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Probabilistic Stability of Traffic Load Balancing on Wireless Complex Networks

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***Abstract*—Load balancing between adjacent base stations (BSs) is important for balancing load distributions and improving service provisioning. Whilst load balancing between any given pair of BSs is beneficial, cascade load sharing can cause network level instability that is hard to predict. The relationship between each BS's load balancing dynamics and the network topology is not understood. In this seminal work on stability analysis, we consider a frequency re-use network with no interference, whereby load balancing dynamics doesn't perturb the individual cells' capacity. Our novelty is to show an exact analytical and also a probabilistic relationship for stability, relating generalized local load balancing dynamics with generalized network topology, as well as the uncertainty we have in load balancing parameters due to noisy channel or network sensing. We prove that the stability analysis is valid for any generalized load balancing dynamics and topological cell deployment and we believe this general relationship can inform the joint design of both the load balancing dynamics and the neighbour list of the network. The probabilistic framework provides uncertainty quantification and stability prediction for Digital Twins of wireless infrastructure.**

***Index Terms*—Wireless Networks, Reliability and Survivability, Stability, Complex Network, Load Balancing, Digital Twin, Uncertainty Quantification**

I. Introduction

Load balancing is an important aspect of current and future cellular network operations, homogenizing traffic demand and interference patterns [1]–[4]. In each base station (BS), load balancing typically involves tuning the transmit power and active radio elements to match the traffic demand. When overloaded with time-sensitive demand, BSs can offload demand to neighbouring BSs, if their demand is relatively low. Load balancing can be implemented between active BSs [5], provide support for sleep mode BSs [6], user equipments (UEs) in a D2D underlay [7], and in wireless sensor networks [8].

Current literature focuses on the algorithms of load balancing and doing so in a multi-RAT/spectrum co-existence setting, and doesn't consider cascade effects across large-scale and hyper-dense networks. We know from other coupled optimisation systems that runaway cascades are possible, see: power control in pairwise coupled BSs [9], [10] and in routing [11]. In the case of load balancing, this would mean that users are shifted constantly between BSs, without a significant improvement in the quality of service, but at the cost of significant spectral inefficiency and coordination signaling. Unstable behaviour would be the introduction of new users that cause endless load balancing between BSs. For example, node A offloading to neighbouring node B can cause congestion in node B and further offloading to node C, and so fourth. This can cause fewer users being satisfied overall across the whole network and endless load sharing actions propagating across the network. Cascade effects on large scale complex networks (i.e. no. of nodes N is large) that affect stability and equilibrium solutions are difficult to quantify analytically.

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A. Open Challenges

Compared to conventional analysis, there are 2 aspects of complexity largely unconsidered in wireless literature: (1) from a network topology perspective, the non-regular structure of the load balancing network means that the number of connections (degree) of each node is distributed over a range as opposed to a single number for a regular lattice, and (2) from a dynamics perspective, not only is the load balancing dynamics at each BS node linked to the number of connections (degree), but also can take any form in this paper.

Recent breakthroughs have shown that there indeed can exist a relationship between local dynamical behaviour and global network structure by compressing the N-dimensional dynamics to a 1-dimensional average behaviour approximation [12]. However, their work examines the average effective behaviour of the whole network and an explicit relationship does not exist universally at the node level. This suffers from covering up discrepancies at the node level. Our own more recent work shows that sequential equilibrium substitution can reveal node level behavioural dynamics [13], but caveats exist in the application to network topology (e.g. low clustering coefficient).

B. Contribution

Our contribution is to show an exact analytical and also a probabilistic relationship for stability, relating generalized local load balancing dynamics with generalized network topology, as well as the uncertainty we have in the load balancing parameters due to noisy channel or network sensing. We prove that the stability analysis given is valid for any generalized load balancing dynamics and topological cell deployment and we believe this general relationship can inform the joint design of both the load balancing dynamics and the neighbour list of the network. The proposed probabilistic framework that links sensor accuracy with network dynamics provides uncertainty quantification and stability prediction for Digital Twins of wireless infrastructure.

