Matroids: h-vectors, Zonotopes, and Lawrence Polytopes

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Doctoral program in Applied Mathematics

Matroids: h-vectors, Zonotopes, and Lawrence Polytopes

Thesis submitted by

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for the degree of Doctor of Mathematics in the

Universitat Politècnica de Catalunya

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Barcelona, January 2015

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This thesis is dedicated to the memory of the dead who made it possible and is written for the living, who make it worthwhile.

Acknowledgements

Many people have been instrumental in seeing this dissertation come to fruition and it is with great pleasure that I acknowledge them here.

First I would like to thank my advisor, Julian Pfeifle, for sharing his time and mathematical ability with me. His enthusiasm and good spirits (sometimes in the face of much hardship), as well as his guidance in all things mathematical, have been invaluable to my development, both personal and professional, during my years in Barcelona.

Next I extend my gratitude to the following mathematicians: Emanuele Delucci, Raman Sanyal, and Francisco Santos for their helpful comments and for agreeing to sit on my thesis committee; Eric Miranda and Arnau Padrol for introducing me to matroid theory, as well as Anna de Mier for sharing her insights on the subject; and Matthias Beck for sharing his love of polytopes and Ehrhart theory with me in my early mathematical days in San Francisco. I would also like to thank Mareike Massow and the late Ferran Hurtado for all their help in making my initial move to Barcelona possible.

Special dissertation production thanks go out to Liza Cox and Cristina Tomàs for applying keen editorial eyes to a late draft, to Laura Hernández for the translation help, and to *die Familie Kohlweis* for their generosity with the printing.

For helping make the unbearable days bearable and the bland days delicious, I thank all of the people who have shared so much with me during my stay in Barcelona including Léa Mevel, Steven Beijer, Eliza Kalfa, Sandra Unterweger, Fee Reeger and the whole Canvis Nous crew. Most of all, I'd like to thank Anna Kohlweis for all of the love, the laughter, and the *brezeling*.

Finally, I would like to thank my family for their strong love and support that consistently reaches me no matter the distances between us. In particular, I thank my parents, Norbert and Stevie, for showing me the value of an inquisitive and critical mind and for encouraging me to follow mine where it leads.

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Chapter 0

Summary

The main objects of study in this thesis are matroids. In particular we will be interested in three particular classes. They are

1. ***regular matroids***, i.e, any matroid representable (over any field) by a unimodular matrix;

2. ***arithmetic matroids***, i.e., a pair consisting of matroid and a multiplicity function; and

3. ***internally perfect matroids***, i.e., a matroid whose internal order poset is sufficiently uniform (see below).

Regular matroids are well-known objects with a wide array of applications. Arithmetic matroids are relatively new structures that simultaneously capture combinatorial and geometric invariants of rational vector configurations. We introduce the class of perfect matroids to enable us to use the structure of the internal order of such a matroid to prove Stanley's conjecture that (under a certain assumption) any h-vector of a matroid is a pure $\mathcal{O}$-sequence in this case.

The thesis is structured as follows. In Chapter 1 we fix notation, define basic objects, and provide results from the literature which we will in later chapters.

In Chapter 2 we give a new proof of a generalization of Kirchoff's matrix-tree theorem. Recall that the classical matrix-tree theorem for graphs states that, for a graph G on n vertices, the number of spanning trees of G is equal to the product of the nonzero eigenvalues of the Laplacian matrix of G divided by n. This theorem generalizes to regular matroids as follows. Let $\mathcal{M}$ be a regular matroid on n elements of rank r represented by an $r \times n$ unimodular matrix M and let $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$ be the Laplacian of $\mathcal{M}$. Then n times the number of bases of $\mathcal{M}$ is equal to the product of the eigenvalues of L.

The classical proof of the matrix tree theorem relies on the Cauchy-Binet theorem from linear algebra, and a similar proof holds in the case of regular matroids. Recently, a proof has been given using divisor theory on tropical curves. In contrast, our proof avoids both the Cauchy-Binet theorem and divisor theory by recasting the problem into the world of polyhedral geometry via the zonotopes $\mathsf{Z}(\mathsf{M})$ and $\mathsf{Z}(\mathsf{L})$ generated by M and L, respectively. We show that the volumes of these zonotopes are equal by providing an explicit bijection between the points in them (up to a set of measure zero). This is the content of Theorem 2.3. We then generalize to the weighted case by proving that the volumes of $\mathsf{Z}(\mathsf{M}D\mathsf{M}^\top)$ and $\mathsf{Z}(\mathsf{M}D)$ are equal, where D is a diagonal matrix with entries in $\mathbb{R}$. We conclude by showing that our technique can be used to reprove the classical matrix-tree theorem by working out the details when the matrix M has rank-plus-one many rows. This chapter is joint work with Julian Pfeifle.

In Chapter 3 we exploit a well-known connection between the zonotope $\mathsf{Z}(\mathcal{A})$ and Lawrence polytope $\Lambda(\mathcal{A})$ generated by an integer representation $\mathcal{A}$ of a rational matroid $\mathcal{M}$ to prove relations between various polynomials associated to them. First we prove in Theorem 3.7 that the Ehrhart polynomial $\mathcal{E}_{\mathcal{A}}(k)$ of $\mathsf{Z}(\mathcal{A})$ and the numerator of the Ehrhart series $\delta_{\Lambda(\mathcal{A})}(k)$ of $\Lambda(\mathcal{A})$ are related via

$$\mathcal{E}_{\mathcal{A}}(k) = \sum_{i=0}^{r} \delta_i t^i (t+1)^{r-i},$$

where r is the rank of $\mathcal{A}$ and δ_i is the coefficient on k^i in the standard basis representation. On the level of arithmetic matroids, this relation allows us to view the δ-polynomial of the Lawrence polytope $\Lambda(\mathcal{A})$ as the arithmetic matroid analogue of the usual matroid h-vector of $\mathcal{A}$. After proving the previous result, we use it to give a new interpretation of the coefficients of the arithmetic Tutte polynomial $\mathcal{T}_{\mathcal{A}}(x, y)$ of $\mathcal{A}$ evaluated at $y = 1$ by showing that $\mathcal{T}(x, 1) = \sum_{i=0}^{d} \delta_i x^{d-i}$. Finally, we give a new proof that the h-vector of the matroid $\mathcal{M}(\mathcal{A})$ and the vector $\delta_{\Lambda(\mathcal{A})}$ coincide when $\mathcal{A}$ is unimodular.

In Chapter 4 we consider a new class of matroids consisting of those matroids whose internal order makes them especially amenable to proving Stanley's conjecture. Stanley's conjecture states that for any rank r matroid $\mathcal{M}$ with h-vector $(h_0, h_1, \ldots, h_r)$ there is a pure order ideal $\mathcal{O} \subset \mathbb{N}^s$ (for some $s \in \mathbb{N}$) such that the number of elements in $\mathcal{O}$ with coordinate sum i is precisely h_i. While this conjecture has been proven for various classes of matroids (e.g., cographic matroids, cotransversal matroids, matroids with rank at most 4) the general case is still open. We give a brief review of known results in Section 4.1 before turning to ordered matroids and the internal order in Section 4.2.

An ***ordered matroid*** is a matroid $\mathcal{M}$ together with a linear ordering on the ground set such that the first r elements form a basis. Let $\mathcal{M}$ be a rank r ordered matroid on n elements and identify the ground set with with $[n]$. For a basis B write $\mathrm{IA}(B)$ for the internally active elements of B and $\mathrm{IP}(B) = B \setminus \mathrm{IA}(B)$ for the internally passive elements. The internal order poset $\mathcal{P}(\mathcal{B}, \prec)$ of $\mathcal{M}$ with respect to the fixed linear ordering is the poset on the bases of $\mathcal{M}$ with $B \prec B'$ if and only if $\mathrm{IP}(B) \subset \mathrm{IP}(B')$.

For a basis B we write $B = (S, T, A)$ where the sets S, T, and A are defined as follows:

$$\begin{aligned} S &= \mathrm{IP}(B) \setminus [r]; \\ T &= \mathrm{IP}(B) \cap [r]; \\ A &= \mathrm{IA}(B). \end{aligned}$$

Such a basis is called ***internally perfect*** if the set T decomposes into a disjoint union of sets that allow us to assign to each element $t \in T$ a unique element $s \in S$ in a way that is congruous with the internal order of the ordered matroid. An ordered matroid is ***internally perfect*** if all of its bases are. In Section 4.3 we first prove preliminary results about internally perfect bases culminating in Theorem 4.11 in which we show that, under a certain assumption, any internally perfect matroid satisfies Stanley's conjecture. Moreover, we conjecture that the assumption in the previous sentence holds for all internally perfect matroids.

Chapter 1

Preliminaries

1.1 Basic Notation

Throughout this thesis we let $\mathbb{C}, \mathbb{R}$ and $\mathbb{Q}$ denote the fields of complex, real, and rational numbers respectively. An arbitrary field will be denoted $\mathbb{F}$. The ring of integers and the semigroup of natural numbers will be denoted by $\mathbb{Z}$ and $\mathbb{N} = \mathbb{Z}_{\geq 0}$. For $n \in \mathbb{N}_{\geq 0}$ write $[n] := \{1, 2, \ldots, n\}$. Given two sets, S and T, we write $S \sqcup T$ for their disjoint union.

Let R be a ring and M be an R-module. We will typically denote the elements in M in boldface and, given a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ for M and an element $\mathbf{v} \in M$, we write $\mathbf{v} = (v_1, \ldots, v_d)$. The only exception to this notation is when we discuss polynomials. In this case we assume that $R = \mathbb{F}$ is a field and write $R[x_1, \ldots, x_d]$ for the polynomial ring in d indeterminates. When no confusion can arise we write $S = R[\mathbf{x}]$ for $R[x_1, \ldots, x_d]$ and, for $p \in S$ we write $p = p(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{N}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ where $c_{\mathbf{a}} = 0$ for all but finitely many elements of $\mathbb{N}^d$. The coefficient $c_{\mathbf{a}}$ will be written $[p(\mathbf{x})]_{\mathbf{a}}$.

The (Euclidean) inner product of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}^d$ is

$$\langle \mathbf{v}, \mathbf{w} \rangle := \sum_{i=1}^{d} v_i w_i.$$

Given a vector $\mathbf{v} = (v_1, v_2, \ldots)$ in a (possibly infinite dimensional) vector space, the ***generating function*** of $\mathbf{v}$ is the univariate formal power series defined by $v(t) := \sum_{i>0} v_i t^i$.

1.2 Matrices, Posets, and Graphs

Matrices

Let $\mathbb{F}$ be a field and let M be an $m \times n$ matrix with entries in $\mathbb{F}$. The row space of M will be denoted by $\langle M \rangle_{\mathbb{F}}$ and the column space of M by ${}_{\mathbb{F}}\langle M \rangle \cong \langle M^\top \rangle_{\mathbb{F}}$, where $M^\top$ is the transpose of M. We write $\operatorname{rank}(M)$ and $\operatorname{corank}(M)$ for the rank and corank of M, respectively. When $\mathbb{Z} \subset \mathbb{F}$ we write $\langle M \rangle_{\mathbb{Z}}$ for the integer span of the rows. For a vector $\mathbf{v} \in \mathbb{F}^m$ we write $M' = [M|\mathbf{v}]$ for the matrix whose first n columns are the columns of M and whose last column is $\mathbf{v}$. In particular, a vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{F}^n$ is identified with the $n \times 1$ matrix whose i^{th} row is v_i. More generally, if N is an $m \times n'$ matrix, then $[M|N]$ denotes the concatenation of M and N. Given a subset $S \subset [n]$, we write M_S for the matrix consisting of those columns of M indexed by S. When $S = \{i\}$ is a singleton, we write M_i in place of $M_{\{i\}}$.

The matrix M is an ***integer matrix*** if all of its entries are in $\mathbb{Z}$. An integer matrix M of rank r is ***unimodular*** if all of its nonzero $r \times r$ minors are equal to ± 1, and is ***totally unimodular*** if all of its minors are in the set $\{-1, 0, 1\}$. Note that if M is a (totally) unimodular matrix then so are the matrices $[M|\mathbf{0}]$, $[M|\mathbf{e}_i]$, and $[M|M_j]$, where $\mathbf{0}$ is a column of zeros, $\mathbf{e}_i$ is the i^{th} standard unit vector, and M_j is the j^{th} column of M.

A ***Gale dual*** of M is any matrix N that makes the following sequence exact

$$0 \longrightarrow \mathbb{F}^{m-n} \xrightarrow{N} \mathbb{F}^n \xrightarrow{M} \mathbb{F}^m \longrightarrow 0.$$

Equivalently, a Gale dual of M is the transpose of a matrix whose columns are a basis for the kernel of M.

Posets

We now give the basic terminology for partially ordered sets, essentially following [62]. A ***partially ordered set***, or ***poset***, is a pair $\mathcal{P} = (S, \preccurlyeq)$ consisting of a ground set S (usually assumed to be finite) and a binary relation $\preccurlyeq$ that is reflexive, anti-symmetric, and transitive. Symbolically, the pair $\mathcal{P} = (S, \preccurlyeq)$ is a poset if

1. $x \preccurlyeq x$ for all $x \in S$;
2. if $x, y \in S$ with $x \preccurlyeq y$, then $y \not\preccurlyeq x$; and
3. if $x, y, z \in S$ with $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$.

The symbols $\prec, \succeq$ and $\succ$ are taken to have the obvious meanings. Two posets $\mathcal{P}_1 = (S_1, \preccurlyeq_1)$ and $\mathcal{P}_2 = (S_2, \preccurlyeq_2)$ are ***isomorphic*** if there is an order-preserving bijection $\phi : S_1 \to S_2$ on their ground sets such that ϕ^{-1} is also order-preserving.

Let $\mathcal{P} = (S, \preceq)$. For $x, y \in S$, y is said to ***cover*** x if $x \prec y$ and there is no $z \in S$ such that $x \prec z \prec y$. In this case we write $x \lessdot y$. The poset $\mathcal{P}$ is ***graded*** if there is a function $\mathrm{ht} : \mathcal{P} \to \mathbb{N}$ such that the following two conditions hold:

1. if $x \prec y$, then $\mathrm{ht}(x) < \mathrm{ht}(y)$; and
2. if $x \lessdot y$, then $\mathrm{ht}(x) = \mathrm{ht}(y) + 1$.

Given a subset $T \subseteq S$, a ***lower bound*** for T in $\mathcal{P}$ is an element $x \in S$ such that $x \preceq t$ for all $t \in T$. A lower bound x of T is the ***meet*** of T if it is $\prec$-greater than all other lower bounds of T. Similarly, x is an ***upper bound*** for T if it is $\prec$-greater than all elements of T. Moreover, x is the ***join*** of T if it is $\prec$-greater than all other upper bounds of T. In general, a subset $T \subseteq S$ may have neither a meet nor a join. In the case when every subset $T \subseteq S$ has both a meet and a join, the poset $\mathcal{P}$ is called a ***lattice***.

We pause here to advise the reader to use caution when encountering the word "lattice" in this thesis as it may refer to either a poset with the property just mentioned or to a finitely-generated free abelian subgroup of $\mathbb{R}^d$ (see Section 1.7). Though the intended meaning will usually be clear from the context, every effort will be made to be explicit whenever confusion may arise.

The ***Hasse diagram*** of a poset $\mathcal{P} = (S, \prec)$ is a pictorial representation of $\mathcal{P}$ in which one node is drawn for each element of the ground set and, given any two nodes m, n, m is below n in the Hasse diagram if $m \prec n$ and there is an edge connecting m and n if $m \lessdot n$. For example, in Figure 1.1 the Hasse diagram is given of a poset of a family S of subsets of [8] ordered by inclusion.

Given two posets $\mathcal{P}_1 = (S_1, \prec_2)$ and $\mathcal{P}_2 = (S_2, \prec_2)$, their ***product*** is the poset $\mathcal{P}_1 \times \mathcal{P}_2$ whose ground set $S = \{(x, y) \mid x \in S_1, y \in S_2\}$ is ordered by $(x_1, y_1) \prec (x_2, y_2)$ if $x_1 \prec_1 x_2$ and $y_1 \prec_1 y_2$.

Graphs

A graph $G = (V, E)$ consists of a finite set of vertices V together with a finite multiset of edges E consisting of pairs of elements in V. We assume that both V and E are ordered, and if V has n elements we identify it with the set $[n] := \{1, \ldots, n\}$ with the usual ordering. An edge $\{v, w\}$ in a graph G

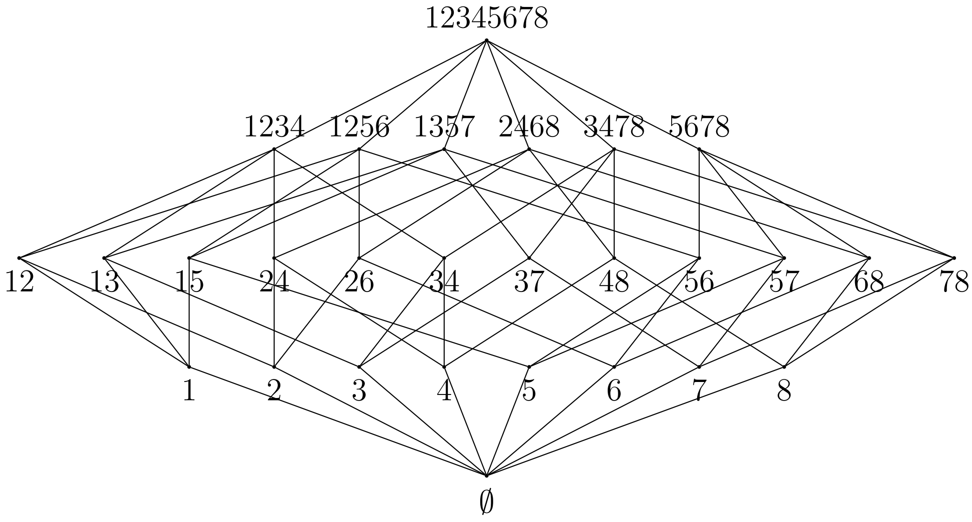


Figure 1.1: the Hasse diagram of a lattice $(S, \subseteq)$

will be written vw (or wv). A ***weighted graph*** is a graph G together with an assignment of a weight (a real number in this thesis) to each edge. An ***orientation*** of G assigns a direction to each edge, i.e., assigns to each edge vw one of two ordered pairs (v, w) or (w, v), where the first (respectively, second) element is called the ***tail*** (respectively, ***head***) of the edge. A graph with an orientation is called a ***directed graph***, or digraph for short. If G is a digraph and $e = (v, w)$ is an edge in G directed from v to w, then we write $e = \overrightarrow{vw}$.

Let $G = (V, E)$ be a graph and $V = \{v_1, \ldots, v_n\}$. The ***degree*** of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges $e \in E$ such that $v \in e$. The ***degree sequence*** of G is the vector $\delta(G) \in \mathbb{N}^n$ whose i^{th} entry is $\deg(v_i)$. A ***loop*** in G is an edge of the form $\{v, v\}$ and two edges are ***parallel*** if they are equal (as sets). A graph $H = (V', E')$ is a ***subgraph*** of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. For two vertices $v, w \in G$, a ***path*** from v to w in G is a subgraph P with vertices

$$V(P) = \{v = u_0, u_1, \ldots, u_{n-1}, w = u_n\} \subset V(G)$$

and edges $E(P) = \{e_1, \ldots, e_n\}$ where $e_i = u_{i-1}u_i$. A ***cycle*** in G is a collection of edges $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_0\}$ such that $v_i \neq v_j$ for all $i \neq j$. The graph G is ***connected*** if there is a path between any two vertices v and w. A ***tree*** is a connected, cycle-free graph and a ***forest*** is any disjoint union of trees. A ***spanning tree*** T of a connected graph G is a tree of G such that $V(T) = V(G)$.

In algebraic graph theory one studies the graph invariants encoded in matrices defined from a given graph G. Let $G = (V, E)$ with $V = [n]$ and $|E| = m$. The ***degree matrix*** of G, written $D(G)$, is the $n \times n$ diagonal matrix with the degree sequence $\delta(G)$ on the main diagonal. The ***adjacency matrix*** $A = A(G)$ is the $n \times n$ matrix with $A_{ij} = 1$ if $ij \in E(G)$ and 0 otherwise. The ***characteristic polynomial*** of G is the characteristic polynomial of the adjacency matrix A. The ***Laplacian*** $L = L(G)$ of G is the $n \times n$ matrix $L := D - A$. Given some fixed orientation of G, the ***vertex-edge incidence matrix*** $\mathsf{N}(G)$ (with respect to the orientation) is the matrix with one row for each vertex, one column for each edge, and where the entry $\mathsf{N}_{v,e}$ is 1 (respectively, -1) if v is the head (resp. tail) of e and 0 otherwise. It is easy to prove that for any orientation of G we have the following relationship between the Laplacian and the vertex-edge incidence matrix: $\mathsf{L} = \mathsf{N}\mathsf{N}^\top$. If G is a weighted digraph with edge weights $\omega = (w_1, \ldots, w_m)$ then the ***weighted incidence matrix*** of G is $\mathsf{N}_w(G) := \mathsf{N}(G)D$, where D is the $m \times m$ diagonal matrix with $D_{ii} = w_i$. The ***weighted Laplacian*** is then $\mathsf{L}_w := \mathsf{N}_w\mathsf{N}^\top = \mathsf{N}D\mathsf{N}^\top$.

1.3 Simplicial Complexes

An (abstract) ***simplicial complex*** on the ground set $[n]$ is a nonempty collection Δ of subsets of $[n]$, called the ***faces*** of Δ, such that

1. if $S \in \Delta$ and $T \subset S$, then $T \in \Delta$, and
2. if $S, S' \in \Delta$, then $S \cap S' \in \Delta$.

The ***dimension*** of a face S of a simplicial complex Δ is the cardinality of S. If $S \in \Delta$ has dimension k, then we say that S is a k-face of Δ. The ***dimension*** of Δ is the maximal dimension of a face in Δ. The ***empty complex*** is the simplicial complex on $[n]$ having no faces. Any non-empty simplicial complex has a unique face of dimension 0 corresponding to the empty set. The ***vertices*** of Δ are the 1-faces of Δ, i.e., those subsets of Δ consisting of a single element. The ***facets*** of Δ are those subsets not properly contained in any other face.

