CYLINDRICAL CONCRETE SHELL ROOFS

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INTRODUCTION

Concrete is not a new building material. Records of its extensive use date back to the time of the Roman Empire. Although concrete has been used over these hundreds of years by man in his construction work the complete knowledge of its properties and therefore its capabilities are still being discovered. A large change in its application came with the introduction of reinforcing steel to improve its ability to resist a bending load. Concrete is of such a nature that it is strong in compression but very weak in tension. When the steel is bonded to the concrete, so the two act as a unit, the steel carries the tension and the concrete the compression stresses. This allows the combined section to resist large bending moments.

Another method of concrete construction which has been developed in the past few years and is still in the infant stage employs the principle of prestressing. This use of concrete is based on the principle of loading the concrete prior to its use so that the entire beam will be in compression. When the load is applied it actually subtracts from the compressive stress in a portion of the concrete. The beams are designed to be in compression across the entire section when the full design load is applied. In this type of structure the construction techniques play a very large part in the final capacity of the beam and close inspection with numerous tests and rigid control of the concrete mix are required to insure end results that agree with the design calculations.

The use of thin shell concrete, although comparatively new, does not employ any new design principles. The fact that a curved member can transmit a load by direct stress with little bending, was employed in the construction of the large domes during the Roman Period. Thin shells are structures in which the loads are sustained by thin curved slabs. When the slab is curved in two directions it is called a dome. The recent improvement of construction techniques coupled with the shortage of steel in Europe has been responsible for its development to a point where it can compete with other materials in roof construction.

The design of thin shell structures involves spending innumerable hours on tedious numerical computations. The American Society of Civil Engineers appointed a committee to simplify this process so the design could be performed by the structural designer with greater ease. The results of the work of this committee were published in a manual entitled "Design of Cylindrical Concrete Shell Roofs" and includes numerous tables and examples of design. It is the purpose of this report to explain the procedure in using these tables and show an example of their application in the hope that a greater use of this type of construction will be realized.

DEFINITION

Thin shells may be defined as structures in which the loads are carried by thin curved slabs. This type of construction includes domes, which are slabs curved in two direction and barrels, which are slabs curved in one direction only. In this report the barrel type whose curvature is constant over the entire width of the section will be investigated. These barrels are supported by and are integral with transverse stiffners (Plate I) .

Shell action is characterised by the transmission of loads primarily by direct stress with relatively small bending stresses. Plate I shows the difference in the action of a shell and a flat plate. The peculiar property of a cylindrical shell comes from the behavior of the shell in the transverse direction

EXPLANATION OF PLATE I

Fig. 1. Simply supported cylindrical shell with edge beam and transverse stiffeners.

Fig. 2. Forces on free body cut from a flat plate.

Fig. 3. Forces on free body cut from cylindrical shell.

PLATE I

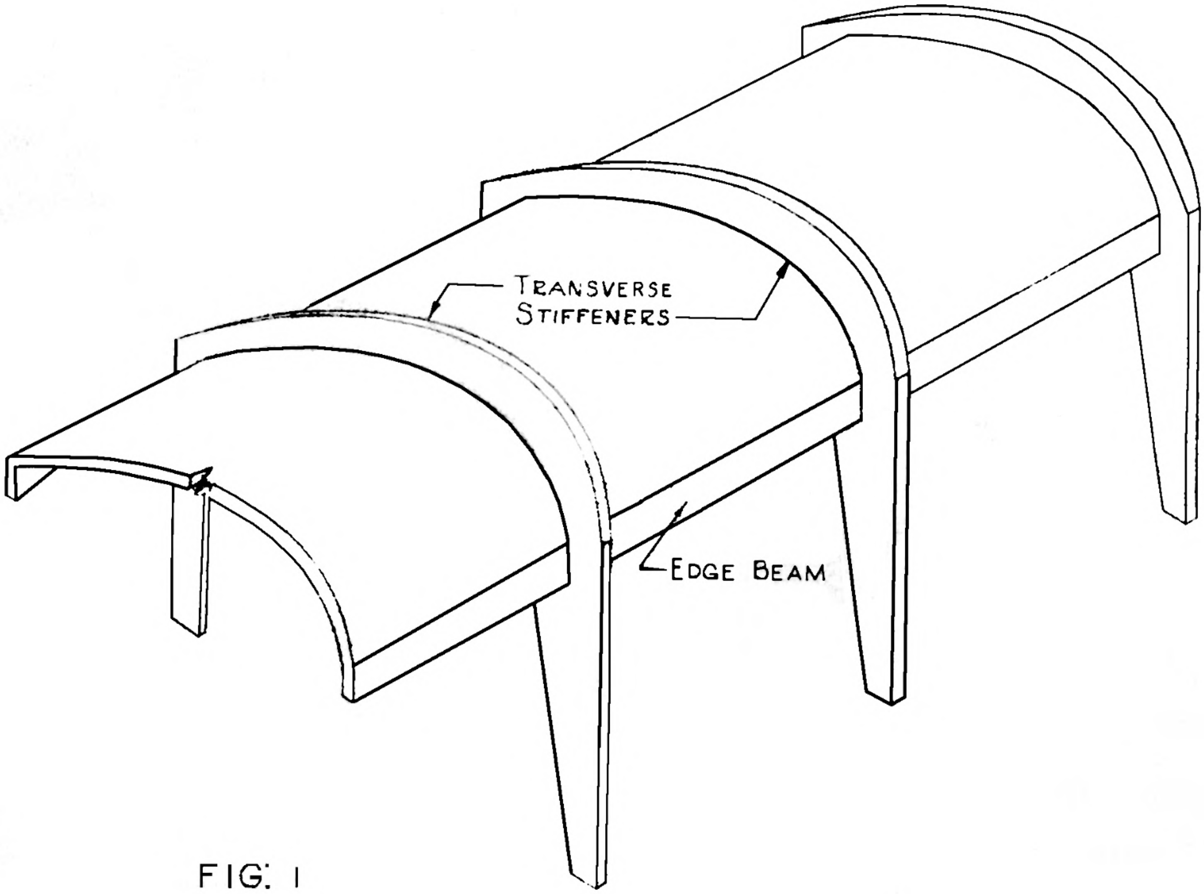


FIG: 1

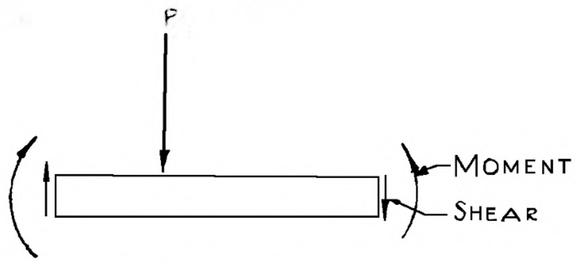


FIG:2

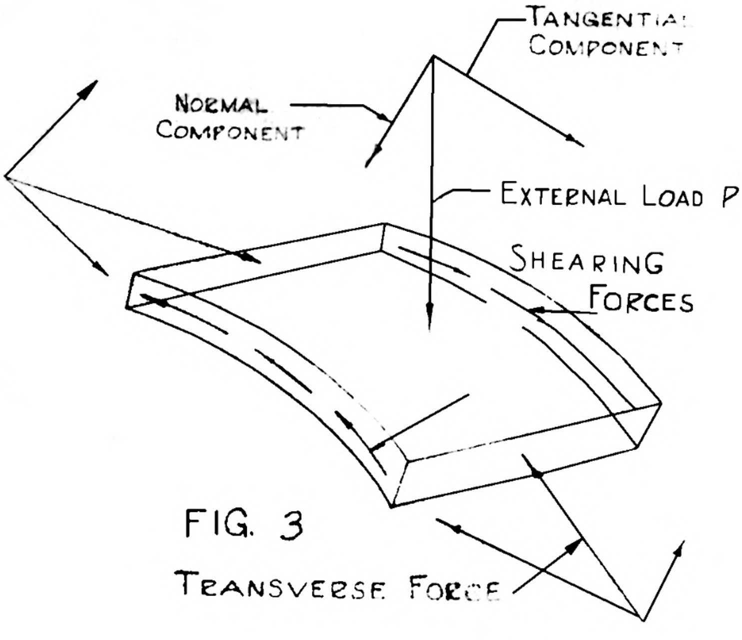


FIG. 3

Looking at Fig. 1 in Plate I as a free body, it is seen that shearing forces and moments are required to maintain the external load in equilibrium. But in the strip of Fig. 3 the normal component of the external load is resisted by the transverse forces (normal stresses) on the radial sections, while the other component (tangential) is resisted by shearing forces on the transverse sections. It is these shearing forces that distinguish shell action from arch action.

The barrels are broken down into two types, the short barrel whose ratio of length between stiffners to the radius is less than five to one and the long barrel with ratio of length between stiffners to the radius of greater than five to one. The long barrel can be examined in the transverse direction in the same manner as a beam of curved cross-section, using the simple beam theory. For instance, by equating the moment of all the internal forces acting on a section cut transversely through a shell to the moment of the external forces and reactions, the intensity of the extreme fiber stresses, even for complicated shell shapes, can be determined under the assumption of a straight line distribution of stress. For the short barrel the distance between supports grows smaller and the longitudinal fiber stress tend to become larger than those given by the simple beam theory. Because of this increase in stress in the region of the free edges(non-supported), the problem of shell design has come to be regarded as a problem of supplying necessary forces at the edges for equilibrium.

To illustrate the spanning capabilities of thin shell concrete one needs only to look at some of the other famous domes in history. The domes constructed by the Romans are probably the most famous but they would certainly not stand a favorable comparison with the thin shell concrete construction of today. The dome of Saint Peters Cathedral, made of stone, spans 131 feet and weighs 10,000 tons; the shell domes in the market hall in Leipzig span 240 feet and weigh only 2,160 tons.

HISTORY OF SHELL DESIGN THEORIES

The credit for the first analytical approach to thin shell design goes to G. Lame and E. Clapeyron in 1828. These Frenchmen used a theory which was based entirely on direct stresses and therefore had to be limited to shells supported on all sides. In 1892 A.E.H. Love included radial shearing forces and moments in his theory, providing the basis for later theoretical developments. Carl Zeiss receives credit for the first application of Love's equations when, in 1924, he used them in the design of a small concrete shell roof for the Zeiss Works in Jena, Germany.

In 1930 U. Finsterwalder presented a new approximate method which gives displacements in fair agreement with experimental results, and a few years later this was improved upon by Mr. Dischinger.

In this country the leading contributer has been H. Scherer, who in 1936, further simplified Mr. Finsterwalder's solution.

EXAMPLES OF NOTE

Foreign

Grand Market Hall at Frankfurt, Germany. The building is 720 feet long by 167 feet wide and consists of fifteen shells, of the long cylindrical type, each 121 feet long by 46 feet wide. The shells at the crown are 2.75 inches thick and at the exterior are slightly thicker.

Rail Station on the Chessingter Line of the Southern Railway, England. The structure is composed of cantilever shells covering a loading platform; the shell spans 25 feet between cantilever frames. The thickness of the shell is 3 inches.

Exhibition Hall in Turin, Italy. In this structure pre-cast thin shell elements were used in combination with cast-in-place concrete. The precision casting of the elements on the ground and the use of the forms repeatedly enabled the roof to be more economical than wood or steel, and at the same time provided a striking architectural appearance, with windows in each panel to provide lighting for the interior. The clear area is 312 feet wide and 250 feet long.

