Universitat Politècnica de Catalunya
Departament de Matemàtica Aplicada I

PhD Thesis

Singular phenomena in the length spectrum of analytic convex curves

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Barcelona, May 2015

Memòria presentada per aspirar el grau de Doctor en Matemàtiques per la Univeristat Politècnica de Catalunya. Programa de doctorat de Matemàtica Aplicada.

A l’Oslo i a l’Adrià, que han mastegat i s’han empassat el meu dia a dia.

Agraïments / Acknowledgements

A l'Uri.

Al Rafa i al Pau.

A la resta del Dinar del Mes. Als Satanets.

A la gent del MA1 de la Sala del Cafè; molt especialment a la Marta, la Gemma, el Jordi, l'Amadeu, el Tomás i la Tere.

Als meus pares.

A la Montse i a la Lidia. I a la Carti.

Als avis de Masrampinyo i Bifurca.

I a tothom, que aquesta secció l'he escrita sent molt feliç ja!

Thanks to Vassily Gelfreich, Robert MacKay, Laurent Niederman, Sônia Pinto-de-Carvalho, Tere M. Seara, and Carles Simó for very useful remarks and comments. Thanks to Pablo Roldán and specially to Albert Granados in the use of the UPC Applied Math cluster for our experiments.

I was supported by the PhD grant FPU-UPC from February 2010 to September 2010. I was supported by the PhD grant BES-2010-031460 from the project MTM2009-06973 from October 2010 to September 2014.

Abstract

Let Q be an analytic closed strictly convex curve and consider the billiard map defined inside Q. Given $q \geq 3$ and $0 < p \leq q$ relatively prime integers, there exist at least two (p, q)-periodic trajectories inside Q. The main goal of this thesis is to study the maximal difference of lengths among (p, q)-periodic trajectories on the billiard, $\Delta^{(p,q)}$.

The quantity $\Delta^{(p,q)}$ gives some dynamical and geometrical information. First, it characterizes part of the length spectrum of the curve Q and, by doing so, relates to Kac's question, "Can one hear the shape of a drum?". Second, $\Delta^{(p,q)}$ is an upper bound of Mather's $\Delta W_{p/q}$ and so it quantifies the chaotic dynamics of the billiard table.

We first focus on the study of the maximal difference of lengths among $(1, q)$-periodic orbits. The $(1, q)$-periodic orbits are orbits that approach the boundary of the billiard table as q tends to infinity. The study of $\Delta^{(1,q)}$ is twofold: an analytic upper bound and some numerically obtained asymptotic formulas.

On the one hand, we obtain an exponentially small upper bound in the period q for $\Delta^{(1,q)}$. The result is obtained on the more general framework of the maximal difference of (p, q)-periodic actions among (p, q)-periodic orbits on analytic area-preserving twist maps.

Precisely, we are able to establish an exponentially small upper bound for the differences of (p, q)-periodic actions when the map is analytic on a (m, n)-resonant rotational invariant curve (resonant RIC) and p/q is "sufficiently close" to m/n. The exponent in this upper bound is closely related to the analyticity strip width of a suitable angular variable. The result is obtained in two steps. First, we prove a Neishtadt-like theorem, in which the n-th power of the twist map is written as an integrable twist map plus an exponentially small remainder on the distance to the RIC. Second, we apply the MacKay-Meiss-Percival action principle.

This result implies that the lengths of all the $(1, q)$-periodic billiard trajectories inside analytic strictly convex domains are exponentially close in the period q. This improves the classical result of Marvizi and Melrose about the smooth case. But it also has several other applications in both classical and dual billiards which we also describe. For instance, we also show that the areas of all the $(1, q)$-periodic dual billiard trajectories outside analytic strictly convex domains are exponentially close in the period q, which also improves a classical result about the smooth case stated by Tabachnikov.

On the other hand, we discuss some exponentially small asymptotic formulas for $\Delta^{(1,q)}$

when the billiard table is a generic axisymmetric analytic strictly convex curve. In this context, we conjecture that the differences behave asymptotically like an exponentially small factor $q^{-3}\mathrm{e}^{-rq}$ times either a constant or an oscillating function. Besides, the exponent r is half of the radius of convergence of the Borel transform of the well-known asymptotic series for the lengths of the $(1,q)$-periodic trajectories. This conjecture is strongly supported by numerical experiments. Our computations require a multiple-precision arithmetic and have been programmed in PARI/GP.

The experiments are restricted to some perturbed ellipses and circles, which allow us to compare the numerical results with some analytical predictions obtained by Melnikov methods and also to detect some non-generic behaviors due to the presence of extra symmetries.

The asymptotic formulas we obtain resemble the ones obtained for the splitting of separatrices on many analytic maps, where the behavior of the splitting size is of order $h^{-m}\mathrm{e}^{-r/h}$. In such cases, the parameter $h > 0$ is small and continuous whereas here q is large and discrete. This is the reason the formulas are exponentially small in $1/h$ and q respectively. The exponent r in the former case has been proved to be (or is strongly numerically supported, depending on the map studied) 2π times the distance to the real axis of the set of complex singularities of the homoclinic solution of a limit Hamiltonian flow. We discuss this analogy and propose and study an equivalent limit problem in the billiard setting.

Next, we give some insight on how $\Delta^{(p,q)}$ behaves when the (p,q)-periodic orbits do not tend to the boundary of the billiard table but to other regions of the phase space. Namely, we consider the cases of (p,q)-periodic orbits such that $p/q \to \vartheta \in (0,1) \setminus \mathbb{Q}$ and (p,q)-periodic orbits approaching to a (P,Q)-resonance. The study of $\Delta^{(p,q)}$ is not as detailed as in the previous chapters. It consists of a phenomenological study based on some numerical results.

*K*eywords: billiards, length spectrum, exponentially small phenomena, dual billiards, twist maps, numerical experiments, high-precision computations

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1 Introduction

Billiards as a dynamical system go back to Birkhoff [Bir66]. Let Q be a closed smooth strictly convex curve in the Euclidean plane. Its Birkhoff billiard models the motion of a particle inside the region enclosed by Q. The particle moves with unit velocity and without friction following a straight line; it reflects elastically when it hits Q. Therefore, billiard trajectories consist of polygonal lines inscribed in Q whose consecutive sides obey to the rule "the angle of reflection is equal to the angle of incidence." Such trajectories are sometimes called broken geodesics. See [KT91, KH95, Tab95a] for a general description.

Let $0 < p < q$ be relatively prime integers. A (p,q)*-periodic billiard trajectory* forms a closed polygon with q sides that makes p turns inside Q. The period of this orbit is q whereas p is the number of turns. Birkhoff [Bir66] stated the existence of at least two geometrically different of such (p,q)-periodic trajectories.

The *length spectrum* of Q is the subset of $\mathbb{R}_+$ defined as

$$\mathcal{LS}(Q) = l\mathbb{N} \cup \bigcup_{(p,q)} \Lambda^{(p,q)}\mathbb{N},$$

where $l = Length(Q)$ and $\Lambda^{(p,q)} \subset \mathbb{R}_+$ is the set of the lengths of all (p,q)-periodic billiard trajectories inside Q. The *maximal* (p,q)*-periodic length difference* is the non-negative quantity

$$\Delta^{(p,q)} = \sup \Lambda^{(p,q)} - \inf \Lambda^{(p,q)}.$$

Many geometric and dynamical properties are encoded in the length spectrum $\mathcal{LS}(Q)$ and the differences $\Delta^{(p,q)}$.

Kac [Kac66] formulated the inverse spectral problem for planar domains. By the suggesting question *Can one hear the shape of a drum?*, he wanted to study how much geometric information about the domain enclosed by Q can be obtained from the Laplacian spectrum with homogeneous Dirichlet conditions on Q. Andersson and Melrose [AM77] gave an explicit relation between the length spectrum and the Laplacian spectrum so that the previous question is restated as: *Does the set $\mathcal{LS}(Q)$ allow one to reconstruct the convex curve Q?* We refer to the book [Sib04] for some results on this question.

The difference $\Delta^{(p,q)}$ is important from a dynamical point of view, because it is an upper bound of Mather's $\Delta W_{p/q}$, which, in its turn, is equal to the flux through the (p,q)-resonance of the corresponding billiard map [MMP84, Mat86, Mei92, MF94]. Thus, the variation of $\Delta^{(p,q)}$ in terms of the rotation number $p/q \in (0,1)$ gives information about the size of the different chaotic zones of the billiard map. See Section 2.2 for a more complete description of these ideas.

In this thesis, our main goal is to gain some insight into the billiard dynamics by determining the asymptotic behavior of the maximal differences of lengths $\Delta^{(p,q)}$ as q tends to infinity in some specific contexts. The quantities $\Delta^{(p,q)}$ were already studied by Marvizi and Melrose [MM82] and Colin de Verdière [Col84] for smooth tables.

Marvizi and Melrose produced an asymptotic expansion of the length of any (p,q)-periodic billiard trajectories approaching the boundary Q when $q \to +\infty$. Let $L^{(p,q)} \in \Lambda^{(p,q)}$ be the length of some (p,q)-periodic billiard trajectory inside Q. It does not matter which one. Fix p and let Q be smooth and strictly convex. Then, there exist some asymptotic coefficients $l_j = l_j(p,Q)$ such that

$$L^{(p,q)} \asymp \sum_{j\geq 0} l_j q^{-2j}, \qquad q \to \infty. \tag{1.1}$$

The symbol $\asymp$ means that the series in the right hand side is asymptotic to $L^{(p,q)}$. The asymptotic coefficients l_j can be explicitly written as integrals over Q of suitable algebraic expressions of κ and its derivatives. For instance, $l_0 = pl = p\,Length(Q)$ and $l_1 = -\left(p\int_Q \kappa^{2/3}\mathrm{d}s\right)^3/24$, where κ is the curvature of Q and $\mathrm{d}s$ is its length element. The explicit formulas for the first five coefficients can be found in [Sor15]. Since the asymptotic series (1.1) coincides for any (p,q)-periodic trajectory,

$$\lim_{q\to\infty} q^k \Delta^{(p,q)} = 0, \quad \forall k > 0$$

for smooth strictly convex tables when p is fixed and $q \to +\infty$. That is, the differences $\Delta^{(p,q)}$ are *beyond all order in* q.

Colin de Verdière studied the lengths of periodic trajectories close to an elliptic $(1,2)$-periodic trajectory on a smooth symmetric billiard table, and found that the quantities $\Delta^{(p,q)}$ are again beyond all order with respect to q.

These works suggest that the maximal length differences $\Delta^{(p,q)}$ are exponentially small in the period q for analytic strictly convex tables in some specific contexts. In the same setting than the one of Marvizi and Melrose, once added the hypothesis of analyticity to the billiard curve, we are able to find an exponentially small upper bound for (p,q)-periodic trajectories approaching the boundary. Indeed, if Q is analytic and p is a fixed positive integer, then there exist $K, q_*, \alpha > 0$ such that

$$\Delta^{(p,q)} \leq K\mathrm{e}^{-2\pi\alpha q/p}, \tag{1.2}$$

for all integer $q \geq q_*$ relatively prime with p. The exponent α is related to the width of a complex strip where a certain 1-periodic angular coordinate is analytic. A more precise statement is given in Theorem 6.

This result is a direct application of a more general one. Billiard maps are exact twist maps. Exact twist maps are defined on an open cylinder, satisfy a Lagrangian formulation and have at least two different (p, q)-periodic orbits for any relatively prime integers p and q such that p/q belongs to the twist interval of the map. See [Bir66, Mei92, KH95] for references. In the general framework of exact twist maps, the maximal difference of lengths of (p, q)-periodic trajectories on the billiard table should be substituted by the maximal difference of (p, q)-periodic actions of the (p, q)-periodic orbits. Let us denote this difference by $\Delta^{(p,q)}$ too.

Let f be an analytic exact twist map. Assume the map has a (m, n)-resonant rotational invariant curve. We obtain an exponentially small upper bound for $\Delta^{(p,q)}$ when its rotation number, p/q, is "sufficiently close" to the one of the invariant curve, m/n. In a more accurate way, given $L \geq 1$, we prove that there exist $K, \alpha, q_* > 0$ such that

$$\Delta^{(p,q)} \leq K \exp\left(-\frac{2\pi\alpha q}{|np - mq|}\right), \tag{1.3}$$

for any relatively prime integers p and q such that $1 \leq |np - mq| \leq L$ and $q \geq q_*$. The exponent α is closely related to the analyticity strip width of a suitable angular variable. A more precise statement is given in Theorem 4.

The proof mainly consists of two steps. The first one is to write the n-th power of the twist map as an integral part plus an exponentially small residue. The arguments we use go back to Neishtadt [Nei81]. As for the second one, we use the MacKay-Meiss-Percival principle [MMP84] which relates $\Delta^{(p,q)}$ with the flux through the (p, q)-periodic resonance.

MacKay [Mac92] (respectively, Delshams and de la Llave [DdlL00]) already proved the existence of exponentially small upper bounds for the Greene residue [Gre79] of (p, q)-periodic orbits when they approach a rotational invariant curve of Diophantine rotation number on analytic area-preserving twist (respectively, non-twist) maps. Their result can be adapted to similar exponentially small upper bounds for the quantity $\Delta^{(p,q)}$ we are studying. However, note that we consider (p, q)-periodic trajectories that approach an invariant curve with a rational rotation number (instead of a Diophantine one).

The general upper bound (1.3) on analytic exact twist maps we obtain gives rise to other exponentially small upper bounds in different billiard contexts apart from the one where the periodic trajectories approach the boundary of the billiard and have the bound (1.2). For instance, for (p, q)-periodic billiard trajectories inside analytic strictly convex tables of constant width when $p/q \to 1/2$. Also, for (p, q)-periodic billiard trajectories inside analytic strictly convex tables in surfaces of constant curvature when $p/q \to 0$. The particular upper bound on such cases can be found in Section 3.3.

The general upper bound (1.3) adapts to different contexts on the dual billiard problem as well. In order to keep this introduction focused on the key issues, let us omit a further explanation on dual billiards and the explicit exponentially small upper bounds on some contexts there, which can be found in Section 3.4.

All the upper bounds presented here can be found in the preprint [MRT14]. With respect to this thesis, they are exposed and proved in Chapter 3.

A similar exponentially small upper bound was obtained in [FS90] in the setting of the splitting of separatrices of weakly hyperbolic fixed points of analytic area-preserving maps. The upper bounds for the separatrix splitting are valid under quite general assumptions about the properties of the map. The natural question is to find asymptotic formulas as well. However, finding lower bounds is more complicated so that exponentially small asymptotic formulas for the splitting of separatrices are "known" for a number of specific families of special form (known must be read as proved in some cases and known must be read as numerically checked in others). There are many references on the splitting of separatrices of analytic maps [GLT91, DR98, DR99, Gel99, GS01, Ram05, GS08, MSS11a, MSS11b, BM12]. The survey [GL01] is a good starting point.

Let us briefly recall the properties of the asymptotic formulas obtained in the context of the splitting of separatrices on analytic maps:

1. These splittings are exponentially small in a continuous small parameter $h > 0$, which is the characteristic exponent of the hyperbolic fixed point whose separatrices split.

2. In many analytic maps, the splitting size is asymptotically equal to $A(1/h)h^{-m}\mathrm{e}^{-r/h}$ as $h \to 0^+$ for an exponent $r > 0$, a power $m \in \mathbb{R}$, and a function $A(1/h)$ which is either a constant or oscillatory.

3. $r = 2\pi\delta$, where δ is the distance to the real axis of the complex singularities of the homoclinic solution of a limit Hamiltonian flow related to the analytic map.

4. $r = \rho/2$, where ρ is the radius of convergence of the Borel transform of the divergent series that approach the perturbed separatrices. See, for instance, [DR99, GS01, Ram05, GS08].

5. In the case of polynomial standard maps contained in [GLT91, GS08], the function $A(1/h)$ seems to behave as

$$A(1/h) = \mu a/2 + a\sum_{j=1}^{J}\cos(2\pi\beta_j/h + \varphi_j), \tag{1.4}$$

for $\mu \in \{0,1\}$, an amplitude $a \neq 0$, and some phases $\varphi_j \in \mathbb{R}$, when the complex singularities of the homoclinic solution of the Hamiltonian limit flow closest to the real axis are $\pm\delta\mathrm{i}$ (if and only if $\mu = 1$), $\pm\beta_1 \pm \delta\mathrm{i}, \ldots, \pm\beta_J \pm \delta\mathrm{i}$. This claim is based on numerical experiments.

Our next goal is to find such asymptotic formulas on our billiard setting. Let us focus on the case that the (p,q)-periodic orbits approach the boundary. Also, we set $p = 1$ for simplicity. The other cases are partially discussed in Chapter 5.

In this thesis, we present several numerical experiments and some analytical results performed on the model tables

$$Q = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2/b^2 + \epsilon y^n = 1\right\}. \tag{1.5}$$

Here, $b \in (0,1]$ is the semi-minor axis, $0 \neq \epsilon \in \mathbb{R}$ is the perturbative parameter, and $n \in \mathbb{N}$, with $3 \leq n \leq 8$, is the degree of the perturbation. We say that Q is a *perturbed ellipse* when $0 < b < 1$ and is a *perturbed circle* when $b = 1$. The choice of these model tables relies on the following comments.

First, all the billiard tables (1.5) are *nonintegrable* for $n \geq 3$ and $0 < \epsilon \ll 1$, and so the dynamics inside them should be far from trivial. Birkhoff conjectured that only ellipses and circles are integrable smooth convex billiard tables [Por50]. The question of which perturbed ellipses (resp., circles) give rise to integrable billiards is addressed in [DR96] (resp., [ASK14]).

Second, the tables (1.5) allow us to use some *Melnikov methods* that already appeared in [Ram06, PR13]. In fact, since $\Delta^{(1,q)} = 0$ for any $q \geq 3$ when $\epsilon = 0$ (due to integrability of elliptic billiards) and the difference $\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon)$ is analytic in ϵ, we can write

$$\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon) = \epsilon\Delta_1^{(1,q)} + \mathrm{O}(\epsilon^2),$$

for some coefficient $\Delta_1^{(1,q)} \in \mathbb{R}$. We are able to explicitly compute $\Delta_1^{(1,q)}$ and find that, if $0 < b < 1$, then

$$\Delta_1^{(1,q)} \asymp M_n q^{m_n} \mathrm{e}^{-cq}, \qquad q \to +\infty,$$

for some Melnikov exponent $c > 0$ not depending on n, some Melnikov power $m_n \in \mathbb{Z}$, and some Melnikov constant $M_n \neq 0$. These three Melnikov quantities can be explicitly computed for all $n \in \mathbb{N}$. However, we have only done so for the cubic ($n = 3$) and quartic ($n = 4$) perturbations for the sake of brevity. Besides, $\lim_{b \to 1} c = +\infty$ (which makes it more difficult to work with perturbed circles). The Melnikov method provides no information when n is odd and q even; $\Delta^{(1,q)} = 0$ in such case. See Proposition 22 for details. We see that the Melnikov method fails to predict the singular behavior of $\Delta^{(1,q)}$ as it also happens for the standard maps and most of its generalizations.

Third, the existence of *symmetries* on the tables (1.5) makes computing the periodic trajectories simpler. Note that Q is symmetric with respect the y-axis. We say that it is *axisymmetric*. Numerically, we find two different $(1,q)$-periodic which are axisymmetric and compute the signed difference D_q between them. However, there are many cases where we can prove that $|D_q| = \Delta^{(1,q)}$. See Proposition 22. Also, when n is even, there is an extra symmetry (with respect to the x-axis) and we say Q is *bi-axisymmetric*. This extra symmetry modifies the asymptotic behavior of $\Delta^{(1,q)}$.

Fourth and last, such simple billiard tables allow us to reduce the *computational effort*. In particular, we limit the perturbation degree to $n \leq 8$. Besides, since we deal with exponentially small behaviors, we need a multiple-precision arithmetic to compute D_q. We perform the computations on the open source PARI/GP system [BBB+06] with sometimes more than 12000 digits of precision. Similar computations in the setting of splitting of separatrices of analytic maps can be found in [DR99, Ram05, GS08].

Both by looking at our billiard problem from the perspective of the results about the splitting of separatrices stated before and by the numerical results obtained on the model tables (1.5), we have conjectured some properties on the asymptotic formulas for $\Delta^{(1,q)}$ on billiard tables defined by closed analytic strictly convex curve. We have also added the hypothesis that Q is a generic axisymmetric algebraic curve. On the one hand, we do so because our model tables are so. On the other hand, this hypothesis allows us the comparison with the results of the splitting of separatrices on polynomial standard maps.

In order to make the analogy with the case of splitting of separatrices in analytic maps clear, we list them in the same order on what follows. Conjecture 21 gathers this same information in a more precise way.

1. The maximal differences $\Delta^{(1,q)}$ are exponentially small in the period, q, which is a discrete big parameter $q \geq 3$.

2. If Q is a generic axisymmetric algebraic curve, then $\Delta^{(1,q)} \asymp A(q)q^{-3}\mathrm{e}^{-rq}$, as $q \to \infty$, for some exponent $r > 0$ and some function $A(q)$, which is constant or oscillatory. If Q is also bi-axisymmetric, the singular behavior is slightly different when q is odd and we obtain $\Delta^{(1,q)} \asymp B(q)q^{-2}\mathrm{e}^{-2rq}$, as $q \to \infty$, for the same r of the case q even and for some constant or oscillatory function $B(q)$.

3. We try to find a limit problem on the billiard setting comparable to the limit Hamiltonian flow for the splitting of separatrices on analytic maps in such a way that the exponent $r = 2\pi\delta$, where δ is the distance to the real axis of some complex singularities on the limit problem. We do not have a completely satisfactory answer yet, but we do have a candidate. Let $\kappa(s)$ be the curvature of Q in some arc-length parameter $s \in \mathbb{R}/l\mathbb{Z}$. Let $\xi \in \mathbb{R}/\mathbb{Z}$ be a new angular variable defined by

   $$C\frac{\mathrm{d}\xi}{\mathrm{d}s} = \kappa^{2/3}(s), \qquad C = \int_Q \kappa^{2/3}\mathrm{d}s.$$

   We define δ as the distance of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis. This choice relies on the Taylor expansions of the billiard map close to the boundary that Lazutkin wrote in [Laz73]. A more precise exposition on the reasons for this choice of δ can be found in Section 4.2.

   Our candidate δ is such that $r \leq 2\pi\delta$. However, the following reasons seem to reinforce our choice. First, recall that the exponent α in (1.2) is related to the width of a complex strip where a certain 1-periodic angular coordinate is analytic. This analyticity width is closely related to our choice of δ (see Subsection 3.3.2). Second, the Melnikov exponent is $c = 2\pi\delta$ on the ellipse (see Proposition 24). Third, on the perturbed circle (1.5) with $b = 1$, there exist some constants $\chi_n, \eta_n \in \mathbb{R}$, $\chi_n \leq \eta_n$, such that

   $$r = \frac{|\log \epsilon|}{n} + \chi_n + \mathrm{o}(1), \qquad 2\pi\delta = \frac{|\log \epsilon|}{n} + \eta_n + \mathrm{o}(1),$$

as $\epsilon \to 0^+$ (see Section 4.5). Thus, our candidate exactly captures the logarithmic growth of the exponent r for perturbed circles. Fourth and last, we numerically obtain $r = 2\pi\delta$ when $b = 1$, $n \in \{5, 7\}$, and $\epsilon \in (0, 1/10)$ (see Section 4.5).

4. If Q is a closed strictly convex curve, $r = \rho/2$, where ρ is the radius of the Borel transform of the asymptotic series for the lengths $L^{(1,q)}$ presented in (1.1).

5. If Q is a generic axisymmetric algebraic curve, then function $A(q)$ is either constant, $A(q) = a/2 \neq 0$, or oscillatory:

$$A(q) = \mu a/2 + a \sum_{j=1}^{J} \cos(2\pi\beta_j q),$$

with $\mu \in \{0, 1\}$, $a \neq 0$, $J \geq 1$, and $0 < \beta_1 < \cdots < \beta_J$. Besides, the cases $A(q) = a/2$ and $A(q) = a\cos(2\pi\beta)$ take place in open sets of the space of axisymmetric algebraic curves. All the other cases are phenomena of co-dimension one.

As for the case when Q is also bi-axisymmetric, the main difference of $B(q)$ with respect to $A(q)$ is that the frequencies β_j are doubled.

Note that, compared to the formula (1.4) for $A(1/h)$, $A(q)$ does not have phases φ_j. However, this is not new. For instance, the asymptotic formulas for the exponentially small splittings of generalized standard maps with trigonometric polynomials do not have phases either [GS08].

Chapter 4 is composed of the asymptotic formulas for the maximal difference of lengths among $(1, q)$-periodic trajectories on perturbed ellipses and circles and the extended discussion to axisymmetric tables. These results have also been presented in the preprint [MRT15].

Finally, in Chapter 5, we discuss the behavior of $\Delta^{(p,q)}$ in cases different from the previous ones. First, we focus on the study of $\Delta^{(p,q)}$ when p/q tends to an irrational rotation number. That is, the (p, q)-periodic orbits approach a rotational invariant curve or a cantori of irrational rotation number. Second, we study how $\Delta^{(p,q)}$ behaves when p/q tends to a rational rotation number P/Q. The cases close to the boundary ($P/Q = 0$) or close to a (m, n)-resonant rotational invariant curve ($P/Q = m/n$) fall in this category but they are already studied in the previous chapters. In order to discard them, we add an extra condition which is the existence of a (P, Q)-resonance on the phase space. In both settings, we restrict ourselves to a phenomenological study based on some numerical results. The study in these settings can be extended in many ways. The results we present should be seen as an initial starting point for a deeper study.

This thesis has the following structure. In Chapter 2, we give some necessary background and notations on exact analytic twist maps and billiard maps. Chapter 3 is devoted to prove the exponentially small upper bound for the maximal difference of (p, q)-periodic actions among (p, q)-periodic orbits $\Delta^{(p,q)}$ in different contexts on the billiards (maximal difference of lengths) and dual billiards (maximal difference of areas). In Chapter 4, we specifically study the maximal difference of lengths among $(1,q)$-periodic orbits for axisymmetric billiard tables and search for some asymptotic formulas. We deal with $\Delta^{(p,q)}$ when p/q tends to a limit other than zero in Chapter 5.

2 Exact twist maps and billiards

On what follows, we recall some results about exact twist maps and billiards. We refer to the books [KT91, Tab95a, KH95] and the surveys [Mei92, MF94] for a more detailed exposition.

We make special emphasis on the dynamical interpretation of the maximal difference among lengths of (p, q)-periodic orbits through Mather's $\Delta W_{p/q}$ and its interpretation as a flux.

2.1 Twist maps, actions, and Mather's $\Delta W_{p/q}$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $I = (r_-, r_+) \subset \mathbb{R}$, for some $-\infty \leq r_- < r_+ \leq +\infty$. We will use the coordinates (s, r) for both $\mathbb{T} \times I$ and its universal cover $\mathbb{R} \times I$. We refer to s as the angular coordinate and to r as the radial coordinate. If h is a real-valued smooth function, $\partial_i h$ denotes the derivative with respect to the i-th variable.

A smooth diffeomorphism $f : \mathbb{T} \times I \to \mathbb{T} \times I$ is an *exact twist map* when it preserves an exact symplectic form ω on the open cylinder, has zero flux, and satisfies the classical *twist condition* $\partial_2 s_1(s, r) > 0$, where $F(s, r) = (s_1, r_1)$ is a lift of f.

Henceforth, let $f : \mathbb{T} \times I \to \mathbb{T} \times I$ be an exact twist map with the lift fixed to $F : \mathbb{R} \times I \to \mathbb{R} \times I$ and the exact symplectic form on the open cylinder fixed to $\omega = -\mathrm{d}\lambda$ with $\lambda = \nu(r)\mathrm{d}s$ for some smooth function $\nu : (r_-, r_+) \to \mathbb{R}$. In particular, $\nu'(r) > 0$.

Note that by taking $t = \nu(r)$, then $\lambda = t\mathrm{d}s$, $\omega = \mathrm{d}s \wedge \mathrm{d}t$, and f preserves the canonical area in the *global Darboux coordinates* (s, t).

A *rotational invariant curve (RIC)* of f is a closed loop $C \subset \mathbb{T} \times I$ homotopically non trivial such that $f(C) = C$. If f is an exact twist map and C is a RIC, $C = \mathit{graph}\, c$ for some Lipschitz function $c : \mathbb{T} \to I$. It follows directly from Birkhoff's Theorem [Bir22].

If $f : \mathbb{T} \times I \to \mathbb{T} \times I$ is a diffeomorphism preserving an exact symplectic form ω and r_- and r_+ are finite, the zero flux condition is satisfied. If the map preserves ω and has a RIC, the zero flux condition is also satisfied.

Let r_- and r_+ be finite. We say that the exact twist map $f : \mathbb{T} \times I \to \mathbb{T} \times I$ *continuously extends to the boundaries* $C_- = \mathbb{T} \times \{r_-\}$ and $C_+ = \mathbb{T} \times \{r_+\}$ as rigid rotations of angles ϑ_- and ϑ_+ if $\lim_{r \to r_-} F(s,r) = (s + \vartheta_-, r_-)$ and $\lim_{r \to r_+} F(s,r) = (s + \vartheta_+, r_+)$.

We know that $\vartheta_- < \vartheta_+$ from the twist condition. The *twist interval* is $(\vartheta_-, \vartheta_+)$. Let $E = \{(s, s_1) \in \mathbb{R}^2 : \vartheta_- < s_1 - s < \vartheta_+\}$. Then there exists a function $h : E \to \mathbb{R}$ such that $h(s+1, s_1+1) = h(s, s_1)$ and

$$F^*\lambda - \lambda = \mathrm{d}h.$$

This function is called *Lagrangian* or *generating function*. It is determined modulo an additive constant.

Twist maps satisfy the following classical Lagrangian formulation. Their orbits are in one-to-one correspondence with the (formal) stationary configurations of the *action functional*

$$\mathbb{R}^{\mathbb{Z}} \ni \mathbf{s} = (s_j)_{j \in \mathbb{Z}} \mapsto W[\mathbf{s}] = \sum_{j \in \mathbb{Z}} h(s_j, s_{j+1}).$$

Note that, although the series for $W[\mathbf{s}]$ may be divergent, $\frac{\partial W}{\partial s_j}$ only involves two terms of the series, and so ∇W is well defined.

Let p and q be two relatively prime integers such that $\vartheta_- < p/q < \vartheta_+$ and $q \geq 1$. A point $(s,r) \in \mathbb{R} \times I$ is (p,q)-*periodic* when $F^q(s,r) = (s+p, r)$. The corresponding point $(s,r) \in \mathbb{T} \times I$ is a q-periodic point of f that is translated p units in the base by the lift. A (p,q)-periodic orbit is *Birkhoff* when it is ordered around the cylinder in the same way that the orbits of the rigid rotation of angle p/q. The Poincaré-Birkhoff Theorem states that there exist at least two different Birkhoff (p,q)-periodic orbits [KH95, Mei92].

If $O = \{(s_j, r_j)\}_{j \in \mathbb{Z}}$ is a (p,q)-periodic orbit of f, then

$$h(s_{j+q}, s_{j+q+1}) = h(s_j + p, s_{j+1} + p) = h(s_j, s_{j+1}),$$

so there are only q different terms in the action functional W which encode the (p,q)-periodic dynamics. In particular, any (p,q)-periodic orbit $O = \{(s_j, r_j)\}_{j \in \mathbb{Z}}$ is in correspondence with a stationary configuration $\mathbf{s} = (s_0, \ldots, s_{q-1}) \in \mathbb{R}^{q-1}$ of the (p,q)-*periodic action*

$$W^{(p,q)}[\mathbf{s}] = h(s_0, s_1) + h(s_1, s_2) + \cdots + h(s_{q-1}, s_0 + p).$$

We say that $W^{(p,q)}[O] = W^{(p,q)}[\mathbf{s}]$ is the (p,q)-periodic action of the (p,q)-periodic orbit O. In fact, the Birkhoff (p,q)-periodic orbits provided by the Poincaré-Birkhoff Theorem correspond to the minimizing and minimax stationary configurations of $W^{(p,q)}$.

Mather defined the quantity $\Delta W_{p/q} \geq 0$ as the action of the minimax periodic orbit minus the action of the minimizing one [Mat86]. He proved that $\Delta W_{p/q}$ can be used as a criterion to guarantee the existence of RICs of given irrational rotation numbers. MacKay, Meiss, and Percival provided a new dynamical interpretation of $\Delta W_{p/q}$ in [MMP84]. It is known

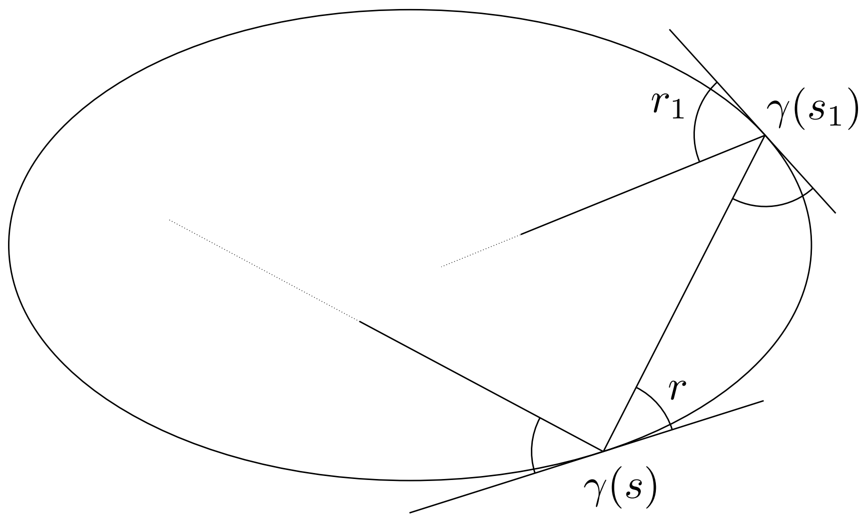


Figure 2.1: The billiard map $f(s,r) = (s_1, r_1)$.

as the *MacKay-Meiss-Percival action principle* and states that $\Delta W_{p/q}$ is equal to the *flux* through any homotopically non trivial curve without self-intersections passing through all the points of both the minimizing and the minimax (p,q)-periodic orbits. Thus, $\Delta W_{p/q}$ gives a rough estimation of the size of the (p,q)-resonance of the twist map.

2.2 Convex billiards, length spectrum, and $\Delta^{(p,q)}$

Let Q be a smooth strictly convex curve in the Euclidean plane. This curve Q is the boundary of the billiard table. Let $l = Length(Q)$. Consider $\mathbb{T} := \mathbb{R}/l\mathbb{Z}$ and let $\gamma : \mathbb{T} \to Q$, $s \mapsto \gamma(s)$, be an arc-length counterclockwise parametrization. The bounce position of a free particle inside the billiard can be determined in terms of the variable s. The direction of motion is measured by the variable r, which is the angle between the incident line at the impact point $\gamma(s)$ and the tangent vector $\gamma'(s)$. The movement can only be inwards, so $r_- = 0$, $r_+ = \pi$, and $I = (0, \pi)$. Let

$$f : \mathbb{T} \times I \to \mathbb{T} \times I, \qquad f(s,r) = (s_1, r_1), \tag{2.1}$$

be the corresponding *billiard map*. Let $F : \mathbb{R} \times I \to \mathbb{R} \times I$. $F(s,r) = (s_1, r_1)$ be a fixed lift of the billiard map. Note that we are using the same coordinates on the lift. The coordinates (s,r) are called the *Birkhoff coordinates*. Figure 2.1 illustrates the billiard construction.

The billiard map is an exact map with respect to the exact symplectic form $\omega = \sin r \, \mathrm{d}s \wedge \mathrm{d}r$. Also, the twist condition is satisfied. Indeed, let $\ell(s, s_1) = |\gamma(s) - \gamma(s_1)|$ be the Euclidean distance between two impact points on Q, then $\partial_r s_1 = \ell(s, s_1)/\sin r_1 > 0$. Thus, billiard maps are exact twist maps.

The billiard map analytically extends to the boundaries components $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{\pi\}$ (see Proposition 5) and $F(s, 0) = (s, 0)$ and $F(s, \pi) = (s + l, \pi)$. In the billiard context,

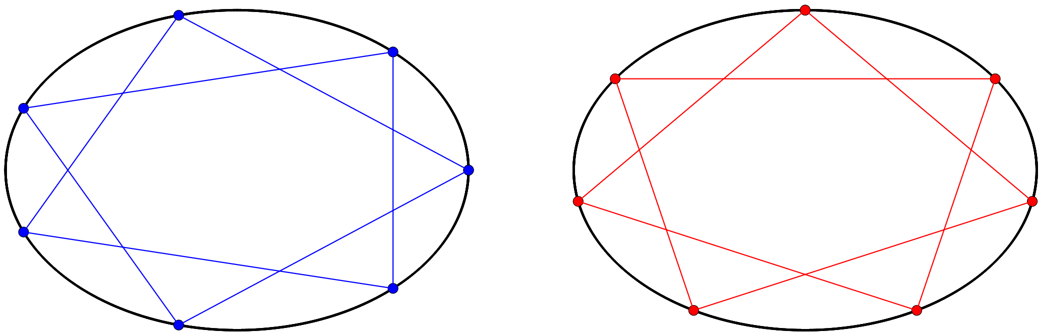


Figure 2.2: Two different (2,7)-periodic orbits (or (5,7)-periodic orbits if traveled clockwise) of the billiard map inside $Q = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2/0.9^2 = 1\}$.

it is usual to take the rotation number modulo l, so we say that the boundary components are rigid rotations of angles $\vartheta_- = 0$ and $\vartheta_+ = 1$.

Let $0 < p < q$ be relatively prime integers. A (p, q)-*periodic billiard trajectory* forms a closed polygon with q sides that makes p turns inside Q. The period of this orbit is q whereas p is the number of turns. Let $O = \{(s_j, r_j)\}_{j\in\mathbb{Z}}$ be a (p, q)-periodic orbit. Its coordinates satisfy

$$s_{j+q} = s_j + pl, \qquad r_{j+q} = r_j,$$

for all $j \in \mathbb{Z}$. The *rotation number* of a (p, q)-periodic orbit is p/q. Note that any (p, q)-periodic billiard trajectory gives rise to a $(q-p, q)$-periodic one by inverting the direction of motion. In Figure 2.2, we show an example of periodic orbits inside a billiard table.

Given a smooth strictly convex billiard, for any relatively prime integers $0 < p < q$, we denote by $\Lambda^{(p,q)}$ the set of the lengths of all the (p, q)-periodic orbits on the billiard. The *length spectrum* is defined as

$$\mathcal{LS}(Q) = l\mathbb{N} \cup \bigcup_{(p,q)} \Lambda^{(p,q)}\mathbb{N}.$$

There exists a relation between the length spectrum and the Laplacian spectrum with homogeneous Dirichlet conditions on the boundary for the region delimited by Q [AM77]. Thus, $\mathcal{LS}(Q)$ has a geometric interest from the point of view of Kac's celebrated question about how much information on the domain Q is given by the Laplacian spectrum with homogeneous Dirichlet conditions on the boundary [Kac66].

Let $\Delta^{(p,q)}$ be the maximal difference among the lengths in $\Lambda^{(p,q)}$. The generating function of the billiard map is $h(s, s_1) = -\ell(s, s_1)$ and so, the action of a periodic billiard trajectory is, up to the sign, its length. Thus, in terms of the (p, q)-periodic actions, $\Delta^{(p,q)}$ can be defined as

$$\Delta^{(p,q)} = \sup_{\mathbf{s}\in\Xi} W^{(p,q)}[\mathbf{s}] - \inf_{\mathbf{s}\in\Xi} W^{(p,q)}[\mathbf{s}],$$

where $\Xi \subset \mathbb{R}^{q-1}$ is the set of all stationary configurations and $\mathbf{s} = (s_0, \cdots, s_{q-1}) \in \Xi$. It is clear that the maximal difference $\Delta^{(p,q)}$ is an upper bound of the Mather's $\Delta W_{p/q}$ for

(p,q)	$H^{(p,q)}$	$E^{(p,q)}$	$\Delta W_{p/q}$
$(1,2)$	4.000000	3.828482	0.171577
$(1,4)$ and $(3,4)$	5.594652	5.536901	0.057751
$(1,3)$ and $(2,3)$	5.115169	5.112940	0.002229
$(3,8)$ and $(5,8)$	14.773311	14.772302	0.001009
$(5,12)$ and $(7,12)$	23.151909	23.150969	0.000940
$(3,10)$ and $(7,10)$	15.925337	15.924445	0.000892
$(1,6)$ and $(5,6)$	5.904338	5.903527	0.000811
$(2,5)$ and $(3,5)$	9.366997	9.366503	0.000494
$(1,8)$ and $(7,8)$	6.024507	6.024232	0.000275
$(3,7)$ and $(4,7)$	13.455442	13.455236	0.000206
$(1,5)$ and $(4,5)$	5.785133	5.785011	0.000122

Table 2.1: The biggest Mather's $\Delta W_{p/q}$ for the billiard inside $x^2 + y^2 + y^4/10 = 1$.

billiard maps. Precisely, this relation is an important reason for the study of the quantity $\Delta^{(p,q)}$.

Applied to the billiard map, Mather's $\Delta W_{p/q}$ is the length of the (p,q)-periodic billiard trajectory that minimizes the action (and so, maximizes the length) minus the length of the minimax one. Generically, the hyperbolic (respectively, elliptic) periodic orbits in a given resonance are minimizing (respectively, minimax) [MM83]. Thus, let us write

$$\Delta W_{p/q} = H^{(p,q)} - E^{(p,q)},$$

where $H^{(p,q)}$ and $E^{(p,q)}$ are the lengths of the hyperbolic and elliptic (p,q)-periodic billiard trajectories inside Q. For instance, $H^{(1,2)} = 4a$, $E^{(1,2)} = 4b$, and $\Delta W_{1/2} = 4(a-b)$ for the billiard inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ with $0 < b < a$.

Since an orbit can be traveled in both directions, $\Delta W_{p/q} = \Delta W_{(q-p)/q}$ for all $p/q \in (0, 1/2)$.

We have listed the biggest Mather's $\Delta W_{p/q}$ for the billiard inside the perturbed circle $x^2 + y^2 + y^4/10 = 1$. See Table 2.1. The rest of Mather's $\Delta W_{p/q}$ are smaller that 10^{-4}. The values in the table suggest that the $(1,2)$-resonance and both $(p,4)$-resonances should be the most important ones. This prediction is confirmed in Figure 2.3, where we display the biggest resonances of the billiard map inside $x^2 + y^2 + y^4/10 = 1$.

Mather's $\Delta W_{p/q}$ allow us to single out the most important resonances, but they do not give an exact measure of the size of resonances. To begin with, there is not a unique way to define such size. A choice is the area $A_{p/q}$ of the Birkhoff instability region that contains the (p,q)-resonance. A *Birkhoff instability region* is a region of the phase space delimited by two rotational invariant curves (RICs) without any other RIC in its interior. If we have a twist map with a (p,q)-resonant RIC, then $\Delta W_{p/q} = \mathrm{O}(\epsilon)$ and $A_{p/q} = \mathrm{O}(\epsilon^{1/2})$ under generic perturbations of order $\mathrm{O}(\epsilon)$. See [Olv01]. This shows up a clear difference between these two quantities. For instance, the billiard map inside the circle $x^2 + y^2 = 1$

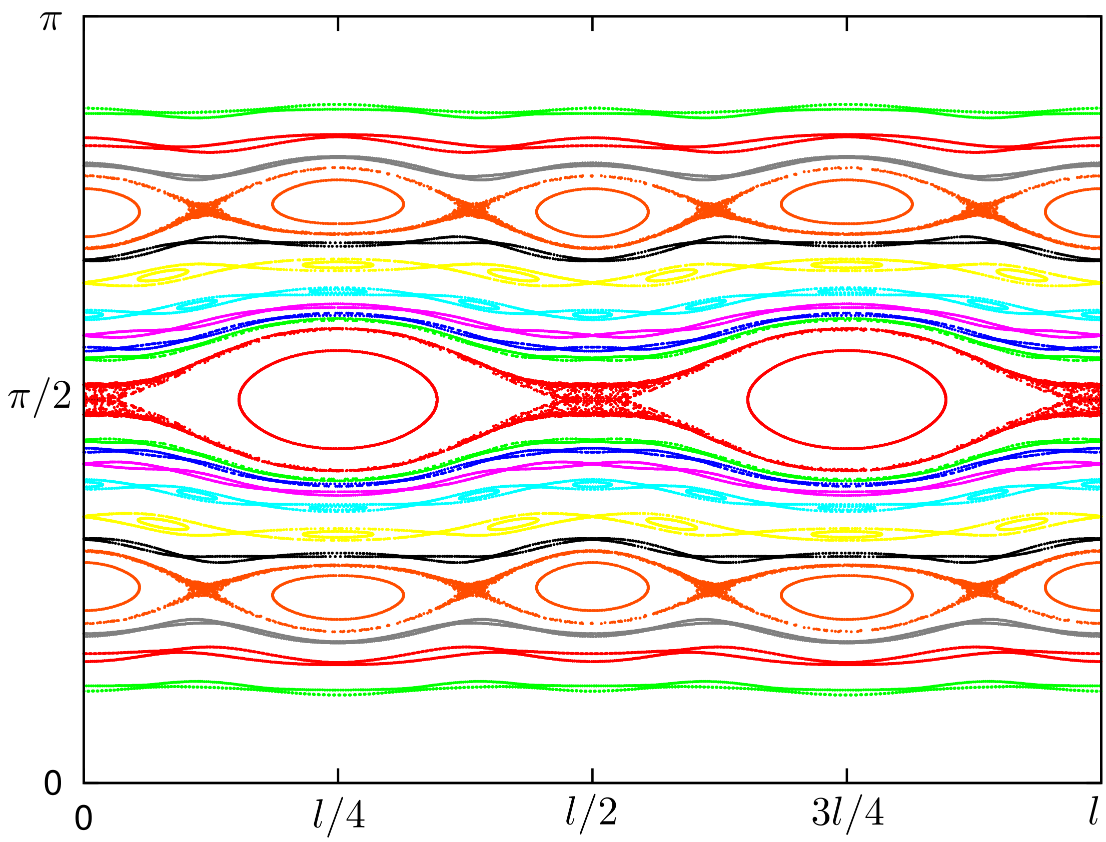


Figure 2.3: The biggest (p,q)-resonances of the billiard map $f(s,r) = (s_1,r_1)$ inside the perturbed circle $Q = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + y^4/10 = 1\}$. We recall that $l = Length(Q)$. All (p,q)-resonances with odd period q have $2q$ elliptic islands due to the bi-axisymmetric character of the curve. From bottom to top: $(1,8)$, $(1,6)$, $(1,5)$, $(1,4)$, $(3,10)$, $(1,3)$, $(3,8)$, $(2,5)$, $(5,12)$, $(3,7)$, $(1,2)$, and their $(q-p,q)$ symmetric counterparts.

has a $(1,2)$-resonant RIC, which is destroyed under the perturbation $x^2 + y^2/(1-\epsilon)^2 = 1$. However, this perturbed billiard table is integrable (it is an ellipse), so both quantities can be analytically computed: $\Delta W_{1/2} = 4\epsilon$ and $A_{1/2} = 8\epsilon^{1/2}$. We omit the details.

3 Exponentially small upper bounds for the length and area spectrum

Abstract of the chapter. Area-preserving twist maps have at least two different (p, q)-periodic orbits and every (p, q)-periodic orbit has its (p, q)-periodic action for suitable couples (p, q). We establish an exponentially small upper bound for the differences of (p, q)-periodic actions when the map is analytic on a (m, n)-resonant rotational invariant curve (resonant RIC) and p/q is "sufficiently close" to m/n. The exponent in this upper bound is closely related to the analyticity strip width of a suitable angular variable. The result is obtained in two steps. First, we prove a Neishtadt-like theorem, in which the n-th power of the twist map is written as an integrable twist map plus an exponentially small remainder on the distance to the RIC. Second, we apply the MacKay-Meiss-Percival action principle. We apply our exponentially small upper bound to several billiard problems. The resonant RIC is a boundary of the phase space in almost all of them. For instance, we show that the lengths (respectively, areas) of all the $(1, q)$-periodic billiard (respectively, dual billiard) trajectories inside (respectively, outside) analytic strictly convex domains are exponentially close in the period q. This improves some classical results of Marvizi, Melrose, Colin de Verdière, Tabachnikov, and others about the smooth case.

3.1 Introduction

Let us recall some concepts about periodic trajectories on billiard maps so that the exposition is fluent and complete. Let Q be a smooth strictly convex curve in the plane, oriented counterclockwise, and let Ω be the billiard table enclosed by Q. A (p, q)-periodic billiard trajectory forms a closed polygon with q sides that makes p turns inside Q. Birkhoff [Bir66] proved that there are at least two different Birkhoff (p, q)-periodic billiard trajectories inside Ω for any relatively prime integers p and q such that $1 \leq p \leq q$.

Let $\mathcal{L}^{(p,q)}$ be the supremum of the absolute values of the differences of the lengths of all such trajectories. The quantities $\mathcal{L}^{(p,q)}$ were already studied by Marvizi and Melrose [MM82] and Colin de Verdière [Col84] for smooth tables. The former authors produced an asymptotic expansion of the lengths for (p, q)-periodic billiard trajectories approaching Q when p is fixed and $q \to +\infty$. They saw that there exists a sequence $(l_k)_{k\geq 1}$, depending only on p and Q, such that, if $L^{(p,q)}$ is the length of any (p,q)-periodic trajectory, then

$$L^{(p,q)} \asymp p\,\mathrm{Length}(Q) + \sum_{k\geq 1} \frac{l_k}{q^{2k}}, \qquad q \to \infty,$$

where $l_1 = l_1(Q,p) = -\frac{1}{24}\left(p\int_Q \kappa^{2/3}(s)\mathrm{d}s\right)^3$, and $\kappa(s)$ is the curvature of Q as a function of the arc-length parameter s. The symbol $\asymp$ means that the series in the right hand side is asymptotic to $L^{(p,q)}$. The asymptotic coefficients $l_k = l_k(Q,p)$ can be explicitly written in terms of the curvature $\kappa(s)$. For instance, the explicit formulas for l_1, l_2, l_3, and l_4 can be found in [Sor15]. Since the expansion of the lengths in powers of q^{-1} coincides for all these (p,q)-periodic trajectories, $\mathcal{L}^{(p,q)} = \mathrm{O}(q^{-\infty})$ for smooth strictly convex tables when p is fixed and $q \to +\infty$. Colin de Verdière studied the lengths of periodic trajectories close to an elliptic $(1,2)$-periodic trajectory on a smooth symmetric billiard table and found that the quantities $\mathcal{L}^{(p,q)}$ are again beyond all order with respect to q.

