assumption 2.5 requires that (A, B) are *stabilizable*. Detectability of (C, A) is not required since P_+ may exist in the case where detectability is not given; see, e.g., [28, 19].

Lemma 2.7. *Under Assumption* 2.5, *the solution to the steady-state optimal control problem is given as*

$$x_s = A_+^{-1}BB^*A_+^{-*}C^*y_c \quad \textit{and} \quad u_s = -B^*P_+x_s - B^*w_s,$$

where $w_s = A_+^{-*}C^*y_c$.

Proof. For Problem 2.2 as a linear-quadratic constrained optimization problem, it holds that (x_s, u_s) is an optimal solution if and only if there exists a *Lagrange multiplier* λ_s such that

$$0 = Ax_s + Bu_s, \tag{2.2a}$$

$$0 = A^*\lambda_s + C^*(Cx_s - y_c), \tag{2.2b}$$

$$0 = B^*\lambda_s + u_s. \tag{2.2c}$$

One can confirm directly (see [15, proof of Lem. 2.7]) that with $\lambda_s := P_+x_s + w_s$, with P_+ solving the ARE (2.1), and with $A_+ = A - BB^*P_+$ invertible, the triple (x_s, λ_s, u_s) fulfills these conditions. ∎

The turnpike property is intimately linked to the convergence of the solution P to the differential Riccati equation toward the stabilizing solution P_+ of the associated algebraic Riccati equation. In fact, this convergence appears as a necessary condition for the turnpike property in linear quadratic systems in the fundamental work by Porretta and Zuazua [33, Cor. 2.7]. On the other hand, basic system theoretic investigations of the convergence of P toward P_+, as presented in [8], resulted in the formula

$$\|x_h(t) - e^{\{tA_+\}}x_0\| \leq \text{const}\, e^{\{\lambda t_1\}}e^{\{\lambda(t_1-t)\}}, \tag{2.3}$$

where x_h is the solution to Problem 2.1 with $y_c = 0$ and $y_e = 0$; cf. [8, Thm. 4]. Since $e^{\{tA_+\}}x_0$ goes to zero exponentially with rate λ and since $x_s = 0$ for the homogeneous problem with $y_c = 0$, by means of (2.3), one can directly infer the turnpike property for the homogeneous case:

$$\begin{aligned}\|x_h(t) - 0\| &\le \|x_h(t) - e^{\{tA_+\}}x_0\| + \|e^{\{tA_+\}}x_0\| \\ &\le \text{const}\, e^{\{\lambda t_1\}}e^{\{\lambda(t_1-t)\}} + \text{const}\, e^{\{\lambda t\}} \le \text{const}(e^{\{\lambda(t_1-t)\}} + e^{\{\lambda t\}}).\end{aligned} \tag{2.4}$$

In view of extending this result from [8] to the affine case, we recall well-known links between the solution to the finite time optimal control problem and the differential Riccati equation combined with a feedforward term.

THEOREM 2.8 ([27, Ch. 3.1]). *The solution* (x, u) *to Problem* 2.1 *is given via*

$$u(t) = -B^*(P(t)x(t) + w(t)),$$

where P *is the unique solution to the differential Riccati equation (DRE)*

$$-\dot{P}(t) = A^*P(t) + P(t)A - P(t)BB^*P(t) + C^*C, \quad P(t_1) = F^*F, \tag{2.5}$$

where w *is the solution to*

$$-\dot{w}(t) = (A^* - P(t)BB^*)w(t) - C^*y_c, \quad w(t_1) = -F^*y_e, \tag{2.6}$$

and via x *as the solution to*

$$\dot{x}(t) = (A - BB^*P(t))x(t) - BB^*w(t), \quad x(0) = x_0.$$

In what follows, we will use the abbreviation $S = F^*F$.

Note that in Theorem 2.8, which characterizes the optimal controls for finite times, stability does not play a role so that the coefficients (A, B, C) and F can be arbitrary. In order to link to the steady state, however, we will require Assumption 2.5 to hold. In this case, namely, if P_+ exists, the following quantities are well-defined; see (see [8, Lems. 1 and 5]):

1. the *closed-loop reachability Gramian:*
$$W := \int_0^\infty e^{\{sA_+\}}BB^*e^{\{sA_+^*\}}\,\mathrm{d}s\ ; \tag{2.7}$$
2. the *closed loop reachability Gramian* on $[0, \tau]$:
$$W(\tau) = \int_0^\tau e^{\{sA_+\}}BB^*e^{\{sA_+^*\}}\,\mathrm{d}s\ = W - e^{\{\tau A_+\}}We^{\{\tau A_+^*\}}; \tag{2.8}$$
3. the *sliding terminal condition*:
$$\tilde{S}(\tau) := (S - P_+)[I + W(\tau)(S - P_+)]^{-1}. \tag{2.9}$$

For the latter, the following lemma is relevant.

LEMMA 2.9 ([8, Lem. 5]). *Let Assumption* 2.5 *hold, and consider* W, $W(\tau)$, *and* $\tilde{S}$ *as defined in* (2.7), (2.8), *and* (2.9). *If* $[I+W(S-P_+)]$ *is invertible, then* $\tau \to \tilde{S}(\tau)$ *is a decreasing function and for any* $\tau \ge 0$, *meaning that*

$$S - P_+ = \tilde{S}(0) \ge \tilde{S}(\tau) \ge \tilde{S}(\infty) = [I + W(S - P_+)]^{-1}.$$

Moreover, for the spectral norm, it holds that

$$K(\tilde{S}) := \sup_{\tau\ge0} \|\tilde{S}(\tau)\| = \max\{\|S - P_+\|, \|\tilde{S}(\infty)\|\}.$$

Remark 2.10. For the spectral norm of the Gramians, it holds that

$$\|W(\tau)\| \leq \|W\|.$$

The condition that $[I + W(S - P_+)]$ is invertible was shown to be necessary and sufficient for the convergence of $P(t) \to P_+$ as $t_1 \to \infty$; see [8, Thm. 2]. For what follows, we will assume that this condition holds.

Assumption 2.11. If Assumption 2.5 holds and W, $W(\tau)$, and $\tilde{S}$ are as defined in (2.7), then, with $S := F^*F$, the matrix

$$[I + W(S - P_+)]$$

is invertible, where F defines the terminal constraint in Problem 2.1.

Remark 2.12. Most literature on turnpike properties of optimal control problems assume that (C, A) is detectable, which is a sufficient condition for Assumption 2.11. However, the undetectable subspace for (C, A) can be compensated for if the nullspace of the terminal cost F only has the trivial intersection with it, which provides a necessary and sufficient condition for the convergence of $P(t)$ toward P_+; see [8, Thm. 2].

Remark 2.13. In [13, Thm. 8.4], the turnpike property of an LQ optimization problem with an undetectable pair (C, A) has been established by imposing state constraints and the absence of unobservable modes on the imaginary axis. These assumptions exclude oscillatory modes or unstable modes that are not detected by C. In our case, unobservable unstable modes are *fixed* by the terminal constraint; cf. Assumption 2.11. With the assumption of stabilizability, which, in particular, excludes uncontrollable modes on the imaginary axis, the assumptions of [13, Thm. 8.4] on (A, B, C) are equivalent to assuming the existence of a stabilizing solution to the ARE as in Assumption 2.5.

We illustrate the implications of Remark 2.12 in a numerical example. Consider Problem 2.1 with $t_1 = 10$, $y_c = 0$, and $y_e = 1$ and with the coefficients

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \sqrt{3} \end{bmatrix}, \tag{2.10}$$

borrowed from an example in [28, p. 31]. Here, (A, B) is controllable and, thus, stabilizable, while (C, A) is not detectable. Still, a stabilizing solution to the associated algebraic Riccati equation (2.1) exists.

Then, as implied by the conditions laid out in Remark 2.12, the solution to the differential Riccati equation that starts in F^*F converges to the stabilizing solution if and only if the nullspace of F and the space that is not detected by (C, A)—which in this case is spanned by $\begin{bmatrix} 1 & 0 \end{bmatrix}^*$—intersect only trivially.

Accordingly, with the choice $F = C$, the solution of the differential Riccati equation converges to a symmetric positive definite solution to the ARE that, however, is not stabilizing. Also, the associated optimal state x does not satisfy the turnpike property, as can be seen from the logarithmic plot of $|x|$ in the first row of Figure 1.

Vice versa, with the choice of $F = \begin{bmatrix} \sqrt{3} & 0 \end{bmatrix}$, Assumption 2.11 holds, the solution to the DRE converges to a stabilizing solution of the ARE, and the optimal state x satisfies the turnpike estimate; cf. the second row of Figure 1.

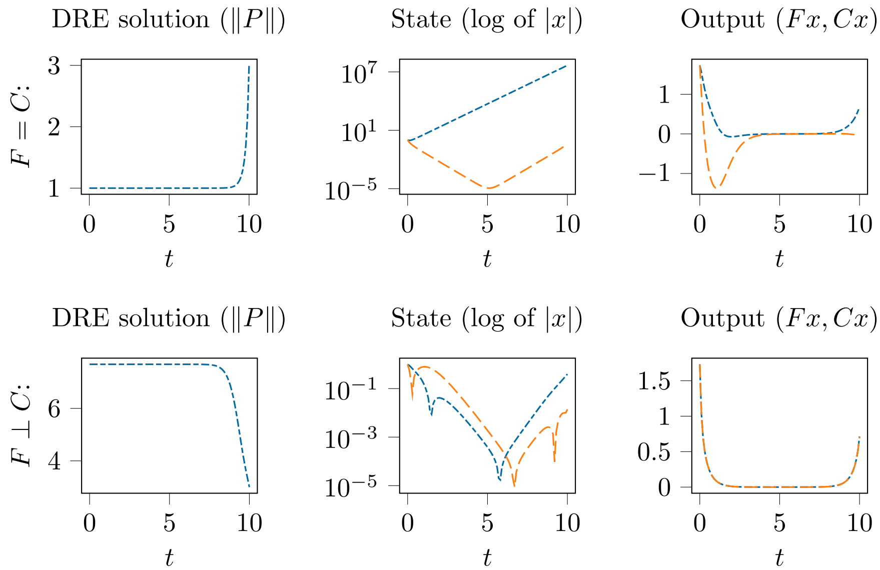

FIG. 1. *Example simulation of the optimal control problem (Problem* 2.1*) with coefficients as in* (2.10)*, with the initial value* $x(0) = [1\ 1]^*$ *and choices of the endpoint constraint* F *that illustrate the sufficiency and necessity of Assumption* 2.11 *for the turnpike property as in Definition* 2.3*.*

3. Explicit formulas and turnpike estimates for solutions to the affine problem. In this section, we use an explicit formula of the state transition matrices to derive formulas for the solution to the finite time optimal control problem.