Two simplicial complexes, Δ and Δ', are ***isomorphic*** if there is a bijection ϕ between their ground sets such that $S, T \in \Delta$ with $S \subset T$ if and only if $\phi(S) \subset \phi(T)$ in Δ'. We now discuss some well-known combinatorial invariants of a d-dimensional simplicial complex Δ.

The first is the ***f-vector*** of Δ, written

$$f = f(\Delta) := (f_0, f_1, \ldots, f_d) \in \mathbb{N}^{d+1},$$

where f_i is the number of i-faces of Δ. The literature on f-vectors is extensive (see [61] and the references within), with a central result being the famous Kruskal-Katona theorem characterizing those vectors that occur as f-vectors of simplicial complexes; see [33, 36].

The next invariant we consider is the ***h*-vector** of Δ. This is the vector $h(\Delta) \in \mathbb{Z}^d$ defined via the equation

$$\sum_{i=0}^{d} f_i (t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}. \tag{1.1}$$

It is clear from the definition that the f-vector and h-vector determine one another. While the f-vector always has positive entries (except when Δ is the empty complex), the h-vector may have entries less than or equal to zero. To see this observe that obtaining the h-vector from the f-vector using (1.1) is equivalent to computing the image of the f-vector under the linear automorphism given by the $(d+2 \times d+2)$-matrix M with entries

$$M_{ij} = (-1)^{i-j} \binom{d-j}{d-i}.$$

It follows that the inverse of M has entries $M_{ij}^{-1} = \binom{d-j}{d-i}$ and for small or simple examples one can compute one of the vectors from the other by hand. If we consider, for example, the d-dimensional simplicial complex Δ consisting of all subsets of $[d+1]$, we see that $f_i(\Delta) = \binom{d}{i} = M^{-1}\mathbf{e}_1$ and so $h(\Delta)$ has $h_0 = 1$ and $h_i = 0$ for all $1 \le i \le d+1$.

The entries of the h-vector of a d-dimensional simplicial complex Δ appear as the coefficients of the numerator of the Hilbert series of the Stanley-Reisner ring of Δ; see Chapter 1 of [46]. More explicitly, let Δ be a simplicial complex on $[n]$ and identify each $S \in \Delta$ with its indicator vector $\sigma(S) \in \{0,1\}^n$. Fix a field $\mathbb{F}$ and let $R = \mathbb{F}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{F}$ in n indeterminates. Letting $\mathbf{x}$ be the monomial $x_1 \cdots x_n$, we obtain a square-free monomial $\mathbf{x}^{\sigma(T)}$ for every subset $T \subset [n]$. The ***Stanley-Reisner ideal*** of the simplicial complex Δ, denoted I_Δ, is the square-free monomial ideal generated by the nonfaces of Δ:

$$I_\Delta := \langle \mathbf{x}^\tau \mid \tau \notin \Delta \rangle.$$

The quotient ring R/I_Δ is the ***Stanley-Reisner ring*** of Δ. It follows that the (coarsely graded) Hilbert series of R/I_Δ is the rational function

$$\begin{aligned} H(R/I_\Delta;t) &= \frac{1}{(1-t)^n}\sum_{i=0}^{d} f_i t^i (1-t)^{n-i} \\ &= \frac{1}{(1-t)^d}\sum_{i=0}^{d} h_i x^i, \end{aligned}$$

where f_i and h_i are the i^{th} entry in the f- and h-vector of Δ, respectively. We call the polynomial $\sum_{i=0}^{d} h_i x^i$ the h***-polynomial*** of Δ.

A simplicial complex Δ on $[n]$ is ***pure*** if every facet of Δ has the same dimension. A ***shelling*** of a pure simplicial complex is a linear ordering $F_1, F_2, \ldots, F_k$ of the facets of Δ such that the complex $(\cup_{i<j} F_i) \cap F_j$ is a pure $(\dim(F_j)-1)$-dimensional simplicial complex for all $j = 2, 3, \ldots, k$. A simplicial complex is ***shellable*** if it admits a shelling.

The two types of simplicial complexes we will encounter often in this thesis are matroid complexes (see Section 1.4) and the faces of a triangulation of a polytope (see Section 1.6). Both types of complexes are trivially pure and, while matroid complexes are generally shellable [8], there exist polytopes having non-shellable triangulations [51, 72].

1.4 Matroids

Matroids and oriented matroids play central roles in this thesis. In the next two sections we give the pertinent definitions and facts about matroids and oriented matroids, essentially following [49] and [9], respectively. In this section we handle the unoriented case.

A ***matroid*** $\mathcal{M} = (E, \mathcal{I})$ is an ordered pair consisting of a finite ground set E and a collection $\mathcal{I}$ of subsets of E that satisfy the following ***independent set axioms***:

I1 $\emptyset \in \mathcal{I}$;

I2 $\mathcal{I}$ is closed with respect to taking subsets; and

I3 if $I_1, I_2 \in \mathcal{I}$ with $|I_1| \leq |I_2|$, then there is some $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

An ***ordered matroid*** is a matroid $\mathcal{M}$ together with a linear ordering $<$ of its ground set E. The set of ***bases*** $\mathcal{B} = \mathcal{B}(\mathcal{M})$ of a matroid $\mathcal{M}$ is the subset of $\mathcal{I}$ consisting of independent sets of maximal size. Moreover, every subset of a basis of a matroid $\mathcal{M}$ is an independent set, and every independent set is a subset of some basis.

A subset of E that is not independent in $\mathcal{M}$ is a ***dependent set*** and a dependent set that is minimal with respect to inclusion is a ***circuit***. The set of all circuits of $\mathcal{M}$ will be denoted $\mathcal{C} = \mathcal{C}(\mathcal{M})$. A matroid $\mathcal{M}$ is ***connected*** if, for every pair of elements $e \neq f$ in E, there is a circuit $C \in \mathcal{C}$ containing both. A ***loop*** is a circuit consisting of one element. If two elements $e, f \in E$ form a two-element circuit then they are said to be ***parallel***. A maximal collection of elements of E containing no loops such that the elements are pairwise parallel in $\mathcal{M}$ is called a ***parallel class*** of $\mathcal{M}$.

The ***rank*** of a subset $S \subseteq E$, denoted $\operatorname{rank}_{\mathcal{M}}(S)$, is the cardinality of the maximal independent set of $\mathcal{M}$ contained in S. It is easy to see from the definition that every basis of $\mathcal{M}$ has the same rank r, called the ***rank*** of $\mathcal{M}$ and written $\operatorname{rank} \mathcal{M}$. When the matroid under consideration is clear from the context, we typically drop it from the notation.

Let $\mathcal{M}$ be a rank r matroid. It follows from the independent set axiom **I2** that the set of independent sets of $\mathcal{M}$ form a simplicial complex $\Delta(\mathcal{M})$ on E, called the ***matroid*** (or ***independence***) ***complex*** of $\mathcal{M}$. The dimension of the matroid complex $\Delta(\mathcal{M})$ is the rank of $\mathcal{M}$. The ***f-vector*** and ***h-vector*** of a matroid $\mathcal{M}$ are the f- and h-vector of its matroid complex, respectively. It is easy to see that the matroid complex of any matroid is pure, and with more work one can show it is shellable [8]. The ***shelling polynomial*** of $\mathcal{M}$ is the polynomial

$$h_{\Delta(\mathcal{M})}(x) = \sum_{i=0}^{r} h_i x^{r-i},$$

where $r = \operatorname{rank} \mathcal{M}$.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, there are numerous constructions for producing new matroids from it. The matroid ***dual*** $\mathcal{M}^*$ of $\mathcal{M}$ is the matroid whose bases are the complements of bases in $\mathcal{M}$. The bases of the dual matroid are called ***cobases***. More generally, we prepend the prefix "co-" to any object associated to a matroid to indicate that we are discussing the corresponding dual object. For example, a ***coloop*** of $\mathcal{M}$ is a loop in the dual matroid $\mathcal{M}^*$. Equivalently, a coloop is an element of the ground set that is in every basis of $\mathcal{M}$. Two other useful constructions are deletion and contraction, defined as follows. Let $T \subseteq E$. The ***deletion*** of $\mathcal{M}$ at T, written as $\mathcal{M} \setminus T$, is the matroid whose independent sets are $I \setminus T$ for each $I \in \mathcal{I}$, while the ***contraction*** of $\mathcal{M}$ at T is the matroid defined by $\mathcal{M}/T := (\mathcal{M}^* \setminus T)^*$. A ***minor*** of $\mathcal{M}$ is any matroid that can be obtained from $\mathcal{M}$ by a sequence of deletions and contractions.

For $i \in \{1, 2\}$, let $\mathcal{M}_i = (E_i, \mathcal{B}_i)$ be a matroid. The 1-***sum*** (or ***direct sum***) of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the matroid

$$\mathcal{M}_1 \oplus \mathcal{M}_2 := (E_1 \sqcup E_2, \mathcal{B}_1 \sqcup \mathcal{B}_2).$$

It is routine to check that a matroid $\mathcal{M}$ is isomorphic to the 1-sum of two matroids if and only if $\mathcal{M}$ is disconnected. Every matroid can be written as the 1-sum of connected matroids, $\mathcal{M} = \bigoplus \mathcal{M}_i$, where the summands $\mathcal{M}_i$ are called the ***connected components*** of $\mathcal{M}$. A related construction is the 2-sum of two matroids, which we now describe. If $E_1 \cap E_2 = \{p\}$, then the 2-***sum*** of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the matroid $\mathcal{M}_1 \oplus_2 \mathcal{M}_2$ on $E = (E_1 \cup E_2) \setminus p$ whose circuits consist of

- the circuits of $\mathcal{M}_1$ not containing p;
- the circuits of $\mathcal{M}_2$ not containing p; and
- every set $(C_1 \cup C_2) \setminus p$, where C_i is a circuit of $\mathcal{M}_i$ containing p.

If the element p is not a coloop in at least one of $\mathcal{M}_1$ and $\mathcal{M}_2$, then a subset B of E is a basis of $\mathcal{M}_1 \oplus \mathcal{M}_2$ if and only if $B = B_1 \oplus B_2$ where B_i is a basis in $\mathcal{M}_i$ (see Proposition 7.1.13 in [49]).

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid with $\operatorname{rank}(\mathcal{M}) = r$. The ***Tutte polynomial***, $T_{\mathcal{M}}(x, y)$, of $\mathcal{M}$ is the bivariate polynomial defined as

$$T_{\mathcal{M}}(x, y) := \sum_{S \subseteq E} (x-1)^{r - \operatorname{rank}(S)} (y-1)^{|S| - \operatorname{rank}(S)}.$$

The Tutte polynomial of the dual matroid $\mathcal{M}^*$ is obtained by exchanging the roles of x and y in the Tutte polynomial of the primal matroid $\mathcal{M}$:

$$T_{\mathcal{M}^*}(x, y) = T_{\mathcal{M}}(y, x).$$

Many invariants of a matroid are obtained by evaluating its Tutte polynomial at certain values of x and y (see, e.g., [71]). For example, evaluating the Tutte polynomial at $y = 1$ yields the shelling polynomial of $\mathcal{M}$ [8], from which it follows that $T_{\mathcal{M}}(1, 1)$ is the number of bases of $\mathcal{M}$.

The Tutte polynomial was first introduced as a powerful tool for studying graphs in [69], and was later shown to be as useful in the more general context of matroids in [19]. To see the connection, let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the collection of all (edge sets of) forests of G. Then the pair $(E, \mathcal{I})$ is a matroid and any matroid arising from a graph G in this way is called a ***graphic*** matroid, and is written $\mathcal{M}(G)$. One advantage gained by viewing graphs in the more general context of matroid theory is that a given matroid always has a dual, while the same cannot be said of every graph. It is easy to show that if G is a planar graph and G^* its dual graph, then $\mathcal{M}(G^*) = \mathcal{M}^*(G)$. A compact way to represent the graphic matroid $\mathcal{M}(G)$ is via its vertex-edge incidence matrix $\mathsf{N}(G)$ viewed as a matrix over an arbitrary field $\mathbb{F}$. To see this, note that a set $S \subset E$ is independent in $\mathcal{M}(G)$ if and only if the corresponding collection of columns of $\mathsf{N}(G)$ are linearly independent in $\mathbb{F}^{|V|}$.

The previous construction of a graphic matroid $\mathcal{M}(G)$ from the incidence matrix of a graph G is easily generalized to arbitrary matrices. Given any $m \times n$ matrix M with entries in a field $\mathbb{F}$, the archetypal construction of an independent set matroid is $\mathcal{M} = ([n], \mathcal{I}(M))$ where $[n] := \{1, \ldots, n\}$ is an indexing set for the columns of M and $\mathcal{I}(M)$ consists of all subsets of (indices of) columns of M that are linearly independent in the m-dimensional vector space $\mathbb{F}^m$ over $\mathbb{F}$. In this case we write $\mathcal{M} = \mathcal{M}(M)$ and call $\mathcal{M}(M)$ the ***vector matroid*** of M. A matroid $\mathcal{M}$ is called $\mathbb{F}$-***representable*** if there exists a matrix M with entries in $\mathbb{F}$ such that $\mathcal{M} = \mathcal{M}(M)$. If $\mathcal{M}$ is a representable matroid represented by the matrix M, then M is a ***full-rank representation*** of $\mathcal{M}$ if the matroid rank of $\mathcal{M}$ is the row rank of M.

Regular matroids are an important subclass of representable matroids consisting of those matroids representable over any field. Regular matroids have many characterizations of which we give a sampling in Theorem 1.1. Before stating the theorem we need the following definitions. Consider the circuits $\mathcal{C} = \{C_1, \ldots, C_m\}$ of $\mathcal{M}$. Then the ***circuit incidence matrix*** of $\mathcal{M}$ is the $m \times n$ matrix $A(\mathcal{C})$ with entry a_{ij} equal to 1 if j is in C_i and equal to 0 otherwise. The cocircuit incidence matrix $A(\mathcal{C}^*)$ is defined analogously. The matroid $\mathcal{M}$ is ***orientable*** if one can replace some of the nonzero entries of $A'(\mathcal{C})$ and $A'(\mathcal{C}^*)$ by -1 such that, if a circuit C and cocircuit C^* have nonempty intersection, then both of the sets $\{i \in [n] \mid \mathbf{a}_C(i) = \mathbf{a}_{C^*}(i) \neq 0\}$ and $\{i \in [n] \mid \mathbf{a}_C(i) = -\mathbf{a}_{C^*}(i) \neq 0\}$ are nonempty.

Theorem 1.1 ([49])**.** *For a matroid $\mathcal{M}$ the following are equivalent:*

(1) *$\mathcal{M}$ is regular;*

(2) *$\mathcal{M}$ is $\mathbb{F}_2$-representable and orientable;*

(3) *$\mathcal{M}$ is representable over $\mathbb{R}$ by a unimodular matrix;*

(4) *$\mathcal{M}$ is representable over $\mathbb{R}$ by a totally unimodular matrix; and*

(5) *the dual of $\mathcal{M}$ is regular.*

Moreover, if $\mathcal{M}$ is regular, M is a totally unimodular matrix that represents $\mathcal{M}$ over $\mathbb{R}$ and $\mathbb{F}$ is any other field, then M is an $\mathbb{F}$-representation of $\mathcal{M}$ when viewed as a matrix over $\mathbb{F}$.

As noted in Section 1.2, a unimodular matrix M remains unimodular after adding either a column of zeros or a copy of the column M_j. In matroid terminology, adding a column of zeros corresponds to adding a loop to the matroid $\mathcal{M}(\mathsf{M})$ while adding a copy of a column gives a parallel element.

If $\mathcal{M}$ is an $\mathbb{R}$-representable matroid on n elements with representation M and if D is any $n \times n$ diagonal matrix with nonzero real entries on the diagonal, then the matroids $\mathcal{M} = \mathcal{M}(M)$ and $\mathcal{M}(MD)$ are isomorphic. By way of analogy with the graphical case, we write $M_{\mathbf{w}}$ for MD, where the i^{th} coordinate of $\mathbf{w}$ is $w_i = D_{ii}$, and we call $M_{\mathbf{w}}$ a ***weighted representation*** of $\mathcal{M}$. We also define the ***weighted Laplacian*** of $\mathcal{M}(M)$ with respect to D to be the matrix $L_w := MDM^\top$.

1.5 Oriented Matroids

We now turn our attention from matroids to oriented matroids. First we give the abstract definition of an oriented matroid in terms of covectors following [9] and then we concentrate on oriented matroids arising from vector configurations.

Let E be a finite set and let $\Sigma = \{+, -, 0\}^{|E|}$ be the set of all sign vectors of length $|E|$. For $\sigma \in \Sigma$, the ***opposite*** of σ is the sign vector $-\sigma$ where $(-\sigma)_e$ takes the value $+$ (respectively, $-, 0$) if $\sigma_e = -$ (respectively, $+, 0$). The ***support*** of $\sigma \in \Sigma$ is the set $\operatorname{supp}(\sigma) = \{e \in E \mid \sigma_e \neq 0\}$. The ***zero vector*** is the sign vector with empty support. For two sign vectors $\sigma_1, \sigma_2 \in \Sigma$ the ***composition*** $\sigma_1 \circ \sigma_2$ is the sign vector given by

$$(\sigma_1 \circ \sigma_2)_e = \begin{cases} (\sigma_1)_e \text{ if } (\sigma_1)_e \neq 0, \\ (\sigma_2)_e \text{ otherwise} \end{cases}$$

and the ***separation set*** $S(\sigma_1, \sigma_2)$ is the set

$$S(\sigma_1, \sigma_2) = \{e \in E \mid (\sigma_1) = -(\sigma_2) \neq 0\}.$$

Two sign vectors are said to be ***conformal*** if their separation set is empty.

An ***oriented matroid*** $\mathcal{M} = (E, \mathcal{V}^*)$ is a pair consisting of a ground set E together with a set of sign vectors $\mathcal{V}^* \subseteq \{+, -, 0\}^{|E|}$ satisfying the following axioms:

V1 The zero vector is in $\mathcal{V}^*$;

V2 If $\sigma \in \mathcal{V}^*$, then so is $-\sigma$;

V3 If $\sigma_1, \sigma_2 \in \mathcal{V}^*$, then so is their composition; and

V4 If $\sigma_1, \sigma_2 \in \mathcal{V}^*$ and $e \in S(\sigma_1, \sigma_2)$, there is a sign vector $\sigma \in \mathcal{V}^*$ with $(\sigma)_e = 0$ and $(\sigma)_f = (\sigma_1 \circ \sigma_2)_f = (\sigma_2 \circ \sigma_1)_f$ for all $f \notin S(\sigma_1, \sigma_2)$.

When $\mathcal{M} = (E, \mathcal{V}^*)$ satisfies the above conditions, then $\mathcal{V}^*$ is the set of ***covectors*** of the oriented matroid $\mathcal{M}$.

In the previous section we saw how a vector configuration gives rise to an unoriented matroid. Now we will see that every such matrix also gives rise to an oriented matroid. For simplicity we restrict ourselves to vector spaces over $\mathbb{R}$ as this is the only case we encounter in the sequel.

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{R}^d$ be a vector configuration that spans the vector space $\mathbb{R}^d$. Then the ***covectors*** of the oriented matroid $\mathcal{M}(\mathcal{A})$ are the elements of the set

$$\begin{aligned}\mathcal{V}^* := &\{(\operatorname{sign} f(\mathbf{a}_1), \ldots, \operatorname{sign} f(\mathbf{a}_n)) \mid f : \mathbb{R}^d \to \mathbb{R} \text{ linear functional}\} \\ &\subseteq \{-1, 0, 1\}^n.\end{aligned}$$

The oriented matroid $\mathcal{M}(\mathcal{A})$ is called the oriented matroid ***generated by*** $\mathcal{A}$. The ***cocircuits*** of the oriented matroid $\mathcal{M}(\mathcal{A})$ are the minimal elements of the poset $(\mathcal{V}^*, \prec)$, where the relation $\prec$ is defined by extending $0 \prec \pm 1$ componentwise. We will see how this poset is related to the face poset of the zonotope generated by the matrix M whose columns are the $\mathbf{a}_i$ when we discuss polyhedra in the next section. We now recall how to retrieve the covectors of the oriented matroid $\mathcal{M}(\mathcal{A})$ from a certain subspace arrangement in $\mathbb{R}^n$.

Let M be the matrix whose columns are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Then the covectors of $\mathcal{M}(\mathcal{A})$ can be read off from the hyperplane arrangement induced by the coordinate hyperplanes of $\mathbb{R}^n$ in the row space of M. To see this, first consider the n columns of M as elements of the dual vector space $(\mathbb{R}^d)^*$. Then each $\mathbf{a}_i$ $(i \in [n])$ defines a hyperplane in $\mathbb{R}^d$ given by

$$\mathcal{H}_i := \{x \in \mathbb{R}^d \mid \langle \mathbf{a}_i, x \rangle = 0\}.$$

Let $\mathcal{H}_i^+ := \{x \in \mathbb{R}^d \mid \langle \mathbf{a}_i, x \rangle > 0\}$ and $\mathcal{H}_i^- = \mathbb{R}^d \setminus (\mathcal{H}_i \cup \mathcal{H}_i^+)$, and assign to each $x \in \mathbb{R}^d$ a sign vector $\sigma(x) \in \{-1, 0, 1\}^n$ whose i^{th} coordinate is 1 (respectively, $0, -1$) if x is in $\mathcal{H}_i^+$ (respectively, $\mathcal{H}_i, \mathcal{H}_i^-$). The set of all points in $\mathbb{R}^d$ that receive the same sign vector σ form a relatively open topological cell (which we label with σ) and the union of all such cells is $\mathbb{R}^d$. The sign vectors that occur are precisely the covectors of the oriented matroid $\mathcal{M}(\mathcal{A})$, and the sign vectors that label 1-dimensional cells are the cocircuits.