United States

American Airlines Hangar, Chicago Municipal Airport. This roof structure proved more economical than any steel design for the hangars at the time of construction. The hangar was built during the war period when steel prices were high and delivery of the steel was hard to forecast. The span is 257 feet and the shell is 3½ inches thick at the crown and 6 inches thick at the edge beam. The supporting ribs are spaced 29 feet on centers and the over all clear height is 58.4 feet. In this design the shell was placed at the neutral axis of the rib to reduce the moments in the rib at the crown and springing line. The form centering was quite expensive but it was reused six times to bring its cost per square foot down. When the cost of fire proofing the steel was added to the cheapest steel design, the resulting cost was considerably more than that of the concrete structure.

Hangar at Rapid City Air Force Base in South Dakota. The clear span is 340 feet; the crown of the hangar is 5 inches thick but commencing 60 feet from the springing line it increases to 7 inches at the spring line. Reinforced concrete stiffening arches are 23 feet apart; length of the hangar is 300 feet.

Army Quartermaster Warehouse in Columbus, Ohio. Each warehouse unit is approximately 180 feet wide by 1600 feet long and the project covers 2,676,000 sq. feet. The span of the shell is 45 feet; thickness at the crown is $3\frac{1}{2}$ inches which increases to $4\frac{1}{2}$ inches at the edges.

Hershey Sports Arena in Hershey, Pennyslvania. The clear span in this structure is 222 feet with a rise of 54 feet. The shell thickness varies from $3\frac{1}{2}$ inches at the crown to 6 inches at the edges. The arched rib stiffeners are 39 feet on center.

Livestock Colosseum in Montgomery Alabama. The clear span is 286 feet with a rise of 48 feet. The shell is $3\frac{1}{2}$ inches thick at the crown and $5\frac{1}{2}$ inches thick at the edges. The space between arches is 28 feet.

Onondaga County War Memorial-Syracuse, New York. The arch spans a distance of 160 feet starting from a cantilever on each side making the total clear span 210 feet. The shell thickness varies from $3\frac{1}{2}$ inches at the crown to 5 inches at the edge. The cantilever and arch frames are spaced on 20 foot centers.

DESIGN PROCEDURE

Although the design of shells appears to be more complicated than the design of ordinary indeterminate structures, it is handled in a very similar manner. As a matter of fact since in most cases the thickness of the shell in concrete construction is determined by some construction detail, such as depth required for reinforcement, a single investigation is generally sufficient.

In the analysis of an ordinary indeterminate structure it is the usual procedure to reduce it to a statically determinate structure by releasing end restraints, thereby allowing rotation and displacement of the ends to occur. Then reactions are applied to bring the boundary back to its original or desired position. The final stresses are the algebraic sum of the stresses found in the statically determinate position and those caused by the end restraint.

In thin shell design it is first assumed that the surface load is transmitted to the supports solely by direct stresses, sometimes called "membrane stresses." In the preliminary step called "membrane analysis" there are displacements and reactions along the longitudinal edges of the shell that do not comply with the boundary requirements. To satisfy the requirements, equal forces are added but of the opposite direction; these forces are called line loads. In contrast with general surface loading, these line loads create bending stresses as well as direct stresses in the shell. The stresses produced by the line loads must be added to the direct stress in the shell to obtain the final stress pattern. Thus it is seen that the design of thin shells can be divided into two parts. The first step is to find the internal stresses and edge forces created by the surface loads, using the membrance analysis, and the second step is to add the stresses due to the edge line loading.

In the following procedure reference is made to certain tables in the A.S.C.E. manual on the "Design of Cylindrical Concrete Shell Roofs." All of these tables do not appear in this report. Only those required in working out the example that will follow are included but an attempt will be made to explain each table as it appears in the design procedure.

The first step is to determine the stresses due to surface loads. Since this is always possible for continuous loads, the stresses created by any type of continuous load distribution expressible as a function of the longitudinal and radial coordinates, x and ϕ can be determined. For most conditions, the surface loads will be uniform in the longitudinal direction of the shell. The force and displacement components produced by two such loads are presented in Plate II.

In the discussion the following symbols will be used: E is the modulus of elasticity of concrete; P_u is the intensity of uniform load on unit area; and P_d is the intensity of dead load on unit area.

Although it is possible to obtain the internal forces produced by any surface loads, because of mathematical difficulties the corrective line loads applied along the longitudinal edges are expressible only as functions of sin $\frac{n\pi x}{L}$ or cos $\frac{n\pi x}{L}$ in which n is any integer. To avoid confusion in the application of the line loads under step two, it is expedient to regard the surface load on the shell as the sum of the partial loads, the intensity of which is defined as $A_n \sin \frac{n\pi x}{L}$. By a suitable selection of the values of A_n which will vary with each n, it is possible to approximate with any degree of accuracy any variation of loading. This representation of a known function by a series of sines or cosines is called "Fourier Analysis" and the series is called the "Fourier Series." The Fourier series for a uniform load equals:

$$Px = \frac{4P}{\pi}\left[\sin \frac{\pi x}{L} + 1/3 \sin \frac{3\pi x}{L} + 1/5 \sin \frac{5\pi x}{L}\right]$$

or

$$Px = \frac{4P}{\pi} \sum_{n=1,3,5} 1/N \sin \frac{n\pi x}{L}$$

The sum of all the terms of the series gives the straight line P=1. In general only the first few terms of this series are needed to achieve sufficient accuracy. The internal force and displacements produced by the sinusoidal loading represented by the first two terms are given in Tables 1B and 1C in the A.S.C.E. handbook.

It is evident from Tables 1A, 1B, and 1C that, for shells whose subtended angle is less than 180°, the membrane analysis gives transverse and shearing forces acting on and along the longitudinal edge. If the necessary marginal forces can be provided by the edge reactions and at the same time if the

EXPLANATION OF PLATE II

T_ϕ = the direct force component per unit length of shell in the transverse direction, considered positive when tensile.

T_x = the direct force component per unit length of shell in the longitudinal direction, considered positive when tensile.

S = the tangential shearing force per unit length of shell, considered positive when it creates tension in the direction of increasing values of x and ϕ.

v = the vertical displacement, considered positive when it is in the downward direction.

h = the horizontal displacement, considered positive when it is directed inward.

L = length of shell between supports.

r = centerline radius of shell.

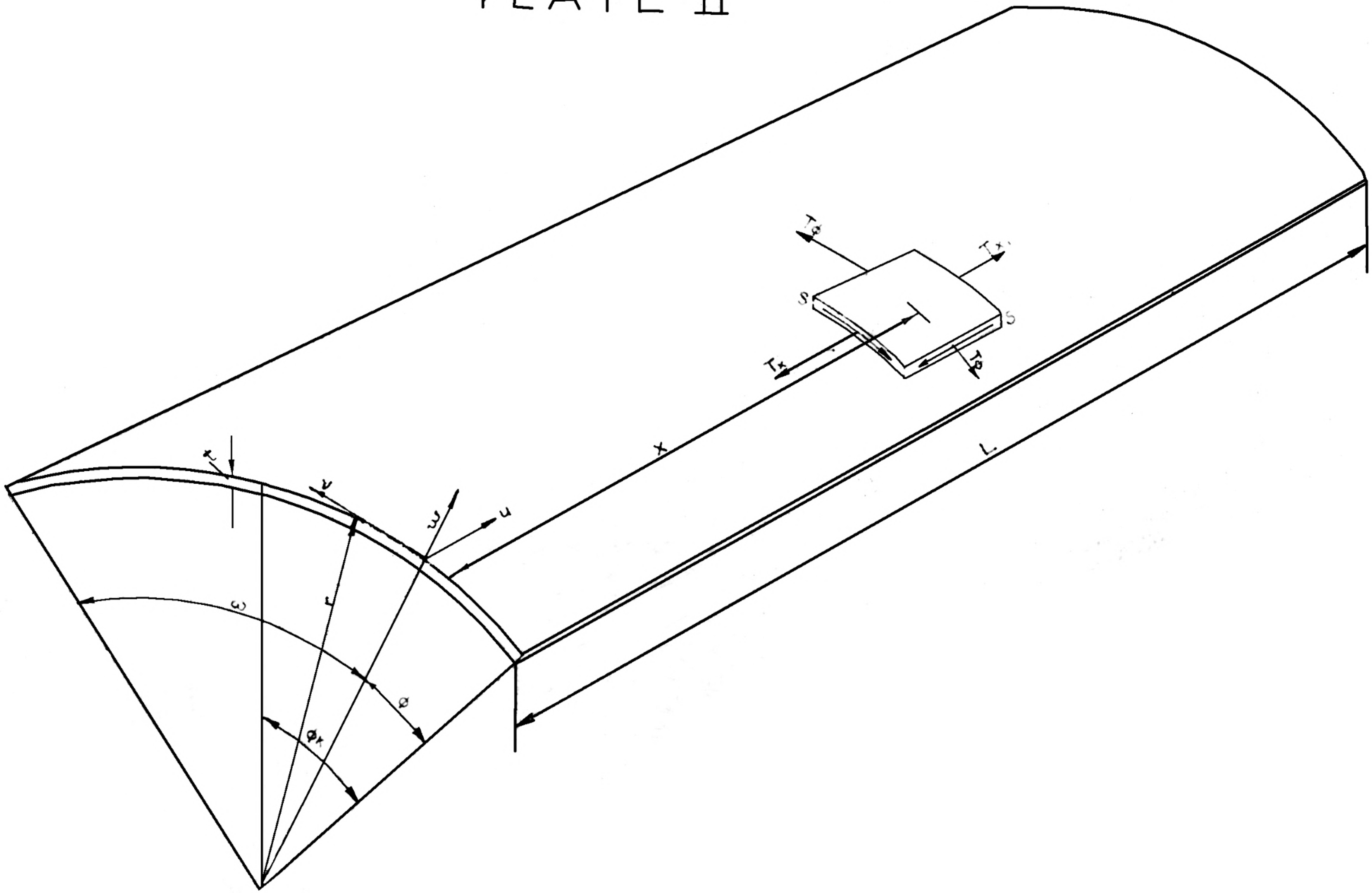
t = thickness of shell.

x = longitudinal distance measured from the left support.

ϕ = angle measured from the right edge of shell.

ϕ_k = angle subtended by the edge of shell measured from the centerline axis.

PLATE II



displacements of the supporting members correspond to the calculated displacements of the shell, the membrane analysis suffices; but to fulfill these requirements, deep edge members are often needed. In most cases these are neither possible nor desirable. On the contrary, in current practice the longitudinal edge beams are frequently omitted or reduced to a minimum size to exploit the full strength of the concrete shell. This reduction in the size of the edge beams means that the boundary conditions required by the membrane analysis are unfulfilled. To satisfy the requirements of statics, line loads must be applied.

The second step in shell design consists of finding the stresses due to line loads along the longitudinal edges. Four separate line loads, a radial shearing force, a longitudinal shearing force, a tangential transverse force and a moment, can be applied along each edge. The line loads along one edge can be different from those acting along the other edge. With these eight line loads, four on each side, any edge requirement can be satisfied.