LEMMA 3.1. *Under Assumption* 2.5*, the fundamental solution matrix* U *to*

$$\dot{U} = (A - BB^*P(t))U, \quad U(t_1) = I,$$

where P *solves the DRE with* $P(t_1) = S$ *given as*

$$U(t) = e^{\{-(t_1-t)A_+\}}\big(I - [W - e^{\{(t_1-t)A_+\}}We^{\{(t_1-t)A_+^*\}}](P_+ - S)\big). \tag{3.1}$$

Proof. This formula has been used in the literature in a more or less explicit way. A direct derivation is provided in [4, proof of Thm. 3.4]. ∎

COROLLARY 3.2 (of Lemma 3.1). *Given initial conditions* α*,* β *and an inhomogeneity* f*, with* U *as in* (3.1)*, the state transition for the forward evolution of*

$$\dot{x}(t) = (A - BB^*P(t))x + f(t), \quad x(t_0) = \alpha,$$

is given as

$$x(t) = U(t)U(t_0)^{-1}\alpha + \int_{t_0}^{t} U(t)U(s)^{-1}f(s)ds,$$

and the backward propagation of

$$-\dot{y}(t) = (A^* - P(t)BB^*)y(t) + f(t), \quad y(t_0) = \beta$$

is given as

$$y(t) = U(t)^{-*}U(t_0)^*\beta - \int_{t_0}^{t} U(t)^{-*}U(s)^*f(s)ds.$$

For the forward and backward state transition maps, one can derive (see [15, Lem. 3.3]) the expressions

$$U(t)U(s)^{-1} = e^{\{(t-s)A_+\}} - W(t-s)e^{\{(t_1-t)A_+^*\}}\tilde{S}(t_1-s)e^{\{(t_1-s)A_+\}} \tag{3.2}$$

and

$$U(t)^{-*}U(s)^{*} = e^{\{(s-t)A_+^*\}} - e^{\{(t_1-t)A_+^*\}}\tilde{S}(t_1-t)e^{\{(t_1-s)A_+\}}W(s-t), \tag{3.3}$$

which relate the evolutions to the functions W and $\tilde{S}$ as defined in (2.7) and (2.9).

Corollary 3.2 and identity (3.3) can then be used for expressing the feedforward w (cf. (2.6)) as

$$w(t) = w_h(t) + w_p(t) := -U(t)^{-*}U(t_1)^*F^*y_e + \int_{t_1}^{t} U(t)^{-*}U(s)^*C^*y_c \, \mathrm{d}s$$

with

$$w_h(t) = -e^{\{(t_1-t)A_+^*\}}[I - \tilde{S}(t_1-t)W(t_1-t)]F^*y_e, \tag{3.4}$$

which is the feedforward induced by the nonzero terminal constraint y_e and with

$$\begin{aligned} w_p(t) &= A_+^{-*}C^*y_c - e^{\{(t_1-t)A_+^*\}}C^*y_c \\ &\quad - e^{\{(t_1-t)A_+^*\}}\tilde{S}(t_1-t)\Big[A_+^{-1}\big[I - e^{\{(t_1-t)A_+\}}\big]W \\ &\qquad + e^{\{(t_1-t)A_+\}}W\big[I - e^{\{(t_1-t)A_+^*\}}\big]A_+^{-*}\Big]C^*y_c, \end{aligned} \tag{3.5}$$

which describes the inhomogeneity induced by y_c.

Similarly, with (3.2), the solution x can be written as

$$\begin{aligned} x(t) &= U(t)U(0)^{-1}x_0 - \int_0^t U(t)U(s)^{-1}BB^*w(s) \, \mathrm{d}s \\ &= x_h(t) - \int_0^t e^{\{(t-s)A_+\}}BB^*w(s) \, \mathrm{d}s \\ &\quad + \int_0^t W(t-s)e^{\{(t_1-t)A_+^*\}}\tilde{S}(t_1-s)e^{\{(t-s)A_+\}}BB^*w(s) \, \mathrm{d}s, \end{aligned} \tag{3.6}$$

where x_h coincides with the solution of the homogeneous problem where $w = 0$. The explicit formulas for w and x can be exploited for stating the following result.

Proposition 3.3 ([15, Prop. 3.5]). *Let Assumptions* 2.5 *and* 2.11 *hold. Then the solution x to the finite time optimal control problem, i.e., Problem* 2.1, *is given as*

$$x(t) = x_h(t) + x_s - e^{\{tA_+\}}A_+^{-1}C^*y_c + g(t,t_1),$$

where x_h solves Problem 2.1 *for $y_c = 0$ and $y_e = 0$, where x_s is the solution to the steady-state optimal control problem (i.e., Problem* 2.2*), and where $g(t,t_1)$ can be estimated as*

$$\|g(t,t_1)\| \le \mathrm{const}\, e^{\{(t_1-t)\lambda\}},$$

where $\lambda < 0$ is the spectral abscissa of A_+.

Proposition 3.3 directly implies the turnpike property for the solutions to Problem 2.1; cf. Definition 2.3.

COROLLARY 3.4. *Under Assumptions* 2.5 *and* 2.11, *for the solutions* (x,u) *to Problem* 2.1 *and* (x_s,u_s) *to Problem* 2.2, *it holds that*

$$\|x(t)-x_s\| \leq \text{const}(e^{\{t\lambda\}}+e^{\{(t_1-t)\lambda\}})$$

and

$$\|u(t)-u_s\| \leq \text{const}(e^{\{t\lambda\}}+e^{\{(t_1-t)\lambda\}})$$

for a constant const >0 *independent of* t_1 *and* $\lambda<0$ *being the spectral abscissa of* A_+.

Proof. By Proposition 3.3, we have that

$$\begin{aligned}\|x(t)-x_s\| &\leq \|x_h(t)-e^{\{tA_+\}}A_+^{-1}C^*y_c+g(t,t_1)\|\\ &\leq \|x_h(t)\|+\|e^{\{tA_+\}}A_+^{-1}C^*y_c\|+\|g(t,t_1)\| \leq \text{const}(e^{\{t\lambda\}}+e^{\{(t_1-t)\lambda\}}).\end{aligned}$$

For the input, we recall that by Theorem 2.8, the optimal input is given as $u(t)=-B^*(P(t)x(t)+w(t))$, where P is the solution to the differential Riccati equation (2.5) and w solves (2.6). With $P(t)=P_+ + P_\Delta(t)$ and with the formulas (3.4) and (3.5) for w, we find that

$$u(t) = -B^*P_+x(t) - B^*P_\Delta x(t) - B^*A_+^*C^*y_c - B^*g_w(t,t_1),$$

where $g_w(t,t_1)$ collects all reminder terms of w and which is readily estimated by the decay of $e^{\{(t_1-t)A_+^*\}}$. With $u_s=-B^*P_+x_s - B^*A_+^*C^*y_c$ (see Lemma 2.7), we directly estimate

$$\begin{aligned}\|u(t)-u_s\| &= \|-B^*P_+(x(t)-x_s)-B^*P_+x_s\\ &\quad -B^*A_+^*C^*y_c - B^*P_\Delta x(t) - B^*g_w(t,t_1)-u_s\|\\ &\leq \|B^*P_+(x(t)-x_s)\|+\|B^*P_\Delta(t)x(t)\|+\|B^*g_w(t,t_1)\|\\ &\leq \|B^*P_+\|\|x(t)-x_s\|+\|B^*\|\|P_\Delta(t)\|(\|x_s\|+\|x(t)-x_s\|)\\ &\quad +\|B^*\|\|g_w(t,t_1)\|,\end{aligned} \tag{3.7}$$

from where the turnpike estimate follows directly by the turnpike estimate for $x(t)-x_s$, the exponential decay of $g_w(t,t_1)$ with $t_1\to\infty$, and the exponential decay of $P_\Delta(t)$ as $t_1\to\infty$; see [8, Thm. 3]. ∎

4. Linear quadratic optimal control for descriptor systems.

We now consider optimal control problems with DAEs of the form

$$\mathcal{E}\dot{x}(t) = \mathcal{A}x(t)+\mathcal{B}u(t), \quad \mathcal{E}x(0)=\mathcal{E}x_0, \tag{4.1}$$

as constraints. If the coefficient matrix $\mathcal{E}$ is not invertible, then (4.1) will be made of differential and algebraic equations for x, hence the name DAE.

Problem 4.1 (finite horizon optimal control problem). For coefficients $\mathcal{A}$, $\mathcal{E}\in\mathbb{R}^{n\times n}$, $\mathcal{B}\in\mathbb{R}^{n\times m}$, $\mathcal{C}\in\mathbb{R}^{k\times n}$, and $\mathcal{F}\in\mathbb{R}^{\ell\times n}$, for an initial value $x_0\in\mathbb{R}^n$, for target outputs $y_c\in\mathbb{R}^k$ and $y_e\in\mathbb{R}^\ell$ and a terminal time $t_1>0$, consider

$$\frac{1}{2}\int_0^{t_1}\|\mathcal{C}x(s)-y_c\|^2+\|u(s)\|^2 \,\mathrm{d}s \;+\frac{1}{2}\|\mathcal{F}x(t_1)-y_e\|^2 \to \min_u$$

subject to the DAE (4.1).

Problem 4.1 is a convex problem with affine linear constraints, which implies that if a candidate solution satisfies first-order necessary optimality conditions, then it is an optimal solution. For $y_c = 0$ and $y_e = 0$, the *formal* first-order necessary conditions [23] for Problem 4.1 read

$$\begin{bmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^* \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{A} & -\mathcal{B}\mathcal{B}^* \\ -\mathcal{C}^*\mathcal{C} & -\mathcal{A}^* \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \quad \mathcal{E}x(0) = \mathcal{E}x_0, \quad \mathcal{E}^*p(t_1) = \mathcal{F}^*\mathcal{F}x(t_1), \tag{4.2}$$

and define the optimal control as $u(t) = -\mathcal{B}^*p(t)$.