These works suggest that the supremum length differences $\mathcal{L}^{(p,q)}$ are exponentially small in the period q for analytic strictly convex tables. Indeed, we have proved that if Q is analytic and p is a fixed positive integer, then there exist $K, q_*, \alpha > 0$ such that

$$\mathcal{L}^{(p,q)} \leq K\mathrm{e}^{-2\pi\alpha q/p}, \tag{3.1}$$

for all integer $q \geq q_*$ relatively prime with p. The exponent α is related to the width of a complex strip where a certain 1-periodic angular coordinate is analytic. A more precise statement is given in Theorem 6.

Similar exponentially small upper bounds hold in other billiard problems. We mention two examples. First, for (p,q)-periodic billiard trajectories inside strictly convex analytic tables of constant width when $p/q \to 1/2$. Second, for (p,q)-periodic billiard trajectories inside strictly convex analytic tables in surfaces of constant curvature when $p/q \to 0$.

The billiard dynamics close to the boundary has also been studied from the point of view of KAM theory. Lazutkin [Laz73] proved that there are infinitely many caustics inside any C^{555} strictly convex table. These caustics accumulate at the boundary of the table, and have Diophantine rotation numbers. Douady [Dou82] improved the result to C^7 billiard tables.

A special remark on the relevance of these results is the following. Kac [Kac66] formulated the inverse spectral problem for planar domains. That is, to study how much geometric information about Ω can be obtained from the Laplacian spectrum with homogeneous Dirichlet conditions on Q. Andersson and Melrose [AM77] gave an explicit relation between the length spectrum and the Laplacian spectrum. The length spectrum of Ω is the union of all the integer multiplels of the lengths of all its (p,q)-periodic billiard trajectories and all the integer multiples of $Length(Q)$. See also [MM82, Col84].

Our results also apply to the dual billiards introduced by Day [Day47] and popularized by Moser [Mos79] as a crude model for planetary motion. Some general references are [GK95, Boy96, Tab95b, Tab95a]. Let $\mho$ be unbounded component of $\mathbb{R}^2 \setminus Q$. The dual billiard map $f : \mho \to \mho$ is defined as follows: $f(z)$ is the reflection of z in the tangency point of the oriented tangent line to Q through z. Billiards and dual billiards are projective dual in the sphere [Tab95b].

A (p,q)-periodic dual billiard trajectory forms a closed circumscribed polygon with q sides that makes p turns outside Q. The area of a (p,q)-periodic trajectory is the area enclosed by the corresponding polygon, taking into account some multiplicities if $p \geq 2$. There are at least two different Birkhoff (p,q)-periodic dual billiard trajectories outside Q for any relatively prime integers p and q such that $q \geq 3$ and $0 < p < q$.

Tabachnikov [Tab95a, Tab95b] studied the supremum $\mathcal{A}^{(1,q)}$ of the absolute value of the differences of the areas enclosed by such $(1,q)$-periodic trajectories for smooth tables. He proved that there is a sequence $(a_k)_{k \geq 1}$, depending only on Q, such that, if $A^{(1,q)}$ is the area enclosed by any $(1,q)$-periodic dual billiard trajectory, then

$$A^{(1,q)} \asymp Area(\Omega) + \sum_{k \geq 1} \frac{a_k}{q^{2k}}, \qquad q \to \infty, \tag{3.2}$$

where $a_1 = a_1(Q) = \frac{1}{24} \int_Q \kappa^{1/3}(s) \mathrm{d}s$. Hence, the expansion of the areas in powers of q^{-1} coincides for all these $(1,q)$-periodic trajectories, and so, $\mathcal{A}^{(1,q)} = \mathrm{O}(q^{-\infty})$ for smooth strictly convex dual tables when $q \to +\infty$. Douady [Dou82] found the existence of infinitely many invariant curves outside any C^7 strictly convex dual table. These invariant curves accumulate at the boundary of the dual table and have Diophantine rotation numbers.

In a completely analogous way to (classical) billiards, we have proved that, once fixed any positive integer p, if Q is analytic, then there exist $K, q_*, \alpha > 0$ such that

$$\mathcal{A}^{(p,q)} \leq K \mathrm{e}^{-2\pi\alpha q/p}, \tag{3.3}$$

for all integer $q \geq q_*$ relatively prime with p. Once more, the exponent α is related to the width of a complex strip where a certain 1-periodic angular coordinate is analytic. The precise statement is given in Theorem 9.

Still in the context of dual billiards, the points at infinity can be seen as $(1,2)$-periodic points, hence they form a $(1,2)$-resonant RIC. Douady [Dou82] found the existence of infinitely many invariant curves outside any C^8 strictly convex dual table. These invariant curves accumulate at infinity and have Diophantine rotation numbers. We have proved that, once fixed any constant $L \geq 1$, if Q is analytic, then there exist $K, q_*, \alpha > 0$ such that

$$\mathcal{A}^{(p,q)} \leq K \exp\left(-\frac{\pi\alpha}{|p/q - 1/2|} \right), \tag{3.4}$$

for all relatively prime integers p and q such that $1 \leq |2p - q| \leq L$ and $q \geq q_*$. See Theorem 10.

The three exponents α that appear in the exponentially small upper bounds (3.1), (3.3), and (3.4) may be different, since each one is associated to a different analyticity strip width. Besides, all of these upper bounds follow directly from a general upper bound about analytic area-preserving twist maps with analytic resonant RICs. Let us explain it.

Classical and dual billiard maps are exact twist maps defined on an open cylinder when written in suitable coordinates. Exact twist maps have been vastly studied. They satisfy a Lagrangian formulation and their orbits are stationary points of the action functional. See for instance [Bir66, Mei92, KH95].

Birkhoff [Bir66] showed that the minima and minimax points of the (p, q)-periodic action correspond to two different Birkhoff (p, q)-periodic orbits of the twist map. A Birkhoff (p, q)-periodic orbit is an orbit such that, after q iterates, performs exactly p revolutions around the cylinder and its points are ordered in the base $\mathbb{T}$ as the ones following a rigid rotation of angle p/q. Since there exist at least two different Birkhoff (p, q)-periodic orbits, we consider the supremum $\Delta^{(p,q)}$ of the absolute value of the differences of the actions among all of them. The quantity $\Delta^{(p,q)}$ coincides with $\mathcal{L}^{(p,q)}$ and $\mathcal{A}^{(p,q)}$ for classical and dual billiards, respectively.

Let $\Delta W_{p/q}$ be the difference of actions between the minimax and minima (p, q)-periodic orbits. Note that $\Delta^{(p,q)}$ is an upper bound of $\Delta W_{p/q}$. Mather [Mat86] used $\Delta W_{p/q}$ as a criterion to prove the existence of RICs of given irrational rotation numbers. More concretely, he proved that there exists a RIC with irrational rotation number ϑ if and only if $\lim_{p/q\to\vartheta} \Delta W_{p/q} = 0$.

Another criterion related to the destruction of RICs, in this case empirical, was proposed by Greene. The destruction of a RIC with Diophantine rotation number ϑ under perturbation is related to a "sudden change from stability to instability of the nearby periodic orbits" [Gre79]. The stability of a periodic orbit is measured by the residue. MacKay [Mac92] proved the criterion in some contexts. In particular, for an analytic area-preserving twist map, the residue of a sequence of periodic orbits with rotation numbers that tend to ϑ decays exponentially as $-dq$ for some positive $d > 0$. The same proof leads to a similar exponentially small bound of Mather's $\Delta W_{p/q}$ as $p/q \to \vartheta$. Delshams and de la Llave [DdlL00] studied similar problems for analytic area-preserving non-twist maps.

Generically, RICs with a rational rotation number break under perturbation [Ram06, PR13]. Nevertheless, there are situations in which some distinguished resonant RICs always exist. See Sections 3.3 and 3.4 for several examples related to billiard and dual billiard maps.

Let us assume that we have an analytic exact twist map with a (m, n)-resonant RIC. That is, a RIC whose points are (m, n)-periodic. Then, there exist some variables (x, y) in which the resonant RIC is located at $\{y = 0\}$ and the n-th power of the exact twist map is a small perturbation of the integrable twist map $(x_1, y_1) = (x + y, y)$. To be precise, it has the form

$$x_1 = x + y + \mathrm{O}(y^2), \qquad y_1 = y + \mathrm{O}(y^3).$$

Since the n-th power map is real analytic, it can be extended to a complex domain of the form

$$D_{a_*,b_*} := \{(x,y) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C} : |\Im x| < a_*, |y| < b_*\}.$$

The quantity a_* plays a more important role than b_*. To be precise, we have proved that, once fixed any $\alpha \in (0, a_*)$ and $L \geq 1$, there exist $K, q_* > 0$ such that

$$\Delta^{(p,q)} \leq K \exp\left(-\frac{2\pi\alpha q}{|np - mq|}\right),$$

for any relatively prime integers p and q such that $1 \leq |np - mq| \leq L$ and $q \geq q_*$. See Theorem 4 for a more detailed statement. This upper bound is optimal because $\alpha \in (0, a_*)$. That is, the exponent α can be taken as close to the analyticity strip width a_* as desired. The constant K may explode when α tends to a_*, so, in general, we can not take $\alpha = a_*$. A similar optimal exponentially small upper bound was obtained in [FS90] in the setting of the splitting of separatrices of weakly hyperbolic fixed points of analytical area-preserving maps. The proof of this optimal bound adds some extra technicalities, but we feel that the effort is worth it.

The proof is based on two facts. First, we write the n-th power of the exact twist map as the integrable twist map $(x_1, y_1) = (x+y, y)$ plus an exponentially small remainder on the distance to the RIC. See Theorem 3. The size of the remainder is reduced by performing a finite sequence of changes of variables, but the number of such changes increases when we approach the resonant RIC. This is a classical Neishtadt-like argument [Nei81]. Second, we apply the MacKay-Meiss-Percival action principle [MMP84], in which the difference of (p,q)-periodic actions is interpreted as an area on the phase space.

The structure of this chapter is the following. Section 3.2 is devoted to state our results in the general context of analytic exact twist maps. We briefly reintroduce some concepts about exact twist maps already exposed in Section 2.1 so that this section is self-contained. In Section 3.3 (resp., Section 3.4), we review the billiard map (resp., we present the dual billiard) and adapt the results obtained on the general context of exact twist maps to some different settings on this map. Sections 3.5 and 3.6 contain the technical proofs.

3.2 Main theorems

3.2.1 Dynamics close to an analytic resonant RIC

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $I = (r_-, r_+) \subset \mathbb{R}$, for some $-\infty \leq r_- < r_+ \leq +\infty$. We will use the coordinates (s, r) for both $\mathbb{T} \times I$ and its universal cover $\mathbb{R} \times I$. We refer to s as the angular coordinate. Let $\omega = -\mathrm{d}\lambda$ be an *exact symplectic form* on the open cylinder $\mathbb{T} \times I$ such that $\lambda = \nu(r)\mathrm{d}s$ and $\omega = \nu'(r)\mathrm{d}s \wedge \mathrm{d}r$ for some smooth function $\nu : (r_-, r_+) \to \mathbb{R}$. In particular, $\nu'(r) > 0$. If h is a real-valued smooth function, $\partial_i h$ denotes the derivative with respect to the i-th variable.

Definition 1. A smooth diffeomorphism $g : \mathbb{T} \times I \to \mathbb{T} \times I$ is an *exact twist map* when it preserves the exact symplectic form $\omega = -\mathrm{d}\lambda$, has zero flux, and satisfies the classical *twist condition* $\partial_2 s_1(s, r) > 0$, where $G(s, r) = (s_1, r_1)$ is a lift of g.

Henceforth, the exact symplectic form $\omega = -\mathrm{d}\lambda = \nu'(r)\mathrm{d}s \wedge \mathrm{d}r$ and the lift G remain fixed. We will assume that $\nu(r)$ is analytic when we deal with analytic maps. If g preserves ω and $t = \nu(r)$, then $\lambda = t\mathrm{d}s$, $\omega = \mathrm{d}s \wedge \mathrm{d}t$, and g preserves the canonical area in the *global Darboux coordinates* (s, t). It is worth to remark that certain billiard maps can be analytically extended to the boundaries of their phase spaces in (s, r) variables, but not in global Darboux coordinates. For this reason we consider the coordinates (s, r) and the above exact symplectic forms. In fact, we could deal with any exact symplectic form, but we do not need it for the problems we have in mind.

Definition 2. Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be a continuous map. The map g has the *intersection property* on $\mathbb{T} \times I$ if the image of any closed homotopically non trivial loop of the cylinder $\mathbb{T} \times I$ intersects the loop.

The intersection property is preserved under global changes of variables.

Definition 3. A *rotational invariant curve (RIC)* of g is a closed loop $C \subset \mathbb{T} \times I$ homotopically non trivial such that $g(C) = C$. Let C be a RIC of g. Let m and n be two relatively prime integers such that $n \geq 1$. We say that C is (m, n)*-resonant* when $G^n(s, r) = (s + m, r)$ for all $(s, r) \in C$, and we say that C is *analytic* when $C = graph\, c := \{(s, c(s)) : s \in \mathbb{T}\}$ for some analytic function $c : \mathbb{T} \to I$.

If $g : \mathbb{T} \times I \to \mathbb{T} \times I$ is a diffeomorphism preserving the exact symplectic form $\omega = -\mathrm{d}\lambda$ and has a RIC, both the zero flux condition and the intersection property are automatically satisfied.

Let us study the dynamics of an analytic exact twist map in a neighbourhood of an analytic (m, n)-resonant RIC. First, we note that all points on a (m, n)-resonant RIC of g remain fixed under the power map $f = g^n$. Second, we adapt a classical lemma that appears in several papers about billiards [Laz73, Tab95b] to our setting.

Lemma 1. *If $g : \mathbb{T} \times I \to \mathbb{T} \times I$ is an analytic exact twist map with an analytic (m, n)-resonant RIC $C \subset \mathbb{T} \times I$, then there exist an analytic strip width $a_* > 0$, an analytic radius $b_* > 0$, and some analytic coordinates (x, y) such that $C \equiv \{y = 0\}$ and the power map $f = g^n$ satisfies the following properties:*

1. *It is real analytic on $\mathbb{T} \times (-b_*, b_*)$ and can be analytically extended to the complex domain*
$$D_{a_*,b_*} = \{(x, y) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{C} : |\Im x| < a_*,\ |y| < b_*\}\,; \tag{3.5}$$
2. *It has the intersection property on the cylinder $\mathbb{T} \times (-b_*, b_*)$; and*

3. *It has the form* $(x_1, y_1) = f(x, y)$*, with*

$$x_1 = x + y + \mathrm{O}(y^2), \qquad y_1 = y + \mathrm{O}(y^3). \tag{3.6}$$

Proof. If $C = graph\, c$ and $v = r - c(s)$, then $C \equiv \{v = 0\}$ and $(s_1, v_1) = f(s, v)$, with

$$s_1 = s + \varphi(s)v + \mathrm{O}(v^2), \qquad v_1 = v + \psi(s)v^2 + \mathrm{O}(v^3), \tag{3.7}$$

for some real analytic 1-periodic functions $\varphi(s)$ and $\psi(s)$. The twist condition on the RIC implies that $\varphi(s)$ is positive, since any power of a twist map is locally twist on its smooth RICs [PR13, Lemma 2.1]. The preservation of ω implies that $2\mu\psi = -(\mu\varphi)'$, where $\mu(s) = \nu'(c(s)) > 0$. Next, we consider the analytic coordinates (x, y) defined by

$$x = k\int_0^s \sqrt{\frac{\mu(s)}{\varphi(s)}}\mathrm{d}t, \qquad y = k\sqrt{\mu(s)\varphi(s)}v, \qquad k^{-1} = \int_0^1 \sqrt{\frac{\mu(t)}{\varphi(t)}}\mathrm{d}t.$$

The constant k has been determined in such a way that the new angular coordinate x is defined modulus one: $x \in \mathbb{T}$. Clearly, $C \equiv \{y = 0\}$. Thus, the coordinates (x, y) cover an open set containing $\mathbb{T} \times \{0\}$, since they are defined in a neighbourhood of C. In particular, f can be analytically extended to the complex domain D_{a_*,b_*} for some $a_*, b_* > 0$. Besides, f has the intersection property on $\mathbb{T} \times (-b_*, b_*)$ because the integral of the area form ω over the region enclosed between the RIC C and any closed homotopically non trivial loop should be preserved. Finally, a straightforward computation shows that f has the form (3.6) in the coordinates (x, y). $\square$

Lemma 1 has, at a first glance, a narrow scope of application because resonant RICs are destroyed under generic perturbations. However, the boundaries of the cylinder can be considered resonant RICs of the extended twist map in many interesting examples.

Let us precise this idea.

Definition 4. Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be a continuous map. If r_- is finite, we say that $C_- = \mathbb{T} \times \{r_-\}$ is a *rigid rotation boundary* when g can be continuously extended to $\mathbb{T} \times [r_-, r_+)$ and its extended lift satisfies that $G(s, r_-) = (s + \vartheta_-, r_-)$ for some *boundary rotation number* $\vartheta_- \in \mathbb{R}$, and we say that C_- is a (m, n)*-resonant boundary* when $\vartheta_- = m/n$.

Definition 5. Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be a smooth diffeomorphism. We say that the *twist condition holds on the boundary* C_- when g can be smoothly extended to $\mathbb{T} \times [r_-, r_+)$ and its extended lift satisfies that $\partial_2 s_1(s, r_-) > 0$ for all $s \in \mathbb{T}$.

Analogous definitions can be written for the upper boundary $C_+ = \mathbb{T} \times \{r_+\}$ when r_+ is finite. We recall that $G(s, r) = (s_1, r_1)$ is a lift of g. Next, we present a version of Lemma 1 for the boundaries of the cylinder. The only remarkable difference is that the exact symplectic form $\omega = -\mathrm{d}\lambda = \nu'(r)\mathrm{d}s \wedge \mathrm{d}r$ may vanish on the boundaries.

Lemma 2. *If $g : \mathbb{T} \times I \to \mathbb{T} \times I$ is an analytic exact twist map such that r_- is finite, $C_- = \mathbb{T} \times \{r_-\}$ is a (m,n)-resonant boundary, g can be analytically extended to $\mathbb{T} \times [r_-, r_+)$, $\nu(r)$ can be analytically extended to $[r_-, r_+)$, and the twist condition holds on C_-, then there exist an analytic strip width $a_* > 0$, an analytic radius $b_* > 0$, and some analytic coordinates (x,y) such that $C_- \equiv \{y = 0\}$ and the power map $f = g^n$ satisfies the properties (i)–(iii) given in Lemma 1.*

An analogous result holds for the upper boundary $C_+ = \mathbb{T} \times \{r_+\}$ when r_+ is finite.

Proof. If $v = r - r_-$, then $C_- \equiv \{v = 0\}$ and the extended power map $(s_1, v_1) = f(s, v)$ has the form (3.7) for some real analytic 1-periodic functions $\varphi(s)$ and $\psi(s)$. The twist condition on C_- implies that $\varphi(s)$ is positive. Since $\nu(r)$ is analytic in $[r_-, r_+)$ and $\nu'(r)$ is positive in (r_-, r_+), we deduce that $\nu'(r) = \nu_* v^j + \mathrm{O}(v^{j+1})$ for some $\nu_* > 0$ and some integer $j \geq 0$. The preservation of ω implies that $\psi = -\varphi'/(j+2)$. Next, we consider the analytic coordinates (x,y) defined by

$$x = k \int_0^s \varphi^{-(j+1)/(j+2)}(t) \mathrm{d}t, \qquad y = k \varphi^{1/(j+2)}(s) v, \tag{3.8}$$

where the constant k is determined in such a way that the angular coordinate x is defined modulus one. The rest of the proof follows the same lines as in Lemma 1. We just note that the intersection property on a (real) neighbourhood of the boundary C_- follows by analytic extension, since f preserves the exact symplectic form ω for negative values of v too. □

We unify the resonant RICs studied in Lemma 1 and the resonant boundaries studied in Lemma 2 as a single object for the sake of brevity.

Definition 6. Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be an analytic map. Let m and $n \geq 1$ be two relatively prime integers. Let $a_* > 0$ and $b_* > 0$. We say that $C \subset \mathbb{T} \times I$ (respectively, $C = \mathbb{T} \times \{r_-\}$ for a finite r_- or $C = \mathbb{T} \times \{r_+\}$ for a finite r_+) is a (a_*, b_*)-*analytic* (m,n)-*resonant RIC* when C is an analytic (m,n)-resonant RIC (respectively, (m,n)-resonant boundary) and there exist some coordinates (x,y) such that $C \equiv \{y = 0\}$ and the power map $f = g^n$ satisfies the properties (i)–(iii) stated in Lemma 1.

The map (3.6) can be viewed as a perturbation of the integrable twist map

$$x_1 = x + y, \qquad y_1 = y. \tag{3.9}$$

We want to reduce the size of the nonintegrable terms $\mathrm{O}(y^2)$ and $\mathrm{O}(y^3)$ as much as possible. We can reduce them through normal form steps up to any desired order; see Lemma 11. Thus, the nonintegrable part of the dynamics is beyond all order in y; that is, in the distance to the resonant RIC (or resonant boundary). A general principle in conservative dynamical systems states that beyond all order phenomena are often exponentially small in the analytic category. Our goal is to write the map as an exponentially small perturbation in y of the integrable twist map (3.9). The final result is stated in the following theorem.

Theorem 3. *Let $f : \mathbb{T} \times I \to \mathbb{T} \times I$ be an analytic map with a (a_*, b_*)-analytic RIC C of fixed points such that $C \subset \mathbb{T} \times I$, $C = \mathbb{T} \times \{r_-\}$ for a finite r_-, or $C = \mathbb{T} \times \{r_+\}$ for a finite r_+. Let $m \geq 2$ be an arbitrary order. Let $\alpha \in (0, a_*)$. There exist constants $K > 0$ and $b'_* \in (0, b_*)$ such that, if $b \in (0, b'_*)$, then there exists an analytic change of variables $(x, y) = \Phi(\xi, \eta)$ such that:*

1. *It is uniformly (with respect to b) close to the identity on $\mathbb{T} \times (-b, b)$. That is,*

$$x = \xi + \mathrm{O}(\eta), \qquad y = \eta + \mathrm{O}(\eta^2), \qquad \det[\mathrm{D}\Phi(\xi, \eta)] = 1 + \mathrm{O}(\eta),$$

   for all $(\xi, \eta) \in \mathbb{T} \times (-b, b)$, where the $\mathrm{O}(\eta)$ and $\mathrm{O}(\eta^2)$ terms are uniform in b; and

2. *The transformed map $(\xi, \eta) \mapsto (\xi_1, \eta_1)$ is real analytic and has the intersection property on the cylinder $\mathbb{T} \times (-b, b)$. Besides, it has the form*

$$\xi_1 = \xi + \eta + \eta^m g_1(\xi, \eta), \qquad \eta_1 = \eta + \eta^{m+1} g_2(\xi, \eta), \tag{3.10}$$

   where $|g_j(\xi, \eta)| \leq K \mathrm{e}^{-2\pi\alpha/b}$ and $|\partial_i g_j(\xi, \eta)| \leq K b^{-2}$ for all $(\xi, \eta) \in \mathbb{T} \times (-b, b)$.

The proof can be found in Section 3.5.

Consider a perturbed Hamiltonian system which is close to an integrable system. It is known that, under the appropriate nondegeneracy conditions, the measure of the set of tori which decompose under the perturbation can be bounded from above by a quantity of order $\sqrt{\epsilon}$, ϵ being the perturbation parameter [Nei81, Pös82]. Neishtadt [Nei81] also considered a context where the perturbation becomes exponentially small in some parameter ϵ and hence the measure of the complementary set which is cut out from phase space by the invariant tori is of order $\mathrm{e}^{-c/\epsilon}$, c being a positive constant. This argument could be applied to our context. First, any neighbourhood of an analytic resonant RIC (or resonant boundary) of an analytic exact twist map contains infinitely many RICs. Second, the area of the complementary of the RICs in any of such neighbourhoods is exponentially small in the size of the neighbourhood. Third, the gaps between the RICs are exponentially small in their distance to the resonant RIC. The first result follows from the classical Moser twist theorem [SM95]. The others follow from the ideas explained above.

3.2.2 Difference of periodic actions

Let $\omega = -\mathrm{d}\lambda$, with $\lambda = \nu(r)\mathrm{d}s$, be a fixed exact symplectic form on the open cylinder $\mathbb{T} \times I$, with $I = (r_-, r_+)$. Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be an exact twist map that can be extended as rigid rotations of angles ϑ_- and ϑ_+ to the boundaries $C_- = \mathbb{T} \times \{r_-\}$ and $C_+ = \mathbb{T} \times \{r_+\}$, respectively. We know that $\vartheta_- < \vartheta_+$ from the twist condition. Let $G : \mathbb{R} \times I \to \mathbb{R} \times I$ be a fixed lift of g. Let $E = \{(s, s_1) \in \mathbb{R}^2 : \vartheta_- < s_1 - s < \vartheta_+\}$. Then there exists a function $h : E \to \mathbb{R}$, determined modulo an additive constant, such that

$$G^*\lambda - \lambda = \mathrm{d}h.$$

The function h is called the *Lagrangian* or *generating function* of g.

Let p and q be two relatively prime integers such that $\vartheta_- < p/q < \vartheta_+$ and $q \geq 1$. A point $(s,r) \in \mathbb{R} \times I$ is (p,q)*-periodic* when $G^q(s,r) = (s+p,r)$. The corresponding point $(s,r) \in \mathbb{T} \times I$ is a periodic point of period q by g that is translated p units in the base by the lift. A (p,q)-periodic orbit is *Birkhoff* when it is ordered around the cylinder in the same way that the orbits of the rigid rotation of angle p/q. See [KH95] for details. The Poincaré-Birkhoff Theorem states that there exist at least two different Birkhoff (p,q)-periodic orbits [KH95, Mei92].

Let $O = \{(s_k, r_k)\}_{k\in\mathbb{Z}}$ be a (p,q)-periodic orbit. Its (p,q)*-periodic action* is

$$W^{(p,q)}[O] = h(s_0,s_1) + h(s_1,s_2) + \cdots + h(s_{q-1}, s_0 + p).$$

Our goal is to establish an exponentially small bound for the non-negative quantity

$$\Delta^{(p,q)} = \sup_{O,\bar{O}\in\mathcal{O}_g^{(p,q)}} \left| W^{(p,q)}[\bar{O}] - W^{(p,q)}[O] \right|,$$

where $\mathcal{O}_g^{(p,q)}$ denotes the set of all Birkhoff (p,q)-periodic orbits of the exact twist map g. The difference of (p,q)-periodic actions can be interpreted as the ω-area of certain domains.

Let us explain it.

Let $O = \{(s_k,r_k)\}_{k\in\mathbb{Z}}$ and $\bar{O} = \{(\bar{s}_k,\bar{r}_k)\}_{k\in\mathbb{Z}}$ be two (p,q)-periodic orbits. We can assume, without loss of generality, that $0 < \bar{s}_0 - s_0 < 1$. Let L_0 be a curve from (s_0,r_0) to $(\bar{s}_0,\bar{r}_0)$ contained in $\mathbb{R}\times I$. Set $L_k = G^k(L_0)$. The curves L_0 and L_q have the same endpoints in $\mathbb{T}\times I$. Let us assume that these two curves have no topological crossing on the cylinder $\mathbb{T}\times I$ and let $B \subset \mathbb{T}\times I$ be the domain enclosed between them.

Observe that $\int_{L_{k+1}} \lambda - \int_{L_k} \lambda = \int_{L_k} (G^*\lambda - \lambda) = \int_{L_k} \mathrm{d}h = h(\bar{s}_k,\bar{s}_{k+1}) - h(s_k,s_{k+1})$. Hence, $\sum_{k=0}^{q-1} \big(h(\bar{s}_k,\bar{s}_{k+1}) - h(s_k,s_{k+1})\big) = \int_{L_q}\lambda - \int_{L_0}\lambda = \int_{L_q - L_0}\lambda = \pm\int_B \omega$, where the sign $\pm$ depends on the orientation of the closed path $g^q(L_0) - L_0$, but we do not need it, because we take absolute values in both sides of the previous relation:

$$\left| W^{(p,q)}[\bar{O}] - W^{(p,q)}[O] \right| = \int_B \omega =: \mathrm{Area}_\omega(B).$$

These arguments go back to the *MacKay-Meiss-Percival action principle* [MMP84, Mei92]. If the curves L_0 and $g^q(L_0)$ have some topological crossing, then the domain B has several connected components, in which case $\left| W^{(p,q)}[\bar{O}] - W^{(p,q)}[O] \right| \leq \int_B \omega =: \mathrm{Area}_\omega(B)$, because the sign in front of the integral of the area form ω depends on the connected component.

If the analytic exact twist map g has a (m,n)-resonant RIC, then

$$\Delta^{(m,n)} = 0.$$

Indeed, we can take a segment of the RIC as the curve L_0 used in the previous construction in such a way that $g^n(L_0) = L_0$ and $\mathrm{Area}_\omega(B) = 0$.

It turns out that the differences of (p,q)-periodic actions of g are exponentially small when p/q is "sufficiently close" to m/n. The meaning of "sufficiently close" is clarified in the following theorem. See also Remark 3.

Theorem 4. *Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be an analytic exact twist map that can be extended as rigid rotations of angles ϑ_- and ϑ_+ to the boundaries $C_- = \mathbb{T} \times \{r_-\}$ and $C_+ = \mathbb{T} \times \{r_+\}$, respectively. We also assume that g has a (a_*, b_*)-analytic (m,n)-resonant RIC $C \subset \mathbb{T} \times I$, $C = \mathbb{T} \times \{r_-\}$ for a finite r_-, or $C = \mathbb{T} \times \{r_+\}$ for a finite r_+. Let $\alpha \in (0, a_*)$ and $L \geq 1$. There exist $K, q_* > 0$ such that*

$$\Delta^{(p,q)} \leq K \exp\left(-\frac{2\pi\alpha q}{|np - mq|}\right), \tag{3.11}$$

for all relatively prime integers p and q with $1 \leq |np - mq| \leq L$ and $q \geq q_*$.

The proof has been placed at Section 3.6.

Remark 1. If $C = \mathbb{T} \times \{r_-\}$ (respectively, $C = \mathbb{T} \times \{r_+\}$), the bound (3.11) only makes sense as p/q tends to the boundary rotation number $\vartheta_- = m/n$ (respectively, $\vartheta_+ = m/n$) from the right: $p/q \to (m/n)^+$ (respectively, from the left: $p/q \to (m/n)^-$). We will see several examples of this situation in Sections 3.3 and 3.4.

Remark 2. Infinite values of $r_\pm$ can also be dealt with. For instance, if $r_+ = +\infty$, then we consider the coordinate $v = 1/r$ and we assume that: 1) $C_+ = \{v = 0\}$ is a (m,n)-resonant boundary; 2) g can be analytically extended to $v \geq 0$; 3) $\omega = \beta(v)\mathrm{d}s \wedge \mathrm{d}v$ with $\beta(v) = \beta_* v^j + \mathrm{O}(v^{j+1})$ for some real number $\beta_* \neq 0$ and some integer $j \neq -2$; 4) $f = g^n$ has the form (3.7) for some real analytic 1-periodic functions $\varphi(s)$ and $\psi(s)$; and 5) $\varphi(s)$ is negative. Then the change $(s,r) \mapsto (s, v = 1/r)$ transforms the infinite setup into a finite one, so Theorem 4 still holds. We just mention two subtle points. First, the change (3.8) is not well-defined when $j = -2$. Thus, $\beta(v)$ can have a pole at $v = 0$, provided it is not a double one. Second, $\varphi(s)$ is negative because the change $(s,r) \mapsto (s, v = 1/r)$ reverses the orientation, and so, it changes the sign of the twist.

Remark 3. Condition $|np - mq| \leq L$ implies that $|p/q - m/n| = \mathrm{O}(1/q)$ as $q \to +\infty$.

Remark 4. There are infinitely many pairs of relatively prime integers p and q such that $1 \leq |np - mq| \leq L$ and $q \geq q_*$. This is a consequence of Bézout's identity because m and n are also relatively prime integers.

Remark 5. In many applications, C is a RIC of fixed points; that is, a $(0,1)$-resonant RIC. Then Theorem 4 implies that $\Delta^{(p,q)}$ is exponentially small in the period q when p remains uniformly bounded. To be precise, if $\alpha \in (0, a_*)$ and $L \geq 1$, there exist $K, q_* > 0$ such that

$$\Delta^{(p,q)} \leq K \mathrm{e}^{-2\pi\alpha q/|p|}$$

for all relatively prime integers p and q with $q \geq q_*$ and $0 \neq |p| \leq L$.

3.3 On the length spectrum of analytic convex domains

3.3.1 Convex billiards

We briefly review some of the results about billiards stated in Section 2.2.

Let Q be a closed strictly convex curve in the Euclidean plane $\mathbb{R}^2$. We assume, without loss of generality, that Q has length one. Let $s \in \mathbb{T}$ be an arc-length parameter on Q. Set $I = (r_-, r_+) = (0, \pi)$. Let $f : \mathbb{T} \times I \to \mathbb{T} \times I$, $(s_1, r_1) = f(s, r)$, be the map that models the billiard dynamics inside Q using the *Birkhoff coordinates* (s, r), where $s \in \mathbb{T}$ determines the impact point on the curve, and $r \in I$ denotes the angle of incidence.

The map f preserves the exact symplectic form $\omega = \nu'(r)\mathrm{d}s \wedge \mathrm{d}r = \sin r \mathrm{d}s \wedge \mathrm{d}r$ and has the intersection property. Indeed, $f : \mathbb{T} \times I \to \mathbb{T} \times I$ is an exact twist map with boundary rotation numbers $\vartheta_- = 0$ and $\vartheta_+ = 1$. Besides, its Lagrangian is minus the distance between consecutive impact points. Finally, f is analytic when Q is analytic.

Any (p, q)-periodic orbit on the billiard map forms a a closed inscribed polygon with q sides that makes p turns inside Q. Since the Lagrangian of the billiard map is minus the distance between consecutive impact points, the periodic action of a periodic orbit is just the total length of the corresponding polygon up to the sign. Therefore, the supremum action difference among (p, q)-periodic billiard orbits is the supremum length difference among inscribed billiard (p, q)-polygons.

3.3.2 Study close to the boundary of the billiard table

Let us check that the boundary $C_- = \mathbb{T} \times \{0\}$ satisfies the hypotheses stated in Theorem 4.

Proposition 5. *If f is the billiard map associated to an analytic strictly convex curve Q, the boundary C_- is a (a_*, b_*)-analytic $(0,1)$-resonant RIC of f for some $a_* > 0$ and $b_* > 0$.*

Proof. The lower boundary C_- is a $(0, 1)$-resonant because $\vartheta_- = 0$. Clearly, $r_- = 0$ is finite and the function $\nu(r) = \sin r$ can be analytically extended to $[0, \pi)$. Hence, in order to end the proof we only need to prove that the billiard map f can be analytically extended to $\mathbb{T} \times [0, \pi)$ and the twist condition holds on C_-; see Lemma 2.

Let $\gamma : \mathbb{T} \to Q$, $\kappa : \mathbb{T} \to (0, +\infty)$, and $n : \mathbb{T} \to \mathbb{R}^2$ be an arc-length parametrization, the curvature, and a unit normal vector of the analytic strictly convex curve linked by the relation $\gamma''(s) = \kappa(s)n(s)$.

We write the angle of incidence as a function of consecutive impact points: $r = r(s, s_1)$. If $s_1 \neq s$, then $r(s, s_1)$ is analytic, $r(s, s_1) \in (0, \pi)$, and $\partial_2 r(s, s_1) > 0$. The last property follows from the twist character of the billiard map. Let us study what happens on C_-; or, equivalently, what happens in the limit $s_1 \to s^+$. We note that r is the angle between $\gamma'(s)$ and $\gamma(s_1) - \gamma(s)$. Hence,

$$\tan r = \frac{\sin r}{\cos r} = \frac{\det\left(\gamma'(s), \gamma(s_1) - \gamma(s)\right)}{\langle \gamma'(s), \gamma(s_1) - \gamma(s) \rangle} = \frac{\int_0^1 \det\left(\gamma'(s), \gamma'(s + t(s_1 - s))\right) \mathrm{d}t}{\int_0^1 \langle \gamma'(s), \gamma'(s + t(s_1 - s)) \rangle \mathrm{d}t}.$$

This expression shows that $r(s, s_1)$ is analytic when $s_1 = s$ and $r(s, s) = 0$. Besides,

$$\lim_{s_1 \to s^+} \partial_2 r(s, s_1) = \lim_{s_1 \to s^+} \cos^2 r(s, s_1) \partial_2 \tan r(s, s_1) = \kappa(s)/2 > 0, \tag{3.12}$$

for all $s \in \mathbb{T}$, which implies that $(s, s_1) \mapsto (s, r)$ is an analytic diffeomorphism that maps a neighbourhood of the diagonal in $\mathbb{T}^2$ to a neighborhood of the lower boundary $C_- = \mathbb{T} \times \{0\}$. Next, we write the billiard map $(s_1, r_1) = f(s, r)$ as the composition of three maps:

$$(s, r) \mapsto (s, s_1) \mapsto (s_1, s) \mapsto (s_1, r_1).$$

We have already seen that the first map is analytic in a neighborhood of C_-. Clearly, the second map is analytic. The third one is also analytic, because $r_1 = r_1(s, s_1) = \pi - r(s_1, s)$ by definition of billiard map. Thus, f can be analytically extended to $\mathbb{T} \times [0, \pi)$. The twist condition on C_- follows from inequality (3.12). $\square$

We can compare the coordinates (3.8) with the coordinates defined by Lazutkin in [Laz73]. We note that $\nu'(r) = \sin r = r + \mathrm{O}(r^2)$, so $j = 1$ in the change (3.8). If $\varrho(s) = 1/\kappa(s)$ is the radius of curvature of Q, then the Taylor expansion around $r = 0$ of the billiard map is

$$\begin{cases} s_1 = s + 2\varrho(s) r + 4\varrho(s)\varrho'(s) r^2/3 + \mathrm{O}(r^3), \\ r_1 = r - 2\varrho'(s) r^2/3 + \left(4(\varrho'(s))^2/9 - 2\varrho(s)\varrho''(s)/3\right) r^3 + \mathrm{O}(r^4). \end{cases}$$

From this Taylor expansion, Lazutkin deduced that the billiard map takes the form

$$x_1 = x + y + \mathrm{O}(y^3), \qquad y_1 = y + \mathrm{O}(y^4)$$

in the analytic *Lazutkin coordinates* (x, y) defined by

$$x = C^{-1} \int_0^s \varrho^{-2/3}(t) \mathrm{d}t, \qquad y = 4C^{-1} \varrho^{1/3}(s) \sin(r/2), \quad C = \int_0^1 \varrho^{-2/3}(t) \mathrm{d}t.$$

The constant C is sometimes called the *Lazutkin perimeter* and has been determined in such a way that the new angular coordinate x is defined modulus one. Lazutkin's results are more refined, he wrote the billiard map as a smaller perturbation of the integrable twist map $(x_1, y_1) = (x + y, y)$, but we do not need it.

By direct application of Proposition 5 and Theorem 4, we get the following exponentially small upper bound of the quantities $\mathcal{L}^{(p,q)}$ defined in the introduction.

Theorem 6. *Let Q be an analytic strictly convex curve in the Euclidean plane. Let $a_* > 0$ be the analyticity strip width of the lower boundary C_-. Let $\alpha \in (0, a_*)$ and $L \geq 1$. There exist a constant $K > 0$ and a period $q_* \geq 1$ such that*

$$\mathcal{L}^{(p,q)} \leq K \mathrm{e}^{-2\pi\alpha q/p},$$

for all relatively prime integers p and q with $q \geq q_*$ and $0 < p \leq L$.

The same exponentially small upper bound holds for analytic geodesically strictly convex curves on surfaces of constant curvature, where the billiard trajectories are just broken geodesics. Billiard maps on the Klein model of the hyperbolic plane $\mathbb{H}^2$ and on the positive hemisphere $\mathbb{S}^2_+$ have been studied, for instance, in [Cou14], where it is shown that they are exact twist maps with the same boundary rotation numbers as in the Euclidean case. Therefore, by local isometry arguments, we can write a version of Theorem 6 on any surface of constant curvature.

3.3.3 Billiard tables of constant width

Definition 7. A smooth closed convex curve is of *constant width* if and only if it has a chord in any direction perpendicular to the curve at both ends.

Billiards inside convex curves of constant width have a nice property [Kni98, Gut12]. Let us explain it.

The billiard map associated to a smooth convex curve of constant width has the horizontal line $\mathbb{T}\times\{\pi/2\}$ as a resonant $(1,2)$-RIC. Any trajectory belonging to that RIC is orthogonal to the curve at its two endpoints. Due to the variational formulation, all the $(1,2)$-periodic orbits are extrema of the $(1,2)$-periodic action and, thus, all $(1,2)$-periodic trajectories have the same length, which is the reason we refer to them as constant width curves.

Another characterization of constant width curves is the following. We reparametrize the curve by using the angle $\varphi \in (0, 2\pi)$ between the tangent vector at a point in the curve and some fixed line. Let $\varrho(\varphi)$ be the radius of curvature at this point. The curve has constant width if and only if the Fourier series of $\varrho(\varphi)$ contains no other even coefficients than the constant term. Thus, the space of analytic constant width curves has infinite dimension and codimension. We show an example of a constant width billiard table in Figure 3.1.

Theorem 4 can be directly applied in this context and we have the following result.

Theorem 7. *Let $g : \mathbb{T} \times I \to \mathbb{T} \times I$ be the billiard map of an analytic strictly convex curve of constant width. Let $a_* > 0$ be the analyticity strip width of the $(1,2)$-resonant RIC of g. Let $\alpha \in (0, a_*)$ and $L \geq 1$. There exist a constant $K > 0$ and a period $q_* \geq 1$ such that*

$$\mathcal{L}^{(p,q)} \leq K \exp\left(-\frac{\pi\alpha}{|p/q - 1/2|}\right),$$

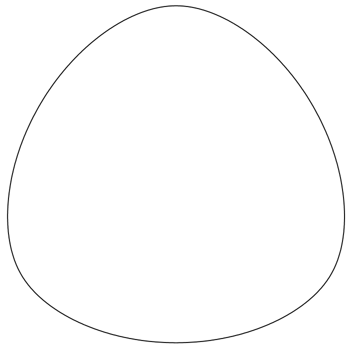


Figure 3.1: Billiard table of constant width. Its radius of curvature is the function $\varrho(\varphi) = 1 + \cos(3\varphi)/2$.

for all relatively prime integers p and q such that $1 \leq |2p - q| \leq L$ and $q \geq q_*$.

One could try to generalize constant width billiards, where $\mathbb{T} \times \{\pi/2\}$ is a $(1, 2)$-resonant RIC, to *constant angle tables*, where $\mathbb{T} \times \{r_0\}$ is assumed to be a (m, n)-resonant RIC. However, the only table such that $\mathbb{T} \times \{r_0\}$ is a (m, n)-resonant RIC, with $(m, n) \neq (1, 2)$, is the circle. See [Gut12, Cyr12]. By the way, Theorem 4 applies to this case but, since the circular billiard is integrable, $\mathcal{L}^{(m,n)} \equiv 0$, for all (m, n). In fact, the circular billiard map is globally conjugated to the integrable twist map (3.9).

There are more billiard tables with resonant RICs, but their RICs are not horizontal. For instance, the elliptic table has all possible (m, n)-resonant RICs, but the $(1, 2)$-resonant one. Hence, in this case, $\mathcal{L}^{(m,n)} = 0$. Baryshnikov and Zharnitsky [BZ06] proved that an ellipse can be infinitesimally perturbed so that any chosen resonant RIC will persist. Innami [Inn88] found a condition on the billiard table that guarantees the existence of a $(1, 3)$-resonant RIC. However, Theorem 4 can not be applied in such cases, because both the Baryshnikov-Zharnitsky and the Innami constructions are done in the smooth category, where we can only claim that $\mathcal{L}^{(p,q)}$ is beyond all order in the difference between rotation numbers.

3.4 On the area spectrum of analytic convex domains

3.4.1 Dual billiards

We recall some well-known facts about dual billiards that can be found in [Boy96, GK95, Tab95a].

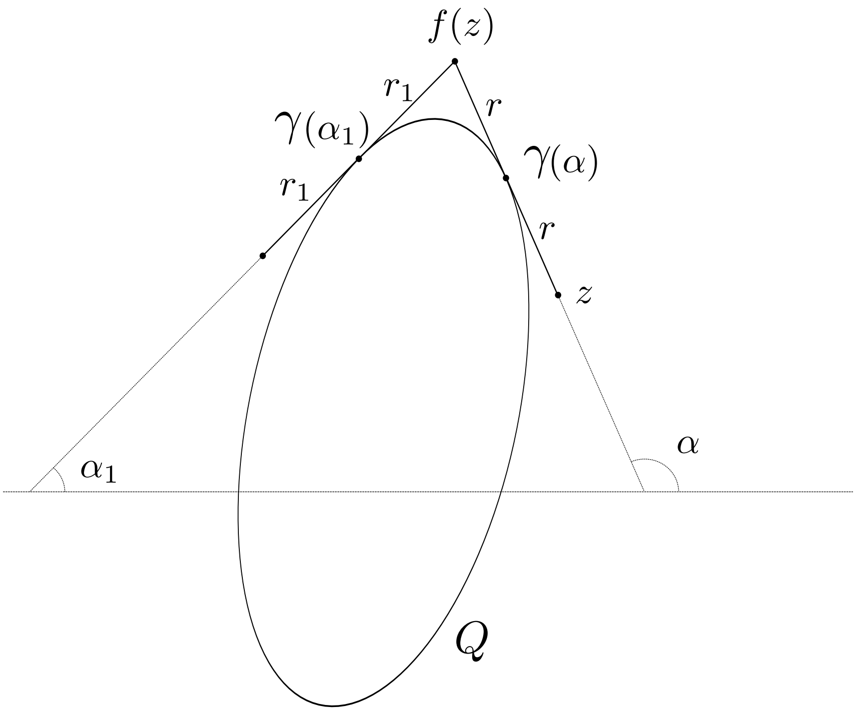


Figure 3.2: The envelope coordinates (α, r) and the dual billiard map $f : \mho \to \mho$.

Let Q be a strictly convex closed curve in the Euclidean plane $\mathbb{R}^2$. Let $\mho$ be the unbounded component of $\mathbb{R}^2 \setminus Q$. The dual billiard map $f : \mho \to \mho$ is defined as follows: $f(z)$ is the reflection of z in the tangency point of the oriented tangent line to Q through z. This map is area-preserving. Next, we introduce the *envelope coordinates* $(\alpha, r) \in \mathbb{T}_\star \times I$ of a point $z \in \mho$. In this section, $\mathbb{T}_\star = \mathbb{R}/2\pi\mathbb{Z}$ and $I = (0, +\infty)$. We recall that $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Given a point $z \in \mho$, let $\alpha \in \mathbb{T}_\star$ be the angle made by the positive tangent line to Q in the direction of z with a fixed direction of the plane, and let $r \in I$ be the distance along this line from Q to z. See Figure 3.2.

The dual billiard map preserves the exact symplectic form $\omega = \nu'(r)\mathrm{d}\alpha\mathrm{d}r = r\mathrm{d}\alpha\mathrm{d}r$, and it has the intersection property. Indeed, $f : \mathbb{T}_\star \times I \to \mathbb{T}_\star \times I$, $(\alpha_1, r_1) = f(\alpha, r)$, is an exact twist map with boundary rotation numbers $\vartheta_- = 0$ and $\vartheta_+ = \pi$. Its Lagrangian is the area enclosed by Q and the tangent lines through the points on Q with coordinates α and α_1.

Any (p, q)-periodic orbit on the dual billiard map forms a closed circumscribed polygon with q sides that makes p turns outside Q. Since the Lagrangian of the dual billiard map is the above-mentioned area, the periodic action of a periodic orbit is just the area enclosed between the corresponding polygon and Q, taking into account some multiplicities when $p \geq 2$. Therefore, the supremum action difference among (p, q)-periodic dual billiard orbits is the supremum area difference among circumscribed dual billiard (p, q)-polygons.

3.4.2 Study close to the curve

We note that $r \to 0^+$ when the point $z \in \mho$ approaches to the curve Q. Therefore, in order to study the dual billiard dynamics close to Q, we must study the dual billiard map f in a neighbourhood of the lower boundary $C_- = \mathbb{T}_\star \times \{0\}$ of $\mathbb{T}_\star \times I$. Let us check that C_- satisfies the hypotheses stated in Theorem 4.

Proposition 8. *If f is the dual billiard map associated to an analytic strictly convex curve Q, the lower boundary C_- is an (a_*, b_*)-analytic $(0,1)$-resonant RIC of f for some $a_* > 0$ and $b_* > 0$.*

Proof. The lower boundary C_- is a $(0,1)$-resonant because $\vartheta_- = 0$. Clearly, $r_- = 0$ is finite and the function $\nu(r) = r$ can be analytically extended to $[0, +\infty)$. Hence, in order to end the proof we only need to prove that the dual billiard map f can be analytically extended to $\mathbb{T}_\star \times [0, +\infty)$ and the twist condition holds on C_-; see Lemma 2.

We write the distance r as a function of consecutive tangent points: $r = r(\alpha, \alpha_1)$. We know that $r(\alpha, \alpha_1)$ is analytic, $r(\alpha, \alpha_1) > 0$, and $\partial_2 r(\alpha, \alpha_1) > 0$ when $\alpha_1 \notin \{\alpha, \alpha + \pi\}$.[1] The last property follows from the twist character of the dual billiard map.