Now consider the subspace arrangement in the row space of M induced by the coordinate hyperplane arrangement in $\mathbb{R}^n$ (oriented in the natural way), which we denote by $\mathscr{H}(M)$. A point y in the row space of M satisfies $y_i = 0$ (respectively, $y_i > 0, y_i < 0$) if and only if any point $x \in \mathbb{R}^d$ with $y = M^\top x$ lies on the hyperplane $\mathcal{H}_i$ (respectively, in $\mathcal{H}_i^+, \mathcal{H}_i^-$) as defined in the previous paragraph. So the oriented matroid coming from the hyperplane arrangement in the row space of M induced by the coordinate hyperplanes in $\mathbb{R}^n$ is exactly $\mathcal{M}(\mathcal{A})$.

The preceding discussion tells us that the row space of M intersects exactly those cells of the coordinate hyperplane arrangement labeled by the covectors of $\mathcal{M}(M)$. It should be noted, however, that in general the covectors themselves do not lie in the row space of M even when M is a totally unimodular matrix (see Remark 2.6 in Section 2.1). We show in Theorem 2.5 that, when M is a unimodular matrix, every cocircuit of the oriented matroid $\mathcal{M}(\mathsf{M})$ does lie in the row space of M, and in fact, the set of cocircuits is a spanning set for the free abelian subgroup ${}_{\mathbb{Z}}\langle \mathsf{M}^\top \rangle$.

To each oriented matroid $\mathcal{M}$ one associates the underlying unoriented matroid $\underline{\mathcal{M}}$ whose cocircuits are obtained from the cocircuits of $\mathcal{M}$ by forgetting signs, i.e., if C^* is a cocircuit of $\mathcal{M}$, then $\underline{C}^*$ is a cocircuit of $\underline{\mathcal{M}}(M)$ where $(\underline{C}^*)_i = |(C^*)_i|$. An oriented matroid is ***regular*** if its underlying unoriented matroid is. Many statistics of an orientable matroid (e.g., the number of bases or the number of independent sets) remain invariant after orientation, and so when discussing these properties with respect to a given matroid $\mathcal{M}(M)$, we often disregard the difference between the oriented matroid and the underlying unoriented matroid when no confusion can arise.

1.6 Polyhedra

For any affine hyperplane $\mathcal{H} \subset \mathbb{R}^d$ there is a linear function $\mathbf{v} \in (\mathbb{R}^d)^*$ and a scalar $a \in \mathbb{R}$ such that

$$\mathcal{H} = \mathcal{H}(\mathbf{v}, a) := \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{v}, \mathbf{x} \rangle = a\}.$$

Each affine hyperplane $\mathcal{H} = \mathcal{H}(\mathbf{v}, a)$ has two closed ***halfspaces***, $\mathcal{H}^+$ and $\mathcal{H}^-$, associated to it, with $\mathcal{H}^+ := \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{v}, \mathbf{x} \rangle \geq a\}$, and $\mathcal{H}^-$ defined analogously. A ***polyhedron*** $P \subseteq \mathbb{R}^d$ is a nonempty intersection of finitely many closed halfspaces. Equivalently, $P \subseteq \mathbb{R}^d$ is a polyhedron if there is a real $d \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^n$ such that $P = \{\mathbf{p} \in \mathbb{R}^d \mid A\mathbf{p} \leq \mathbf{b}\}$. In general, there are many pairs $(A, \mathbf{b})$ that realize the polyhedron P there is a unique pair (up to multiplication by scalars) that minimizes the number of rows involved. This minimal description is called the ***halfspace representation*** of P.

A ***face*** F of a polyhedron $P \subseteq \mathbb{R}^d$ is a set of the form $F = P \cap \mathcal{H}$ where $\mathcal{H}$ is an affine hyperplane such that P is completely contained in $\mathcal{H}^+$ or $\mathcal{H}^-$ but not both. We allow the case that $\mathcal{H}$ is the degenerate hyperplane

$$\mathcal{H} = \mathcal{H}(\mathbf{0}, 0) = \mathbb{R}^d,$$

from which it follows that P is always a face of itself. The ***dimension*** of a face $F \subseteq P$, written $\dim(F)$ is the dimension of its affine hull.

Given a finite set of vectors $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d$, each of the following sets is a polyhedron:

$$\operatorname{aff}(V) = \left\{ \sum_{i \in [n]} \alpha_i \mathbf{v}_i \;\middle|\; \alpha_i \in \mathbb{R} \text{ for all } i \in [n], \sum_{i \in [n]} \alpha_i = 1 \right\};$$

$$\operatorname{pos}(V) = \left\{ \sum_{i \in [n]} \alpha_i \mathbf{v}_i \;\middle|\; \alpha_i \in \mathbb{R}_{\geq 0} \text{ for all } i \in [n] \right\};$$

$$\operatorname{conv}(V) = \left\{ \sum_{i \in [n]} \alpha_i \mathbf{v}_i \;\middle|\; \alpha_i \in \mathbb{R}_{\geq 0} \text{ for all } i \in [n], \sum_{i \in [n]} \alpha_i = 1 \right\}.$$

These polyhedron are the ***affine hull***, ***positive hull***, and ***convex hull*** of the set V, respectively. A ***strongly convex polyhedral cone*** $C \subset \mathbb{R}^d$ is a set of the form $\operatorname{pos}(V)$, where V is a finite set of vectors in $\mathbb{R}^d$, such that the dimension of the largest linear subspace contained in C is 0.

When studying a polyhedron it is often useful to subdivide it into a collection of "smaller" polyhedra that are glued together in a nice way and which, taken individually, are easy to study. We will be interested in one such type of subdivisions that are defined (in close analogy to the definition of simplicial complexes) as follows: A ***polyhedral complex*** in $\mathbb{R}^d$ is a finite collection $\mathscr{P}$ of polyhedra in $\mathbb{R}^d$ such that (a) if $Q \in \mathscr{P}$ and F is a face of Q, then $F \in \mathscr{P}$, and (b) if $Q, Q' \in \mathscr{P}$, then $Q \cap Q' \in \mathscr{P}$ and is a face of both. Let $\mathscr{P}$ be a polyhedral complex. The ***vertices*** (respectively, ***edges***) of $\mathscr{P}$ are the 0- (resp. 1-) faces of $\mathscr{P}$. The ***facets*** of $\mathscr{P}$ are the proper faces of $\mathscr{P}$ not properly contained in any other proper face. The ***support*** of $\mathscr{P}$ is the union of all polyhedra in $\mathscr{P}$, written $\mathrm{supp}(\mathscr{P})$.

Given a polyhedron P, write $\mathscr{P}(P)$ for the polyhedral complex consisting of P and all of its faces. The ***boundary complex*** of P is the polyhedral complex $\mathscr{P}(P) \setminus P$. This complex is denoted ∂P. The ***relative interior*** of P is the set $P \setminus \partial P$.

When working with polyhedral complexes it is often useful to consider "polyhedra with some facets removed". More precisely, let P be a polyhedron with the halfspace description $(A, \mathbf{b})$. Then a subset Q of P is a ***partially-open polyhedron*** if Q is obtained from P by replacing some of the inequalities defining P with strict inequalities.

By definition, a polyhedron $P \subseteq \mathbb{R}^d$ is a closed convex subset with respect to the usual topology on $\mathbb{R}^n$, though in general it need not be compact. A ***polytope*** is a bounded (and hence compact) polyhedron. In particular, throughout this thesis every polytope is assumed to be convex. Every polytope P can be written as the convex hull of a finite set of points, and the the elements of the minimal set $V = V(P)$ such that $P = \mathrm{conv}(V)$ are the ***vertices*** of P. When $\dim P = d$ we call P a d-polytope.

The ***face lattice*** of a polytope P is the poset $\mathcal{P} = (\mathcal{F}, \subseteq)$ consisting of all faces of P ordered by inclusion. Two polytopes P and P' are ***combinatorially equivalent*** if their face lattices are isomorphic as unlabeled posets. Two combinatorially equivalent polytopes are said to be of the same ***combinatorial type***. For example, all 3-dimensional parallelepideds are combinatorially equivalent and their common face lattice is given in Figure 1.1 without the labelings.

Two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are ***affinely equivalent*** if there is an affine map $\phi : \mathbb{R}^d \to \mathbb{R}^e$ that restricts to a bijection between P and Q. For $d \in \mathbb{N}$, the ***standard d-simplex*** Δ_d is the polytope defined by

$$\Delta_d := \mathrm{conv}\{\mathbf{0}_d, \mathbf{e}_1, \ldots, \mathbf{e}_d\}.$$

A ***simplex*** is a polytope that is affinely equivalent to the standard simplex Δ_d for some d. Equivalently, a d-simplex is the convex hull of $d+1$ affinely independent points.

A ***polytopal complex*** is a polyhedral complex where each maximal face is a polytope. A ***subdivision*** of a polytope P is a polytopal complex whose support is P. A subdivision of P into simplicies is called a ***triangulation*** of P.

A ***fan*** is a polyhedral complex consisting of finitely many strongly convex polyhedral cones. A fan Σ ***refines*** a fan Σ' if they have the same support and every cone in Σ is contained in some cone of Σ'. In this case we say Σ is a refinement of Σ' and, conversely, that Σ' is a ***coarsening*** of Σ. A fan Σ is ***simplicial*** if for every cone $\sigma \in \Sigma$ we have $\sigma = \mathrm{pos}(V)$ where V is a linearly independent set of vectors. A simplicial refinement of a fan Σ is called a ***triangulation***. Given a finite set $A \subset \mathbb{R}^d$, the ***chamber complex*** of A is the coarsest fan with support $\mathrm{pos}(A)$ that is a common refinement of every triangulation of $\mathrm{pos}(A)$.

When $P \subset \mathbb{R}^n$ is a full-dimensional polytope ($\dim P = d$), then we write $\mathrm{vol}(P)$ for the Euclidean volume of P. When P is not full-dimensional (as is often in the case in the polytopes appearing in Chapters 2 and 3 for example), we let $\mathrm{vol}(P)$ denote the volume of P with respect to its affine hull. The ***barycenter*** $\beta(P)$ of a polytope P with vertices V is

$$\beta(P) := \frac{1}{|V|}\sum_{v\in V} v.$$

In the next two subsections we discuss two families of polytopes that play central roles in this thesis: *zonotopes* and *Lawrence polytopes.*

Zonotopes

A ***zonotope*** is a Minkowski sum of line segments, i.e., a polytope $\mathsf{Z} \subset \mathbb{R}^d$ is a zonotope if there is a family of line segments $L_1, \dots, L_n$ in $\mathbb{R}^d$ such that

$$\begin{aligned} \mathsf{Z} &= L_1 + \cdots + L_n \\ &= \left\{ \sum_{i\in[n]} l_i \;\middle|\; l_i \in L_i \text{ for all } i \right\}, \end{aligned}$$

and we say that Z is ***generated*** by the line segments $\{L_i\}_{i\in[n]}$. For a fixed positive integer d, the unit d-cube, $\square_d \subset \mathbb{R}^d$, is the zonotope generated by the unit line segments $L_i = \mathrm{conv}\{\mathbf{0}_d, \mathbf{e}_i\}$ for $i \in [d]$ and it is trivial to see that a polytope is a zonotope if and only if it is an affine projection of $\square_d$ for some d. Every zonotope Z is centrally symmetric with respect to its barycenter β, and we define the zonotope $\mathsf{Z}_0 := \mathsf{Z} - \frac{1}{2}\beta$ to be the translation of Z whose barycenter is the origin.

Given a $d \times n$ matrix M with entries in $\mathbb{R}$, the ***zonotope generated by*** M is the zonotope $\mathsf{Z}(M)$ generated by the segments $\mathrm{conv}\{\mathbf{0}_d, M_i\}$ where M_i is the i^{th} column of M. A ***parallelepiped*** is a zonotope generated by a matrix with linearly independent columns. A parallelepiped is ***half-open*** if it is the Minkowski sum of half-open line segments. The next result, due to Stanley, illustrates the close connection between the zonotope generated by a matrix M and the corresponding vector matroid $\mathcal{M}(M)$ by giving a decomposition of a zonotope into half-open parallelepipeds generated by the independent sets of $\mathcal{M}$.

Theorem 1.2 ([60] Lemma 2.1)**.** *Let M be a rank d matrix and $\mathcal{I}$ be the independent sets of the matroid $\mathcal{M}(M)$. Then the zonotope $\mathsf{Z}(M)$ is the disjoint union of half-open parallelepipeds*

$$\Pi_I := \left\{ \sum_{i \in I} \alpha_i \widetilde{M_i} \;\middle|\; \alpha_i \in [0,1) \right\},$$

as I ranges over $\mathcal{I}$, where $\widetilde{M_i}$ is either M_i or $-M_i$.

The half-open parallelepipeds of maximal dimension in the above theorem are generated by maximal independent subsets of the columns of M. As their union covers $\mathsf{Z}(M)$ up to a set of measure zero, it follows that the volume of the zonotope is the sum of the volumes of the parallelepipeds generated by the bases of $\mathcal{M}(M)$. For a fixed basis $B \in \mathcal{B}(M)$, the volume of Π_B is the absolute value of the determinant of (the matrix whose columns are elements of) B. When the matrix M is unimodular each of these determinants is ± 1 and so we have the following well-known corollary:

Corollary 1.3. *The volume of a zonotope generated by a unimodular matrix M is equal to the number of bases of the regular matroid $\mathcal{M}(\mathsf{M})$, i.e.,*

$$\mathrm{vol}(\mathsf{Z}(\mathsf{M})) = |\mathcal{B}(\mathsf{M})|.$$

Let $\mathsf{Z} = \mathsf{Z}(M)$ be the zonotope generated by the matrix M. A polytopal subdivision $\mathscr{P}$ of Z is called ***zonotopal*** (respectively, ***cubical***) if every polytope in $\mathscr{P}$ is a zonotope (respectively, parallelepiped). A cubical subdivision $\mathscr{P}$ of Z is ***maximal*** if there is a bijection between maximal faces of $\mathscr{P}$ and the bases of the matroid $\mathcal{M}(M)$. For example, the polytopal complex consisting of the closures of the parallelepipeds occuring in Theorem 1.2 is a maximal cubical subdivision of Z. The interplay between the geometry of the zonotope Z and the matroid $\mathcal{M}(M)$ will be a common theme in Chapters 2 and 3. Next we explain the close connection between the combinatorics of Z and the oriented matroid generated by the matrix M.

Let M be a real matrix and let $\mathcal{V}^*$ be the covectors of the oriented matroid $\mathcal{M} = \mathcal{M}(M)$ generated by M. Let $\mathcal{P} := (\mathcal{V}^*, \prec)$ be the poset of covectors of $\mathcal{M}(M)$ where $\prec$ is the component-wise extension of $0 \prec \pm 1$, and let $\mathcal{F}$ be the poset whose elements are the faces of $\mathsf{Z}_0(M)$ ordered by inclusion. Then $\mathcal{P}$ is anti-isomorphic to $\mathcal{F}$ as witnessed by the order-reversing bijection that sends a covector $\mathbf{v} = (v_1, \ldots, v_n)$ to the face

$$F_{\mathbf{v}} = \sum_{i:v_i=1} \tfrac{1}{2} M_i - \sum_{i:v_i=-1} \tfrac{1}{2} M_i + \sum_{i:v_i=0} S_i,$$

where $S_i = \operatorname{conv}\left\{-\frac{1}{2}M_i, \frac{1}{2}M_i\right\}$. Note that the facets of $\mathsf{Z}_0(M)$ correspond to the cocircuits of the oriented matroid $\mathcal{M}(M)$. Now consider the barycenters $\pm\beta_1, \ldots, \pm\beta_r$ of the facets $\pm F_1, \ldots, \pm F_r$ of $\mathsf{Z}_0(M)$. If $\mathcal{C}_i^*$ is the cocircuit corresponding to the facet F_i, then it is clear from the above expression that $\beta_i = \frac{1}{2} M \mathcal{C}_i^*$. For the formulation of Corollary 2.7 below, it turns out to be more appropriate to work with the ***scaled barycenter matrix*** $B = B(M)$ whose columns are the β_i scaled by a factor of 2.

A zonotope $\mathsf{Z}(M) \subset \mathbb{R}^d$ generated by a representation M of a regular matroid $\mathcal{M}$ can be used to tile its affine span. More precisely, a polytope P is said to ***tile*** its affine span S if there is a polyhedral subdivision of S whose maximal cells are translates of P. The next theorem, due to Shepard [57], tells us that a zonotope tiles its affine span exactly when the underlying matroid is regular.

Theorem 1.4. *A zonotope* $\mathsf{Z}(M)$ *tiles its affine span if and only if the matroid* $\mathcal{M}(M)$ *is regular.*

Note that in the above theorem M is not required to be unimodular but only a representation over $\mathbb{R}$ of a regular matroid. This distinction will become important later on when we discuss the space-tiling properties of the zonotope generated by the Laplacian of a connected graph which, though not itself a unimodular matrix, is nevertheless a representation of the regular matroid $\mathcal{M}(\mathsf{N}^\top)$, where N is the signed vertex-edge incidence matrix of the graph.

Any k-dimensional zonotope $\mathsf{Z} \subseteq \mathbb{R}^n$ can be viewed as the projection of the unit n-cube. Moreover, in [43] one finds the following theorem in which the (Euclidean) d-dimensional volume of Z and the $(n-d)$-dimensional volume of a certain zonotope $\overline{\mathsf{Z}}$ in the orthogonal complement of the linear hull of Z are shown to be the same.

Theorem 1.5. *If Z and $\overline{\mathsf{Z}}$ are images of the unit cube in $\mathbb{R}^n$ under orthogonal projection onto orthogonal subspaces of dimension d and $n-d$, respectively, and* vol_k *denotes the k-dimensional Euclidean volume form, then*

$$\mathrm{vol}_d(\mathsf{Z}) = \mathrm{vol}_{n-d}(\overline{\mathsf{Z}}).$$

Lawrence Polytopes

In Section 1.6 we saw that the zonotope generated by a matrix M is a natural geometric construction that encodes invariants of both the oriented matroid $\mathcal{M} = \mathcal{M}(M)$ and its underlying unoriented matroid $\underline{\mathcal{M}}$. We now discuss another polytopal construction that captures matroid invariants of $\mathcal{M}$.

Given a real $d \times n$ matrix M, the ***Lawrence matrix*** of M is the matrix

$$\begin{pmatrix} M & \mathbf{0} \\ I_n & I_n \end{pmatrix}$$

where $\mathbf{0}$ is the $d \times n$ matrix with all entries equal to zero and I_n is the $n \times n$ identity matrix. The ***Lawrence polytope*** generated by M is the convex hull of the columns of the Lawrence matrix and is denoted $\Lambda(M)$. The vertices of $\Lambda(M)$ are precisely the columns of the Lawrence matrix and $\dim \Lambda(M) = \mathrm{rank}(M) + n - 1$.

Lawrence polytopes are a specific case of a more general construction initially introduced by Lawrence (unpublished but see [7, 52]) to study arbitrary oriented matroids. Given a rank r oriented matroid $\mathcal{M}$ on n elements, a fundamental question in oriented matroid theory is to decide the realizability of $\mathcal{M}$. Lawrence's approach to answering this question was to construct a rank $r+n$ oriented matroid $\mathcal{L}(\mathcal{M})$ on $2n$ elements whose face lattice is polytopal if and only if $\mathcal{M}$ is realizable. The fact that the oriented matroid $\mathcal{M}$ is representable if and only if the oriented matroid $\mathcal{L}(\mathcal{M})$ can be represented by the Lawrence matrix of a matrix M representing $\mathcal{M}$ is essentially one of the characterizations of Lawrence polytopes given in Theorem 2.1 of [4]. In that paper the following equivalent definition of Lawrence polytopes was given: A polytope P is a Lawrence polytope if and only if every Gale diagram of P (i.e., every Gale dual of $\mathrm{vert}(P)$) is centrally symmetric.

There is an intimate relationship between Lawrence polytopes and zonotopes. This relationship is a particular example of a more general phenomenon known as the *polyhedral Cayley trick*, which we now breifly review. Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^d$ and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis for $\mathbb{R}^n$. For each $i \in [n]$ define the map $\mu_i : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^n$ by $\mu_i(x) = (x, \mathbf{e}_i)$. Then the ***Cayley embedding*** $\mathscr{C} = \mathscr{C}(P_1, \ldots, P_n)$ is the polytope defined by

$$\mathscr{C}(P_1, \ldots, P_n) := \mathrm{conv}\left(\bigcup_{i \in [n]} \mu_i(P_i)\right).$$

The Minkowski sum $\Sigma = \Sigma(P_1, \ldots, P_n)$ is the polytope consisting of all points of the form $\mathbf{p}_1 + \cdots + \mathbf{p}_n$ where $\mathbf{p}_i \in P_i$ for each $i \in [n]$. A polyhedral subdivision of Σ is called ***mixed*** if every full-dimensional polytope in the subdivision can be written as a Minkowski sum $\Sigma(Q_1, \ldots, Q_n)$ such that, for all $i \in [n]$, we have $Q_i \subseteq P_i$. A polytope combinatorially equivalent to the Minkowski sum Σ can be obtained by intersecting the Cayley embedding $\mathscr{C}$ with the affine subspace $\mathbb{R}^d \times \mathbf{1}_n$ (see Lemma 3.2 in [30]). This leads to the following surprising connection between polyhedral subdivsions of the Cayley embedding $\mathscr{C}$, and mixed subdivisions of the Minkowski sum Σ ([65, 53, 30]).

Theorem 1.6 (The Polyhedral Cayley Trick)**.** *Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^d$. Then the poset of polyhedral subdivisions of the Cayley embedding $\mathscr{C}(P_1, \ldots, P_n)$ ordered by refinement is isomorphic to the poset of mixed subdivisions of the Minkowski sum $\Sigma(P_1, \ldots, P_n)$, also ordered by refinement.*

Note that if each of the $P_i = \mathrm{conv}\{\mathbf{0}, \mathbf{v}_i\}$ is a line segment, then the corresponding Cayley embedding (with respect to the standard basis) is the Lawrence polytope generated by the matrix M whose columns are the $\mathbf{v}_i$. The corresponding Minkowski sum is the zonotope generated by M. In this case, Theorem 1.6 tells us that polyhedral subdivisions of $\Lambda(M)$ correspond to zonotopal subdivisions of $\mathsf{Z}(M)$.