Remark 4.2. The optimality conditions (4.2) are called formal because they are formally derived through a variation of the original problem formulation. However, it is known that the optimal control problem can have a solution, while the formal optimality conditions do not have a solution [23]. Thus, one should either use an equivalent reformulation of the optimal control problem (as proposed in [23]) or make sure that the formal optimality conditions are solvable [14]. In this work, we will pursue the second approach.

One can confirm directly that, if $\mathcal{P}$ solves the gDRE

$$-\mathcal{E}^*\dot{\mathcal{P}} = \mathcal{A}^*\mathcal{P} + \mathcal{P}^*\mathcal{A} - \mathcal{P}^*\mathcal{B}\mathcal{B}^*\mathcal{P} + \mathcal{C}^*\mathcal{C} = 0, \quad \mathcal{E}^*\mathcal{P}(t_1) = \mathcal{F}^*\mathcal{F}, \tag{4.3}$$

then the ansatz $p = \mathcal{P}x$ decouples the optimality conditions (4.2) and defines a solution.

As in the ODE case, we will consider stabilizing solutions of an associated gARE

$$\mathcal{A}^*X + X^*\mathcal{A} - X^*\mathcal{B}\mathcal{B}^*X + \mathcal{C}^*\mathcal{C} = 0, \quad \mathcal{E}^*X = X^*\mathcal{E}. \tag{4.4}$$

Next, we provide the basic nomenclature and fundamental results for DAEs with inputs and outputs.

DEFINITION 4.3. *A matrix pair $(\mathcal{E}, \mathcal{A})$ or a matrix pencil $s\mathcal{E} - \mathcal{A}$ is called* regular *if there exists an $s \in \mathbb{C}$ such that $s\mathcal{E} - \mathcal{A}$ is invertible.*

To introduce the stability concepts, we rely on a direct consequence of a canonical form that can be derived for regular DAEs; see [21, Thm. I.2.7].

LEMMA 4.4. *If $(\mathcal{E}, \mathcal{A})$ is regular, then the associated DAE is equivalent to the decoupled system*

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_{\text{sss}} \\ x_{\text{fss}} \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{\text{sss}} \\ x_{\text{fss}} \end{bmatrix} + \begin{bmatrix} B_{\text{sss}} \\ B_{\text{fss}} \end{bmatrix} u, \tag{4.5}$$

where J and N are square matrices in Jordan canonical form *and where N is* nilpotent*, which means that there exists a $\nu \in \mathbb{N}$ such that $N^\nu = 0$.*

The part with the state x_{sss} is called the *slow subsystem* or the *finite dynamics*, and the part of x_{fss} is called the *fast subsystem.*

DEFINITION 4.5 (finite dynamics stability). *Let $(\mathcal{E}, \mathcal{A})$ be regular.*

1. *The DAE (4.1) with coefficients $(\mathcal{E}, \mathcal{A})$ is called* finite dynamics stable *if its* slow subsystem *is stable, i.e., if all eigenvalues of J in the associated canonical form (4.5) have negative real part.*
2. *The triple $(\mathcal{E}, \mathcal{A}, \mathcal{B})$ is called* finite dynamics stabilizable *if there exists a feedback matrix $\mathcal{K}$ such that $(\mathcal{E}, \mathcal{A} - \mathcal{B}\mathcal{K})$ is finite dynamics stable.*

For equivalent algebraic characterizations and for the duality with detectability, see [10, Chap. 3-1.2].

From the solution formula ([21, Lem. I.2.8]) for the *fast subsystem*

$$x_{\mathrm{fss}}(t) = -\sum_{i=1}^{\nu-1} N^i B_{\mathrm{fss}} \frac{d^i}{dt^i} u(t), \tag{4.6}$$

one finds that an initial condition for x_{fss} that does not equal the expression of (4.6) at $t = 0$ generates impulses in the solution; cf. [10, eq. (2-2.9)]. If any such impulse can be compensated by an input that is piecewise $\nu - 1$-times differentiable, then the system is called *impulse controllable*. To circumvent the technicalities that come with distributions and to express *impulse controllability* in terms of the original coefficients $(\mathcal{E}, \mathcal{A}, \mathcal{B})$, we will use an equivalent algebraic characterization; see [10, Thm. 2-2.3].

DEFINITION 4.6. *Let* $(\mathcal{E}, \mathcal{A})$ *be regular. The DAE* (4.1) *with coefficients* $(\mathcal{E}, \mathcal{A}, \mathcal{B})$ *is called* impulse controllable *if*

$$\operatorname{rank} \begin{bmatrix} \mathcal{E} & 0 & 0 \\ \mathcal{A} & \mathcal{E} & \mathcal{B} \end{bmatrix} = n + \operatorname{rank} \mathcal{E}.$$

A matrix pencil $s\mathcal{E} - \mathcal{A}$ is called *impulse free* if no impulses occur in the DAE solution regardless of the initial value. This means that it is trivially *impulse controllable*. In line with Definition 4.6, this can be characterized as follows.

DEFINITION 4.7. *Let* $(\mathcal{E}, \mathcal{A})$ *be regular. The DAE* (4.1) *with coefficients* $(\mathcal{E}, \mathcal{A})$ *is called* impulse free *if*

$$\operatorname{rank} \begin{bmatrix} \mathcal{E} & 0 \\ \mathcal{A} & \mathcal{E} \end{bmatrix} = n + \operatorname{rank} \mathcal{E}.$$

Remark 4.8. As for standard systems, the notions of *finite time detectability* and *impulse observability* of a DAE with output matrix $\mathcal{C}$ can be defined by duality, i.e., via the *finite time stability* and *impulse controllability* of $(\mathcal{E}^*, \mathcal{A}^*, \mathcal{C}^*)$; see [10, Thm. 2-4.1].

Next, we state the underlying assumptions for our analysis and some immediate consequences.

Assumption 4.9. The coefficients $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in Problem 4.1 and (4.1) are such that

1. the pair $(\mathcal{E}, \mathcal{A})$ is regular;
2. the gARE (4.4) has a stabilizing solution $\mathcal{P}_+$; i.e., the pair

$$(\mathcal{E}, \mathcal{A}_+) := (\mathcal{E}, \mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}_+)$$

is regular, *impulse free*, and *finite dynamics stable*.

Assumption 4.9 implies that the system $(\mathcal{E}, \mathcal{A}, \mathcal{B})$ is *finite dynamics stabilizable* and *impulse controllable*. We will show below that *impulse observability* of $(\mathcal{E}, \mathcal{A}, \mathcal{B})$ is a necessary condition for the existence of a stabilizing Riccati solution $\mathcal{P}_+$. As for the ODE case, *(finite time) detectability* is not necessary for the existence of $\mathcal{P}_+$; cf. Remark 2.6.

For the existence of solutions to the necessary optimality conditions (4.2) in general, we make the following assumption.

Assumption 4.10. The matrices $\mathcal{E}$ and $\mathcal{F}$ in Problem 4.1 are compatible in the sense that

$$\operatorname{range} \mathcal{F}^* \subset \operatorname{range} \mathcal{E}^*.$$

Looking at the terminal condition $\mathcal{E}^* p(t_1) = \mathcal{F}^* \mathcal{F} x(t_1)$, one can find that Assumption 4.10 is necessary for the existence of solutions to the optimality conditions (4.2). Also, it is an implicit assumption made in [5, cf. eq. (1)] and the base for more general results (see, e.g., [22, Thm. 13]). Still it is not a necessary condition for the existence of optimal solutions; cf. Remark 4.2.

In order to simplify the formulas, we further assume that $\mathcal{E}$ has a *semiexplicit* structure, which means that the differentiated variables appear explicitly.

Assumption 4.11. The matrix $\mathcal{E} \in \mathbb{R}^{n,n}$ in (4.1) is of the form

$$\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.7}$$

where $I \in \mathbb{R}^{d,d}$ is the identity matrix.

Remark 4.12. In theory, this assumption is not restrictive since a regular transformation of the system can always provide such a form of $\mathcal{E}$. In practice, to actually solve the Riccati equations, this semiexplicit realization of the state equations might be helpful. While in the general and, in particular, large-scale case such a transformation may destroy sparsity and introduce systematic errors to the model, we want to remark that for many applications, the invertible part of $\mathcal{E}$ can be inferred from the structure so that it can be transformed into the semiexplicit form in a computationally feasible way.

Remark 4.13. Assumption 4.11 and the symmetry constraint $\mathcal{E}^* \mathcal{P}_+ = \mathcal{P}_+^* \mathcal{E}$ imply that $\mathcal{P}_+$ is a block lower-triangular matrix, i.e.,

$$\mathcal{P}_+ = \begin{bmatrix} P_{+;1} & 0 \\ P_{+;21} & P_{+;2} \end{bmatrix} \tag{4.8}$$

with $P_{+;1}$ being symmetric, i.e., $P_{+;1}^* = P_{+;1}$. Moreover, Assumptions 4.11 and 4.10 together imply that

$$\mathcal{F}^* \mathcal{F} = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.9}$$

Remark 4.14. For $\mathcal{E}$ in semiexplicit form, several concepts can be made more explicit. Let

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{4.10}$$

Then $(\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C})$ are impulse controllable or impulse observable if and only if

$$\begin{bmatrix} A_{22} & B_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_{22}^* & C_2^* \end{bmatrix} \tag{4.11}$$

has full rank, respectively. And $(\mathcal{E}, \mathcal{A})$ is impulse-free if and only if A_{22} is invertible.

Moreover, for semiexplicit *impulse-free* systems, *finite dynamics stability* can be characterized as follows.

Lemma 4.15 ([15, Lem. 4.9]). *Let*

$$\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)$$

be a regular impulse-free matrix pair. Then it is finite dynamics stable *if and only if* $A_{11} - A_{12}A_{22}^{-1}A_{12}$ *is stable.*

The following lemma relates the associated Hamiltonian matrix pencil to the existence of stabilizing solutions of the gARE (4.4). Although it is the direct extension of the standard state space result, it has not been stated explicitly so far.

Lemma 4.16. *Let* $(\mathcal{E}, \mathcal{A})$ *be regular. The gARE* (4.4) *has a stabilizing solution if and only if* $(\mathcal{E}, \mathcal{A}, \mathcal{B})$ *is finite dynamics stabilizable and the matrix pencil*

$$\mathcal{H}(s) = \begin{bmatrix} -s\mathcal{E} + \mathcal{A} & -\mathcal{B}\mathcal{B}^* \\ -\mathcal{C}^*\mathcal{C} & -s\mathcal{E}^* - \mathcal{A}^* \end{bmatrix}$$

is regular, is impulse free, and has no finite eigenvalues on the imaginary axis.