II. System Model

A. Model Assumptions

Consider a geographic area covered by N BSs. There are two time scales: long term traffic variations (traffic variation time scale T, e.g. seconds), and short term load dynamics under some constant traffic demand (symbol period time scale t, e.g. milliseconds). We are primarily concerned with the latter time scale. Each BS i has a load defined by $l_i(t) = d_i(T)/c_i(t)$, the ratio between: (1) the quasi-static long-term traffic demand aggregated across all users u in cell i, $d_i(T) = \sum_u d_{i,u}(T)$; and (2) the BS aggregated area capacity over all users u in cell i, $c_i(t) = \sum_u c_{i,u}(t)$.

In this seminal paper on stability, we assume that the capacity of each cell is stationary, in that load balancing changes do not dramatically affect the cell capacity (i.e., different from inter-cell interference based load balancing optimisation [14]). This can be justified with frequency reuse patterns and coordinated inter-cell cooperation designed to eliminate inter-cell interference, both of which are actively researched and utilized technologies in hyper-dense scenarios [15]. We do not consider user-level experience in this initial paper, and rather focus on network level stability.

We are interested in the transient dynamics and stability of load balancing at time scale t, for a particular demand $d_i(T = T_1)$ - see Figure 1b. As such, we do not yet examine the user-level aspects of demand change, scheduling and propagation dynamics, nor user flow [16]. Suffice to say, we acknowledge that the wireless capacity depends on user location and PHY/MAC protocols, but for this seminal paper, we simply model (via differential equations) each BS's load dynamics as being under a certain random demand value and is able to deliver a certain capacity profile to meet the demand to the best of its ability.

B. Linear Example of Load Balancing Dynamics

Within the quasi-static traffic demand regime, the BS capacity $c_i(t)$ reacts to the demand $d(T = T_1)$ using adaptive modulation and coding (AMC) - see Figure 1b1. However, the mutual information of discrete modulation constellations will saturate [17], and therefore as the load exceeds 1, load balancing is necessary in order to avoid outage (see Figure 1b2).

As a demonstration example, we assume an ideal and simple linear load scaling between capacity and demand. In Section III, we show that our results hold for **any general dynamics**. To demonstrate, the load dynamics in cell i can be described by (see Figure 1c):

$$\dot{l}_i = f(l_i) = \beta(1 - l_i), \tag{1}$$

where a desirable equilibrium for *maximum service efficiency* is at $l_i = 1$ (fully loaded). When the BS load not at equilibrium, the parameter β controls how strong the load is being pushed away. When the BS is overloaded $l_i > 1$, the load will attempt to be reduced ($\dot{l}_i < 0$); and when it is underloaded $l_i < 1$, the load will attempt to be increased ($\dot{l}_i > 0$).

Each BS may have a list of adjacent BSs neighbours that it can share load with, and we can think of this virtual coupling of loads as through the a_{ji} connectivity matrix (e.g. a BS load sharing neighbour list). The dynamics of the offloading process can be described by the difference in the BSs' loads (see Figure 1d):