For a single barrel with no edge member, subject to a symmetrical load the corrective line loads will consist simply of a tangential transverse force and a longitudinal shearing force equal to but opposite in direction to $T\phi$ and S given by the membrane analysis. When the edge conditions are complicated by the presence of longitudinal edge beams or adjoining shells, in addition to the above loads, a radial shearing force and a moment normal to the edge must be applied. The relative magnitude of each force will depend on the strains and rotation produced by each load. The procedure consists of establishing a number of simultaneous equations fulfilling known edge requirements.

The effect of edge line loads in a shell is basically different from surface loads, the difference being the fact that the line loads produce bending as well as direct forces. The various forces produced by line loads are illustrated in Plate III.

The effect of line loads is most pronounced in the vicinity of the edge at which the line load is applied, with the intensity of the internal forces generally diminishing as the distance from the applied forces is increased. When the chord width is small compared to the longitudinal span, the internal resisting forces produced by the line loads applied at one edge do not diminish fast enough so that the effect of a line load applied at one edge is felt at the farther edge. For this condition, the magnitude of angle subtended by the shell plays an important role in the distribution of the forces and must be considered; but, when the chord width is large compared with the longitudinal span, the effect of line loads on one edge is negligible on the other edge, and therefore line loads can be treated separately. There is no definite line of demarcation as to when the effect of line loads applied to one edge can be neglected at the other edge. However, computed values seem to indicate that, when r/L exceeds 0.6 with ϕk more than 30°, each longitudinal edge can be treated separately.

Because of this indication, data showing the effect of line loads are presented in two groups of tables in the A.S.C.E. manual. The first group (Tables 2A and 2B) gives the force distribution and displacements at the edge in long barrels for symmetrical line loads represented by the first term of the Fourier series. Since the angle subtended by the shell plays an important role, this set of results is presented in terms of three parameters r/L, r/t and ϕk. It should be noted that in addition to the three force components $T\phi$, S, Tx, the value of $M\phi$ is given. The value of the other bending moments and force components are omitted because they are insignificant in most cases.

In compiling this set of tables (2A and 2B), the radial and tangential line loads have been resolved into vertical and horizontal line loads. For shells

EXPLANATION OF PLATE III

N_ϕ = the radial shearing force on the radial face per unit length of shell, considered positive when it acts outwardly on the face facing the negative ϕ direction.

N_x = the radial shearing force on the longitudinal face per unit length of shell, considered positive when it acts outwardly on the face facing the negative x axis.

M_ϕ = bending moment on the radial face per unit length of shell, considered positive when it produces tension in the inner fibers.

M_x = bending moment on the transverse face per unit length of shell, considered positive when it produces tension in the inner fibers.

M_t = torsional moment per unit length of shell, considered positive when it produces tension in the inner fibers in the direction of increasing values of x and ϕ.

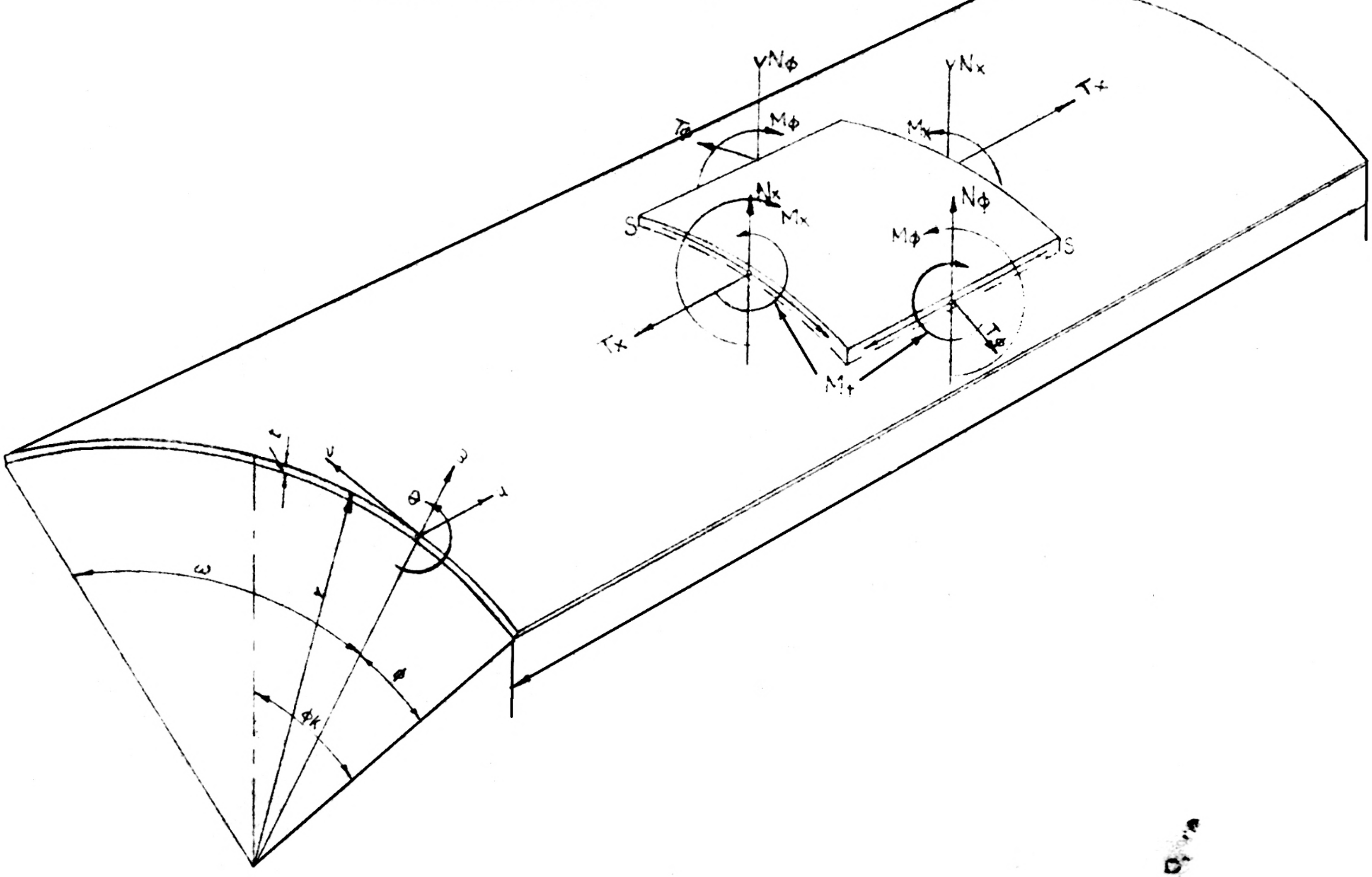
u = the displacement of the shell in the longitudinal direction, considered positive in the direction of increasing values of x.

v = the tangential displacement of the shell, considered positive in the direction of increasing values of ϕ.

w = the radial displacement of the shell, considered positive in the outward direction.

θ = the rotation of the shell, considered positive when the section rotates clockwise.

PLATE III



with edge members or multiple barrel shells, the horizontal and vertical components are preferable to tangential and radial line loads.

The second group (Tables 3A and 3B) covers shells whose ratio r/L exceeds 0.6 and gives force components produced by line loads on one edge. For this range the behavior of the shell can be expressed as a function of the parameter $\frac{rt}{L}$. The tables in the A.S.C.E. manual have been computed for two sinusoidal loadings, one with n=1 and one with n=3, since the second partial loading becomes of increasing importance as the value of r/L increases.

The derivation of the three equations of equilibrium for the membrane analysis is shown below.

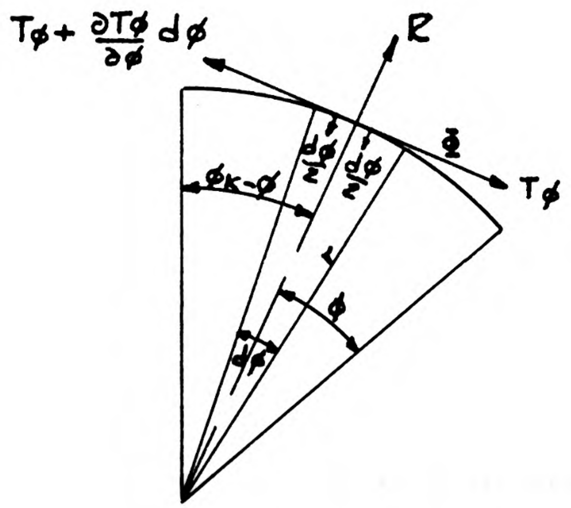


Fig. 4 Transverse view of shell element

The surface load is broken down into two components. R is the load per unit area acting radially and is considered positive when it acts outwardly. Φ is the load per unit area acting tangentially and is considered positive when it acts clockwise.

Summation of the forces in the radial direction gives the following equations.

F radial = 0

$$R\, r d\phi = \left(T\phi + \frac{\partial T\phi}{\partial \phi} d\phi\right) d\phi/2 + T\phi \;\; d\phi/2$$

$$R\, r = \frac{T\phi}{2} + \frac{\partial T\phi}{\partial \phi} \;\; \frac{d\phi}{2} + \frac{T\phi}{2}$$

$$R\, r = T\phi + \frac{\partial T\phi}{\partial \phi} \;\; \frac{d\phi}{2}$$

drop $\frac{\partial T\phi}{\partial \phi} \;\; \frac{d\phi}{2}$ because it is small

Therefore

$$T\phi - Rr = 0$$

A summation of the forces in the tangential direction is presented below.

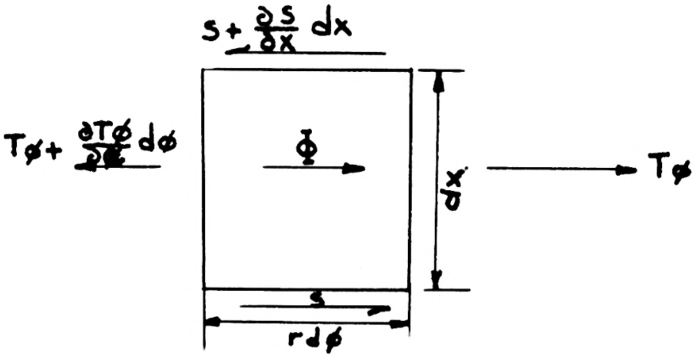


Fig. 5 Sketch of shell element showing tangential forces

$$\left(T\phi + \frac{\partial T\phi}{\partial \phi} d\phi\right) dx + \left(S + \frac{\partial S}{\partial x} dx\right) r d\phi = \Phi r d\phi dx + S r d\phi + T\phi dx$$

$$\frac{\partial T\phi}{\partial \phi} \;\; d\phi dx + \frac{\partial S}{\partial x} \;\; dx r d\phi = \Phi r d\phi dx$$

Therefore

$$\frac{\partial T\phi}{\partial \phi} + \frac{\partial S}{\partial x} \;\; r - \Phi r = 0$$

A summation of forces in the longitudinal direction gives the third equilibrium equation.