1.7 Lattices

A (group theoretical) ***lattice*** $\mathscr{L} \subset \mathbb{R}^n$ is a finitely-generated free abelian subgroup of $\mathbb{R}^n$. The ***rank*** of a lattice $\mathscr{L}$ is the dimension of the $\mathbb{R}$-vector space $\mathscr{L} \otimes_{\mathbb{Z}} \mathbb{R}$. A set $A \subset \mathbb{R}^n$ ***generates*** $\mathscr{L}$ if every element of $\mathscr{L}$ is an linear combination of elements of A with integer coefficients, i.e., $\mathscr{L} = {}_{\mathbb{Z}}\langle A \rangle$. Moreover, if $\mathscr{L}$ has rank r then there exists a set A with cardinality r such that A generates $\mathscr{L}$. Such a set is called a $\mathbb{Z}$-***basis*** for $\mathscr{L}$.

A polytope $P \subset \mathbb{R}^n$ is an $\mathscr{L}$-***polytope*** if the vertices of P are elements of $\mathscr{L}$ and, when the lattice $\mathscr{L}$ is clear from the context, we call P a ***lattice polytope***. Two $\mathscr{L}$-polytopes P and Q are ***lattice equivalent*** if there is a unimodular automorphism on $\mathbb{R}^n$ that maps P bijectively onto Q. Given any two $\mathbb{Z}$-bases A and A' of a $\mathscr{L}$, there is a unimodular transformation taking A to A'. It follows that the parallelepipeds $\mathsf{Z}(A)$ and $\mathsf{Z}(A')$ are lattice equivalent and hence that their volumes coincide. Any translation by a lattice vector of a parallelepiped generated by a $\mathbb{Z}$-basis for $\mathscr{L}$ is called a ***fundamental parallelepiped*** for $\mathscr{L}$. The ***index*** of $\mathscr{L}$ is the volume of any fundamental parallelepiped.

We will study volumes of a certain class of lattice zonotopes more closely in Chapter 2. In Chapter 3 we study another invariant of a lattice polytope P, the *Ehrhart polynomial* of P, which we now define (see [6] for an excellent introduction to Ehrhart theory).

For any set $S \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, the k^{th} ***dilate*** of S, written kS, is the set $kS := \{ks \mid s \in S\}$. The ***Ehrhart function*** of S with respect to a given lattice $\mathscr{L}$ is the function $\mathcal{E}(S; \mathscr{L}) : \mathbb{N} \to \mathbb{N}$ that counts the number of lattice points in the k^{th} dilate of S. When the lattice $\mathscr{L}$ is clear from the context we write $\mathcal{E}_S(k)$ for the Ehrhart function of S with respect to $\mathscr{L}$. The following famous theorem, due to Ehrhart [23], states that the Ehrhart function of a lattice polytope is a polynomial.

Theorem 1.7. *Let P be a d-polytope whose vertices lie in a lattice $\mathscr{L}$. Then there is a degree d polynomial that evaluates to $\mathcal{E}_S(k)$ for all $k \in \mathbb{N}$.*

The polynomial in the above theorem is called the ***Ehrhart polynomial*** of the lattice polytope P, and is written as $\mathcal{E}_P(x)$. It has coefficients in $\frac{1}{d!}\mathbb{Z}$ when written in the standard basis. Though the coefficients of the Ehrhart polynomial $\mathcal{E}_P(x) = \sum_{i=0}^{d} c_i x^i$ of a lattice d-polytope P are generally not well understood, there are the following exceptions:

- the leading coefficient c_d is the relative volume of P;
- the second leading coefficient c_{d-1} is the surface area of P divided by 2; and
- the constant coefficient $c_0 = 1$ is the Euler characteristic of P

The ***Ehrhart series*** of a lattice polytope P is the generating function

$$\mathrm{Ehr}_P(z) := \sum_{n \geq 0} \mathcal{E}_P(n) z^n$$

On the level of rational functions we have

$$\mathrm{Ehr}_P(z) = \frac{\delta_P(z)}{(1-z)^{d+1}},$$

where $d = \dim P$ and $\delta_P(z)$ is the (Ehrhart) ***δ-polynomial*** of P. The δ-polynomial of P is a polynomial of degree at most $\dim P$ with integer coefficients.

Now we consider oriented matroids that arise in lattice theory. To each lattice $\mathscr{L}$ one associates the oriented matroid $\mathcal{M}(\mathscr{L})$ whose covectors are exactly $\mathcal{V}^* = \{\mathrm{sign}(\mathbf{v}) \mid \mathbf{v} \in \mathscr{L}\}$. The ***support*** of a vector $\mathbf{v} \in \mathscr{L}$ is the set $\mathrm{supp}(\mathbf{v}) = \{i \in [n] \mid \mathbf{v}_i \neq 0\}$. A nonzero vector $\mathbf{v} \in \mathscr{L}$ is ***elementary*** if its coordinates lie in $\{-1, 0, 1\}$ and it has minimal support in $\mathscr{L} \setminus \mathbf{0}$. Two vectors in $\mathscr{L}$ are ***conformal*** if their component-wise product is in $\mathbb{R}^n_{\geq 0}$.

A ***zonotopal lattice*** is a pair $(\mathscr{L}, \langle\cdot,\cdot\rangle)$ where $\mathscr{L} \subset \mathbb{Z}^n$ is a lattice, $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^n$ such that the canonical basis vectors are pairwise orthogonal, and such that for every $\mathbf{v} \in \mathscr{L} \setminus \{\mathbf{0}\}$ there is an elementary vector $\mathbf{u} \in \mathscr{L}$ such that $\mathrm{supp}(\mathbf{u}) \subseteq \mathrm{supp}(\mathbf{v})$. The next proposition (Lemma 3.2 in [70]) tells us that zonotopal lattices are generated by the cocircuits of their oriented matroids in an especially nice way.

Proposition 1.8. *The elementary vectors of a zonotopal lattice $\mathscr{L}$ are exactly the cocircuits of the oriented matroid $\mathcal{M}(\mathscr{L})$. Moreover, every vector $\mathbf{v} \in \mathscr{L}$ is the sum of pairwise conformal elementary vectors, and if the support of $\mathbf{v}$ equals the support of some elementary vector $\mathbf{u}$, then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$.*

As noted in Remark 4.2 of [70], the oriented matroid of a zonotopal lattice is regular. Historically this was taken as the definition of a regular matroid (see Section 1.2 of [68]). We reestablish this connection and give a modern proof for the fact that, for a regular oriented matroid $\mathcal{M}(\mathsf{M})$ with cocircuits $\mathcal{C}^*$ and M unimodular, the lattices generated by $\mathsf{M}^\top$ and $\mathcal{C}^*$ coincide (see Theorem 2.5).

Chapter 2

A Polyhedral Proof of the Matrix Tree Theorem

The matrix tree theorem is a classical result in algebraic graph theory relating the number of spanning trees of a connected graph G with the product of the nonzero eigenvalues of the Laplacian matrix of G. Also known as Kirchoff's Theorem as it is already implicit in his 1847 paper [34] on electrical systems, the exact relationship is as follows.

Theorem 2.1 (Kirchoff [34])**.** *Let G be a connected graph on n vertices with s spanning trees and whose Laplacian L has nonzero eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then*

$$\prod_{i \in [n-1]} \lambda_i = ns.$$

Applications of this theorem and its variants are myriad [29] and include studying distance relationships in social networks [13], proving the existence of unique stationary distributions of Markov chains [56], and studying certain optimal designs of statistical experiments [14].

A classical proof of this theorem proceeds as follows (see [1], [12]). First one shows that every principal minor of L is equal to the sum of the squares of the maximal minors of the signed vertex-edge incidence matrix of G with one row removed. Then one computes that such a minor is equal to ± 1 if the edges of G corresponding to the columns of the submatrix span a tree. Finally, one verifies that the maximal principal minor is precisely $1/n$ times the product of the nonzero eigenvalues of L using the characteristic polynomial.

Many generalizations of the theorem exist including for weighted graphs, simplicial complexes, and regular matroids. In the last case, the regular matroid matrix tree theorem is the following particular case of Theorem 3 in [40].

Theorem 2.2. *Let $\mathcal{M}$ be a rank d regular matroid represented by a $d \times n$ unimodular matrix M of full rank, and let $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$. Then the number of bases of $\mathcal{M}$ is $\lambda_1 \cdots \lambda_d$, where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of L .*

In this chapter we recast this result into the domain of polyhedral geometry by considering the zonotopes generated by the columns of the matrices M and L. Although the combinatorial structures of these zonotopes are in general vastly different, we obtain a new proof of Theorem 2.2 by showing that their volumes coincide.

Theorem 2.3. *Let M be a unimodular matrix of full rank, and $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$. Then the volume of the zonotope $\mathsf{Z}(\mathsf{M})$, the volume of the zonotope $\mathsf{Z}(\mathsf{L})$, and the product of the eigenvalues of L are all equal.*

When M has full row rank, then so does L. Moreover, as L is a symmetric matrix it has rank-many nonzero real eigenvalues. While the combinatorial structure of $\mathsf{Z}(\mathsf{M})$ can be complicated, the structure of $\mathsf{Z}(\mathsf{L})$ is always simple: it is a d-dimensional parallelepiped. It follows that the volume of $\mathsf{Z}(\mathsf{L})$ is exactly the determinant of L, which in turn is the product of the eigenvalues of L. This shows that the last two quantities in Theorem 2.3 are equal, and so the crucial part of the proof is to show that the zonotopes have the same volume.

We now plot the course for this chapter. In Section 2.1 we prove Theorem 2.3 via a novel dissect-and-rearrange argument. Then we generalize our proof in Section 2.2 to the case of weighted regular matroids. Finally, in Section 2.3 we give a new polyhedral proof of the classical Matrix Tree Theorem that, while similar to the general proof for full rank matrices in the previous section, copes with the fact that the defining matrices M and L do not have full rank. The classical proof of the matrix tree theorem involves matrix calculations that rely on the total unimodularity of the signed vertex-edge incidence matrix of a graph G, i.e., that one has a totally unimodular representation of the matroid $\mathcal{M}(G)$. Our polyhedral approach works even when the representation of $\mathcal{M}(G)$ is only unimodular. Moreover, we make no use of the Binet-Cauchy Theorem (as in the classical proof) nor of divisor theory on graphs as in [2].

This chapter is the result of joint work with Julian Pfeifle.

2.1 The Full-Rank Case

Let $\mathcal{M}$ be a regular rank d matroid. If M is a unimodular representation of $\mathcal{M}$, then by Corollary 1.3 the volume of the zonotope generated by M is equal to the number of bases of $\mathcal{M}$, $\mathrm{vol}(\mathsf{Z}(\mathsf{M})) = |\mathcal{B}(\mathcal{M})|$. When M has full-row rank then so does the square matrix L, and so the zonotope $\mathsf{Z}(\mathsf{L})$ is a parallelepiped with volume $\det(\mathsf{L}) = \lambda_1 \cdots \lambda_d$, where the λ_i are the eigenvalues of L. Using row operations that preserve unimodularity and then deleting any rows of zeros, any unimodular represention M of $\mathcal{M}$ can be transformed into a full row-rank unimodular representation of $\mathcal{M}$, so we now assume M is such a representation of $\mathcal{M}$. The proof of Theorem 2.3 will be complete once we show that the zonotopes $\mathsf{Z}(\mathsf{M})$ and $\mathsf{Z}(\mathsf{L})$ have the same volume.

Remark 2.4. When M has nontrivial corank (as is the case, for example, when $\mathsf{M} = \mathsf{N}(G)$ is the signed incidence matrix of a graph), the zonotope $\mathsf{Z}(\mathsf{L})$ is no longer a parallelepiped. This means that some care must be taken when showing that the volume of $\mathsf{Z}(\mathsf{L})$ is the product of its nonzero eigenvalues. We sweep this detail under the rug in this section for ease of exposition, but deal with it in detail in Section 2.3 where we use our techniques to prove the graphical matrix tree theorem.

Our first goal is to see that, when M is a unimodular representation of a regular matroid, the lattices generated by L and the scaled barycenter matrix B coincide. (Note that we do not require M to have full rank nor to be totally unimodular.) This fact is an immediate corollary of the following theorem.

Theorem 2.5. *Let $\mathcal{M}$ be a regular oriented matroid on n elements and M be a unimodular matrix representing $\mathcal{M}$ over $\mathbb{R}$. Then the lattices ${}_{\mathbb{Z}}\langle \mathsf{M}^\top \rangle$ and ${}_{\mathbb{Z}}\langle \mathcal{C}^* \rangle$, generated by the columns of $\mathsf{M}^\top$ and by the cocircuits of $\mathcal{M}$, respectively, coincide.*

Proof. Recall that the subspace arrangement $\mathscr{H} = \mathscr{H}(\mathsf{M}) \subset \mathbb{R}^n$ is obtained by intersecting the row space of M with the coordinate hyperplane arrangement in $\mathbb{R}^n$. Clearly, the closure of any cell of $\mathscr{H}$ is the positive hull of the rays of $\mathscr{H}$ it contains and the sign vector of a cell is conformal to each of the rays contained in its closure. By the discussion in Section 1.5, the cocircuits of $\mathcal{M}(\mathsf{M})$ are the sign vectors that label the rays of this arrangement. Let ρ be such a ray, labeled with the sign vector σ. We claim $\rho = \mathrm{pos}(\sigma)$.

Consider the polytope $\rho \cap [-1,1]^n$. The equations for the row space of M are given by the kernel of M and it follows from Theorem 1.1 that one can find a unimodular basis for $\ker \mathsf{M}$ (see [5] Lemma 2.10 for details when M is a full-rank totally unimodular matrix). Thus, the line segment $\rho \cap [-1,1]^n$ is the intersection of hyperplanes and halfspaces whose normal vectors can be viewed as the rows of a unimodular matrix. Moreover, the equations and inequalities of the segment all have integer (in fact $\{0,\pm 1\}$) right-hand sides. Thus, by Theorem 19.2 in [54], we obtain that $\rho \cap [-1,1]^n = \operatorname{conv}\{\mathbf{0},\mathbf{v}\}$ is a lattice segment with $\mathbf{v} \in \{-1,0,1\}^n$. But then $\mathbf{v} = \operatorname{sign}(\mathbf{v}) = \sigma$, and so $\rho = \operatorname{pos}(\sigma)$. In particular, every cocircuit C^* of $\mathcal{M}(\mathsf{M})$ is in the row space of M, and hence ${}_{\mathbb{Z}}\langle \mathcal{C}^* \rangle \subseteq {}_{\mathbb{Z}}\langle \mathsf{M} \rangle$.

For the opposite inclusion, let $\mathbf{w} \in {}_{\mathbb{Z}}\langle \mathsf{M} \rangle$ with sign vector $\sigma_{\mathbf{w}}$. Then, as the cell of $\mathscr{H}$ labelled with $\sigma_{\mathbf{w}}$ is the positive hull of the rays it contains and the labels on these rays have minimal support, for any such ray ρ we have $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma_{\mathbf{w}})$. It follows immediately that ${}_{\mathbb{Z}}\langle \mathsf{M} \rangle$ together with the standard inner product on $\mathbb{R}^n$ is a zonotopal lattice. Moreover, the elementary vectors of ${}_{\mathbb{Z}}\langle \mathsf{M} \rangle$ are those $\{-1,0,1\}$-vectors in the row space of M that have minimal support, i.e., lie on a ray of the arrangement induced by the coordinate hyperplane arrangement. It follows that the elementary vectors of ${}_{\mathbb{Z}}\langle \mathsf{M} \rangle$ are the cocircuits of $\mathcal{M}(\mathsf{M})$ and, since elementary vectors of a zonotopal lattice span the lattice by Proposition 1.8, the theorem follows. $\square$

Remark 2.6. As we already mentioned, the covectors of $\mathcal{M}$ do not always lie in the row space of M. Consider the totally unimodular matrix

$$\mathsf{N}_{K_3} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By Theorem 2.5, the row space of N_{K_3} has a basis of cocircuits, for example

$$\mathcal{C}^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The lattice point $(1,2,1)$ lies in the row space of $\mathcal{C}^*$. However, taking signs yields the covector $(1,1,1)$ of $\mathcal{M}$, which does not lie in the row space of $\mathcal{C}^*$. $\square$

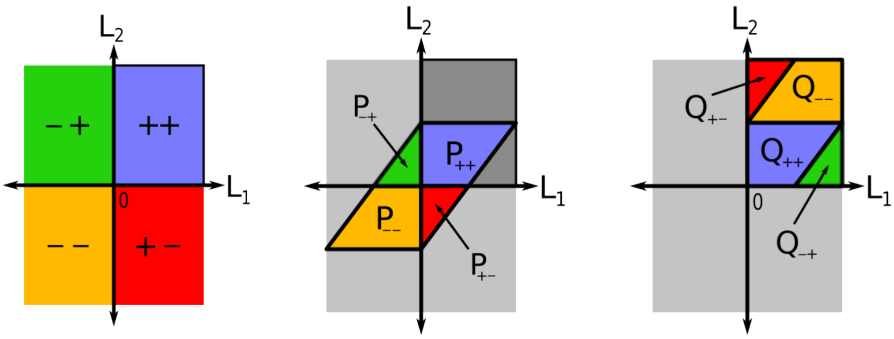
Recall that for an arbitrary matrix M, the columns of the scaled barycenter matrix $B = B(M)$ are the barycenters $\beta_i = \frac{1}{2} M \mathcal{C}_i^*$ of the facets of $\mathsf{Z}_0(M)$, scaled by 2.

Corollary 2.7. *Let $\mathcal{M}$ be a regular oriented matroid on n elements and M be a unimodular matrix representing $\mathcal{M}$ over $\mathbb{R}$. Then the lattices generated by the columns of L and the columns of B are equal.*

Proof. Theorem 2.5 tells us that the lattices generated respectively by $\mathsf{M}^\top$ and $\mathcal{C}^*$ coincide, and therefore the lattices generated by the images $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$ and $B = \mathsf{M}\mathcal{C}^*$ of these matrices under M coincide as well. □

We now use the fact that the columns of L are an $\mathbb{R}$-basis for $\mathbb{R}^d$ to define a subdivision of $\mathsf{Z}_0(\mathsf{M})$. For each sign vector $\epsilon \in \{+,-\}^d$ we define the following objects:

- the simplicial cone $\sigma_\epsilon := \mathrm{pos}\{\epsilon_i \mathsf{L}_i \mid i \in [d]\}$ (see Figure 2.1a);
- the vector $v_{\epsilon^-} := \sum_{i:\,\epsilon_i=-} \mathsf{L}_i$;
- the polytope $P_\epsilon := \sigma_\epsilon \cap \mathsf{Z}_0(\mathsf{M})$ (see Figure 2.1b);
- the polytope $Q_\epsilon := P_\epsilon + v_{\epsilon^-}$ (see Figure 2.1c).



(a) The cones σ_ϵ (b) $\mathsf{Z}_0(\mathsf{M}) = \bigcup P_\epsilon$ (c) $\mathsf{Z}(\mathsf{L}) = \bigcup Q_\epsilon$

Figure 2.1: The polyhedra induced by sign vectors for the path on three vertices, after a change of coordinates that transforms the columns of L to the standard basis.

Example. Consider the path on three vertices with the edges oriented so that $i \to j$ if $i < j$. A full-rank representation for the independent set matroid $\mathcal{M}(\mathsf{N})$ of the signed incidence matrix of this graph is given by the matrix on the left below, while the corresponding Laplacian is the matrix on the right:

$$\mathsf{M} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad \mathsf{L} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Both of the zonotopes $\mathsf{Z}(\mathsf{M})$ and $\mathsf{Z}(\mathsf{L})$ are two dimensional parallelepipeds and Figure 2.1 illustrates the families σ_ϵ, P_ϵ, and Q_ϵ as ϵ varies over all sign vectors for this example after a suitable coordinate transformation. Note that the zonotope of the Laplacian is the parallelepiped in the positive quadrant that is shaded dark grey. □

Clearly the union of the P_ϵ over all sign vectors is the zonotope $\mathsf{Z}_0(\mathsf{M})$ and the intersection of any two of them is a face of both. We now prove Theorem 2.3 by showing that the union of the Q_ϵ is in fact $\mathsf{Z}(\mathsf{L})$ and that any two Q_ϵ intersect in a set of measure zero.

Theorem 2.3. *Let $\mathcal{M}$ be a regular oriented matroid on n elements, let M be a unimodular matrix representing $\mathcal{M}$ over $\mathbb{R}$, and put $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$. Then the volume of the zonotope $\mathsf{Z}(\mathsf{L})$ equals the volume of the zonotope $\mathsf{Z}(\mathsf{M})$.*

Proof. By Corollary 2.7, the line segment $\operatorname{conv}\{\mathbf{0}, \mathsf{L}_i\}$ intersects some proper face of $\mathsf{Z}_0(\mathsf{M})$ and the point of intersection is the barycenter of both. In particular, the distance between any two points of $\mathsf{Z}_0(\mathsf{M})$ in the direction parallel to L_i is less than or equal to $||\mathsf{L}_i||$, with equality if and only if the points lie in opposite faces of $\mathsf{Z}_0(\mathsf{M})$ intersected by the line ${}_{\mathbb{R}}\langle \mathsf{L}_i \rangle$.

First we show that $\bigcup_\epsilon Q_\epsilon \subseteq \mathsf{Z}(\mathsf{L})$.

Let $H_1 = \langle \mathsf{L}_i \mid i \in \{2, \dots, d\} \rangle_{\mathbb{R}}$ be the hyperplane spanned by all columns of L except for L_1 and let H_1^+ be the open halfspace bounded by H_1 and containing L_1. For $p \in \mathsf{Z}_0(\mathsf{M})$ define $\mathscr{L}_{1,p} := p + \langle \mathsf{L}_1 \rangle$ to be the line through p parallel to L_1 and let $q_1 = \mathscr{L}_{1,p} \cap H_1$ (see Figure 2.2).