Let us study what happens on C_-; or, equivalently, what happens in the limit $\alpha_1 \to \alpha$. Let $\varrho : \mathbb{T}_\star \to (0, +\infty)$ be the radius of curvature of Q in the angular coordinate α. Set $\alpha_1 = \alpha + \delta$. From Boyland [Boy96], we know that

$$r(\alpha, \alpha_1) = \frac{\int_\alpha^{\alpha_1} \sin(v - \alpha)\varrho(v)\mathrm{d}v}{\sin(\alpha_1 - \alpha)} = \frac{\delta}{\sin \delta} \int_0^1 \sin(\delta t)\varrho(\alpha + \delta t)\mathrm{d}t. \tag{3.13}$$

This expression shows that $r(\alpha, \alpha_1)$ is analytic when $\alpha_1 = \alpha$ and $r(\alpha, \alpha) = 0$. By taking derivatives in relation (3.13) with respect to α_1, we get that

$$\partial_2 r(\alpha, \alpha) = \varrho(\alpha)/2 > 0, \tag{3.14}$$

for all $\alpha \in \mathbb{T}_\star$, which implies that $(\alpha, \alpha_1) \mapsto (\alpha, r)$ is an analytic diffeomorphism that maps a neighbourhood of the diagonal in $\mathbb{T}_\star^2$ to a neighborhood of the lower boundary $C_- = \mathbb{T}_\star \times \{0\}$. Next, we write the billiard map $(\alpha_1, r_1) = f(\alpha, r)$ as the composition of three analytic maps:

$$(\alpha, r) \mapsto (\alpha, \alpha_1) \mapsto (\alpha_1, \alpha) \mapsto (\alpha_1, r_1).$$

Thus, f can be analytically extended to $\mathbb{T}_\star \times [0, +\infty)$. The twist condition on C_- follows from inequality (3.14). □

We can compare the coordinates (3.8) with the coordinates defined by Tabachnikov in [Tab95b]. We note that $\nu'(r) = r$, so $j = 1$ in the change (3.8). If $\kappa(\alpha)$ is the curvature of Q, then the Taylor expansion around $r = 0$ of the dual billiard map is

$$\alpha_1 = \alpha + 2\kappa(\alpha)r + \mathrm{O}(r^2), \qquad r_1 = r - 2\kappa'(\alpha)r^2/3 + \mathrm{O}(r^3).$$

[1] Tangent lines through points with coordinates α and $\alpha + \pi$ are parallel, so $\lim_{\alpha_1 - \alpha \to \pi} r(\alpha, \alpha_1) = +\infty$.

(Tabachnikov wrote this Taylor expansion in coordinates (s, r), where s is an arc-length parameter, but his result can be easily adapted.) From this Taylor expansion, one deduces that the dual billiard map takes the form (3.6) in the analytic *Tabachnikov coordinates*

$$x = C^{-1} \int_0^\alpha \kappa^{-2/3}(v) \mathrm{d}v, \qquad y = 2C^{-1} \kappa^{1/3}(\alpha) r, \quad C = \int_0^{2\pi} \kappa^{-2/3}(v) \mathrm{d}v.$$

The constant C has been determined in such a way that x is defined modulus one: $x \in \mathbb{T}$.

We get the following exponentially small upper bound of the quantities $\mathcal{A}^{(p,q)}$ defined in the introduction by direct application of Proposition 8 and Theorem 4.

Theorem 9. *Let Q be an analytic strictly convex curve in the Euclidean plane. Let $a_* > 0$ be the analyticity strip width of the lower boundary C_-. Let $\alpha \in (0, a_*)$ and $L \geq 1$. There exist a constant $K > 0$ and a period $q_* \geq 1$ such that*

$$\mathcal{A}^{(p,q)} \leq K \mathrm{e}^{-2\pi \alpha q / p},$$

for all relatively prime integers p and q with $q \geq q_*$ and $0 < p \leq L$.

Tabachnikov [Tab02] studied the dual billiard map in the hyperbolic plane $\mathbb{H}^2$, and extended the asymptotic expansion (3.2) to that new setting. He also claimed that there exists an analogous formula for dual billiards on the unit sphere $\mathbb{S}^2$. Therefore, by local isometry arguments, we can write a version of Theorem 9 on any surface of constant curvature.

3.4.3 Study far away from the curve

We note that $r \to +\infty$ when the point $z \in \mho$ moves away from the curve Q. We use the coordinates (α, v) to work at infinity, where $v = 1/r$ and $(\alpha, r) \in \mathbb{T}_\star \times I$ are the coordinates introduced in Subsection 3.4.1. The exact symplectic form $\omega = r \mathrm{d}\alpha \mathrm{d}r$ becomes $\omega = -v^{-3} \mathrm{d}\alpha \mathrm{d}v$ in coordinates (α, v). Tabachnikov [Tab95b] realized that the dual billiard map at infinity can be seen as a map defined in a neighbourhood of the $(1, 2)$-resonant RIC $C_+ \equiv \{v = 0\}$. To be precise, he saw that the dual billiard map can be analytically extended to $v \geq 0$, its square has the form

$$\alpha_1 = \alpha + \varphi(\alpha) v + \mathrm{O}(v^2), \qquad v_1 = v + \psi(\alpha) v^2 + \mathrm{O}(v^3),$$

for some real analytic 1-periodic functions $\varphi(\alpha)$ and $\psi(\alpha)$, and $\varphi(\alpha)$ is negative. Hence, the following result is deduced from Remark 2.

Theorem 10. *Let Q be an analytic strictly convex curve in the Euclidean plane. Let $a_* > 0$ be the analyticity strip width of the boundary C_+. Let $\alpha \in (0, a_*)$ and $L \geq 1$. There exist a constant $K > 0$ and a period $q_* \geq 1$ such that*

$$\mathcal{A}^{(p,q)} \leq K \exp\left(-\frac{\pi \alpha}{|p/q - 1/2|} \right),$$

for all relatively prime integers p and q such that $1 \leq |2p - q| \leq L$ and $q \geq q_*$.

3.5 Proof of Theorem 3

3.5.1 Spaces, norms, and projections

Let $\mathcal{X}_{a,b}$, with $a > 0$ and $b > 0$, be the space of all analytic functions g defined on the open set

$$D_{a,b} = \{(x, y) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{C} : |\Im x| < a,\ |y| < b\}$$

with bounded Fourier norm

$$\|g\|_{a,b} = \sum_{k\in\mathbb{Z}} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|a},$$

where $\hat{g}_k(y)$ denotes the k-th Fourier coefficient of the 1-periodic function $g(\cdot, y)$ and

$$|\hat{g}_k|_b = \sup\{|\hat{g}_k(y)| : y \in B_b\}$$

denotes its sup-norm on the complex open ball $B_b = \{y \in \mathbb{C} : |y| < b\}$. Let

$$|g|_{a,b} = \sup\{|g(x, y)| : (x, y) \in D_{a,b}\}$$

be the sup-norm on $D_{a,b}$.

Let $\mathcal{X}_{a,b,m}$ be the space of all vectorial functions $G : D_{a,b} \to \mathbb{C}^2$ of the form

$$G(x, y) = (y^m g_1(x, y), y^{m+1} g_2(x, y))$$

such that $g_1, g_2 \in \mathcal{X}_{a,b}$. The space $\mathcal{X}_{a,b,m}$ is a Banach space with the Fourier norm $\|G\|_{a,b,m} = \max\{\|g_1\|_{a,b}, \|g_2\|_{a,b}\}$. The sup-norm $|\cdot|_{a,b,m}$ on $\mathcal{X}_{a,b,m}$ is defined analogously.

Let $g_2^*(y) = \int_0^1 g_2(x, y)\mathrm{d}x$ be the average of $g_2(x, y)$. Let $\mathcal{X}_{a,b,m} = \mathcal{X}^*_{a,b,m} \oplus \mathcal{X}^\bullet_{a,b,m}$ be the direct decomposition where $\mathcal{X}^*_{a,b,m}$ is the vectorial subspace of the elements of the form $G^*(x, y) = (0, y^{m+1} g_2^*(y))$, whereas $\mathcal{X}^\bullet_{a,b,m}$ is the one of the elements with $g_2^*(y) = 0$. Let $\pi^* : \mathcal{X}_{a,b,m} \to \mathcal{X}^*_{a,b,m}$ and $\pi^\bullet : \mathcal{X}_{a,b,m} \to \mathcal{X}^\bullet_{a,b,m}$ be the associated projections. Thus, any $G \in \mathcal{X}_{a,b,m}$ can be decomposed as $G = G^* + G^\bullet$, where

$$G^* = \pi^*(G) = (0, y^{m+1} g_2^*(y)) \in \mathcal{X}^*_{a,b,m}, \qquad G^\bullet = \pi^\bullet(G) \in \mathcal{X}^\bullet_{a,b,m}.$$

Obviously, $\|G^*\|_{a,b,m} \leq \|G\|_{a,b,m}$ and $\|G^\bullet\|_{a,b,m} \leq \|G\|_{a,b,m}$

We will always denote the scalar functions in $\mathcal{X}_{a,b}$ with lower-case letters, and the vectorial functions in $\mathcal{X}_{a,b,m}$ with upper-case letters. Asterisk and bullet superscripts in upper-case letters stand for the π^*-projections and $\pi^\bullet$-projections of vectorial functions in $\mathcal{X}_{a,b,m}$, respectively. Asterisk superscripts in lower-case letters denote averages of scalar functions in $\mathcal{X}_{a,b}$. We will always write the couple of scalar functions associated to any given vectorial function of $\mathcal{X}_{a,b,m}$ with the corresponding lower-case letter and the subscripts $j = 1, 2$. Hat symbols denote Fourier coefficients.

3.5.2 The averaging and the iterative lemmas

Henceforth, let $A(x, y) = (x + y, y)$ be the integrable twist map introduced in (3.9). Let $F = F_2$ be a map satisfying the properties listed in Lemma 1, so $F_2 = A + G_2$ for some $G_2 \in \mathcal{X}_{a_2,b_2,2}$, where $a_2 = a_*$ is the analyticity strip width in the angular variable x, and $b_2 = b_*$ is the analyticity radius in y. Hence, F_2 is a perturbation of A of order two. The following lemma allows us to increase that order as much as we want by simply losing as little analyticity strip width as we want. It is based on classical averaging methods. In particular, we see that F is a perturbation beyond all order of A.

Lemma 11 (Averaging Lemma)**.** *Let $F_2 = A + G_2$, with $G_2 \in \mathcal{X}_{a_2,b_2,2}$ and $a_2 > 0$ and $b_2 > 0$, be a real analytic map with the intersection property on the cylinder $\mathbb{T}\times(-b_2, b_2)$. Let $m \geq 3$ be an integer. Let a_m be any analyticity strip width such that $a_m \in (0, a_2)$.*

There exist an analyticity radius $b_m \in (0, b_2)$ and a change of variables of the form $\Phi_m = \mathrm{I}+\Psi_m$ for some $\Psi_m \in \mathcal{X}_{a_m,b_m,1}$ such that the transformed map $F_m = \Phi_m^{-1}\circ F_2\circ\Phi_m$ is real analytic, has the intersection property on the cylinder $\mathbb{T} \times (-b_m, b_m)$, and has the form $F_m = A + G_m$ for some $G_m \in \mathcal{X}_{a_m,b_m,m}$.

Besides, the change of variables Φ_m is close to the identity on $\mathbb{T} \times (-b_m, b_m)$. That is,

$$\Phi_m(x, y) = \big(x + \mathrm{O}(y), y + \mathrm{O}(y^2)\big), \qquad \det[\Phi_m(x, y)] = 1 + \mathrm{O}(y), \tag{3.15}$$

uniformly for all $(x, y) \in \mathbb{T} \times (-b_m, b_m)$.

Proof. The change Φ_m is the composition of $m - 2$ changes of the form

$$\tilde{\Phi}_l = \mathrm{I} + \tilde{\Psi}_l, \qquad \tilde{\Psi}_l \in \mathcal{X}_{a_l,b_l,l-1}, \qquad 2 \leq l < m,$$

where $a_l = a_2 - (l - 2)\epsilon$, $\epsilon = (a_2 - a_m)/(m - 2)$, $(b_l)_{2\leq l<m}$ is a positive decreasing sequence, and $\tilde{\Psi}_l$ is constructed as follows to increase the order of the perturbation from l to $l + 1$.

Let us suppose that we have a real analytic map with the intersection property on $\mathbb{T} \times (-b_l, b_l)$ of the form $F_l = A + G_l$, for some $G_l \in \mathcal{X}_{a_l,b_l,l}$ with $a_l, b_l > 0$ and $l \geq 2$.

We begin with a formal computation. We write

$$F_l(x, y) = \big(x + y + y^l h_1(x) + \mathrm{O}(y^{l+1}), y + y^{l+1}h_2(x) + \mathrm{O}(y^{l+2})\big),$$

where the functions $h_1(x)$ and $h_2(x)$ are 1-periodic and analytic on the open complex strip $\{x \in \mathbb{C}/\mathbb{Z} : |\Im x| < a_l\}$. We will see, by using an *a posteriori* reasoning, that the intersection property implies that $h_2(x)$ has zero average; that is, $h_2^* = \int_0^1 h_2(x)\mathrm{d}x = 0$. Nevertheless, we can not prove it yet. Thus, we will keep an eye on h_2^* in what follows.

If we take the change of variables

$$\tilde{\Phi}_l(x, y) = \big(x + y^{l-1}\psi_1(x), y + y^l\psi_2(x)\big)$$

for some functions $\psi_1(x)$ and $\psi_2(x)$, then, after a straightforward computation, the map $F_{l+1} = (\tilde{\Phi}_l)^{-1} \circ F_l \circ \tilde{\Phi}_l$ has the form

$$F_{l+1}(x,y) = \left(x + y + y^l k_1(x) + \mathrm{O}(y^{l+1}), y + y^{l+1} k_2(x) + \mathrm{O}(y^{l+2})\right),$$

with $k_1 = \psi_2 + h_1 - \psi_1'$ and $k_2 = h_2 - \psi_2'$. Therefore, we take

$$\psi_2(x) = \int_0^x \left(h_2(s) - h_2^*\right)\mathrm{d}s - h_1^*, \qquad \psi_1(x) = \int_0^x \left(\psi_2(s) + h_1(s)\right)\mathrm{d}s,$$

so that $k_1(x) = 0$ and $k_2(x) = h_2^*$. These functions $\psi_2(x)$ and $\psi_1(x)$ are 1-periodic, because $h_2(x) - h_2^*$ and $\psi_2(x) + h_1(x)$ have zero average. Besides, $\psi_1(x)$ and $\psi_2(x)$ are analytic in the open complex strip $\{x \in \mathbb{C}/\mathbb{Z} : |\Im x| < a_l\}$. Indeed, $\tilde{\Phi}_l = \mathrm{I} + \tilde{\Psi}_l$ with $\tilde{\Psi}_l \in \mathcal{X}_{a_l, b_l, l-1}$.

Next, we control the domain of definition of the map F_{l+1}. The inverse change is

$$(\tilde{\Phi}_l)^{-1}(x,y) = \left(x - y^{l-1}\psi_1(x) + \mathrm{O}(y^l), y - y^l \psi_2(x) + \mathrm{O}(y^{l+1})\right).$$

Thus, the maps $\tilde{\Phi}_l$, F_l, and $(\tilde{\Phi}_l)^{-1}$ have the form $(x,y) \mapsto (x + \mathrm{O}(y), y + \mathrm{O}(y^2))$, since $l \geq 2$. Consequently, if $b_{l+1} \leq b_l/2$ is small enough, then

$$D_{a_{l+1}, b_{l+1}} \xrightarrow{\tilde{\Phi}_l} D_{a_l - 2\epsilon/3, 4b_{l+1}/3} \xrightarrow{F_l} D_{a_l - \epsilon/3, 5b_{l+1}/3} \xrightarrow{(\tilde{\Phi}_l)^{-1}} D_{a_l, 2b_{l+1}} \subset D_{a_l, b_l},$$

so $F_{l+1} = (\Phi_l)^{-1} \circ F_l \circ \Phi_l$ is well-defined on $D_{a_{l+1}, b_{l+1}}$. Now, let us check that $h_2^* = 0$. At this moment, we only know that

$$F_{l+1}(x,y) = \left(x + y + \mathrm{O}(y^{l+1}), y + y^{l+1} h_2^* + \mathrm{O}(y^{l+2})\right),$$

since the change of variables has not eliminated the average h_2^*. The map F_{l+1} has the intersection property on the cylinder $\mathbb{T} \times (-b_{l+1}, b_{l+1})$, because the intersection property is preserved by changes of variables. This implies that $h_2^* = 0$. On the contrary, the image of the loop $\mathbb{T} \times \{y_0\}$ does not intersect itself when $0 < y_0 \ll 1$.

Finally, properties (3.15) follow directly from the fact that we have performed a finite number of changes, all of them satisfying these same properties. □

Next, the following theorem provides the exponentially small bound for the $\pi^\bullet$-projection of the residue provided an initial order m big enough. It is the main tool to prove Theorem 3.

Theorem 12. *Let $m \geq 6$ be an integer, $\bar{a} > 0$, $\bar{d} > 0$, and $r \in (0,1)$. There exist constants $\bar{b} = \bar{b}(m, \bar{a}, \bar{d}, r) > 0$ and $c_j = c_j(r) > 0$, $j = 1,2,3$, such that, if*

$$\bar{F} = A + \bar{G}, \quad \bar{G} \in \mathcal{X}_{\bar{a}, \bar{b}, m}, \quad \bar{d}^* = \|\pi^*(\bar{G})\|_{\bar{a}, \bar{b}, m}, \quad \bar{d}^\bullet = \|\pi^\bullet(\bar{G})\|_{\bar{a}, \bar{b}, m}, \tag{3.16}$$

and

$$0 < \breve{a} < \bar{a}, \qquad 0 < \breve{b} \leq \bar{b}\sqrt{r}, \qquad \bar{d}^* + (1 + c_2)\bar{d}^\bullet \leq \bar{d},$$

then there exists a change of variables $\breve{\Phi} = \mathrm{I} + \breve{\Psi}$ satisfying the following properties:

1. $\breve{\Psi} \in \mathcal{X}_{\breve{a},\breve{b},m-1}$ *with* $|\breve{\Psi}|_{\breve{a},\breve{b},m-1} \leq c_1 \bar{d}^{\bullet}$*; and*

2. *The transformed map* $\breve{F} = \breve{\Phi}^{-1} \circ \bar{F} \circ \breve{\Phi}$ *is real analytic, has the intersection property on the cylinder* $\mathbb{T} \times (-\breve{b}, \breve{b})$*, and has the form* $\breve{F} = A + \breve{G}$*,* $\breve{G} \in \mathcal{X}_{\breve{a},\breve{b},m}$*,*

$$\|\pi^*(\breve{G})\|_{\breve{a},\breve{b},m} \leq \bar{d}^* + c_2 \bar{d}^{\bullet}, \qquad \|\pi^{\bullet}(\breve{G})\|_{\breve{a},\breve{b},m} \leq c_3 \mathrm{e}^{-2\pi r(\bar{a}-\breve{a})/\breve{b}} \bar{d}^{\bullet}.$$

Theorem 12 is proved in Subsection 3.5.5.

In order to present the main ideas of the proof, let us try to completely get rid of the remainder of the map of the form $F = A + G$, for some $G \in \mathcal{X}_{a,b,m}$, with a change of variables of the form $\Phi = \mathrm{I} + \Psi$, for some $\Psi \in \mathcal{X}_{a,b,m-1}$. Concretely, we look for Φ such that $A = \Phi^{-1} \circ F \circ \Phi$, or, equivalently, we look for Ψ such that

$$\Psi \circ A - A\Psi = G \circ (\mathrm{I} + \Psi).$$

It is not possible to solve this equation in general. Instead, we consider the linear equation

$$\Psi \circ A - A\Psi = G.$$

This vectorial equation reads as

$$\begin{cases} \psi_1(x+y,y) - \psi_1(x,y) = y\left(\psi_2(x,y) + g_1(x,y)\right), \\ \psi_2(x+y,y) - \psi_2(x,y) = y g_2(x,y). \end{cases}$$

Therefore, we need to solve two linear equations of the form

$$\psi(x+y,y) - \psi(x,y) = y g(x,y), \tag{3.17}$$

where $g \in \mathcal{X}_{a,b}$ is known. If the average of $g(x,y)$ is different from zero: $g^*(y) = \hat{g}_0(y) \neq 0$, then this equation can not be solved. Besides, it is a straightforward computation to check that, if $\hat{g}_0(y) = 0$, the formal solution of this equation in the Fourier basis is

$$\hat{\psi}_k(y) = \frac{y}{\mathrm{e}^{2\pi k y \mathrm{i}} - 1} \hat{g}_k(y), \qquad \forall k \neq 0, \tag{3.18}$$

whereas the zero-th coefficient $\hat{\psi}_0(y)$ can be chosen arbitrarily. From (3.18), it is clear that (3.17) can not be solved unless g has only a finite number of harmonics and zero average. For this reason, given a function $g(x,y)$ with zero average, we define its *K-cut off* as

$$g^{<K}(x,y) = \sum_{|k|<K} \hat{g}_k(y) \mathrm{e}^{2\pi k x \mathrm{i}}. \tag{3.19}$$

Let K be such that $|2\pi k y| < 2\pi$ for all $|y| < b$ and $|k| < K$. Hence, we will take $K = s/b$ for some fixed $s \in (0,1)$, and we will actually solve truncated linear equations of the form

$$\psi(x+y,y) - \psi(x,y) = y g^{<K}(x,y). \tag{3.20}$$

The Fourier norm is specially suited to analyze this kind of equations; see Lemma 15.

Summarizing these ideas, we look for a change of variables of the form $\Phi = \mathrm{I} + \Psi$, where Ψ satisfies the truncated linear vectorial equation

$$\Psi \circ A - A\Psi = (G^{\bullet})^{<K}, \tag{3.21}$$

where $(G^{\bullet})^{<K}$ denotes the K-cut off of $G^{\bullet} = \pi^{\bullet}(G)$. The average of the first component of $(G^{\bullet})^{<K}$ may be non-zero. Equation (3.21) is studied in Lemma 16. This close to the identity change of variables $\Phi = \mathrm{I} + \Psi$ does not completely eliminate the remainder. However, if b is small enough, it reduces the size of the $\pi^{\bullet}$-projection of the remainder as the following lemma shows.

Lemma 13 (Iterative lemma)**.** *Let $m \geq 6$ be an integer, $\bar{a} > 0$, $\bar{d} > 0$, $\mu > 0$, and $\rho \in (0,1)$. There exists a constant $\bar{b} > 0$ such that if*

$$F = A + G, \quad G \in \mathcal{X}_{a,b,m}, \quad d^* = \|\pi^*(G)\|_{a,b,m}, \quad d^{\bullet} = \|\pi^{\bullet}(G)\|_{a,b,m},$$

with

$$0 < a \leq \bar{a}, \qquad 0 < b \leq \min\{\bar{b}, a/6\}, \qquad d^* + d^{\bullet} \leq \bar{d},$$

then there exists a change of variables $\Phi = \mathrm{I} + \Psi$ satisfying the following properties:

1. *$\Psi \in \mathcal{X}_{a,b,m-1}$ is a solution of the truncated linear equation (3.21) such that*

   $$|\Psi|_{a,b,m-1} \leq \|\Psi\|_{a,b,m-1} \leq \Omega d^{\bullet},$$

   where $\Omega = \Omega(\sqrt{\rho})$ is defined in Lemma 16; and

2. *The transformed map $\tilde{F} = \Phi^{-1} \circ F \circ \Phi$ is real analytic, has the intersection property on the cylinder $\mathbb{T} \times (-\tilde{b}, \tilde{b})$, and has the form $\tilde{F} = A + \tilde{G}$, $\tilde{G} \in \mathcal{X}_{\tilde{a},\tilde{b},m}$,*

   $$\|\pi^*(\tilde{G})\|_{\tilde{a},\tilde{b},m} \leq d^* + \mathrm{e}^{-12\pi\rho} d^{\bullet}, \qquad \|\pi^{\bullet}(\tilde{G})\|_{\tilde{a},\tilde{b},m} \leq \mathrm{e}^{-12\pi\rho} d^{\bullet},$$

   where $\tilde{a} = a - 6b$ and $\tilde{b} = b - \mu b^2$.

Remark 6. If $\tilde{a} = a - 6b$, then $\mathrm{e}^{-2\pi\rho(a-\tilde{a})/b} = \mathrm{e}^{-12\pi\rho}$.

The proof of this lemma is found in Subsection 3.5.4. Some technicalities in the proof require the use of the sup-norm, which forces us to deal with both the Fourier norm and the sup-norm. The relations between them are stated in Lemma 14.

Finally, Theorem 12 is obtained by means of a finite sequence of changes of variables like the ones described in the iterative lemma. We want to perform as many of such changes as possible because each change reduces the size of the $\pi^{\bullet}$-projection of the remainder by the factor $\mathrm{e}^{-12\pi\rho}$. Since the loss of analyticity in the angular variable is $6b = \mathrm{O}(b)$, then we can at most perform a number $\mathrm{O}(1/b)$ of such changes. This idea goes back to Neishtadt [Nei84].

The intersection property is used neither in the proof of the iterative lemma nor in the proof of Theorem 12, but will be essential to control the size of the π^*-projections of the remainders in terms of the size of their $\pi^{\bullet}$-projections later on.

3.5.3 Technical lemmas

Lemma 14. *Let $0 < \alpha < \min\{a, 1/2\pi\}$, $b > 0$, and $g \in \mathcal{X}_{a,b}$. Let $g^{\geq K} = g - g^{<K}$, with $g^{<K}$ the K-cut off of g, defined in (3.19). Then:*

1. $\|g^{<K}\|_{a,b} \leq \|g\|_{a,b}$,
2. $\|g^{\geq K}\|_{a-\alpha,b} \leq \mathrm{e}^{-2\pi K\alpha}\|g\|_{a,b}$,
3. $|g|_{a,b} \leq \|g\|_{a,b}$, *and*
4. $\|g\|_{a-\alpha,b} \leq \alpha^{-1}|g|_{a,b}$.

If $m \in \mathbb{N}$, then these bounds also hold for any vectorial function $G \in \mathcal{X}_{a,b,m}$.

Proof. First, the Fourier norm of $g^{<K}$ is a partial sum of the Fourier norm of g. Second, $\|g^{\geq K}\|_{a-\alpha,b} = \sum_{|k|\geq K} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|(a-\alpha)} = \mathrm{e}^{-2\pi K\alpha} \sum_{|k|\geq K} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|a} \leq \mathrm{e}^{-2\pi K\alpha}\|g\|_{a,b}$. Third, $|g(x,y)| \leq \sum_{k\in\mathbb{Z}} |\hat{g}_k(y)||\mathrm{e}^{2\pi kx\mathrm{i}}| \leq \sum_{k\in\mathbb{Z}} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|a} = \|g\|_{a,b}$, for all $(x,y) \in D_{a,b}$. Fourth, we recall that the Fourier coefficients of the analytic function g satisfy the inequality $|\hat{g}_k|_b \leq \mathrm{e}^{-2\pi|k|a}|g|_{a,b}$ for all $k \in \mathbb{Z}$. Hence,

$$\|g\|_{a-\alpha,b} = \sum_{k\in\mathbb{Z}} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|(a-\alpha)} \leq 2|g|_{a,b} \sum_{k\geq 0} \mathrm{e}^{-2\pi k\alpha} \leq \alpha^{-1}|g|_{a,b},$$

where we have used that $\sum_{k\geq 0} \mathrm{e}^{-kt} = (1-\mathrm{e}^{-t})^{-1} \leq \mathrm{e}/t < \pi/t$ for all $t \in (0,1)$. The last part follows from the definition of the norms $\|\cdot\|_{a,b,m}$ and $|\cdot|_{a,b,m}$. □

Lemma 15. *If $s \in (0,1)$, $K = s/b$, and $g \in \mathcal{X}_{a,b}$ is a function with zero average, then the truncated linear equation (3.20) has a unique solution $\psi \in \mathcal{X}_{a,b}$ with zero average and $\|\psi\|_{a,b} \leq \omega\|g\|_{a,b}$, where*

$$\omega = \omega(s) = \frac{1}{2\pi} \cdot \max_{|z|\leq 2\pi s} \left| \frac{z}{\mathrm{e}^z - 1} \right|. \tag{3.22}$$

Proof. The Fourier coefficients of ψ must satisfy (3.18). We note that $\omega < \infty$ for all $s \in (0,1)$, since the function $z/(\mathrm{e}^z - 1)$ is analytic on the open ball $|z| < 2\pi$. Moreover,

$$|\hat{\psi}_k|_b \leq \left(\max_{|y|\leq b} \left| \frac{y}{\mathrm{e}^{2\pi k y\mathrm{i}} - 1} \right| \right) |\hat{g}_k|_b \leq \frac{\omega}{|k|} |\hat{g}_k|_b \leq \omega|\hat{g}_k|_b,$$

for all $0 < |k| < K = s/b$. Finally, we recall that $\hat{\psi}_0(y) \equiv 0$. Then we obtain that $\|\psi\|_{a,b} = \sum_{0<|k|<K} |\hat{\psi}_k|_b \mathrm{e}^{2\pi|k|a} \leq \omega \sum_{k\in\mathbb{Z}} |\hat{g}_k|_b \mathrm{e}^{2\pi|k|a} = \omega\|g\|_{a,b}$. □

Remark 7. We will denote by $\psi = \mathcal{G}_K(g^{<K})$ the linear operator that sends the independent term $g^{<K}$ of the truncated linear equation (3.20) to the solution ψ with zero average. Note that the solution ψ has no harmonics of order $\geq K$.

Lemma 16. *If $m \geq 1$, $s \in (0,1)$, $K = s/b$, and $G \in \mathcal{X}_{a,b,m}$, then the truncated linear equation (3.21) has a solution $\Psi \in \mathcal{X}_{a,b,m-1}$ such that*

$$\|\Psi\|_{a,b,m-1} \leq \Omega \|G^\bullet\|_{a,b,m},$$

where $G^\bullet = \pi^\bullet(G)$, $\Omega = \Omega(s) = (\omega(s)+1)\max\{1, \omega(s)\}$, and $\omega(s)$ is defined in (3.22).

Proof. Let $G = (y^m g_1, y^{m+1} g_2)$ and $\Psi = (y^{m-1}\psi_1, y^m \psi_2)$. Then the vectorial equation $\Psi \circ A - A\Psi = (G^\bullet)^{<K}$ reads as

$$\begin{cases} \psi_1(x+y,y) - \psi_1(x,y) = y\left(\psi_2(x,y) + g_1^{<K}(x,y)\right), \\ \psi_2(x+y,y) - \psi_2(x,y) = y(g_2^{<K}(x,y) - g_2^*(y)). \end{cases}$$

Let $\psi_2 = \mathcal{G}_K(g_2^{<K} - g_2^*) - g_1^*$ and $\psi_1 = \mathcal{G}_K(\psi_2 + g_1^{<K})$. These operations are well-defined since both $g_2 - g_2^*$ and $\psi_2 + g_1$ have zero average. As for the bounds,

$$\begin{aligned} \|\psi_2\|_{a,b} &\leq \omega\|g_2 - g_2^*\|_{a,b} + \|g_1\|_{a,b} \leq \Omega\|G^\bullet\|_{a,b,m}, \\ \|\psi_1\|_{a,b} &\leq \omega\|\psi_2 + g_1^{<K}\|_{a,b} \\ &\leq \omega\|\psi_2 + g_1^*\|_{a,b} + \omega\|g_1^{<K} - g_1^*\|_{a,b} \\ &\leq \omega^2\|g_2 - g_2^*\|_{a,b} + \omega\|g_1\|_{a,b} \leq \Omega\|G^\bullet\|_{a,b,m}, \end{aligned}$$

where we have used Lemma 15. □

Lemma 17. *Let $l, n \in \mathbb{N}$, $0 < \alpha < \min\{a/3, 1/2\pi\}$, $0 < \beta < b/2$, $c_1, c_2 > 0$, and $c = c_1 + c_2$, such that*

$$b + b^n c < \alpha + \beta, \qquad b^{n+1} c < \beta. \tag{3.23}$$

Let $M = M(\alpha, b, \beta, c_1, c_2, l, n) = (1 + b^n c)^{l+1} b^{l-1}\left(\alpha^{-1} + b\beta^{-1} + l + 1\right)$. Let $\Delta \in \mathcal{X}_{a,b,l}$. Let $\Gamma_1, \Gamma_2 \in \mathcal{X}_{a-2\alpha,b-2\beta,n}$ with $\|\Gamma_j\|_{a-2\alpha,b-2\beta,n} \leq c_j$. Let $L(x,y) = (x + \eta y, y)$ with $|\eta| \leq 1$. Then,

1. *$\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2) \in \mathcal{X}_{a-3\alpha,b-2\beta,n+l}$,*
2. *$|\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2)|_{a-2\alpha,b-2\beta,n+1} \leq M|\Delta|_{a,b,l}|\Gamma_1 - \Gamma_2|_{a-2\alpha,b-2\beta,n}$, and*
3. *$\|\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2)\|_{a-3\alpha,b-2\beta,n+1} \leq M\alpha^{-1}\|\Delta\|_{a,b,l}\|\Gamma_1 - \Gamma_2\|_{a-2\alpha,b-2\beta,n}$.*

Proof. Let $\Gamma = \Gamma_1 - \Gamma_2$. Then $|\Gamma|_{a-2\alpha,b-2\beta,n} \leq \|\Gamma\|_{a-2\alpha,b-2\beta,n} \leq c$ and

$$\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2) = \int_0^1 \left(\mathrm{D}\Delta \circ (L + t\Gamma)\right) \cdot \Gamma \, \mathrm{d}t.$$

Let $(x_t, y_t) = (L + t\Gamma)(x,y) = (x + \eta y + t y^n \gamma_1(x,y), y + t y^{n+1}\gamma_2(x,y))$, with $t \in [0,1]$ fixed. We deduce from conditions (3.23) that (x_0, y_0) and (x_1, y_1) belong to $D_{a-\alpha,b-\beta}$ for all $(x,y) \in D_{a-2\alpha,b-2\beta}$. Therefore, $(x_t, y_t) \in D_{a-\alpha,b-\beta}$ by convexity of the domain, and so, the composition $\Delta \circ (L + t\Gamma)$ is well-defined on the domain $D_{a-2\alpha,b-2\beta}$.

A simple computation shows that the product $\mathrm{D}\Delta(x_t, y_t) \cdot \Gamma(x, y)$ is equal to

$$\begin{pmatrix} y_t^{l-1} y^{n+1} \left(y^{-1} y_t \partial_1 \delta_1(x_t, y_t) \gamma_1(x, y) + (y_t \partial_2 \delta_1(x_t, y_t) + l \delta_1(x_t, y_t)) \gamma_2(x, y) \right) \\ y_t^l y^{n+1} \left(y^{-1} y_t \partial_1 \delta_2(x_t, y_t) \gamma_1(x, y) + (y_t \partial_2 \delta_2(x_t, y_t) + (l+1) \delta_2(x_t, y_t)) \gamma_2(x, y) \right) \end{pmatrix}.$$

Let us bound the elements above. On the one hand, $|\gamma_i(x, y)| \leq |\Gamma|_{a-2\alpha, b-2\beta, n} \leq c$, $|y_t| = |y + t y^{n+1} \gamma_2(x, y)| \leq (1 + b^n c)|y|$, and $|y| < b$ for all $(x, y) \in D_{a-2\alpha, b-2\beta}$. On the other hand, $|\delta_i(x_s, y_s)| \leq |\Delta|_{a,b,l}$ and the Cauchy estimates imply that

$$|\partial_1 \delta_i(x_t, y_t)| \leq \alpha^{-1} |\Delta|_{a,b,l}, \qquad |\partial_2 \delta_i(x_t, y_t)| \leq \beta^{-1} |\Delta|_{a,b,l}.$$

From the previous bounds and the definitions of both norms, we deduce that

$$|(\mathrm{D}\Delta \circ (L + t\Gamma)) \cdot \Gamma|_{a-2\alpha, b-2\beta, n+l} \leq M' |\Delta|_{a,b,l} |\Gamma|_{a-2\alpha, b-2\beta, n},$$

for all $t \in [0, 1]$, where $M' = (1 + b^n c)^{l+1} \left(\alpha^{-1} + b\beta^{-1} + l + 1 \right)$. Thus,

$$\begin{aligned} |\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2)|_{a-2\alpha, b-2\beta, n+l} &\leq M' |\Delta|_{a,b,l} |\Gamma|_{a-2\alpha, b-2\beta, n}, \\ \|\Delta \circ (L + \Gamma_1) - \Delta \circ (L + \Gamma_2)\|_{a-3\alpha, b-2\beta, n+l} &\leq \alpha^{-1} M' \|\Delta\|_{a,b,l} \|\Gamma\|_{a-2\alpha, b-2\beta, n}. \end{aligned}$$

This proves the first item. The other items follow from the bounds $|\cdot|_{a,b,n+1} \leq b^{l-1} |\cdot|_{a,b,n+l}$ and $\|\cdot\|_{a,b,n+1} \leq b^{l-1} \|\cdot\|_{a,b,n+l}$, since $M = M' b^{l-1}$. □

Lemma 18. *Let $n \in \mathbb{N}$, $0 < \alpha < \min\{a/3, 1/2\pi\}$, $0 < \beta < b/2$, and $p > 0$ such that conditions (3.23) hold with $c = 2p$. Let $\Phi = \mathrm{I} + \Psi$, with $\Psi \in \mathcal{X}_{a,b,n}$ and $\|\Psi\|_{a,b,n} \leq p$. Then $\Phi(D_{a',b'}) \subset D_{a'+\alpha, b'+\beta}$ for all $0 < a' \leq a$ and $0 < b' \leq b$.*

Let $M_* = M(\alpha, b, \beta, p, p, n, n)$, where M is defined in Lemma 17. If $M_* bp < 1$, then Φ is invertible and the inverse change Φ^{-1} satisfies the following properties:

1. *$\Phi^{-1} = \mathrm{I} + \Upsilon$ for some $\Upsilon \in \mathcal{X}_{a-2\alpha, b-2\beta, n}$ such that $|\Upsilon|_{a-2\alpha, b-2\beta, n} \leq |\Psi|_{a,b,n}$,*
2. *$\Phi^{-1}(D_{a',b'}) \subset D_{a'+\alpha, b'+\beta}$ for all $0 < a' \leq a - 2\alpha$ and $0 < b' \leq b - 2\beta$, and*
3. *$\|\Upsilon + \Psi\|_{a-3\alpha, b-2\beta, n+1} \leq M_* \alpha^{-1} \|\Psi\|^2_{a,b,n}$.*

Proof. Note that $|\Psi|_{a,b,n} \leq \|\Psi\|_{a,b,n} \leq p$. Conditions (3.23) imply that $b^n p < \alpha$ and $b^{n+1} p < \beta$. Therefore, $\Phi(D_{a',b'}) \subset D_{a'+\alpha, b'+\beta}$ for all $0 < a' \leq a$ and $0 < b' \leq b$. Analogously, if $\Upsilon \in \mathcal{X}_{a-2\alpha, b-2\beta, n}$ and $|\Upsilon|_{a-2\alpha, b-2\beta, n} \leq p$, then $(\mathrm{I} + \Upsilon)(D_{a',b'}) \subset D_{a'+\alpha, b'+\beta}$ for all $0 < a' \leq a - 2\alpha$, $0 < b' \leq b - 2\beta$.

We denote by $\mathcal{B}$ the closed ball in $\mathcal{X}_{a-2\alpha, b-2\beta, n}$ of radius p in the sup-norm. Let us prove that the functional $\mathcal{P} : \mathcal{B} \to \mathcal{B}$, $\mathcal{P}(\Upsilon) = -\Psi \circ (\mathrm{I} + \Upsilon)$, is a well-defined contraction with Lipschitz constant

$$\mathrm{Lip}\, \mathcal{P} \leq M_* b |\Psi|_{a,b,n} \leq M_* bp < 1. \tag{3.24}$$

First, we observe that

$$|\mathcal{P}(\Upsilon)|_{a-2\alpha, b-2\beta, n} \leq |\Psi|_{a,b,n} \leq p, \qquad \forall \Upsilon \in \mathcal{B}, \tag{3.25}$$

so $\mathcal{P}(\mathcal{B}) \subset \mathcal{B}$. Second, we bound $\mathcal{P}(\Upsilon) - \mathcal{P}(\Xi) = \Psi \circ (\mathrm{I} + \Xi) - \Psi \circ (\mathrm{I} + \Upsilon)$ as follows:

$$\begin{aligned} |\mathcal{P}(\Upsilon) - \mathcal{P}(\Xi)|_{a-2\alpha,b-2\beta,n} &\leq b|\mathcal{P}(\Upsilon) - \mathcal{P}(\Xi)|_{a-2\alpha,b-2\beta,n+1} \\ &\leq M_* b|\Psi|_{a,b,n}|\Xi - \Upsilon|_{a-2\alpha,b-2\beta,n}. \end{aligned} \tag{3.26}$$

The first inequality is direct, and the second comes from Lemma 17 with $\Delta = \Psi$, $\Gamma_1 = \Xi$, $\Gamma_2 = \Upsilon$, $L = \mathrm{I}$, $c_j = p$, and $l = n$. This proves that $\mathcal{P}$ is a contraction with Lipschitz constant (3.24). Thus, $\mathcal{P}$ has a unique fixed point $\Upsilon \in \mathcal{B}$ which satisfies that

$$(\mathrm{I} + \Psi) \circ (\mathrm{I} + \Upsilon) = \mathrm{I} + \Upsilon + \Psi \circ (\mathrm{I} + \Upsilon) = \mathrm{I} + \Upsilon - \mathcal{P}(\Upsilon) = \mathrm{I}$$

on $D_{a-2\alpha,b-2\beta}$. Therefore, the inverse map Φ^{-1} exists and equals $\mathrm{I} + \Upsilon$. Furthermore, $|\Upsilon|_{a-2\alpha,b-2\beta,n} \leq |\Psi|_{a,b,n}$ follows from (3.25). Finally,

$$\begin{aligned} \|\Upsilon + \Psi\|_{a-3\alpha,b-2\beta,n+1} &\leq \alpha^{-1}|\Upsilon + \Psi|_{a-2\alpha,b-2\beta,n+1} \\ &\leq \alpha^{-1}|\mathcal{P}(\Upsilon) - \mathcal{P}(\mathbf{0})|_{a-2\alpha,b-2\beta,n+1} \\ &\leq M_* \alpha^{-1}|\Psi|_{a,b,n}|\Upsilon|_{a-2\alpha,b-2\beta,n} \\ &\leq M_* \alpha^{-1}|\Psi|^2_{a,b,n} \leq M_* \alpha^{-1}\|\Psi\|^2_{a,b,n}. \end{aligned}$$

We have used the second inequality of equation (3.26) with $\Xi = \mathbf{0}$. □

3.5.4 Proof of Lemma 13

We recall that $F = A + G$ with $G = \pi^*(G) + \pi^\bullet(G) = G^* + G^\bullet \in \mathcal{X}_{a,b,m}$, $d^* = \|G^*\|_{a,b,m}$, $d^\bullet = \|G^\bullet\|_{a,b,m}$, $d^* + d^\bullet \leq \bar{d}$, and $m \geq 6$. Let $s = \sqrt{\rho} \in (\rho, 1)$. Let $\Omega = \Omega(s) = \Omega(\sqrt{\rho}) > 0$ be the constant introduced in Lemma 16 and $\sigma = 1 + 2\Omega$. Let $\Phi = \mathrm{I} + \Psi$ be the change of variables where $\Psi \in \mathcal{X}_{a,b,m-1}$ is the solution given in Lemma 16 of the truncated linear equation $\Psi \circ A - A\Psi = (G^\bullet)^{<K}$ with $K = s/b$, so that

$$|\Psi|_{a',b',m-1} \leq \|\Psi\|_{a',b',m-1} \leq \Omega\|G^\bullet\|_{a',b',m} \leq \Omega d^\bullet, \tag{3.27}$$

for all $0 < a' \leq a$ and $0 < b' \leq b$. Let $\Phi^{-1} = \mathrm{I} + \Upsilon$ be the inverse change studied in Lemma 18. Let $\tilde{F} = \Phi^{-1} \circ F \circ \Phi$ be the transformed map. Let $\tilde{G} = \tilde{F} - A$ be the new remainder.

Henceforth, we will assume that α, b, and β are some positive constants such that

$$b \leq \alpha < \min\{a/6, 1/2\pi\}, \quad 0 < \beta < b/4, \quad b + \sigma b^{m-1}\bar{d} < \alpha + \beta, \quad \sigma b^m \bar{d} < \min(\alpha, \beta). \tag{3.28}$$

We split the proof in four steps.

Step 1: Control of the domains. Note that $\tilde{F}(D_{a',b'}) \subset D_{a'+4\alpha,b'+3\beta}$ for all $0 < a' \leq a - 4\alpha$ and $0 < b' \leq b - 4\beta$. Indeed,

$$\tilde{F} : D_{a',b'} \xrightarrow{\Phi} D_{a'+\alpha,b'+\beta} \xrightarrow{F} D_{a'+3\alpha,b'+2\beta} \xrightarrow{\Phi^{-1}} D_{a'+4\alpha,b'+3\beta}.$$

The behaviors of the changes Φ and Φ^{-1} follow directly from Lemma 18, which can be applied since conditions (3.28) are more restrictive than the ones required in Lemma 18 when $p = \Omega\bar{d}$ and $n = m-1$. We also need that $a' + 2\alpha \leq a - 2\alpha$ and $b' + 2\beta \leq b - 2\beta$ in order to control the inverse Φ^{-1}, which explains the restrictions on a' and b'.

The behaviour of the map $F = A + G$ follows from the bound

$$|G|_{a,b,m} \leq \|G\|_{a,b,m} \leq \|G^*\|_{a,b,m} + \|G^\bullet\|_{a,b,m} = d^* + d^\bullet \leq \bar{d}$$

and conditions $b + b^m\bar{d} < 2\alpha$ and $b^{m+1}\bar{d} < \beta$, which are also a consequence of (3.28).

Step 2: Decomposition of the new remainder. It turns out that $\tilde{G} = G^* + \sum_{j=1}^4 \tilde{G}_j$, where

$$\begin{array}{ll} \tilde{G}_1 = (G^\bullet)^{\geq K} = G^\bullet - (G^\bullet)^{<K}, & \tilde{G}_2 = G \circ \Phi - G, \\ \tilde{G}_3 = \Psi \circ A - \Psi \circ F \circ \Phi, & \tilde{G}_4 = (\Upsilon + \Psi) \circ (F \circ \Phi). \end{array}$$

Indeed, $G^* + \tilde{G}_1 + \tilde{G}_2 = G \circ \Phi - (G^\bullet)^{<K}$ and $\tilde{G}_3 + \tilde{G}_4 = \Psi \circ A + \Upsilon \circ (F \circ \Phi)$, so

$$\begin{aligned} G^* + \textstyle\sum_{j=1}^4 \tilde{G}_j &= G \circ \Phi + A\Psi + \Upsilon \circ (F \circ \Phi) \\ &= (F - A) \circ \Phi + A(\Phi - \mathrm{I}) + (\Phi^{-1} - \mathrm{I}) \circ (F \circ \Phi) \\ &= \Phi^{-1} \circ F \circ \Phi - A = \tilde{G}. \end{aligned}$$

Finally, let $\tilde{G}^* = \pi^*(\tilde{G}) = G^* + \sum_{j=2}^4 \pi^*(G_j)$ and $\tilde{G}^\bullet = \pi^\bullet(\tilde{G}) = \sum_{j=1}^4 \pi^\bullet(G_j)$.

Step 3: Bounds of the projections of the new remainder. Lemma 14 and the bound (3.27) will be used several times in what follows. Below, we apply Lemma 17 (twice) and Lemma 18 (once). The required hypotheses in each case are satisfied due to conditions (3.28).

- If $\tilde{a} < a$ and $\tilde{b} \leq b$, then

  $$\|\tilde{G}_1\|_{\tilde{a},\tilde{b},m} = \|(G^\bullet)^{\geq K}\|_{\tilde{a},\tilde{b},m} \leq \mathrm{e}^{-2\pi K(a-\tilde{a})}\|G^\bullet\|_{a,\tilde{b},m} \leq \mathrm{e}^{-2\pi K(a-\tilde{a})} d^\bullet.$$

- If $\tilde{a} \leq a - 3\alpha$ and $\tilde{b} \leq b - 2\beta$, then

  $$\begin{aligned} \|\tilde{G}_2\|_{\tilde{a},\tilde{b},m} &\leq M_2\alpha^{-1}\|G\|_{\tilde{a}+3\alpha,\tilde{b}+2\beta,m}\|\Psi\|_{\tilde{a}+\alpha,\tilde{b},m-1} \\ &\leq \Omega M_2\alpha^{-1}\|G\|_{\tilde{a}+3\alpha,\tilde{b}+2\beta,m}\|G^\bullet\|_{\tilde{a}+\alpha,\tilde{b},m} \\ &\leq \Omega M_2\alpha^{-1}\bar{d}d^\bullet. \end{aligned}$$

  The first inequality follows from Lemma 17 with $\Delta = G$, $L = \mathrm{I}$, $\Gamma_1 = \Psi$, $\Gamma_2 = \mathbf{0}$, $l = m$, and $n = m - 1$, so that $M_2 = M(\alpha, b, \beta, \Omega\bar{d}, 0, m, m-1)$.