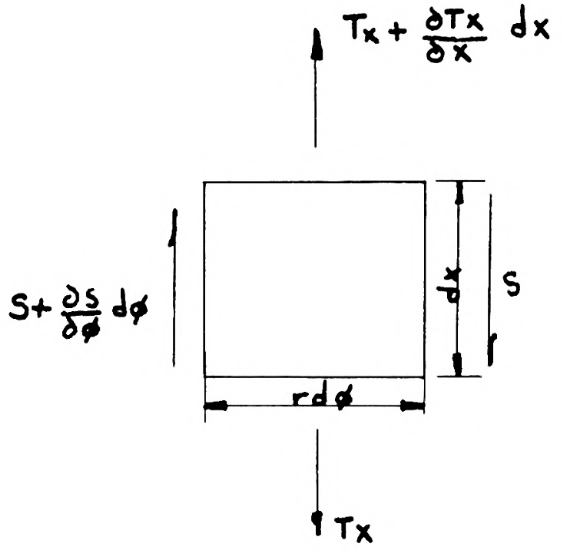


Fig. 6 Sketch of shell element showing longitudinal forces.

$$\left(Tx + \frac{\partial Tx}{\partial x} dx\right) rd\phi \quad \left(S + \frac{\partial S}{\partial \phi} d\phi\right) dx = Sdx + Txrd\phi$$

$$\frac{\partial Tx}{\partial x} dxrd\phi + \frac{\partial S}{\partial \phi} d\phi dx$$

Therefore

$$\frac{\partial Tx}{\partial x} r + \frac{\partial S}{\partial \phi} = 0$$

These three equations can be written in the following form:

$$T\phi = Rr$$

$$S = \int \Phi \, dx - 1/r \int \frac{\partial T\phi}{\partial \phi} dx + f_1(\phi)$$

$$Tx = -1/r \int \frac{\partial S}{\partial \phi} dx + f_2(\phi)$$

From the three equations of equilibrium one can study the stresses caused by different types of loading. The first loading to be studied is a uniform transverse load called Pu in which the weight is evenly distributed per horizontal foot.

$$R = -Pu \cos^2 (\phi k - \phi)$$

$$\Phi = Pu \cos (\phi k - \phi) \sin (\phi k - \phi)$$

$T\phi = -Pu\ r\cos^2(\phi k-\phi)$

$\frac{\partial T\phi}{\partial\phi} = -Pu\ r\ 2\cos(\phi k-\phi)\sin(\phi k-\phi)$

When substituted in the equation for shear the following is obtained:

$S = \int 2Pu\cos(\phi k-\phi)\sin(\phi k-\phi)\,dx + \int Pu\cos(\phi k-\phi)\sin(\phi k-\phi)\,dx + f_1(\phi)$

$S = 3Pu\cos(\phi k-\phi)\sin(\phi k-\phi)\,x + f_1(\phi)$ when $x = 1/2$ $s = 0$

$0 = 3Pu\cos(\phi k-\phi)\sin(\phi k-\phi)\,L/2 + f_1(\phi)$

$f_1\phi = -\frac{3L}{2}Pu\cos(\phi k-\phi)\sin(\phi k-\phi)$

$S = 3Pu\cos(\phi k-\phi)\sin(\phi k-\phi)\,x - \frac{3l}{2}Pu\cos(\phi k-\phi)\sin(\phi k-\phi)$

$S = (x - L/2)\ 3Pu\cos(\phi k-\phi)\sin(\phi k-\phi)$

which can be written in the form

$S = -Pu\ r\ (L/r)\ 3/2\cos(\phi k-\phi)\sin(\phi k-\phi)(1-2x/L)$

Next this is substituted in the formula for Tx

$Tx = -L/r\int\frac{\partial S}{\partial\phi}\,dx + f_2(\phi)$

$\frac{\partial S}{\partial\phi}dx = Pu\ r\ L/r\ (1-2x/L)\left[-3/2\cos^2(\phi k-\phi) + 3/2\sin^2(\phi k-\phi)\right]dx$

$Tx = \int Pu\ L/r\ (1-2x/L)\left[-3/2\cos^2(\phi k-\phi) + 3/2\sin^2(\phi k-\phi)\right]dx + f_2(\phi)$

$Tx = (x/L - x^2/L^2)\left[3/2\sin^2(\phi k-\phi) - 3/2\cos^2(\phi k-\phi)\right]Pur\ L^2/r + f_2(\phi)$

$Tx = 0$ when $x = 0$ for all (ϕ)

Therefore

$f_2(\phi) = 0$

The other loading considered is a variable transverse load varying as the weight of the shell. The dead load Pd is a loading of this type. The radial and tangential components of the load are:

$R = -Pd\cos(\phi k-\phi)$

$\Phi = Pd\sin(\phi k-\phi)$

In a similar manner one arrives at the equilibrium equations:

$S = -Pd\ r\ (L/r)\sin(\phi k-\phi)(1-2x/L)$

$T\phi = -Pd\ r \cos(\phi k - \phi)$

$Tx = -Pd\ r\ (L/r)^2 \cos(\phi k - \phi)\ x/L\ (1 - x/L)$

It will be noted in the preceding derivation that the surface loads with the same transverse distribution but of different longitudinal distribution will likewise have the same transverse distribution of stresses. Also loads with like longitudinal distribution, but with different transverse distribution, will produce the same longitudinal distribution of stresses. Therefore, surface loads, uniform in the transverse direction but varying as the sin $\frac{n\pi x}{L}$ in the longitudinal direction, a distribution employed to approximate other load conditions, will have the same stress distribution in the transverse direction as a surface load uniform in both directions. If the force and displacement components created by a surface load in the transverse direction, are varied as the sin $\frac{n\pi x}{L}$ in the longitudinal direction we obtain the following equations:

$T\phi = -Pu\ r \cos^2(\phi k - \phi) \sin \frac{n\pi x}{L}$

$S = -Pu\ r\ \frac{3}{n\pi}\ (L/r) \cos(\phi k - \phi) \sin(\phi k - \phi) \cos \frac{n\pi x}{L}$

$Tx = -Pu\ r\ \frac{3}{n^2\pi^2}\ (L/r)^2 \left[\cos^2(\phi k - \phi) - \sin^2(\phi k - \phi)\right] \sin \frac{n\pi x}{L}$

$u = Pu\ r\ r/Et\ (L/r)^3\ \frac{3}{n^3\pi^3} \left[\cos^2(\phi k - \phi) - \sin^2(\phi k - \phi)\right] \cos \frac{n\pi x}{L}$

$v = -Pu\ r\ r/Et\ (L/r)^4\ \frac{6}{n^4\pi^4} \left[\cos(\phi k - \phi)\ x \sin(\phi k - \phi)\right] \left[2 + \left(\frac{n\pi r}{L}\right)^2\right] \sin \frac{n\pi x}{L}$

$w = -Pu\ r\ r/Et\ (L/r)^4\ \frac{6}{n^4\pi^4} \left\{\left[\cos^2(\phi k - \phi) - \sin^2(\phi k - \phi)\right] x \left[2 + \left(\frac{n\pi r}{L}\right)^2\right] + 1/6 \left(\frac{n\pi r}{L}\right)^4 \cos^2(\phi k - \phi)\right\} \sin \frac{n\pi x}{L}$

in which u, v, and w are displacements in the x, ϕ, and radial directions.

The force components created by a surface load varying as weight of shell in the transverse direction and as sin $\frac{n\pi x}{L}$ in the longitudinal direction are as follows:

$$T\phi = -Pd\ r \cos(\phi k-\phi) \sin \frac{n\pi x}{L}$$

$$S = -Pd\ r\ L/r\ \frac{2}{n\pi} \sin(\phi k-\phi) \cos \frac{n\pi x}{L}$$

$$Tx = -Pd\ r\ (L/r)^2 \frac{2}{n^2\pi^2} \cos(\phi k-\phi) \sin \frac{n\pi x}{L}$$

$$u = Pd\ r\ r/Et\ (L/r)^3 \frac{2}{n^3\pi^3} \cos(\phi k-\phi) \cos \frac{n\pi x}{L}$$

$$v = -Pd\ r\ r/Et\ (L/r)^4 \frac{2}{n^4\pi^4} \left[1+2n^2\pi^2 (r/L)^2 \sin(\phi k-\phi)\right] \sin \frac{n\pi x}{L}$$

$$w = -Pd\ r\ r/Et\ (L/r)^4 \frac{2}{n^4\pi^4} \left[1+2n^2\pi^2 (r/L)^2 \frac{n^4\pi^4}{2} (r/L)^4\right] \cos(\phi k-\phi) \sin \frac{n\pi x}{L}$$

The general equations just reviewed are independent of the longitudinal boundary conditions. Hence, the values for the internal forces obtained from the final expressions for these forces are applicable to all cylindrical shells irrespective of the central angle 2ϕ subtended by the shell.

The marginal forces existing along the longitudinal edge must be corrected to agree with the actual conditions by the application of edge line loads. Since these edge loads produce bending and direct stresses on the shell, the analysis of the effect of edge loads involves the bending theory of shells.

It is not the purpose of this report to derive all the equilibrium equations in the bending theory of shells but a few comments on the method seem appropriate. The strains and rotations expressible as functions of the displacements u, v, and w are used to provide the required relationships to solve the equations. Three equilibrium equations can be written in terms of the unknown displacement components u, v, and w and their derivatives, as functions of x and ϕ. To solve these three equations simultaneously, two of the three unknowns must be eliminated and this requires successive differentiations. When one finally arrives at a solution of this equation, it involves an eighth order homogeneous differential equation. The final equations contain four arbitrary constants; their evaluation depends on the specified edge conditions.

The force and moment components can now be written in terms of u, v, and w, and substituted into the equilibrium equations.

EXAMPLE OF DESIGN PROCEDURE OF SIMPLY SUPPORTED SINGLE-BARREL SHELL

A snow load of $Py = 25$ #/□′ is assumed as live load. The dimensions are as shown on Plate IV. With these dimension $r/t = \frac{31.12}{3.75} = 99.2$ call it (100); $r/L = \frac{31}{62}$ or 0.5; and $Pd = \frac{3.75}{12.00} 150 = 47$ #/□′.

The first step in the analysis is to determine by the membrane theory the internal forces produced by the surface loads. In this example, for the purpose of greater clarity, only the first term of the Fourier series was used. The coefficients came from Table 1B in the A.S.C.E. manual, which is derived in the following manner. The surface load is broken into two components, R, in pounds per unit area acting radially, and Φ per unit area acting tangentially. The three equilibrium equations are rewritten below.

$$T\phi = Rr$$

$$S = -L/r \int \frac{\partial T\phi}{\partial \phi} dx + \int \Phi dx + f_1(\phi)$$

$$Tx = -L/r \int \frac{\partial S}{\partial \phi} dx + f_2(\phi)$$

The terms $f_1(\phi)$ and $f_2(\phi)$ represent functions of the variable ϕ and their values depend on the boundary conditions. By successive differentiations and integrations the values for the internal forces can be found.