Proof. The necessity is stated and proved in the first lines of the proof of [17, Lem. 1]. The sufficiency follows by the arguments of [29, sect. 3] as follows. With $\mathcal{E}$ in semiexplicit form, the absence of impulses in $\mathcal{H}(s)$ implies that

$$\begin{bmatrix} A_{22} & -B_2B_2^* \\ -C_2^*C_2 & A_{22}^* \end{bmatrix}$$

is invertible so that $(\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C})$ must be impulse controllable and impulse observable; cf. (4.11). Thus, the conditions of [29, Thm. 3.2] are fulfilled up to the finite dynamics observability of $(\mathcal{E}, \mathcal{A}, \mathcal{C})$. Still, one can apply [29, Lem. 3.10] since the needed invertibility is guaranteed by $\mathcal{H}(s)$ having no finite modes on the imaginary axis, which implies that $\begin{bmatrix} \mathcal{C}^* & \mathcal{A}^* \end{bmatrix}$ has full rank (see [28, Thm. 4]), as it is needed in the proof of [29, Lem. 3.8]. Thus, the existence of the relevant stabilizing solution, which is denoted by Y in [29], follows as laid out in [29, sect. 3.2]. ◻

5. Existence and asymptotic behavior of structured solutions to the differential Riccati solution. In this section, we establish the existence of a particularly structured solution to the gDRE (4.3) under the assumption that the associated algebraic Riccati equation gARE (4.4) has a stabilizing solution.

We start with adding another assumption that will be shown to be a sufficient criterion for the existence of solutions to the gDRE (4.3) and that we will justify by considering its necessity for particular cases.

Assumption 5.1. Let Assumptions 4.9 and 4.11 hold, and let $(\mathcal{E}, \mathcal{A}, \mathcal{C})$ and the stabilizing solution $\mathcal{P}$ to the gARE (4.4) be partitioned as in (4.8) and (4.10). Then with the $P_{+;2}$ block of $\mathcal{P}$ and with $K_2 := A_{22}^* - P_{+;2}B_2B_2^*$ being regular (cf. [17, Lem. 1]),

$$\begin{aligned} \tilde{Q} := C_1^*C_1 - (A_{21}^*P_2 + C_1^*C_2)K_2^{-*}A_{21} - A_{21}^*K_2^{-1}(C_2^*C_1 + P_2^*A_{21}) \\ - (C_1^*C_2 + A_{21}^*P_{+;2})K_2^{-*}B_2B_2^*K_2^{-1}(C_2^*C_1 + P_{+;2}^*A_{21}) \geq 0. \end{aligned} \tag{5.1}$$

Remark 5.2. We note that the $P_{+;2}$ block of $\mathcal{P}_+$ is not uniquely defined; it only has to fulfill the quadratic equation

$$A_{22}^*P_2 + P_2^*A_{22} - P_2^*B_2B_2^*P_2 + C_2^*C_2 = 0 \tag{5.2}$$

and the condition that $A_{22}^* - P_2^* B_2 B_2$ is regular. See [17, Lem. 1] or the solution representation provided in [29, Lem. 3.10] for the semiexplicit $\mathcal{E}$. Also, compare the nonuniqueness of the feedback law provided in [16, eq. (3.43)].

Before we state the global existence of solutions to the gDREs, we show that Assumption 5.1 generalizes the general assumption of a positive definite cost functional for problems that are impulse free.

For that, we consider a system that is impulse free and that, without loss of generality, can be assumed in the form of

$$\mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & -I \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{5.3}$$

Like for every system with $(\mathcal{E}, \mathcal{A})$ impulse free, Problem 4.1 with system (5.3) as constraint is equivalent to a standard LQR problem. In this case, if the state $x = (x_1, x_2)$ is partioned accordingly, one can express x_2 as $x_2 = B_2 u$ and find that the cost functional is positive definite if and only if

$$C_1^* C_1 - C_1^* C_2 B_2 B_2^* C_2^* C_1 \geq 0 \quad \text{or} \quad I - C_2 B_2 B_2^* C_2^* \geq 0,$$

which is equivalent to the largest singular vector of $C_2 B_2$ being less than one, i.e., $\bar{\sigma}(C_2 B_2) \leq 1$. Thus, for systems in the form (5.3), the condition $\bar{\sigma}(C_2 B_2) \leq 1$ is needed in the standard theory (cf. [42, eq. (14.2)] or [27, Rem. 3.4]).

LEMMA 5.3. *Consider a system* $(\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C})$ *in the form of* (5.3)*. There exists a* P_2 *that solves* (5.2) *such that* $A_{22}^* - P_2^* B_2 B_2^*$ *is regular and such that* (5.1) *holds if and only if* $\bar{\sigma}(C_2 B_2) \leq 1$*.*

Proof. Sufficieny: For a system in the form of (5.3), i.e., $A_{22} = -I$, by standard theory, the Riccati equation (5.2) has a symmetric positive definite or positive semidefinite solution P_2 such that $-I - P_2^* B_2 B_2^*$ is stable and, thus, invertible. With $A_{21} = 0$, condition (5.1) reads

$$C_1^* C_1 - C_1^* C_2 (-I - B_2 B_2^* P_2)^{-1} B_2 B_2^* (-I - P_2^* B_2 B_2^*)^{-1} C_2^* C_1 \geq 0$$

or, with the identity $(I + B_2 B_2^* P_2)^{-1} B_2 = B_2 (I + B_2^* P_2 B_2)^{-1}$,

$$I - C_2 B_2 (I + B_2^* P_2 B_2)^{-1} (I + B_2^* P_2^* B_2)^{-1} B_2^* C_2^* \geq 0. \tag{5.4}$$

With $P_2 \geq 0$, all eigenvalues of $I + B_2^* P_2 B_2$ are larger than one, and, accordingly, all eigenvalues of $(I + B_2^* P_2 B_2)^{-1}$ are smaller than one. Since for symmetric positive definite matrices the eigenvalues coincide with the singular values and since $\bar{\sigma}$ has all the properties of a norm, we can estimate

$$\begin{aligned} \bar{\sigma}(C_2 (I + B_2 B_2^* P_2)^{-1} B_2) &= \bar{\sigma}(C_2 B_2 (I + B_2 B_2^* P_2)^{-1}) \\ &\leq \bar{\sigma}(C_2 B_2) \bar{\sigma}((I + B_2 B_2^* P_2)^{-1}) \leq \bar{\sigma}(C_2 B_2) \leq 1, \end{aligned}$$

from where we conclude (5.4).

Necessity: The multiplication of the corresponding Riccati equation by B_2^* and B_2 from the left and the right, respectively, and the completion of the square yield

$$(I - B_2^* P_2 B_2)^* (I - B_2^* P_2 B_2) = I - B_2^* C_2^* C_2 B_2,$$

which, by the positive definiteness of the left-hand side, can only hold if $\bar{\sigma}(C_2 B_2) \leq 1$. ∎

By Lemma 5.3, for impulse-free systems, validity of Assumption 5.1 is implied by the assumption that the underlying cost functional is positive definite.

THEOREM 5.4. *Consider the DAE* (4.1) *and Problem* 4.1. *Assume that $\mathcal{E}$ is semiexplicit and that $\mathcal{F}$ is compatible (Assumptions* 4.11 *and* 4.10*). Assume that $(\mathcal{E}, \mathcal{A})$ is regular and that a stabilizing solution to the gARE exists (Assumption* 4.9*). Let Assumption* 5.1 *hold. Then the gDRE* (4.3) *has a solution $\mathcal{P}$ for $t \leq t_1$.*

Proof. Since $\mathcal{E}^*\dot{\mathcal{P}}(t)$ is symmetric and $\mathcal{E}^*\mathcal{P}(t_1) = \mathcal{F}^*\mathcal{F}$ is symmetric, it holds that $\mathcal{E}^*\mathcal{P}(t)$ is symmetric or, due to the semiexplicit form of $\mathcal{E}$, that

$$\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ P_{21} & P_2 \end{bmatrix}$$

with $P_1(t)$ being symmetric. With this block triangular structure and the partition of the coefficients $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as in (4.10), the gDRE (4.3) can be written in terms of the following four coupled matrix valued equations:

$$\begin{aligned} -\dot{P}_1 = {} & A_{11}^* P_1 + A_{21}^* P_{21} + P_1 A_{11} + P_{21}^* A_{21} \\ & - P_1 B_1 B_1^* P_1 - P_{21}^* B_2 B_1^* P_1 - P_1 B_1 B_2^* P_{21} - P_{21}^* B_2 B_2^* P_{21} + C_1^* C_1, \end{aligned}$$

$$P_1(t_1) = S, \tag{5.5a}$$

$$0 = A_{12}^* P_1 + A_{22}^* P_{21} + P_2^* A_{21} - P_2^* B_2 B_1^* P_1 - P_2^* B_2 B_2^* P_{21} + C_2^* C_1, \tag{5.5b}$$

$$0 = P_1 A_{12} + P_{21}^* A_{22} + A_{21}^* P_2 - P_1^* B_1 B_2^* P_2 - P_{21}^* B_2 B_2^* P_2 + C_1^* C_2, \tag{5.5c}$$

$$0 = A_{22}^* P_2 + P_2^* A_{22} - P_2^* B_2 B_2^* P_2 + C_2^* C_2. \tag{5.5d}$$

Note that (5.5c) is the transpose of and, thus, equivalent to (5.5b).