- If $\tilde{a} \leq a - 2\alpha$ and $\tilde{b} \leq b - \beta$, then $\|F \circ \Phi - A\|_{\tilde{a},\tilde{b},m-1} \leq \sigma\|G\|_{a,b,m}$. Indeed, $F \circ \Phi - A = A\Psi + G \circ \Phi$ and

  $$\begin{aligned} \|A\Psi\|_{\tilde{a},\tilde{b},m-1} &\leq 2\|\Psi\|_{\tilde{a},\tilde{b},m-1} \leq 2\Omega\|G^\bullet\|_{\tilde{a},\tilde{b},m} \leq 2\Omega d^\bullet \leq 2\Omega\bar{d}, \\ \|G \circ \Phi\|_{\tilde{a},\tilde{b},m-1} &\leq \alpha^{-1}|G \circ \Phi|_{\tilde{a}+\alpha,\tilde{b},m-1} \leq \alpha^{-1}|G|_{\tilde{a}+2\alpha,\tilde{b}+\beta,m-1} \\ &\leq \alpha^{-1}\|G\|_{\tilde{a}+2\alpha,\tilde{b}+\beta,m-1} \leq b\alpha^{-1}\|G\|_{\tilde{a}+2\alpha,\tilde{b}+\beta,m} \\ &\leq b\alpha^{-1}\|G\|_{a,b,m} \leq \|G\|_{a,b,m} \leq \bar{d}. \end{aligned}$$

We have used that $\Phi(D_{\tilde{a}+\alpha,\tilde{b}}) \subset D_{\tilde{a}+2\alpha,\tilde{b}+\beta}$ to bound $|G \circ \Phi|_{\tilde{a},\tilde{b},m-1}$.

- If $\tilde{a} \le a - 3\alpha$ and $\tilde{b} \le b - 2\beta$, then

$$\begin{aligned} \|\tilde{G}_3\|_{\tilde{a},\tilde{b},m} &\le M_3\alpha^{-1}\|\Psi\|_{\tilde{a}+3\alpha,\tilde{b}+2\beta,m-1}\|F\circ\Phi - A\|_{\tilde{a}+\alpha,\tilde{b},m-1} \\ &\le M_3\alpha^{-1}\Omega\|G^\bullet\|_{\tilde{a}+3\alpha,\tilde{b}+2\beta,m}\sigma\bar{d} \\ &\le \Omega\sigma M_3\alpha^{-1}\bar{d}d^\bullet. \end{aligned}$$

The first inequality follows from Lemma 17 with $\Delta = \Psi$, $L = A$, $\Gamma_1 = \mathbf{0}$, $\Gamma_2 = F\circ\Phi - A$, $l = m-1$, and $n = m-1$, so that $M_3 = M(\alpha, b, \beta, 0, \sigma\bar{d}, m-1, m-1)$.

- If $\tilde{a} \le a - 6\alpha$ and $\tilde{b} \le b - 4\beta$, then

$$\begin{aligned} \|\tilde{G}_4\|_{\tilde{a},\tilde{b},m} &\le \alpha^{-1}|(\Upsilon+\Psi)\circ(F\circ\Phi)|_{\tilde{a}+\alpha,\tilde{b},m} \le \alpha^{-1}|\Upsilon+\Psi|_{\tilde{a}+3\alpha,\tilde{b}+2\beta,m} \\ &\le \alpha^{-1}\|\Upsilon+\Psi\|_{\tilde{a}+\alpha,\tilde{b},m} \le M_4\alpha^{-2}\|\Psi\|^2_{\tilde{a}+6\alpha,\tilde{b}+4\beta,m-1} \\ &\le M_4\alpha^{-2}\left(\Omega\|G^\bullet\|_{\tilde{a}+6\alpha,\tilde{b}+4\beta,m}\right)^2 \\ &\le \Omega^2 M_4\alpha^{-2}(d^\bullet)^2 \le \Omega^2 M_4\alpha^{-2}\bar{d}d^\bullet. \end{aligned}$$

The second inequality uses the inclusion $(F\circ\Phi)(D_{\tilde{a}+\alpha,\tilde{b}}) \subset D_{\tilde{a}+3\alpha,\tilde{b}+2\beta}$. The fourth one follows from Lemma 18 with $M_4 = M(\alpha, b, \beta, \Omega\bar{d}, \Omega\bar{d}, m-1, m-1)$. We need to verify the hypothesis $M_* bp < 1$ in this last lemma. It turns out that $M_* bp = M_4 b\Omega\bar{d} = \mathrm{O}(b^{m-2})$, so it suffices to take $0 < b \le \bar{b}$, with $\bar{b}$ small enough.

- If $\tilde{a} \le a - 6\alpha$ and $\tilde{b} \le b - 4\beta$, then

$$\begin{aligned} \|\tilde{G}^*\|_{\tilde{a},\tilde{b},m} &\le \|G^*\|_{\tilde{a},\tilde{b},m} + \textstyle\sum_{j=2}^4 \|\tilde{G}_j\|_{\tilde{a},\tilde{b},m} \le d^* + \tilde{M}\bar{d}d^\bullet, \\ \|\tilde{G}^\bullet\|_{\tilde{a},\tilde{b},m} &\le \textstyle\sum_{j=1}^4 \|\tilde{G}_j\|_{\tilde{a},\tilde{b},m} \le \left(\mathrm{e}^{-2\pi K(a-\tilde{a})} + \tilde{M}\bar{d}\right) d^\bullet, \end{aligned}$$

where $\tilde{M} = \Omega\alpha^{-1}\left(M_2 + \sigma M_3 + \Omega\alpha^{-1}M_4\right)$ and the constants M_j, $j = 2, 3, 4$, have been defined previously.

Step 4: Choice of the loss of analyticity domain. We set $\alpha = b$ and $\beta = \mu b^2/4$. If $\bar{b} > 0$ is small enough, then conditions (3.28) hold for all $0 < b \le \bar{b}$. In addition,

$$M = M(\alpha, b, \beta, c_1, c_2, l, n) = \mathrm{O}(b^{l-2}) \qquad \text{as } b \to 0^+,$$

where M is the expression introduced in Lemma 17. If we take $\tilde{a} = a - 6\alpha$ and $\tilde{b} = b - 4\beta$, then the bounds of the previous step imply that

$$\|\tilde{G}^*\|_{\tilde{a},\tilde{b},m} \le d^* + \tilde{M}\bar{d}d^\bullet, \qquad \|\tilde{G}^\bullet\|_{\tilde{a},\tilde{b},m} \le \left(\mathrm{e}^{-12\pi K b} + \tilde{M}\bar{d}\right) d^\bullet,$$

where $\tilde{M} = \tilde{M}(b; d, m, s) = \Omega b^{-1}\left(M_2 + \sigma M_3 + \Omega b^{-1}M_4\right) = \mathrm{O}(b^{m-5})$. We recall that $m \ge 6$, $0 < \rho < s < 1$, and $K = s/b$. Hence, if $0 < b \le \bar{b}$ and $\bar{b}$ is small enough, then

$$\|\tilde{G}^*\|_{\tilde{a},\tilde{b},m} \le d^* + \mathrm{e}^{-12\pi\rho}d^\bullet, \qquad \|\tilde{G}^\bullet\|_{\tilde{a},\tilde{b},m} \le \mathrm{e}^{-12\pi\rho}d^\bullet.$$

Indeed, $\tilde{M}\bar{d} \le \mathrm{e}^{-12\pi\rho} - \mathrm{e}^{-12\pi s} \le \mathrm{e}^{-12\pi\rho}$ if we take a small enough value of $\bar{b}$.

This ends the proof of the iterative lemma.

3.5.5 Proof of Theorem 12

Set $\rho = \sqrt{r} \in (0,1)$, $\mu = 6(1-\rho)/(\bar{a}-\breve{a})$, and $\Omega = \Omega(\sqrt{\rho})$, where the function $\Omega(s)$ is defined in Lemma 16. Let $\bar{b}$ be the positive constant associated to the integer $m \geq 6$ in Lemma 13, the numbers $\bar{a}, \bar{d}, \mu > 0$, and the exponent $\rho \in (0,1)$. Let $c_1 = c_1(r) = \Omega \sum_{n\geq 0} \mathrm{e}^{-12\pi\rho n}$, $c_2 = c_2(r) = \sum_{n\geq 1} \mathrm{e}^{-12\pi\rho n}$, and $c_3 = c_3(r) = \mathrm{e}^{12\pi\rho}$.

Let us check that $\bar{b}$, c_1, c_2, and c_3 satisfy the properties given in Theorem 12.

Let $a_0 = \bar{a}$, $d_0^* = \bar{d}^*$, $d_0^\bullet = \bar{d}^\bullet$, $0 < b_0 = \breve{b}/\rho \leq \bar{b}$, $F_0 = \bar{F} = A + \bar{G}$ be the map given in (3.16), $G_0^* = \pi^*(\bar{G})$, and $G_0^\bullet = \pi^\bullet(\bar{G})$. By recursively applying Lemma 13, we obtain a sequence of changes of variables $\Phi_n = \mathrm{I} + \Psi_n$, with $\Psi_n \in \mathcal{X}_{a_{n-1},b_{n-1},m-1}$, and a sequence of maps $F_n = A + G_n$, with $G_n = G_n^* + G_n^\bullet$, $G_n^* \in \mathcal{X}^*_{a_n,b_n,m}$ and $G_n^\bullet \in \mathcal{X}^\bullet_{a_n,b_n,m}$, such that

$$\|\Psi_n\|_{a_{n-1},b_{n-1},m-1} \leq \Omega d_{n-1}^\bullet, \quad \|G_n^*\|_{a_n,b_n,m} \leq d_n^*, \quad \|G_n^\bullet\|_{a_n,b_n,m} \leq d_n^\bullet,$$

with $a_{n+1} = a_n - 6b_n$, $b_{n+1} = b_n - \mu b_n^2$, $d_{n+1}^* = d_n^* + \mathrm{e}^{-12\pi\rho} d_n^\bullet$, and $d_{n+1}^\bullet = \mathrm{e}^{-12\pi\rho} d_n^\bullet$.

Let N be the biggest integer satisfying $N b_0 \leq (\bar{a}-\breve{a})/6$. The sequences $(a_n)_{0\leq n\leq N}$, $(b_n)_{0\leq n\leq N}$, and $(d_n^\bullet)_{0\leq n\leq N}$ are decreasing. The sequence $(d_n^*)_{0\leq n\leq N}$ is increasing. Indeed,

$$\begin{aligned}
a_N &= a_{N-1} - 6b_{N-1} \geq a_{N-1} - 6b_0 \geq \cdots \geq a_0 - 6Nb_0 \geq \breve{a}, \\
b_N &= b_{N-1} - \mu b_{N-1}^2 \geq b_{N-1} - \mu b_0^2 \geq \cdots \geq (1-\mu N b_0) b_0 \geq \rho b_0 = \breve{b}, \\
d_N^\bullet &\leq \mathrm{e}^{-12\pi\rho} d_{N-1}^\bullet \leq \cdots \leq \mathrm{e}^{-12\pi\rho N} d_0^\bullet \leq c_3 \mathrm{e}^{-2\pi r(\bar{a}-\breve{a})/\breve{b}} \bar{d}^\bullet, \\
d_N^* &\leq d_{N-1}^* + \mathrm{e}^{-12\pi\rho} d_{N-1}^\bullet \leq \cdots \leq d_0^* + \left(\textstyle\sum_{n=1}^N \mathrm{e}^{-12\pi\rho n}\right) d_0^\bullet \leq \bar{d}^* + c_2 \bar{d}^\bullet,
\end{aligned}$$

and $d_n^* + d_n^\bullet \leq d_N^* + d_0^\bullet \leq \bar{d}^* + (1+c_2)\bar{d}^\bullet \leq \bar{d}$ for all $n = 0, \ldots, N$.

We can apply N times the iterative lemma. Let $\breve{F} = A + \breve{G} = A + G_N = F_N$ be the map obtained after those N steps. Then

$$\begin{aligned}
\|\pi^*(\breve{G})\|_{\breve{a},\breve{b},m} &\leq \|\pi^*(G_N)\|_{a_N,b_N,m} \leq d_N^* \leq \bar{d}^* + c_2 \bar{d}^\bullet, \\
\|\pi^\bullet(\breve{G})\|_{\breve{a},\breve{b},m} &\leq \|\pi^\bullet(G_N)\|_{a_N,b_N,m} \leq d_N^\bullet \leq c_3 \mathrm{e}^{-2\pi r(\bar{a}-\breve{a})/\breve{b}} \bar{d}^\bullet.
\end{aligned}$$

Finally, let $\breve{\Phi} = \Phi_N \circ \cdots \circ \Phi_1$ be the change of variables such that $\breve{F} = \breve{\Phi}^{-1} \circ \bar{F} \circ \breve{\Phi}$. We want to check that $\breve{\Phi} = \mathrm{I} + \breve{\Psi}$ for some $\breve{\Psi} \in \mathcal{X}_{\breve{a},\breve{b},m-1}$ such that $|\breve{\Psi}|_{\breve{a},\breve{b},m-1} \leq c_1 \bar{d}^\bullet$. We note that

$$\breve{\Psi} = \Psi_1 + \cdots + \Psi_N,$$

where each term of the above summation is evaluated at a different argument. Nevertheless, those arguments are not important when computing the sup-norm:

$$|\breve{\Psi}|_{\breve{a},\breve{b},m-1} \leq \sum_{n=1}^{N} |\Psi_n|_{a_{n-1},b_{n-1},m-1} \leq \Omega \sum_{n=0}^{N-1} d_n^\bullet \leq \Omega d_0^\bullet \sum_{n=0}^{N-1} \mathrm{e}^{-12\pi\rho n} = c_1 \bar{d}^\bullet.$$

This ends the proof of Theorem 12.

3.5.6 Proof of Theorem 3

Let us begin with a simple, but essential, chain of inequalities associated to certain analyticity strip widths that will appear along the proof. If $\alpha \in (0, a_*)$, then there exists $r \in (0,1)$, $\bar{b} > 0$, and some analyticity strip widths a_2, $\bar{a} = a_m$, and $\breve{a} = \bar{b}$, such that

$$0 < \bar{b} =: \breve{a} := \bar{a} - (1+\bar{b})\alpha/r < \bar{a} := a_m < a_2 < a_*. \tag{3.29}$$

The first two reductions (that is, from a_* to a_2 and from a_2 to a_m) are as small as we want. The third reduction (from $\bar{a} = a_m$ to $\breve{a} = \bar{a} - (1+\bar{b})\alpha/r$) should be a little bigger than α in order to get the desired exponentially small upper bound with the exponent α. The fourth reduction (that is, from $\breve{a} = \bar{b}$ to 0) is also small, since $\bar{b}$ can be taken as small as necessary.

This decreasing positive sequence of analyticity strip widths is associated to a similar sequence of analyticity radii. To be precise, we will construct a sequence of the form

$$b < \breve{b} < \bar{b} \leq b_m < b_2 < b_*, \qquad \breve{b} := b + b^2 < \bar{b}\sqrt{r}.$$

The inequality $\bar{b} \leq b_m$ does not correspond to a true reduction, but to a restriction on the size of $\bar{b}$. Note that we have consumed all the analyticity strip width after the last reduction, but we still keep a positive analyticity radius b.

We split the proof in the eight steps.

Step 1: Control of the Fourier norm. If the analytic map f satisfies the properties (i)–(iii) listed in Lemma 1, then the map $F_2 = A + G_2 := f$ is real analytic and has the intersection property on the cylinder $\mathbb{T} \times (-b_*, b_*)$, can be extended to the complex domain D_{a_*,b_*}, and has the form (3.6). The Fourier norm $\|G_2\|_{a_*,b_*,2}$ may be infinite, but $\|G_2\|_{a_2,b_2,2} < \infty$ for any $a_2 \in (0, a_*)$ and $b_2 \in (0, b_*)$.

Step 2: Application of the averaging lemma. Once fixed an integer $m \geq 6$ and any $a_m \in (0, a_2)$, we know from Lemma 11 that there exist an analytical radius $b_m \in (0, b_2)$ and a change of variables of the form $\Phi_m = \mathrm{I} + \Psi_m$ for some $\Psi_m \in \mathcal{X}_{a_m,b_m,1}$ such that the transformed map $F_m = \Phi_m^{-1} \circ F_2 \circ \Phi_m$ is real analytic, has the intersection property on the cylinder $\mathbb{T} \times (-b_m, b_m)$, and has the form $F_m = A + G_m$ for some $G_m \in \mathcal{X}_{a_m,b_m,m}$.

Step 3: Application of Theorem 12. Let $r \in (0,1)$ be the number that appears in (3.29). Set $\bar{F} = A + \bar{G} = A + G_m$, $\bar{a} = a_m$, and $\bar{d} = \|\pi^*(\bar{G})\|_{\bar{a},b_m,m} + (1 + c_2(r))\|\pi^\bullet(\bar{G})\|_{\bar{a},b_m,m}$.

Let $\bar{b} = \bar{b}(m, \bar{a}, \bar{d}, r) > 0$ be the constant stated in Theorem 12. We can assume that $\bar{b} \leq b_m$ and the condition (3.29) holds, by taking a smaller $\bar{b} > 0$ if necessary. Let $b'_* \in (0, b_*)$ be defined by $b'_* + (b'_*)^2 = \bar{b}\sqrt{r}$. Fix any $b \in (0, b'_*)$. Set $\breve{a} = \bar{a} - (1+\bar{b})\alpha/r$ and $\breve{b} = b + b^2 \leq b'_* + (b'_*)^2 = \bar{b}\sqrt{r}$.

If $\bar{d}^*$ and $\bar{d}^\bullet$ are the norms defined in (3.16), then $\bar{d}^* + (1 + c_2\bar{d}^\bullet) \leq \bar{d}$. Hence, we can apply Theorem 12 to obtain a change of variables $\breve{\Phi} = \mathrm{I} + \breve{\Psi}$, with $\breve{\Psi} \in \mathcal{X}_{\breve{a},\breve{b},m}$ and

$|\breve{\Psi}|_{\breve{a},\breve{b},m-1} \leq c_1 \bar{d}^{\bullet} \leq c_1 \bar{d}$, and a transformed map $\breve{F} = A + \breve{G} = \breve{\Phi}^{-1} \circ \bar{F} \circ \breve{\Phi}$, with $\breve{G} \in \mathcal{X}_{\breve{a},\breve{b},m}$, $|\pi^{\bullet}(\breve{G})|_{\breve{a},\breve{b},m} \leq \|\pi^{\bullet}(\breve{G})\|_{\breve{a},\breve{b},m} \leq c_3 \mathrm{e}^{-2\pi r(\bar{a}-\breve{a})/\breve{b}} \bar{d}^{\bullet} \leq c_3 \mathrm{e}^{-2\pi\alpha(1+\bar{b})/\breve{b}} \bar{d} \leq c_3 \mathrm{e}^{-2\pi\alpha/b} \bar{d}$, and $|\pi^*(\breve{G})|_{\breve{a},\breve{b},m} \leq \|\pi^*(\breve{G})\|_{\breve{a},\breve{b},m} \leq \bar{d}$.

Step 4: Uniform estimates on the change $\Phi = \Phi_m \circ \breve{\Phi}$. By construction, $\breve{\Phi} = \mathrm{I} + \breve{\Psi}$, with $\breve{\Psi} \in \mathcal{X}_{\breve{a},\breve{b},m-1}$ and $|\breve{\Psi}|_{\breve{a},\breve{b},m-1} \leq \breve{M}$, where the constant $\breve{M} := c_1 \bar{d}$ does not depend on b. Thus,

$$\breve{\Phi}(x,y) = \Big(x + y^{m-1}\breve{\psi}_1(x,y), y + y^m \breve{\psi}_2(x,y)\Big)$$

for some functions $\breve{\psi}_j(x,y)$ analytic on $D_{\breve{a},\breve{b}} = D_{\bar{b},b+b^2}$ such that $|\breve{\psi}_j|_{\bar{b},b+b^2} \leq \breve{M}$. The Cauchy estimates imply that

$$|\breve{\psi}_j(x,y)| \leq \breve{M}, \qquad |\partial_1 \breve{\psi}_j(x,y)| \leq \bar{b}^{-1}\breve{M}, \qquad |\partial_2 \breve{\psi}_j(x,y)| \leq b^{-2}\breve{M},$$

for all $(x,y) \in \mathbb{T} \times B_b$ and, in particular, for all $(x,y) \in \mathbb{T} \times (-b,b)$. Hence,

$$\breve{\Phi}(x,y) = \left(x + \mathrm{O}(y^{m-1}), y + \mathrm{O}(y^m)\right), \qquad \det\left[\breve{\Phi}(x,y)\right] = 1 + \mathrm{O}(y^{m-2}),$$

for all $(x,y) \in \mathbb{T} \times (-b,b)$, where the $\mathrm{O}(y^{m-2})$, $\mathrm{O}(y^{m-1})$, and $\mathrm{O}(y^m)$ terms are uniform in b. We recall that $m \geq 6$ and the change Φ_m satisfies properties (3.15), so the complete change $\Phi = \Phi_m \circ \breve{\Phi}$ satisfies the properties stated in Theorem 3.

Step 5: Exponentially small bound on the remainder G. After all these changes of variables, we have the map $F = A + G := \breve{F}$, with $G := \breve{G} \in \mathcal{X}_{\breve{a},\breve{b},m}$, $|\pi^*(G)|_{\breve{a},\breve{b},m} \leq \bar{d}$ and $|\pi^{\bullet}(G)|_{\breve{a},\breve{b},m} \leq c_3 \bar{d} \mathrm{e}^{-2\pi\alpha/b}$. We can bound $G^* = \pi^*(G)$ by using the bound on $G^{\bullet} = \pi^{\bullet}(G)$ and the intersection property of F on the cylinder $\mathbb{T} \times (-b,b)$. We recall that if

$$G(\xi,\eta) = (\eta^m g_1(\xi,\eta), \eta^{m+1} g_2(\xi,\eta)),$$

for some $g_1, g_2 \in \mathcal{X}_{a,b}$, then

$$G^*(\xi,\eta) = (0, \eta^{m+1} g_2^*(\eta)), \qquad G^{\bullet}(\xi,\eta) = (\eta^m g_1(\xi,\eta), \eta^{m+1} g_2^{\bullet}(\xi,\eta)),$$

where $g_2^*(\eta)$ is the average of $g_2(\xi,\eta)$ and $g_2^{\bullet} = g_2 - g_2^*$. Fix any $\eta_0 \in (-b,b)$. We know that

$$F(\mathbb{T} \times \{\eta_0\}) \cap \big(\mathbb{T} \times \{\eta_0\}\big) \neq \emptyset.$$

Therefore, there exists $\xi_0 \in \mathbb{T}$ such that $g_2^*(\eta_0) + g_2^{\bullet}(\xi_0,\eta_0) = 0$, and so

$$|g_2^*(\eta)| \leq \sup_{\mathbb{T} \times B_b} |g_2^{\bullet}| \leq |G^{\bullet}|_{\breve{a},\breve{b},m} \leq c_3 \bar{d} \mathrm{e}^{-2\pi\alpha/b}, \qquad \forall \eta \in (-b,b).$$

This implies that $|g_j(\xi,\eta)| \leq |G|_{\breve{a},\breve{b},m} \leq |G^*|_{\breve{a},\breve{b},m} + |G^{\bullet}|_{\breve{a},\breve{b},m} \leq 2c_3 \bar{d} \mathrm{e}^{-2\pi\alpha/b}$ for all $(\xi,\eta) \in \mathbb{T} \times (-b,b)$.

Step 6: Exponentially small bounds on some derivatives of the remainder. We recall that

$$\max\{|g_1(\xi,\eta)|, |g_2^{\bullet}(\xi,\eta)|\} \leq |G^{\bullet}|_{\breve{a},\breve{b},m} \leq c_3 \bar{d} \mathrm{e}^{-2\pi\alpha/b}$$

for all $(\xi,\eta)\in D_{\breve a,\breve b}=D_{\bar b,b+b^2}$. Thus, we get from $\partial_1 g_2=\partial_1 g_2^\bullet$ and the Cauchy estimates that

$$\begin{aligned} |\partial_1 g_j(\xi,\eta)| &\le c_3\bar d\,\bar b^{-1}\mathrm{e}^{-2\pi\alpha/b},\\ \max\{|\partial_2 g_1(\xi,\eta)|,|\partial_2 g_2^\bullet(\xi,\eta)|\} &\le c_3\bar d b^{-2}\mathrm{e}^{-2\pi\alpha/b}, \end{aligned}$$

for all $(\xi,\eta)\in\mathbb{T}\times(-b,b)$.

Step 7: A crude bound on the derivative of g_2^*. We recall that $G^*(\xi,\eta)=(0,\eta^{m+1}g_2^*(\eta))$, so

$$|g_2^*(\eta)|\le|G^*|_{\breve a,\breve b,m}\le\bar d,\qquad\forall\eta\in B_{\breve b}=B_{b+b^2}.$$

Therefore, the Cauchy estimates imply that $|(g_2^*)'(\eta)|\le b^{-2}\bar d$ for all $\eta\in B_b$ and, in particular, for all $\eta\in(-b,b)$.

Step 8: Computation of the constant K. By combining the inequalities obtained in Steps 5–7, we get that $|g_j(\xi,\eta)|\le K\mathrm{e}^{-2\pi\alpha/b}$ and $|\partial_i g_j(\xi,\eta)|\le Kb^{-2}$ for all $\mathbb{T}\times(-b,b)$, provided

$$K=\bar d\max\{2c_3,c_3+1\}.$$

This ends the proof of Theorem 3.

3.6 Proof of Theorem 4

3.6.1 A space of matrix functions

Henceforth, let $I_b=(-b,b)\subset\mathbb{R}$ and $S_b=\mathbb{T}\times I_b$ with $b>0$. Let S be any compact subset of S_b. Let $\mu\in\mathbb{N}$. Let $\mathcal{M}_{S,\mu}$ be the set of all matrix functions $\Gamma:S\to\mathcal{M}_{2\times2}(\mathbb{R})$ of the form

$$\Gamma(\xi,\eta)=\begin{pmatrix}\eta^\mu\gamma_{11}(\xi,\eta) & \eta^{\mu-1}\gamma_{12}(\xi,\eta)\\ \eta^{\mu+1}\gamma_{12}(\xi,\eta) & \eta^\mu\gamma_{22}(\xi,\eta)\end{pmatrix},$$

for some continuous functions $\gamma_{ij}:S\to\mathbb{R}$. The set $\mathcal{M}_{S,\mu}$ is a Banach space with the norm

$$\|\Gamma\|_{S,\mu}=\max\{|\gamma_{ij}(\xi,\eta)|:(\xi,\eta)\in S,\ 1\le i,j\le2\}.$$

Lemma 19. *Let $S\subset S_b$, $\Gamma\in\mathcal{M}_{S,\mu}$, $\Delta\in\mathcal{M}_{S,\nu}$, and $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. Let $k\in\mathbb{N}$ and $j\in\mathbb{Z}$. Let $f:S\to S$ be a map of the form (3.10) with $|g_2(\xi,\eta)|\le K_0$ for all $(\xi,\eta)\in S$. Then:*

1. *$\Gamma\Delta\in\mathcal{M}_{S,\mu+\nu}$ and $\|\Gamma\Delta\|_{S,\mu+\nu}\le2\|\Gamma\|_{S,\mu}\|\Delta\|_{S,\nu}$.*

2. *$A^k\Gamma\in\mathcal{M}_{S,\mu}$ and $\|A^k\Gamma\|_{S,\mu}\le(1+bk)\|\Gamma\|_{S,\mu}$.*

3. $\Gamma A^k \in \mathcal{M}_{S,\mu}$ *and* $\|\Gamma A^k\|_{S,\mu} \leq (1+bk)\|\Gamma\|_{S,\mu}$.

4. $\Gamma \circ f^j \in \mathcal{M}_{S,\mu}$ *and* $\|\Gamma \circ f^j\|_{S,\mu} \leq (1+K_0 b^m)^{(\mu+1)|j|}\|\Gamma\|_{S,\mu}$.

Proof. It is a straightforward computation. □

3.6.2 A technical lemma

Let f be an analytic map of the form (3.10). Let p and q be two relatively prime integers. There exist two curves $R = \mathit{graph}\,\zeta$ and $\hat{R} = \mathit{graph}\,\hat{\zeta}$ and two RICs $R_\pm = \mathit{graph}\,\zeta_\pm$ with Diophantine rotation numbers $\omega_- < p/q < \omega_+$, all four contained in a small neighborhood of $\mathbb{T} \times \{p/q\}$, such that f^q projects R onto $\hat{R}$ along the vertical direction and R and $\hat{R}$ are contained in the strip of the cylinder enclosed by the RICs $R_\pm$. Following Birkhoff [Bir66, Section VI] and Arnold [AA68, Section 20], all (p,q)-periodic points of f are contained in $R \cap \hat{R}$. Besides, we will see in the next subsection that the (geometric) area enclosed between R and $\hat{R}$ is an upper bound of the quantities $\Delta^{(p,q)}$. These are the reasons for the study of R and $\hat{R}$.

Let us prove that these four curves exist for big enough periods q. In this case, "big enough" only depends on the size of the nonintegrable terms of f, the size of the neighborhood of $\mathbb{T} \times \{p/q\}$, the exponent m, and the winding number p. On the contrary, it *does not depend* on the particular map at hand. Therefore, every time that we ask q to be "big enough" along the proof of the following lemma, it only depends on the quantities $K_0 > 0$, $c > 1$, $m \geq 4$, $p \in \mathbb{Z}\setminus\{0\}$, and $q'_* \in \mathbb{N}$ fixed at the first line of the next statement.

Lemma 20. *Let $K_0 > 0$, $c > 1$, $m \geq 4$, and $p, q'_* \in \mathbb{N}$. Let $q \geq q'_*$ be an integer relatively prime with p. Set $b = c^2 p/q$. Let $f : S_b \to \mathbb{T} \times \mathbb{R}$ be an analytic map of the form (3.10) such that $|g_j(\xi,\eta)| \leq K_0$ and $|\partial_i g_j(\xi,\eta)| \leq K_0 b^{-2}$ for all $(\xi,\eta) \in S_b$. Let $(\xi_q,\eta_q) = f^q(\xi,\eta)$. Let $I = (p/c^2 q, c^2 p/q)$, $I_- = (p/c^2 q, p/cq)$, and $I_+ = (cp/q, c^2 p/q)$. There exists $q''_* = q''_*(K_0, c, m, p, q'_*) \geq q'_*$ such that, if $q \geq q''_*$, the following properties hold:*

1. *The map f has two RICs $R_\pm \subset \mathbb{T} \times I_\pm \subset S_b$ whose internal dynamics is conjugated to a rigid rotation of angles $\omega_\pm \in I_\pm$, respectively;*

2. *If S is the compact subset of S_b enclosed by R_- and R_+, then*

$$\frac{\partial \xi_q}{\partial \eta}(\xi,\eta) > 0, \qquad \forall (\xi,\eta) \in S; \tag{3.30}$$

3. *There exist two unique analytic functions $\zeta : \mathbb{T} \to I$ and $\hat{\zeta} : \mathbb{T} \to I$ such that*

$$f^q(\xi,\zeta(\xi)) = (\xi,\hat{\zeta}(\xi)), \qquad \forall \xi \in \mathbb{T},$$

and all the (p,q)-periodic points of the restriction $f|_S$ are contained in $\mathrm{graph}\,\zeta$.

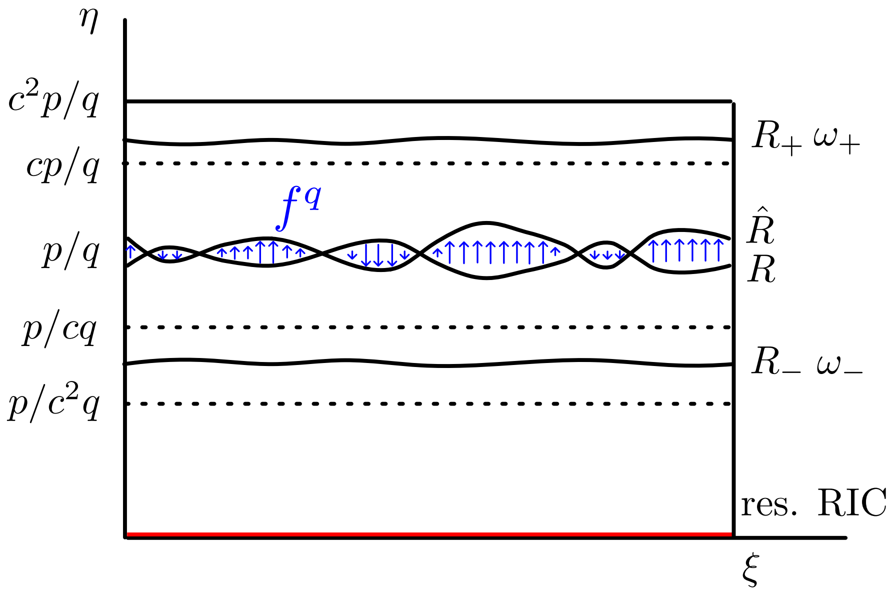


Figure 3.3: We schematically show the position of the curves $R = graph\, \zeta$ and $\hat{R} = graph\, \hat{\zeta}$ and the RICs R_+ and R_- appearing in Lemma 20 when $p > 0$. $\{\eta = 0\}$ is a (0,1)-resonant RIC.

The same statement holds if p is a negative integer, $b = c^2|p|/q$, $I = (c^2p/q, p/c^2q)$, $I_- = (c^2p/q, cp/q)$, and $I_+ = (p/cq, p/c^2q)$.

In Figure 3.3, we schematically show the position and some properties of the curves $\zeta(\xi), \hat{\zeta}(\xi), R_\pm$ whose existence is proved in this lemma.

Proof. Let us assume $p > 0$. The case $p < 0$ is analogous.

First, the existence of the RICs R_- and R_+ follows from some quantitative estimates in KAM theory established by Lazutkin [Laz73, Theorem 2]. To be precise, Lazutkin proved that there exists $b'_* = b'_*(K_0) > 0$ such that if $\omega \in (-b'_*, b'_*)$ satisfies the Diophantine condition

$$2|\omega - i/j| \geq |i|j^{-4} \tag{3.31}$$

for all integers $j \geq 1$ and i, then f has a RIC $R = \{\eta = \omega + \mathrm{O}(\omega^m)\}$ whose internal dynamics is C^l-conjugated to a rigid rotation of angle ω, for a suitable $l \geq 1$. The conjugation is $\mathrm{O}(1/q^{m-1})$-close to the identity. Item (i) follows directly from this estimate, because there exist some real numbers $\omega_+ \in (c^{4/3}p/q, c^{5/3}p/q) \subset I_+$ and $\omega_- \in (p/c^{5/3}q, p/c^{4/3}q) \subset I_-$ satisfying the Diophantine condition (3.31), provided q is big enough.

Second, let us check that the power map f^q satisfies (3.30). The compact subset $S \subset S_b$ is invariant by f, because it is delimited by RICs. Thus, all powers $(\xi_j, \eta_j) = f^j(\xi, \eta)$ are well defined on S. We write $\mathrm{D}f(\xi, \eta) = A + \Gamma(\xi, \eta)$, where A was introduced in

Lemma 19. Next, we compute the differential of the power map:

$$\mathrm{D}f^q = (A+\Gamma_q)\cdots(A+\Gamma_1) = A^q + \Delta_1 + \cdots + \Delta_q, \tag{3.32}$$

where $\Gamma_j = \Gamma \circ f^j$ and Δ_l is the sum of all the products of the form $A^{k_1}\Gamma_{j_1}\cdots A^{k_l}\Gamma_{j_l}A^{k_{l+1}}$, with $k_i \geq 0$, $q \geq j_1 > j_2 > \cdots > j_l \geq 1$, and $q = l + \sum_{i=1}^{l+1} k_i$. These products are elements of $\mathcal{M}_{S,ml}$, because $\Gamma \in \mathcal{M}_{S,m}$. Indeed, if $C = \|\Gamma\|_{S,m}$, then

$$\begin{aligned}\left\|A^{k_1}\Gamma_{j_1}\cdots A^{k_l}\Gamma_{j_l}A^{k_{l+1}}\right\|_{S,ml} &\leq 2^{l-1}\|A^{k_1}\Gamma_{j_1}\|_{S,m}\cdots\|A^{k_{l-1}}\Gamma_{j_{l-1}}\|_{S,m}\|A^{k_l}\Gamma_{j_l}A^{k_{l+1}}\|_{S,m}\\ &\leq 2^{l-1}C^l \textstyle\prod_{i=1}^{l+1}(1+bk_i)\prod_{i=1}^{l}(1+K_0b^m)^{(m+1)j_i}\\ &\leq 2^{l-1}C^l \exp\left(\textstyle\sum_{i=1}^{l+1} bk_i + \sum_{i=1}^{l}(m+1)K_0b^m j_i\right)\\ &\leq 2^{l-1}C^l \mathrm{e}^{c^2p+(m+1)K_0c^{2m}p^m} = C'(2C)^l,\end{aligned}$$

where we have used Lemma 19, inequality $1+x \leq \mathrm{e}^x$ for $x \geq 0$, $\sum_{i=1}^{l+1} k_i \leq q$, $l \leq q$, $j_i \leq q$, $b = c^2p/q$, and $m \geq 4$. We have also defined $C' = \mathrm{e}^{c^2p+(m+1)K_0c^{2m}p^m}/2$.

The matrix Δ_l is the sum of the products with precisely l factors Γ_j. This shows that there are $\binom{q}{l}$ terms inside Δ_l. Therefore, $\Delta_l \in \mathcal{M}_{S,ml}$ and

$$\|\Delta_l\|_{S,ml} \leq \binom{q}{l}\left\|A^{k_1}\Gamma_{j_1}\cdots A^{k_l}\Gamma_{j_l}A^{k_{l+1}}\right\|_{S,ml} \leq C'(2Cq)^l. \tag{3.33}$$

The element of the first row and second column of A^q is equal to q, so

$$\begin{aligned}\left|\frac{\partial \xi_q}{\partial \eta}(\xi,\eta) - q\right| &\leq C'\sum_{l=1}^{q}(2Cq)^l b^{ml-1} \leq \frac{C'}{b}\sum_{l=1}^{q}\left(2K_0qb^{m-2}\right)^l\\ &\leq 4C'K_0qb^{m-3} \leq 4C'K_0c^{2(m-3)}p^{m-3},\end{aligned}$$

for all $(\xi,\eta) \in S \subset S_b$, which implies the twist condition (3.30) provided that q is big enough. Here, we have used relation (3.32), bound (3.33), $b = c^2p/q$, and $m \geq 4$. We have also used that $C \leq K_0b^{-2}$ and $2K_0qb^{m-2} \leq 1/2$, provided q is big enough.

Third, we establish the existence of the functions $\zeta, \hat{\zeta} : \mathbb{T} \to I$. We know from Lazutkin [Laz73] that $R_\pm = graph\,\zeta_\pm$ for some differentiable functions $\zeta_\pm : \mathbb{T} \to I_\pm$. We work with the lifts F, Ξ_q, and $Z_\pm$ of the objects f, ξ_q, and $\zeta_\pm$. The RICs are invariant, so $F^q(\xi, Z_\pm(\xi)) = (\Xi_\pm(\xi), Z_\pm(\Xi_\pm(\xi)))$ for some differentiable functions $\Xi_\pm : \mathbb{R} \to \mathbb{R}$. If we prove that there exist two unique analytic 1-periodic functions $Z, \hat{Z} : \mathbb{R} \to I$ such that

$$F^q(\xi, Z(\xi)) = (\xi + p, \hat{Z}(\xi+p)), \qquad \forall \xi \in \mathbb{R}, \tag{3.34}$$

then item (iii) follows. Since the dynamics of F^q on R_+ is C^l-conjugated to a rigid rotation of angle $q\omega_+$ through a $\mathrm{O}(1/q^{m-1})$-close to the identity conjugation,

$$\Xi_+(\xi) = \xi + q\omega_+ + \mathrm{O}(1/q^{m-1}) \geq \xi + cp + \mathrm{O}(1/q^{m-1}) > \xi + p$$

provided that q is big enough. Analogously, we obtain $\Xi_-(\xi) < \xi + p$. That is,

$$\Xi_q(\xi, Z_-(\xi)) = \Xi_-(\xi) < \xi + p < \Xi_+(\xi) = \Xi_q(\xi, Z_+(\xi)), \qquad \forall \xi \in \mathbb{R}.$$

Since $\Xi_q(\xi,\eta)$ is analytic and strictly increasing for $\eta \in (Z_-(\xi), Z_+(\xi)) \subset I$, we deduce that there exists a unique function $Z : \mathbb{R} \to I$ such that $\Xi_q(\xi, Z(\xi)) = \xi + p$.

The function $S \ni (\xi,\eta) \mapsto G(\xi,\eta) := \Xi_q(\xi,\eta) - \xi - p$ is analytic and $\frac{\partial G}{\partial \eta}(\xi,\eta) > 0$, so Z is analytic by the Implicit Function Theorem. The 1-periodicity of Z follows from the uniqueness and the property $F^q(\xi+1,\eta) = F^q(\xi,\eta) + (1,0)$. Function $\hat{Z} : \mathbb{R} \to I$ is defined by means of relation (3.34). Finally, functions $\zeta, \hat{\zeta} : \mathbb{T} \to I$ are the projections of $Z, \hat{Z} : \mathbb{R} \to I$. □

3.6.3 Proof of Theorem 4: Case $(m,n) = (0,1)$

If $(m,n) = (0,1)$, by hypothesis, the map $g : \mathbb{T} \times I \to \mathbb{T} \times I$, $(s,r) \mapsto (s_1,r_1)$, is an analytic exact twist map with a (a_*,b_*)-analytic $(0,1)$-resonant RIC, such that $\vartheta_- \le 0 \le \vartheta_+$. The map $f = g^n = g$ satisfies the properties (i)–(iii) listed in Lemma 1 in some suitable coordinates (x,y). Let $(s,r) = \tilde{\Phi}(x,y)$ be the associated change of variables. Let $\tilde{f} = \tilde{\Phi}^{-1} \circ f \circ \tilde{\Phi}$ be the new map defined in the domain (3.5). Note that the (a_*,b_*)-analytic $(0,1)$-resonant RIC is $C \equiv \{y = 0\}$ in the (x,y) coordinates.

Let p be an integer such that $1 \le |p| \le L$. Let $c \in (1,2)$ such that $\alpha < c^2\alpha < a_*$. We take $c^2\alpha$ as the α appearing in Theorem 3, $m = 4$, and $b = c^2|p|/q$, provided that q is relatively prime with p and is large enough so that $c^2|p|/q < b'_* = b'_*(\alpha)$. That is, $q > q'_* := c^2|p|/b'_*$.

Hence, there exist $K_0, K_1 > 0$, both independent of q, and a change of coordinates $(x,y) = \Phi(\xi,\eta)$ such that $\bar{f} = \Phi^{-1} \circ \tilde{f} \circ \Phi : S_b \to \mathbb{T} \times \mathbb{R}$ is an analytic map of the form (3.10) such that $|g_j(\xi,\eta)| \le K_0 \mathrm{e}^{-2\pi c^2 \alpha/b} = K_0 \mathrm{e}^{-2\pi\alpha q/|p|} \le K_0$, $|\partial_i g_j(\xi,\eta)| \le K_0 b^{-2}$, and $\sup\{|\det[\mathrm{D}\Phi(\xi,\eta)]|\} \le K_1$ for all $(\xi,\eta) \in S_b$.

The map $\bar{f} : S_b \to \mathbb{T} \times \mathbb{R}$ satisfies the hypotheses of Lemma 20 for any $q \ge q'_*$. Let q_* be the maximum value of q''_* among the integers $0 < |p| \le L$. Let $R_\pm$ be the RICs with rotation numbers $\omega_\pm$ given in Lemma 20. Let S be the compact subset of S_b enclosed by R_- and R_+. Since f is globally twist and $\vartheta_- < \omega_- < p/q < \omega_+ < \vartheta_+$, all the Birkhoff (p,q)-periodic orbits of f are contained in S. By Lemma 20, any (p,q)-periodic orbit in S lies on $R = graph\,\zeta$. Let $\Omega \subset S$ be the domain enclosed by the curves $R = graph\,\zeta$ and $\hat{R} = graph\,\hat{\zeta}$. Let $B = (\tilde{\Phi} \circ \Phi)(\Omega)$. Let K_2 be the supremum of $|\det[\mathrm{D}\tilde{\Phi}]|$ in the compact set $\mathbb{T} \times [-b'_*, b'_*]$. Let $K = 4K_0K_1K_2L(b'_*)^3$. Then, following the arguments contained in Subsection 3.2.2 about the difference of periodic actions, we get that

$$
\begin{aligned}
\Delta^{(p,q)} &\le Area(B) \le K_1K_2\,Area(\Omega) = K_1K_2 \int_{\mathbb{T}} \left|\hat{\zeta}(\xi) - \zeta(\xi)\right| \mathrm{d}\xi \\
&\le K_1K_2 q b^4 K_0 \mathrm{e}^{-2\pi\alpha q/|p|} \le K \mathrm{e}^{-2\pi\alpha q/|p|},
\end{aligned}
\tag{3.35}
$$

for all relatively prime integers p and q with $1 \le |p| \le L$ and $q \ge q_*$. We have used expression (3.10), $b = c^2|p|/q \le b'_*$, $c^2 < 4$, the bounds on the nonintegrable terms $g_j(\xi,\eta)$, and the bounds on the Jacobians of the changes of variables Φ and $\tilde{\Phi}$.

This ends the proof of Theorem 4 when $(m,n)=(0,1)$.

3.6.4 Proof of Theorem 4: General case

We reduce the general case to the previous one. We split the argument in four steps.

Step 1: About the rational rotation numbers. If C is a (m,n)-resonant RIC and (s,r) is a (p,q)-periodic point of g, then C is a $(m,1)$-resonant RIC and (s,r) is a (p',q')-periodic point of the power map $f=g^n$, where

$$p'=\frac{np}{\gcd(n,q)},\qquad q'=\frac{q}{\gcd(n,q)}.$$

By taking the suitable lift F of f, we can assume that C is a $(0,1)$-resonant RIC and (s,r) is a (p'',q'')-periodic point of f, with $p''/q''=p'/q'-m$. That is,

$$p''=p'-mq'=\frac{np-mq}{\gcd(n,q)},\qquad q''=q'=\frac{q}{\gcd(n,q)}. \tag{3.36}$$

If p and q are relatively prime integers such that $1\leq|np-mq|\leq L$ and $q\geq q_*$, p'' and q'' are relatively prime integers such that $|p''|\leq L/\gcd(n,q)\leq L$ and $q''\geq q_*/\gcd(n,q)\geq q_*/n$.

Step 2: About the Lagrangians. Let G and $F=G^n$ be the lifts of g and $f=g^n$ we are dealing with. If $G^*\lambda-\lambda=\mathrm{d}h$, then

$$F^*\lambda-\lambda=\sum_{j=0}^{n-1}\left[\left(G^{j+1}\right)^*\lambda-\left(G^j\right)^*\lambda\right]=\sum_{j=0}^{n-1}\mathrm{d}(h\circ G^j)=\mathrm{d}\left(\sum_{j=0}^{n-1}h\circ G^j\right),$$

so $\ell(s_0,s_n):=h(s_0,s_1)+h(s_1,s_2)+\cdots+h(s_{n-1},s_n)$ is a Lagrangian of f. This Lagrangian is well defined in a neighborhood of the resonant RIC C, because f is twist on C.

Step 3: About the periodic actions. Let O be the (p,q)-periodic orbit of g through the point (s,r), being $W^{(p,q)}[O]$ its (p,q)-periodic action. Let O'' be the (p'',q'')-periodic orbit of f through the same point, being $W''^{(p'',q'')}[O'']$ its (p'',q'')-periodic action. We deduce from the previous steps and a straightforward computation that

$$W''^{(p'',q'')}[O'']=\frac{n}{\gcd(n,q)}W^{(p,q)}[O]. \tag{3.37}$$

Step 4: Final bound. The result follows directly from the bound (3.35) taking into account relations (3.36) and (3.37). We just note that

$$\mathrm{e}^{-2\pi\alpha q''/|p''|}=\exp\left(-\frac{2\pi\alpha q}{|np-mq|}\right).$$

This ends the proof of Theorem 4.

4 Exponentially small asymptotic formulas for the length spectrum

Abstract of the chapter. Let $q \geq 3$ be a period. There are at least two $(1,q)$-periodic trajectories inside any smooth strictly convex billiard table, and all of them have the same length when the table is an ellipse or a circle. We quantify the chaotic dynamics of axisymmetric billiard tables close to their boundaries by studying the asymptotic behavior of the differences of the lengths of their axisymmetric $(1,q)$-periodic trajectories as $q \to +\infty$. Based on numerical experiments, we conjecture that, if the billiard table is a generic axisymmetric analytic strictly convex curve, then these differences behave asymptotically like an exponentially small factor $q^{-3}\mathrm{e}^{-rq}$ times either a constant or an oscillating function, and the exponent r is half of the radius of convergence of the Borel transform of the well-known asymptotic series for the lengths of the $(1,q)$-periodic trajectories. Our experiments are restricted to some perturbed ellipses and circles, which allow us to compare the numerical results with some analytical predictions obtained by Melnikov methods and also to detect some non-generic behaviors due to the presence of extra symmetries. Our computations require a multiple-precision arithmetic and have been programmed in PARI/GP.

4.1 Introduction

Let us recall some concepts about periodic trajectories on billiard maps so that the exposition is fluent and complete. A (p,q)-periodic billiard trajectory forms a closed polygon of q sides that makes p turns before closing. Birkhoff [Bir66] proved that there are at least two different Birkhoff (p,q)-periodic billiard trajectories inside Q for any relatively prime integers p and q such that $1 \leq p < q$.

The *length spectrum* of Q is the subset of $\mathbb{R}_+$ defined as

$$\mathcal{LS}(Q) = l\mathbb{N} \cup \bigcup_{(p,q)} \Lambda^{(p,q)}\mathbb{N},$$

where $l = Length(Q)$ and $\Lambda^{(p,q)} \subset \mathbb{R}_+$ is the set of the lengths of all (p,q)-periodic billiard trajectories inside Q. The *maximal difference* among lengths of (p,q)-periodic trajectories is the non-negative quantity

$$\Delta^{(p,q)} = \sup \Lambda^{(p,q)} - \inf \Lambda^{(p,q)}.$$

Many geometric and dynamical properties are encoded in the length spectrum $\mathcal{LS}(Q)$ and the differences $\Delta^{(p,q)}$.