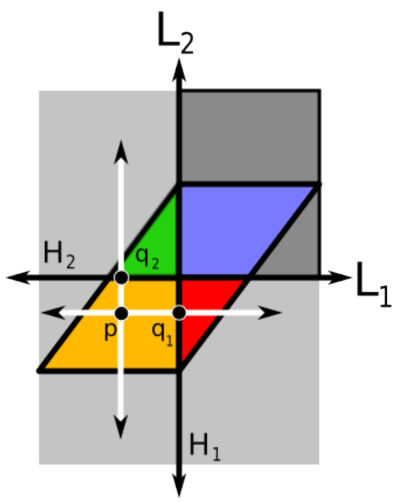


Figure 2.2: A point $p \in \mathsf{Z}_0(\mathsf{M}) \cap \sigma_{(-,-)}$, the hyperplanes H_i (black), and the lines $\mathcal{L}_{i,p}$ (white)

Since the width of $\mathsf{Z}_0(\mathsf{M})$ parallel to L_i is at most $||\mathsf{L}_i||$, it follows that we may express p as $p = \sum \alpha_i \mathsf{L}_i$ for some unique set of α_i with $|\alpha_i| \leq 1$. For example, when $i = 1$ we have $||p - q_1|| \leq ||\mathsf{L}_1||$ and, as $p - q_1$ is parallel to L_1 by construction, it follows that $p - q_1 = \alpha_1 \mathsf{L}_1$ where

$$\alpha_1 := \pm \frac{||p - q_1||}{||\mathsf{L}_1||}$$

is positive (respectively negative, zero) if and only if we have $p \in H_1^+$ (respectively $p \in H_1^-$, $p \in H_1$).

Given $p = \sum \alpha_i \mathsf{L}_i$, define the sign vector ϵ by

$$\epsilon_i = \begin{cases} \mathrm{sign}(\alpha_i) & \text{if } \alpha_i \neq 0 \\ + & \text{else.} \end{cases}$$

Then each δ_k in the expression

$$p + v_\epsilon = \sum_{i \in [d]} \alpha_i \mathsf{L}_i + \sum_{j:\, \epsilon_j = -} \mathsf{L}_j = \sum_k \delta_k \mathsf{L}_k$$

is in $[0, 1]$ and it follows that $Q_\epsilon \subseteq \mathsf{Z}(\mathsf{L})$ (see Figure 2.3).

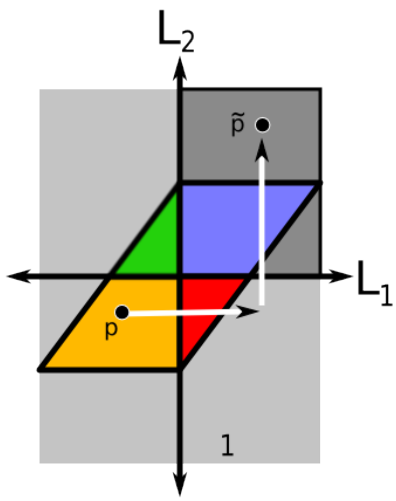


Figure 2.3: The point p and its shift $\tilde{p} = p + \mathsf{L}_1 + \mathsf{L}_2$ into $\mathsf{Z}(\mathsf{L})$.

Now we prove $\mathsf{Z}(\mathsf{L}) \subseteq \bigcup_\epsilon Q_\epsilon$. For this, let $q = \sum_{i \in [d]} \gamma_i \mathsf{L}_i \in \mathsf{Z}(\mathsf{L})$, so that $\gamma_i \in [0, 1]$ for all i by definition. Since facet-to-facet shifts of $\mathsf{Z}_0(\mathsf{M})$ tile the column space of M, the point q lies in some translate of $\mathsf{Z}_0(\mathsf{M})$. Since to pass from one tile to a neighboring one through a facet is to add some vector w in ${}_{\mathbb{Z}}\langle B\rangle = {}_{\mathbb{Z}}\langle \mathsf{L}\rangle$ (Corollary 2.7), we have $q \in \mathsf{Z}_0(\mathsf{M}) + \sum_{i\in[d]} a_i \mathsf{L}_i$, where the $a_i \in \mathbb{Z}$, so that

$$q = \sum_{i\in[d]} \alpha_i \mathsf{L}_i + \sum_{i\in[d]} a_i \mathsf{L}_i$$

with $\alpha_i \in [-1, 1]$. Moreover, all $a_i \geq 0$ because q lies in the positive hull of the L_i's. Comparing coefficients in the two expressions for q, and using the fact that the L_i form a basis, yields $\alpha_i + a_i = \gamma_i$. Since a_i is a nonnegative integer and $\gamma_i \in [0, 1]$, we have $a_i \in \{0, 1\}$ (notice that the degenerate case $\alpha_i = -1$ and $\gamma_i = 1$, in which case a_i would equal 2, cannot occur), and

$$a_i = \begin{cases} 1 \text{ if } \alpha_i \in [-1, 0) \text{ and} \\ 0 \text{ if } \alpha_i \in (0, 1]. \end{cases}$$

Let ϵ be the sign vector defined by $\epsilon_i = -$ (respectively $+$) if $a_i = 1$ (respectively, 0). Then $q \in Q_\epsilon$ and hence $\mathsf{Z}(\mathsf{L}) \subseteq \bigcup_\epsilon Q_\epsilon$.

Finally, we show that for any two sign vectors ϵ, ϵ' the intersection of the relative interiors of Q_ϵ and $Q_{\epsilon'}$ is empty. Let $\phi : \bigcup(\operatorname{relint} P_\epsilon) \to \bigcup(\operatorname{relint} Q_\epsilon)$ be the map that sends $\operatorname{relint} P_\epsilon \to \operatorname{relint} Q_\epsilon$. There are two points $p \neq p'$ in $\mathsf{Z}_0(\mathsf{M})$ with $\phi(p) = \phi(p') =: q$ if and only if

$$q \in (\mathsf{Z}_0(\mathsf{M}) + v_\epsilon) \cap (\mathsf{Z}_0(\mathsf{M}) + v_{\epsilon'})$$

for two sign vectors ϵ and ϵ'. So q lies on the boundary of both translates of $\mathsf{Z}_0(\mathsf{M})$. But then p and p' both lie on the boundary of $\mathsf{Z}_0(\mathsf{M})$ which contradicts the fact that they were in the relative interior of their respective cells. Thus ϕ is a bijective map onto

$$\mathsf{Z}(\mathsf{L}) \setminus \left(\partial \mathsf{Z}(\mathsf{L}) \cup \bigcup_\epsilon \partial Q_\epsilon \right).$$

So we have produced a volume-preserving bijection between $\mathsf{Z}(\mathsf{L})$ and $\mathsf{Z}(\mathsf{M})$ (up to a set of measure zero), which completes the proof. □

Note that all our proofs in this section go through in the case that the regular matroid $\mathcal{M}$ has loops or parallel elements by taking a unimodular representation M of $\mathcal{M}$ where the columns corresponding to loops are columns consisting only of zeros and the columns corresponding to parallel elements are all equal. Moreover, the zonotope $\mathsf{Z}(\mathsf{M})$ is equal to the zonotope generated by the matrix M' whose columns are the distinct nonzero columns of M scaled by their multiplicity. This shows that, after an appropriate modification to the definition of L, Theorem 2.3 is valid even after scaling the columns of the unimodular matrix M by integers.

Corollary 2.8. *Let $\mathcal{M}$ be a regular matroid on n elements represented by the unimodular matrix M of full row rank, D be a a $n \times n$ diagonal matrix with integer entries, and $\mathsf{M}' = \mathsf{M}D$. Then the volume of the zonotope $\mathsf{Z}(\mathsf{M}')$ equals the volume of $\mathsf{Z}(\mathsf{L}')$, where $\mathsf{L}' = \mathsf{M}D\mathsf{M}^\top$.*

This result generalizes to the case that D is a diagonal matrix with real entries, as we show in the next section.

2.2 Weighted Regular Matroids

Let $\mathcal{M}$ be a regular matroid on n elements and let M be a unimodular representation of $\mathcal{M}$. Let D be a diagonal matrix with diagonal $\omega = (w_1, \ldots, w_n) \in \mathbb{R}^n$ and $\mathsf{M}_\omega = \mathsf{M}D$ and $\mathsf{L}_\omega = \mathsf{M}D\mathsf{M}^\top$ be as in Section 1.2. As scaling columns of M does not affect the matroid $\mathcal{M}$, we have $\mathcal{M}(\mathsf{M}) \cong \mathcal{M}(\mathsf{M}_\omega)$. In particular, scaling the columns of M does not affect the cocircuits, and so $_{\mathbb{Z}}\langle \mathcal{C}^* \rangle = {}_{\mathbb{Z}}\langle \mathsf{M}^\top \rangle$.

Let F be a facet of $\mathsf{Z}_0(\mathsf{M}_w)$ corresponding to the cocircuit $C^\star$. Then F is given by

$$F_{C^*} = \sum_{i:C_i^*=1} \tfrac{1}{2} w_i \mathsf{M}_i - \sum_{i:C_i^*=-1} \tfrac{1}{2} w_i \mathsf{M}_i + \sum_{i:C_i^*=0} w_i S_i,$$

from which it is clear that the barycenter of F_{C^*} is $\frac{1}{2} M_\omega C^*$. It follows that the lattice spanned by L_ω equals the lattice spanned by $\mathsf{M}_\omega \mathcal{C}^*$, generalizing Corollary 2.7. Replacing M and L in the proof of Theorem 2.3 by M_ω and L_ω, respectively, proves the following version of the matrix tree theorem for weighted regular matroids.

Theorem 2.9. *Let $\mathcal{M}$ be a regular matroid on n elements with full-rank unimodular representation M and let $D = \operatorname{diag}(\omega)$ be an $n \times n$ diagonal matrix with real entries. Then $\operatorname{vol}(\mathsf{Z}(\mathsf{L}_\omega)) = \operatorname{vol}(\mathsf{Z}(\mathsf{M}_\omega))$.*

This result gives a new proof for Theorem 5.5 in [2] while simultaneously generalizing it from weighted graphs to regular matroids. Moreover, by Theorem 1.5 and the fact that duals of regular matroids are regular, our result implies the dual version of the matrix tree theorem (see Theorem 5.2 in [2]), generalized to regular matroids. All of this is done without use of the Cauchy-Binet Theorem nor divisor theory on graphs.

2.3 The Graphical Case

Let $G = ([n], E)$ be a connected graph on n vertices with signed vertex-edge incidence matrix N and Laplacian L. The rank of N (and hence of L) is equal to the maximal size of a linearly independent subset of the columns of N. This is exactly the number of edges in a spanning tree of G, i.e., $\operatorname{rank} \mathsf{N} = \operatorname{rank} \mathsf{L} = n - 1$. It follows that 0 is an eigenvalue of L of multiplicity 1, and it is easy to check that the all-ones vector $\mathbf{1}_n$ is a corresponding eigenvector. So the zonotope $\mathsf{Z}(\mathsf{L})$ is no longer a parallelepiped and its volume is no longer obtained by computing the determinant of L, as was the case in the previous section. Nonetheless, we now modify our techniques from the previous section to obtain a polyhedral proof of the classical matrix tree theorem.

Recall from the introduction that the original formulation for the matrix tree theorem states that, for G and $\mathsf{L} = \mathsf{N}\mathsf{N}^\top$ as in the previous paragraph, the number of vertices times the number of spanning trees is equal to the product of the nonzero eigenvalues of L. The classical proof of this version of the matrix tree theorem proceeds in three steps. First one uses the fact that 0 is an eigenvalue of L of multiplicity 1 with corresponding eigenvector $\mathbf{1}_n$ to show that all n of the maximal principal minors of L are equal and that the coefficient c_1 on the linear term of the characteristic polynomial of L is equal to n times any maximal principal minor. Then one uses the Cauchy-Binet theorem and the total unimodularity of N to prove that each of these minors equals the number of spanning trees of G. Finally the theorem follows from the observation that, since L is symmetric and 0 is an eigenvalue of multiplicity 1, the characteristic polynomial of L factors over $\mathbb{R}$ and hence the coefficient c_1 is the product of the nonzero eigenvalues of L. Our polyhedral proof of the matrix tree theorem follows a similar tack.

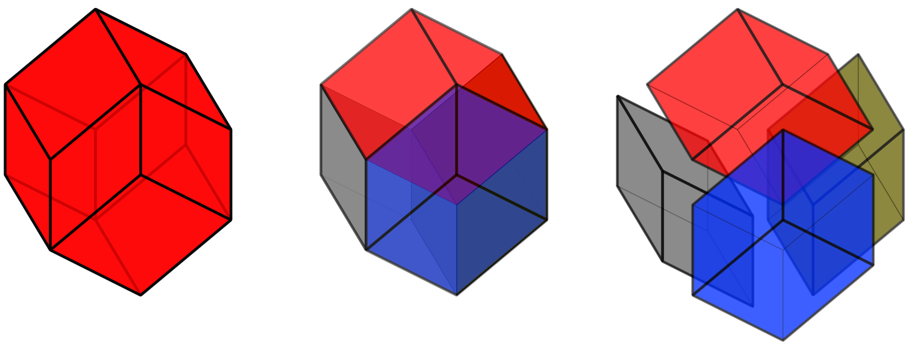
First we show in Proposition 2.10 that the zonotope $\mathsf{Z}(\mathsf{L})$ decomposes into n parallelepipeds all having the same volume. Then we explain how results from the previous sections show that the volume of one (and hence any) of these parallelepipeds is equal to the number of spanning trees of G. Finally we show that the volume of $\mathsf{Z}(\mathsf{L})$ is the product of the nonzero eigenvalues of L as follows: First we construct two full-dimensional zonotopes, one having d-dimensional volume equal to n times the $(d-1)$-dimensional volume of $\mathsf{Z}(\mathsf{L})$

and the other having volume equal to n times the product of the nonzero eigenvalues of L. Then we show that these two zonotopes have the same volume using a proof technique reminiscent of that used to prove Theorem 2.3. Moreover, we prove these results in greater generality whenever possible.

Our first goal is to see how the factor of n in the Matrix Tree Theorem manifests itself in the polyhedral set-up, the idea being that the zonotope of the Laplacian of G is the union of n zonotopes all having the same volume. We formalize this in the following result which holds in the more general case that the matrix M is only ***unimodular***, i.e., it has all *maximal* minors in $\{-1, 0, 1\}$.

Proposition 2.10. *Let* M *be a unimodular matrix and let* $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$. *Then the zonotope* $\mathsf{Z}(\mathsf{L})$ *decomposes into* $|\mathcal{B}(\mathsf{M}^\top)|$ *top dimensional parallelepipeds all having the same volume.*

Proof. As $\operatorname{im}(\mathsf{M}^\top)$ is orthogonal to $\ker \mathsf{M}$, an independent set in the matroid $\mathcal{M}(\mathsf{M}^\top)$ remains independent after multiplication by M, i.e., $\mathcal{M}(\mathsf{M}^\top)$ and $\mathcal{M}(\mathsf{L})$ are isomorphic matroids. As M is unimodular, so is $\mathsf{M}^\top$, and so any set of columns B of $\mathsf{M}^\top$ corresponding to a basis of its matroid is a $\mathbb{Z}$-basis for the lattice $\mathscr{L} = \mathbb{Z}^n \cap \operatorname{im}(\mathsf{M}^\top)$, that is, $\mathsf{Z}(B)$ is a fundamental parallelepiped of $\mathscr{L}$. It follows that every top dimensional parallelepiped in a maximal cubical decomposition of $\mathsf{Z}(\mathsf{L})$ is a fundamental parallelepiped for the lattice $\mathsf{M}\mathscr{L} = {}_{\mathbb{Z}}\langle \mathsf{L} \rangle$, the image of $\mathscr{L}$ under M. The result now follows from the fact that the volume of a fundamental parallelepiped of a lattice is a lattice invariant. $\square$



(a) $\mathsf{Z}(\mathsf{L}) \subset \mathbb{R}^4$ of K_4 (b) The decomposition (c) An exploded view

Figure 2.4: Proposition 2.10 at work on $\mathsf{Z}(\mathsf{L})$ of the complete graph K_4.

Example. Consider the complete graph K_4 on four vertices with edges oriented so that $i \to j$ if $i < j$. The signed vertex-edge incidence matrix N and the Laplacian L are

$$\mathsf{N} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \qquad \mathsf{L} = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$

The three dimensional zonotope $\mathsf{Z}(\mathsf{N}) \subset \mathbb{R}^4$ is a translate of the classical permutahedron obtained by taking the convex hull of all points obtained from $[1, 2, 3, 4]$ by permuting coordinates.

The zonotope $\mathsf{Z}(\mathsf{L})$ is the cubical zonotope (all of its facets are 2-cubes) displayed in Figure 2.4a. By Proposition 2.10 it is the union of four parallelepipeds of equal volume; see Figure 2.4b for the subdivision of $\mathsf{Z}(\mathsf{L})$ into parallelepipeds and Figure 2.4c for an exploded view of the subdivision. □

In the graphical case, Proposition 2.10 tells us that the zonotope $\mathsf{Z}(\mathsf{L})$ decomposes into n parallelepipeds in ${}_{\mathbb{R}}\langle \mathsf{N} \rangle$ all having the same volume. More explicitly the decomposition is $\mathsf{Z}(\mathsf{L}) = \bigcup_i \Pi_i$ where, for $i \in [n]$, the parallelepiped Π_i is generated by all of the columns of L save for the i^{th}. We now show that the volume of one (and hence, any) of these parallelepipeds is equal to the number of spanning trees of G. To see this first note that Theorem 1.3 holds regardless of the corank of the unimodular matrix involved, so in our case the volume of $\mathsf{Z}(\mathsf{N})$ equals the number of spanning trees of G (recall here that volume is taken with respect to the affine hull of the columns of N). Also independent of the corank of the defining matrix is Lemma 2.7, in which we showed that the lattice generated by the columns of the Laplacian is equal to the lattice generated by the matrix B whose columns are the barycenters of the facets of $\mathsf{Z}(\mathsf{N})$ scaled by a factor of 2. Since any $n - 1$ columns of L form a lattice basis for ${}_{\mathbb{Z}}\langle \mathsf{L} \rangle = {}_{\mathbb{Z}}\langle B \rangle$, we only need to check that an appropriate modification of Theorem 2.3 still holds when we drop the full-rank condition. Indeed, in the proof of the theorem the full-rank condition guaranteed that the columns of L formed a basis for their $\mathbb{Z}$-span, whereas when the corank of L is greater than 0 the columns over-determine the $\mathbb{Z}$-span. Nonetheless, the proof of Theorem 2.3 at the end of Section 2.1 goes through verbatim for the following theorem in which M is allowed to have arbitrary corank.

Theorem 2.11. *Let $\mathcal{M}$ be a regular matroid and M be a unimodular representation of $\mathcal{M}$ over $\mathbb{R}$. Let $\mathsf{L} = \mathsf{M}\mathsf{M}^\top$ and let $\overline{\mathsf{L}}$ be the matrix obtained by taking* *any basis for* ${}_{\mathbb{Z}}\langle \mathsf{L} \rangle$ *from among the columns of* L. *Then the volume of* $\mathsf{Z}(\mathsf{M})$ *equals the volume of* $\mathsf{Z}\left(\overline{\mathsf{L}}\right)$. $\square$

In the graphical case, taking Proposition 2.10 and Theorem 2.11 together shows that the volume of $\mathsf{Z}(\mathsf{L})$ is n times the number of spanning trees. So all that remains is to show that the volume of $\mathsf{Z}(\mathsf{L})$ is the product of nonzero eigenvalues of L. We will achieve this by defining two new full-dimensional zonotopes $\mathsf{Z}(\Lambda)$ and $\mathsf{Z}(\Gamma)$ and then showing that

(i) $\operatorname{vol} \mathsf{Z}(\Lambda) = n\lambda_1 \cdots \lambda_{n-1}$,

(ii) $\operatorname{vol} \mathsf{Z}(\Gamma) = n \operatorname{vol} \mathsf{Z}(\mathsf{L})$, and

(iii) $\operatorname{vol} \mathsf{Z}(\Lambda) = \operatorname{vol} \mathsf{Z}(\Gamma)$.

To construct these new zonotopes, define the matrices Λ and Γ by setting $\Lambda_{ij} = \mathsf{L}_{ij} + 1$ and letting $\Gamma = [\mathsf{L}|\mathbf{1}]$ be the matrix obtained from L by appending a column of ones.

To prove (i), observe that the columns of Λ arise by summing the vector $\mathbf{1}$ to each column of the rank $(n-1)$ matrix L, and that $\mathbf{1}$ is orthogonal to each of these columns. In consequence, the columns of the $n \times n$ matrix Λ are linearly independent. Thus, the zonotope $\mathsf{Z}(\Lambda)$ is an n-dimensional parallelopiped with volume equal to the product of the eigenvalues of Λ. If $\lambda \in \operatorname{Spec}(\mathsf{L})$ is a nonzero eigenvalue with eigenvector v, then the sum of the coordinates of v is zero. It follows that $\Lambda v = \mathsf{L}v = \lambda v$, and so λ is also an eigenvalue of Λ. Since $\mathbf{1} \in \ker L$, it follows that $\Lambda\mathbf{1} = n\mathbf{1}$, and so $\operatorname{Spec} \Lambda = (\operatorname{Spec}(\mathsf{L}) \setminus \{0\}) \cup \{n\}$, and $\operatorname{vol} \mathsf{Z}(\Lambda) = n\, \lambda_1 \cdots \lambda_{n-1}$.