Since (5.5d) does not differ from the left-lower block of the gARE (4.4), Assumption 4.9 implies the existence of a (constant) matrix (function) P_2 that solves (5.5d) such that

$$K_2 := A_{22}^* - P_2^* B_2 B_2^*$$

is invertible. Accordingly, the matrix valued function P_{21} is defined by virtue of (5.5c) or (5.5b) as

$$P_{21}(t) = -K_2^{-1}(A_{12}^* P_1(t) + P_2^* A_{21} - P_2^* B_2 B_1^* P_1(t) + C_2^* C_1), \tag{5.6}$$

and the existence of $\mathcal{P}$ relies on the existence of a Riccati solution to (5.5a), which, with P_{21} expressed in terms of P_1 and a constant term as in (5.6), reads

$$-\dot{P}_1 = \tilde{A}^* P_1 + P_1 \tilde{A} - P_1 \tilde{R} P_1 + \tilde{Q}, \quad P_1(t_1) = S, \tag{5.7}$$

where $\tilde{Q}$ is as in Assumption 5.1 and where

$$\begin{aligned} \tilde{A} &:= A_{11} - (A_{21} - B_1 B_2^* P_2) K_2^{-*} A_{21} + B_1 B_2^* K_2^{-1}(P_2 A_{21} + C_2^* C_1), \\ \tilde{R} &:= \begin{bmatrix} B_1 & -(B_1 B_2^* P_2 - A_{12}^*) K_2^{-*} B_2 \end{bmatrix} \begin{bmatrix} B_1 & -(B_1 B_2^* P_2 - A_{12}^*) K_2^{-*} B_2 \end{bmatrix}^*. \end{aligned}$$

With $\tilde{R} \geq 0$, for arbitrary $S \geq 0$, the global existence of the unique solution to (5.7) is ensured (cf. [1, Thm. 4.1.6]) if also $\tilde{Q}$ is positive semidefinite, which it is by Assumption 5.1 and with the choice $P_2 = P_{+;2}$. ∎

For this solution to the gDRE, in analogy to the standard ODE case [8], we can show the exponential decay toward the gARE solution.

To prepare the arguments, we consider the associated boundary value problem

$$\begin{bmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^* \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & -\mathcal{B}\mathcal{B}^* \\ -\mathcal{C}^*\mathcal{C} & -\mathcal{A}^* \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \mathcal{E}V_1(t_1) = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathcal{E}V_2(t_1) = \begin{bmatrix} S \\ 0 \end{bmatrix}, \tag{5.8}$$

with $V_1(t), V_2(t) \in \mathbb{R}^{n,d}$, where d is the size of identity block in $\mathcal{E}$ (cf. Assumption 4.11) and where the initial conditions already anticipate the semiexplicit form of $\mathcal{E}$.

By standard arguments, we can write (5.8) as

$$\begin{bmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^* \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_+ & -\mathcal{B}\mathcal{B}^* \\ 0 & -\mathcal{A}_+^* \end{bmatrix} \begin{bmatrix} V_1 \\ \tilde{V}_2 \end{bmatrix}, \quad \mathcal{E}V_1(t_1) = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathcal{E}\tilde{V}_2(t_1) = \begin{bmatrix} S_1 - P_{+;1} \\ 0 \end{bmatrix}, \tag{5.9}$$

with

$$\begin{bmatrix} V_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\mathcal{P}_+ & I \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

In line with the semiexplicit structure of $\mathcal{E}$, we further differentiate

$$\mathcal{A}_+ = \begin{bmatrix} A_{+;1} & A_{+;12} \\ A_{+;21} & A_{+;2} \end{bmatrix}, \quad \mathcal{B}\mathcal{B}^* = \begin{bmatrix} B_1B_1^* & B_1B_2^* \\ B_2B_1^* & B_2B_2^* \end{bmatrix}, \quad \begin{bmatrix} V_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} V_{11} \\ V_{12} \\ \tilde{V}_{21} \\ \tilde{V}_{22} \end{bmatrix}. \tag{5.10}$$

With this partitioning and since $\mathcal{A}_+$ is impulse free, we can use that the block $A_{+;2}$ is invertible and the relation

$$\begin{aligned} \begin{bmatrix} V_{12} \\ \tilde{V}_{22} \end{bmatrix} &= -\begin{bmatrix} A_{+;2} & -B_2B_2^* \\ 0 & -A_{+;2}^* \end{bmatrix}^{-1} \begin{bmatrix} A_{+;21} & -B_2B_1^* \\ 0 & -A_{+;12}^* \end{bmatrix} \begin{bmatrix} V_{11} \\ \tilde{V}_{21} \end{bmatrix} \\ &= -\begin{bmatrix} A_{+;2}^{-1}A_{+;21} & -A_{+;2}^{-1}B_2B_1^* + A_{+;2}^{-1}B_2B_2^*A_{+;2}^{-*}A_{+;12}^* \\ 0 & A_{+;2}^{-*}A_{+;12}^* \end{bmatrix} \begin{bmatrix} V_{11} \\ \tilde{V}_{21} \end{bmatrix} \end{aligned} \tag{5.11}$$

to get the reduced system

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{11} \\ \tilde{V}_{21} \end{bmatrix} = \begin{bmatrix} \bar{A} & -\bar{B}\bar{B}^* \\ 0 & -\bar{A}^* \end{bmatrix} \begin{bmatrix} V_{11} \\ \tilde{V}_{21} \end{bmatrix}, \quad V_{11}(t_1) = I, \quad \tilde{V}_{21}(t_1) = S - P_{+;1}, \tag{5.12}$$

where

$$\bar{A} := A_{+;1} - A_{+;12}A_{+;2}^{-1}A_{+;21} \quad \text{and} \quad \bar{B} := [B_1 - A_{+;12}A_{+;2}^{-1}B_2].$$

THEOREM 5.5. *Consider the gDRE* (4.3) *with $\mathcal{E}$ semiexplicit as in Assumption* 4.11*, and let $\mathcal{F}$ be compatible as in Assumption* 4.10. *Let the coefficients $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be partitioned as in* (4.10) *in accordance with $\mathcal{E}$. Let Assumption* 4.9 *hold, let $\mathcal{P}_+$ be the stabilizing solution to the gARE* (4.4)*, and let $\mathcal{A}_+ := \mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}_+$ be partitioned as in* (5.10)*. Let Assumption* 5.1 *hold, and let $\mathcal{P}$ be the solution to the gDRE* (4.3)*. Then system* (5.9) *has a unique solution*

$$\begin{bmatrix} V_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} V_{11} \\ V_{12} \\ \tilde{V}_{21} \\ \tilde{V}_{22} \end{bmatrix}$$

with $V_{11}(t)$ invertible for $t \leq t_1$ and such that for $\mathcal{P}_\Delta(t) := \mathcal{P}(t) - \mathcal{P}_+$ it holds that

$$\begin{aligned} \mathcal{P}_\Delta(t) = \begin{bmatrix} P_{\Delta;1}(t) & 0 \\ P_{\Delta;21}(t) & 0 \end{bmatrix} &= \begin{bmatrix} P_{\Delta;1}(t) & 0 \\ -A_{+;2}^{-*}A_{+;12}^*P_{\Delta;1}(t) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{V}(t)_{21}V(t)_{11}^{-1} & 0 \\ -A_{+;2}^{-*}A_{+;12}^*\tilde{V}(t)_{21}V(t)_{11}^{-1} & 0 \end{bmatrix}. \end{aligned} \tag{5.13}$$

Proof. By Assumption 2.5 and Theorem 5.4, the existence of $\mathcal{P}_\Delta$ is ensured for $t \leq t_1$. A direct computation reveals that $\mathcal{P}_\Delta$ solves the gDRE (cf. [8, p. 994]), i.e.,

$$-\mathcal{E}^*\dot{\mathcal{P}}_\Delta = \mathcal{A}_+^*\mathcal{P}_\Delta + \mathcal{P}_\Delta^*\mathcal{A}_+ - \mathcal{P}_\Delta^*\mathcal{B}\mathcal{B}^*\mathcal{P}_\Delta, \quad \mathcal{E}^*\mathcal{P}_\Delta(t_1) = \mathcal{F}^*\mathcal{F} - \mathcal{E}^*\mathcal{P}_+.$$

Since the second block column of $\mathcal{P}_\Delta$ is zero—as it follows directly from $\mathcal{P}_\Delta = \mathcal{P} - \mathcal{P}_+$—and $A_{+;2}$ is invertible, (4.3) implies that the left-lower block of $\mathcal{P}_\Delta$ is given as $-A_{+;2}^{-*}A_{+;12}^*P_{\Delta;1}(t)$ and that $P_{\Delta;1}$ is the solution to the Riccati equation

$$-\dot{P}_{\Delta;1} = \bar{A}^*P_{\Delta;1} + P_{\Delta;1}\bar{A} - P_{\Delta;1}\bar{B}\bar{B}^*P_{\Delta;1}, \quad P_{\Delta;1}(t_1) = S - P_{+;1},$$

which in particular means that this Riccati equation has a global solution for $t \leq t_1$ despite the possibly indefinite initial condition.

From the existence of the Riccati solution $P_{\Delta;1}$, we can infer the invertibility of $V_{11}(t)$ from the relation

$$-\dot{V}_{11}(t) = (\bar{A} - \bar{B}\bar{B}^*P_{\Delta;1}(t))V_{11}(t), \quad V_{11}(t_1) = I,$$

which is a consequence of *Radon's lemma* (see, e.g., [1, Thm. 4.1.1]).

Uniqueness follows from V_{11} and $\tilde{V}_{21}$ being solutions to an ordinary linear differential equation, namely, (5.12), so that $\mathcal{P}_\Delta$, as defined in (5.13), is well-defined.

Following the lines of a proof for a standard result for ODEs [7, 180 Thm.], one can directly confirm that $\mathcal{P}_\Delta$ solves the gDRE (4.3); see the proof in [15, Thm. 5.5] for the technical details. ∎

From Theorem 5.5, we can directly deduce necessary conditions for the convergence of $\mathcal{P}(t)$ toward $\mathcal{P}_+$ as $t_1 \to \infty$; cf. Lemma 2.9 for the standard ODE case.

Corollary 5.6 (of Theorem 5.5). *The left-upper block $P_{\Delta;1}$ of $\mathcal{P}_\Delta$ as defined in* (5.13) *satisfies the relation*

$$P_{\Delta;1}(t) = e^{\{(t_1-t)\bar{A}^*\}}\bar{\bar{S}}(t_1 - t)e^{\{(t_1-t)\bar{A}\}}, \tag{5.14}$$

where

$$\bar{\bar{S}}(\tau) := (S - P_{+;1})[I + \bar{W}(\tau)(S - P_{+;1})]^{-1} \tag{5.15}$$

with

$$\bar{W}(\tau) := \int_0^\tau e^{\{s\bar{A}\}}\bar{B}\bar{B}^*e^{\{s\bar{A}^*\}} \,\mathrm{d}s$$

being well-defined. Moreover, $P_{\Delta;1}(t) \to 0$ exponentially as $t_1 \to \infty$ if and only if

$$I + \lim_{\tau\to\infty} \bar{W}(\tau)(S - P_{+;1})$$

is nonsingular.