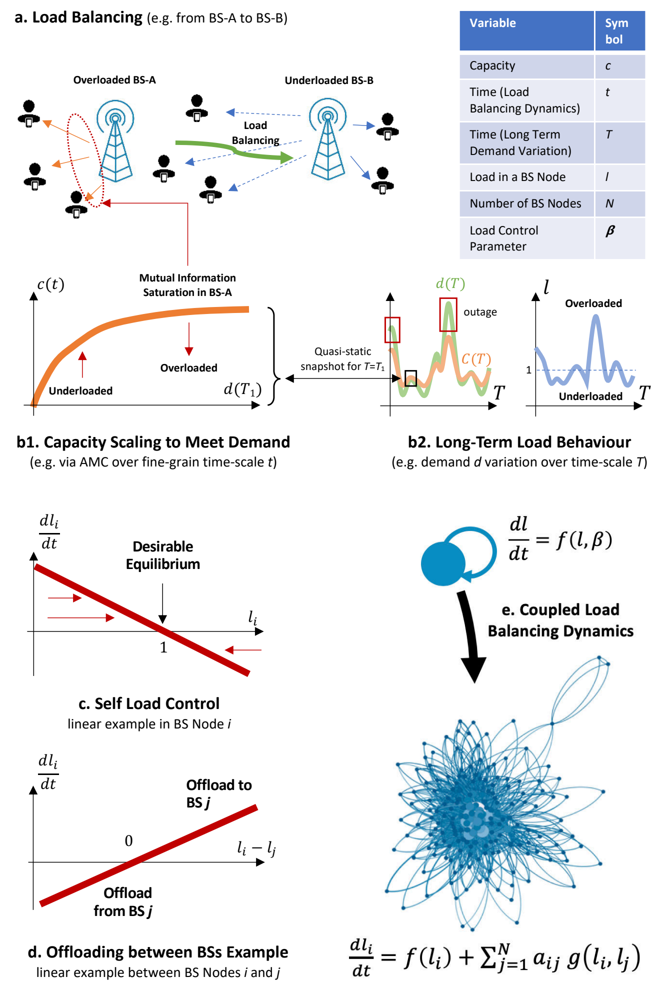

Fig. 1. **Wireless Network Load Balancing Dynamics: a) illustration of load balancing between frequency-reuse cells, b) capacity saturates for high loads (due to discrete modulation scheme) and drives load balancing requirement, c) an example of the load dynamic control for a single cell, d) an example of the load balancing action between two cells, and e) a complex network of load balancing between N cells.**

$$\dot{l}_i = g(l_i, l_j) = \gamma(l_j - l_i), \tag{2}$$

with an offloading rate γ.

The overall network load balancing dynamics is the linear combination of the local intra-node level load processing dynamics (Eq. (1)) and inter-node load balancing dynamics (Eq. (2)). They are coupled together via the load sharing network, described by the adjacency matrix a_{ji} (see Figure 1e):

$$\dot{l}_i = \beta(1 - l_i) + \sum_{j=1}^{N} a_{ji}\gamma(l_j - l_i), \tag{3}$$

where $a_{ji} = (A)_{ji}$. Here, we note that the dynamics due to cascades is N-dimensional, which makes direct prediction on stability challenging when N is large.

III. Stability Analysis of General Dynamics

Setting aside the linear dynamics example in Section II, here we look at the general case

$$\dot{l}_i = f(l_i) + \sum_{j=1}^{N} a_{ji} g(l_j - l_i), \tag{4}$$

where f, g are twice differentiable functions and $g(0) = 0$. In order to understand stability, we need to look at the linearization of the system. As such, we write $g(x) = \gamma x + O(x^2)$ as the Taylor expansion, where $O(\cdot)$ is the big O notation which bounds asymptotic behaviour at $x = 0$.

We denote $L = (l_1, \ldots, l_N)$ and we write equation (4) as

$$\dot{L} = F(L). \tag{5}$$

Let $\mathbf{1} = (1, \ldots, 1)$, then it is straightforward to check that if r is a root of f, i.e. $f(r) = 0$; then $r\mathbf{1}$ is an equilibrium of the dynamical system.

For any load balancing dynamics stated in Eq.(4), we know that there exists an equilibrium solution at $r\mathbf{1}$. For linear dynamics (see Section II), this is the **only** equilibrium solution. As such, we provide the analysis for stability around this equilibrium solution.

In order to determine the stability of the equilibrium we compute the eigenvalues of the Jacobian at the equilibrium. Let F_i be the i-th component of the function F of equation (4), then we have

$$\begin{aligned} \frac{\partial}{\partial l_i} F_i(L)\Big|_{L=r\mathbf{1}} &= f'(r) - \sum_{j=1}^{N} a_{ji} g'(0) \\ &= f'(r) - \gamma w_i. \end{aligned} \tag{6}$$

where we have defined $w_i = \sum_{j=1}^{N} a_{ji}$ and by the assumptions on g it holds that $g'(0) = \gamma$.