For (ii), first observe that $\det(\mathsf{N}_{P_n}|\mathbf{1}) = n$, where N_{P_n} is the signed incidence matrix of the path on n vertices. Thus, the volume of any zonotope that is a prism $\mathsf{Z}(M|\mathbf{1}) = \mathsf{Z}(M) \times \mathbf{1}$ over a unimodular cube $\mathsf{Z}(M)$ is n. Our claim $\operatorname{vol} \mathsf{Z}(\Gamma) = n \operatorname{vol} \mathsf{Z}(\mathsf{L})$ now follows from the following general fact:

Proposition 2.12. *Let* $P \subseteq \mathbb{R}^n$ *be an* $(n-1)$*-dimensional lattice polytope with affine span* S *and let* $\mathcal{L} = S \cap \mathbb{Z}^n$ *be the induced lattice. For* $v \in \mathbb{Z}^n \setminus S$ *let* Q *be the prism* $P \times v$. *Then*

$$\operatorname{vol}(Q) \;=\; h_S(v) \operatorname{vol}_S(P),$$

where $h_S(v)$ *is the lattice height of* v *from* S *and* vol_S *is the induced volume form on* $\operatorname{aff} S$.

Proof. Without loss of generality we may assume that $\mathbf{0} \in S$ so that S is a linear hyperplane with primitive normal vector $u \in \mathbb{Z}^d$, say. For any $i \in \mathbb{Z}$ define S_i to be the parallel translate of S given by $\{x \in \mathbb{R}^n \mid \langle x, u \rangle = i\}$. Then for every $v \in \mathbb{Z}^n$ there is an $i \in \mathbb{Z}$ such that $v \in S_i$, and this is precisely the lattice height of v with respect to S, $h_S(v) = i$.

Suppose $v \in \mathbb{Z}^n$ satisfies $h_S(v) = 1$. In the k^{th} dilate of Q, the only lattice points of $\mathbb{Z}^n$ lie on the sections $Q \cap H_i$ where H_i is the hyperplane defined by $H_i = \{x \in \mathbb{R}^n : h_S(x) = i\}$ for $0 \leq i \leq k$. Moreover, the distribution of lattice points is the same in each section $Q \cap H_i$. Thus, the number of lattice points in the k^{th} dilate of $Q = P \times v$ is exactly

$$\begin{aligned} \#\left(Q \cap \frac{1}{k}\mathbb{Z}^n\right) &= (k\, h_S(v) + 1)\ \#\left(P \cap \frac{1}{k}\mathcal{L}\right) \\ &= (k+1)\ \#\left(P \cap \frac{1}{k}\mathcal{L}\right). \end{aligned}$$

So in this case we have

$$\begin{aligned} \mathrm{vol}(Q) &= \lim_{k\to\infty} \frac{1}{k^n}\#\left(Q \cap \frac{1}{k}\mathbb{Z}^n\right) \\ &= \lim_{k\to\infty} \frac{k+1}{k}\frac{1}{k^{n-1}}\#\left(P \cap \frac{1}{k}\mathcal{L}\right) \\ &= \lim_{k\to\infty} \frac{1}{k^{n-1}}\#\left(P \cap \frac{1}{k}\mathcal{L}\right) \\ &= \mathrm{vol}_S(P). \end{aligned}$$

Since Q is a full-dimensional prism, its lattice volume and Euclidean volume coincide. It follows that $\mathrm{vol}(P \times \mathbf{v}) = \mathrm{vol}_S(P)$ for any $\mathbf{v} \in \mathbb{R}^n$ such that $h_S(\mathbf{v}) = 1$.

For an arbitrary $\mathbf{v} \in \mathbb{Z}^n$ with $h_S(\mathbf{v}) = i$, the prism Q decomposes into i (typically rational) polytopes which are slices of Q sitting between the affine hyperplanes S_{j-1} and S_j where $j \in [i]$. Each of these slices is a translated copy of a height-one prism over P and hence has volume $\mathrm{vol}(P)$. As there are $h_S(\mathbf{v})$ many of them, the result follows. □

The missing claim (iii), $\mathrm{vol}\,\mathsf{Z}(\Lambda) = \mathrm{vol}\,\mathsf{Z}(\Gamma)$, is true in much greater generality, and it is this generalization that we state in Theorem 2.13, the proof of which uses a technique analogous to the proof of Theorem 2.3.

Theorem 2.13. *For any set $B = \{b_1, \dots, b_n\}$ of points that linearly span $\mathbb{R}^n$, let $\beta = \frac{1}{n}\sum_{i\in[n]} b_i$ be their barycenter and let $\Pi = \mathsf{Z}(B)$ be the zonotope they generate. Let P be the zonotope generated by β together with the points $b_i - \beta$ for $i \in [n]$. Then* $\operatorname{vol}\Pi = \operatorname{vol} P$.

Note that we obtain claim (iii) as a special case by taking B, Π, and P to be the columns of Λ, the zonotope $\mathsf{Z}(\Lambda)$, and the zonotope $\mathsf{Z}(\Gamma)$, respectively. Before proceeding with the proof in the general case, let us illustrate the techniques to be used:

Example. For the complete graph K_3 on three vertices, the zonotope $\mathsf{Z}(\Gamma)$ is the prism over the hexagon $\mathsf{Z}(\mathsf{L})$ shown in blue in Figure 2.5 intersecting the red parallelepiped $\mathsf{Z}(\Lambda)$.

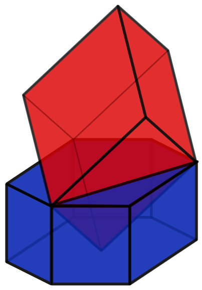


Figure 2.5: The zonotopes $P = \mathsf{Z}(\Gamma)$ and $\Pi = \mathsf{Z}(\Lambda)$ appearing in Theorem 2.13 in the case of the graph K_3.

For each sign vector $\epsilon \in \{+,-\}^3$, the simplicial cone spanned by the set $\epsilon\mathsf{L} = \{\epsilon_i \mathsf{L}_i\}$ intersects $\mathsf{Z}(\Gamma)$ and these intersections are the P_ϵ. By construction, all of the P_ϵ are full-dimensional except for $P_{\{-,-,-\}}$ which consists only of the origin. The seven full-dimensional pieces are illustrated center-left in Figure 2.6.

Six of the seven P_ϵ are visible in the figure, while the colored hexagon beneath the prism suggests the location of the invisible piece. By translating each P_ϵ by the sum of all Λ_i such that ϵ_i is negative, we obtain the union of the Q_ϵ as seen center-right in Figure 2.6. This union is exactly the zonotope of Λ. □

Proof of Theorem 2.13. We prove that there is a decomposition of P into full dimensional polytopal cells and a set of translations (one for each polytope in the decomposition) such that the union of the translated cells is exactly Π and that if two shifted cells intersect, they do so only on their boundaries.

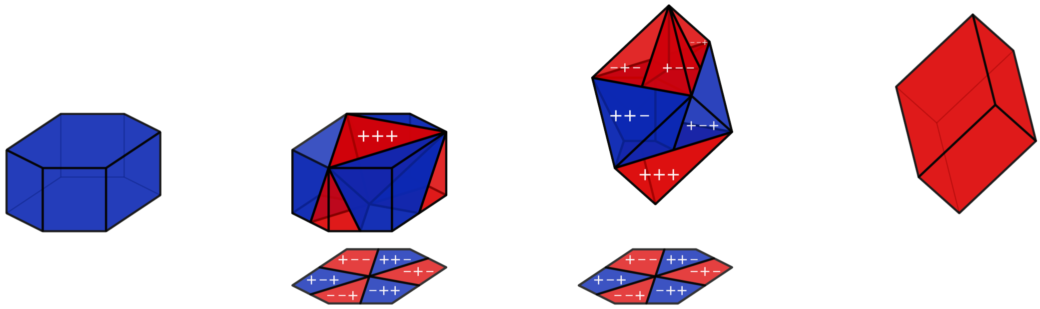


Figure 2.6: The zonotopes of Theorem 2.13 in the case of the graph K_3. From left to right: (i) $P = \mathsf{Z}(\Gamma)$, (ii) the decomposition $Z(\Gamma) = \bigcup P_\epsilon$, (iii) the rearrangement $\bigcup Q_\epsilon$, and (iv) the parallelepiped $\mathsf{Z}(\Lambda) = \bigcup Q_\epsilon$.

First we show that for every point $p \in P$ there is a sign vector $\epsilon = \epsilon(p)$ such that $p \in \mathsf{Z}(\epsilon B)$ where $\epsilon B := \{\epsilon_i b_i \mid i \in [n]\}$. As P is a zonotope, given any $p \in P$ there is an $\alpha \in [0,1]^{n+1}$ such that

$$
\begin{aligned}
p &= \alpha_{n+1}\beta + \sum_{i \in [n]} \alpha_i (b_i - \beta) \\
&= \sum_{i \in [n]} \frac{1}{n} \left(n\alpha_i + \alpha_{n+1} - \sum_{j \in [n]} \alpha_j \right) b_i \\
&= \sum_{i \in [n]} \frac{1}{n} \left((n-1)\alpha_i + \alpha_{n+1} - \sum_{j \in [n] \setminus \{i\}} \alpha_j \right) b_i.
\end{aligned}
$$

Let us abbreviate this last expression to $p = \sum_{i \in [n]} \gamma_i b_i$, where the γ_i are unique because the b_i form a basis of $\mathbb{R}^n$. Since each α_j is in $[0,1]$, it follows that $\gamma_i \in [-1,1]$ for all i. Therefore, setting $\epsilon_i = \operatorname{sign} \gamma_i$ if $\gamma_i \neq 0$ (and $\epsilon_i = \pm$ arbitrarily if $\gamma_i = 0$) proves the claim.

For each $\epsilon \in \{+,-\}^n$, define $P_\epsilon := P \cap \mathsf{Z}(\epsilon B)$ and $v_\epsilon = \sum_{i : \epsilon_i = -} b_i$, see Figure 2.5. By the previous paragraph we know that P is the union of the P_ϵ and we now show that the union of the translated polytopes $P_\epsilon + v_\epsilon$ is Π. To see this let $q = \sum_{i \in [n]} \alpha_i b_i \in \mathsf{Z}(B)$. If $q = 0$ then $q \in P$ so we may assume there is a non-negative integer k such that $\sum_{i \in [n]} \alpha_i \in (k, k+1]$. Moreover, we may assume (after permuting indices if necessary) that the α_i are decreasing, i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Now we define ϵ to be the sign vector with $\epsilon_i = -$ if and only if $i \leq k$. It follows that

$$\begin{aligned}
q - v_\epsilon = q - \sum_{i=1}^{k} b_i \\
&= \sum_{i<k} (\alpha_i - 1)(b_i - \beta + \beta) + (\alpha_k - 1) b_k + \sum_{j>k} \alpha_j (b_j - \beta + \beta) \\
&= \sum_{i<k} (\alpha_i - 1)(b_i - \beta) + (\alpha_k - 1) b_k + \\
&\quad + \sum_{j>k} \alpha_j (b_j - \beta) + \left(-(k-1) + \sum_{i \in [n] \setminus k} \alpha_i \right) \beta.
\end{aligned}$$

Since $b_k = n\beta - \sum_{i \neq k} b_i$, we can express the second summand as

$$\begin{aligned}
(\alpha_k - 1) b_k &= -(\alpha_k - 1) \left(-n\beta + \sum_{i \neq k} b_i \right) \\
&= -(\alpha_k - 1) \left(-\beta + \sum_{i \neq k} (b_i - \beta) \right) \\
&= \sum_{i<k} (1 - \alpha_k)(b_i - \beta) + (\alpha_k - 1)\beta + \sum_{j>k} (1 - \alpha_k)(b_j - \beta),
\end{aligned}$$

so that

$$\begin{aligned}
q - v_\epsilon &= \sum_{i<k} (\alpha_i - \alpha_k)(b_i - \beta) + \\
&\quad + \sum_{j>k} (\alpha_j - \alpha_k + 1)(b_j - \beta) + \left(-k + \sum_{i \in [n]} \alpha_i \right) \beta
\end{aligned}$$

and so $q - v_\epsilon \in P$ since $\alpha_i \geq \alpha_k$ if $i \leq k$ and $\alpha_k \geq \alpha_i$ otherwise. Moreover, our choice of k guarantees that all coefficients in this linear combination lie in $[0, 1]$.

Finally, in order to prove that our decomposition and rearrangement preserves volume, we must show that if any two translated cells $P_\epsilon + v_\epsilon$ and $P_{\epsilon'} + v_{\epsilon'}$ intersect, then they do so on a set of measure zero. To see this let $p \in P_\epsilon$ and $p' \in P_{\epsilon'}$ be such that $\epsilon \neq \epsilon'$ and $p + v_\epsilon = p' + v_{\epsilon'} \in \mathsf{Z}(B)$, where the coordinate $v_\epsilon = \sum_{i:\epsilon_i=-} b_i$ as before. Then

$$\begin{aligned}
0 &= p + v_\epsilon - p' - v_{\epsilon'} \\
&= \sum \alpha_i b_i + v_\epsilon - \sum \beta_i b_i - v_{\epsilon'} \\
&= \sum_{i\in\epsilon^-\setminus\epsilon'^-} (\alpha_i - \beta_i + 1) b_i + \sum_{j\in\epsilon'^-\setminus\epsilon^-} (\alpha_j - \beta_j - 1) b_j + \sum_{k:\epsilon_k=\epsilon'_k} (\alpha_k - \beta_k) b_k.
\end{aligned}$$

Since B is a basis it follows that the coefficient on any b_i in the final expression equals zero. Therefore $\alpha_k = \beta_k$ if $\epsilon_k = \epsilon'_k$, and otherwise we have either $\alpha_i = 0$ and $\beta_i = 1$ or vice versa. It now follows from the definitions of P_ϵ and $P_{\epsilon'}$ that $p \in \partial P_\epsilon$ and $p' \in \partial P_{\epsilon'}$, and the proof is complete. □

Taken together, Proposition 2.12 and Theorem 2.13 tell us that when M is a unimodular matrix with corank 1, we can recover the product of the nonzero eigenvalues of L by constructing a certain full-rank matrix Λ associated to L and analyzing the zonotope it generates. This construction essentially replaces the eigenvalue 0 of L with the eigenvalue n while fixing the other eigenvalues. We suspect that this can be strengthened to allow for unimodular representations of regular matroids of arbitrary corank in the statement of Theorem 2.3. Presently we have no proof for this fact, and so we leave it as a conjecture.

Conjecture 2.14. *Let $\mathcal{M}$ be a regular matroid and M a unimodular $m \times n$ representation of $\mathcal{M}$ with corank greater than 1. Then there exists an $m \times m$ matrix Λ with full rank such that every nonzero eigenvalue of L is an eigenvalue of Λ and every other eigenvalue of Λ depends only on the ambient dimension m.*

Chapter 3

The Arithmetic h-vector of a Lattice Point Configuration

Arithmetic matroids were recently introduced to study the interplay of matroidal and geometric properties of a configuration of lattice points. Building on the study of arithmetic Tutte polynomials in [47], arithmetic matroids were axiomatized in [21] and further studied in [11, 20, 24]. In this chapter we add to this growing body of research by providing a natural arithmetic analogue of the classical h-vector of a matroid in the case when an arithmetic matroid is represented by an integer matrix. In this case, we obtain the "arithmetic h-vector" by comparing certain lattice invariants of the zonotope and Lawrence polytope generated by an integer matrix.

The chapter is organized as follows. In Section 3.1, we provide the necessary background in Ehrhart theory of zonotopes and Lawrence polytopes, as well as providing the axiomatization of arithmetic matroids as introduced in [21]. Then in Section 3.2 we prove an identity relating the Ehrhart polynomial of a lattice zonotope and the Ehrhart δ-polynomial of the corresponding Lawrence polytope (Theorem 3.7). We then show how that identity leads to a natural generalization of the classical matroid h-vector in the arithmetic setting.

3.1 Background

The results in this chapter tie certain lattice properties of the zonotope generated by an integer matrix M together with properties of the Lawrence polytope and the arithmetic matroid generated by M. In this section we review each of these constructions and give the pertinent results from the literature.

Zonotopes and Ehrhart Theory

Zonotopes played the starring role in the previous chapter and will be a central object of study in the present one. Recall from Section 1.6 that, given a matrix M over $\mathbb{R}$ with n columns, the zonotope $\mathsf{Z}(M)$ generated by M is the Minkowski sum of the line segments $\operatorname{conv}\{\mathbf{0}, M_i\}$ ($i \in [n]$), where M_i is the i^{th} column of M. The dimension of $\mathsf{Z}(M)$ equals the rank of M and, when M is an integer matrix, the zonotope it generates is a lattice polytope.

Let M be a matrix with integer entries and consider the Ehrhart polynomial $\mathcal{E} = \mathcal{E}_{\mathsf{Z}(M)}(t)$ of the zonotope generated by M. The following expression for $\mathcal{E}$—first stated in [59] without proof—makes clear the relationship between the zonotope $\mathsf{Z}(M)$, the vector matroid $\mathcal{M}(M)$ (taken over $\mathbb{Q}$), and the choice of M as a representation for $\mathcal{M}(M)$.

Theorem 3.1 ([60])**.** *Let M be a $d\times n$ integer matrix and let $\mathsf{Z}(M)$ be the zonotope generated by the columns of M. Then the Ehrhart polynomial of $\mathsf{Z}(M)$ is given by*

$$\mathcal{E}_{\mathsf{Z}(M)}(t) = \sum_{I\in\mathcal{I}} g(M_I)t^{|I|}, \tag{3.1}$$

where the sum is over all independent subsets of columns of M and $g(M_I)$ is the greatest common divisor of the maximal minors of the matrix M_I.

We give a sketch of the proof of Theorem 3.1 as given in [60]. First apply Theorem 1.2 to obtain a decomposition of $\mathsf{Z}(M)$ into a disjoint union of half-open parallelepipeds, $\mathsf{Z}(M) = \bigsqcup_I \Pi_I$, where $I \in \mathcal{I}$ runs over all independent subsets of $\mathcal{M}(M)$. Then the Ehrhart polynomial $\mathcal{E}_{\mathsf{Z}(M)}(t)$ equals the sum of the Ehrhart polynomials of the constituent parallelepipeds Π_I. For every $I \in \mathcal{I}$, the lattice $\mathscr{L}_1(I) := {}_{\mathbb{Z}}\langle M_I\rangle$ is a sublattice of $\mathscr{L}_2(I) := {}_{\mathbb{R}}\langle M_I\rangle \cap \mathbb{Z}^d$ with index $[\mathscr{L}_1(I) : \mathscr{L}_2(I)] = g(M_I)$. Moreover, the parallelepiped Π_I is a fundamental domain for $\mathscr{L}_1(I)$. Thus the Ehrhart polynomial of Π_I is $g(I)t^{|I|}$ and the theorem follows.

Recall that for an arbitrary lattice d-polytope the coefficients $c_0, c_1, \ldots, c_d$ of its Ehrhart polynomial are generally rational numbers and that they are not well-understood except when $i \in \{0, d-1, d\}$. In contrast, Theorem 3.1 gives explicit formulae for the coefficients of the Ehrhart polynomial of a lattice zonotope that imply that each coefficient is, in fact, an integer that depends on the underlying matroid and the geometry of the particular choice of representation. Moreover, if the matrix M is unimodular, then Theorem 3.1 implies that the Ehrhart polynomial of $\mathsf{Z}(M)$ is exactly the f-polynomial of the regular matroid $\mathcal{M}(M)$. The converse is also easy to see and so we obtain the following corollary.

Corollary 3.2. *Let M be a rank d matrix and $c_i = [\mathcal{E}_{\mathsf{Z}(M)}]_i$ be the coefficient on t^i in the Ehrhart polynomial of the zonotope generated by M. Then the vector $(c_0, c_1, \ldots, c_d)$ is the f-vector of the matroid $\mathcal{M}(M)$ if and only if M is unimodular.*

Thus we may think of the Ehrhart polynomial of the zonotope generated by an arbitrary integer matrix as a "weighted f-vector" of the corresponding vector matroid.

Now that we have seen how the Ehrhart polynomial of a lattice zonotope $\mathsf{Z}(M)$ is related to the f-vector of the matroid $\mathcal{M} = \mathcal{M}(M)$, we turn our attention to the δ-polynomial of the Lawrence polytope $\Lambda(M)$ and its connection to the h-vector $h(\mathcal{M})$.

Lawrence polytopes and Ehrhart Theory

Recall from Section 1.6 that the Lawrence polytope $\Lambda(M)$ associated to a $d \times n$ matrix M is the convex hull of the columns of the matrix

$$\begin{pmatrix} M & \mathbf{0}_{d\times n} \\ I_n & I_n \end{pmatrix},$$

where $\mathbf{0}_{d\times n}$ is the $d\times n$ matrix with every entry equal to zero and I_n is the $n\times n$ identity matrix. The following theorem, due to Stapledon, was proved in [64].

Theorem 3.3. *Let M be a $d \times n$ integer matrix and $\mathcal{M}$ be the corresponding vector matroid (viewed as a matroid over $\mathbb{Q}$, say). Let $\mathscr{L} = {}_{\mathbb{Z}}\langle M\rangle$ be the lattice spanned by the columns of M. Then the Ehrhart δ-polynomial of the Lawrence polytope $\Lambda(M)$ with respect to $\mathscr{L}$ is*

$$\delta_{\Lambda(M)}(t) = \sum_{I \in \mathcal{I}} \mathcal{E}_{\mathsf{Z}(I)}(-1) t^{\dim I} h_I(t), \tag{3.2}$$

where $h_I(t)$ is the h-polynomial of the contraction $\mathcal{M}/I$.

This theorem follows by applying a result in [63] expressing the δ-vector of an arbitrary lattice polytope as the sum of dimensions of orbifold cohomology groups of a related toric variety to the work of various authors (see [28, 32, 25]) on the orbifold cohomology of hypertoric varieties.

Though we will only need the statements of Theorems 3.1 and 3.3, we will have an opportunity in Section 3.3 to discuss some objects lurking in the background of the latter. For that reason we now give a more detailed exposition of the work on which it is based. We assume a basic familiarity with manifolds, toric varieties, and homological algebra; see [67], [18], and [46] respectively for more on these topics.