Proof. Relation (5.14) and (5.15) follow from the *variation of constants* formula applied to (5.12) that gives

$$\tilde{V}_{21}(t) = e^{\{-(t-t_1)\bar{A}^*\}}(S - P_{+;1})$$

and

$$\tilde{V}_{11}(t) = e^{\{(t-t_1)\bar{A}\}}\left(I + \int_{t_1}^t e^{\{-(s-t_1)\bar{A}\}}\bar{B}\bar{B}^*e^{\{-(s-t_1)\bar{A}^*\}} \,\mathrm{d}s\ (S - P_{+;1})\right)$$

and the invertibility of $V_{11}(t)$. By stability of $\bar{A}$ (cf. Lemma 4.15), well-posedness of the improper integral that defines $\bar{W}$ is ensured, and the equivalence of $P_{\Delta;1}(t) \to 0$ and invertibility of $I + \bar{W}(S - P_{+;1})$ follow by [8, Lem. 3]. ∎

The condition for the convergence motivates the following assumption.

Assumption 5.7. Consider Problem 4.1, let Assumption 4.9 hold, let $\mathcal{E}$ be semiexplicit as in Assumption 4.11, and $\mathcal{A}_+$ and $\mathcal{B}\mathcal{B}^*$ be partitioned as in (5.10), and let $\mathcal{F}$ be compatible with $\mathcal{E}$ as in Assumption 4.10. The matrix

$$I + \bar{W}(S - P_{+;1})$$

is nonsingular, where

$$\bar{W} := \int_0^\infty e^{\{s\bar{A}\}} \bar{B}\bar{B}^* e^{\{s\bar{A}^*\}} \,\mathrm{d}s$$

is the closed-loop *finite dynamics* reachability Gramian and where S is the left-upper block of $\mathcal{F}^*\mathcal{F}$; cf. (4.9).

Another outcome of the existence of this structured solution to the gDRE is that the corresponding closed-loop system does not generate impulses. Since the closed-loop system is time varying, the definition of *impulse freeness* (Definition 4.7) does not apply. Instead, we directly derive a representation of the closed-loop system in which the differential and algebraic parts are decoupled.

COROLLARY 5.8 (of Theorem 5.5). *The closed-loop system*

$$\mathcal{E}\dot{x}(t) = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}(t))x(t), \quad \mathcal{E}x(0) = \mathcal{E}x_0,$$

can be written in decoupled form for $x = (x_1, x_2)$*:*

$$\dot{x}_1(t) = \hat{\mathcal{A}}_1(t)x_1(t), \quad 0 = x_2(t) - \hat{\mathcal{A}}_2(t)x_1(t), \quad \mathcal{E}x(0) = \mathcal{E}x_0. \tag{5.16}$$

Proof. By the particular structure of P_Δ, it follows that the left-lower block of $\mathcal{A} - \mathcal{B}\mathcal{B}^*P(t) = \mathcal{A} - \mathcal{B}\mathcal{B}^*(P_+ + P_\Delta)$ equals the left-lower block of A_+, namely, $A_{+;2}$. Since $A_{+;2}$ is invertible, with the help of corresponding Schur complement, the representation (5.16) of the DAE follows with $\hat{\mathcal{A}}_1(t) := (\bar{A} - \bar{B}\bar{B}^*P_{\Delta;1}(t))$ and $\hat{\mathcal{A}}_2(t) := -A_{+;2}^{-1}A_{+;21} + A_{+;2}^{-1}B_2\bar{B}^*P_{\Delta;1}(t)$. ∎

Remark 5.9. From the representation (5.16), one can read off that an inconsistent initial value will make in x_2 discontinuous at $t = 0$ but will not induce impulses. In a more general context, the existence of such a decoupled representation is referred to as *strangeness free*; cf. [21, Thm. 3.17].

Next, we show the turnpike property of the homogeneous optimal control problem with DAE constraints.

THEOREM 5.10. *Under the assumptions of Theorem* 5.5 *and under Assumption* 5.7*, Problem* 4.1 *with* $y_c = 0$ *and* $y_e = 0$ *has a solution* (x, u) *which fulfills the estimate*

$$\|x(t)\| \leq \mathrm{const}(e^{\{t\bar{\sigma}\}} + e^{\{(t_1-t)\bar{\sigma}\}})$$

and

$$\|u(t)\| \leq \mathrm{const}(e^{\{t\bar{\sigma}\}} + e^{\{(t_1-t)\bar{\sigma}\}})$$

with const *independent of* t_1 *and where* $\bar{\sigma} < 0$ *is the spectral abscissa of* $\bar{A}$.

Proof. With $(V_{11}, V_{12}, \tilde V_{21}, \tilde V_{22})$ solving (5.9), with the relation (5.11), and with $V_{11}(t)$ being invertible, we have that

$$\begin{bmatrix} x_1 \\ x_2 \\ \tilde p_1 \\ \tilde p_2 \end{bmatrix} = \begin{bmatrix} V_{11} \\ V_{12} \\ \tilde V_{21} \\ \tilde V_{22} \end{bmatrix} V_{11}^{-1}(t_0)x_0 = \begin{bmatrix} I & 0 \\ s_{11} & s_{12} \\ 0 & I \\ 0 & s_{22} \end{bmatrix} \begin{bmatrix} V_{11} \\ \tilde V_{21} \end{bmatrix} V_{11}^{-1}(t_0)x_0$$
$$= \begin{bmatrix} I & 0 \\ A_{+;2}^{-1}A_{+;21} & -A_{+;2}^{-1}B_2B_1^* + A_{+;2}^{-1}B_2B_2^*A_{+;2}^{-*}A_{+;12}^* \\ 0 & I \\ 0 & A_{+;2}^{-*}A_{+;12}^* \end{bmatrix} \begin{bmatrix} V_{11} \\ \tilde V_{21} \end{bmatrix} V_{11}^{-1}(t_0)x_0$$

defines the solution to (5.9), and, in particular, the optimal state as in (5.8). Multiplication of the first row gives that

$$V_{11}^{-1}(t)x(t) = V_{11}^{-1}(t_0)x(t_0)$$

so that, with $P_{\Delta;1} = V_{21}V_{11}^{-1}$ (cf. Theorem 5.5), we get the following formula for the optimal state:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_{11}(t)V_{11}(t_0)^{-1}x_1(t_0) \\ A_{+;2}^{-1}A_{+;21}x_1(t) - [A_{+;2}^{-1}B_2B_1^* + A_{+;2}^{-1}B_2B_2^*A_{+;2}^{-*}A_{+;12}^*]P_{\Delta;1}(t)x_1(t) \end{bmatrix}.$$

Then the turnpike property for x_1 follows by the arguments for the ODE case [8, Thm. 4] as follows: From

$$\|x_1(t) - e^{\{\bar A t\}}\| \le \text{const}\, e^{\{t_1\bar\lambda\}}e^{\{(t_1-t)\bar\lambda\}}$$

(cf. [8, eq. (63)]), we conclude with $e^{\{t_1\bar\lambda\}} \le 1$ independent of t_1, $\bar A$ being stable, and an application of the triangle inequality as in (2.4) that

$$\|x_1(t)\| \le \text{const}(e^{\{(t_1-t)\bar\lambda\}} + e^{\{t\bar\lambda\}}).$$

The same type of estimate for x_2 follows from the turnpike property of x_1 since $P_{\Delta;1}$ is bounded by Corollary 5.6. Finally, the turnpike estimate for $u(t) = -\mathcal{B}\mathcal{B}^*\mathcal{P}(t)x(t) = -\mathcal{B}\mathcal{B}^*\mathcal{P}_+x(t) - \mathcal{B}\mathcal{B}^*\mathcal{P}_\Delta(t)x(t)$ follows as in (3.7). ∎

We summarize and comment on the assumptions and the results of this paper on the optimal control of the linear descriptor system (4.1) with a quadratic cost functional defined in Problem 4.1.

Assumptions:

1. Without loss of generality: $\mathcal{E}$ is semiexplicit (Assumption 4.11).
2. To enable the existence of solutions to the first-order optimality conditions (4.2): compatibility of $\mathcal{F}$ and $\mathcal{E}$ (Assumption 4.10).
3. In line with the basic assumption for the ODE case: existence of a stabilizing solution to the gARE (4.4) including regularity of the matrix pair (Assumption 4.9).
4. To ensure the existence of global solutions to the reduced closed-loop Riccati equation (5.7): a spectral condition on the coefficients (Assumption 5.1).
5. In line with the relevant condition in the ODE case: compatibility of the terminal constraint S, the solution to the gARE (4.4), and the relevant *reachability Gramian* (Assumption 5.7).

Certainly, Assumption 5.1 is somewhat unpleasant and because of its dependency on P_2 not readily confirmed or discarded for a given system. Nonetheless, it generalizes the standard assumption for ODE systems that ensures the definiteness of the cost functional in the presence of cross terms in the costs or a feedthrough term in the system.

With these assumptions, the following results have been derived:

Summary of results:
1. Existence of solutions to the gDRE (Theorem 5.4).
2. Representation of the difference $\mathcal{P}(t) - \mathcal{P}_+$ that implies that the closed-loop system is *impulse free* (Theorem 5.5 and Corollary 5.8).
3. Convergence of $\mathcal{P}(t) \to \mathcal{P}_+$ as $t_1 \to \infty$ and turnpike property of the *homogeneous* optimal control problem with DAE constraints (Corollary 5.6 and Theorem 5.10).

6. The affine DAE LQR problem. In this section, we study the optimal control problem with nonzero target states y_c and y_e and get back to the question of what *the* steady-state optimal control problem is for a descriptor system.