When $k \neq i$ we have

$$\frac{\partial}{\partial l_k} F_i(L)\Big|_{L=r\mathbf{1}} = \sum_{j=1}^{N} \delta_{jk} a_{ji} g'(0) = \gamma a_{ki},$$

where δ_{ki} is the Kronecker delta. This equation together with equation (6) shows that the Jacobian has the form

$$J(r\mathbf{1}) = f'(r)\mathrm{I} - \gamma D + \gamma A^T = f'(r)\mathrm{I} - \gamma \Lambda^T, \tag{7}$$

where I is the identity matrix, D is the weighted in-degree matrix and Λ the weighted in-Laplacian of the graph and Λ^T its transpose. Notice that the spectrum of $J(r\mathbf{1})$ is a spectral shift of the spectrum of $\gamma\Lambda^T$. Remember that Λ and Λ^T have the same spectrum.

A. Gershgorin Circle Theorem

For the Laplacian it is known that 0 is an eigenvalue and that all eigenvalues have non-negative real part. The first assertion is a direct implication of the relation $\Lambda^T \cdot \mathbf{1} = 0$. The second assertion a consequence of Gershgorin circle theorem, [18]. For each row of the matrix we construct the disc that has the diagonal element as centre and the sum of the absolute values of the remaining elements as radius, we call each of these discs a *Gershgorin disc*. Gershgorin's theorem states that each Gershgorin disc contains at least one eigenvalue of the matrix.

Because a matrix and its transpose have the same eigenvalues, we can do the same with the columns instead of the rows. In the case of the Laplacian matrix, since the sum of a row is zero and the diagonal elements are all non-negative, each disc has centre on the positive real axis and is tangent to the imaginary axis.

B. Stability Scenarios for Various Dynamics

Let μ_i denote the eigenvalues of $J(r\mathbf{1})$ and λ_i denote the eigenvalues of Λ, the relation between them is $\mu_i = f'(r) - \gamma\lambda_i$. The equilibrium $r\mathbf{1}$ is stable if $\mathrm{Re}(\mu_i) > 0$ for all i. Then from the discussion on the eigenvalues of Λ we deduce the following:

- **Default Load Balancing:** If $f'(r) < 0$ and $\gamma \geq 0$, then the equilibrium $r\mathbf{1}$ is asymptotically *stable*. This scenario is the default load balancing setup. As such, in this default case, the dynamics (e.g. $f(\cdot), \gamma$) only affect how resilient the stable system is to faults and how fast it approaches the equilibrium, but not the stability itself.
  Since the system (3) is linear and we know that the largest eigenvalue of the Jacobian is $-\beta$, we know that regardless the initial condition system approaches the equilibrium with rate $e^{-\beta t}$.
- If $f'(r) < 0$ and $\gamma < 0$, then the equilibrium $r\mathbf{1}$ is asymptotically *stable* if $|f'(r)| > |\gamma|\,\rho$ and asymptotically unstable if $|f'(r)| < |\gamma|\,\rho$, where $\rho = \max\{\mathrm{Re}(\lambda_i)\}$. This is appropriate for *sleep mode* operations, where BS nodes with a lighter load tend to be switched off and their load transferred to neighbouring heavy loaded BS nodes $\gamma < 0$. This is in order to conserve energy [6].
- If $f'(r) = 0$ and $\gamma < 0$, then the equilibrium $r\mathbf{1}$ is asymptotically *unstable*. This scenario is similar to the above *sleep mode* case.
- If $f'(r) = 0$ and $\gamma \geq 0$, then we *cannot determine the stability* of the equilibrium $r\mathbf{1}$ just by looking at the eigenvalues of the Jacobian. In *multi-hop routing*, one is generally not motivated to push demand away based on the demand itself, but by other motivations (e.g. need to move information from one geographic area to another). Therefore, the action is entirely inter-node based ($f'(r) = 0$).
- If $f'(r) > 0$, then the equilibrium $r\mathbf{1}$ is asymptotically *unstable*. Here, the BS attempts to attract load when it is over-loaded and remove load when under-loaded, which is against the purpose of serving customer demand. As such, this scenario is not applicable to most telecommunication dynamics.