Let Pu be the uniform load per unit area and ϕk be the angle subtended by a radial plane through the edge of shell and a vertical plane. Then $R = -Pu \cos^2 (\phi k-\phi)$ and $\Phi = Pu \cos (\phi k-\phi) \sin (\phi k-\phi)$. These equations and the nes for $T\phi$, Tx, and S can be written in the following form. $T\phi = -Pu\ r \cos^2 (\phi k-\phi)$ in which the term $-\cos^2 (\phi k-\phi)$ is tabulated in column 3 of Table 1A. Likewise $S = -Pu\ r (L/r)\ 3/2 \cos (\phi k-\phi) \sin (\phi k-\phi) (1-2x/L)$ in which the term $3/2 \cos (\phi k-\phi) \sin$

EXPLANATION OF PLATE IV

Simply supported shell with the following dimensions:

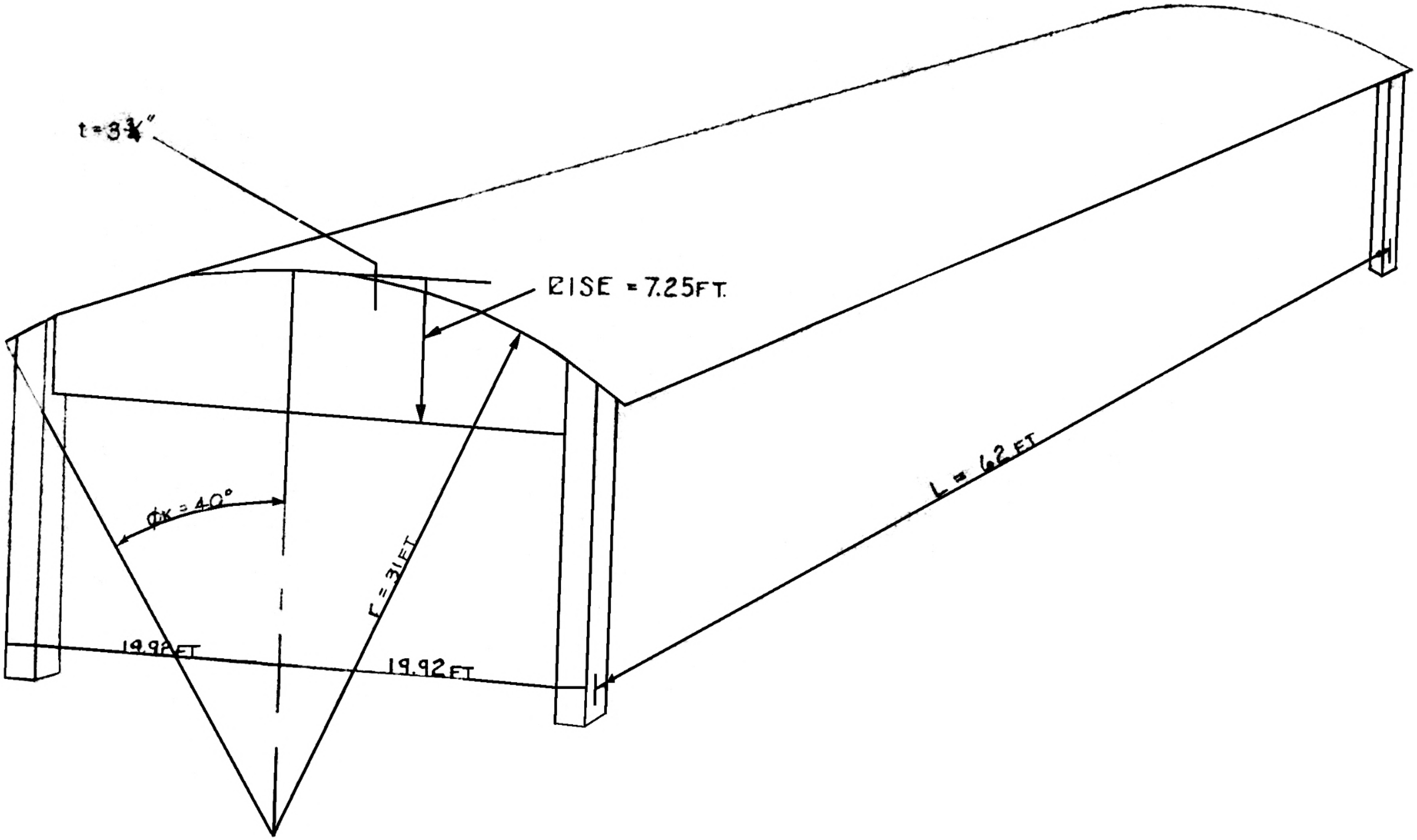
Length L= 62 feet

Radius r= 31 feet

Thickness t= 3 3/4 inches

Subtended angles from edge to center line $\phi_k = 40^\circ$

PLATE IV

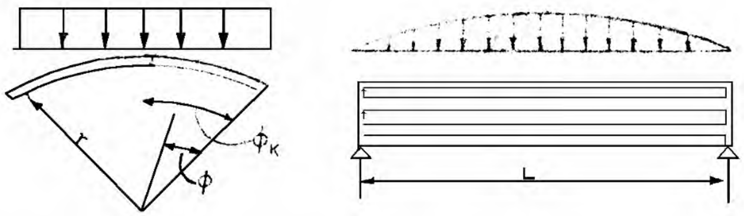


EXPLANATION OF PLATE V

Membrane forces and displacements in simply supported cylindrical shells; loads varying longitudinally from zero at the ends to maximum positive at the middle.

PLATE V

(a) Uniform Transverse Load



LONGITUDINAL FORCE T_x —

$$P_u r\left[\left(\frac{L}{r}\right)^2 \times \text{Col. }(1)\right]\sin\frac{\pi x}{L}$$

SHEARING FORCE S —

$$P_u r\left[\left(\frac{L}{r}\right) \times \text{Col. }(2)\right]\cos\frac{\pi x}{L}$$

TRANSVERSE FORCE $T\phi$ —

$$P_u r \times \text{Col. }(3) \times \sin\frac{\pi x}{L}$$

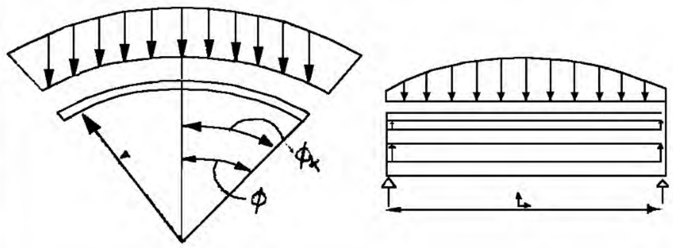
VERTICAL DISPLACEMENT Δ_V —

$$P_u r\frac{L^4}{r^3tE}\left[\left(1+\tfrac{1}{2}(\pi r/L)^2+\tfrac{1}{12}(\pi r/L)^4 \times \text{Col. }(4)\right]\sin\frac{\pi x}{L}$$

HORIZONTAL DISPLACEMENT Δ_H —

$$+P_u r\frac{L^4}{r^3tE}\left\{\left(\frac{r}{L}\right)^4 \times \text{Col. }(5)+\left[1+\tfrac{1}{2}\left(\frac{\pi r}{L}\right)^2+\tfrac{1}{12}\left(\frac{\pi r}{L}\right)^4\right]\times \text{Col. }(6)\right\} \times \sin\frac{\pi x}{L}$$

(b) Dead Weight Load



LONGITUDINAL FORCE T_x —

$$P_d r\left[\left(\frac{L}{r}\right)^2 \times \text{Col. }(7)\right]\sin\frac{\pi x}{L}$$

SHEARING FORCE S —

$$P_d r\left[\left(\frac{L}{r}\right) \times \text{Col. }(8)\right]\cos\frac{\pi x}{L}$$

TRANSVERSE FORCE $T\phi$ —

$$P_d r \times \text{Col. }(9) \times \sin\frac{\pi x}{L}$$

VERTICAL DISPLACEMENT Δ_V —

$$P_d r\frac{L^4}{r^3tE}\left[\left(\frac{2r}{\pi L}\right)^2+\frac{8}{\pi^4}+\left(\frac{r}{L}\right)^4 \times \text{Col. }(10)\right]\sin\frac{\pi x}{L}$$

HORIZONTAL DISPLACEMENT

$$+P_d r\frac{L^4}{r^3tE}\left[\left(\frac{r}{L}\right)^4 \times \text{Col. }(11)\right] \times \sin\frac{\pi x}{L}$$

$\phi_K-\phi$	T_x	S	$T\phi$	Δ_V	Δ_H	Δ_H	T_x	S	$T\phi$	Δ_V	Δ_H
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
0	-0.3040	0	-1.0000	0.12319	0	0	-0.2026	0	-1.000	1.0000	0
5	-0.2993	-0.0829	-0.9924	0.12180	0.0872	-0.00009	-0.2019	-0.0555	-0.9962	0.9924	0.0868
10	-0.2856	-0.1633	-0.9698	0.11766	0.1736	-0.00064	-0.1996	-0.1105	-0.9848	0.9698	0.1710
15	-0.2623	-0.2387	-0.9330	0.11102	0.2588	-0.00213	-0.1975	-0.1648	-0.9659	0.9330	0.2500
20	-0.2329	-0.3069	-0.8830	0.10222	0.3420	-0.00490	-0.1904	-0.2178	-0.9397	0.8830	0.3214
25	-0.1954	-0.3658	-0.8214	0.09170	0.4226	-0.00930	-0.1837	-0.2690	-0.9063	0.8214	0.3830
30	-0.1520	-0.4140	-0.7500	0.08001	0.5000	-0.01539	-0.1754	-0.3183	-0.8660	0.7500	0.4330
35	-0.1040	-0.4487	-0.6710	0.06771	0.5736	-0.02325	-0.1660	-0.3652	-0.8191	0.6710	0.4698
40	-0.0528	-0.4702	-0.5868	0.05537	0.6428	-0.03272	-0.1552	-0.4092	-0.7660	0.5868	0.4924
45	0	-0.4715	-0.5000	0.04355	0.7071	-0.04355	-0.1433	-0.4502	-0.7071	0.5000	0.5000
50	0.0528	-0.4702	-0.4132	0.03272	0.7660	-0.05537	-0.1302	-0.4877	-0.6428	0.4132	0.4924
55	0.1040	-0.4487	-0.3290	0.02325	0.8191	-0.06771	-0.1162	-0.5215	-0.5736	0.3290	0.4698
60	0.1520	-0.4140	-0.2500	0.01539	0.8660	-0.08001	-0.1013	-0.5513	-0.5000	0.2500	0.4330
65	0.1954	-0.3658	-0.1786	0.00930	0.9063	-0.09170	-0.0856	-0.5769	-0.4226	0.1786	0.3830
70	0.2329	-0.3069	-0.1170	0.00490	0.9397	-0.10222	-0.0693	-0.5982	-0.3420	0.1170	0.3214
75	0.2623	-0.2387	-0.0669	0.00213	0.9659	-0.11102	-0.0524	-0.6149	-0.2588	0.0669	0.2500
80	0.2856	-0.1633	-0.0301	0.00064	0.9848	-0.11766	-0.0351	-0.6269	-0.1736	0.0301	0.1710
85	0.2993	-0.0829	-0.0076	0.00009	0.9962	-0.12180	-0.0177	-0.6342	-0.0872	0.0076	0.0868
90	0.3040	0	0	0	1.0000	-0.12319	0	-0.6366	0	0	0

EXPLANATION OF PLATE VI

Symmetrical edge loads on simply supported cylindrical shells.