Similarly to the ODE case, the feedthrough w is defined via

$$-\mathcal{E}\dot{w} = (\mathcal{A}^* - \mathcal{P}^*(t)\mathcal{B}\mathcal{B}^*)w - \mathcal{C}^* y_c, \quad \mathcal{E}^* w(t_1) = -\mathcal{F}^* y_e,$$

which we rewrite as

$$-\mathcal{E}\dot{w} = \mathcal{A}_+^* w - \mathcal{P}_\Delta^*(t)\mathcal{B}\mathcal{B}^* w - \mathcal{C}^* y_c, \quad \mathcal{E}^* w(t_1) = -\mathcal{F}^* y_e.$$

We partition the variables and coefficients

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \mathcal{C}^* = \begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix}, \quad \text{and } \mathcal{F}^* = \begin{bmatrix} F_1^* \\ 0 \end{bmatrix}$$

in accordance with $\mathcal{E}$, $\mathcal{A}_+$, $\mathcal{B}\mathcal{B}^*$, and $\mathcal{P}_\Delta$, as in (4.7), (5.10), and (5.13), respectively, and write

$$\begin{aligned} \mathcal{A}_+^* - \mathcal{P}_\Delta^*(t)\mathcal{B}\mathcal{B}^* &= \begin{bmatrix} A_{+;1}^* & A_{+;21}^* \\ A_{+;12}^* & A_{+;2}^* \end{bmatrix} - \begin{bmatrix} P_{\Delta;1}(t) & -P_{\Delta;1}(t)A_{+;12}A_{+;2}^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1B_1^* & B_2B_1^* \\ B_1B_2^* & B_2B_2^* \end{bmatrix} \\ &= \begin{bmatrix} A_{+;1}^* - P_{\Delta;1}(t)\bar{B}B_1^* & A_{+;21}^{-*} - P_{\Delta;1}(t)\bar{B}B_2^* \\ A_{+;12}^* & A_{+;2}^* \end{bmatrix}. \end{aligned}$$

In what follows, we will omit the time dependency of $\mathcal{P}_\Delta$. With the relation

$$w_2 = -A_{+;2}^{-*}A_{+;21}^* w_1 + A_{+;2}^{-*}C_2^* y_c, \tag{6.1}$$

we can eliminate w_2 from the equations and consider only

$$-\dot{w}_1 = (\bar{A}^* - P_{\Delta;1}(t)\bar{B}\bar{B}^*)w_1 - \bar{C}^* y_c + P_{\Delta;1}(t)\bar{B}B_2^* A_{+;2}^{-*}C_2^* y_c, \tag{6.2}$$

with the abbreviations

$$\bar{A} := A_{+;1} - A_{+;12}A_{+;2}^{-1}A_{+;21}, \quad \bar{B} := B_1 - A_{+;12}A_{+;2}^{-1}B_2, \quad \text{and } \bar{C} := C_1 - C_2A_{+;2}^{-1}A_{+;21}$$

as they have been used before.

By the same procedure, we derive the expressions for the parts of the optimal state as

$$x_2 = -A_{+;2}^{-1}A_{+;21}x_1 + A_{+;2}^{-1}B_2\bar{B}^*(P_{\Delta;1}x_1 + w_1) + A_{+;2}^{-1}B_2B_2^*A_{+;2}^{-*}C_2y_c \tag{6.3}$$

and

$$\dot{x}_1 = (\bar{A} - \bar{B}\bar{B}^*P_{\Delta;1}(t))x_1 - \bar{B}\bar{B}^*w_1 - \bar{B}B_2^*A_{+;2}^{-*}C_2y_c. \tag{6.4}$$

The preceding derivations show that if the Riccati solution exists, then the optimal states $x = (x_1, x_2)$ decouple such that x_1 reads like a solution to an optimal control problem with ODE constraints and x_2 is in a direct algebraic relation with x_1. Accordingly, we can state the turnpike property for Problem 4.1 with similar arguments as for the standard LQR case.

THEOREM 6.1. *Consider Problem* 4.1 *with the costs defined through* t_1, $\mathcal{C}$, $\mathcal{F}$, *and target states* y_c *and* y_e *and subject to the DAE* (4.1) *with coefficients* $(\mathcal{E}, \mathcal{A}, \mathcal{B})$.

Assume that $\mathcal{E}$ *is semiexplicit, that* $(\mathcal{E}, \mathcal{A})$ *is regular, and that the gARE* (4.4) *has a stabilizing solution* $\mathcal{P}_+$ *(Assumptions* 4.9 *and* 4.11*).*

Assume that $\mathcal{F}$ *is compatible with* $\mathcal{E}$ *so that with Assumptions* 5.1 *the gDRE* (4.3) *has a solution* $\mathcal{P}$ *with* $\mathcal{P}_\Delta = \mathcal{P} - \mathcal{P}_+$ *as in* (5.13).

Assume that the relevant part of $\mathcal{F}$ *is compatible with the relevant part of* $\mathcal{P}_+$ *such that Assumption* 5.7 *is fulfilled.*

Then the optimal control has a solution (x, u) *with* x *and* u *satisfying the estimates*

$$\|x(t) - x_s\| \leq \text{const}(e^{\{t\bar{\lambda}\}} + e^{\{(t_1-t)\bar{\lambda}\}})$$

and

$$\|u(t) - u_s\| \leq \text{const}(e^{\{t\bar{\lambda}\}} + e^{\{(t_1-t)\bar{\lambda}\}}),$$

with const *independent of* t_1 *and where* $\bar{\lambda} < 0$ *is the spectral abscissa of* $\bar{A}$ *and*

$$x_s = \begin{bmatrix} x_{s;1} \\ -A_{+;2}^{-1}A_{+;21}x_{s;1} + A_{+;2}^{-1}B_2(\bar{B}^*\bar{A}^{-*}\bar{C}^* + B_2^*A_{+;2}^{-*}C_2^*)y_c \end{bmatrix} \tag{6.5}$$

with $x_{s;1} := (\bar{A}^{-1}\bar{B}\bar{B}^*\bar{A}^{-*}\bar{C}^* + \bar{A}^{-1}\bar{B}B_2^*A_{+;2}^{-*}C_2^*)y_c$ *and*

$$u_s = -\mathcal{B}^*\mathcal{P}_+x_s - \bar{B}^*\bar{A}^{-*}\bar{C}^*y_c - B_2^*A_{+;2}^{-*}C_2^*y_c. \tag{6.6}$$

Proof. We consider (6.2) and (6.4) for the parts w_1 and x_1, respectively. As laid out in the proof of Theorem 5.5, the part $\mathcal{P}_\Delta$ solves the differential Riccati equation

$$-\dot{P}_{\Delta;1} = \bar{A}^*P_{\Delta;1} + P_{\Delta;1}\bar{A} - P_{\Delta;1}\bar{B}\bar{B}^*P_{\Delta;1}, \quad P_{\Delta;1}(t_1) = S_1 - P_{+;1}$$

and exists for all $t \leq t_1$. Thus, Lemma 3.1 applies and provides the formulas for the relevant fundamental solution as

$$U(t) = e^{\{-(t_1-t)\bar{A}\}}\big(I - [\bar{W} - e^{\{(t_1-t)\bar{A}\}}\bar{W}e^{\{(t_1-t)\bar{A}^*\}}](P_{+;1} - S_1)\big).$$

As for the ODE case, we conclude that the feedthrough can be written as

$$w_1(t) = \bar{A}^{-*}\bar{C}^* y_c + g(t,t_1) \tag{6.7}$$

with a remainder term $g(t,t_1)$ that is dominated by $e^{\{(t_1-t)\bar{A}\}}$; cf. (3.4) and (3.5). The only difference to the ODE case lies in the additional term $P_{\Delta;1}(t)\bar{B}A_{+;2}^{-*}C_2^* y_c$ to w_1 as defined (6.7), which, however, can be included in the estimates by the decaying behavior of $P_{\Delta;1}$ as it is ensured by Assumption 5.7 and Corollary 5.6. Those parts of x_1 that cannot be bounded by $e^{\{(t_1-t)\bar{\sigma}\}}$, where $\bar{\sigma}$ is the spectral abscissa of $\bar{A}$, are given by the integral over the constant parts in the right-hand side of (6.4) (cf. also (3.6)). Thus, the turnpike for x_1 is given by the constant part of

$$\begin{aligned} -\int_0^t e^{\{(t-s)\bar{A}\}}\big(\bar{B}\bar{B}^*\bar{A}^{-*}\bar{C}^* y_c + \bar{B}B_2^* A_{+;2}^{-*} C_2 y_c\big)\,\mathrm{d}s \\ = (I - e^{\{t\bar{A}\}})\bar{A}^{-1}\big[\bar{B}\bar{B}^*\bar{A}^{-*}\bar{C}^* y_c + \bar{B}B_2^* A_{+;2}^{-*} C_2 y_c\big], \end{aligned}$$

which is as in the first component of (6.5). The turnpike for x_2 as in the second component of (6.5) follows from formula (6.3) in combination with the decaying behavior of $P_{\Delta;1}$ and the estimate for w_1 given in (6.7).

With the formulas (6.7) and (6.1) for w_1 and w_2, the optimal control writes as

$$\begin{aligned} u(t) = -\mathcal{B}^*(\mathcal{P}x(t) + w(t)) &= -\mathcal{B}^*\mathcal{P}_+ x(t) - \mathcal{B}^*\mathcal{P}_\Delta(t)x(t) - B_1^* w_1(t) - B_2^* w_2(t) \\ &= -\mathcal{B}^*\mathcal{P}_+(x(t) - x_s) - \bar{B}^* P_{\Delta;1}(t)x_1(t) - \bar{B}^* g(t,t_1) \\ &\quad - \bar{B}^*\bar{A}^{-*}\bar{C}^* y_c - B_2^* A_{+;2}^{-*} C_2^* y_c - \mathcal{B}^*\mathcal{P}_+ x_s \\ &= -\mathcal{B}^*\mathcal{P}_+(x(t) - x_s) - \bar{B}^* P_{\Delta;1}(t)x_1(t) - \bar{B}^* g(t,t_1) + u_s, \end{aligned}$$

from where the estimate (6.6) for $u(t) - u_s$ follows by the same arguments as for (3.7). ∎

With that, we have established the existence of the turnpike for a class of linear-quadratic optimization problems with DAE constraints.

It remains to investigate whether, like in the ODE case, the turnpike can be defined as the solution to an associated steady-state optimal control problem. For that, we observe that with $p_s := \mathcal{P}_+ x_s + w_s$, with w_s being the time constant parts of w and with u_s as defined in Theorem 6.1, the triple (x_s, p_s, u_s) is a critical point of the *Lagrange function*

$$\mathcal{L}(x,p,u) = \frac{1}{2}\|\mathcal{C}x - y_c\|^2 + \frac{1}{2}\|u\|^2 + p^*(\mathcal{A}x + \mathcal{B}u).$$

Still, since the omission of the time derivative discards the DAE structure, the turnpike is not that solution that arises from

$$\frac{1}{2}\|\mathcal{C}x - y_c\|^2 + \frac{1}{2}\|u\|^2 \to \min_u \text{ subject to } \mathcal{A}x + \mathcal{B}u = 0.$$

In particular, the feedback structure $u_s = -\mathcal{B}\mathcal{B}^*\mathcal{P}_+ x_s$ that makes the left-lower block of $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}_+$ regular is not ensured.