For the purpose of load balancing as described in this paper, we are only interested in the case of $f'(r) < 0$ and $\gamma > 0$, which when referring to the system considered in Eq.(3), it implies that the equilibrium is always asymptotically stable.

IV. Probabilistic Uncertainty

In realistic networks, each BS can have different load balancing mechanisms. The general models provided here are a baseline for Digital Twins [19] and sensors at the channel and network level inform the actual performance of individual BSs. Given the challenges in wireless sensing and flaws in sensors, measurement of the parameters and load flow are subject to measurement noise and can affect stability [10].

We investigate this by adding a random variable to the load scaling gradient β and load balancing rate γ (see Eq.(3)). As such, the dynamical system is $\dot{l}_i = (\beta + \zeta_i)(1 - l_i) + \sum_{j=1}^N a_{ji}(\gamma + \xi_{ji})(l_j - l_i)$, where ζ_i and ξ_{ji} are random variables of known distribution.

The entries of the Jacobian matrix are

$$(J)_{ii} = -\beta - \zeta_i - \sum_{j=1}^N a_{ji}(\gamma + \xi_{ji})$$
$$(J)_{ij} = a_{ji}\gamma + a_{ji}\xi_{ji}.$$

We define $\theta_i = \sum_{j=1}^N a_{ji}\xi_{ji}$, then the Jacobian becomes

$$J = -\beta\,\mathrm{I} - \gamma\,\Lambda^T - Z - \Theta - \Xi,$$

with $Z = \mathrm{diag}(\zeta_1, \ldots, \zeta_N)$, $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_N)$ and $(\Xi)_{ij} = a_{ji}\xi_{ji}$.

Just like before, the system is stable if and only if all the eigenvalues of J have negative real part. We use the Gershgorin circle theorem to get a bound on that probability. Let

$$s_i = -\beta - \zeta_i + \sum_{j=1}^N a_{ji}(|\gamma + \xi_{ji}| - \gamma - \xi_{ji}),$$

then if for all i, $s_i < 0$ then the system is stable.

In this case, the *stability is probabilistic*, i.e., the probability that the system is stable is bounded from below by $\prod_{i=1}^N \mathbb{P}(s_i > 0)$. The exact form of this bound will depend on the distribution of the random variables ζ_i and ξ_{ji}.

As an example we will look at the case of Default Load Balancing where the we know the values of β and γ up to a uniform measurement error. Let $\beta > 0$, $\gamma > 0$, $\zeta_i \sim \mathrm{Uniform}[-b, b]$ and $\xi_{ji} \sim \mathrm{Uniform}[-c, c]$, with $c > \gamma$. Let $X_i = |\gamma + \xi_{ji}| - \gamma - \xi_{ji}$ Since $\mathbb{P}(\gamma + \xi_{ji} < 0) = \frac{1}{2}(1 - \gamma/c)$, the random variables X_i take the values

$$X_i \sim \begin{cases} 0, & \text{with probability } \frac{1}{2}(1 + \gamma/c) \\ \mathrm{Uniform}[0, 2(c-\gamma)], & \text{with probability } \frac{1}{2}(1 - \gamma/c). \end{cases}$$