Plate VI

BASIC FORMULAS AND LOADING DIAGRAMS

VERTICAL EDGE LOAD

LONGITUDINAL FORCE T_x — $V_L\left[\left(\frac{L}{r}\right)^2 \times \text{Col.}(1)\right] \sin\frac{\pi x}{L}$

SHEARING FORCE S — $V_L\left[\frac{L}{r} \times \text{Col.}(2)\right] \cos\frac{\pi x}{L}$

TRANSVERSE FORCE T_ϕ — $V_L \times \text{Col.}(3) \times \sin\frac{\pi x}{L}$

TRANSVERSE MOMENT M_ϕ — $V_L\left[r \times \text{Col.}(4)\right] \sin\frac{\pi x}{L}$

HORIZONTAL EDGE LOAD

LONGITUDINAL FORCE T_x — $H_L\left[\left(\frac{L}{r}\right)^2 \times \text{Col.}(5)\right] \sin\frac{\pi x}{L}$

SHEARING FORCE S — $H_L\left[\frac{L}{r} \times \text{Col.}(6)\right] \cos\frac{\pi x}{L}$

TRANSVERSE FORCE T_ϕ — $H_L \times \text{Col.}(7) \times \sin\frac{\pi x}{L}$

TRANSVERSE MOMENT M_ϕ — $H_L\left[r \times \text{Col.}(8)\right] \sin\frac{\pi x}{L}$

SHEAR EDGE LOAD

LONGITUDINAL FORCE T_x — $S_L\left[\left(\frac{L}{r}\right)^2 \times \text{Col.}(9)\right] \sin\frac{\pi x}{L}$

SHEARING FORCE S — $S_L\left[\frac{L}{r} \times \text{Col.}(10)\right] \cos\frac{\pi x}{L}$

TRANSVERSE FORCE T_ϕ — $S_L \times \text{Col.}(11) \times \sin\frac{\pi x}{L}$

TRANSVERSE MOMENT M_ϕ — $S_L\left[r \times \text{Col.}(12)\right] \sin\frac{\pi x}{L}$

EDGE MOMENT LOAD

LONGITUDINAL FORCE T_x — $\frac{M_L}{r}\left[\left(\frac{L}{r}\right)^2 \times \text{Col.}(13)\right] \sin\frac{\pi x}{L}$

SHEARING FORCE S — $\frac{M_L}{r}\left[\frac{L}{r} \times \text{Col.}(14)\right] \cos\frac{\pi x}{L}$

TRANSVERSE FORCE T_ϕ — $\frac{M_L}{r} \times \text{Col.}(15) \times \sin\frac{\pi x}{L}$

TRANSVERSE MOMENT M_ϕ — $M_L \times \text{Col.}(16) \times \sin\frac{\pi x}{L}$

ϕ	T_x (1)	S (2)	T_ϕ (3)	M_ϕ (4)	T_x (5)	S (6)	T_ϕ (7)	M_ϕ (8)	T_x (9)	S (10)	T_ϕ (11)	M_ϕ (12)	T_x (13)	S (14)	T_ϕ (15)	M_ϕ (16)
							t = 100 AND [illegible]									
ϕ_K=35: 35	+2.471	0	-1.677	-0.1188	-2.776	0	+0.395	+0.0816	-0.2558	0	0.0962	-0.0039	14.99	0	-3.345	+0.3692
30	+1.493	+0.587	-1.758	-0.1758	-2.223	-0.7101	+0.495	+0.0816	-0.2425	-0.0837	-0.0875	-0.0037	-12.12	-4.668	-2.786	+0.3888
20	-4.209	-0.060	-2.002	-0.1480	+1.246	-1.062	+1.056	+0.0775	-0.2679	-0.1706	-0.0227	-0.0018	+6.40	-5.906	+0.453	+0.5266
10	-4.426	-3.090	-1.055	-0.0837	+2.986	+0.3737	+1.252	+0.0535	+0.4733	-0.0921	-0.0629	+0.0000	+17.12	+1.978	+1.931	-0.7345
0	+22.54	0	+0.574	0	-7.555	0	+0.819	0	+1.867	+0.5000	0	0	-44.56	0	0	+1.000
ϕ_K=40: 40	+5.472	0	-0.358	-0.1324	-4.282	0	-0.220	+0.0682	-0.1143	0	-0.0775	-0.0034	-17.17	0	-5.060	+0.2033
30	+1.404	+2.232	-1.063	-0.1354	-1.839	-1.889	-0.363	+0.0754	-0.1406	-0.0683	-0.0585	-0.0034	-7.86	-7.678	-2.645	+0.2912
20	-6.661	+0.757	-2.062	-0.1254	+3.336	-1.488	+1.415	+0.0819	-0.0853	-0.1408	+0.0027	-0.0014	+13.14	-6.372	+1.904	+0.5001
10	-5.826	-3.503	-1.214	-0.0751	+4.196	+1.017	+1.528	+0.0593	+0.4218	-0.0802	+0.0780	+0.0003	+18.72	+4.129	+2.821	+0.7345
0	+26.12	0	+0.643	0	-12.17	0	+0.766	0	+1.908	+0.5000	0	0	-54.72	0	0	+1.000
ϕ_K=45: 45	+66.55	0	+0.771	-0.0750	-4.823	0	-0.923	+0.0405	+0.0153	0	-0.0477	-0.0038	-15.73	0	-6.128	+0.0523
40	+57.49	+1.741	+0.524	-0.0793	-4.625	-1.375	-0.727	+0.0448	-0.0054	+0.0023	-0.0480	-0.0037	-14.17	-4.167	-5.525	+0.0765
30	-0.652	+3.364	-1.055	-0.1039	-0.365	-2.902	+0.573	+0.0709	-0.1266	-0.0318	-0.0417	-0.0028	-2.11	-9.122	-1.418	+0.2487
20	-8.443	+0.730	-2.362	-0.1125	+5.266	-1.497	+1.929	+0.0901	-0.1395	-0.1180	+0.0012	-0.0011	+16.12	-5.253	+3.200	+0.5015
10	-5.758	-4.033	-1.326	-0.0708	+4.705	+1.781	+1.798	+0.0670	+0.3807	-0.0882	+0.0726	+0.0003	+17.30	+5.658	+3.294	+0.7426
0	+27.26	0	+0.707	0	-1612	0	+0.707	0	+2.002	+0.5000	0	0	-58.57	0	0	+1.000
ϕ_K=50: 50	+6.074	0	+1.416	-0.0253	-5.122	0	-1.462	+0.0088	+0.1028	0	-0.0139	-0.0027	-11.38	.0	-6.094	-0.0520
40	+3.738	+2.900	+0.542	-0.0469	-3.478	-2.507	-0.713	+0.0302	+0.0147	+0.0398	-0.0269	-0.0027	-8.59	-5.742	-4.363	+0.0339
30	-2.427	+3.394	-1.371	-0.0900	+1.103	-3.275	+1.025	+0.0759	-0.1664	-0.0015	-0.0410	-0.0023	+0.92	-8.355	-0.093	+0.2533

$(\phi k-\phi)$ is tabulated in column 2 of Table 1A. If the shell is simply supported, then Tx at $x = 0$ is equal to zero. Therefore, $f_2(\phi)$ is equal to zero. Therefore $Tx = -Pu\, r\, (L/r)^2\, 3/2 \left[\cos^2 (\phi k-\phi) - \sin^2 (\phi k-\phi)\right] x/L\, (1-x/L)$ in which the term $3/2 \left[\cos^2 (\phi k-\phi) - \sin^2 (\phi k-\phi)\right]$ has been evaluated and makes up the coefficients in column 1 of Table 1A.

The formulas just mentioned are for a loading that is constant in the transverse direction. For a loading that varies with the weight of the shell, such as dead load, the following expressions exist. $R = -Pd \cos (\phi k-\phi)$ and $\Phi = Pd \sin (\phi k-\phi)$ in which Pd is the unit weight of the shell. Now one can write $T\phi = -Pd\, r \cos (\phi k-\phi)$ in which the value for $-\cos (\phi k-\phi)$ can be found in column 9 of Table 1A. Similarly $S = -Pd\, r\, (L/r) \sin (\phi k-\phi)\, (1-2x/L)$. The term $-\sin (\phi k-\phi)$ makes up column 8 of 1A; $Tx = -Pd\, r\, (L/r)^2 \cos (\phi k-\phi)\, x/L\, (1-x/L)$; column 7 of the table is compiled by evaluating $-\cos (\phi k-\phi)$.

Realizing that Tables 1B and 1C are made up of partial loading one can adjust the terms just derived by putting in a factor of $\sin \frac{n\pi x}{L}$, in which $n = 1$ for Table 1B and $n = 3$ for 1C.

Our equations for a load uniform in the transverse direction but varying as the $\sin \frac{n\pi x}{L}$ in the longitudinal direction are as follows: $Tx = -Pu\, r \cos^2 (\phi k-\phi)$ is presented in column 3 of tables 1B and 1C. $S = -Pu\, r\, \frac{3}{n\pi} (L/r) \cos (\phi k-\phi) \sin (\phi k-\phi) \cos \frac{n\pi x}{L}$, column 2 is $\frac{3}{n\pi} \left[-\cos (\phi k-\phi) \sin (\phi k-\phi)\right]$ in both tables 1B and 1C, $n = 1$ in 1B and $n = 3$ in 1C. $Tx = -Pu\, r\, \frac{3}{n^2\pi^2} (L/r)^2 \left[\cos^2 (\phi k-\phi) - \sin^2 (\phi k-\phi)\right] \sin \frac{n\pi x}{L}$. Column 1 is the evaluation of $\frac{3}{n^2\pi^2} \left[-\cos^2 (\phi k-\phi) - \sin (\phi k-\phi)\right]$.

The force components created by a surface load varying as the weight of the shell in the transverse direction and as $\sin \frac{n\pi x}{L}$ in the longitudinal direction are as follows:

$T\phi = -Pd\ r \cos(\phi k-\phi) \sin \frac{n\pi x}{L}$, $\left[-\cos(\phi k-\phi)\right]$ is compiled in column 9.

$S = -Pd\ r\ L/r\ \frac{2}{n\pi} \sin(\phi k-\phi) \cos \frac{n\pi x}{L}$, $\left[-\frac{2}{n\pi} \sin(\phi k-\phi)\right]$ is compiled in column 8.

$Tx = -Pd\ r\ (L/r)^2 \frac{2}{n^2\pi^2} \cos(\phi k-\phi) \sin \frac{n\pi x}{L}$, $\left[-\frac{2}{n^2\pi^2} \cos(\phi k-\phi)\right]$ makes up column 7.

Since these tabulated values represent the effect of a unit load, these coefficients must be multiplied by the magnitude of the applied loads which are 25 for snow load, and 47 for dead load. They also must be multiplied by both the factor indicated in table 1B and $4/\pi$ as required by the Fourier analysis. The values are listed in rows 1 and 2 of table 1. For example, row 1 for part (a), the solution of $T\phi$ at $x = L/2$ (pound per foot) is obtained in the following manner. The equation from Table 1B for $T\phi$ is $T\phi = Pu\ r\ 4/\pi$ coefficient $\sin \frac{\pi x}{L}$. So at $x = L/2$ the $\sin \frac{\pi L/2}{L} = 1$, leaving $4/\pi$ (Fouriers Series) 25(Pu) 31(r) coefficient (column 3 Table 1B). For $\phi = 40^{\circ}$, $\phi k-\phi = 0$ therefore coefficient $= 1.000$, giving an answer of -987. Likewise at $\phi = 30$ the coefficient would be found at $\phi k-\phi = 10$ which is -.9698 when this is multiplied by 4/ .25.31 it is equal to -957. Line 2 gives similar results for the dead load using 47 as Pd and the coefficient from column 9 in Table 1B. The values for S and Tx are found in a similar manner using the following formulas: $S = -Py\ r\ L/r\ 4/\pi$ (coefficient) $\cos \frac{\pi x}{L}$ and $Tx = Pu\ r\ (L/r)^2\ 4/\pi$ (coefficient) $\sin \frac{\pi x}{L}$.