7. Conclusion and discussion.

The presented results show that classical system theoretic results well apply to prove turnpike properties for LQR problem constraint by standard linear state space systems. Under the assumption of impulse controllability, a descriptor system can be controlled such that it is basically an ODE with an additional but well-separated algebraic part so that similar arguments can be used to confer turnpike properties of LQR problems with DAE constraints.

The characterization of the turnpike for DAE problems as an optimal steady state is still undecided. For time-dependent problems, we succeeded in establishing an *underlying Hamiltonian ODE system* for the optimality conditions (cf. (5.12)). In the steady-state regime, similar operations did not lead to a system that allowed for a characterization as an optimality system like (2.2). A possible approach would be to establish results for impulse-free matrix pairs $(\mathcal{E}, \mathcal{A})$, e.g., through the equivalence to ODE systems with feedthrough, and then extend to impulse-controllable systems, e.g., through the *feedback equivalence form* (see [34]).

In line with the literature on turnpike phenomena in control systems, natural extensions of the presented results could consider periodic orbits as turnpikes (as in [2, 37]), PDE formulations (as in, e.g., [12]), or particular nonlinear phenomena (as in [31, 36]) for the differential algebraic case.

As for the theory of control of DAEs, an immediate strengthening of the results could be achieved by removing the assumption on impulse controllability. A more general framework will also consider indefinite cost functionals and suitable replacements for the Riccati equations, as they are used in recent works on singular feedback control [6] or infinite time horizon problems [35]. The adaption of the concepts of *Lur'e equations*—as they are a key tool in [35]—to tracking problems on finite time horizons has not been investigated yet. However, the related concept of *storage functions* and their connection to turnpike properties, as discussed in [13], may well be used for a general theory for DAEs that does not resort to Riccati equations.

REFERENCES

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*, Birkhäuser, Basel, Switzerland, 2003.

[2] Z. Artstein and A. Leizarowitz, *Tracking periodic signals with the overtaking criterion*, IEEE Trans. Automat. Control, 30 (1985), pp. 1123–1126, https://doi.org/10.1109/TAC.1985.1103851.

[3] A. Backes, *Optimale Steuerung der linearen DAE im Fall Index* 2, preprint 2003-04, Institut für Mathematik, Humboldt-Universität, Berlin, 2003.

[4] M. Behr, P. Benner, and J. Heiland, *Invariant Galerkin Ansatz Spaces and Davison-Maki Methods for the Numerical Solution of Differential Riccati Equations*, arXiv:1910.13362 2019.

[5] D. J. Bender and A. J. Laub, *The linear-quadratic optimal regulator for descriptor systems*, IEEE Trans. Automat. Control, 32 (1987), pp. 672–688.

[6] C. Bhawal and D. Pal, *Almost every single-input LQR optimal control problem admits a PD feedback solution*, IEEE Control Syst. Lett., 3 (2019), pp. 452–457, https://doi.org/10.1109/LCSYS.2019.2898388.

[7] F. M. Callier and C. A. Desoer, *Linear System Theory*, Springer-Verlag, Berlin, 1991.

[8] F. M. Callier, J. Winkin, and J. L. Willems, *Convergence of the time-invariant Riccati differential equation and LQ-problem: Mechanisms of attraction*, Internat. J. Control, 59 (1994), pp. 983–1000, https://doi.org/10.1080/00207179408923113.

[9] D. Cobb, *Descriptor variable systems and optimal state regulation*, IEEE Trans. Automat. Control, 28 (1983), pp. 601–611.

[10] L. Dai, *Singular Control Systems*, Lecture Notes in Control and Information Sciences, no. 118, Springer-Verlag, Berlin, 1989.

[11] T. Faulwasser, K. Flasskamp, S. Ober-Blöbaum, and K. Worthmann, *A dissipativity characterization of velocity turnpikes in optimal control problems for mechanical systems*, CoRR, abs/2002.04388, 2020.

[12] L. Grüne, M. Schaller, and A. Schiela, *Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control*, SIAM J. Control Optim., 57 (2019), pp. 2753–2774, https://doi.org/10.1137/18M1223083.

[13] L. Grüne and R. Guglielmi, *On the relation between turnpike properties and dissipativity for continuous time linear quadratic optimal control problems*, Math. Control Relat. Fields, 11 (2021), pp. 169–188, https://doi.org/10.3934/mcrf.2020032.

[14] J. Heiland, *A differential-algebraic Riccati equation for applications in flow control*, SIAM J. Control Optim., 54 (2016), pp. 718–739, https://doi.org/10.1137/151004963.

[15] J. Heiland and E. Zuazua, *Classical System Theory Revisited for Turnpike in Standard State Space Systems and Impulse Controllable Descriptor Systems*, Technical report, 2020, https://arxiv.org/abs/2007.13621.

[16] T. Katayama and K. Minamino, *Linear quadratic regulator and spectral factorization for continous-time descriptor systems*, in Proceedings of the 31st IEEE Conference on Decision and Control, Vol. 1, 1992, pp. 967–972, https://doi.org/10.1109/CDC.1992.371582.

[17] A. Kawamoto, K. Takaba, and T. Katayama, *On the generalized algebraic Riccati equation for continuous-time descriptor systems*, Linear Algebra Appl., 296 (1999), pp. 1–14, https://doi.org/10.1016/S0024-3795(99)00079-8.

[18] P. Kokotovic and R. Yackel, *Singular perturbation of linear regulators: Basic theorems*, IEEE Trans. Automat. Control, 17 (1972), pp. 29–37.

[19] V. Kucera, *A review of the matrix Riccati equation*, Kybernetika, 9 (1973), pp. 42–61, http://www.kybernetika.cz/content/1973/1/42.

[20] P. Kunkel and V. Mehrmann, *The linear quadratic optimal control problem for linear descriptor systems with variable coefficients*, Math. Control Signals Syst., 10 (1997), pp. 247–264, https://doi.org/10.1007/BF01211506.

[21] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution*, European Mathematical Society Publishing House, Zürich, Switzerland, 2006.

[22] P. Kunkel and V. Mehrmann, *Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index*, Math. Control Signals Syst., 20 (2008), pp. 227–269.

[23] P. Kunkel and V. Mehrmann, *Formal adjoints of linear DAE operators and their role in optimal control*, Electron. J. Linear Algebra, 22 (2011), pp. 672–693.

[24] P. Kunkel and V. Mehrmann, *Optimal control for linear descriptor systems with variable coefficients*, in Numerical Linear Algebra in Signals, Systems and Control, P. Van Dooren, S. P. Bhattacharyya, R. H. Chan, V. Olshevsky, and A. Routray, eds., Springer-Verlag, Berlin, 2011, pp. 313–339, https://doi.org/10.1007/978-94-007-0602-6.

[25] G. A. Kurina, *Upravlenie s obratnoj svjaz'ju dlja linejnyh sistem, ne razreshennyh otnositel'no proizvodnoj*, Automat. Remote Control, 45 (1984), pp. 713–717.

[26] G. A. Kurina and R. März, *Feedback solutions of optimal control problems with DAE constraints*, SIAM J. Control Optim., 46 (2007), pp. 1277–1298, https://doi.org/10.1137/050637352.

[27] A. Locatelli, *Optimal Control: An Introduction*, Birkhäuser, Basel, Switzerland, 2001.

[28] K. Mårtensson, *On the matrix Riccati equation*, Inf. Sci., 3 (1971), pp. 17–49, https://doi.org/10.1016/S0020-0255(71)80020-8.

[29] J. Möckel, T. Reis, and T. Stykel, *Linear-quadratic Gaussian balancing for model reduction of differential-algebraic systems*, Internat. J. Control, 84 (2011), pp. 1627–1643, https://doi.org/10.1080/00207179.2011.622791.

[30] L. Pandolfi, *On the regulator problem for linear degenerate control systems*, J. Optim. Theory Appl., 33 (1981), pp. 241–254.

[31] D. Pighin, *Nonuniqueness of Minimizers for Semilinear Optimal Control Problems*, preprint, arXiv:2002.04485, 2020.

[32] D. Pighin and N. Sakamoto, *The Turnpike with Lack of Observability*, preprint, arXiv:2007.14081, 2020.

[33] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim., 51 (2013), pp. 4242–4273, https://doi.org/10.1137/130907239.

[34] T. Reis, O. Rendel, and M. Voigt, *The Kalman-Yakubovich-Popov inequality for differential-algebraic systems*, Linear Algebra Appl., 485 (2015), pp. 153–193.

[35] T. Reis and M. Voigt, *Linear-quadratic optimal control of differential-algebraic systems: The infinite time horizon problem with zero terminal state*, SIAM J. Control Optim., 57 (2019), pp. 1567–1596, https://doi.org/10.1137/18M1189609.

[36] N. Sakamoto, D. Pighin, and E. Zuazua, *The turnpike property in nonlinear optimal control—A geometric approach*, in Proceedings of 58th IEEE Conference on Decision and Control, 2019, pp. 2422–2427.

[37] E. Trélat, C. Zhang, and E. Zuazua, *Steady-state and periodic exponential turnpike property for optimal control problems in Hilbert spaces*, SIAM J. Control Optim., 56 (2018), pp. 1222–1252, https://doi.org/10.1137/16M1097638.

[38] E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, J. Differential Equations, 258 (2015), pp. 81–114.

[39] A. J. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Vol. 80, Springer-Verlag, Berlin, 2006.

[40] A. J. Zaslavski, *Turnpike Theory of Continuous-time Linear Optimal Control Problems*, Vol. 104, Springer-Verlag, Berlin, 2015, https://doi.org/10.1007/978-3-319-19141-6.

[41] C. Zhaolin, Y. Jiuxi, and M. Zhaobo, *The linear-quadratic optimal regulator of descriptor systems with the terminal state constrained*, IFAC Proceedings Volumes, 23 (1990), pp. 265–270, https://doi.org/10.1016/S1474-6670(17)52106-0, 11th IFAC World Congress on Automatic Control, Tallinn, 1990, Vol. 1, Tallinn, Finland.

[42] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.