An old geometric question is: *Does the set $\mathcal{LS}(Q)$ allow one to reconstruct the convex curve Q?* The length spectrum and the Laplacian spectrum with Dirichlet boundary conditions are closely related [AM77]. Therefore, the question above can be colorfully restated as [Kac66]: *Can one hear the shape of a drum?* We refer to the book [Sib04] for some results on this question.

The difference $\Delta^{(p,q)}$ is important from a dynamical point of view, because it is an upper bound of Mather's $\Delta W_{p/q}$. In its turn, $\Delta W_{p/q}$ is equal to the flux through the (p,q)-resonance of the corresponding billiard map [MMP84, Mat86, Mei92, MF94]. Thus, the variation of $\Delta^{(p,q)}$ in terms of the rotation number $p/q \in (0,1)$ gives information about the size of the different chaotic zones of the billiard map. See Section 2.2 for a more complete description of these ideas.

Here, our main goal is to gain some insight into the billiard dynamics close to the boundary of the billiard table. We focus on the $(1,q)$-periodic billiards trajectories; that is, we set $p = 1$. We want to determine the asymptotic behavior of

$$\Delta^{(1,q)} = \sup \Lambda^{(1,q)} - \inf \Lambda^{(1,q)}$$

as $q \to +\infty$.

Let $L^{(1,q)} \in \Lambda^{(1,q)}$ be the length of a $(1,q)$-periodic billiard trajectory inside Q. It does not matter which one. Marvizi and Melrose [MM82] proved that if Q is smooth and strictly convex, then there exist some asymptotic coefficients $l_j = l_j(Q)$ such that

$$L^{(1,q)} \asymp \sum_{j\geq 0} l_j q^{-2j}, \qquad q \to \infty. \tag{4.1}$$

For instance, $l_0 = l = Length(Q)$ and $l_1 = -\frac{1}{24}\left(\int_Q \kappa^{2/3} \mathrm{d}s\right)^3$, where κ and $\mathrm{d}s$ are the curvature and the length element of Q, respectively. The symbol $\asymp$ means that the series in the right hand side is asymptotic to $L^{(1,q)}$. The asymptotic coefficients l_j can be explicitly written as integrals over Q of suitable algebraic expressions of κ and its derivatives. The first five coefficients can be found in [Sor15]. The asymptotic series (4.1) does not depend on the choice $L^{(1,q)} \in \Lambda^{(1,q)}$, so

$$\lim_{q\to+\infty} q^k \Delta^{(1,q)} = 0, \qquad \forall k > 0.$$

That is, the differences $\Delta^{(1,q)}$ are *beyond all order* in q. In fact, they satisfy the following *exponentially small upper bound* in the analytic case (recall Theorem 6). If Q is a closed analytic strictly convex curve, then there exist constants $K, q_0, \alpha > 0$ such that

$$\Delta^{(1,q)} \leq K \mathrm{e}^{-2\pi\alpha q}, \qquad \forall q \geq q_0.$$

The exponent α is related to the width of the complex strip where a certain 1-periodic angular coordinate is analytic. If a billiard map (or any analytic exact twist map) has a rotational invariant circle of Diophantine rotation number ϑ, then there exist other exponentially small upper bounds for $\Delta^{(p,q)}$ (or for the residues of (p,q)-periodic orbits) when $p/q \to \vartheta$. See [Gre79, Mac92, DdlL00].

Similar singular behaviors have been observed in problems about the splitting of separatrices of analytic maps [FS90, GLT91, DR98, DR99, Gel99, GL01, GS01, Ram05, GS08, MSS11a, MSS11b, BM12]. All these splittings are not exponentially small in a discrete big parameter $q \in \mathbb{N}$, but in a continuous small parameter $h > 0$. Namely, h is the characteristic exponent of the hyperbolic fixed point whose separatrices split. Thus, we may think that $h = 1/q$ for comparison purposes. The splitting size in many analytic maps satisfies the exponentially small asymptotic formula

$$\text{“splitting size”} \asymp A(1/h) h^{-m} \mathrm{e}^{-r/h}, \qquad h \to 0^+, \tag{4.2}$$

for some exponent $r > 0$, some power $m \in \mathbb{R}$, and some function $A(1/h)$ that is either constant or oscillating. The exponent r and the function $A(1/h)$ are determined by looking at the complex singularities closest to the real axis of the homoclinic solution of a limit Hamiltonian flow related to the map. Such methodology has been rigorously established for the standard map [Gel99], the Hénon map [GS01], and some perturbed McMillan maps [DR98, MSS11a, MSS11b]. It has also been numerically checked in certain billiard maps [Ram05] and several polynomial maps [GS08], but there are other maps where it fails [BM12]. Let us briefly recall some claims about polynomial standard maps contained in [GLT91, GS08]. First, $r = 2\pi\delta$, where δ is the distance of these singularities to the real axis. Besides,

$$A(1/h) = \mu a/2 + a \sum_{j=1}^{J} \cos(2\pi\beta_j/h + \varphi_j), \tag{4.3}$$

for some $\mu \in \{0,1\}$, some amplitude $a \neq 0$, and some phases $\varphi_j \in \mathbb{R}$, when these singularities are

$$\pm\delta\mathrm{i} \text{ (if and only if } \mu = 1), \pm\beta_1 \pm \delta\mathrm{i}, \ldots, \pm\beta_J \pm \delta\mathrm{i}.$$

For instance, the limit Hamiltonian flow for the standard map is a pendulum, so $\pm\pi\mathrm{i}/2$ are the closest singularities to the real axis and the “splitting size” is the so-called *Lazutkin constant* $\omega_0 \simeq 1118.827706$ times $h^{-2}\mathrm{e}^{-\pi^2/h}$, see [Gel99].

It is also known that, usually, $r = \rho/2$, where ρ is the radius of convergence of the Borel transform of the divergent asymptotic series that approaches the separatrices [DR99, GS01, Ram05, GS08].

By looking at our billiard problem from the perspective of those results (and others not mentioned here for the sake of brevity), it is natural to make the following conjecture. This conjecture is strongly supported by our numerical experiments.

Conjecture 21. *If Q is a closed analytic strictly convex curve, but it is neither a circle nor an ellipse, the asymptotic series (4.1) diverges for all period $q \in \mathbb{N}$, but it is Gevrey-1, so its Borel transform*

$$\sum_{j \geq 0} \hat{l}_j z^{2j-1}, \qquad \hat{l}_j = \frac{l_j}{(2j-1)!}, \tag{4.4}$$

has a radius of convergence $\rho \in (0,+\infty)$. Set $r = \rho/2$.

If Q is a generic axisymmetric algebraic curve, then

$$\Delta^{(1,q)} \asymp |A(q)| q^{-3} \mathrm{e}^{-rq}, \qquad q \to +\infty, \tag{4.5}$$

for some function $A(q)$ that is either constant: $A(q) = a/2 \neq 0$, or oscillatory:

$$A(q) = \mu a/2 + a \sum_{j=1}^{J} \cos(2\pi \beta_j q), \tag{4.6}$$

with $\mu \in \{0,1\}$, $a \neq 0$, $J \geq 1$, and $0 < \beta_1 < \cdots < \beta_J$. The cases $A(q) = a/2$ and $A(q) = a\cos(2\pi\beta)$ take place in open sets of the space of axisymmetric algebraic curves. All the other cases are phenomena of co-dimension at least one.

If Q is a generic bi-axisymmetric algebraic curve, $\Delta^{(1,q)}$ has the previous asymptotic behavior when q is even and $q \to +\infty$, but $\Delta^{(1,q)} = \mathrm{O}(q^{-2}\mathrm{e}^{-2rq})$ when q is odd and $q \to +\infty$.

We stress that the oscillating function (4.3) has some phases, but there are no phases in the oscillating function (4.6). This phenomenon is not new. The asymptotic formulas for the exponentially small splittings of generalized standard maps with trigonometric polynomials do not have phases either [GS08].

A curve is *axisymmetric* when it is symmetric with respect to a line, and *bi-axisymmetric* when it is symmetric with respect to two perpendicular lines. A planar curve is *algebraic* when its points are the zeros of some polynomial in two variables. We require strict convexity, since it is already an essential hypothesis in the smooth setup. We only consider algebraic curves by comparison with the above results about polynomial standard maps. Our algebraic curves have no singular points, because we ask them to be closed and analytic.

If Q is a circle of radius r_0, all its (p,q)-periodic billiard trajectories have length $2r_0 q \sin(\pi p/q)$, so $\Delta^{(p,q)} = 0$ for all $p/q \in (0,1)$, and the asymptotic series (4.1) becomes

$$L^{(1,q)} = 2r_0 q \sin(\pi/q) = 2r_0 \sum_{j\geq 0} \frac{(-1)^j \pi^{2j+1}}{(2j+1)!} q^{-2j},$$

which converges for all q. In particular, $\rho = +\infty$. Ellipses have analogous properties. This has to do with the fact that elliptic and circular billiards are integrable. A conjecture attributed to Birkhoff claims that the only integrable smooth convex billiard tables are ellipses and circles [Por50]. Following the discussion on the Mather's β-function contained in [Sor15], this old conjecture is reformulated as: *The series in (4.1) converges for some period $q \in \mathbb{N}$ if and only if Q is an ellipse or a circle.*

In this chapter, we present several numerical experiments and some analytical results that support Conjecture 21. For the sake of simplicity, all numerical experiments are carried out using the model tables

$$Q = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2/b^2 + \epsilon y^n = 1 \right\}. \tag{4.7}$$

Here, $b \in (0, 1]$ is the semi-minor axis, $\epsilon \in \mathbb{R}$ is the perturbative parameter, and $n \in \mathbb{N}$, with $3 \leq n \leq 8$, is the degree of the perturbation. We will refer to Q as a perturbed ellipse when $0 < b < 1$ and as a perturbed circle when $b = 1$. Next, we explain the four main reasons for this choice of billiard tables.

As a first reason, we know that all the billiard tables (4.7) are *nonintegrable* for $n \geq 3$ and $0 < \epsilon \ll 1$, and so the dynamics inside them should be far from trivial. The question of which perturbed ellipses give rise to integrable billiards is addressed in [DR96]. Theorem 4.1 of that paper imply that the tables (4.7) are nonintegrable if $0 < b < 1$, $n \geq 4$ is even, and ϵ is small enough. This result can be extended, after some technicalities, to odd degrees. Furthermore, all integrable deformations of ellipses of small eccentricities —this includes, of course, circles— are ellipses [ASK14], so the tables (4.7) are nonintegrable if $b = 1$, $n \geq 3$, and ϵ is small enough.

The second reason is that we want to use some *Melnikov methods* that are well suited for the study of billiards inside perturbed ellipses and perturbed circles [Ram06, PR13]. We recall that $\Delta^{(1,q)} = 0$ for any $q \geq 3$ in elliptic billiards. Thus, since the difference $\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon)$ is analytic in ϵ and vanishes at $\epsilon = 0$, we know that

$$\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon) = \epsilon \Delta_1^{(1,q)} + \mathrm{O}(\epsilon^2),$$

for some coefficient $\Delta_1^{(1,q)} \in \mathbb{R}$ that can be computed explicitly. To be precise, it turns out that if $0 < b < 1$ then

$$\Delta_1^{(1,q)} \asymp M_n q^{m_n} \mathrm{e}^{-cq}, \qquad q \to +\infty, \tag{4.8}$$

for some Melnikov exponent $c > 0$ not depending on n, some Melnikov power $m_n \in \mathbb{Z}$, and some Melnikov constant $M_n \neq 0$. These three Melnikov quantities can be explicitly computed, but we have carried out the computations only for the cubic ($n = 3$) and quartic ($n = 4$) perturbations for the sake of brevity. Besides, $\lim_{b \to 1} c = +\infty$. The Melnikov method provides no information when n is odd and q even; $\Delta^{(1,q)} = 0$ in such case. See Proposition 22 for details.

Which is the relation between the asymptotic formula (4.5) and the first order Melnikov computation (4.8)? The answer is that $r \neq c$ and $m_n \neq -3$, so *the Melnikov method does not accurately predict the singular behavior of* $\Delta^{(1,q)}$. Nevertheless, $\lim_{\epsilon \to 0} r = c$, so some information can be retrieved from the Melnikov method, at least for perturbed ellipses.

The case of perturbed circles is harder. See Section 4.5.

Symmetries are another reason for the choice of tables (4.7). On the one hand, symmetries greatly simplify the computation of periodic trajectories. To be precise, we just compute the signed difference D_q between two particular axisymmetric $(1, q)$-periodic trajectories, instead of $\Delta^{(1,q)}$ or $\Delta W_{1/q}$. Clearly, $|D_q| \le \Delta^{(1,q)}$. Often, $|D_q| = \Delta^{(1,q)} = \Delta W_{1/q}$. See Proposition 22. On the other hand, bi-axisymmetric curves are a very particular class of axisymmetric curves, so our model tables may display other asymptotic behaviors when n is even. We will check that this expectation is fulfilled. Concretely,

$$\Delta^{(1,q)} \asymp |B(q)| q^{-2} \mathrm{e}^{-2rq}, \qquad q \to +\infty,$$

for some constant or oscillating function $B(q)$ when n is even and q is odd. This asymptotic behavior has several differences with respect to the generic one conjectured in (4.5). Both the exponent in e^{-rq} and (if any) the frequencies $0 < \beta_1 < \cdots < \beta_J$ are doubled, the power in q^{-3} is increased by one, etcetera. We think that this new asymptotic behavior is generic among bi-axisymmetric algebraic curves when the period q is odd.

The last reason for the choice of such simple billiard tables is to reduce the *computational effort* as much as possible. In particular, we limit the degree of the perturbation to the range $3 \le n \le 8$ for this reason. Recall that each set $\Lambda^{(1,q)}$ is contained in an exponentially small (in q) interval, so the computation of $\Delta^{(1,q)}$ (or D_q) gives rise to very strong cancellations. This forces us to use a multiple-precision arithmetic to compute them. We have performed some computations with more than twelve thousand digits, based on the open source PARI/GP system [BBB+06]. Similar computations in the setting of splitting of separatrices of analytic maps can be found in [DR99, Ram05, GS08].

Finally, we recall that the exponent r is found by looking at the complex singularities of the homoclinic solution of a limit Hamiltonian flow in many cases of splitting of separatrices. *Does such kind of limit problem exist in our billiard setting?* Unfortunately, we do not have a completely satisfactory answer yet, but we propose a candidate in Section 4.2. It is empirically derived by using the Taylor expansions of the billiard dynamics close to the boundary given by Lazutkin in [Laz73]. Let $\kappa(s)$ be the curvature of Q in some arc-length parameter $s \in \mathbb{R}/l\mathbb{Z}$. Let $\xi \in \mathbb{R}/\mathbb{Z}$ be a new angular variable defined by

$$C\frac{\mathrm{d}\xi}{\mathrm{d}s} = \kappa^{2/3}(s), \qquad C = \int_Q \kappa^{2/3} \mathrm{d}s. \tag{4.9}$$

The constant C is the *Lazutkin perimeter*. Let δ be the distance of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis. We thought that $r = 2\pi\delta$, but our experiments disprove it. We have only obtained that $r \le 2\pi\delta$, the equality being an infrequent situation. But there are some good news about our candidate. First, the Melnikov exponent is $c = 2\pi\delta$, when $Q = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2/b^2 = 1\}$, with $0 < b < 1$. See Proposition 24. Second, we have also seen that, if $b = 1$ and $n \ge 3$ is fixed, then there exist some constants $\chi_n, \eta_n \in \mathbb{R}$, $\chi_n \le \eta_n$, such that

$$r = \frac{|\log \epsilon|}{n} + \chi_n + \mathrm{o}(1), \qquad 2\pi\delta = \frac{|\log \epsilon|}{n} + \eta_n + \mathrm{o}(1),$$

as $\epsilon \to 0^+$. The second formula is proved in Proposition 27, the first one is numerically checked in Section 4.5. Therefore, our candidate exactly captures the logarithmic growth of the exponent r for perturbed circles. Third, our experiments suggest that $r = 2\pi\delta$ when $b = 1$, $n \in \{5, 7\}$, and $\epsilon \in (0, 1/10)$.

This chapter has the following structure. We discuss our candidate to limit problem in Section 4.2. The axisymmetric tables and their axisymmetric periodic billiard trajectories are presented in more detail in Section 4.3. The main results about perturbed ellipses and perturbed circles are described in Sections 4.4 and 4.5, respectively. All proofs have been relegated to the last three sections, 4.6, 4.7, and 4.8.

4.2 A candidate for limit problem

To begin with, we recall how to obtain the limit problem for the splitting of separatrices of the generalized standard map $f(x, y) = (x_1, y_1)$ given by

$$x_1 = x + y_1, \qquad y_1 = y + \epsilon p(x). \tag{4.10}$$

For simplicity, we assume that $p(x)$ is a polynomial, $p(0) = 0$, and $p'(0) = 1$, so the origin is a hyperbolic fixed point of f with eigenvalues $\lambda = \mathrm{e}^h$ and $\lambda^{-1} = \mathrm{e}^{-h}$, where $\epsilon = 4\sinh^2(h/2)$. There is numerical evidence that the splitting size in this kind of polynomial standard maps satisfies the asymptotic formula (4.2) for some exponent $r > 0$, some power $m \in \mathbb{R}$, and some constant or oscillating function $A(1/h)$. We determine the exponent following [GL01].

First, we transform the original map into the map

$$x_1 = x + \mu z_1, \qquad z_1 = z + \mu p(x)$$

by means of the scaling $z = y/\mu$, where $\mu = \sqrt{\epsilon}$. Note that $\mu \asymp h$ as $\epsilon \to 0^+$. The dynamics of this map for small μ resembles the dynamics of the μ-time flow of the Hamiltonian $H_0(x, z) = z^2/2 - \int p(x)\mathrm{d}x$. Besides, the origin is a hyperbolic equilibrium point of the Hamiltonian system

$$x' = \partial_z H_0(x, z) = z, \qquad z' = -\partial_x H_0(x, z) = p(x).$$

If the singular level set $\{(x, z) \in \mathbb{R}^2 : H_0(x, z) \equiv H_0(0, 0)\}$ contains a separatrix to the origin, then we compute the flow on it and we get a homoclinic solution $(x_0(\xi), z_0(\xi))$ that can be seen as the limit of the map on its separatrices when $\epsilon \to 0^+$. Such homoclinic solution is determined, up to a constant time shift, by imposing

$$x_0''(\xi) = p(x_0(\xi)), \qquad \lim_{\xi \to \pm\infty} x_0(\xi) = 0.$$

It turns out that there exists $\delta > 0$ such that $x_0(\xi)$ is analytic in the open complex strip $\mathcal{I}_\delta = \{\xi \in \mathbb{C} : |\Im \xi| < \delta\}$ and has singularities on the boundary of $\mathcal{I}_\delta$. Then $r = 2\pi\delta$. This claim is contained in [GL01], although a complete proof is still pending. However, Fontich and Simó proved the following exponentially small upper bound in [FS90]. If $\alpha \in (0, \delta)$, then there exist some constants $K, h_0 > 0$ such that

$$\text{“splitting size”} \leq K\mathrm{e}^{-2\pi\alpha/h}, \qquad \forall h \in (0, h_0].$$

We want to emphasize an essential, but sometimes forgotten, hypothesis of the Fontich-Simó Theorem. Let

$$\sigma_0(\xi) = (x_0(\xi), y_0(\xi))$$

be the original homoclinic solution. The generalized standard map (4.10) should have an analytic extension to a complex neighborhood in $\mathbb{C}^2$ of $\sigma_0(\overline{\mathcal{I}_\alpha})$. If $p(x)$ is a polynomial, then f can be extended to the whole $\mathbb{C}^2$ and this hypothesis is automatically fulfilled. On the contrary, it remains to be checked when $p : \mathbb{R} \to \mathbb{R}$ is just a real analytic function.

Next, we adapt these ideas to our billiard problem.

Let Q be an analytic strictly convex curve in the Euclidean plane. Set $l = \mathrm{L}ength(Q)$. Let $\kappa(s)$ be the curvature of Q in some arc-length parameter $s \in \mathbb{R}/l\mathbb{Z}$. Note that $\kappa(s) > 0$ for all $s \in \mathbb{R}/l\mathbb{Z}$. Let $\varrho(s) = 1/\kappa(s)$ be the radius of curvature. We are interested in the dynamics of the billiard map (2.1) when the angle of incidence r tends to zero. More precisely, we consider that $r = \mathrm{O}(1/q)$ and $q \to +\infty$.

Lazutkin [Laz73] gave the Taylor expansion

$$s_1 = s + 2\varrho(s)r + \mathrm{O}(r^2), \qquad r_1 = r - 2\varrho'(s)r^2/3 + \mathrm{O}(r^3)$$

for the dynamics of the billiard map (2.1) around $r = 0$. Once fixed a period $q \gg 1$, we take $\mu = 1/q \ll 1$ as the small parameter. Then we transform the previous expansion into

$$s_1 = s + \mu\varrho(s)v^{1/2} + \mathrm{O}(\mu^2), \quad v_1 = v - \frac{2}{3}\mu\varrho'(s)v^{3/2} + \mathrm{O}(\mu^2),$$

by means of the change of variables $\sqrt{v} = 2r/\mu$. The billiard dynamics for small μ resembles the dynamics of the μ-flow of the Hamiltonian $H_0(s,v) = \frac{2}{3}\varrho(s)v^{3/2}$. That is, the μ-flow of the Hamiltonian system

$$s' = \varrho(s)v^{1/2}, \qquad v' = -\frac{2}{3}\varrho'(s)v^{3/2}.$$

We compute the flow on the level set $\mathcal{H}_C := \{H_0(s,v) \equiv \frac{2}{3}C^3\}$, for some constant $C > 0$. If $(s,v) \in \mathcal{H}_C$, then the first equation of the Hamiltonian system reads as

$$\frac{\mathrm{d}s}{\mathrm{d}\xi} = s' = \varrho(s)v^{1/2} = C\varrho^{2/3}(s),$$

or, equivalently, as

$$C\frac{\mathrm{d}\xi}{\mathrm{d}s} = \kappa^{2/3}(s). \tag{4.11}$$

We only need the following observations to determine C. We are looking at the $(1,q)$-periodic trajectories inside Q. We have approximated the billiard dynamics by the μ-time of the Hamiltonian flow with $\mu = 1/q$. Any $(1,q)$-periodic trajectory gives one turn after q iterates of the billiard map, so the variable ξ should be increased by one if s is increased by $l = \mathrm{L}ength(Q)$. Therefore,

$$C = C\int_0^l \frac{\mathrm{d}\xi}{\mathrm{d}s}\mathrm{d}s = \int_0^l \kappa^{2/3}(s)\mathrm{d}s = \int_Q \kappa^{2/3}\mathrm{d}s. \tag{4.12}$$

Relation (4.9) is obtained by joining equations (4.11) and (4.12). Let $s = s_0(\xi)$ be the inverse of the solution $\xi = \xi_0(s)$ of the differential equation (4.9) determined, for the sake of definiteness, by the initial condition $\xi_0(0) = 0$. By abusing the notation, let $\kappa(\xi) = \kappa(s_0(\xi))$ be the curvature in the new angular variable $\xi \in \mathbb{R}/\mathbb{Z}$. Then $\kappa(\xi)$ is a 1-periodic real analytic function which does not vanish on the reals. Let us assume that there exists $\delta > 0$ such that $\kappa(\xi)$ is analytic and does not vanish on the open complex strip $\mathcal{I}_\delta$ and has singularities and/or zeros on the boundary of $\mathcal{I}_\delta$. Note that we are avoiding not only singularities but also zeros of the curvature $\kappa(\xi)$. On the one hand, the results found by Marvizi and Melrose only hold for smooth strictly convex curves, so the zeros of the curvature are a source of potential problems. On the other hand, several positive and negative fractional powers of the curvature appear in the previous computations (see also below), and such powers are not analytic at the zeros of the curvature.

Following the numerical evidences in the splitting problems of the polynomial standard maps, we thought that $r = 2\pi\delta$, but our experiments disprove it. We have obtained that $r \leq 2\pi\delta$, the equality being an infrequent situation.

An explanation of such discrepancy is the following one. Set $\sigma_0(\xi) = (s_0(\xi), r_0(\xi))$, $r_0(\xi) = \mu\sqrt{v_0(\xi)}/2 = C\kappa^{1/3}(\xi)/2q$. We know that the billiard map (2.1) can be analytically extended to $(\mathbb{R}/l\mathbb{Z}) \times [0, \pi)$; see Proposition 5. However, we do not know whether it can be analytically extended to a complex neighborhood in $(\mathbb{C}/l\mathbb{Z}) \times \mathbb{C}$ of $\sigma_0(\overline{\mathcal{I}_\alpha})$ as $\alpha \to \delta^-$ and $q \to +\infty$ or not. Hence, the inequality $r \leq 2\pi\delta$ does not look so bad in the light of the previous discussion about the Fontich-Simó Theorem. In fact, it is commonly accepted that the magnitude involved in the exponent of the exponentially small formulas for splitting problems is not the minimum distance to the real line of the set of singularities of the time parametrization of the separatrix but the minimum distance to the real line of the set of singularities *of the perturbation of the system when evaluated on the time parametrization of the separatrix*. See [GS12, BM12] for some examples. It seems reasonable to think that one has to compute the singularities of the Lagrangian evaluated on the solution of (4.9), which, in its turn, reduces to the study of the singularities of $\gamma(s_0(\xi))$. This is a work in progress.

4.3 Model tables

We restrict our study to the perturbed ellipses and perturbed circles given implicitly in (4.7). To be precise, the algebraic curve $x^2 + y^2/b^2 + \epsilon y^n = 1$ has several real connected components when n is odd. Henceforth, we only consider the one that tends to the ellipse (or circle) $x^2 + y^2/b^2 = 1$ as ϵ tends to zero.

Let $\epsilon_n = \epsilon_n(b)$ be the maximal positive parameter such that

$$Q \text{ is analytic and strictly convex for all } \epsilon \in I_n := (0, \epsilon_n). \tag{4.13}$$

On the one hand, $I_n = (0, +\infty)$ when n is even. In such cases, we will reach the value $\epsilon = 1$ in some numerical computations. On the other hand, if n is odd, the algebraic curve defined by $x^2 + y^2/b^2 + \epsilon y^n = 1$ has a singular point on the y-axis when

$$\epsilon = \bar{\epsilon}_n = \bar{\epsilon}_n(b) := 2(n-2)^{n/2-1} n^{-n/2} b^{-n}. \tag{4.14}$$

Thus, Q is no longer analytic when $\epsilon = \bar{\epsilon}_n$. Our computations suggest that $\epsilon_n = \bar{\epsilon}_n$ so we restrict our experiments to the range $0 < \epsilon < \bar{\epsilon}_n$. We note that $\bar{\epsilon}_3(b) \approx 0.3849/b^3$, $\bar{\epsilon}_5(b) \approx 0.1859/b^5$, and $\bar{\epsilon}_7(b) \approx 0.1232/b^7$. We also restrict our experiments to the degrees $3 \leq n \leq 8$.

The symmetries of our model tables simplify the search of some periodic trajectories. If n is even, Q is symmetric with respect to both axis of coordinates, so Q is bi-axisymmetric. If n is odd, Q is symmetric with respect to the y-axis only, so Q is axisymmetric but not bi-axisymmetric. We say that a billiard trajectory is *axisymmetric* when its corresponding polygon is symmetric with respect to some axis of coordinates. We only compute axisymmetric periodic trajectories, APTs for short.

First, let us focus on the case odd n. The axisymmetric trajectories inside Q are characterized as the ones with an impact point on or with a segment perpendicular to the y-axis. The APTs are characterized as the ones satisfying twice the former condition. Thus, there are four kinds of APTs inside Q. Besides, only two of these kinds are possible depending on the (parity of the) period q.

The classification for even n is richer because the symmetry with respect to the x-axis plays the same role. See Table 4.1.

We wanted to study the differences $\Delta^{(1,q)}$ and the Mather's $\Delta W_{1/q}$, but instead we will compute the signed differences D_q between the lengths of the $(1,q)$-APTs. Clearly, $|D_q| \leq \Delta^{(1,q)}$. In some cases, all periodic trajectories are axisymmetric, and so $\Delta^{(1,q)} = \Delta W_{1/q} = |D_q|$. See Proposition 22.

We will fix the semi-minor axis b and the degree n in our numerical experiments. That is, we will study the dependence of $D_q = D_q(\epsilon)$ on the perturbative parameter ϵ and the period q. The quantity $D_q(\epsilon)$ is analytic at $\epsilon = 0$ because all $(1,q)$-APTs are so. On the contrary, the period q is a singular parameter of this problem because D_q is exponentially small in q. Thus, we will deal with:

- The *regular* case, where we study the asymptotic behavior of $D_q(\epsilon)$ when $\epsilon \to 0$ and $q \geq 3$ is fixed; and

- The *singular* case, where we study the asymptotic behavior of $D_q(\epsilon)$ when $q \to +\infty$ and $\epsilon \in \mathbb{R}$ is fixed.

We will see that the classical Melnikov method is suitable to study the regular case but it is not so to study the singular one. Besides, the Melnikov method gives more information on perturbed ellipses than on perturbed circles. The singular case is only studied numerically.

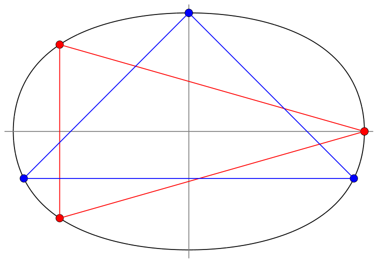
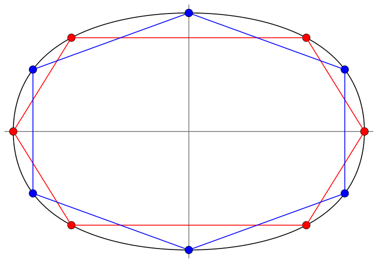
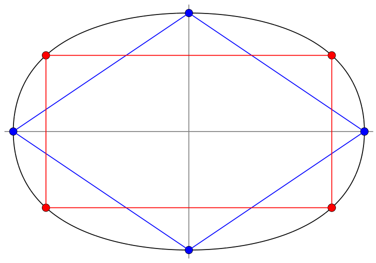
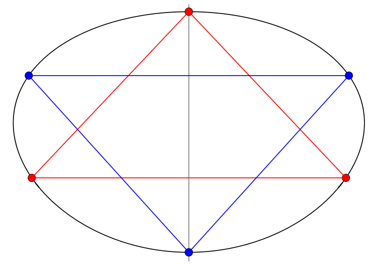
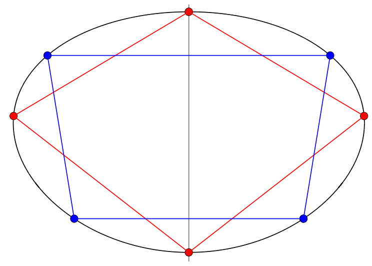
n	q	Examples of APTs with minimal periods
even	$2k+1$	
even	$4k+2$	
even	$4k$	
odd	$2k+1$	
odd	$2k$	

Table 4.1: Classification of $(1, q)$-APTs inside bi-axisymmetric and axisymmetric billiard tables Q. In each case, the difference D_q is the length of the $(1, q)$-APT in red minus the length of the $(1, q)$-APT in blue. The gray lines denote the axis of symmetry.

4.4 Perturbed ellipses

In this section we restrict ourselves to the case $0 < b < 1$. We begin with the regular case, so the semi-minor axis b, the degree $n \geq 3$, and the period $q \geq 3$ are fixed, whereas $\epsilon \to 0^+$. Since the quantity $\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon)$ is analytic and vanishes at $\epsilon = 0$, then

$$\Delta^{(1,q)} = \epsilon \Delta_1^{(1,q)} + \mathrm{O}(\epsilon^2), \tag{4.15}$$

for some coefficient $\Delta_1^{(1,q)} \in \mathbb{R}$. This coefficient can be computed by using a standard Melnikov method. In fact, the model tables (4.7) have been chosen in such a way that the asymptotic behavior of $\Delta_1^{(1,q)}$ can be determined. The analytical results for $\Delta_1^{(1,q)}$ in the cubic and quartic perturbations are stated below, but we need to introduce some notation first.

Given $m \in [0,1)$, the *complete elliptic integral of the first kind* is

$$K = K(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} \mathrm{d}\theta.$$

We also write $K' = K'(m) = K(1-m)$.

Proposition 22. *If $b \in (0,1)$ and $q \geq 3$, the following properties hold.*

1. $\Delta_1^{(1,q)} = 0$, *for odd n and even q.*

2. *There exist some constants $c, M_3, M_4, K_4 > 0$, depending only on b, such that*

$$\Delta_1^{(1,q)} \asymp \begin{cases} M_3 \mathrm{e}^{-cq}, & \text{for } n = 3 \text{ and odd } q, \\ K_4 q \mathrm{e}^{-2cq}, & \text{for } n = 4 \text{ and odd } q, \\ M_4 q \mathrm{e}^{-cq}, & \text{for } n = 4 \text{ and even } q, \end{cases} \tag{4.16}$$

when $q \to +\infty$. Besides, $K_4 = 2M_4$, and

$$c = \frac{\pi K(b^2)}{2K(1-b^2)} = \frac{\pi K'(1-b^2)}{2K(1-b^2)}. \tag{4.17}$$

3. *If $n = 3$ and q is odd or if $n = 4$, then there exists $\tilde{\epsilon}_n = \tilde{\epsilon}_n(b,q) \in I_n$ such that all $(1,q)$-periodic billiard trajectories inside* (4.7) *are axisymmetric when $\epsilon \in (0, \tilde{\epsilon}_n)$. In particular, $\Delta^{(1,q)} = \Delta W_{1/q} = |D_q|$ for all $\epsilon \in (0, \tilde{\epsilon}_n)$.*

See Section 4.6 for the proof. The explicit values of M_3 and M_4 can be found in (4.34). Related computations can be found in [PR13].

Remark 8. Similar results hold for any degree $n \geq 5$, although it is more cumbersome to compute the *Melnikov constants* M_n (and K_n if n is even) and the *Melnikov powers* m_n such that

$$\Delta_1^{(1,q)} \asymp \begin{cases} M_n q^{m_n} \mathrm{e}^{-cq}, & \text{for odd } n \text{ and odd } q, \\ K_n q^{m_n} \mathrm{e}^{-2cq}, & \text{for even } n \text{ and odd } q, \\ M_n q^{m_n} \mathrm{e}^{-cq}, & \text{for even } n \text{ and even } q, \end{cases}$$

as $q \to +\infty$. The *Melnikov exponent* c does not depend on n.

From the first order formula (4.15), we deduce that

$$\lim_{\epsilon \to 0} \left[\Delta^{(1,q)} / \epsilon \Delta_1^{(1,q)} \right] = 1,$$

for any fixed $q \geq 3$. Next, we wonder whether the roles of ϵ and q are interchangeable; that is, if

$$\lim_{q \to +\infty} \left[\Delta^{(1,q)} / \epsilon \Delta_1^{(1,q)} \right] = 1, \tag{4.18}$$

for any fixed but small enough $\epsilon > 0$.

We should compute $\Delta^{(1,q)}/\epsilon\Delta_1^{(1,q)}$ for big periods q in order to answer this question, but instead we compute $|D_q|/\epsilon\Delta_1^{(1,q)}$. Both quotients coincide if ϵ is small enough, see Proposition 22. We do not compute $|D_q|/\epsilon\Delta_1^{(1,q)}$ for the cubic perturbation and even periods, because $\Delta_1^{(1,q)} = 0$ for $n = 3$ and even q.

We show the results obtained for the cubic and quartic perturbations in Figure 4.1. These figures are obtained by taking the semi-minor axis $b = 4/5$. Other values for the semi-minor axis give rise to similar figures.

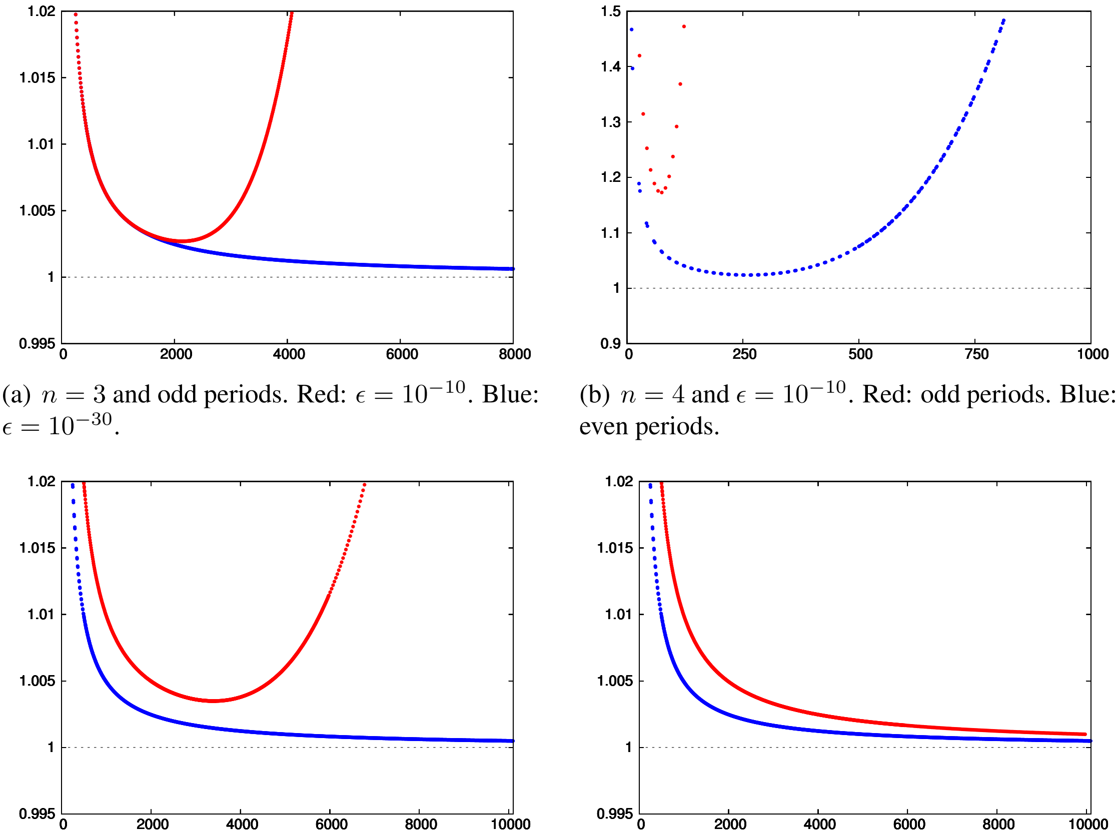
The Melnikov method does not predict the asymptotic behavior of $\Delta^{(1,q)}$ in the singular case. That is, limit (4.18) does not hold. Indeed, if we fix any $\epsilon > 0$, then the quotient $|D_q|/\epsilon\Delta_1^{(1,q)}$ drifts away from one as q grows. The drift appears earlier for odd periods in the case of the quartic perturbation. As ϵ gets smaller, the drift appears at larger periods q. Since the computing time grows quickly when q grows, the computations to see that drift when ϵ is very small are unfeasible with our resources. This happens, for instance, when $n = 4$ and $\epsilon = 10^{-30}$. See Figure 4.1.

Based on these numerical experiments, we guess that there exist some critical exponents $\nu_n > 0$ such that

$$\Delta^{(1,q)} = \Delta^{(1,q)}(\epsilon) \asymp \begin{cases} M_n \epsilon q^{m_n} \mathrm{e}^{-cq}, & \text{for odd } n \text{ and odd } q, \\ K_n \epsilon q^{m_n} \mathrm{e}^{-2cq}, & \text{for even } n \text{ and odd } q, \\ M_n \epsilon q^{m_n} \mathrm{e}^{-cq}, & \text{for even } n \text{ and even } q, \end{cases}$$

when $\epsilon = \mathrm{O}(q^{-\nu})$, $q \to +\infty$, and $\nu > \nu_n$. Here, M_n, K_n, m_n, and c are the Melnikov quantities introduced in Proposition 22 and Remark 8. We do not give an asymptotic behavior when n is odd and q is even because we do not have any Melnikov prediction for that case. Results about exponentially small asymptotic behaviors based on Melnikov predictions are common in the literature. For instance, the rapidly forced pendulum is studied in [DS92, DS97, GL01, GOS10, GS12] and some perturbed McMillan maps are studied in [DR96, DR98, MSS11a, MSS11b].

Nevertheless, we are interested in a more natural problem. Namely, the asymptotic behavior of $\Delta^{(1,q)}$ when $q \to +\infty$ and ϵ is fixed. As we have said before, we compute the



(a) $n = 3$ and odd periods. Red: $\epsilon = 10^{-10}$. Blue: $\epsilon = 10^{-30}$.

(b) $n = 4$ and $\epsilon = 10^{-10}$. Red: odd periods. Blue: even periods.

(c) $n = 4$ and $\epsilon = 10^{-20}$. Red: odd periods. Blue: even periods.

(d) $n = 4$ and $\epsilon = 10^{-30}$. Red: odd periods. Blue: even periods.

Figure 4.1: The quotient $|D_q|/\epsilon\Delta_1^{(1,q)}$ versus the period q for $b = 4/5$.

signed difference D_q instead of $\Delta^{(1,q)}$. We have numerically checked that, if ϵ is small enough, then there exist a constant $A \neq 0$, a power $m \in \mathbb{Z}$, and an exponent $r > 0$ such that

$$D_q \asymp A q^m \mathrm{e}^{-rq}, \tag{4.19}$$

as $q \to +\infty$. In fact, the real behavior is slightly more complicated, since these three quantities depend on the parity of q. We summarize our results as follows.

Numerical Result 23. *Fix $b \in (0,1)$ and $n \geq 3$. Let I_n be the maximal interval defined in (4.13). There exists $\hat{\epsilon}_n = \hat{\epsilon}_n(b) \in I_n$ such that the billiard inside* (4.7) *satisfies the following properties for all $\epsilon \in (0, \hat{\epsilon}_n)$. The Borel transform (4.4) has a radius of convergence $\rho \in (0,+\infty)$. Set $r = \rho/2$. There exist two constants $A, B \neq 0$ such that*

$$D_q \asymp \begin{cases} Bq^{-2}\mathrm{e}^{-2rq}, & \textit{for even } n \textit{ and odd } q, \\ Aq^{-3}\mathrm{e}^{-rq}, & \textit{otherwise}, \end{cases} \tag{4.20}$$

as $q \to +\infty$. The quantities ρ, r, A, and B depend on b, ϵ, and n. The constant B is defined only when n is even. Besides, $\lim_{\epsilon \to 0} r = c$, where c is the Melnikov exponent defined in (4.17).

We stated in Conjecture 21 that the function $A(q)$ that appears in the exponentially small asymptotic formula (4.6) is constant when the billiard table belongs to a certain open set of the space of axisymmetric algebraic curves. Thus, the previous numerical result fits perfectly into the conjecture.

It is interesting to compare the Melnikov formulas (4.16) with the asymptotic formulas (4.20). The asymptotic behavior of D_q does not depend on the parity of q when n is odd. The exponents c and r play the same role. Finally, the factors q^{-2} and q^{-3} in (4.20) can not be directly guessed from the Melnikov formulas.

Let us describe our numerical experiments. First, once the exponent r is determined (see next paragraph), we compute the normalized differences

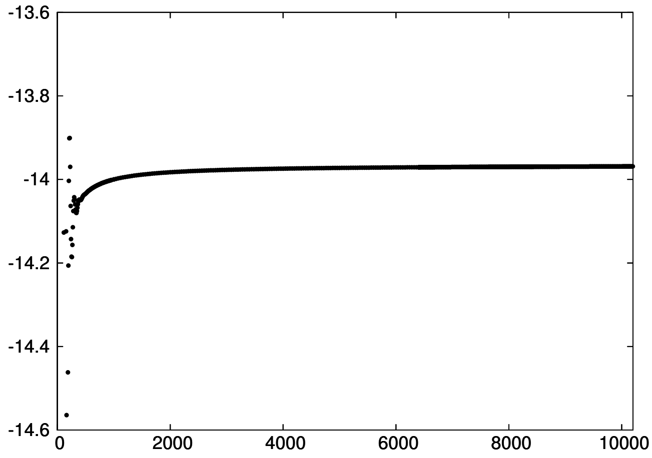
$$\hat{D}_q = \begin{cases} q^2 \mathrm{e}^{2rq} D_q, & \text{for even } n \text{ and odd } q, \\ q^3 \mathrm{e}^{rq} D_q, & \text{otherwise}. \end{cases} \tag{4.21}$$

We have checked that these normalized differences $\hat{D}_q$ tend to some constant as $q \to +\infty$ in the ranges $1/2 \leq b \leq 9/10$ and $0 < \epsilon \leq 1/10$. Figure 4.2 shows this behavior on three different scenarios for $b = 9/10$ and $\epsilon = 1/10$.

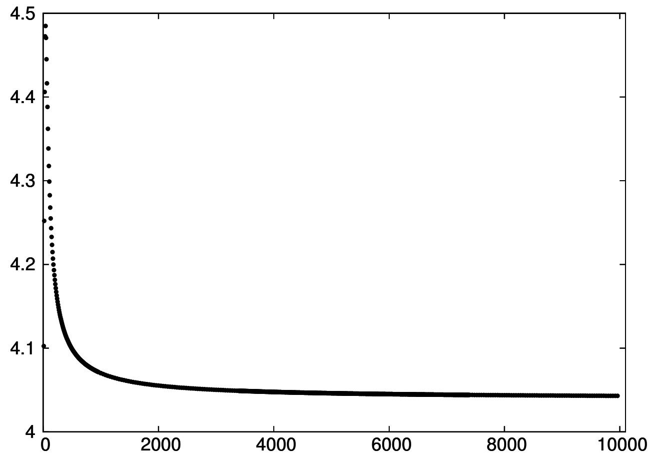
Let us explain how to compute the exponent $r = r(b, \epsilon, n)$. First, we assume that the exponentially small asymptotic formula (4.19) can be refined as

$$D_q \asymp q^m \mathrm{e}^{-rq} \sum_{j \geq 0} d_j q^{-2j},$$

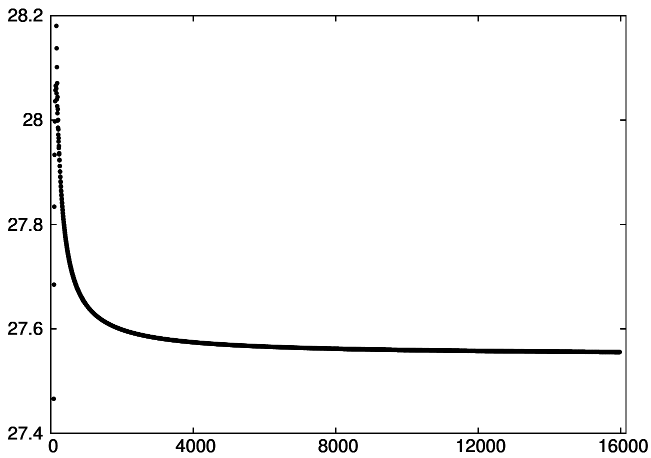
for some asymptotic coefficients $d_j \in \mathbb{R}$ with $d_0 = A \neq 0$. This assumption is based on similar refined asymptotic formulas for the splitting of separatrices of analytic



(a) $n = 3$, $b = 9/10$, and $\epsilon = 1/10$.



(b) $n = 4$, $b = 9/10$, $\epsilon = 1/10$, and odd periods.



(c) $n = 4$, $b = 9/10$, $\epsilon = 1/10$, and even periods.

Figure 4.2: The normalized differences $\hat{D}_q$ tend to a constant when $q \to +\infty$ in the ranges $1/2 \leq b \leq 9/10$ and $0 < \epsilon \leq 1/10$ for the cubic and quartic perturbations. If n is even, then we have to study the even and odd periods separately.

maps [Gel99, MSS11a]. By taking logarithms, we find the asymptotic expansion

$$\frac{1}{q}\log\left(q^{-m}D_q\right) \asymp -r + \frac{1}{q}\log\left(\sum_{j\geq 0}\frac{d_j}{q^{2j}}\right) \asymp -r + \sum_{j\geq 0}\frac{\alpha_j}{q^{2j+1}},$$

for some coefficients $\alpha_j \in \mathbb{R}$. Therefore, we can compute r by using a Neville extrapolation method from a sequence of differences D_q. The longer the sequence, the more correct digits in r. We obtain 15 correct digits with the following choices. We fix the perturbed ellipse Q, that is, we fix $b \in (0,1)$, $\epsilon \in \mathbb{R}$, and $n \geq 3$. Second, we fix the class of periods q, so that we are on one of the cases of Table 4.1. That is, $q = q(k) = 2k+1$, $q = q(k) = 4k+2$, $q = q(k) = 4k$, or $q = q(k) = 2k$. Then, we compute D_q with at least 400 correct digits on an increasing sequence of 500 periods $q_i = q(k_i)$, with $k_i = k_0 + 10i$. The initial period q_0 is chosen to be big enough so that $|D_{q_0}| \leq 10^{-3000}$. In fact, we perform the Neville extrapolation with two different sequences of 500 periods each which allows us to determine the number of correct digits in the final result. The power $m \in \{-2,-3\}$ is found by trial-and-error.

In Figure 4.3, we display the exponent $r = r(\epsilon)$ for several values of b for the cubic and quartic perturbations. We also depict the Melnikov exponent c at $\epsilon = 0$ in full circles. Note that $\lim_{\epsilon\to 0} r = c$ and r is decreasing in ϵ.

Next, let us relate the exponent r with the radius of convergence ρ of the Borel transform (4.4). Once fixed $b \in (0,1)$, $\epsilon \in \mathbb{R}$, and $n \geq 3$, we compute $\rho = \rho(b,\epsilon,n)$ as follows.

First, we compute the length $L^{(1,q)}$ of one of the $(1,q)$-APTs inside Q for the same sequences of periods (q_i) used for computing D_q. We use a precision of 3000 correct digits in these computations. The choice of the APT does not matter, since $|D_{q_i}| \leq 10^{-3000}$ for any period $q_i \geq q_0$. Second, we obtain the first asymptotic coefficients l_j in the expansion (4.1) by using the Neville extrapolation method again. Third, we determine the number of correct digits in each coefficient l_j by comparing the results obtained with two different sequences of periods. The number of correct digits in l_j decreases as j grows. We always get at least 1500 correct digits in l_0 and at least 40 correct digits in l_{450}.