A ***toric variety*** is an algebraic variety X whose geometry is determined by the combinatorics and arithmetics of a polyhedral fan $\Sigma(X)$ whose rays are elements of a lattice $\mathscr{L}$. A toric variety is a ***toric orbifold*** if the corresponding polyhedral fan is simplicial. More generally, an ***orbifold*** is, loosely speaking, a topological space that locally looks like the quotient of a Euclidean space by the linear action of a finite group ([66]). A general orbifold is a manifold that typically has complicated gluing maps, though this difficulty is avoided in the toric case where the gluing maps are determined by the fan $\Sigma(X)$.

A toric variety X is ***projective*** if there is an associated lattice polytope $P = P(X)$ (assumed to be full-dimensional) with halfspace description $(A, \mathbf{b})$ where the combinatorics of the face structure of P determine a fan $\Sigma(P)$ such that $\Sigma(P) = \Sigma(X)$, and the vector $\mathbf{b}$ determines the projective embedding of X. Given a lattice polytope P as above, one typically takes the fan $\Sigma(P)$ to be the ***inner normal fan*** of P defined as follows. For each facet F of P, let the inner normal vector of F be denoted $\mathbf{a}_F$. Every face $F \subset P$ is the intersection of a set of $\mathcal{F}$ of facets of P, and we let σ_F be the strongly-convex rational polyhedral cone $\sigma_F := \operatorname{pos}\{\mathbf{a}_F \mid F \in \mathcal{F}\}$ generated by the facets of P containing F. The inner normal fan of P is then the union of the σ_F over all faces of P.

There are toric varieties that are not projective. A toric variety X determined by a (not necessarily bounded) polyhedron is called ***semi-projective***. More generally, X is a ***semi-projective*** toric variety if it is determined by a (not necessarily bounded) polyhedron. Given a lattice $(d-1)$-polytope P in $\mathbb{R}^d$, one can construct a semi-projective toric orbifold as follows. Consider the inner normal fan $\Sigma = \Sigma(\operatorname{pos}(P))$ of the cone over P. Any triangulation of P induces a triangulation of $\operatorname{cone}(P)$ which in turn induces a simplicial refinement $\widetilde{\Sigma}$ of the fan Σ. Thus the toric variety X associated with the fan $\widetilde{\Sigma}$ is a toric orbifold. We now consider certain semi-projective toric orbifolds arising from Lawrence polytopes.

Let A be an integer $d \times n$ matrix such that $A^{\pm} = [A, -A]$ generates $\mathbb{Z}^d$ as a semigroup. Since $A^{\pm}$ is centrally symmetric there is an integer matrix B such that the Lawrence matrix of B is a Gale dual of $A^{\pm}$. Moreover, by Corollary 5.4.9 of [31], there is a bijection between open cells in the chamber complex of $A^{\pm}$ and regular polyhedral subdivisions of the Lawrence polytope $\Lambda(B)$. Under this bijection, a maximal open cell of the chamber complex of $A^{\pm}$ corresponds to a regular triangulation of $\Lambda(B)$. Thus for any vector $\mathbf{v}$ in a maximal open cell C of the chamber complex of $A^{\pm}$ we obtain a semi-projective toric variety $X(A^{\pm}, \mathbf{v})$ from the cone over the regular triangulation of the Lawrence polytope $\Lambda(B)$ corresponding to the cell C. Any semi-projective toric variety obtained in this way is a ***Lawrence toric variety***. The following fact relating the Betti numbers of a Lawrence toric variety $X(A^{\pm}, \mathbf{v})$ and the matroid h-vector of the Gale dual of A is the content of Corollary 4.6 in [28].

Theorem 3.4. *Let A and $A^{\pm}$ be as above and let B be the Gale dual of A. Then, for $i \in \{0, 1, \dots, d\}$, the Betti numbers of the Lawrence toric variety $X = X(A^{\pm}, \mathbf{v})$ are given by*

$$\dim_{\mathbb{Q}} H^{2i}(X(A^{\pm}, \mathbf{v}); \mathbb{Q}) = h_i(\mathcal{M}(B)),$$

and vanish otherwise. In particular, the Betti numbers of X are independent of the choice of the regular triangulation used to construct X.

Arithmetic Matroids

The connection between the Ehrhart polynomial of $\mathsf{Z}(M)$ and the Ehrhart δ-polynomial of $\Lambda(M)$ is nicely expressed in terms of the arithmetic Tutte polynomial of the underlying arithmetic matroid $\mathcal{T}(M)$. The arithmetic Tutte polynomial was introduced in [47] as a tool for studying toric hyperplane arrangments. Arithmetic matroids were then axiomatized in [21] and were subsequently studied in [11, 20, 24]. For completeness, we now give the axioms for general arithmetic matroids following [21] and then explain how this specializes to the case of $\mathbb{Q}$-representable matroids represented by an integer matrix.

Let M be a $d \times n$ matrix with entries in a field and let $\mathcal{M}(M)$ be the vector matroid of (the columns of) M. A ***multiplicity function*** on M is a map $m : \mathbb{P}(M) \to \mathbb{N} \setminus \{0\}$ that satisfies all of the following properties

1. if $S \subseteq [n]$ and if $i \in \mathrm{cl}(S)$, then $m(S \cup \{i\})$ divides $m(S)$;
2. if $S \subseteq [n]$ and if $i \notin \mathrm{cl}(S)$, then $m(S)$ divides $m(S \cup \{i\})$;

3. if $S \subseteq T \subseteq [n]$ and T can be written as a disjoint union $T = S \sqcup F \sqcup G$ in such a way that for all A with $S \subseteq A \subseteq T$ the equality $\mathrm{rank}(A) = \mathrm{rank}(S) + |A \cap F|$ holds, then
$$m(S) \cdot m(T) = m(A \cup F) \cdot m(A \cup G);$$
4. if $S \subseteq T$ such that $\mathrm{rank}(S) = \mathrm{rank}(T)$ then
$$\mu_T(S) := \sum_{S \subseteq A \subseteq T} (-1)^{|A|-|S|} m(A) \geq 0;$$
5. if $S \subseteq T$ such that $\mathrm{corank}(S) = \mathrm{corank}(T)$ then
$$\mu_T^*(S) := \sum_{S \subseteq A \subseteq T} (-1)^{|A|-|S|} m([n] \setminus T) \geq 0.$$

An ***arithmetic matroid*** is a pair $(\mathcal{M}, m)$ where $\mathcal{M} = \mathcal{M}(M)$ is the vector matroid of some matrix M and m is a multiplicity function on M. The arithmetic matroid $\mathcal{A} = (\mathcal{M}, m)$ inherits the basic matroid structure and nomenclature (i.e., bases, dependent set, rank) from the underlying matroid $\mathcal{M}$. The multiplicity function of an arithmetic matroid retains some of the geometric structure that is usually lost when passing from a vector configuration to the corresponding matroid. When the multiplicity function is clear from the context we call the arithmetic matroid $\mathcal{A}(\mathcal{M}(M), m)$ the arithmetic matroid generated by M.

Those arithmetic matroids with which we will be concerned are of the form $(\mathcal{M}, m)$ where the underlying matroid is $\mathbb{Q}$-representable and represented by an integer matrix M. In this case we assume the multiplicity function m is as follows: for any subset S of the column indices of M, the integer $m(S)$ is equal to the greatest common divisor of the minors of M_S of size $\mathrm{rank}(S)$. This fact, viewed in light of Theorem 3.1, suggests a strong connection between the arithmetic matroid of a set of lattice points and the Ehrhart polynomial of the zonotope they generate. To see this connection, consider the following modification of the classical Tutte polynomial to the arithmetic setting. The ***arithmetic Tutte polynomial*** of an arithmetic matroid $\mathcal{A}$ is the bivariate polynomial $\mathcal{T}_{\mathcal{A}}(x, y)$ defined by
$$\mathcal{T}_{\mathcal{A}}(x, y) := \sum_{S \subset E} m(S)(x-1)^{r-\mathrm{rank}(S)}(y-1)^{|S|-\mathrm{rank}(S)},$$

where $r = \operatorname{rank}(\mathcal{A})$. Note that, for any arithmetic matroid generated by a unimodular matrix, the arithmetic Tutte polynomial equals the classical Tutte polynomial.

Theorem 3.5. *[20] Let M be a $d \times n$ integer matrix of rank r and $\mathcal{A}$ be the corresponding arithmetic matroid. Then the Ehrhart polynomial of the zonotope $\mathsf{Z}(M)$ is given by*

$$\mathcal{E}_{\mathsf{Z}(M)}(t) = t^r \mathcal{T}_{\mathcal{A}}(1 + 1/t, 1),$$

where $\mathcal{T}_{\mathcal{A}}$ is the arithmetic Tutte polynomial of $\mathcal{A}$.

Recall from Section 1.7 that the leading coefficient of the Ehrhart polynomial of a lattice polytope is its Euclidean volume. An immediate consequence of this fact, taken together with Theorem 3.5 is the following corollary:

Corollary 3.6 ([47])**.** *The volume of the zonotope $\mathsf{Z}(M)$ is $\mathcal{T}(1, 1)$.*

Theorem 3.5 together with Theorem 3.1 and Corollary 3.2 lead us to think of the Ehrhart polynomial of the zonotope generated by an integer matrix M as the "arithmetic f-vector" of the arithmetic matroid $\mathcal{A} = (M, m)$. It is then a natural question to ask for the arithmetic analogue of the classical matroid h-vector. We answer this question in the next section.

3.2 The Arithmetic h-vector

Let $A \subset \mathbb{Z}^d$ be a finite set of vectors that span $\mathbb{Z}^d$. On the one hand, A gives rise to a matroid $\mathcal{M}$ (over $\mathbb{Q}$) whose f- and h-vectors are related via

$$f_k = \sum_{i=0}^{k} \binom{d-i}{k-i} h_i. \tag{3.3}$$

These relations can be encoded into a single equation as follows

$$f_{\mathcal{M}}(t) = \sum_{i=0}^{r} h_i t^i (t+1)^{d-i}.$$

On the other hand, consider the arithmetic matroid $\mathcal{A}$ generated by A and ask what are natural analogues of the f- and h-vectors in the arithmetic setting. A result of Stanley (see Theorem 3.1) suggests that the Ehrhart polynomial $\mathcal{E}_{\mathsf{Z}(A)}(x)$ of the zonotope generated by A can be viewed as an arithmetic version of the f-vector of $\mathcal{M}$. The first theorem we prove is that (3.3) continues to hold if we replace the f-vector by the Ehrhart polynomial of $\mathsf{Z}(A)$ and the h-vector by the Ehrhart δ-vector of the Lawrence polytope $\Lambda(A)$.

Theorem 3.7. *Let $A \subset \mathbb{Z}^d$ be a finite set of vectors that spans $\mathbb{Z}^d$. Then the Ehrhart polynomial of the zonotope $\mathsf{Z}(A)$ is related to the Ehrhart δ-vector of the Lawrence polytope $\Lambda(A)$ via*

$$\mathcal{E}_{\mathsf{Z}(A)}(t) = \sum_{i=0}^{r} [\delta_{\Lambda(A)}(x)]_i t^i (t+1)^{d-i}. \tag{3.4}$$

Proof. Let $A \subset \mathbb{Z}^d$ be a finite set of vectors spanning $\mathbb{Z}^d$ and let $\mathcal{M}$ denote the matroid $\mathcal{M}(A)$ over $\mathbb{Q}$. For any independent set $I \in \mathcal{M}$ we can express the half-open cube $\mathcal{C}(I)$ as a disjoint union of open zonotopes:

$$\mathcal{C}(I) = \bigsqcup_{J \subseteq I} \mathsf{Z}(J)^{\circ}.$$

By Theorem 3.1 we may write the Ehrhart polynomial of $\mathsf{Z}(A)$ as

$$\begin{aligned}
\mathcal{E}_{\mathsf{Z}(A)} t &= \sum_{I \in \mathcal{M}} \mathcal{E}_{\mathcal{C}(I)}(1) t^{\dim I} \\
&= \sum_{i=0}^{d} \left(\sum_{\substack{I \in \mathcal{M} \\ \dim I = i}} \sum_{J \subseteq I} \mathcal{E}_{\mathsf{Z}(J)}(-1) \right) t^i \\
&= \sum_{i=0}^{d} \left(\sum_{\substack{J \in \mathcal{M} \\ \dim J \leq i}} f_{i-\dim J}(J) \mathcal{E}_{\mathsf{Z}(J)}(-1) \right) t^i,
\end{aligned}$$

where $f_{i-\dim J}(J)$ is the number of independent sets of size $(i - \dim J)$ in $\mathcal{M}/J$, which is exactly the number of independent sets of size i in $\mathcal{M}$ that contain J. So to prove the theorem we need to show that the coefficient on t^l in

$$r_A(t) := \sum_{i=0}^{d} [\delta_{\Lambda(A)}(t)]_j t^j (t+1)^{d-j}$$

is given by

$$[r_A(t)]_l = \sum_{\substack{J \in \mathcal{M} \\ \dim J \leq l}} f_{l-\dim J}(J) \mathcal{E}_{\mathsf{Z}(J)}(-1),$$

for all $l \in \{0, 1, \ldots, d\}$.

By Theorem 3.3 the j^{th} coefficient of $\delta_{\Lambda(A)}(t)$ is given by

$$[\delta_{\Lambda(A)}(t)]_j = \sum_{I \in \mathcal{M}} \mathcal{E}_{\mathsf{Z}(I)}(-1) \sum_{k=0}^{\operatorname{codim} I} f_k(I) \left[(1-t)^{\operatorname{codim} I - k}\right]_{j-k-\dim I} \tag{3.5}$$

$$= \sum_{i=0}^{d} \left(\sum_{\substack{I \in \mathcal{M} \\ \dim I = i}} \mathcal{E}_{\mathsf{Z}(I)}(-1) \sum_{k=0}^{d-i} f_k(I)(-1)^{j-k-\dim I} \binom{d-i-k}{d-j} \right) \tag{3.6}$$

$$= \sum_{i=0}^{j} \left(\sum_{\substack{I \in \mathcal{M} \\ \dim I = i}} \mathcal{E}_{\mathsf{Z}(I)}(-1) \sum_{k=0}^{j-i} f_k(I)(-1)^{j-k-\dim I} \binom{d-i-k}{d-j} \right). \tag{3.7}$$

With this expression in hand we can now we compute the desired coefficient $[r_A(t)]_l$ via the following bird tracks, to which we append a line-by-line explanation to aid the reader. Letting $\alpha_I = \mathcal{E}_{\mathsf{Z}(I)}(-1)$, we have

$$\begin{aligned}
[r_A(t)]_l &= \sum_{j=0}^{d} [\delta_A(t)]_j \left[(t+1)^{d-j}\right]_{l-j} \\
&= \sum_{j=0}^{l} \left(\sum_{i=0}^{j} \sum_{\substack{I\in\mathcal{M} \\ \dim I = i}} \alpha_I \left(\sum_{k=0}^{j-i} f_k(I)(-1)^{j-k-i} \binom{d-i-k}{d-j} \right) \right) \binom{d-j}{d-l} && (3.8) \\
&= \sum_{i=0}^{l} \sum_{\substack{I\in\mathcal{M} \\ \dim I = i}} \alpha_I \left(\sum_{j=i}^{l} \sum_{k=0}^{j-i} f_k(I)(-1)^{j-k-i} \binom{d-i-k}{d-j} \binom{d-j}{d-l} \right) && (3.9) \\
&= \sum_{i=0}^{l} \sum_{\substack{I\in\mathcal{M} \\ \dim I = i}} \alpha_I \left(\sum_{j=0}^{l-i} \sum_{k=0}^{j} f_k(I)(-1)^{j-k} \binom{d-i-k}{d-l} \binom{l-i-k}{l-i-j} \right) && (3.10) \\
&= \sum_{i=0}^{l} \sum_{\substack{I\in\mathcal{M} \\ \dim I = i}} \alpha_I \left(\sum_{k=0}^{l-i} f_k(I) \binom{d-i-k}{d-l} \left(\sum_{j=k}^{l-i} (-1)^{j-k} \binom{l-i-k}{j-k} \right) \right) && (3.11) \\
&= \sum_{i=0}^{l} \sum_{\substack{I\in\mathcal{M} \\ \dim I = i}} \alpha_I \left(\sum_{k=0}^{l-i} f_k(I) \binom{d-i-k}{d-l} \left(\sum_{j=0}^{l-i-k} (-1)^{j} \binom{l-i-k}{j} \right) \right), && (3.12)
\end{aligned}$$

with the following rationale: Equation 3.8 follows from 3.7 and the fact that the $(l-j)^{th}$ coefficient on $(t+1)^{d-j}$ equals 0 if $j > l$; Equation 3.9 is obtained from 3.8 by switching the order of summatation of i and j; Equation 3.10 follows from 3.9 by first using the identity $\binom{u}{v}\binom{v}{w} = \binom{v}{w}\binom{v-w}{v-w}$ and then adjusting the dummy variable j to start at 0; Equation 3.11 comes from first summing over k and then over j; and finally Equation 3.12 is obtained by again adjusting j to start at 0.

The binomial identity

$$\sum_{u=0}^{v} (-1)^u \binom{v}{u} = \begin{cases} 1 & \text{if } v = 0, \\ 0 & \text{for all } v \in \mathbb{Z}_{>0}. \end{cases}$$

implies that the innermost sum in equation 3.12 is

$$\sum_{j=0}^{l-i-k} (-1)^j \binom{l-i-k}{j} = \begin{cases} 1 & \text{if } l = i+k, \\ 0 & \text{else.} \end{cases}$$

It follows from 3.12 that

$$[r_A(t)]_l = \sum_{i=0}^{l} \sum_{\substack{I \in \mathcal{M} \\ \dim I = i}} \mathcal{E}_{\mathsf{Z}(I)}(-1) f_{l-i}(I) \tag{3.13}$$

$$= \sum_{\substack{J \in \mathcal{M} \\ \dim J \leq l}} \mathcal{E}_{\mathsf{Z}(J)}(-1) f_{l-\dim J}(J). \tag{3.14}$$

So we have shown that the l^{th} coefficient of the Ehrhart polynomial of the zonotope generated by A and the transformation $r_A(t)$ of the δ polynomial of the Lawrence polytope of A (when expressed in the standard basis) are equal, completing the proof. □

A well-known feature of the Ehrhart polynomial of a lattice d-polytope P is that the following values coincide:

1. the Euclidean volume $\mathrm{vol}(P)$;
2. the leading coefficient of $\mathcal{E}_P(k)$; and
3. $\frac{1}{d!} \sum_{i=0}^{d} [\delta_P(t)]_i$.

Combining Theorem 3.7 and these basic facts from Ehrhart theory allows us to give a simple proof of the following corollary.

Corollary 3.8. *For $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d$ that spans $\mathbb{Z}^d$, the Euclidean d-dimensional volume of the zonotope $\mathsf{Z}(A)$ equals the normalized $(d+n-1)$-dimensional volume of the Lawrence polytope $\Lambda(A)$:*

$$\mathrm{vol}(\mathsf{Z}(A)) = \frac{1}{\dim(\Lambda(A))!} \mathrm{vol}(\Lambda(A)).$$

Proof. Multipling both sides of equation 3.4 by k^r and taking the limit $k \to \infty$ shows that the leading coefficient $c_r = \mathrm{vol}(\mathsf{Z}(A))$ of the Ehrhart polynomial of $\mathsf{Z}(A)$ equals $\sum_{i=0}^{r} [\delta_{\Lambda(A)}]_i = \dim(\Lambda(A))! \,\mathrm{vol}(\Lambda(A))$, completing the proof. □

Recall from Section 1.6, that by intersecting the Lawrence polytope $\Lambda(A)$ with the affine subspace $S := \mathbb{R}^d \times {}_{\mathbb{R}}\langle \mathbf{1}_n \rangle$ one obtains a polytope P that is combinatorially equivalent to $\mathsf{Z}(A)$. A simple computation actually shows that $P = \frac{1}{n}\mathsf{Z}(A)$.

Recall that for any matroid $\mathcal{M}$ the entries of the h-vector of $\mathcal{M}$ appear as the coefficients of the shelling polynomial of the matroid complex $\Delta_{\mathcal{M}}$ and that the shelling polynomial is an evaluation of the Tutte polynomial $T_{\mathcal{M}}(x, y)$ at $y = 1$:

$$T_{\mathcal{M}}(x, 1) = \sum_{i=0}^{d} h_i x^{d-i}. \tag{3.15}$$

For an arithmetic matroid represented by the vectors $A \subset \mathbb{Z}^d$, Theorem 3.7 tells us that the δ-vector of the Lawrence polytope $\Lambda(A)$ is the analogue of the usual matroid h-vector in the arithmetic case. Our next result strengthens this analogy by proving that equation (3.15) continues to hold if we replace the h-vector with the δ-vector and the Tutte polynomial with the arithmetic Tutte polynomial.

Theorem 3.9. *Let $\mathcal{A}$ be a rank d arithmetic matroid represented by the integer vectors $A \subset \mathbb{Z}^d$, where A spans $\mathbb{Z}^d$, and let $\delta(t)$ be the numerator of the Ehrhart series of $\Lambda(A)$. Then the arithmetic Tutte polynomial of A evaluated at $y = 1$ is*

$$\mathcal{T}_{\mathcal{A}}(x, 1) = \sum_{i=0}^{d} \delta_i x^{d-i}, \tag{3.16}$$

where δ_i is the coefficient on t^i in $\delta(t)$.