We denote $Y = \sum_{i=1}^N X_i$. The random variable Y is the sum of n uniform random variables with probability $\binom{N}{n}(\frac{1}{2}(1-\gamma/c))^n(\frac{1}{2}(1+\gamma/c))^{N-n}$. The sum of independent uniform random variables follows the Irwin-Hall distribution. This implies that the PDF of Y is

$$f_Y(x) = \frac{1}{2^N} \sum_{n=1}^N \binom{N}{n} (1 - \gamma/c)^n (1 + \gamma/c)^{N-n} \times \frac{1}{(n-1)!} \sum_{k=0}^{\left\lfloor \frac{x}{2(c-\gamma)} \right\rfloor} (-1)^k \binom{n}{k} \left(\frac{x}{2(c-\gamma)} - k \right)^{n-1}.$$

We can now write $s_i = -\beta - \zeta_i + Y_i$ and ask what is the probability that s_i is positive. The random variable $-\beta - \zeta_i$ is uniform on $[-\beta - b, -\beta + b]$ and PDF of the sum of two random variables is the convolution of the PDFs, we get

$$\mathbb{P}(s_i < 0) = \frac{1}{2b} \int_{-\infty}^0 \int_{x+\beta-b}^{x+\beta+b} f_Y(s)\, ds\, dx.$$

This integral cannot be written in a simple form, but it can easily be evaluated numerically. Finally, the probability that every s_i is negative is $(\mathbb{P}(s_i < 0))^n$, which bounds from below the probability that the system is stable.

Note that we looked at the case where $c > \gamma$, which corresponds to a measurement error that is of the same order of magnitude as the measurement itself. If we assume that the error is smaller, i.e. $c \leq \gamma$ then the $\gamma + \xi_{ji}$ cannot be negative, which implies that the system is always stable.

V. Capacity Stability Analysis

A smooth invertible transformation of a dynamical system does not change the stability of its equilibria. In particular let $\phi_i : \mathbb{R} \to \mathbb{R}$ be invertible, twice continuously differentiable functions and let us define $c_i = \phi_i(l_i)$. The dynamical system for c_i's is given by the equations using chain rule $\dot{c}_i = \phi_i'(l_i)\dot{l}_i$:

$$\dot{c}_i = \phi_i'\big(\phi_i^{-1}(c_i)\big) \times \left(f\big(\phi_i^{-1}(c_i)\big) + \sum_{j=1}^N a_{ji} g\big(\phi_j^{-1}(c_j) - \phi_i^{-1}(c_i)\big) \right). \tag{8}$$

From the previous discussion we know that the point $(\phi_1(r), \ldots, \phi_N(r))$ is an equilibrium of the system (8) that corresponds to the equilibrium $r\mathbf{1}$ of the system (4). The stability of this equilibrium is the same as the stability of $r\mathbf{1}$.

Now we can apply the previous analysis in the case of of load-balancing of BSs, governed by the dynamical system (3). Notice that this system is linear, which implies that there is only one equilibrium and by the previous analysis we know that when $\gamma,\ \beta > 0$, this equilibrium is asymptotically stable.

We assume that $\phi_i(l_i) = d_i/l_i$. This implies that $\phi_i^{-1}(c_i) = d_i/c_i$ and $\phi_i'(l_i) = -d_i/l_i^2$. Then the system (8) becomes

$$\begin{aligned} \dot{c}_i &= -\frac{c_i^2}{d_i} \left(\beta \left(1 - \frac{d_i}{c_i} \right) + \sum_{j=1}^N a_{ji}\gamma \left(\frac{d_j}{c_j} - \frac{d_i}{c_i} \right) \right) \\ &= \beta c_i \left(1 - \frac{c_i}{d_i} \right) + \sum_{j=1}^N \gamma\, a_{ji} c_i \left(1 - \frac{c_i d_j}{c_j d_i} \right). \end{aligned} \tag{9}$$

At first glance it seems that the above equation implies that the self-dynamics of a BS is given by $f(c_i) = \beta c_i(1 - c_i/d_i)$