It turns out that the coefficients l_j increase at a factorial rate, so the asymptotic series (4.1) is Gevrey-1 and diverges for any q. Indeed, we have found that there exist a radius of convergence $\rho = \rho(b,\epsilon,n) > 0$ and a constant $\gamma = \gamma(b,\epsilon,n) > 0$ such that

$$\hat{l}_j \asymp \gamma j^{-2}\rho^{-2j}, \qquad j \to +\infty,$$

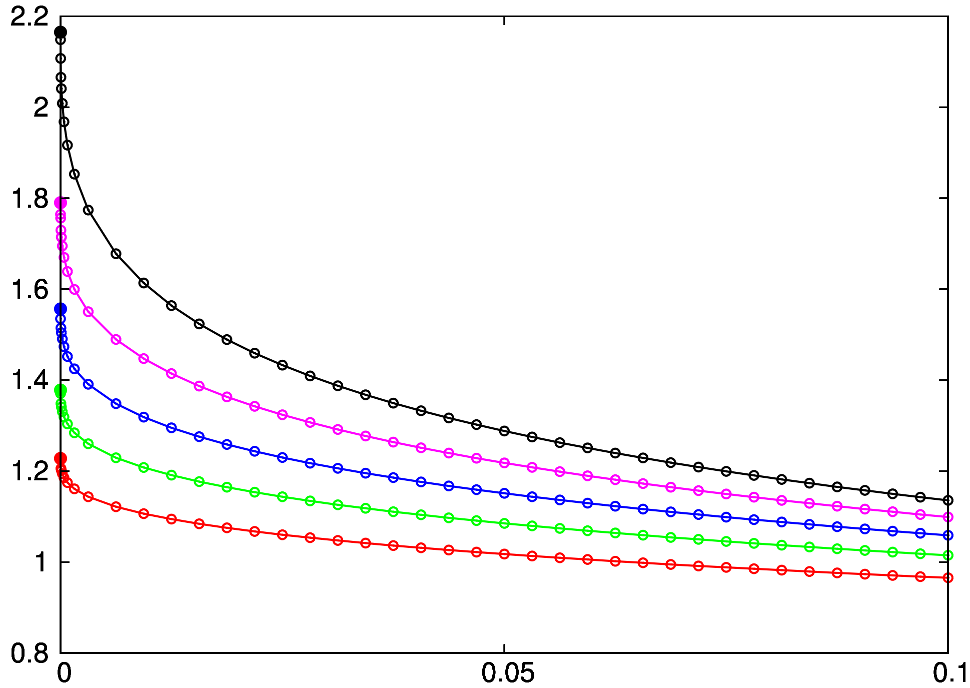
provided ϵ is small enough. That is, the Borel transform (4.4) has a singularity at $z = \rho$. In particular,

$$\rho = \lim_{j\to\infty}\left|\hat{l}_j/\hat{l}_{j+1}\right|^{1/2}.$$

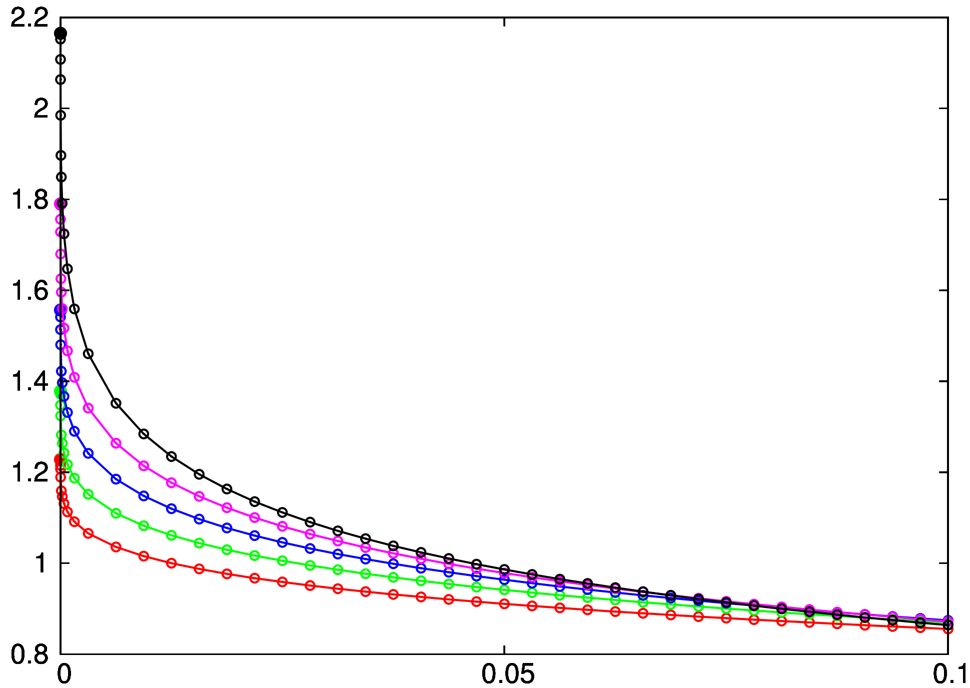
We see this asymptotic behavior in Figure 4.4.

The rough approximation

$$\rho \approx \left|\hat{l}_{449}/\hat{l}_{450}\right|^{1/2}$$



(a) $n = 3$.



(b) $n = 4$.

Figure 4.3: The exponent r versus the perturbative parameter ϵ. We also display the points $(0, c)$ in solid circles, where c is the Melnikov exponent. We note that $\lim_{\epsilon\to 0^+} r = c$. Red: $b = 1/2$. Green: $b = 3/5$. Blue: $b = 7/10$. Magenta: $b = 4/5$. Black: $b = 9/10$.

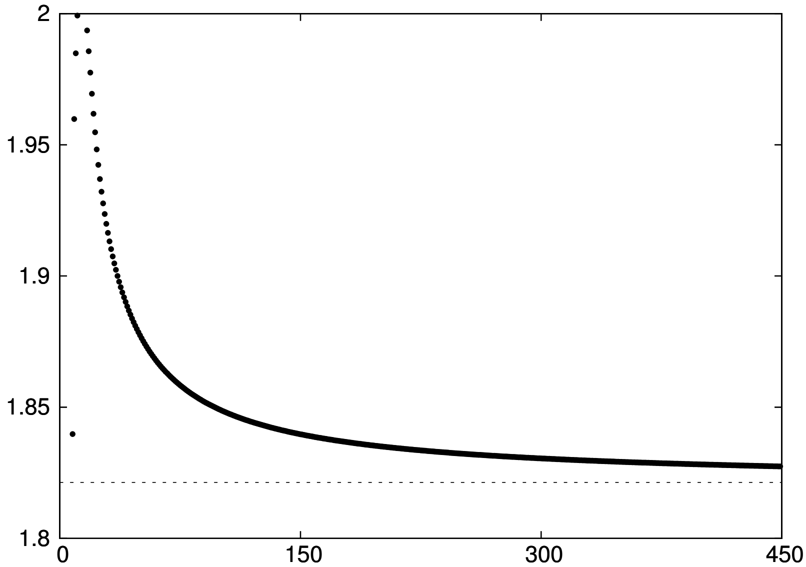


Figure 4.4: $\left|\hat{l}_j/\hat{l}_{j+1}\right|^{1/2}$ versus j for $b = 9/10$, $\epsilon = 1/20$, and $n = 4$. The dashed line corresponds to the limit value ρ obtained by extrapolation.

only gives about 3 correct digits. If we use an extrapolation method based on the asymptotic expansion

$$\left|\hat{l}_j/\hat{l}_{j+1}\right|^{1/2} \asymp \rho + \sum_{i>0} \beta_i j^{-i},$$

the radius of convergence is improved up to 8 correct digits. This is the limit value plotted in Figure 4.4. We stress that this asymptotic expansion in powers of j^{-1} is probably wrong since the extrapolation becomes unstable after a few steps.

The radius of convergence ρ does not depend on the parity of the periods of the sequence (q_i). Thus, the value of ρ obtained by sequences of different parities must coincide. This provides another validation to the number of correct digits of ρ.

Remark 9. Taking into account relation $r = \rho/2$, we have two different ways of computing the exponent r, the direct method and the Borel one. The Borel method is computationally much cheaper. Indeed, the precision required to compute the differences D_{q_i} increases along the periods q_i whereas it is fixed when computing the lengths $L^{(1,q_i)}$.

At this point, we have established the relations among the Melnikov exponent c, the exponent r, and the radius of convergence ρ. Next, we relate c with the distance δ provided by our candidate for limit problem, since we are only able to analytically compute δ for unperturbed ellipses.

Proposition 24. *Let $b \in (0, 1)$. Let $\kappa(s)$ be the curvature of the unperturbed ellipse $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2/b^2 = 1\}$ in some arc-length parameter s. Let $\xi \in \mathbb{R}/\mathbb{Z}$ be the angular variable defined by (4.9). Let δ be the distance of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis. Then $2\pi\delta = c$, where c is the Melnikov exponent defined in* (4.17).

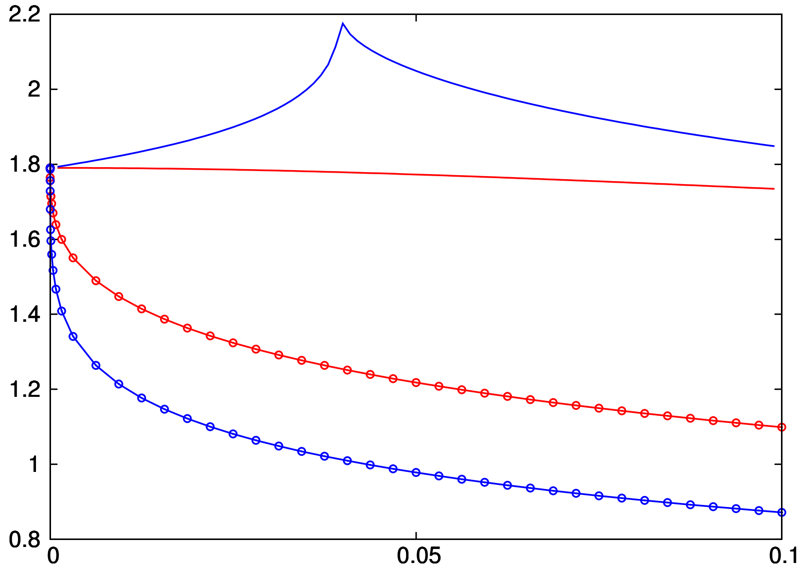


Figure 4.5: The exponent r (continuous lines with points) and the quantity $2\pi\delta$ (continuous lines) versus ϵ for $b = 4/5$. Red: $n = 3$. Blue: $n = 4$.

This proposition is proved in Section 4.7.

We have numerically checked that the inequality $r < 2\pi\delta$ holds in the ranges $1/2 \leq b \leq 9/10$ and $0 < \epsilon \leq 1/10$ for the cubic and quartic perturbations. The case $b = 4/5$ is displayed in Figure 4.5.

Remark 10. The distance δ is numerically computed as follows. First, we write the curvature κ and the length element $\mathrm{d}s$ of the perturbed ellipse (4.7) in terms of the vertical coordinate y. It turns out that there exist three polynomials $r(y)$, $p(y)$ and $q(y)$ such that

$$\kappa^{2/3}\mathrm{d}s = g(y)\mathrm{d}y := \frac{p^{2/3}(y)}{\sqrt{r(y)q(y)}}\mathrm{d}y.$$

For instance, $r(y) = 1 - y^2/b^2 - \epsilon y^n$ and $\deg[p] = \deg[q] = 2n - 2$. Let $y_\pm$ be the roots of $r(y)$ that tend to $\pm b$ when $\epsilon \to 0$. The points $(0, y_\pm)$ are the vertices on the vertical axis of the perturbed ellipse (4.7). Then $\delta = |\Im\xi_\star|/C$, where

$$C = \int_Q \kappa^{2/3}\mathrm{d}s = 2\int_{y_-}^{y_+} g(y)\mathrm{d}y, \qquad \xi_\star = \int_0^{y_\star} g(y)\mathrm{d}y,$$

and $y_\star \neq y_\pm$ is the root of $p(y)$, $q(y)$, or $r(y)$ that gives the closest singularity $\xi_\star \in \mathbb{C}/\mathbb{Z}$ to the real axis. That is, $y_\star$ minimizes δ. The path from $y = 0$ to $y = y_\star$ in the second integral should be contained in an open simply connected subset of the complex plane where the function $g(y)$ is analytic. See Section 4.8 for more details about the function $g(y)$ and their domain of analyticity, although that section deals with perturbed circles only.

We note that the cusp that appears in the graph of $2\pi\delta$ for the quartic perturbation correspond to a perturbative parameter ϵ for which two different roots of $p(y)$, $q(y)$, or $r(y)$ give rise to the same $\delta = |\Im\xi_\star|/C$.

4.5 Perturbed circles

In this section, we take $b = 1$ in the model tables (4.7). This setting is harder than the one of the perturbed ellipses both in the regular and singular cases. Let us explain it.

We begin with the regular case, so we fix the degree $n \geq 3$ and the period $q \geq 3$ whereas ϵ tends to zero. First, we note that the Melnikov exponent c in (4.17) tends to infinity as b tends to one, since $K(0) = \pi/2$ and $\lim_{m \to 1^-} K(m) = +\infty$. This suggests that the Melnikov method gives little information for perturbed circles. In fact, in [Ram06], it is proved that the first order coefficient $\Delta_1^{(1,q)}$ in (4.15) vanishes for every period $q \notin \mathcal{Q}_n$, where

$$\mathcal{Q}_n = \begin{cases} \{3, 5, \ldots, n-2, n\}, & \text{for odd } n, \\ \{2, 4, \ldots, n-2, n\} \cup \{2, 3, \ldots, n/2\}, & \text{for even } n. \end{cases}$$

We might use a higher order Melnikov method to look for an order $k = k(n, q) \in \mathbb{N}$ such that

$$\Delta^{(1,q)} = \epsilon^k \Delta_k^{(1,q)} + \mathrm{O}(\epsilon^{k+1}),$$

with $\Delta_k^{(1,q)} \neq 0$. This Melnikov computation is not easy so we have performed a numerical study instead. As before, we do not study $\Delta^{(1,q)}$ but the difference D_q.

Numerical Result 25. *Set*

$$k = k(n, q) = \begin{cases} 1 + 2\left\lceil \frac{q-n}{2n} \right\rceil, & \textit{for odd } n \textit{ and odd } q, \\ 2\lceil q/2n \rceil, & \textit{for odd } n \textit{ and even } q, \\ \lceil 2q/n \rceil, & \textit{for even } n \textit{ and odd } q, \\ \lceil q/n \rceil, & \textit{for even } n \textit{ and even } q. \end{cases}$$

If $n \geq 3$ and $q \geq 2$, then there exists $d_k = d_k(n, q) \neq 0$ such that

$$D_q(\epsilon) = d_k \epsilon^k + \mathrm{O}(\epsilon^{k+1}). \tag{4.22}$$

This numerical result has two nice consequences on the breakup of the resonant caustics of the circular billiard under the perturbation $x^2 + y^2 + \epsilon y^n = 1$ with any fixed degree $n \geq 3$. First, *all* $(1, q)$*-resonant caustics break up*, because, once fixed the period $q \geq 2$, $\Delta^{(1,q)} \neq 0$ for ϵ small enough. Second, *there are breakups of any order*, because the map $q \mapsto k(n, q) \in \mathbb{N}$ is exhaustive.

We numerically compute the order k in (4.22) by noting that

$$k \simeq \log\left(\frac{D_q(\epsilon)}{D_q(\epsilon/\mathrm{e})}\right).$$

For instance, if $n = 7$, $q = 36$, and $\epsilon = 10^{-10}$, then we obtain the approximation

$$k \simeq 5.99999999999999999401\ldots,$$

so $k = 6$. We have tested the formulas listed in Numerical Result 25 for all degrees $3 \leq n \leq 8$ and all periods $3 \leq q \leq 100$. Note that, once fixed n,

$$k = k(n,q) \asymp \begin{cases} 2q/n, & \text{for even } n \text{ and odd } q, \\ q/n, & \text{otherwise,} \end{cases} \tag{4.23}$$

as $q \to +\infty$. Next, we focus on the singular case.

Numerical Result 26. *Fix $n \geq 3$. Let I_n be the maximal interval defined in (4.13). If $\epsilon \in I_n$, then the Borel transform* (4.4) *has a radius of convergence $\rho \in (0, +\infty)$. Set $r = \rho/2$. There exist two non-zero quasiperiodic functions $A(q)$ and $B(q)$ such that*

$$D_q \asymp \begin{cases} B(q) q^{-2} \mathrm{e}^{-2rq}, & \textit{for even } n \textit{ and odd } q, \\ A(q) q^{-3} \mathrm{e}^{-rq}, & \textit{otherwise}, \end{cases}$$

as $q \to +\infty$. Besides, there exists $\chi_n \in \mathbb{R}$ such that

$$r = \frac{|\log \epsilon|}{n} + \chi_n + \mathrm{o}(1) \tag{4.24}$$

as $\epsilon \to 0$. Finally, there exist a partition $I_n = C_n \cup P_n \cup R_n$ satisfying the following properties.

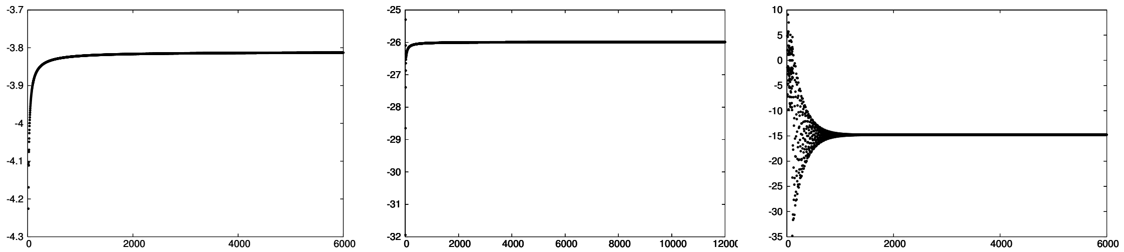
1. *C_n and P_n are open subsets of I_n, whereas R_n is a set of isolated perturbative parameters.*
2. *If $\epsilon \in C_n$, both functions $A(q)$ and $B(q)$ are constant.*
3. *If $\epsilon \in P_n$, both functions $A(q)$ and $B(q)$ are periodic. Namely, they have the form*

$$A(q) = a \cos(2\pi\beta q), \quad B(q) = \bar{b} + b \cos(4\pi\beta q),$$

for some average $\bar{b} \neq 0$, some amplitudes $a, b > 0$, and some "shared" frequency $\beta > 0$. We note that $\bar{b} \neq b/2$.

All these numerical results strongly support Conjecture 21. For instance, we conjectured that the function $A(q)$ is either constant: $A(q) \equiv a/2$, or periodic: $A(q) = a\cos(2\pi\beta q)$ in open sets of the space of axisymmetric algebraic curves, whereas all other cases are phenomena of co-dimension at least one. This claim agrees with the fact that C_n and P_n are open subsets of I_n, whereas R_n only contains the perturbative parameters where a transition between constant and periodic cases takes place.

The functions $A(q)$ and $B(q)$ and the exponent r depend on the degree n and the perturbative parameter ϵ, although $B(q)$ is defined only for even n. Both functions $A(q)$ and $B(q)$ "share" the frequency in the periodic case. To be precise, the frequency of $B(q)$ is twice the frequency of $A(q)$. It makes sense because the exponent in the asymptotic formula containing the function $B(q)$ is also twice the exponent in the one containing $A(q)$.



(a) $\hat{D}_q$ versus q for $n = 4$, $\epsilon = 1$, and odd q. (b) $\hat{D}_q$ versus q for $n = 4$, $\epsilon = 1$, and even q. (c) $\hat{D}_q$ versus q for $n = 7$ and $\epsilon = 1/1280$.

Figure 4.6: Examples with a constant asymptotic behavior of the normalized differences $\hat{D}_q$.

The logarithmic behavior of the exponent r stated in (4.24) is closely related to the asymptotic formula (4.23). Indeed, if we roughly try to fit the regular behavior (4.22) when $\epsilon \to 0$ with the singular behavior $D_q = \mathrm{O}(q^m \mathrm{e}^{-rq})$ when $q \to +\infty$, then we get

$$\mathrm{O}(q^m \mathrm{e}^{-rq}) = D_q = \mathrm{O}(\epsilon^k) \simeq \mathrm{O}(\epsilon^{q/n}) = \mathrm{O}(\mathrm{e}^{-q|\log \epsilon|/n}),$$

so we guess that $r \simeq |\log \epsilon|/n$. This reasoning is informal but it is confirmed by our experiments. Let us describe them.

We have set $\epsilon \in I_n \cap \mathbb{Q}$ in all the experiments. First, we do so because our multiple-precision computations become a bit faster for rational perturbative parameters. There is a second reason for that choice. Namely, we change the precision very often along our computations, and rational values of ϵ are not affected by such changes, because they are stored as exact numbers. We have also tried to deal with “big” perturbations in order to stress that our results are not perturbative, but we recall that ϵ should be smaller than the singular value (4.14) when n is odd.

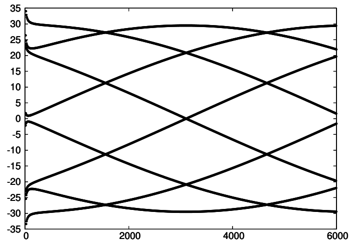
First, we compute the exponent $r = \rho/2$ by using the Borel method, since it is computationally cheaper than the direct one. See Remark 9. Besides, it is not clear how to adapt the direct method when the functions $A(q)$ and $B(q)$ oscillate. We follow the same steps as in the case of perturbed ellipses. However, the Neville extrapolation is more unstable for perturbed circles. In order to overcome this instability, now we take sequences (q_i) of 1000 periods such that $|D_{q_0}| \leq 10^{-5000}$.

Once we find r, we compute the normalized differences $\hat{D}_q$ already introduced in (4.21). We have checked that there exist two non-zero quasiperiodic functions $A(q)$ and $B(q)$ such that

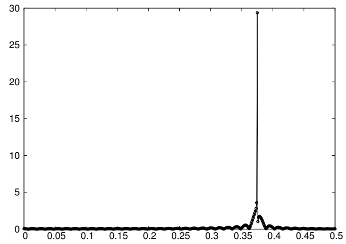
$$\hat{D}_q \asymp \begin{cases} B(q), & \text{for even } n \text{ and odd } q, \\ A(q), & \text{otherwise,} \end{cases}$$

as $q \to +\infty$.

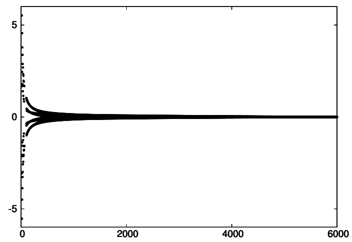
Some paradigmatic examples of the asymptotic behavior of the normalized differences $\hat{D}_q$ are displayed in Figures 4.6 and 4.7. All these examples are generic in the sense that a small change of the perturbative parameter ϵ does not produce any qualitative change in the pictures.



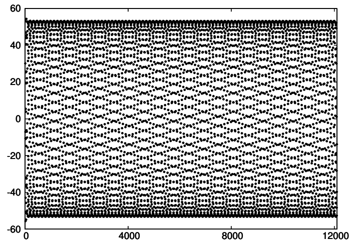
(a) $\hat{D}_q$ versus q for $n = 3$ and $\epsilon = 1/3$.



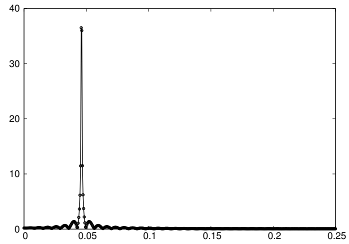
(b) DFT of $\hat{D}_q$ for $n = 3$ and $\epsilon = 1/3$.



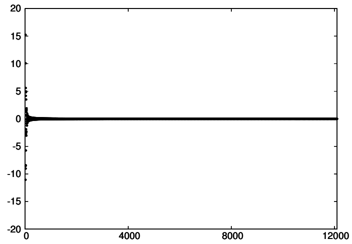
(c) $\hat{D}_q - A(q)$ versus q for $n = 3$ and $\epsilon = 1/3$.



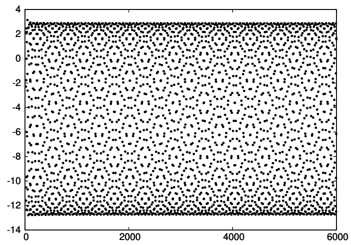
(d) $\hat{D}_q$ versus q for $n = 6$, $\epsilon = 1$, and even q.



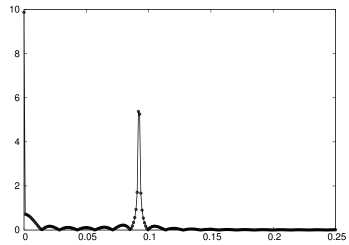
(e) DFT of $\hat{D}_q$ for $n = 6$, $\epsilon = 1$, and even q.



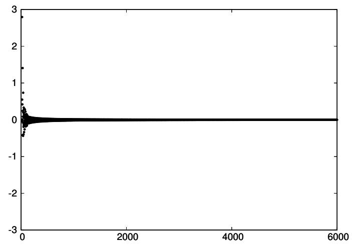
(f) $\hat{D}_q - A(q)$ versus q for $n = 6$, $\epsilon = 1$, and even q.



(g) $\hat{D}_q$ versus q for $n = 6$, $\epsilon = 1$, and odd q.



(h) DFT of $\hat{D}_q$ for $n = 6$, $\epsilon = 1$, and odd q.



(i) $\hat{D}_q - B(q)$ versus q for $n = 6$, $\epsilon = 1$, and odd q.

Figure 4.7: Examples with a periodic asymptotic behavior of the normalized differences $\hat{D}_q$. We recall that $A(q) = a\cos(2\pi\beta q)$ and $B(q) = \bar{b} + b\cos(4\pi\beta q)$. Besides, $a \approx 29.4849$ and $\beta \approx 1/8$ in Figure 4.7(c); $a \approx 53.2369$ and $\beta \approx 0.04614$ in Figure 4.7(f); and $\bar{b} \approx -4.9257$, $b \approx 7.80853$, and $\beta \approx 0.04614$ in Figure 4.7(i).

For instance, we see three examples where $\hat{D}_q$ tends to some constant as $q \to +\infty$ in Figure 4.6. The constant is A in the second and third subfigures, and B in the first one.

We display a first example of periodic asymptotic behavior in Figure 4.7(a) for the cubic perturbation and $\epsilon = 1/3$. This value $\epsilon = 1/3$ is relatively close to the value $\bar{\epsilon}_3(1) \approx 0.3849$ where the algebraic curve $x^2 + y^2 + \epsilon y^3 = 1$ becomes singular. Next, we compute the discrete Fourier transform (DFT) of the last terms of the sequence $\hat{D}_q$. To be precise, the terms in the range $10000 < q \leq 12000$ for $n = 6$ and even q, and in the range $5000 < q \leq 6000$ otherwise. We discard the first terms because $\hat{D}_q \asymp A(q)$ and $\hat{D}_q \asymp B(q)$, so the last normalized differences are closer to the periodic functions we want to determine.

The DFT of the normalized differences $\hat{D}_q$ suggests that the periodic function $A(q)$ has a dominant harmonic with amplitude $a \approx 29.4849$ and frequency $\beta \approx 0.375 = 1/8$ when $\epsilon = 1/3$ and $n = 3$. See Figure 4.7(b). This explains why we see eight waves in Figure 4.7(a), each one with frequency $|\beta - 1/8|$. This situation is a source of problems for the following reason. Let us assume that, due to time or computational restrictions, we only compute the normalized differences for periods of the form $q_i = q_0 + 8i$. In that case, we would only see one wave and we would get a wrong frequency. The moral of this story is that we have to compute the normalized differences for *all* periods. Then we compare the normalized differences $\hat{D}_q$ with the cosine wave $A(q) = a\cos(2\pi\beta q)$ as $q \to +\infty$. The amplitude a and the frequency β are determined by mixing several tools: the DFT, some direct algebraic computations, etcetera. The plot in Figure 4.7(c) shows that

$$\lim_{q \to +\infty} \left(\hat{D}_q - A(q) \right) = 0.$$

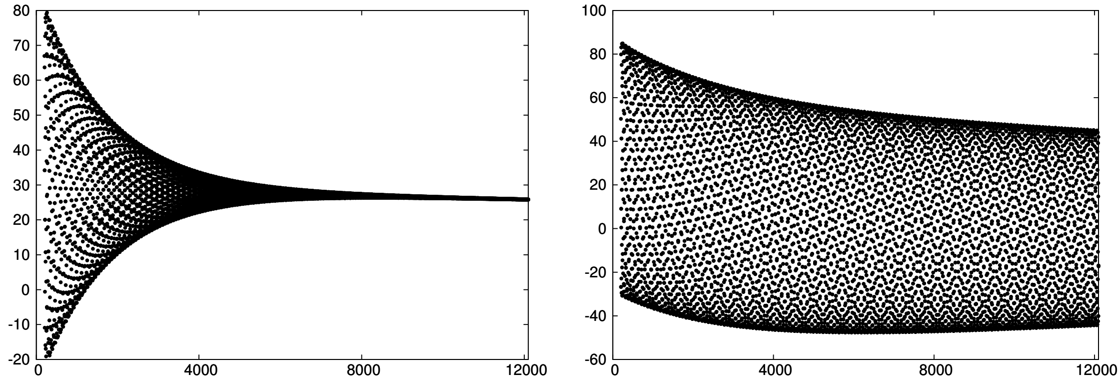
We study the case $n = 6$ and $\epsilon = 1$ in Figures 4.7(d)–4.7(i). The most interesting phenomena shown up by those pictures are the following ones. First, we confirm that the frequency of the periodic function $B(q)$ is twice the frequency of the cosine wave $A(q)$. See Figures 4.7(e) and 4.7(h). Second, the average of $B(q)$ is not zero. This is a surprise, because both the periodic functions obtained in similar splitting problems and the periodic function $A(q)$ obtained in this billiard problem have generically zero average. Third, $B(q) = \bar{b} + b\cos(4\pi\beta q)$, but $\bar{b} \neq b/2$, which sets another difference with the known asymptotic behaviors for splitting problems.

Next, we present some results about the transition between the two generic —“constant” and “periodic”— asymptotic behaviors of the normalized differences $\hat{D}_q$. That is, we intend to visualize what happens at some $\epsilon_* \in \partial C_n \cap \partial P_n \subset R_n$.

We focus our attention on the sixtic perturbation: $n = 6$. Then the normalized differences have “constant” and “periodic” asymptotic behaviors for $\epsilon = 1/10$ and $\epsilon = 1$, respectively. We study the quantities $\hat{D}_q$ in a fine grid of perturbative parameters in the interval $[1/10, 1]$. Both functions $A(q)$ and $B(q)$ change at the same transition value ϵ_*. Indeed,

$$[1/10, 23/200] \subset C_6, \qquad [3/25, 1] \subset P_6,$$

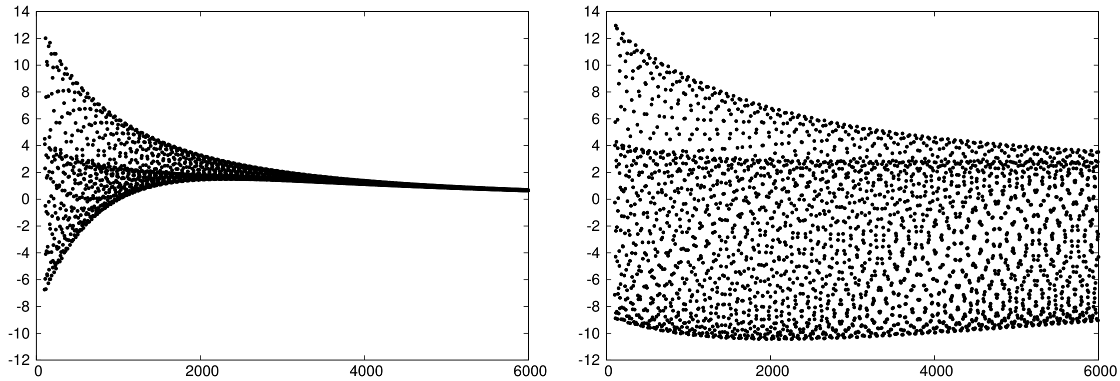
so the transition takes place at some $\epsilon_* \in (23/200, 3/25)$.



(a) $\epsilon = 23/200$.

(b) $\epsilon = 3/25$.

Figure 4.8: Transition of the function $A(q)$ from constant to periodic. We plot the normalized differences $\hat{D}_q$ versus q for $n = 6$ and even periods.



(a) $\epsilon = 23/200$.

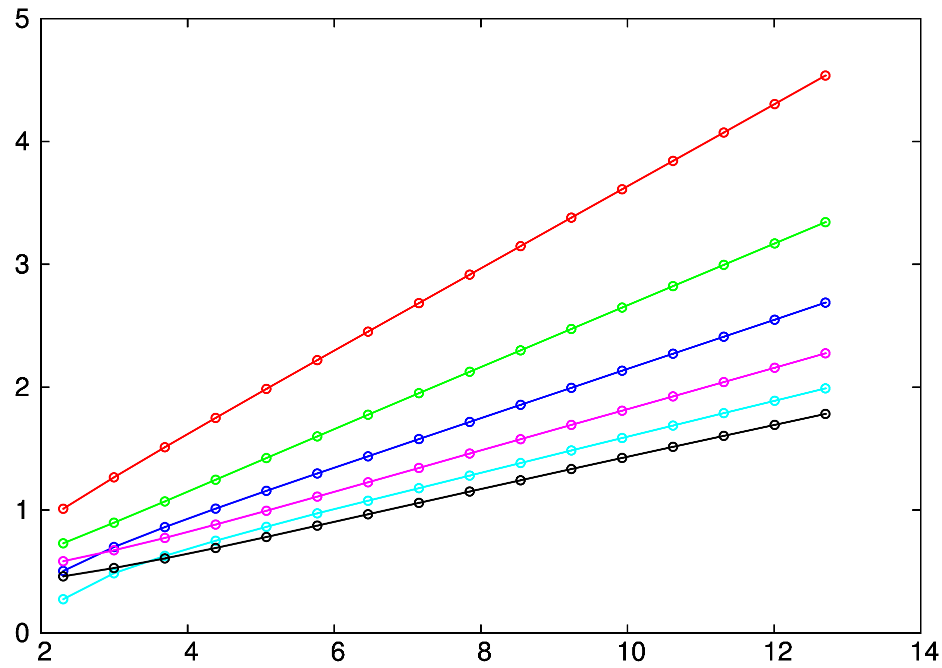
(b) $\epsilon = 3/25$.

Figure 4.9: Transition of the function $B(q)$ from constant to periodic. We plot the normalized differences $\hat{D}_q$ versus q for $n = 6$ and odd periods.

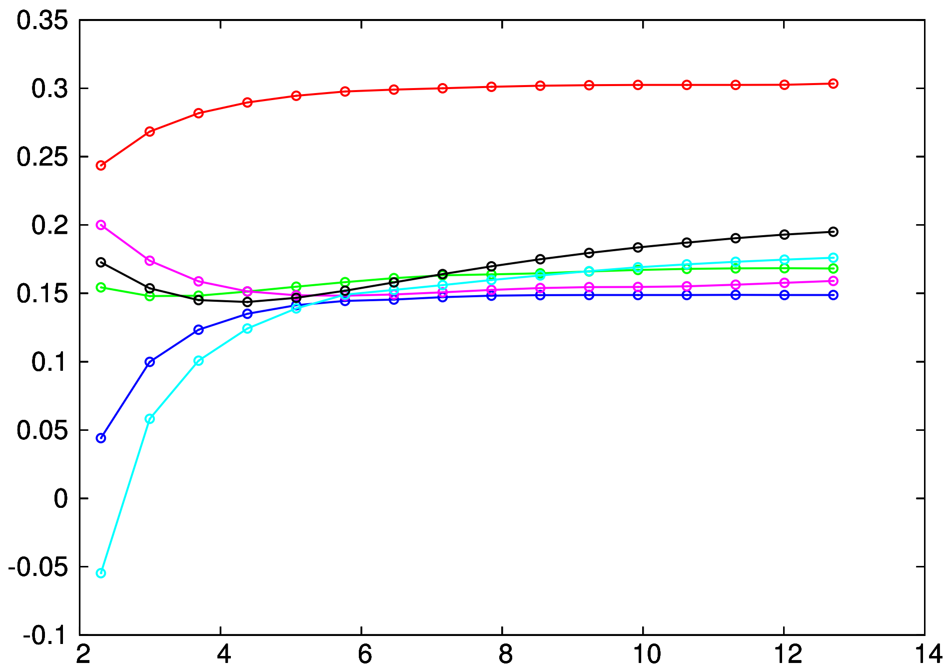
Unfortunately, a more precise computation of ϵ_* is beyond our current abilities, because we do not have a limit problem whose complex singularities allow us to determine analytically the transition values. An example of such analytical computations for splitting problems can be found in [GL01, GS08].

Therefore, we only display the normalized differences $\hat{D}_q$ for $\epsilon = 23/200$ and $\epsilon = 3/25$ in Figures 4.8 and 4.9 to see the transition of the functions $A(q)$ and $B(q)$, respectively.

Let us present some numerical results about the logarithmic growth (4.24) of the exponent r. We have computed the exponent $r = \rho/2$ by using the Borel method for $3 \leq n \leq 8$ in a sequence of perturbative parameters of the form $\epsilon_j = 2^{-j}/10$ with $j \geq 0$. We have plotted the results in Figure 4.10. On the one hand, the curves in Figure 4.10(a) look like straight lines with slopes $1/n$, as expected. On the other hand, the curves in Figure 4.10(b) tend to some constant values $\chi_n > 0$. This ends the numerical study of such phenomenon.



(a) r versus $|\log \epsilon|$.



(b) $r - |\log \epsilon|/n$ versus $|\log \epsilon|$.

Figure 4.10: Logarithmic growth of the exponent r as $\epsilon \to 0^+$. Red: $n = 3$. Green: $n = 4$. Blue: $n = 5$. Magenta: $n = 6$. Cyan: $n = 7$. Black: $n = 8$.

n	χ_n	η_n
3	$0.30\ldots$	$1.1358418243\ldots$
4	$0.17\ldots$	$1.0703321545\ldots$
5	$0.15\ldots$	$0.1488295936\ldots$
6	$0.15\ldots$	$1.0385641059\ldots$
7	$0.18\ldots$	$0.1823551667\ldots$
8	$0.19\ldots$	$1.0332248276\ldots$

Table 4.2: The constants χ_n and η_n, with $\chi_n \leq \eta_n$, that appear in formulas (4.24) and (4.25), respectively.

Finally, we see that our candidate for limit problem captures this logarithmic behavior, although it may not give the exact value of the exponent r.

Proposition 27. *Let $n \geq 3$ and $\epsilon \in I_n$. Let $\kappa(s)$ be the curvature of the strictly convex curve $Q = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + \epsilon y^n = 1\}$ in some arc-length parameter s. Let $\xi \in \mathbb{R}/\mathbb{Z}$ be the angular variable defined by (4.9). Let δ be the distance of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis. There exists $\eta_n \in \mathbb{R}$ such that*

$$2\pi\delta = \frac{|\log\epsilon|}{n} + \eta_n + \mathrm{O}(\epsilon^{2/n}\log\epsilon), \tag{4.25}$$

as $\epsilon \to 0^+$.

The proof of this proposition is placed in Section 4.8.

The constant χ_n in (4.24) is always smaller than (or equal to) the constant η_n in (4.25). We compare both constants in Table 4.2.

Constants χ_n are computed from the numerical data used in Figure 4.10(b). Constants η_n are computed by using the techniques explained in Remark 10. On the one hand, we obtain just two significant digits for the constants χ_n. On the other hand, we can compute η_n with a much higher precision; here we have just written their first ten decimal digits. We see that $\chi_n < \eta_n$ for $n \in \{3,4,6,8\}$. We do not discard the equalities $\chi_5 = \eta_5$ and $\chi_7 = \eta_7$. In order to elucidate them, we compare the exponent r with the quantity $2\pi\delta$, as we have done before for perturbed ellipses at the end of Section 4.4. The results are displayed in Figure 4.11, where we see that our candidate for limit problem gives the exact exponent r in two cases.

To be precise, our numerical results suggest that:

- If $n \in \{3,4,6,8\}$, then $r < 2\pi\delta$ for all $\epsilon \in (0,1/10)$; and
- If $n \in \{5,7\}$, then $r = 2\pi\delta$ for all $\epsilon \in (0,1/10)$.

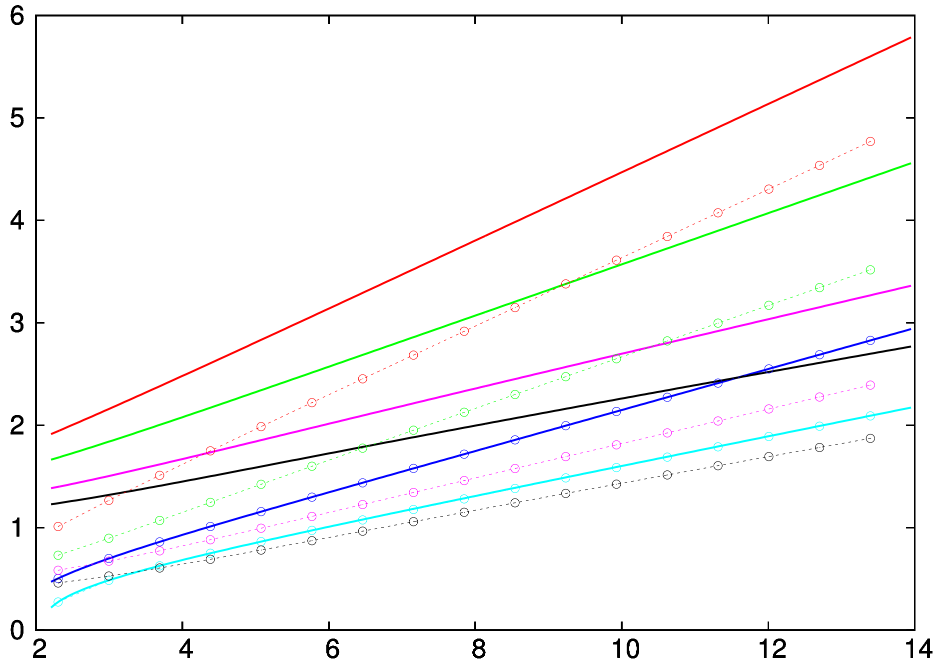


Figure 4.11: The exponent r (dashed lines with points) and the quantity $2\pi\delta$ (continuous lines) versus $|\log \epsilon|$. Red: $n = 3$. Green: $n = 4$. Blue: $n = 5$. Magenta: $n = 6$. Cyan: $n = 7$. Black: $n = 8$.

4.6 Proof of Proposition 22

We will use many properties of elliptic functions listed in the books [AS64, WW96], a couple of technical results about elliptic billiards contained in [CF88, CR11], and the subharmonic Melnikov potential of billiards inside perturbed ellipses introduced in [PR13].

We consider the unperturbed ellipse

$$E = \left\{(x, y) \in \mathbb{R}^2 : x^2/a^2 + y^2/b^2 = 1\right\}, \qquad 0 < b < a. \tag{4.26}$$

It is known that the convex caustics of the billiard inside E are the confocal ellipses

$$C_\lambda = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1\right\}, \qquad 0 < \lambda < b.$$

There is a unique (p, q)-resonant elliptic caustic C_λ for any relatively prime integers p and q such that $1 \leq p < q/2$. The caustic parameter of the (p, q)-resonant elliptic caustic is implicitly determined by means of equation (4.28).

The *complete elliptic integral of the first kind* is

$$K = K(m) = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} \mathrm{d}\theta.$$

Its argument $m \in (0,1)$ is called the *parameter*. We also write $K' = K'(m) = K(1-m)$. The *amplitude* function $\varphi = \mathrm{am}\, t$ is defined through the inversion of the integral

$$t = \int_0^\varphi (1 - m\sin^2\theta)^{-1/2} \mathrm{d}\theta.$$

The *elliptic sine* and *elliptic cosine* associated to the parameter $m \in (0,1)$ are defined by the trigonometric relations

$$\mathrm{sn}\, t = \mathrm{sn}(t,m) = \sin\varphi, \qquad \mathrm{cn}\, t = \mathrm{cn}(t,m) = \cos\varphi.$$

If the angular variable φ changes by 2π, the angular variable t changes by $4K$. Thus, any 2π-periodic function in φ, becomes $4K$-periodic in t. By abuse of notation, we will also denote the $4K$-periodic functions with the name of the corresponding 2π-periodic ones. For example, if $q(\varphi) = (a\cos\varphi, b\sin\varphi)$ is the natural 2π-periodic parameterization of the ellipse E, then $q(t) = (a\,\mathrm{cn}\, t, b\,\mathrm{sn}\, t)$ is the corresponding $4K$-periodic parameterization. The billiard dynamics associated to an elliptic caustic C_λ becomes a rigid rotation $t \mapsto t + \delta$ in the variable t. It suffices to find the shift δ and the parameter m associated to each elliptic caustic C_λ. The parameter m is given in [CF88, Eq. (3.28)] and the constant shift δ is given in [CF88, p. 1543]. We list the formulas in the following lemma.

Lemma 28. *Once fixed an elliptic caustic C_λ with $\lambda \in (0,b)$, the parameter $m \in (0,1)$ and the shift $\delta \in (0,2K)$ are*

$$m = \frac{a^2 - b^2}{a^2 - \lambda^2}, \qquad \delta/2 = \int_0^{\phi/2} (1 - m\sin^2\theta)^{-1/2} \mathrm{d}\theta, \tag{4.27}$$

where $\phi \in (0,\pi)$ is the angle such that $\sin(\phi/2) = \lambda/b$. The segment joining the points $q(t)$ and $q(t+\delta)$ is tangent to C_λ for all $t \in \mathbb{R}$.

From now on, m and δ will denote the parameter and the constant shift defined in (4.27). Observe that the elliptic caustic C_λ is (p,q)-resonant if and only if

$$q\delta = 4Kp. \tag{4.28}$$

This identity has the following geometric interpretation. When a billiard trajectory makes one turn around C_λ, the old angular variable φ changes by 2π, so the new angular variable t changes by $4K$. Besides, we have seen that the variable t changes by δ when a billiard trajectory bounces once. Hence, a billiard trajectory inscribed in E and circumscribed around C_λ makes exactly p turns after q bounces if and only if (4.28) holds.

We consider the elliptic coordinates (μ,φ) associated to the semi-lengths $0 < b < a$. That is, (μ,φ) are defined by relations

$$x = \sigma\cosh\mu\cos\varphi, \qquad y = \sigma\sinh\mu\sin\varphi,$$

where $\sigma = \sqrt{a^2 - b^2}$ is the semi-focal distance of E. The ellipse E in these coordinates reads as $\mu \equiv \mu_0$, where $\cosh\mu_0 = a/\sigma$ and $\sinh\mu_0 = b/\sigma$. Hence, any smooth perturbation of E can be written in elliptic coordinates as

$$\mu = \mu_0 + \epsilon\mu_1(\varphi) + \mathrm{O}(\epsilon^2), \tag{4.29}$$

for some 2π-periodic function $\mu_1 : \mathbb{R} \to \mathbb{R}$.

Lemma 29. *Let p and q be two relatively prime integers such that $1 \leq p < q/2$. Let C_λ be the (p,q)-resonant elliptic caustic of the ellipse (4.26). Let*

$$\Delta^{(p,q)} = \epsilon \Delta_1^{(p,q)} + \mathrm{O}(\epsilon^2)$$

be the maximal difference among lengths of (p,q)-periodic trajectories inside the perturbed ellipse (4.29). Let $\mu_1(t)$ be the $4K$-periodic function associated to the 2π-periodic one $\mu_1(\varphi)$. Let

$$L_1^{(p,q)}(t) = 2\lambda \sum_{j=0}^{q-1} \mu_1(t + j\delta)$$

be the subharmonic Melnikov potential of the caustic C_λ for the perturbed ellipse (4.29). If $L_1^{(p,q)}(t)$ does not have degenerate critical points and $\epsilon > 0$ is small enough, then there is a one-to-one correspondence between the critical points of $L_1^{(p,q)}(t)$ and the (p,q)-periodic billiard trajectories inside (4.29). Besides,

$$\Delta_1^{(p,q)} = \max L_1^{(p,q)} - \min L_1^{(p,q)}.$$

Proof. It follows directly from results contained in [PR13]. □

We will determine the asymptotic behavior of $\Delta_1^{(p,q)}$. First, we study the asymptotic behavior of the (p,q)-resonant caustic C_λ as $p/q \to 0^+$.

Lemma 30. *If C_λ is the (p,q)-resonant elliptic caustic of the ellipse (4.26), then $\lambda \asymp \Xi p/q$ as $p/q \to 0^+$, where*

$$\Xi = \Xi(a,b) := ab \int_{b^2}^{a^2} \big(s(s-b^2)(a^2-s)\big)^{-1/2} \, \mathrm{d}s. \tag{4.30}$$

Proof. It follows directly from [CR11, Proposition 10]. □

Lemma 31. *The following properties hold for $\mu_1(\varphi) = \cos^2\varphi$.*

1. *The Melnikov potential $L_1^{(p,q)}(t)$ has just two real critical points (modulo its periodicity), none of them degenerate.*

2. *There exist an exponent $\zeta = \zeta(\vartheta_*, a, b) > 0$ and a quantity $\Omega_4 = \Omega_4(\vartheta_*, a, b, p, q) > 0$ such that*

$$\Delta_1^{(p,q)} \asymp \begin{cases} 2\Omega_4 \mathrm{e}^{-2\zeta q}, & \text{for odd } q, \\ \Omega_4 \mathrm{e}^{-\zeta q}, & \text{for even } q, \end{cases}$$

   as $p/q \to \vartheta_* \in \{0\} \cup \big((0,1) \setminus \mathbb{Q}\big)$.