It is evident, by inspecting values at $\phi = 0$ in Table 1, that the membrane theory yields reactions along the longitudinal edges. Since these edges are unsupported, corrective line loads of equal but opposite values must be applied. At $x = L/2$ for $T\phi$ and at $x = 0$ for S, the unbalanced forces at $\phi = 0$ are: $T\phi = -579 -1,421 = -2,000$ #/ft; and $S = -928 -1,519 = -2,447$ #/ft.

The vertical and horizontal components of the transverse force are: $V_L = 2,000$

sin $\phi k =$ 1,286 #/ft; $H_L =$ 2,000 cos $\phi k =$ 1,532 #/ft; and $S_L =$ 2,447 #/ft.

Since Table 2A represents the effect of unit line loads on a shell, the coefficients for $\phi k = 40°$ $r/t = 100$ and $r/L = 0.5$ times the factor of r/L or $(r/L)^2$ as specified in the heading are multiplied by the values of V_L, H_L and S_L just determined. The products of this operation are tabulated in rows 3 to 5 of Table 1. Row 3 is found in the following manner. In Table 2A on the page corresponding to the values of $r/t = 100$ and $r/L = 0.5$ find the row for $\phi k = 40°$, then under column 3 ($T\phi$ for a vertical edge load) find the coefficient for $\phi = 40$ to be −0.358 giving the value V_L coefficient = −460. The values for Tx, S, and $M\phi$ are found in a similar manner. The respective formulas are $T\phi = V_L$ column (3) $\sin \frac{\pi x}{L}$, $Tx = V_L \left[(L/r)^2 \text{ column } (1)\right] \sin \frac{\pi x}{L}$; $S = V_L \left[L/r \text{ column } (2)\right] \cos \frac{\pi x}{L}$ and $M\phi = V_L \left[r \text{ column } (4)\right] \sin \frac{\pi x}{L}$.

The resultant forces acting in the shell due to the combination of all the applied loads are equal to the sum of rows 1 to 5, and are tabulated in row 6. It should be noted that both $T\phi$ and S are now equal to zero along the free edge and hence the boundary conditions are satisfied.

The internal forces at any point in the shell can be obtained by multiplying the values of $T\phi$, Tx, $M\phi$ by $\sin \frac{\pi x}{L}$ and S by the $\cos \frac{\pi x}{L}$.

In order to determine the steel necessary to resist the tensile forces, the principal stresses generated by the combined direct stresses and tangential shears must be evaluated from the equation $Tp = \frac{Tx+T\phi}{2} \pm \sqrt{\frac{(Tx-T\phi)^2}{4} + S^2}$ or from Mohr's Circle (Fig. 7) in which Tp = the principal force. The plane on which the first principal force acts is given by $\tan 2\delta = \frac{2S}{Tx-T\phi}$ in which, for positive values of $\tan 2\delta$, δ is measured in a counter clockwise direction from the face on which Tx acts. The second principal stress will be at right angles to the first principal stress.

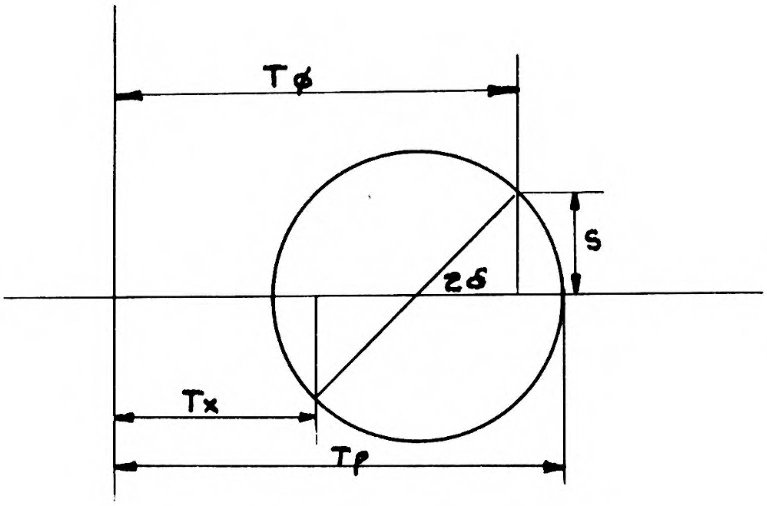


Fig. 7 Mohr's Circle

The principal forces and planes on which they act are given in Table 3. The lines of principal stresses are curvilinear and steel is required to carry the tensile stress.

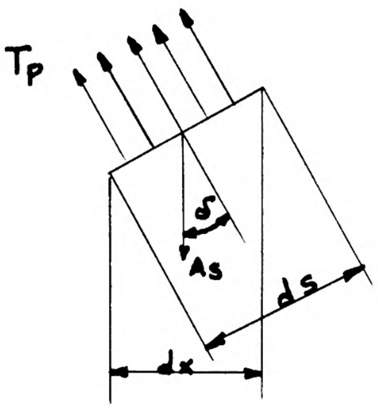


Fig. 8 One way reinforced

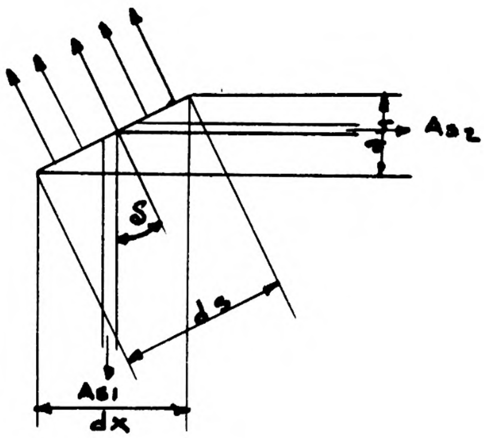


Fig. 9 Two way reinforced

Although concrete can withstand a small tensile force, steel should be provided to resist all the tensile forces. For this to be accomplished theoretically the steel should be bent to follow the curved lines of the principal stresses. Because of the difficulty in bending the bars to follow the definite paths, the bars are placed instead at an angle to the direction of principal stresses. The disadvantage of placing the steel in this manner lies in the fact that the effectiveness of the steel is reduced. This reduction results from the increased distance between the bars and the decreased components of the allowable tension in bars. If the force is summed in the direction of the principal stress Tp the result is $Tp\, ds = As\, fs\, dx \cos\delta$; in which As is the area of steel per unit of length in the x-direction. If this equation is divided through by ds then $Tp = As\, fs \cos\delta \frac{dx}{ds} = As\, fs \cos^2\delta$.

When steel is placed in two directions at right angles to each other, as shown in Fig. 8, the steel stress in one direction can be arbitrarily specified but the stress in the other direction is governed by the direction of the principal strain. If the forces in the direction of Tp are equated, the result is $Tp\, ds = As_1\, fs_1\, dx \cos\delta + As_2\, fs_2\, dy \sin\delta$ in which As_2 is the area of steel per unit of length in the y-direction. From this, it follows that $Tp = As_1\, fs_1 \cos^2\delta + As_2\, fs_2 \sin^2\delta$. The resulting strain in the reinforcement is in the direction of the principal stress; therefore $fs_2 = fs_1 \tan$. Hence, for reinforcement in two directions, $Tp = fs_1 (As_1 \cos^2\delta + As_2 \sin^2\delta \tan\delta)$.

The analysis of the shell has been based on the assumption of homogeneity of the material. To be consistent then the steel stresses should be based on the ratio of steel strain to concrete strain. This arrangement would require the use of a constant percentage of steel throughout the tensile zone.

In indeterminate structures it is customary to analyze the structure on the basis of the constant elastic property of the member and then determine the area of the required steel on the assumption that is is stressed to an allowable design value. The same practice will be followed in shell design with the modification that the stresses in the longitudinal steel will be proportioned according to their distance from the neutral axis. This is the point where $Tx = 0$.

The steel at the bottom edge will be assumed at 20,000 psi. On this basis, from Table 3, the required area along the bottom edge at midspan is computed from the formula as $As = \frac{77,000}{20,000} = 3.85$ square inches.

To furnish the area, one inch round bars spaced on 2 3/8 inch centers are needed. At one foot from the bottom edge, the permissible steel stress will be based on the ratio of the distance to the neutral axis. The neutral axis is three feet from the bottom edge. The permissible stress is therefore 20,000 times 2/3= 13,000 #/sq. inch. The total force per unit of length acting at this point is approximately 44,500 pound per foot. The required area of steel computed by the formula As= 44,500/13,300= 3.32 square inch. Consequently, the spacing can be increased to 2 7/8 inches. In the same manner, the steel apacing at two feet from the bottom is found to be 3 3/8 inches. This spacing is maintained to the neutral axis. Above the neutral axis a nominal amount of reinforcement should be provided.

With the longitudinal forces varying as the sin $\frac{\pi x}{L}$, the steel area is reduced by 25 per cent at x= L/4, and by 50 per cent at x= L/6. Every other bar is continued to the support.

Along the supporting rim, the tensile forces (Table 3) are resisted by continuous diagonal reinforcement. The steel area at $\phi = 10^{\circ}$, and $\phi = 20^{\circ}$, is $\frac{Tp^2}{fs} =$ 8,283/20,000= 0.42 square inches and 4,716/20,000= 0.24 square inches. Using $\frac{1}{2}$ inch round bars, the required spacing normal to the direction of the reinforcement becomes 5/3/4 inches and 10 inches respectively. However since the bars are continuous, the spacing at $\phi = 20^{\circ}$ is governed by the spacing required at $\phi = 10^{\circ}$ and x= L/8. The area at this point is 7,002/20,000= 0.35 square inches. The required spacing of 6 3/4 inches is maintained to the support.

The spacing of the transverse reinforcement is dictated by the transverse moment $M\phi$ and thrust $T\phi$. At the center of the span at the crown, with an effective depth of 2 3/4 inches, there is a moment= 2,297 foot-pounds (Table 1) and a trust of 3,329 pounds (Table 1); the steel required is 0.45 square inch per foot in the top of the shell. Since $M\phi$ decreases as ϕ decreases, half the bars are terminated at approximately $\phi = 20^{\circ}$, where $M\phi$ is approximately half of the crown transverse moment. In the longitudinal direction $M\phi$ varies as the sin $\frac{\pi x}{L}$. Hence the transverse steel area can be reduced by 25 per cent at $x = L/4$ and by 50 per cent at $x = L/6$. Where partial loads are anticipated normal bottom reinforcement should be provided.

CONSTRUCTION TECHNIQUES

Although it is a fact that thin shell construction makes the maximum use of material, giving the designer greater space for the amount of material actually used in the completed structure, at the same time it gives the builder many new problems. The amount of concrete and steel used in the structure accounts for but a small percentage of the total cost of the job. The problem of form design, form handling and labor, becomes a problem of increasing importance. The problem was first tackled in Europe, where for both economic reasons and because of lack of certain structural materials, thin shell construction made its most notable strides. The important fact that an increase in use of material can be offset by a saving in labor and time, has played an important role in construction techniques. Reusing the forms as much as possible and repetition of operations that increase the efficiency and skill of the laborer are also two important aspects of economic construction. It might be said then that the growth in the use of thin shell construction is dependent on the improvements in construction techniques.