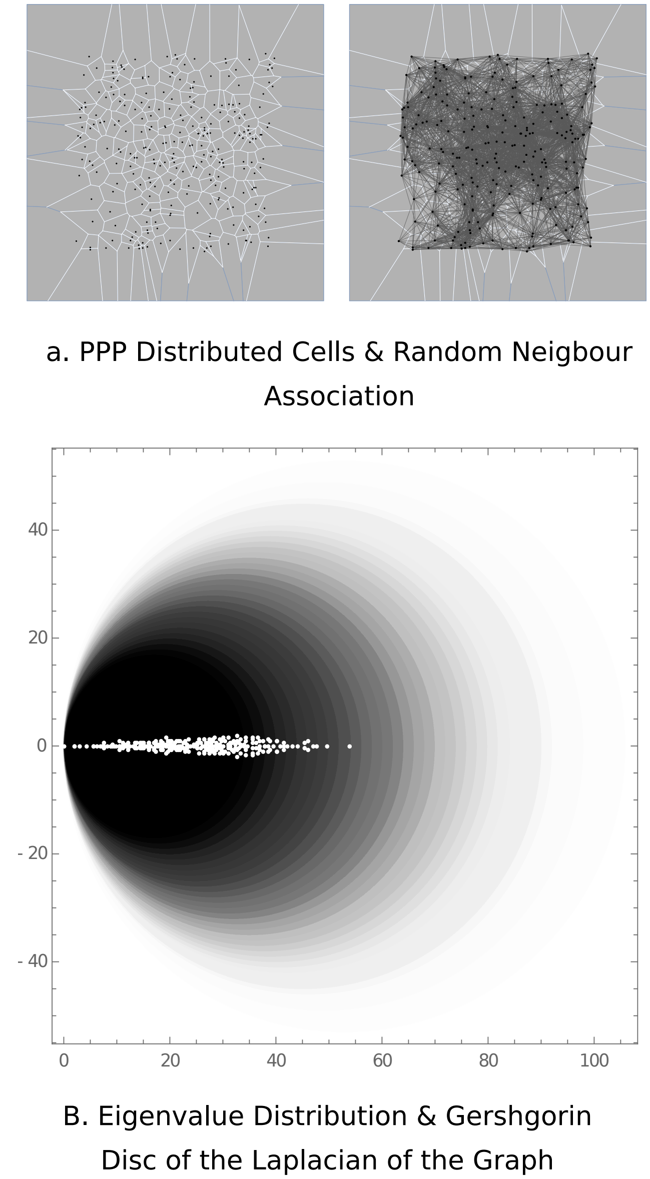

Fig. 2. **Distribution of Eigenvalues for PPP Distributed Network.**

and it has two equilibria, d_i which is stable and 0 which is unstable. The equilibrium $(d_i, \dots, d_N)$ corresponds to the stable equilibrium of the system (3) and from this we deduce that it is not just asymptotically stable but also a global attractor of the system. Moreover, we get that it **textbf**approaches the equilibrium with the same speed, i.e. $e^{-\lambda t}$. The equilibrium 0, however, is not an admittable one because it appears also as a denominator and in this case the right-hand side of (9) cannot be evaluated. In a sense the 0 "equilibrium" of the system (9) corresponds to infinity in the system (3).