3. *There exist $\Gamma_4 = \Gamma_4(\vartheta_*, a, b) > 0$ and $\Theta_4 = \Theta_4(a,b) > 0$ such that*

$$\Omega_4(\vartheta_*, a, b, p, q) = \begin{cases} \Gamma_4 q^2, & \text{if } \vartheta_* \in (0,1) \setminus \mathbb{Q}, \\ \Theta_4 pq, & \text{if } \vartheta_* = 0. \end{cases}$$

4. $\zeta(0,a,b) = \pi K'(1-(b/a)^2)/2K(1-(b/a)^2)$.

Proof. By definition, if $\mu_1(\varphi) = \cos^2\varphi$, then

$$L_1^{(p,q)}(t) = 2\lambda \sum_{j=0}^{q-1} \mathrm{cn}^2(t+j\delta).$$

The square of the elliptic cosine is an elliptic function of order two, periods $2K$ and $2K'\mathrm{i}$, and double poles in the set

$$P = K'\mathrm{i} + 2K\mathbb{Z} + 2K'\mathrm{i}\mathbb{Z}.$$

Besides, the principal part of any pole $\tau \in P$ is $-m^{-1}(t-\tau)^{-2}$. In particular, $L_1^{(p,q)}(t)$ is also an elliptic function of order two, and so, it can be determined (modulo an additive constant) by its periods, poles, and principal parts.

We study the cases odd q and even q separately.

If q is odd, then $L_1^{(p,q)}(t)$ has periods $2K/q$ and $2K'\mathrm{i}$ and double poles with principal parts $-2\lambda m^{-1}(t-\tau)^{-2}$ in the set

$$P_q = K'\mathrm{i} + \frac{2K}{q}\mathbb{Z} + 2K'\mathrm{i}.$$

It is known that $K'(m)/K(m)$ is a decreasing function such that

$$\lim_{m\to 0^+} \frac{K'(m)}{K(m)} = +\infty, \qquad \lim_{m\to 1^-} \frac{K'(m)}{K(m)} = 0.$$

Therefore, there exists a unique $m_q \in (0,1)$ such that

$$\frac{K'_q}{K_q} := \frac{K'(m_q)}{K(m_q)} = q\frac{K'(m)}{K(m)} =: q\frac{K'}{K}.$$

Henceforth, we write that $K = K(m)$, $K' = K'(m)$, $K_q = K(m_q)$, and $K'_q = K'(m_q)$ for short. Thus,

$$L_1^{(p,q)}(t) = \text{const.} + 2\lambda(qK_q/K)^2(m_q/m)\,\mathrm{cn}^2(qK_q t/K, m_q),$$

which has just two real critical points (modulo its periodicity), none of them degenerate. Besides

$$\Delta_1^{(p,q)} = \max L_1^{(p,q)} - \min L_1^{(p,q)} = 2\lambda(qK_q/K)^2(m_q/m).$$

If $p/q \to \vartheta_* \in (0,1)\setminus\mathbb{Q}$, then $q\to+\infty$ and $\lambda\to\lambda_*\in(0,b)$, where C_{λ_*} is the elliptic caustic with rotation number ϑ_*, so

$$m \to m_* := \frac{a^2-b^2}{a^2-\lambda_*^2} \in (0,1), \qquad m_q \to 0^+,$$
$$\frac{K'}{K} \to \frac{K'_*}{K_*} := \frac{K'(m_*)}{K(m_*)} \in (0,+\infty), \qquad K_q \to \frac{\pi}{2}.$$

Using [AS64, 17.3.14 & 17.3.16], we get the asymptotic formula $m_q \asymp 16\mathrm{e}^{-2\zeta q}$, where

$$\zeta := \pi K'_*/2K_*. \tag{4.31}$$

Finally, we obtain that

$$\Delta_1^{(p,q)} \asymp \frac{8\pi^2\lambda_*}{m_* K_*^2} q^2 \mathrm{e}^{-2\zeta q}, \text{ as } p/q \to \vartheta_* \text{ and } q \text{ is odd.}$$

If q is even, then $cn^2(t + q\delta/2, m) = cn^2(t, m)$ and

$$L_1^{(p,q)}(t) = 4\lambda \sum_{j=0}^{q/2-1} \mathrm{cn}^2(t + j\delta, m),$$

so $L_1^{(p,q)}(t)$ has periods $4K/q$ and $2K'\mathrm{i}$. In this case,

$$\Delta_1^{(p,q)} \asymp \frac{4\pi^2\lambda_*}{m_* K_*^2} q^2 \mathrm{e}^{-\zeta q}, \text{ as } p/q \to \vartheta_* \text{ and } q \text{ is even.}$$

Next, we study the case $\vartheta_* = 0$, when the (p,q)-periodic orbits approach the boundary. In this case,

$$\lambda_* = 0, \quad m_* = 1 - (b/a)^2, \quad \zeta = \zeta(0,a,b) = \frac{\pi K'(1-(b/a)^2)}{2K(1-(b/a)^2)}.$$

Since $\lambda_* = 0$, we need the asymptotic behavior of the caustic parameter λ as $p/q \to 0^+$. We recall that $\lambda \asymp \Xi p/q$ in that case, where $\Xi = \Xi(a,b)$ is the integral defined in (4.30). Hence,

$$\Gamma_4 = \frac{4\pi^2\lambda_*}{m_* K_*^2}, \qquad \Theta_4 = \frac{4\pi^2\Xi(a,b)}{(1-(b/a)^2)K(1-(b/a)^2)^2}, \tag{4.32}$$

and this ends the proof of the lemma. □

Lemma 32. *The following properties hold for $\mu_1(\varphi) = -\sin\varphi$.*

1. *If q is even, then $L_1^{(p,q)}(t) \equiv 0$ and $\Delta_1^{(p,q)} = 0$.*

2. *If q is odd, then $L_1^{(p,q)}(t)$ has just two real critical points (modulo its periodicity), none of them degenerate.*

3. *Let $\zeta(\vartheta_*, a, b)$ be the exponent introduced in Lemma 31. If q is even, then there exists $\Omega_3 = \Omega_3(\vartheta_*, a, b, p, q) > 0$ such that*

$$\Delta_1^{(p,q)} \asymp \Omega_3 \mathrm{e}^{-\zeta q}, \qquad p/q \to \vartheta_* \in \{0\} \cup \big((0,1)\setminus\mathbb{Q}\big).$$

4. *There exist $\Gamma_3 = \Gamma_3(\vartheta_*, a, b) > 0$ and $\Theta_3 = \Theta_3(a,b) > 0$ such that*

$$\Omega_3(\vartheta_*, b, a, p, q) = \begin{cases} \Gamma_3 q, & \text{if } \vartheta_* \in (0,1)\setminus\mathbb{Q}, \\ \Theta_3 p, & \text{if } \vartheta_* = 0. \end{cases}$$

Proof. If q is even, then p is odd, $\mathrm{sn}(t+\delta/2) = -\mathrm{sn}\, t$, and $L_1^{(p,q)}(t) = -2\lambda \sum_{j=0}^{q-1} \mathrm{sn}(t + j\delta) \equiv 0$.

The case odd q follows the lines of the proof of Lemma 31. The constants are

$$\Gamma_3 = \frac{8\pi\lambda_*}{\sqrt{m_*}K_*}, \qquad \Theta_3 = \frac{8\pi\Xi(a,b)}{(1-(b/a)^2)^{1/2}K(1-(b/a)^2)}, \tag{4.33}$$

where C_{λ_*} is the elliptic caustic with rotation number ϑ_*, $m_* = (a^2-b^2)/(a^2-\lambda_*^2)$, and $K_* = K(m_*)$. We omit the details. □

Next, we relate the original perturbed ellipses (4.7) written in Cartesian coordinates, to the perturbed ellipses (4.29) written in elliptic coordinates.

Lemma 33. *Set* $0 < b < a$*.*

1. *The perturbed ellipse (4.29) with* $\mu_1(\varphi) = -\sin\varphi$ *has, up to terms of second order in* ϵ*, the implicit equation*
$$\frac{x^2}{a^2} + \frac{(y-\epsilon b^2/a)^2}{b^2} + 2\frac{a^2-b^2}{b^4}\epsilon y^3 = 1.$$

2. *The perturbed ellipse (4.29) with* $\mu_1(\varphi) = \cos^2\varphi$ *has, up to terms of second order in* ϵ*, the implicit equation*
$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + 2\frac{a^2-b^2}{b^5}\epsilon y^4 = 1,$$
for some semi-lengths $\alpha = a + \mathrm{O}(\epsilon)$ *and* $\beta = b + \mathrm{O}(\epsilon)$*.*

Proof. Let $P_1 : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The perturbed ellipse written in Cartesian coordinates as

$$x^2/a^2 + y^2/b^2 + \epsilon P_1(x,y) + \mathrm{O}(\epsilon^2) = 1$$

and the perturbed ellipse written in elliptic coordinates as (4.29) are linked through the relation

$$2(a^2\sin^2\varphi + b^2\cos^2\varphi)\mu_1(\varphi) + abP_1(a\cos\varphi, b\sin\varphi) = 0.$$

The rest of the proof is a tedious, but straightforward, computation. □

Finally, we get the claims stated in Proposition 22 from the previous results by using that $\alpha = a + \mathrm{O}(\epsilon)$ and $\beta = b + \mathrm{O}(\epsilon)$ and by taking $a = 1$. To be precise, then

$$\begin{aligned} c = c(b) &= \zeta(0,1,b) = \frac{\pi K'(1-b^2)}{2K(1-b^2)}, \\ M_3 = M_3(b) &= \frac{b^4\Theta_3(1,b)}{2(1-b^2)} = \frac{4\pi b^4\Xi(1,b)}{(1-b^2)^{3/2}K(1-b^2)}, \\ M_4 = M_4(b) &= \frac{b^5\Theta_4(1,b)}{2(1-b^2)} = \frac{2\pi^2 b^5\Xi(1,b)}{(1-b^2)^2K(1-b^2)^2}, \end{aligned} \tag{4.34}$$

where the elliptic integral $\Xi = \Xi(a,b)$ is defined in (4.30).

4.7 Proof of Proposition 24

We parameterize the ellipse by using the angular variable φ. That is, we use the parametrization $\sigma(\varphi) = (\cos\varphi, b\sin\varphi)$. The curvature of the ellipse E at the point $\sigma(\varphi)$ is

$$\kappa(\varphi) = \frac{b}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} = \frac{1}{b^2(1+\nu\sin^2\varphi)^{3/2}},$$

where $\nu = (1-b^2)/b^2 > 0$. The arc-length parameter s and the angular parameter φ are related by

$$\frac{\mathrm{d}s}{\mathrm{d}\varphi}(\varphi) = \|\sigma'(\varphi)\| = \sqrt{\sin^2\varphi + b^2\cos^2\varphi} = b\sqrt{1+\nu\sin^2\varphi}.$$

First, we compute the constant

$$\begin{aligned} C &= \int_E \kappa^{2/3}\mathrm{d}s = 4b^{-1/3}\int_0^{\pi/2}(1+\nu\sin^2\varphi)^{-1/2}\mathrm{d}\varphi \\ &= 4b^{-1/3}K(-\nu) = 4b^{2/3}K(1-b^2). \end{aligned}$$

We have used [AS64, 17.4.17] in the last equality.

The *incomplete elliptic integral of the first kind* with amplitude $\varphi \in (0,\pi/2)$ and parameter $m \in (0,1)$ is

$$F(\varphi|m) = \int_0^\varphi (1-m\sin^2\theta)^{-1/2}\mathrm{d}\theta.$$

This definition can be extended to complex amplitudes and any real parameter [AS64]. Note that $F(\pi/2|m) = K(m)$.

The curvature $\kappa(\varphi)$ has no complex zeros but has complex singularities at the points such that $\sin^2\varphi = -1/\nu$. This equation becomes $\sinh^2\psi = 1/\nu$ under the change $\varphi = \mathrm{i}\psi$. Let ψ_* be the only positive solution of the previous equation. Any singularity of $\kappa(\varphi)$ has the form

$$\varphi = \varphi_n^\pm := \pm\mathrm{i}\psi_* + n\pi, \qquad n \in \mathbb{Z}.$$

Let $\xi_n^\pm$ be the complex singularity of $\kappa(\xi)$ associated to $\varphi_n^\pm$ through the change of variables

$$\xi = C^{-1}\int_0^s \kappa^{2/3}(t)\mathrm{d}t = C^{-1}\int_0^\varphi \kappa^{2/3}(\theta)\frac{\mathrm{d}s}{\mathrm{d}\varphi}(\theta)\mathrm{d}\theta.$$

The complex path in this integral is the segment from 0 to φ.

Next, we compute the complex singularities ξ_n^+:

$$\begin{aligned}\xi_n^+ &= C^{-1}\int_0^{\varphi_n^+} \kappa^{2/3}(\theta)\frac{\mathrm{d}s}{\mathrm{d}\varphi}(\theta)\mathrm{d}\theta \\ &= C^{-1}b^{-1/3}F(\mathrm{i}\psi_* + n\pi| -\nu) \\ &= 2nC^{-1}b^{-1/3}K(-\nu) + \mathrm{i}C^{-1}b^{-1/3}F(\pi/2|b^2) \\ &= 2nC^{-1}b^{2/3}K(1-b^2) + \mathrm{i}C^{-1}b^{2/3}K(b^2) \\ &= n/2 + \mathrm{i}C^{-1}b^{2/3}K'(1-b^2).\end{aligned}$$

By symmetry, $\xi_n^- = -\xi_{-n}^+$. We have used formula [AS64, 17.4.3] to compute $F(\mathrm{i}\psi_* + n\pi| -\nu)$, formula [AS64, 17.4.8] to compute $F(\mathrm{i}\psi_*| -\nu)$, and formula [AS64, 17.4.15] to compute $F(\pi/2|b^2)$.

Therefore, the distance δ of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis is

$$\delta = C^{-1}b^{2/3}K'(1-b^2) = \frac{K'(1-b^2)}{4K(1-b^2)} = c/2\pi.$$

4.8 Proof of Proposition 27

Fix the integer $n \geq 3$. We consider the perturbed circles

$$Q = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + \epsilon y^n = 1\right\} \tag{4.35}$$

where $0 < \epsilon \ll 1$ is a small perturbative parameter.

Let $C = C(\epsilon)$ be the constant defined in (4.9). If $\epsilon = 0$, then Q is a circle of radius one with curvature $\kappa \equiv 1$, so

$$C(0) = \int_Q \kappa^{2/3}\mathrm{d}s = \int_Q \mathrm{d}s = Length(Q) = 2\pi.$$

We note that (4.35) is a smooth perturbation of a circle of radius one, so $C(\epsilon)$ is smooth at $\epsilon = 0$ and

$$C = C(\epsilon) = C(0) + \mathrm{O}(\epsilon) = 2\pi + \mathrm{O}(\epsilon). \tag{4.36}$$

We introduce the polynomial $r(y) = 1 - y^2 - \epsilon y^n$. Note that $(x,y) \in Q$ if and only if $x^2 = r(y)$. By taking derivatives twice with respect to y the implicit relation $x^2 = r(y)$, we get the auxiliary polynomials

$$\begin{aligned} p(y) &= -x^3\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = \left(\frac{r'(y)}{2}\right)^2 - \frac{r(y)r''(y)}{2} \\ &= 1 + \epsilon p_{n-2}y^{n-2} + \epsilon p_n y^n + \epsilon^2 p_{2n-2}y^{2n-2}, \\ q(y) &= x^2 + \left(x\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 = r(y) + \left(\frac{r'(y)}{2}\right)^2 \\ &= 1 + \epsilon q_n y^n + \epsilon^2 q_{2n-2}y^{2n-2},\end{aligned}$$

whose coefficients are $p_{n-2} = n(n-1)/2$, $p_n = -(n-1)(n-2)/2$, $p_{2n-2} = -n(n-2)/4$, $q_n = n-1$, and $q_{2n-2} = n^2/4$. The length element and the curvature at the point $(x,y) \in Q$ are

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y = \sqrt{\frac{q(y)}{r(y)}}\mathrm{d}y,$$
$$\kappa = -\frac{\mathrm{d}^2 x}{\mathrm{d}y^2}\left(1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right)^{-3/2} = \frac{p(y)}{q^{3/2}(y)}.$$

The curvature should be positive, which explains the minus sign in the formula for $\kappa(y)$. Thus, we can relate any singularity (or any zero) $y_\star \in \mathbb{C}$ of the curvature $\kappa(y)$, with the corresponding singularities (or zeros) $s_\star \in \mathbb{C}/l\mathbb{Z}$ and $\xi_\star \in \mathbb{C}/\mathbb{Z}$ by means of the formula

$$\xi_\star = \int_0^{s_\star} \kappa^{2/3}(s)\mathrm{d}s = \int_0^{y_\star} g(y)\mathrm{d}y,$$

where

$$g(y) := \kappa^{2/3}(y)\frac{\mathrm{d}s}{\mathrm{d}y}(y) = \frac{p^{2/3}(y)}{\sqrt{r(y)q(y)}}.$$

Let $\mathcal{R} \subset \mathbb{C}$ be the union of the complex rays $\{\alpha y_0 : \alpha \geq 0\}$, where y_0 is a root of $p(y)$, $q(y)$ or $r(y)$. The function $g(y)$ is analytic in $\mathbb{C} \setminus \mathcal{R}$, so we will avoid the set $\mathcal{R}$ when computing the integral $\int_0^{y_\star} g(y)\mathrm{d}y$ along complex paths.

Lemma 34. *Let $0 < \epsilon \ll 1$ and $n \in \mathbb{N}$ with $n \geq 3$.*

1. *The polynomial $p(y)$ has n roots of the form*

$$z\epsilon^{-1/n} + \mathrm{O}(\epsilon^{1/n}), \qquad z^n = 2/((n-1)(n-2));$$

and $n-2$ roots of the form

$$z\epsilon^{-1/(n-2)} + \mathrm{O}(\epsilon^{1/(n-2)}), \qquad z^{n-2} = -(n-1)(n-2)/n.$$

2. *The polynomial $q(y)$ has n roots of the form*

$$z\epsilon^{-1/n} + \mathrm{O}(\epsilon^{1/n}), \qquad z^n = -1/(n-1);$$

and $n-2$ roots of the form

$$z\epsilon^{-1/(n-2)} + \mathrm{O}(\epsilon^{1/(n-2)}), \qquad z^{n-2} = -4(n-1)/n^2.$$

3. *The polynomial $r(y)$ has $n-2$ roots of the form*

$$z\epsilon^{-1/(n-2)} + \mathrm{O}(\epsilon^{1/(n-2)}), \qquad z^{n-2} = -1;$$

and two real roots of the form $y_\pm = \pm 1 + \mathrm{O}(\epsilon)$.

Besides, each one of these roots depends on some positive fractional power of ϵ in an analytic way.

Proof. If $w_0(z)$ is a polynomial with a simple root z_0 and $w_1(z)$ is another polynomial, then $w(z) = w_0(z) + \mu w_1(z)$ has some root of the form $z = z_0 + \mathrm{O}(\mu)$ which depends analytically on μ. The roots $y_\pm = \pm 1 + \mathrm{O}(\epsilon)$ of the polynomial $r(y) = 1 - y^2 - \epsilon y^n$ are obtained directly with $w_0(z) = 1 - z^2$, $w_1(z) = -z^n$, and $\mu = \epsilon$.

If we take $\mu = \epsilon^{2/n}$, then

$$\begin{aligned} p(\epsilon^{-1/n} z) &= 1 + p_n z^n + \mu(p_{n-2} z^{n-2} + p_{2n-2} z^{2n-2}), \\ q(\epsilon^{-1/n} z) &= 1 + q_n z^n + \mu q_{2n-2} z^{2n-2}, \end{aligned}$$

and we find the n roots with an $\mathrm{O}(\epsilon^{-1/n})$-modulus of $p(y)$ and the n roots with an $\mathrm{O}(\epsilon^{-1/n})$-modulus of $q(y)$.

If we take $\mu = \epsilon^{2/(n-2)}$, then

$$\begin{aligned} \mu p(\epsilon^{-1/(n-2)} z) &= z^n (p_n + p_{2n-2} z^{n-2}) + \mu(1 + p_{n-2} z^{n-2}), \\ \mu q(\epsilon^{-1/(n-2)} z) &= z^n (q_n + q_{2n-2} z^{n-2}) + \mu, \\ \mu r(\epsilon^{-1/(n-2)} z) &= -z^2 (1 + z^{n-2}) + \mu, \end{aligned}$$

and we find the $n-2$ roots with an $\mathrm{O}(\epsilon^{-1/(n-2)})$-modulus of $p(y)$, the $n-2$ roots with an $\mathrm{O}(\epsilon^{-1/(n-2)})$-modulus of $q(y)$, and the $n-2$ roots with an $\mathrm{O}(\epsilon^{-1/(n-2)})$-modulus of $r(y)$. □

Lemma 35. *If $0 < \epsilon \ll 1$, $n \in \mathbb{N}$ with $n \geq 3$, and $y_\star \in \mathbb{C}$ is a root of $p(y)$ or $q(y)$ with an $\mathrm{O}(\epsilon^{-1/n})$-modulus, then there exists a constant $\eta_\star \in \mathbb{R}$ such that*

$$|\Im \xi_\star| = \frac{|\log \epsilon|}{n} + \eta_\star + \mathrm{O}(\epsilon^{2/n} \log \epsilon),$$

as $\epsilon \to 0^+$.

Proof. For simplicity, we assume that $y_\star$ is a root of $q(y)$ such that $\Re y_\star \leq 0$ and $\Im y_\star \geq 0$. Other cases require minor changes.

If $r_0 = (n-1)^{-1/n}/2$, $r_\star = \epsilon^{1/n} |y_\star|$, and $\theta_\star = \arg y_\star$, then $\pi/2 \leq \theta_\star < n\pi/(n+1)$ and $r_\star = 2r_0 + \mathrm{O}(\epsilon^{2/n})$, because $y_\star = \epsilon^{-1/n} z + \mathrm{O}(\epsilon^{1/n})$ for some $z \in \mathbb{C}$ such that $z^n = -1/(n-1) < 0$.

We compute $\xi_\star = \int_0^{y_\star} g(y) \mathrm{d}y$ by integrating over the path $\sigma_\star = \sigma_1 \cup \sigma_2 \cup \sigma_3$, where

$$\begin{aligned} \sigma_1 &= \{\epsilon^{-1/n} \mathrm{i} t : 0 \leq t \leq r_0\}, \\ \sigma_2 &= \{\epsilon^{-1/n} r_0 \mathrm{e}^{\theta \mathrm{i}} : \pi/2 \leq \theta \leq \theta_\star\}, \\ \sigma_3 &= \{\epsilon^{-1/n} \mathrm{e}^{\theta_\star \mathrm{i}} r : r_0 \leq r \leq r_\star\}. \end{aligned}$$

This path only intersects the set of rays $\mathcal{R}$ at its endpoint $y_\star$, since the $2n$ roots of $p(y)$ and $q(y)$ with an $\mathrm{O}(\epsilon^{-1/n})$-modulus have pairwise different arguments when $\epsilon \to 0^+$.

We write $\xi_\star = \int_0^{y_\star} g(y)\mathrm{d}y = \int_{\sigma_\star} g(y)\mathrm{d}y = \xi_1 + \xi_2 + \xi_3$, where

$$\xi_1 = \int_{\sigma_1} g(y)\mathrm{d}y = \int_0^{r_0} \epsilon^{-1/n}\mathrm{i}g\big(\epsilon^{-1/n}\mathrm{i}t\big)\mathrm{d}t,$$
$$\xi_2 = \int_{\sigma_2} g(y)\mathrm{d}y = \int_{\pi/2}^{\theta_\star} \epsilon^{-1/n}r_0\mathrm{e}^{\theta\mathrm{i}}\mathrm{i}g\big(\epsilon^{-1/n}r_0\mathrm{e}^{\theta\mathrm{i}}\big)\mathrm{d}\theta,$$
$$\xi_3 = \int_{\sigma_3} g(y)\mathrm{d}y = \int_{r_0}^{r_\star} \epsilon^{-1/n}\mathrm{e}^{\theta_\star\mathrm{i}}g\big(\epsilon^{-1/n}\mathrm{e}^{\theta_\star\mathrm{i}}r\big)\mathrm{d}r.$$

In order to study ξ_1, we consider the function

$$h(t) := \epsilon^{-1/n}\sqrt{t^2+\epsilon^{2/n}}g\big(\epsilon^{-1/n}\mathrm{i}t\big) = h_0(t) + \mathrm{O}(\epsilon^{2/n}), \tag{4.37}$$

where $h_0(t) = (1+p_n\mathrm{i}^n t^n)^{2/3}(1+q_n\mathrm{i}^n t^n)^{-1/2}$. The function $h_0(t)$ is smooth in the interval $[0, r_0]$ and $h_0(t) = 1$. Besides,

$$\xi_1 = \mathrm{i}\int_0^{r_0} (t^2+\epsilon^{2/n})^{-1/2}h(t)\mathrm{d}t = \hat{\xi}_1 + \check{\xi}_1 + \tilde{\xi}_1 + \breve{\xi}_1,$$

where

$$\hat{\xi}_1 = \mathrm{i}\int_0^{r_0} \frac{\mathrm{d}t}{\sqrt{t^2+\epsilon^{2/n}}} = \mathrm{i}\,argsinh(\epsilon^{-1/n}r_0)$$
$$= \mathrm{i}\frac{|\log\epsilon|}{n} + \mathrm{i}\log(2r_0) + \mathrm{O}(\epsilon^{2/n}),$$
$$\check{\xi}_1 = \mathrm{i}\int_0^{r_0} \frac{h_0(t)-1}{t}\mathrm{d}t,$$
$$\tilde{\xi}_1 = \mathrm{i}\int_0^{r_0} \frac{h_0(t)-1}{t}\left(\frac{t}{\sqrt{t^2+\epsilon^{2/n}}} - 1\right)\mathrm{d}t = \mathrm{O}(\epsilon^{2/n}\log\epsilon),$$
$$\breve{\xi}_1 = \mathrm{i}\int_0^{r_0} \frac{h(t)-h_0(t)}{\sqrt{t^2+\epsilon^{2/n}}}\mathrm{d}t = \mathrm{O}(\epsilon^{2/n}\log\epsilon).$$

The integral $\hat{\xi}_1$ is immediate. The integral $\check{\xi}_1$ does not depend on ϵ. The integral $\tilde{\xi}_1$ is bounded using ideas from the proof of Lemma 23 in [CR11]. The integral $\breve{\xi}_1$ is bounded using (4.37). Hence, we have already seen that there exists $\eta_1 \in \mathbb{R}$ such that

$$|\Im\xi_1| = \frac{|\log\epsilon|}{n} + \eta_1 + \mathrm{O}(\epsilon^{2/n}\log\epsilon).$$

The study of ξ_2 and ξ_3 is easier, because

$$\xi_2 = \check{\xi}_2 + \mathrm{O}(\epsilon^{2/n}), \qquad \xi_3 = \check{\xi}_3 + \mathrm{O}(\epsilon^{2/n}),$$

for some constants $\check{\xi}_2$ and $\check{\xi}_3$ that do not depend on ϵ.

For instance, ξ_2 depends on $\epsilon^{2/n}$ in an analytic way, because both the integrand $\epsilon^{-1/n} r_0 \mathrm{e}^{\theta \mathrm{i}} \mathrm{i} g\big(\epsilon^{-1/n} r_0 \mathrm{e}^{\theta \mathrm{i}}\big)$ and the argument $\theta_\star$ are analytic in $\epsilon^{2/n}$, and all the singularities of the integrand are far from the integration path. The study of ξ_3 is similar. $\square$

Finally, if δ is the distance of the set of singularities and zeros of the curvature $\kappa(\xi)$ to the real axis, then

$$
\begin{aligned}
2\pi\delta &= \frac{2\pi}{C} \min\big\{ |\Im \xi_\star| : y_\star \text{ is a root with an } \mathrm{O}(\epsilon^{-1/n})\text{-modulus} \big\} \\
&= \frac{|\log \epsilon|}{n} + \eta + \mathrm{O}(\epsilon^{2/n} \log \epsilon),
\end{aligned}
$$

where the constant $\eta = \eta_n \in \mathbb{R}$ is equal to the smallest constant $\eta_\star$ provided by Lemma 35 among all the roots of $p(y)$ and $q(y)$ with an $\mathrm{O}(\epsilon^{-1/n})$-modulus. We have also used relation (4.36) in the last equality.

We do not care about the roots $y_\pm = \pm 1 + \mathrm{O}(\epsilon)$ of $r(y)$, since they correspond to points where y is not a true coordinate over the perturbed circle Q. To be precise, the points $(0, y_\pm)$ are the two vertices of Q over the symmetry line $\{x = 0\}$, and the curvature has a finite positive value at them. Nor do we care about the roots whose modulus is $\mathrm{O}(\epsilon^{-1/(n-2)})$, because

$$
\epsilon^{-1/(n-2)} g(\epsilon^{-1/(n-2)} z) = \epsilon^{-1/(3n-6)} \big(l_0(z) + \mathrm{o}(1) \big),
$$

where

$$
l_0(z) = \frac{z^{n/6-1}(p_n + p_{2n-2} z^{n-2})^{2/3}}{(1 + z^{n-2})^{1/2}(q_n + q_{2n-2} z^{n-2})^{1/2}}.
$$

This implies that, if $y_\star \in \mathbb{C}$ is one of those farther roots, then

$$
|\Im \xi_\star| = \epsilon^{-1/(3n-6)} \big(\nu_\star + \mathrm{o}(1) \big)
$$

for some constant $\nu_\star \in \mathbb{R}$. That is, the farther roots give rise to much bigger imaginary parts.

5 Other limits

5.1 Introduction

Most of this thesis has been dedicated to the study of the maximal difference of lengths $\Delta^{(p,q)}$ for (p,q)-periodic orbits approaching the boundary of the table. In particular, in Chapter 4, we start by fixing $p = 1$ and all the numerical results are for $(1,q)$-periodic orbits close enough to the boundary.

In this chapter, we want to give some insight on how $\Delta^{(p,q)}$ behaves when the (p,q)-periodic orbits do not tend to the boundary of the billiard table but to other regions of the phase space. Namely, we consider the cases of (p,q)-periodic orbits such that $p/q \to \vartheta \in (0,1) \setminus \mathbb{Q}$ and (p,q)-periodic orbits approaching to a (P,Q)-resonance. The study of $\Delta^{(p,q)}$ is not as detailed as in the previous chapters. Here, we restrict ourselves to a phenomenological study based on some numerical results.

The model tables we use in this study are the same as in the previous chapter. That is,

$$\left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2/b^2 + \epsilon y^n = 1\right\},$$

with $0 < b \leq 1$, $3 \leq n \leq 8$ and $\epsilon > 0$. To reduce the computational effort, we only work with axisymmetric periodic trajectories (APTs). APTs are defined in Section 4.3. Let us define $D^{(p,q)}$ as the signed difference of lengths between the two (p,q)-APTs. Just as in the case $p = 1$, we are able to prove that $|D^{(p,q)}| = \Delta^{(p,q)}$ in many cases. See Proposition 36. Henceforth, we will only consider (p,q)-periodic orbits such that $0 < p < q/2$. Recall that any billiard trajectory can be traveled in both directions. Thus, we obtain the information about all the twist interval by only studying the range of rotation numbers $(0, 1/2)$.

We consider first the case when p/q tends to an irrational rotation number. Mather introduced $\Delta W_{p/q}$ as a numerical criterion to discern between RICs and cantori in [Mat86]. Let $\vartheta \in (0,1/2) \setminus \mathbb{Q}$. Whenever $\Delta W_{p/q} \to 0$ as $p/q \to \vartheta$, there exists a RIC of rotation number ϑ. Moreover, MacKay in [Mac92] proved that $\Delta W_{p/q}$ decays exponentially as $-\rho q$ for some positive $\rho > 0$ for (p,q)-periodic orbits that approach an analytic RIC of Diophantine rotation number. On the contrary, if the limit value of $\Delta W_{p/q}$ is different from zero, the structure of rotation number ϑ is a cantori. The numerical results we obtain in this setting seem to indicate that we have chosen irrational rotation numbers ϑ such that there exist RICs of rotation number ϑ. This fact is due to two reasons; both related to integrability phenomena.

On the one hand, our model tables can be seen as perturbations of elliptic and circular tables. Elliptic and circular maps are integrable so we expect that many Diophantine RICs persist under perturbation. In our study, we have mostly used Diophantine numbers, although we also study $D^{(p,q)}$ when p/q tend to a Liouville number.

On the other hand, Lazutkin [Laz73] proved (resp., Douady [Dou82] improved) that there are infinitely many caustics accumulating to the boundary of the billiard for any C^{555} (resp., C^7) strictly convex table. These caustics are in correspondence with rotational invariant curves of Diophantine rotation numbers on the phase space. These curves control the chaotic regions close to the boundary of the billiard table, making it easy to find the periodic orbits on that part of the phase space. In fact, we can not compute (p, q)-periodic orbits for large periods q with our current algorithm when we are far away from the boundary. Thus, we have chosen irrational rotation numbers that generate orbits in the lower part of the phase space. For instance, $\vartheta = 0.10397\ldots$. In Figure 5.1, we show the phase space of the billiard map for some billiard tables. Instead of using the Birkhoff coordinates $(s, r) \in \mathbb{R}/l\mathbb{Z} \times (0, \pi)$, we use the coordinates $(s, I) \in \mathbb{R}/l\mathbb{Z} \times (-1, 1)$, where $I = -\cos r$. We show the region $\mathbb{R}/l\mathbb{Z} \times (-1, 0)$. Recall that the phase space is symmetric with respect to $\{I = 0\}$ so that these figures give all the information of the phase space.

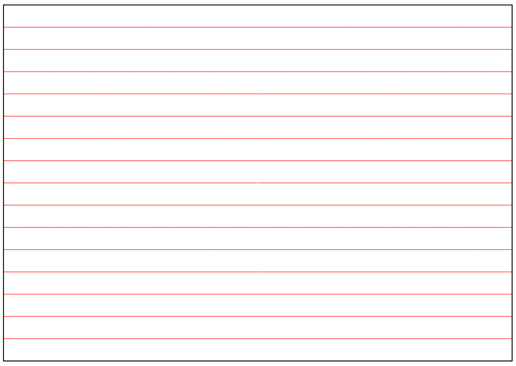
We study the behavior of $D^{(p,q)}$ for (p, q)-periodic orbits such that p/q tends to an irrational rotation number ϑ in Section 5.3. We approach the study of $D^{(p,q)}$ from three different points of view.

First, there are many ways to choose how the sequences of p/q tend to $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$. We study how the different ways of tending to the limit irrational number affect on the behaviour of $D^{(p,q)}$. In particular, we focus on studying the different ways MacKay proposed in [Mac92] when he proved the exponentially small decay of $\Delta W_{p/q}$ for (p, q)-periodic orbits close enough to an analytic RIC with Diophantine rotation number.

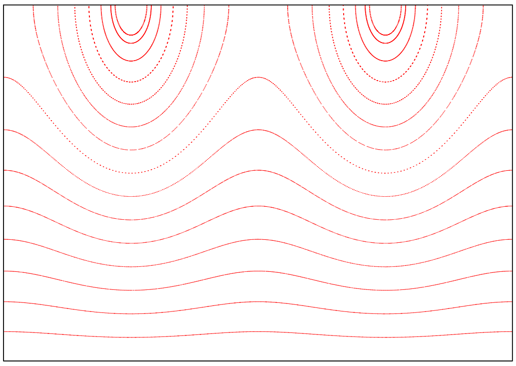
Second, we use the Melnikov method to obtain a prediction for the behavior of $\Delta^{(p,q)}$. See Proposition 36. We compare $D^{(p,q)}$ with the Melnikov prediction. Just as in Chapter 4, we see that the Melnikov method fails to predict the behavior of $D^{(p,q)}$.

Finally, following papers [MMP84, MMP87, Mei92], we perform a study of $D^{(p,q)}$ from the point of view of the Farey tree.

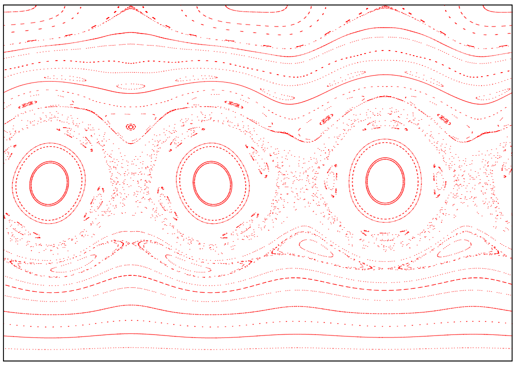
Next, we consider the case when p/q tends to a rational rotation number. MacKay, Meiss, and Percival study the case of (p, q)-periodic orbits approaching a (P, Q)-resonance in [MMP87]. They use $\Delta W_{p/q}$ to measure the flux through the upper and lower separatrices of the (P, Q)-hyperbolic periodic orbit. The pairs (p, q) they use have some particular continued fraction expansion pattern. We study $D^{(p,q)}$ as $p/q \to P/Q$ by choosing the number of turns p such that p/q best approximates P/Q at every step. We also restrict the study of $D^{(p,q)}$ to the pairs (p, q) described in [MMP87]. We compare the results. We support our conclusions by considering two different examples of (P, Q)-resonances in Section 5.4.



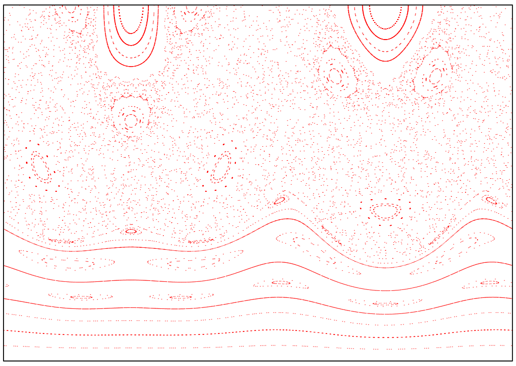
(a) $b = 1$ and $\epsilon = 0$



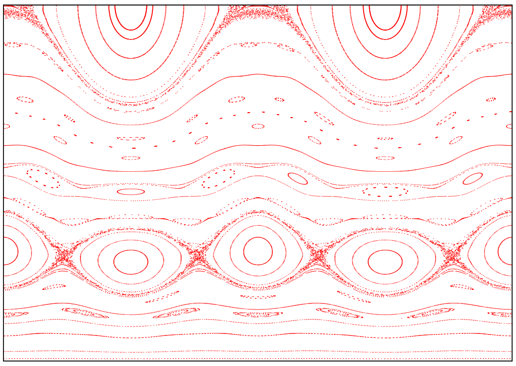
(b) $b = 0.9$ and $\epsilon = 0$



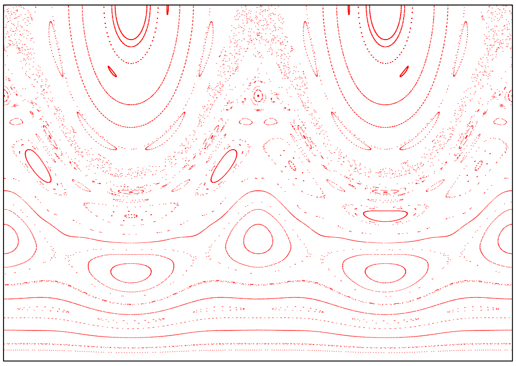
(c) $b = 1$, $n = 3$, and $\epsilon = 10^{-1}$



(d) $b = 0.9$, $n = 3$, and $\epsilon = 10^{-1}$



(e) $b = 1$, $n = 4$, and $\epsilon = 10^{-1}$



(f) $b = 0.9$, $n = 4$, and $\epsilon = 10^{-1}$

Figure 5.1: Partial phase space in coordinates $(s, I) \in \mathbb{R}/l\mathbb{Z} \times (-1, 1)$.

The results in this chapter consist of showing the behavior of $D^{(p,q)}$ in the various settings we have mentioned above. These results may seem incomplete in regard with the ones shown on the previous chapter. Indeed, here we do not apply any type of normalization to $D^{(p,q)}$ whereas in the previous chapter we dealt with the normalized quantity $\hat{D}_q$ in addition to D_q. On what follows, we explain the main reasons behind this fact. First, these are the initial results we obtained and we had not yet developed the whole methodology. Second, there are many different ways to characterize which (p, q)-periodic orbits we choose as $q \to \infty$ and as $p/q \to \vartheta \in (0, 1/2]$. The natural solution in the boundary seems to fix $p = 1$. However, MacKay in [Mac92] proposed different sets of pairs (p, q). Depending on which set we use to study $D^{(p,q)}$, the behavior is different. This is explained in Section 5.3 in a more detailed way.

The structure of this chapter is the following. Prior to the study of the different phenomena, we show the behavior of $D^{(p,q)}$ in a wide range of the phase space in Section 5.2. In Section 5.3, we consider the case where p/q tends to an irrational number; whereas we consider the case where (p, q)-periodic orbits approach a (P, Q)-resonance in Section 5.4.

5.2 A global perspective

The use of the Farey tree to study the dynamics of a particular area-preserving twist map is not new. MacKay, Meiss, and Percival use it for displaying $\Delta W_{p/q}$ for the sawtooth map in [MMP84] and the areas of the resonances for the standard map in [MMP87]. Meiss also uses it to study $\Delta W_{p/q}$ for the standard map in [Mei92].

Let us briefly introduce the continued fraction expansions and the Farey tree. Both representations show some arithmetic properties of the real numbers. See [Khi97] and [HW79] respectively for more details.

The *continued fraction* expansion of a number ϑ is the sequence $[a_0, a_1, a_2, \ldots]$ of integers generated by

$$a_k = \lfloor \vartheta_k \rfloor, \qquad \vartheta_{k+1} = \frac{1}{\vartheta_k - a_k},$$

where $\vartheta_0 = \vartheta$ and $\lfloor \cdot \rfloor$ is the floor function. Equivalently, it can be thought as the representation such that

$$\vartheta = a_0 + \cfrac{1}{a_1 + \frac{1}{\frac{1}{a_2 + \cdots + \frac{1}{a_k + \cdots}}}}.$$

Every rational number has two different finite representations. If $\vartheta = [a_0, \ldots, a_k] \in \mathbb{Q}$, with $a_k > 1$, then it can also be written as $\vartheta = [a_0, \ldots, a_k - 1, 1]$. On the contrary, the sequence for a irrational number is infinite. *Convergents* of a continued fraction are the rationals obtained by truncating the expansion at some level,

$$p_k/q_k = [a_0, a_1, \ldots, a_k],$$

with p_k and q_k relatively prime integers. At every level k, convergents are the best approximation to number ϑ among all the p/q with $1 \le q \le q_k$. *Noble numbers* are Diophantine numbers such that its continued fraction becomes an infinite sequence of ones at some point, $[0, a_1, a_2, \ldots, a_k, \dot{1}]$.

The *Farey tree* is a technique for organizing the rational numbers according to the length of their continued fraction expansions. Given two rationals, p/q and p'/q', such that

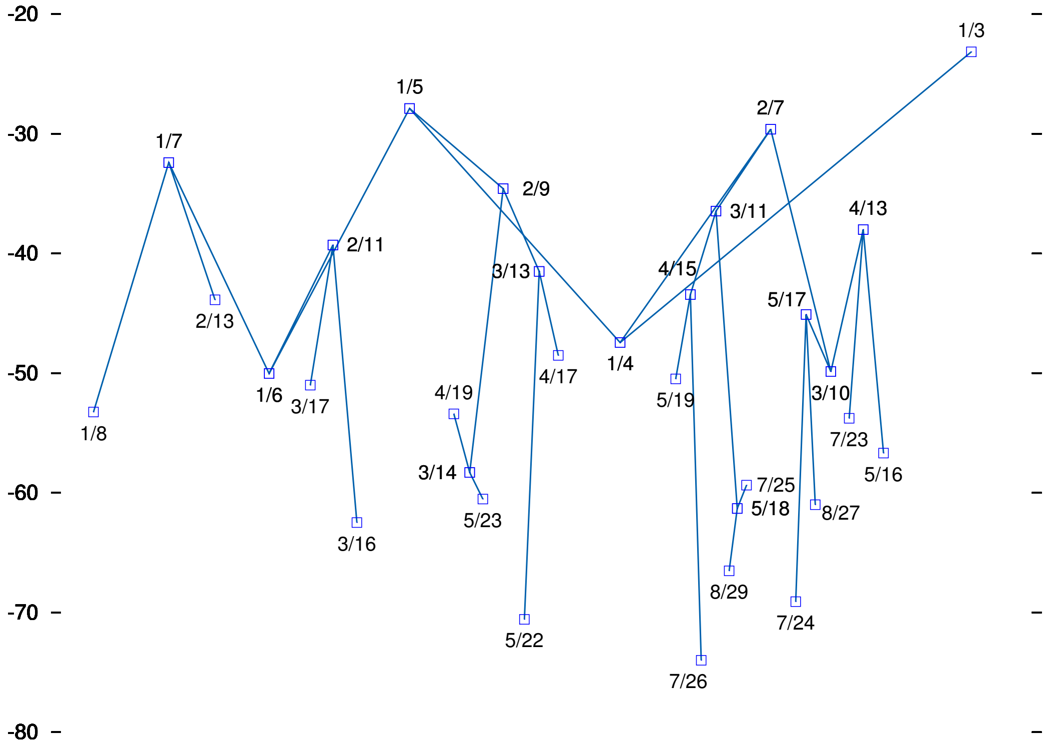
$pq' - p'q = 1$, the construction will give rise to every rational in the interval $(p'/q', p/q)$. The Farey tree is a binary tree. Its root is the *mediant*, $p''/q'' = (p+p')/(q+q')$. At any other stage, from a node corresponding to the rational $[a_0, \ldots, a_k]$, with $a_k \neq 1$, its daughters are $[a_0, \ldots, a_k - 1, 2]$ and $[a_0, \ldots, a_k, 1]$. The left daughter is the smallest of the two values. Infinite paths that, at some point, take only left daughters (or only right daughters) converge to rational numbers whereas paths that never settle down to either one direction or the other approach irrational numbers. In particular, noble numbers have paths that eventually alternate from the left daughter to the right one and vice versa.

In Figure 5.2, we show two Farey trees that cover a wide range of the twist interval. We construct the Farey tree by choosing $p = q' = 1$, $q = 2$, and $p' = 0$. We show up to six levels of the Farey tree on the range of rotation numbers $[1/8, 1/3]$. The ordinate is $\log D^{(p,q)}$ and the abscissa is the rotation number p/q. The model table is such that $b = 9/10$, $n = 3$, and $\epsilon = 10^{-10}$ on Figure 5.2(a), whereas $b = 9/10$, $n = 3$, and $\epsilon = 10^{-1}$ on Figure 5.2(b). From these figures, it is clear that the Farey tree structure varies a lot depending on the size of the perturbation.

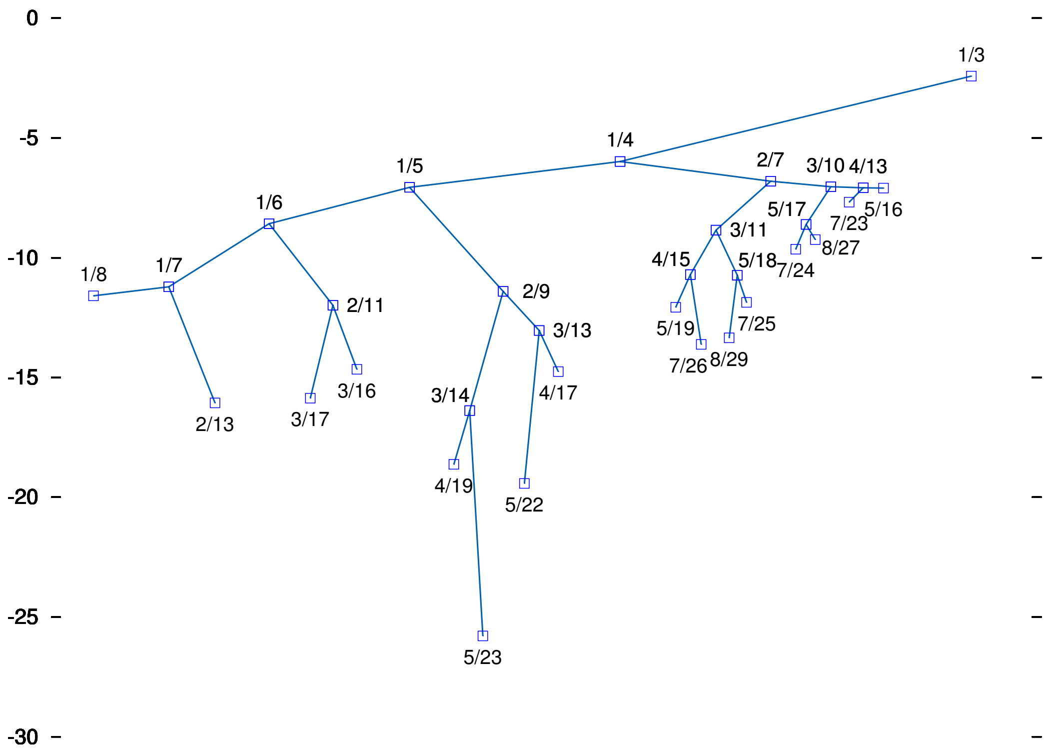
Figure 5.2(b) resembles the figures [MMP84, Fig.9], [MMP87, Fig. 13], and [Mei92, Fig. 56]. On [Mei92], the author uses the Farey tree to study $\log \Delta W_{p/q}$ for the standard map at a critical value of the parameter such that only the last invariant circle exists. Some of his conclusions are that $\log \Delta W_{p/q}$ decreases as one goes down the tree, $\log \Delta W_{p/q}$ tends to a limit as one follows each route down the tree, and alternating the direction at every node is the fastest way to decrease $\Delta W_{p/q}$. Recall that paths obtained when alternating the direction at every node converge to a noble number. Call to mind that $\Delta W_{p/q}$ is the flux through an homotopically nontrivial curve joining all the (p,q)-periodic points (see Chapter 2). Thus, the previous observations reinforce the idea that the main obstructions to the transport in area-preserving twist maps are orbits (RICs or cantori) with irrational rotation numbers. Also, the more Diophantine the rotation number is, the more it limits the transport.