The use of mobile form centering is often advantageous, permitting repeated use of forms by moving them intact to their new location. In Europe where labor is much cheaper, the actual disassembly and assembly of forms by hand has proved economical in some cases. The selection of steel or wood for the construction of the centering is chiefly a problem of availability and the size of the job. While steel is higher in its initial cost, on large jobs it has proved economical, due mainly to its adaptability to be quickly assembled and disassembled. The centering is actually a wooden truss with a curved upper chord designed to carry the load of the wet concrete. It is placed on vertical supports with an adjustable base, usually consisting of a jack and wheel arrangement. Rails are laid in the longitudinal direction and the centering is rolled into place. The jacks are adjusted to give the form the proper elevation. The centering is braced for wind stresses usually by cables or by bracing directly on part of the existing structure. When the form is in place the reinforcing steel is laid in a mat system of a simple grid. Additional steel is placed near the edges and ribs to take care of the bending stresses. Sometimes the position of the bars are painted on the forms for speed in placing the steel. The pour is started on both sides at the same time at the springing line and kept even to prevent any arching action. By measuring the deflection of test beams poured at the same time of the shell it can be determined when it is safe to remove the forms. This is done by lowering the jacks evenly and placing the wheels back down on the rails for movement to the next pour.

ECONOMICS

The first question that will come to most peoples minds in regard to the use of thin shell construction in this country is, how will it compare economically with the types of construction now in use? This is a fair question because it is the economy that is the main obstacle to the use of this type of construction in this country. To be fair to the thin shell technique one should compare Europe and this country separately. It seems to be the practice at home to sacrifice materials for time; and consequently, in a structure such as this, the vast savings in material are more than offset, at the present time, by increased labor cost. However, there have been notable exceptions to this which will be discussed later.

It has already been mentioned that in Europe, where materials are a critical item and labor is relatively cheap, thin shell construction has proved one of the best forms of construction. Necessity is the mother of invention and this is why most of our outstanding examples are found in Europe. Because even in spanning small areas thin shell uses about 40 per cent as much steel as an all steel frame for the same job, it has been given a great deal of study abroad.

The disadvantages in the use of this type of construction can be broken down into three parts. These three parts are the design, placing of steel reinforcing, and formwork. In the design, some of the earlier jobs of complicated shapes have taken as much effort as four engineers working for six months to complete. This factor is being rapidly decreased and for standard shape shells various committees, such as the A.S.C.E. committee on thin shell construction, have simplified the procedure and provided tables to cut down on the design time.

The second disadvantage, that of placing reinforcing steel, has been reduced considerably by the use of woven wire mesh as the main reinforcing.

Still the biggest obstacle is the form work. Yet even this problem is being overcome by the use of traveling formwork. On the construction of some large hangars, a form for one or two bays was constructed and then moved on rails along the length of the building to be used as the form in each successive bay as described earlier. An example of the cost of one of these forms is given by the one used in the Air Force Hangar at Rapid City, South Dakota. The form was 50 feet long, 340 feet wide, and 90 feet high. It cost $85,000 to build, contained 280,000 board feet of lumber and weighed 500 tons.

All three of these disadvantages are being overcome and it may not be long before this type of structure will have a firm place in the building techniques of this country.

Two examples in which the concrete roof was cheaper than the other types of roof construction are the U.S. Army Quartermaster Warehouse, at Columbus, Ohio and the Livestock Coliseum at Montgomery, Alabama. In the warehouse job the concrete was originally bid as an alternate to a steel truss with wood deck roofing. The concrete cost only one cent per square foot more than the steel design and had the additional advantage of being fireproof. For the roof bids on the Livestock Coliseum, the steel bid was $599,000 and the concrete bid was $577,500. Time for completion was 500 days for the concrete as compared with 730 days for the steel.

The disadvantages of thin shell construction can be described, as primarily those of construction techniques in this country. But what are advantages? Maintenance is a good place to start on the long list. Steel trusses have to be cleaned and painted periodically where as there is no such maintenance to concrete. The smooth curved surface is easy to light either artificially or naturally with openings in the roof. Shells may be insulated against both heat and reverberation by applying thermal insulation and sound absorptive materials. The acoustics are easier to control in a series of smaller cross barrels than in a large single shell. The biggest advantage to an owner that needs a large clear area free from columns is the safety of his structure in case of fire.

Another advantage was pointed out in World War II where in Europe some of these structures received heavy bombing and stood up remarkably well. The shells or bombs would make a hole in the shell but the roof would not collapse. A team of assessors sent to Germany after the war by the British Ministry of Works, reported that shell concrete was the most impressive modern building form found in Germany.

CONCLUSION

Thin shell concrete construction is the most economical type of construction as far as getting the greatest use out of the material. The reason that it has not become a popular type of roof construction in this country is chiefly because of the expense and labor involved in forming, and the time required in the design.

It is our belief that the advantages of this type of construction such as, greater utilization of material, high fireproof rating, little maintenance required and the striking architectural appearance will soon outweigh the economic disadvantages due to construction techniques.

The hand book of the A.S.C.E. that has been referred to in this report has helped to reduce the time required to design the shells. As has been pointed out, this manual enables the designer to solve for his stresses without going through the laborious process of solving an eighth ordered differential equation.

With the constant improvement of construction techniques, such as the centering placed on rails for repeated use, the disadvantage will no longer deprive this country of the type of structures which we see and admire in European countries. The inexpensive European labor will be overcome by the "Yankee Ingenuity" and this type of roof construction may soon take its place along side the other types of roof construction now in use in this country. This is a clear example in which the engineer has given the architect a structural element that gives him economy of material, greatest use of space and a striking appearance.

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APPENDIX

Table 1 Internal forces in a simply supported, single-barrel cylindrical shell

(a) Tϕ at X=L/2 (Pounds per Foot)

Row	Loading	Multipliers	Angle ϕ, in Degrees 40	30	20	10	0
1	Uniform	4/π x 25 x 31 x Coef=	- 987	- 957	- 871	- 740	- 579
2	Dead	4/π x 47 x 31 x Coef=	- 1,855	- 1,827	- 1,743	- 1,607	- 1,421
3	V_L	1,286 x Coef=	- 460	- 1,375	- 2,652	- 1,561	+ 827
4	H_L	1,532 x Coef=	- 337	+ 556	+ 2,168	+ 2,341	+ 1,174
5	S_L	2,447 x Coef=	- 190	- 143	+ 7	+ 191	0
6			- 3,829	- 3,746	- 3,091	- 1,376	0

(b) S at X=0 (Pounds per Foot)

Row	Multipliers	Angle ϕ, in Degrees 40	30	20	10	0
1	4/π x 25 x 31 x 2 x Coef=	0	- 322	- 606	- 817	- 928
2	4/π x 47 x 31 x 2 x Coef=	0	- 410	- 808	-1,180	-1,519
3	1,286 x 2 x Coef=	0	+5,741	+1,947	-9,010	0
4	1,532 x 2 x Coef=	0	-5,787	-4,560	+3,116	0
5	2,447 x 2 x Coef=	0	- 334	- 689	- 392	+2,447
6		0	-1,112	-4,716	-8,283	0

(c) Tx at X=L/2 (Pounds per Foot)

Row	Loading	Multipliers	40	30	20	10	0
1	Uniform	4/π x 25 x 31 x 2^2 Coef=	- 1,200	- 1,127	- 919	- 600	- 208
2	Dead	4/π x 47 x 31 x 2^2 Coef=	- 1,503	- 1,481	- 1,413	- 1,301	- 1 ,151
3	V_L	1,286 x 2^2 x Coef=	+28,150	+ 7,222	-34,260	-29,970	+134,300
4	H_L	1,532 x 2^2 x Coef=	-26,238	-11,270	+20,440	+25,710	- 74,610
5	S_L	2,447 x 2^2 x Coef=	- 1,119	- 1,378	- 835	- 4,129	+ 18,670
6			- 1,910	- 8,034	-16,987	- 2,032	+ 77,000

Mϕ at X=L/2 (Pounds per Foot)

Row	Multipliers	40	30	20	10	0
3	1,286 x 31 x Coef=	-5,278	-5,400	-5,000	-2,994	0
4	1,532 x 31 x Coef=	+3,239	+3,581	+3,890	+2,816	0
5	2,447 x 31 x Coef=	- 256	- 258	- 106	+ 23	0
6		-2,297	-2,077	-1,216	- 155	0

Table 2 Force components throughout the simply supported, single-barrel, cylindrical shell, for sundry values of X/L (=0,1/8,1/4,3/8, and 1/2)

	(a) Tϕ, in Pounds per Foot					(b) S, in Pounds per Foot)					(c) Tx, in Pounds per Foot					(d) Mϕ, in Pounds per Foot				
ϕ	0	1/8	1/4	3/8	1/2	0	1/8	1/4	3/8	1/2	0	1/8	1/4	3/8	1/2	0	1/8	1/4	3/8	1/2
40°	0	-1,465	-2,707	-3,537	-3,829	0	0	0	0	0	0	- 731	- 1,350	- 1,765	- 1,910	0	- 880	-1,624	-2,122	-2,297
30°	0	-1,433	-2,649	-3,461	-3,746	-1,112	-1,027	- 786	- 426	0	0	-3,074	- 5,680	- 7,423	- 8,034	0	- 795	-1,469	-1,919	-2,077
20°	0	-1,163	-2,186	-2,856	-3,091	-4,716	-4,357	- 3,334	- 1,805	0	0	-6,501	-12,010	-15,690	-16,987	0	- 465	- 860	-1,123	-1,216
10°	0	- 527	- 973	-1,271	-1,376	-8,283	-7,653	- 5,856	- 3,170	0	0	- 778	- 1,437	- 1,877	- 2,032	0	- 59	- 110	- 143	- 155
0°	0	0	0	0	0	0	0	0	0	0	0	+29,470	+54,460	+71,140	+77,000	0	0	0	0	0

Table 3 Principal forces, T in simply supported, single-barrel, cylindrical shell in example 1

	X/L=0			X/L=1/8			X/L=1/4			X/L=3/8			X/L=1/2		
ϕ	T_{p1}	T_{p2}	δ	T_{p1}	T_{p2}	δ	T_{p1}	T_{p2}	δ	T_{p1}	T_{p2}	δ	T_{p1}	T_{p2}	δ
40°	0	0	45°	-1,465	- 731	90°	- 2,707	- 1,350	90°	- 3,537	- 1,765	90°	- 3,829	- 1,910	90°
30°	-1,112	1,112	45°	-3,560	- 948	26°	-5 ,871	- 2,457	14°	- 7,467	- 3,415	6°	- 8,034	- 3,746	0°
20°	-4,716	4,716	45°	-8,944	1,261	29°	-13,040	- 1,161	17°	-15,940	- 2,607	8°	-16,987	- 3,091	0°
10°	-8,283	8,283	45°	-8,306	7,002	45°	- 7,062	4,652	44°	- 4,758	1,610	42°	- 2,032	- 1,376	0°
0°	0	0	45°	0	29,470	90°	0	54,460	90°	0	71,140	90°	0	77,000	90°