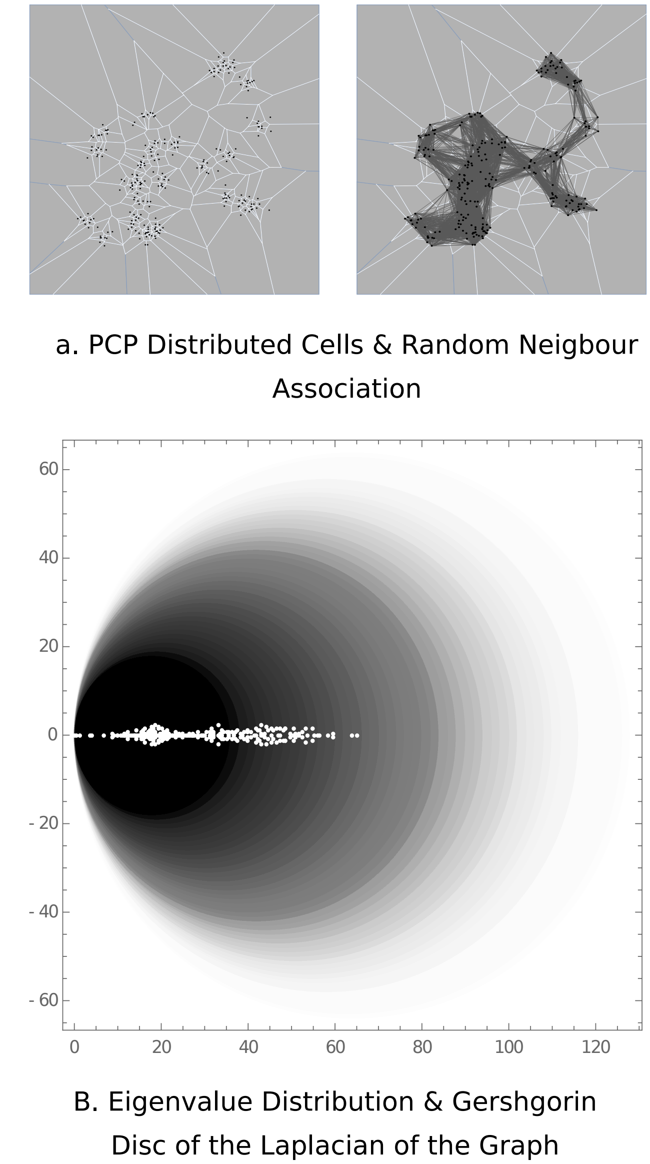

Fig. 3. **Distribution of Eigenvalues for PCP Distributed Network.**

VI. Results

We present results for differing Poisson Point Process (PPP) and Poisson Cluster Process (PCP) generated random complex networks [20], where nodes are omni-directional BS sites and links are offloading relations. We connect the nodes in accordance to a random network, whereby a percolation control parameter R and a probability of connecting P is used to determine if adjacent BSs can offload to each other. Traffic and capacity values are not needed because we know from earlier that the system is always stable and the load dynamics only affect the speed of convergence to the equilibrium and resilience to faults, but not its asymptotic stability. This is an important insight.

Fig.2a show the PPP Voronoi plots along with BS load balancing network. The results in Fig.2b demonstrate that, as we expected, the eigenvalues of the Laplacian are all in the positive real half-plane. Therefore the whole load balancing network is always stable in this case. Fig.3a show the PCP Voronoi plots along with BS load balancing network. The results in Fig.3b demonstrate that, as we expected, the eigenvalues of the Laplacian are all in the positive real half-plane. Therefore the whole load balancing network is always stable in this case.

VII. Conclusion & Future Work

In this paper, we show the mathematical stability criteria that links the generalized load balancing dynamics ($f(\cdot), \gamma$) with the maximum eigenvalue of the weighted in-Laplacian of the adjacency matrix (ρ). We prove that default load balancing networks are always asymptotically stable, irrespective network topology and the balancing dynamics (linear or otherwise). However, we observe that for other forms of balancing actions, the stability is not ensured. We also present the probabilistic stability in the face of heterogeneous uncertainty among the load balancing actions. We showed that given uncertainty in the load balancing actions, as long as the system measurement accuracy is better the underlying noise process, the system is stable. The proposed probabilistic framework that links sensor accuracy with network dynamics provides uncertainty quantification and stability prediction for Digital Twins of wireless infrastructure.

We believe this general relationship can inform the joint design of both the base station (BS) dynamics and the BS interaction network. Whilst this seminal work on stability analysis considered a frequency re-use network with no interference, future work will consider the effects of interference, sleep mode, user entry/exit demand dynamics [16], and their influence on stability.

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