Meiss conjectures that the previous considerations are general properties of smooth area-preserving twist maps for large enough levels on the Farey tree. However, in Figure 5.2(a), we show that $\log D^{(p,q)}$ does not decrease as one goes down the tree. The branches seem to go up and down randomly. Billiard maps and the standard map are both area-preserving twist maps. Yet, billiard maps have an additional property that plays an important role on the Farey tree structure. The results of Lazutkin [Laz73] and Douady [Dou82] on the existence of infinitely many Diophantine RICs with Diophantine rotation number close to the boundary give extra structure to this region of the phase space. In fact, Figure 5.2(a) shows the behavior predicted by Melnikov theory.

The numerical study of $\Delta^{(p,q)}$ is more challenging for convex billiards than for standard maps without RICs. For this reason, MacKay, Meiss, and Percival are able to construct Farey trees of about 8 levels by working with a precision of 10^{-9} whereas we need up to 80 decimals to generate Figure 5.2. Moreover, we need to work with a precision of 600 decimals to construct the Farey trees that appear in the following section.



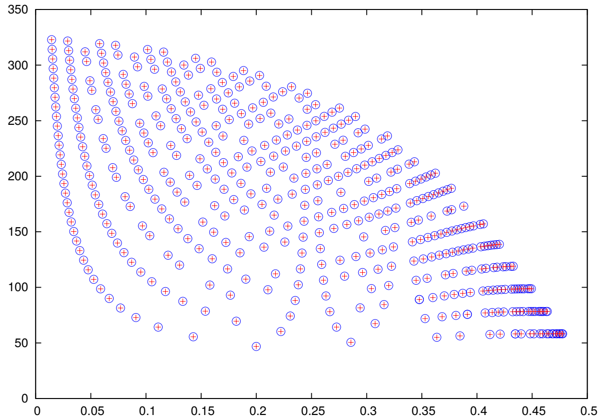
(a) $\epsilon = 10^{-10}$.



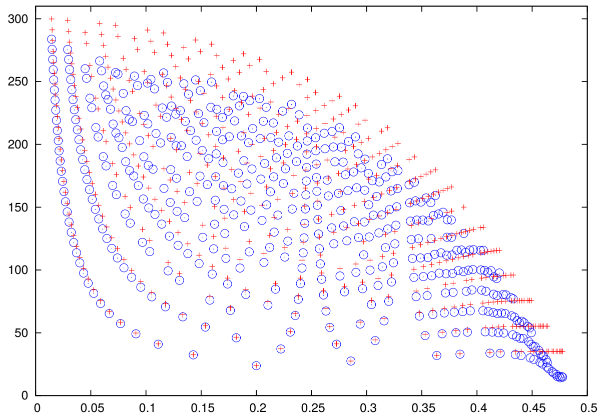
(b) $\epsilon = 1/10$.

Figure 5.2: $\log D^{(p,q)}$ versus $p/q \in [1/8, 1/3]$ for $b = 9/10$ and $n = 3$.

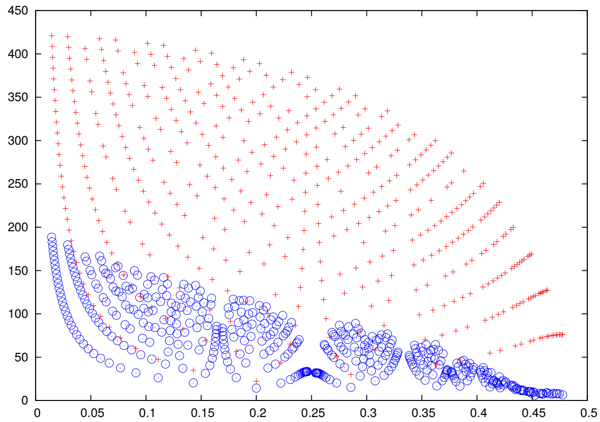
Next, in Figure 5.3, we display the values of the difference $D^{(p,q)}$ for $b = 9/10$, $n = 4$, and all odd $q \in \mathbb{N}$ such that $q < 70$, $0 < p < q/2$ and $\gcd(p,q) = 1$. We compare them to their corresponding Melnikov predictions (see Proposition 36).



(a) $\epsilon = 10^{-15}$.



(b) $\epsilon = 10^{-5}$.



(c) $\epsilon = 10^{-1}$.

Figure 5.3: $-\log D^{(p,q)}$ (in blue) and the Melnikov prediction $-\log \epsilon \Delta_1^{(p,q)}$ (in red) versus p/q for $b = 9/10$ and $n = 4$ for all odd $q \in \mathbb{N}$ such that $q < 70$, $0 < p < q/2$ and $\gcd(p,q) = 1$.

5.3 Limit to irrational rotation numbers

MacKay in [Mac92] proved that $\Delta W_{p/q}$ decays exponentially as $-\rho q$ for some positive $\rho > 0$ when (p, q)-periodic orbits approach an analytic RIC of Diophantine rotation number. The result applied to an analytic strictly convex billiard reads as follows.

Given a rotation number $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$, an exponent $\alpha \in (-1, 1)$ and a factor $\delta > 0$, we define the set

$$S_\vartheta(\alpha, \delta) := \left\{(p, q) \in \mathbb{N}^2 : 0 < p < q/2,\ |p - q\vartheta| \leq \delta q^\alpha/2\right\}.$$

Note that, if $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$, there exists $\alpha_{inf}(\vartheta) \leq -1$ such that, for any $q_0 \in \mathbb{N}$ and $\alpha > \alpha_{inf}(\vartheta)$, the set $S_\vartheta(\alpha, \delta)$ has infinitely many points (p, q) with $q > q_0$. Thus, we restrict to values $\alpha > -1$. The restriction $\alpha < 1$ ensures that the pairs $(p, q) \in S_\vartheta(\alpha, \delta)$ are such that p/q tends to ϑ as $q \to +\infty$.

Let C be a rotational invariant curve of Diophantine rotation number ϑ with analytic conjugacy to a rotation of analyticity width $\rho' > 0$. If $\rho < \rho'$, then there exist constants $K = K(\rho, \delta, \alpha, \vartheta) > 0$ and $q_* = q_*(\rho, \delta, \alpha, \vartheta) \in \mathbb{N}$ such that

$$\Delta^{(p,q)} \leq K\mathrm{e}^{-\rho q},$$

for any relatively prime integers p and q such that $(p, q) \in S_\vartheta(\alpha, \delta)$ and $q > q_*$.

The set $S_\vartheta(\alpha, \delta)$ characterizes how close the (p, q)-periodic orbits are to the RIC. Let us discuss how the sets $S_\vartheta(\alpha, \delta)$ are. Let $\delta > 0$ and $q_0 \in \mathbb{N}$, $q_0 > \delta$. For $\alpha = 0$, the set $S_\vartheta(\alpha, \delta)$ has exactly $\lfloor \delta \rfloor$ pairs (p, q) such that $q = q_0$. By taking $\alpha = 0$ and $\delta = 1$, a pair (p, q) belongs to $S_\vartheta(0, 1)$ if and only if p is such that p/q best approximates ϑ. By taking $\alpha = 0$ and $\delta = 2$, only two different pairs (p, q_0) belong to $S_\vartheta(0, 2)$. Namely, (p, q_0) with p such that p/q_0 best approximates ϑ from below and $(p + 1, q_0)$. Besides, note that the set $S_\vartheta(0, \delta)$ is equivalent to the condition $|Qp - Pq| \leq L$ on Theorem 4 when we take $\vartheta = P/Q$ and $\delta = L/Q$ (see also Remark 3). For $\alpha < 0$, the number of pairs (p, q) shrinks as q grows. Figure 5.4 shows the different sets $S_\vartheta(\alpha, \delta)$.

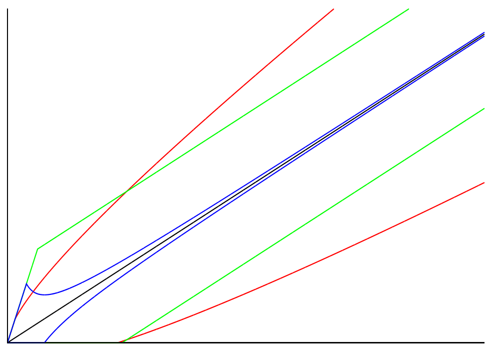
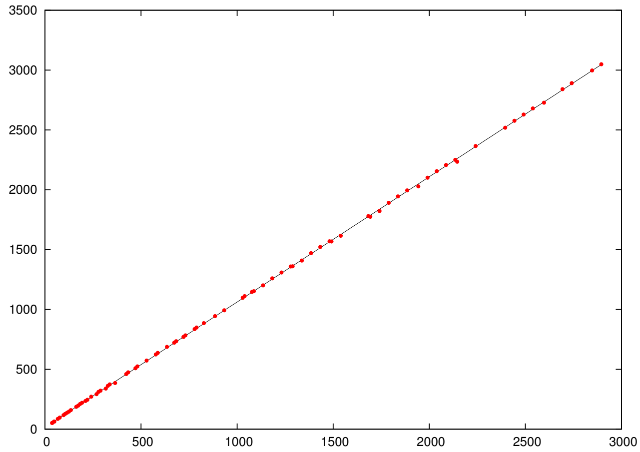
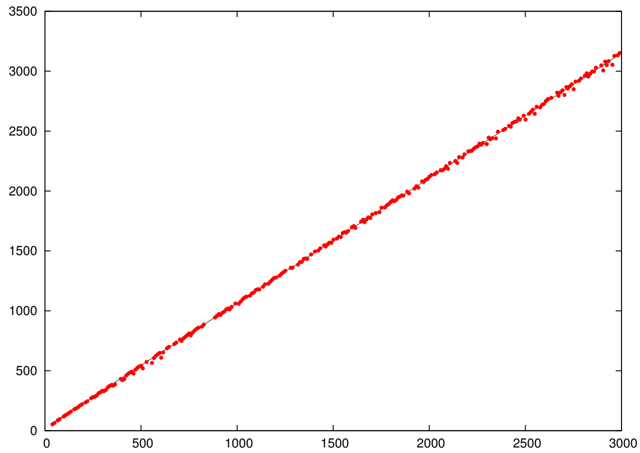


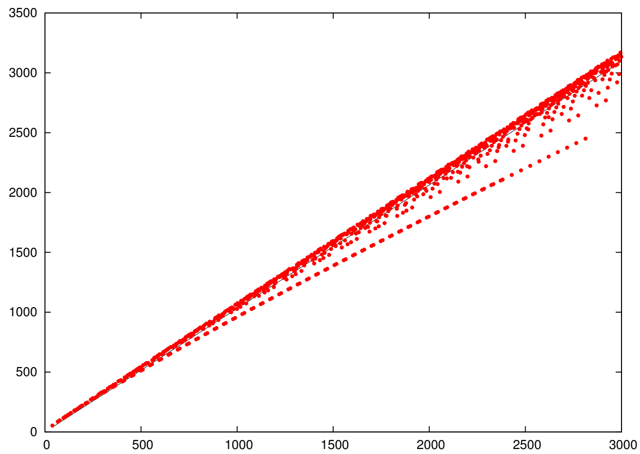
Figure 5.4: Space $(q, p) \in \mathbb{N}^2$. Black $p = q\vartheta$. $S_\vartheta(\alpha, \delta)$ is shown for $\alpha > 0$ (red), $\alpha = 0$ (green), and $\alpha < 0$ (blue).



(a) $\alpha = -0.5$ and $\delta = 1$.



(b) $\alpha = 0$ and $\delta = 10^{-1}$.



(c) $\alpha = 0.5$ and $\delta = 10^{-2}$.

Figure 5.5: $-\log D^{(p,q)}$ versus odd q for $b = 9/10$, $n = 4$, $\epsilon = 1/10$ and $(p,q) \in S_\vartheta(\alpha, \delta)$ with $\gcd(p,q) = 1$ and $\vartheta = [0, 9, \dot{1}]$.

In Figure 5.5, we compute $-\log D^{(p,q)}$ for p/q tending to the noble number $\vartheta = [0, 9, \dot{1}]$ and $(p,q) \in S_\vartheta(\alpha, \delta)$, for different choices of (α, δ).

We observe that there exist two exponents $r_\pm = r_\pm(\delta, \alpha, \vartheta) > 0$ such that

$$r_- q \le -\log D^{(p,q)} \le r_+ q,$$

if q is large enough. By taking exponentials, these inequalities become

$$\mathrm{e}^{-r_+ q} \le D^{(p,q)} \le \mathrm{e}^{-r_- q},$$

if q is large enough. These exponentially small upper and lower bounds collapse when $\alpha \to -1$. That is, there exists an exponent $r = r(\vartheta) \in (r_-, r_+)$ such that

$$\lim_{\alpha \to -1} r_\pm = r.$$

Let us focus on the set of points (p, q) such that their corresponding $-\log D^{(p,q)}$ describe the lowest curve in Figure 5.5(c). This curve seems to stop at some point where the period is about 2800. Although, the points (p, q) on that curve belong to the set $S_\vartheta(\alpha, \delta)$, their rotation number p/q is far from ϑ compared to the other points. Also, their (p, q)-periodic orbits are more influenced by a certain (P, Q)-resonance than by the RIC (or cantori) of rotation number ϑ. Indeed, we affirm that their rotation number p/q is closer to a distinguished P/Q than to ϑ. These last observations become clearer on the following section, where we deal with the case of (p, q)-periodic orbits approaching a (P, Q)-resonance.

In Section 4.6, we derive a Melnikov formula for $\Delta^{(p,q)}$ when p/q approaches any number $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$ on perturbed ellipses. The result follows from a direct application of Lemmas 31, 32, and 33. Let us gather the Melnikov formulas obtained for our model tables when $0 < b < 1$ in the following proposition. Analogously to Section 4.4, we write

$$\Delta^{(p,q)} = \epsilon \Delta_1^{(p,q)} + \mathrm{O}(\epsilon^2).$$

Proposition 36. *Let $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$. If $b \in (0, 1)$ and $q \ge 3$ and $0 < p < q/2$ are relatively prime integers, the following properties hold.*

1. *$\Delta_1^{(p,q)} = 0$, for odd n and even q.*

2. *There exist some constants $\zeta = \zeta(\vartheta, b), \mho_3(\vartheta, b), \mho_4(\vartheta, b) > 0$ such that*

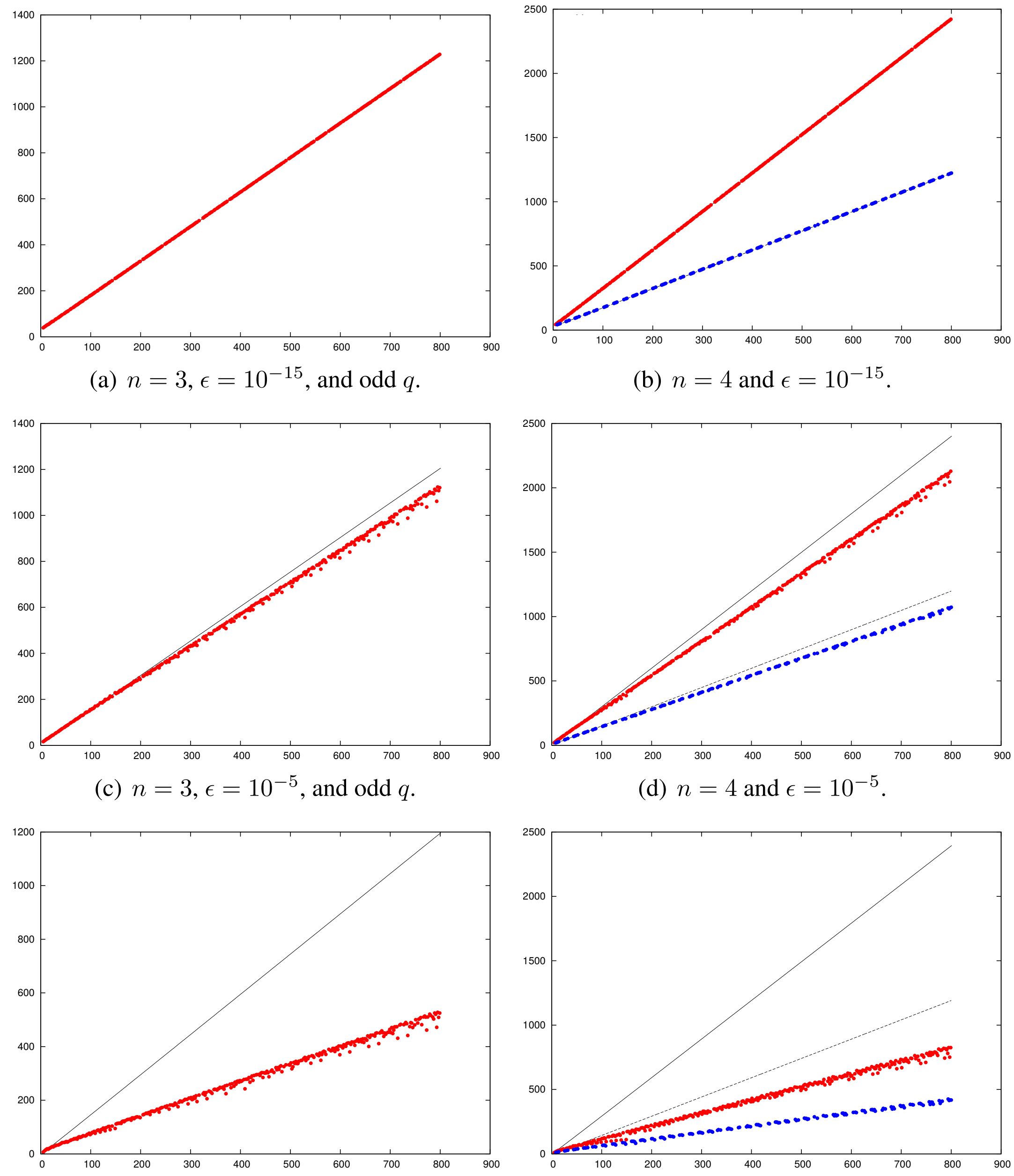
$$\Delta_1^{(p,q)} \asymp \begin{cases} \mho_3 q \mathrm{e}^{-\zeta q}, & \text{for } n = 3 \text{ and odd } q, \\ 2\mho_4 q^2 \mathrm{e}^{-2\zeta q}, & \text{for } n = 4 \text{ and odd } q, \\ \mho_4 q^2 \mathrm{e}^{-\zeta q}, & \text{for } n = 4 \text{ and even } q, \end{cases}$$

   as $p/q \to \vartheta$.

3. *If $n = 3$ and q is odd or if $n = 4$, then there exists $\tilde{\epsilon}_n = \tilde{\epsilon}_n(b, \vartheta, p, q) \in I_n$ such that all (p, q)-periodic billiard trajectories inside (4.7) are axisymmetric when $\epsilon \in (0, \tilde{\epsilon}_n)$. In particular, $\Delta^{(p,q)} = \Delta W_{p/q} = |D^{(p,q)}|$ for all $\epsilon \in (0, \tilde{\epsilon}_n)$.*

Henceforth, fixed $\vartheta \in (0, 1/2) \setminus \mathbb{Q}$, we consider pairs $(p, q) \in S_\vartheta(0, 1)$ to do our numeric computations. So, for every q, we will consider only p such that p/q best approximates ϑ.

In Figure 5.6, we show a comparison between the analytic Melnikov prediction and the numerical computations when $\vartheta = [0, 9, \dot{1}]$ and $(p, q) \in S_\vartheta(0, 1)$ on some perturbed ellipses such that $b = 7/10$. Other values of the semi-minor axis give rise to similar figures. As in the case of $(1, q)$-periodic orbits close to the boundary, we see that the Melnikov method does not predict the asymptotic behavior of $D^{(p,q)}$.



(a) $n = 3$, $\epsilon = 10^{-15}$, and odd q.
(b) $n = 4$ and $\epsilon = 10^{-15}$.
(c) $n = 3$, $\epsilon = 10^{-5}$, and odd q.
(d) $n = 4$ and $\epsilon = 10^{-5}$.
(e) $n = 3$, $\epsilon = 10^{-1}$, and odd q.
(f) $n = 4$ and $\epsilon = 10^{-1}$.

Figure 5.6: $-\log D^{(p,q)}$ and $-\log \epsilon \Delta_1^{(p,q)}$ versus q for $b = 7/10$, $\vartheta = [0, 9, \dot{1}]$ and $(p, q) \in S_\vartheta(0, 1)$ such that $\gcd(p, q) = 1$. Red: $-\log D^{(p,q)}$ for odd q. Blue: $-\log D^{(p,q)}$ for even q. Black line: $-\log \epsilon \Delta_1^{(p,q)}$ for odd q. Discontinuous black line: $-\log \epsilon \Delta_1^{(p,q)}$ for even q.

In Figures 5.6(b), 5.6(d), and 5.6(f), we can also observe that the extra symmetry on our model tables with even n has the same effect that the one we describe close to boundary. Indeed, for quartic perturbations of the ellipse, we see that the slope on $-\log D^{(p,q)}$ for odd q is twice the one on $-\log D^{(p,q)}$ for even q. We observe the same property for $-\log D^{(p,q)}$ on perturbed circles with even n.

Next, we show how the behavior of $D^{(p,q)}$ changes when the rotation number p/q approaches different irrational numbers ϑ.

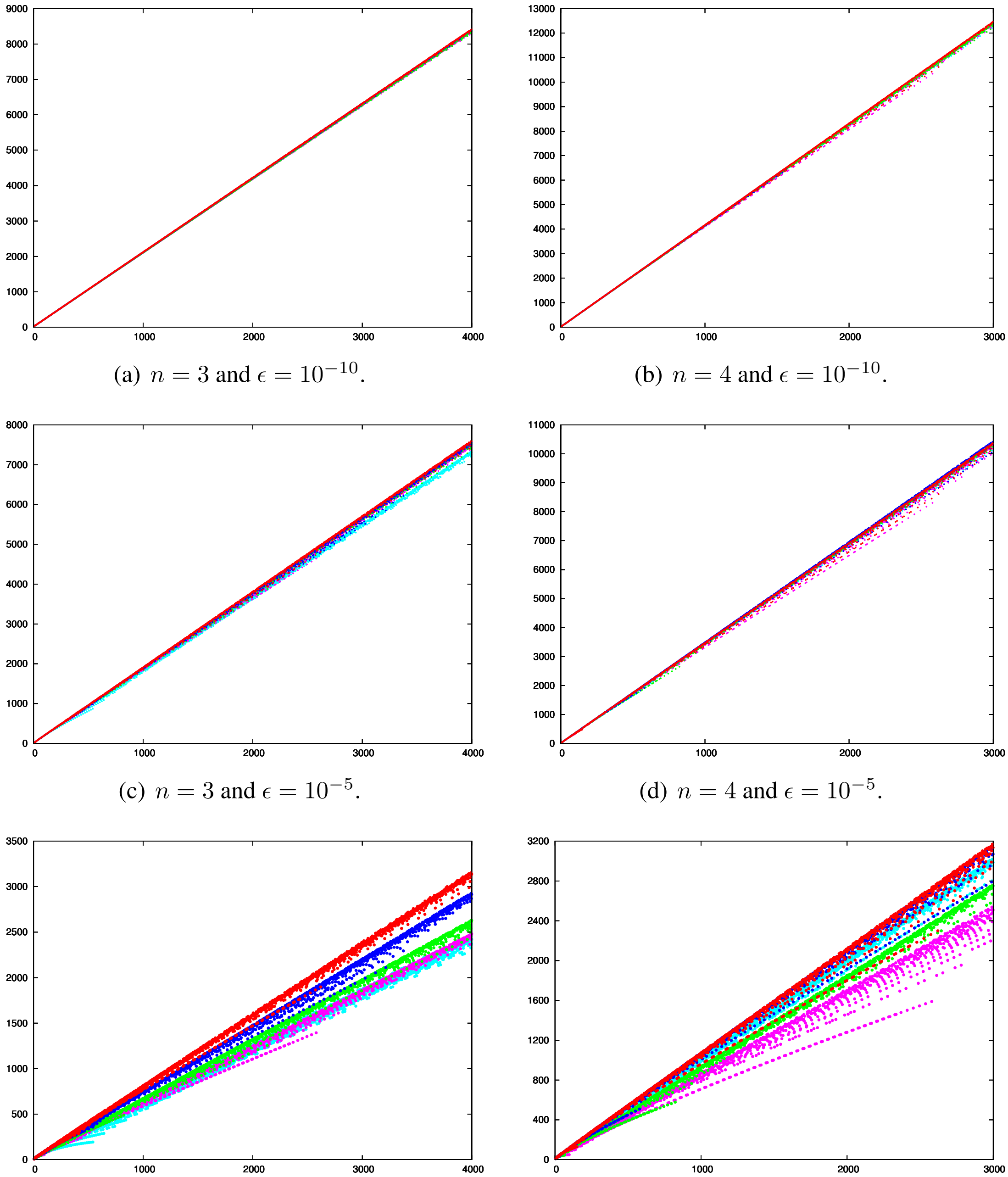
We have used the following Diophantine numbers: $[0, 9, \dot{1}] = 0.10397\ldots$,$[0, 9, \dot{2}] = 0.10622\ldots$, $[0, 8, \dot{1}] = 0.11603\ldots$, and $[0, 8, \dot{2}] = 0.11884\ldots$; and also the Liouville number $\sum_{k\geq 1} 10^{-k!} = 0.1100010\ldots$. We choose these irrational numbers in a small range of the twist interval. We want the cantori or rotational invariant curves with such irrational rotation number to be close in the phase space so that we can discuss how the behavior $D^{(p,q)}$ changes depending on the arithmetic properties of the limit rotation number.

We show the results in Figure 5.7. In particular, we plot $-\log D^{(p,q)}$ versus q when $(p, q) \in S_\vartheta(0, 1)$ for the different rotation numbers ϑ mentioned above. There are several phenomena to be considered in this figure.

Since the different limit rotation numbers are very close together, the slopes $-\log D^{(p,q)}$ are very similar for very small ϵ (10^{-10} or 10^{-5}). The Melnikov exponent $\zeta = \zeta(\vartheta, b)$ is a strictly decreasing function with respect to ϑ. So we should see that the slopes are ordered in the inverse way to their corresponding limit rotation numbers when ϵ is small enough.

However, as seen in Figures 5.6(c) and 5.6(d), the behavior of $D^{(p,q)}$ considerably differs from the Melnikov prediction when $\epsilon = 10^{-5}$. As ϵ grows we would expect that the order on the slopes of the different $D^{(p,q)}$ was given by the arithmetic properties of the limit irrational rotation numbers as we have already seen in Figure 5.2(b). This fact can be observed in Figures 5.7(c) and 5.7(e), where $D^{(p,q)}$ decays at a slower rate for the (p, q)-periodic orbits such that p/q tends to the Liouville rotation number. The case large ϵ and $n = 4$ is somehow different. In Figures 5.7(d) and 5.7(f), the smallest slope on $-\log D^{(p,q)}$ is not for the (p, q)-periodic orbits such p/q tends to the Liouville rotation number but to those such p/q tends to $[0, 8, \dot{2}]$. We believe that there exists a (P, Q)-resonance such that P/Q is close to $[0, 8, \dot{2}]$ and this affects the behavior of $D^{(p,q)}$ as we will discuss in the next section.

Finally, we gather some figures about Farey trees we did when we first studied $D^{(p,q)}$. These Farey trees are obtained by initially choosing $p = q' = 1$, $p = 0$, and $q = 2$, as in Section 5.2. However, we do not show all the nodes on the first stages of the tree but some of them. The reason is that we focus on a certain Diophantine number and we show the parts of the Farey tree that are in the neighborhood of it. In particular, we have fixed $\vartheta = [0, 9, \dot{1}]$ in Figures 5.8 and 5.9(a) and $\vartheta = [0, 9, \dot{2}]$ in Figure 5.9(b).



(a) $n = 3$ and $\epsilon = 10^{-10}$.

(b) $n = 4$ and $\epsilon = 10^{-10}$.

(c) $n = 3$ and $\epsilon = 10^{-5}$.

(d) $n = 4$ and $\epsilon = 10^{-5}$.

(e) $n = 3$ and $\epsilon = 10^{-1}$.

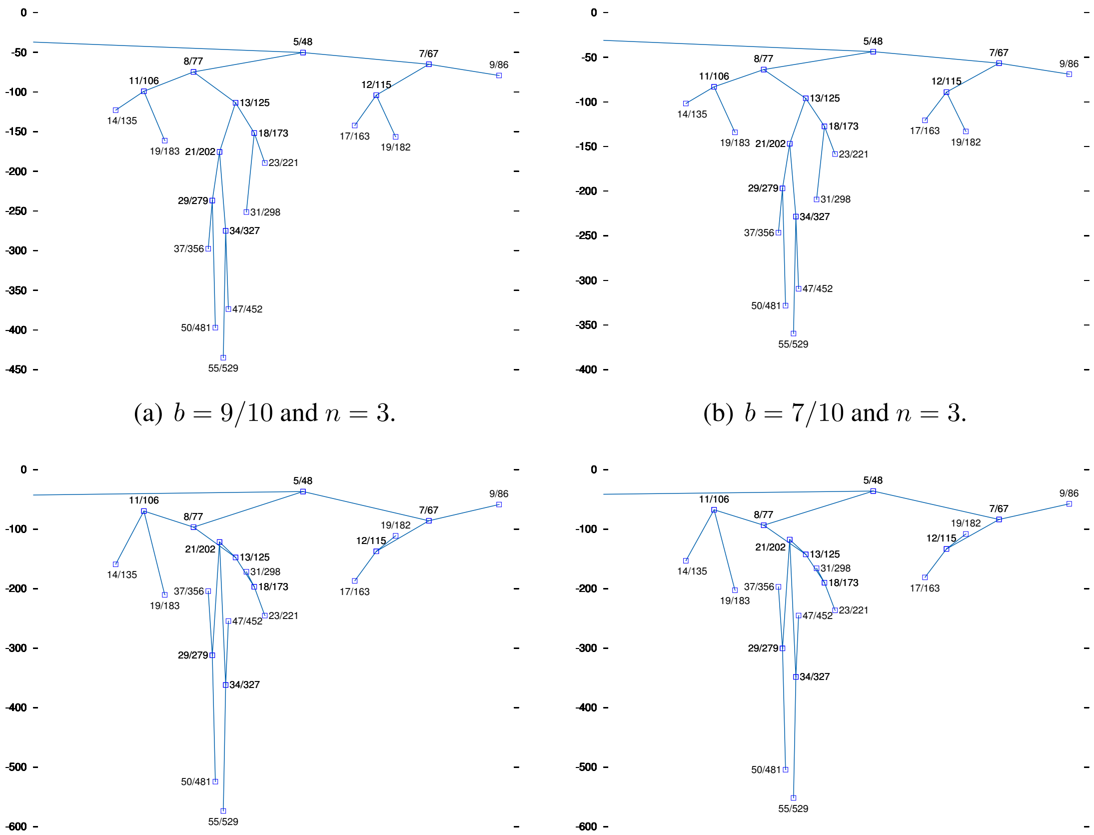
(f) $n = 4$ and $\epsilon = 10^{-1}$.

Figure 5.7: $-\log D^{(p,q)}$ versus q for $b = 9/10$ and p such that $(p, q) \in S_\vartheta(0, 1)$ and $\gcd(p, q) = 1$. Red: $\vartheta = [0, 9, \dot{1}]$. Blue: $\vartheta = [0, 9, \dot{2}]$. Green: $\vartheta = [0, 8, \dot{1}]$. Magenta: $\vartheta = [0, 8, \dot{2}]$. Cyan: $\vartheta = \sum_{k\geq 1} 10^{-k!}$.

Recall that $\vartheta = [0, 9, \dot{1}]$ is a noble number. Noble numbers are such that the path on the Farey tree eventually alternates the direction at each step. Precisely, the fractions $1/9$, $1/10$, $2/29$, $3/39$, $5/48$, $8/77$, $13/125$, $21/202$, $34/327$, and $55/529$ are the initial convergents of $\vartheta = [0, 9, \dot{1}]$. The other irrational number we consider is $\vartheta = [0, 9, \dot{2}]$. The 2 periodic tail on its continued fraction expansion also draws a pattern on the Farey tree. It is obtained by alternating direction not at every step but at every two steps. In particular, the initial convergents of $\vartheta = [0, 9, \dot{2}]$ are the fractions $1/9$, $2/19$, $5/47$, $12/113$, $29/273$ and $70/659$.

In Figure 5.8, we show $\log D^{(p,q)}$ versus p/q for some model tables with $\epsilon = 1/10$. We observe that, when n is fix, the Farey tree is qualitatively equal for different values of the semi-minor axis b. In Figure 5.6, we have observed that the exponent on $D^{(p,q)}$ for odd q is twice the exponent on $D^{(p,q)}$ for even q on quartic perturbations of the ellipse. Figures 5.8(c) and 5.8(d) partially show this phenomena. Branches with a daughter such that q is even go up whereas branches with a daughter such that q is odd go down.

In Figure 5.9, we compare the Farey trees focused on two different limit rotation numbers for the same model table, $b = 9/10$, $n = 3$, and $\epsilon = 10^{-5}$. The ordinate is $\log D^{(p,q)}$ and the abscissa is p/q. In the interval of rotation numbers in Figure 5.9(a) and in Figure5.9(b), we observe that the route down the tree obtained when alternating the direction at each step is the route where the maximum difference of lengths tends fastest to zero.



(a) $b = 9/10$ and $n = 3$.

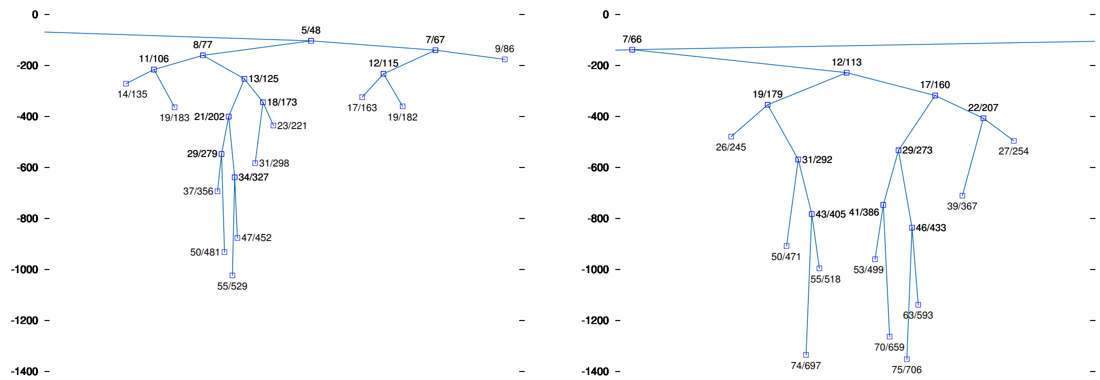
(b) $b = 7/10$ and $n = 3$.

(c) $b = 9/10$ and $n = 4$.

(d) $b = 7/10$ and $n = 4$.

Figure 5.8: $\log D^{(p,q)}$ versus p/q on the Farey tree on the neighborhood of the rotation number $[0, 9, \dot{1}]$ for $\epsilon = 1/10$.

Thus, it seems that the RIC (or cantori if it has already broken) with Diophantine rotation number $[0, 9, \dot{1}]$ is the structure that limits the transport the most on a neighborhood of the rotation number $[0, 9, \dot{1}]$. However, the RIC (or cantori) with Diophantine rotation number $[0, 9, \dot{2}]$ is not the most limiting structure on a neighborhood of $[0, 9, \dot{2}]$.



(a) Limit rotation number $\vartheta = \lfloor 0, 9, \dot{1} \rfloor$.

(b) Limit rotation number $\vartheta = \lfloor 0, 9, \dot{2} \rfloor$.

Figure 5.9: $\log D^{(p,q)}$ versus p/q on the Farey tree close to different limit rotation numbers for $b = 9/10$, $\epsilon = 10^{-5}$, and $n = 3$.

5.4 Limit to rational rotation numbers

In this section we present some figures that show the behavior of $D^{(p,q)}$ when p/q tends to P/Q, with $0 < P < Q/2$ and $\gcd(P, Q) = 1$.

In Chapter 3, we have found an exponentially small upper bound for $\Delta^{(p,q)}$ when the (p, q)-periodic orbits are sufficiently close to a (P, Q)-resonant RIC on an analytic exact twist map. However, Ramírez-Ros in [Ram05] (resp., and Pinto-de-Carvalho in [PR13]) proved that resonant RICs on the circle (resp., ellipse) do not persist under generic perturbations. In fact, a sufficient condition so that a (p, q)-resonant RIC breaks under perturbation is that $\Delta_1^{(p,q)} \neq 0$. Namely, there are no (p, q)-resonant RICs on the tables with $n = 4$ or with $n = 3$ and odd q on perturbed ellipses. See Propositions 22 and 36.

Recall that a (P, Q)-resonance can not contain orbits with a fix rotation number p/q different from P/Q. Indeed, all the orbits with points inside a (P, Q)-resonance are either orbits that do not have a rotation number (they stay in the resonance for a certain time and then escape elsewhere) or are orbits that have the same rotation number (they always stay inside the resonance). Thus, any sequence of (p, q)-periodic orbits such that $p/q \to P/Q$ approaches the limits of the (P, Q)-resonance area on the phase space.

To illustrate this phenomena, we present some numerical results on two different (P, Q)-resonances. It seems that the behavior of $D^{(p,q)}$ is qualitatively equal close to any (P, Q)-resonance. We study the case of the $(1, 8)$-resonance in Figure 5.10. We show the case of (p, q)-periodic orbits close to the $(3, 20)$-resonance in Figure 5.11.

An initial way to approach the (P, Q) resonance is to consider (p, q)-periodic orbits such that $(p, q) \in S_{P/Q}(0, 2)$ and $\gcd(p, q) = 1$. Recall that for any q, there exists two pairs belonging to $S_{P/Q}(0, 2)$: (p, q) where p is such that p/q best approximates P/Q from below and $(p + 1, q)$. In Figures 5.10(a) and 5.10(b), we show $-\log D^{(p,q)}$ versus q for

$P/Q = 1/8$. In Figures 5.11(a) and 5.11(b), we show $-\log D^{(p,q)}$ versus q for $P/Q = 3/20$. We depict the (p,q)-periodic orbits such that $p/q < P/Q$ (resp. $p/q > P/Q$) in red (resp., in blue).

The maximal differences of lengths seem to belong to some distinguished curves. In Figures 5.10(a) and 5.10(b), we see that the different curves seem to meet for q large enough. In Figures 5.11(a) and 5.11(b), we do not see it (we need to compute $D^{(p,q)}$ for larger values of q) but we believe that the pattern is the same. We are not sure whether the cloud of points that appears for q large enough ($q > 2250$) in 5.10(b) is a numerical error or not. We do not have an explanation but its appearance strangely coincides with the moment the second group of curves meets with the first one.

The different curves we can see depend on the arithmetic properties of the pair (p,q). For instance, these curves come in pairs (blue-red pairs). Besides, consider the curves that have the largest values of $D^{(p,q)}$. That is, the curves that first behave as "$-\log D^{(p,q)} =$ constant" in the figures. Let us write P/Q as its finite continued fraction expansion. Say $P/Q = [a_0, a_1, \ldots, a_j]$, with $a_j \neq 1$. The pairs (p,q) in these particular curves belong to one of these two sequences:

$$p_k/q_k = [a_0, a_1, \ldots, a_j, k], \quad \text{and} \quad p_k/q_k = [a_0, a_1, \ldots, a_j - 1, 1, k]. \tag{5.1}$$

Both sequences tend to P/Q and are monotone. One is monotonically increasing and the other is monotonically decreasing.

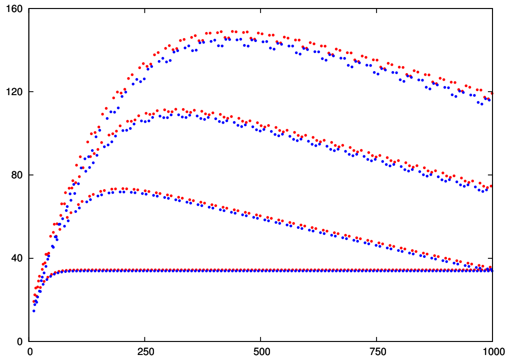
For $P/Q = 1/8 = [0,8]$, the monotonically increasing sequence is $[0,8,k]$ and the monotonically decreasing one is $[0,7,1,k]$. For $P/Q = 3/20 = [0,6,1,2]$, the increasing sequence is $[0,6,1,2,k]$ and the decreasing one is $[0,6,1,1,1,k]$.

In [MMP84], MacKay, Meiss, and Percival already propose to take these sequences p_k/q_k to construct a sequence of (p_k,q_k)-periodic orbits that approach the (P,Q)-resonances. In fact, they use these (p_k,q_k)-periodic orbits to numerically compute the areas of the lobes enclosed by the separatrices of the (P,Q)-hyperbolic periodic points.

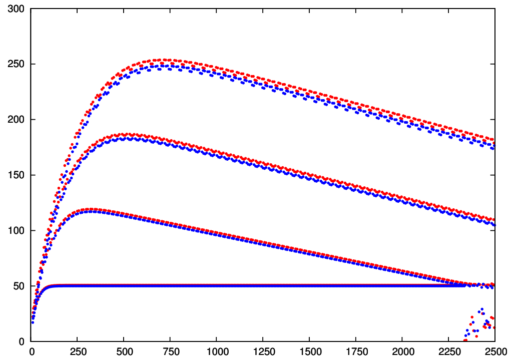
In Figures 5.10(c), 5.10(d), 5.11(c) and 5.11(d), we show $-\log D^{(p_k,q_k)}$ versus q_k for the sequences (p_k,q_k) defined by (5.1). We observe that we obtain two different values of $D^{(p_k,q_k)}$ depending on which sequence we consider. This behavior is already observed in [MMP84].

Finally, in Figures 5.10(e) and 5.10(f), we partially show the phase space that shapes the $(1,8)$-resonance on a perturbed circle and a perturbed ellipse respectively. We mark the elliptic periodic points with solid circles and the hyperbolic periodic points with crosses in the phase space. We show the two $(1,8)$-periodic orbits in green. We draw some orbits inside the $(1,8)$-resonance in black. Finally, we show the two $(51,407)$-periodic orbits (in blue) and the two $(50,401)$-periodic orbits (in red). Their rotation numbers, $50/401 = [0,8,50]$ and $51/407 = [0,7,1,50]$, belong to the monotone sequences (5.1) for $P/Q = 1/8$. Observe that most of the $(51,407)$-periodic points and the $(50,401)$-

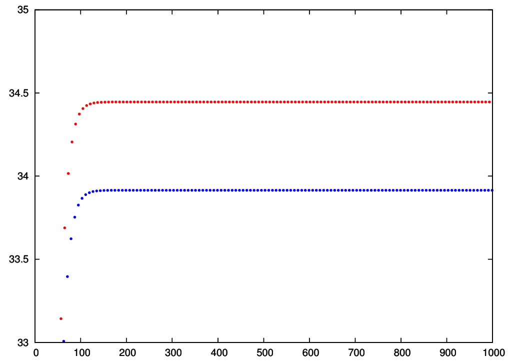
periodic points accumulate to the hyperbolic $(1,8)$-periodic points. So we essentially see the same figure when taking other (p_k, q_k)-periodic orbits with p_k/q_k of the form (5.1).



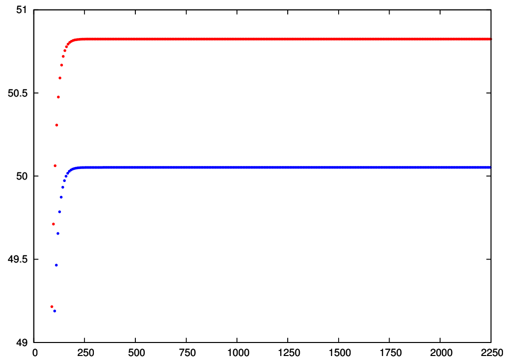
(a) $-\log D^{(p,q)}$ versus q for $b = 1$ and $(p,q) \in S_{1/8}(0,2)$ such that $\gcd(p,q) = 1$. Red: $p/q < 1/8$. Blue: $p/q > 1/8$.



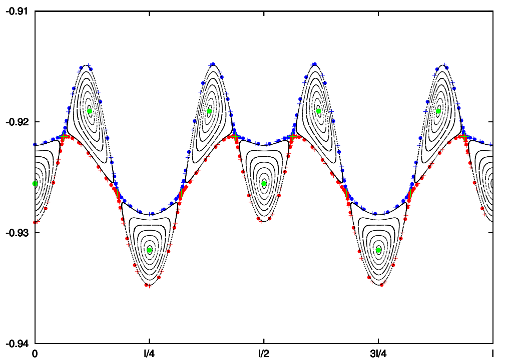
(b) $-\log D^{(p,q)}$ versus q for $b = 0.9$ and $(p,q) \in S_{1/8}(0,2)$ such that $\gcd(p,q) = 1$. Red: $p/q < 1/8$. Blue: $p/q > 1/8$.



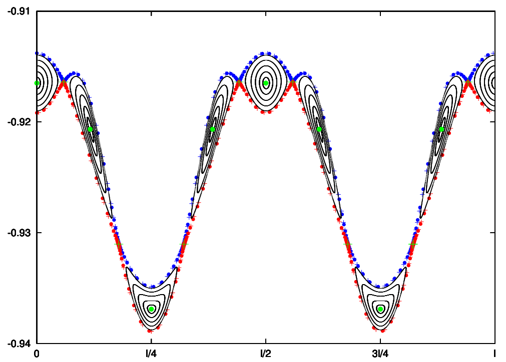
(c) $-\log D^{(p_k,q_k)}$ versus q_k for $b = 1$. Red: $p_k/q_k = [0,8,k]$. Blue: $p_k/q_k = [0,7,1,k]$.



(d) $-\log D^{(p_k,q_k)}$ versus q_k for $b = 0.9$. Red: $p_k/q_k = [0,8,k]$. Blue: $p_k/q_k = [0,7,1,k]$.

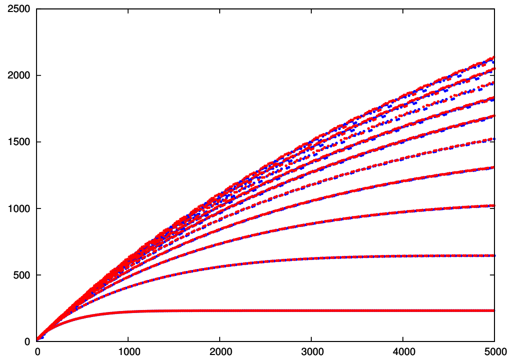


(e) Partial phase space in coordinates (s, I), $I = -\cos r$, for $b = 1$. Red: $(50, 401)$-periodic orbits. Blue: $(51, 407)$-periodic orbits. Green: $(1,8)$-periodic orbits.

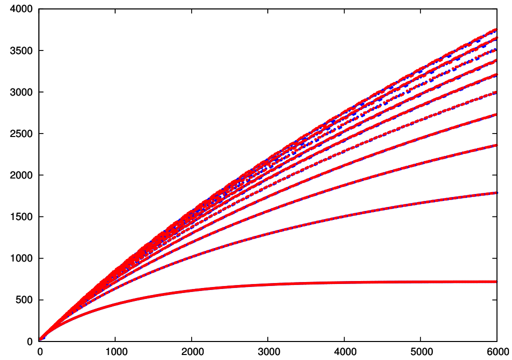


(f) Partial phase space in coordinates (s, I), $I = -\cos r$, for $b = 0.9$. Red: $(50, 401)$-periodic orbits. Blue: $(51, 407)$-periodic orbits. Green: $(1,8)$-periodic orbits.

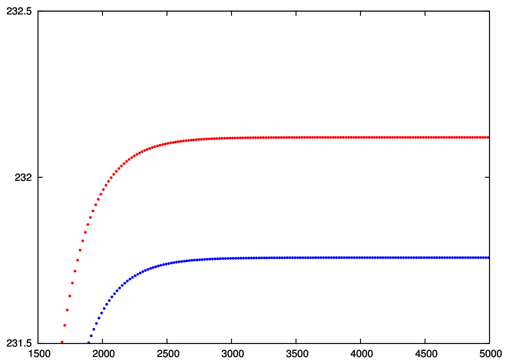
Figure 5.10: Study close to the $(1,8)$-resonance when $n = 4$ and $\epsilon = 1/10$.



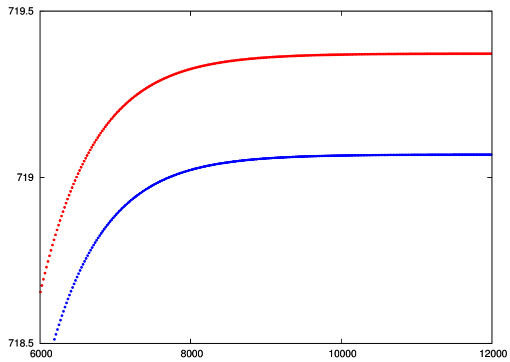
(a) $-\log D^{(p,q)}$ versus q for $b = 1$ and $(p,q) \in S_{3/20}(0,2)$ such that $\gcd(p,q) = 1$. Red: $p/q < 3/20$. Blue: $p/q > 3/20$.



(b) $-\log D^{(p,q)}$ versus q for $b = 0.9$ and $(p,q) \in S_{3/20}(0,2)$ such that $\gcd(p,q) = 1$. Red: $p/q < 3/20$. Blue: $p/q > 3/20$.



(c) $-\log D^{(p_k,q_k)}$ versus q_k for $b = 1$. Red: $p_k/q_k = [0,6,1,1,1,k]$. Blue: $p_k/q_k = [0,6,1,2,k]$.



(d) $-\log D^{(p_k,q_k)}$ versus q_k for $b = 0.9$. Red: $p_k/q_k = [0,6,1,1,1,k]$. Blue: $p_k/q_k = [0,6,1,2,k]$.

Figure 5.11: Study close to the $(3,20)$-resonance when $n = 4$ and $\epsilon = 1/10$.

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