Proof. Combining Theorems 3.5 and 3.7, we obtain the following relationships between the Ehrhart polynomial of the zonotope $\mathsf{Z}(A)$, the arithmetic Tutte polynomial $\mathcal{T}_{\mathcal{A}}$ and the polynomial

$$\sum_{i=0}^{d} \delta_i t^i (t+1)^{d-i}$$

appearing on the right-hand side of equation 3.4:

$$\mathcal{E}_{\mathsf{Z}(A)}(t) = t^d\, \mathcal{T}_{A}(1 + 1/t, 1) = \sum_{i=0}^{d} \delta_i t^i (t+1)^{d-i}.$$

Letting $x = 1 + 1/t$ we obtain

$$\begin{aligned}(x-1)^{-d}\, \mathcal{T}_A(x,1) &= \sum_{i=0}^{d} \delta_i \left(\frac{1}{x-1}\right)^i \left(\frac{x}{x-1}\right)^{d-i} \\ &= (x-1)^{-d} \sum_{i=0}^{d} \delta_i x^{d-i},\end{aligned}$$

and hence $\mathcal{T}_A(x,1) = \sum_{i=0}^{d} \delta_i x^{d-i}$ for all $x \neq 1$. When $x = 1$, the right hand side is the normalized volume of the Lawrence polytope, while Proposition 3.6 tells us that $\mathcal{T}_A(1,1)$ is the Euclidean volume of the zonotope $\mathsf{Z}(A)$. As these two values are equal by Corollary 3.8, it follows that $\mathcal{T}_A(x,1) = \sum_{i=0}^{d} \delta_i x^{d-i}$ for all x as desired. □

In [47], a different expression for $\mathcal{T}_A(x,1)$ was obtained in terms of the lattice points in a shifted copy of the zonotope $\mathsf{Z}(A)$ as follows. Define $\mathcal{H}_0$ to be the collection of all linear hyperplanes spanned by subsets of A and write $\mathcal{H}_{\mathbb{Z}}$ for the collection of affine hyperplanes consisting of all integer translates of the hyperplanes in $\mathcal{H}_0$. Let $\mathbf{v} \in \mathbb{R}^d \setminus \mathcal{H}_{\mathbb{Z}}$ be any vector with $\|\mathbf{v}\| \ll 1$ such that the shifted zonotope $\mathsf{Z}_{\mathbf{v}}(A) := \mathsf{Z}(A) - \mathbf{z}$ contains the origin. Let $P = \mathsf{Z}_{\mathbf{v}}(A) \cap \mathbb{Z}^d$ be the set of lattice points in the shifted zonotope and for $i = 0, 1, \ldots, d$ define P_i as follows: $P_0 := \{\mathbf{0}\}$ and for $i > 0$ the set P_i consists of those nonzero lattice points p in P such that i is the dimension of the smallest face of $\mathsf{Z}_{\mathbf{v}}(A)$ containing p. Then Theorem 4.1 in [47] tells us that the arithmetic Tutte polynomial of A evaluated at $y = 1$ is

$$\mathcal{T}(x,1) = \sum_{i=0}^{d} |P_i| x^{d-i}.$$

Combining this fact with Theorem 3.9 proves the following

Corollary 3.10. *Let $A \subset \mathbb{Z}^d$ be a subset that spans $\mathbb{Z}^d$, let δ be the numerator of the Ehrhart series of $\Lambda(A)$. Then letting P_i ($i = 0, 1, \ldots, d$) be as in the preceding paragraph, yields $\delta_i = |P_i|$.* □

Having given evidence that the δ-vector of the Lawrence polytope $\Lambda(A)$ is a good candidate for the arithmetic matroid analogue of the matroid h-vector, we close this chapter by studying the case when these two vectors coincide.

3.3 Unimodular Arithmetic Matroids

Recall that a regular matroid is a matroid that is representable over $\mathbb{Q}$ by a unimodular matrix A. In this case the Ehrhart polynomial of $\mathsf{Z}(A)$ is

$$\mathcal{E}_{\mathsf{Z}(A)}(x) = \sum_{i=0}^{d} f_i x^i,$$

where $f = (f_0, f_1, \ldots, f_d)$ is the f-vector of $\mathcal{M}$ by Corollary 3.2. The next corollary follows immediately from this fact and Theorem 3.7.

Corollary 3.11. *Let $A \subset \mathbb{Z}^d$ be a set of integer vectors that spans $\mathbb{Z}^d$. Then the Ehrhart δ-vector of the Lawrence polytope $\Lambda(A)$ is the h-vector of $\mathcal{M}(A)$ if and only if A is unimodular.*

This corollary can also be proven from an arithmetic matroid point of view as follows. Unimodular matrices are precisely those giving rise to arithmetic matroids whose arithmetic Tutte polynomial $\mathcal{T}_{\mathcal{A}}(x, y)$ equals the classical Tutte polynomial $T_{\mathcal{M}}(x, y)$. As the h-vector of $\mathcal{M}$ can be read off from $T_{\mathcal{M}}(x, 1)$, applying Theorem 3.7 yields Corollary 3.11.

Recall from Section 3.1 that for an integer matrix A subject to some mild conditions, a Lawrence toric variety was defined to be a semi-projective toric orbifold $X(A^{\pm}, \mathbf{v})$ given by the cone over a regular triangulation of a Lawrence polytope $\Lambda(B)$ generated by the Gale dual of A. Moreover, by Theorem 3.4 we know that, for all $0 \leq i \leq \operatorname{rank}(B)$, the Betti number β_{2i} of $X(A^{\pm}, \mathbf{v})$ satisfies $\beta_{2i} = h_i(\mathcal{M}(B))$ and that all other Betti numbers vanish. So when B is unimodular the Betti numbers of $X(A^{\pm}, \mathbf{v})$ also appear as the Ehrhart δ-vector of the corresponding Lawrence polytope.

Chapter 4

The Internal Order and Pure $\mathcal{O}$-sequences

Consider the poset $\mathcal{P} = (\mathbb{N}^d, \leq)$ where $\mathbf{v} \leq \mathbf{w}$ if $v_i \leq w_i$ for all i. The $\mathcal{O}$-sequence of an order ideal $\mathcal{O}$ in $\mathcal{P}$ is the vector $(h_0, h_1, \dots)$ where h_i is the number of elements $\mathbf{v} \in \mathcal{O}$ with coordinate sum i. A pure $\mathcal{O}$-sequence is the $\mathcal{O}$-sequence of a pure order ideal (see the definition in the next section).

In [58], Stanley proved that the h-vector of a matroid is an $\mathcal{O}$-sequence and conjectured that it is pure $\mathcal{O}$-sequence. This long-standing conjecture has received a great deal of attention in recent years ([15, 16, 22, 27, 35, 44, 45, 48]).

Las Vergnas [37] used the concepts of internal and external activities to define three posets on the bases of an ordered matroid. One of these orders is the *internal order* which Las Vergnas proved is a graded lattice whose height function encodes the h-vector of the underlying unordered matroid. Though we will focus entirely on this order, the other two are of independent interest [3, 10].

In this chapter, we initiate the study of a new class of matroids, called *internally perfect matroids* (see Definition 1 below). Our main theoretical result (Theorem 4.11) states that, assuming the injectivity of a certain map, the internal order of every such matroid is isomorphic to a pure multicomplex, thus proving Stanley's conjecture for this class. Moreover, we conjecture that the map in the previous sentence is injective for every internally perfect matroid.

The chapter unfolds as follows. In Section 4.1 we begin by recalling Stanley's conjecture and giving a brief summary of the known results. In Section 4.2 we review the internal order of a matroid following [37] and introduce the concept of an *internally perfect basis* of a matroid. In Section 4.3, we prove a number of results concerning internally perfect bases of a matroid, introduce a map $\mu : \mathcal{B} \to \mathbb{N}^{h_1}$, and prove Theorem 4.11 which states that if every basis of a matroid is internally perfect and the map μ is injective, then the h-vector of μ is a pure $\mathcal{O}$-sequence and hence satisfies Stanley's conjecture. In Section 4.4 we show that internally perfect matroids are closed with respect to some matroid operations and discuss computational obstacles to finding interesting examples of perfect matroids.

4.1 Background

Recall from Section 1.4 that, given a matroid $\mathcal{M}$, the h-vector of $\mathcal{M}$ is the h-vector of the matroid complex $\Delta(\mathcal{M})$, which is a pure simplicial complex of dimension rank$(\mathcal{M})$.

An ***order ideal*** $\mathcal{P}'$ of a poset $\mathcal{P}$ is a subposet of $\mathcal{P}$ such that if $p \in \mathcal{P}'$ and $q \prec p$ in $\mathcal{P}$ then $q \in \mathcal{P}'$. An order ideal $\mathcal{P}'$ of a graded poset is ***pure*** if all maximal elements of $\mathcal{P}'$ have the same height. A ***multicomplex*** is an order ideal of the poset $(\mathbb{N}^d, \leq)$ where $\leq$ is the dominance order, i.e., for two elements $\mathbf{a}, \mathbf{b} \in \mathbb{N}$ we have $\mathbf{a} \leq \mathbf{b}$ if and only if $a_i \leq b_i$ for all $i \in [d]$. A vector $f = (f_0, f_1, \dots, f_i)$ is a (***pure***) ***$\mathcal{O}$-sequence*** if there is a (pure) multicomplex $\mathcal{O}$ such that f_i is the number of elements of $\mathcal{O}$ with coordinate sum i.

In [58], Stanley proved that h-vector of a matroid is an $\mathcal{O}$-sequence and made the following conjecture:

Conjecture 4.1 (Stanley, 1977)**.** *The h-vector of a matroid is a pure $\mathcal{O}$-sequence.*

Though this conjecture is trivial for some small classes of matroids (e.g., for the uniform matroid $\mathcal{U}_{r,n}$, and for the graphical matroid of the cycle C_n on n vertices), a proof of the general case remains elusive. And while no progress was made in the twenty-three years after the conjecture was stated, since 2001 it has received a flurry of attention that has yielded many partial positive results. We now give a brief summary of these results.

The first positive result was given in [44] where every cographical matroid $\mathcal{M}^*(G)$ was shown to satisfy Stanley's conjecture. To prove this Merino considered the critical configurations of a certain chip-firing game on the underlying graph G. Using the fact that the generating function of these critical configurations is the h-polynomial of the matroid $\mathcal{M}^*(G)$ [38], he was able to prove that a certain transformation of these configurations is a pure order ideal whose $\mathcal{O}$-sequence coincides with the h-vector of $\mathcal{M}^*(G)$.

Stanley's conjecture was shown to hold for all lattice-path matroids in [55] by characterizing internally-active elements of a basis in terms of certain segments in the corresponding lattice path and using this characterization to produce a pure order ideal with the right $\mathcal{O}$-sequence. More generally, Stanley's conjecture was verified for every cotransversal matroid $\mathcal{M}$ in [48] by considering a certain generalized permutahedron $P_{\mathcal{M}^*}$ (in the sense of [50]) associated to the dual matroid $\mathcal{M}^*$. A subset of the lattice points of the polytope $P_{\mathcal{M}^*}$ were shown to be (up to a shift) a pure order ideal with $\mathcal{O}$-sequence equal to $h(\mathcal{M})$.

A matroid $\mathcal{M}$ is paving if every circuit of $\mathcal{M}$ has cardinality equal to either $\text{rank}(\mathcal{M})$ or $\text{rank}(\mathcal{M})+1$. In [41], it is conjectured that asymptotically every matroid is a paving matroid. In [45], paving matroids were shown to satisfy Stanley's conjecture by proving that their h-vectors enjoy a certain rigidity. Thus, if the above conjecture is true, then the result from [45] implies that almost all matroids satisfy Stanley's conjecture.

In [16], the set of all rank r matroids on n elements is partitioned into classes and it is shown that the matroids in each class with component-wise minimal (respectively, maximal) h-vectors satisfy Conjecture 4.1.

Next consider the following types of matroids. A matroid $\mathcal{M}$ is a *truncation* of a rank r matroid $\mathcal{M}'$ if the matroid complex of $\mathcal{M}$ is isomorphic to the k-skeleton of the matroid complex of $\mathcal{M}'$, for some $0 \leq k < r$. A matroid $\mathcal{M}$ is *k-partite* if the number of parallel classes of $\mathcal{M}$ equals k. A matroid $\mathcal{M}$ is of *Cohen-Macaulay type k* if the last nonzero entry in the h-vector of $\mathcal{M}$ is equal to k. In [15], Constantinescu et al. show that if a $\mathcal{M}$ is a truncation, is $(\text{rank}(\mathcal{M})+2)-$*partite*, or is of Cohen-Macaulay type less than or equal to 5, then it satisfies Stanley's conjecture. These results, together with fact that every rank 3 matroid satisfies Stanley's conjecture (see [27]), are proven by studying level Artinian algebras.

In [22], De Loera et al. prove Conjecture 4.1 combinatorially for all matroids with rank no greater than three, as well as for all matroids with corank no more than two. They also prove Stanley's conjecture for all matroids on up to nine elements by direct computation using the database of all such matroids given in [42].

Finally, Klee et al. propose a combinatorial strengthening of Conjecture 4.1 in [35]. They prove their conjecture (and hence Stanley's conjecture) for all matroids having rank at most 4. In the general case they prove that if their conjecture holds for all rank r matroids on no more than $2r$ elements, then it holds for every rank r matroid.

We add to this growing body of work in the following sections by proving

Stanley's conjecture for internally perfect matroids under the assumption that a particular map is injective.

4.2 The Internal Order

We now review the pertinent facts about the internal order associated to an ordered matroid and introduce the class of *internally perfect matroids* that will be the focus of our study in the next section.

Throughout this section we assume $\mathcal{M} = ([n], <, \mathcal{B})$ be a rank $r > 0$ ordered matroid such that the set $[r]$ is a basis unless otherwise stated. Let B be a basis of $\mathcal{M}$ and suppose $e \notin B$. Then there is a unique circuit of $\mathcal{M}$ contained in the set $B \cup e$. This circuit is called the ***fundamental circuit*** of B with respect to e and is denoted $C(B; e)$. Similarly, for an element $f \in B$ the ***fundamental cocircuit*** of B with respect to f is the unique cocircuit $C^*(B; f)$ of $\mathcal{M}$ contained in the set $[n] \setminus B \cup f$. It is a basic fact that for $b \in B$ and $a \notin B$ the following are equivalent:

1. the set $A := B \setminus b \cup a$ is a basis;
2. $b \in C(B; a)$; and
3. $b \in C^*(A; a)$.

An element $f \in B$ is ***internally active*** in B if f is the minimum element (with respect to the ordering of the ground set) of its fundamental cocircuit $C^*(B; f)$. Otherwise, f is ***internally passive***.

For each basis $B \in \mathcal{B}$ write $B = (S, T, A)$ where $S = S(B)$ (respectively, $T = T(B)$) is the set of internally passive elements of B not in $[r]$ (respectively, in $[r]$), and $A = A(B)$ is the set of internally active elements of B. We will write $\mathrm{IP}(B)$ for the set of all internally passive elements of B, i.e., $\mathrm{IP}(B) := S \cup T$.

Example 1. To illustrate these concepts consider the vector matroid $\mathcal{M}$ of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}.$$

The set $B = \{2, 3, 6, 7, 8\}$ is a basis of $\mathcal{M}$ and, for $i \in B$, the fundamental cocircuits $\mathcal{C}^*(B; i)$ are

$$\begin{aligned} C^*(B;2) &= \{1,2,4,5\} && \subseteq \{1,2,4,5\}, \\ C^*(B;3) &= \{3,4,5\} && \subseteq \{1,3,4,5\}, \\ C^*(B;6) &= \{4,5,6\} && \subseteq \{1,4,5,6\}, \\ C^*(B;7) &= \{1,4,7\} && \subseteq \{1,4,5,7\}, \\ C^*(B;8) &= \{5,8\} && \subseteq \{1,4,5,8\}. \end{aligned}$$

Notice that 3 is internally active with respect to B (as $3 = \min C^*(B; 3)$) and that all other elements are internally passive. Therefore, we see that the basis $B = (S, T, A)$ where $S = \{6, 7, 8\}, T = \{2\}$ and $A = \{3\}$.

Now suppose we want to form a basis $A = B \setminus 8 \cup a$. Then as the fundamental circuits $C(B, a)$ for $a \in [8] \setminus B = \{1, 4, 5\}$ are

$$\begin{aligned} C(B;1) &= \{1,2,7\} && \subseteq B \cup 1 \\ C(B;4) &= \{2,3,4,6,7\} && \subseteq B \cup 4 \\ C(B;5) &= \{2,3,5,6,8\} && \subseteq B \cup 5 \end{aligned}$$

and 8 is only in $C(B; 5)$, the only such basis $A = \{2, 3, 5, 6, 7\}$. □

With this terminology in hand we now define the internal order of an arbitrary ordered matroid. The ***internal order*** of the ordered matroid $\mathcal{M}$ is the poset $\mathcal{P}_{\text{int}}(\mathcal{M}) = (\mathcal{B} \cup \widehat{1}, \preccurlyeq_{\text{int}})$ on the bases of $\mathcal{M}$ together with an artificial top element $\widehat{1}$. The relation $\preccurlyeq_{\text{int}}$ is defined by $B_1 \preccurlyeq_{\text{int}} B_2$ if and only if every internally passive element in B_1 is also internally passive in B_2. Equivalently, the internal order is the transitive closure of the relation defined by $B_1 \prec B_2$ if and only if $B_1 = B_2 - e \cup f$ where $f = \min C^*(B_1; f)$ (i.e., f is internally active in B_1) and $e \in C^*(B_1, f) - f$.

The internal order was introduced in [37] together with two closely related orders on the bases of an ordered matroid: the external order and the external/internal order. The internal order $\mathcal{P}_{\text{int}}$ and the external order $\mathcal{P}_{\text{ext}}$ of an ordered matroid $\mathcal{M}$ are related via matroid and poset duality:

$$\mathcal{P}_{\text{int}}(\mathcal{M}) = \mathcal{P}^{\vee}_{\text{ext}}(\mathcal{M}^*).$$

Example 2. The internal order of an ordered matroid depends on the ordering of the ground set. To see this consider the rank-3 ordered matroid $\mathcal{M} = ([6], \mathcal{B})$ whose basis are

$$\mathcal{B} := \{123, 124, 235, 236, 134, 245, 246, 234, 345, 346\}.$$

The h-vector of $\mathcal{M}$ is $(1, 3, 3, 3)$ and the internal order of $\mathcal{M}$ (with respect to the natural ordering) is the poset on the left in Figure 4.1. The poset on the right of the same figure is the internal order of the same matroid with the ground set ordered by $3 < 1 < 2 < 4 < 5 < 6$. (Note that when we draw Hasse diagram of an internal order, we suppress the maximal element $\widehat{1}$.) While these posets are clearly not isomorphic, they do share a number of properties. For example, they are both graded posets with $h_i(\mathcal{M})$ elements at each height. $\square$

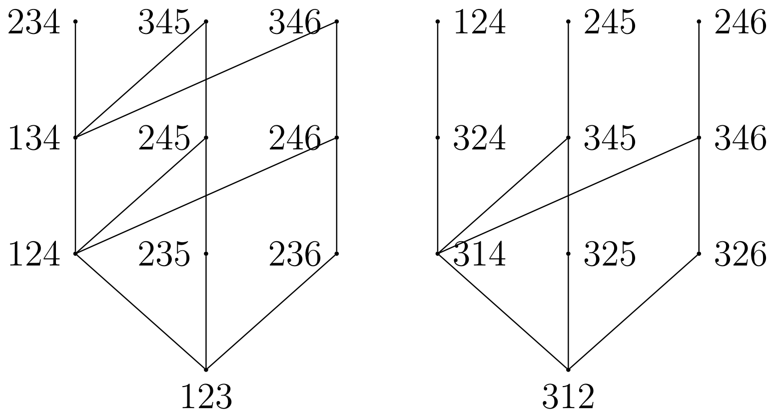


Figure 4.1: Non-isomorphic internal orders for same matroid with different ground set orderings

We now supply the basic facts about the internal order we will need in the sequel; their proofs may all be found in [37]. The first of these is of fundamental importance as it provides a link between the h-vector of a matroid and the internal order for any ordering of its ground set.

Proposition 4.2 ([37] Proposition 5.1)**.** *The internal order of an ordered matroid $\mathcal{M}$ is a graded lattice with height function* $\mathrm{ht}(B) = |\,\mathrm{IP}(B)|$. *The number of bases at height i in $\mathcal{P}_{\mathrm{int}}(\mathcal{M})$ is exactly $h_i(\mathcal{M})$.*

In particular, as $h_0(\mathcal{M}) = 1$ for any matroid, there is only one height-0 basis and, by construction this is $B_0 := [r]$. Moreover, as $h_1(\mathcal{M}) = n - r - l$ where l is the number of loops of $\mathcal{M}$, we have that for every $f \in [n] \setminus [r]$ that is not a loop there is exactly one basis $B_0 - b_f \cup f$ for some $b_f \in [r]$. Exactly which b_f is to be removed is a special case of the following proposition characterizing the cover relation in $\mathcal{P}_{\mathrm{int}}(\mathcal{M})$.

Proposition 4.3 ([37] Proposition 3.1 part 2 dual version)**.** *Let $\mathcal{M}$ be an ordered matroid. A basis $B = (S, T, A)$ covers a basis $B' = (S', T', A')$ in the internal order if and only if $B' = B - e \cup f$ where $e \in C^*(B'; f) - \{f\}$ and $f \in B'$ satisfies*

$$f = \min(C^*(B'; f)) \quad \textit{and}$$
$$f = \max(C(B'; e) \cap A').$$

In this case we have $A' = A \cup f$ and $S' \cup T' \cup e = S \cup T$.

Example 3. Let $\mathcal{M} = \mathcal{U}_{3,6}$ be the rank 3 uniform matroid on $[6]$. The internal order $\mathcal{P}_{\text{int}}(\mathcal{M})$ (less the artificial top element $\widehat{1}$) is given in Figure 4.2 using the following notation: a basis $B = (S, T, A)$ is written S_A^T and a sub- or superscript is omitted if the corresponding set is empty. To clarify the notation consider the basis 45^2 in Figure 4.2. It represents the basis $B = (45, 2, \emptyset)$ and is $\preccurlyeq_{\text{int}}$-greater than any basis $B = (S, T, A)$ such that $S \subseteq \{4, 5\}$ and $T \subseteq \{2\}$. In particular, for the basis $B' = (4, \emptyset, 12)$ we have $B' \preccurlyeq_{\text{int}} B$ and $B' = B - 5 \cup 1$. The fundamental cocircuit of B' with respect to 1 is $C^*(B'; 1) = \{1, 3, 4, 5\}$ and the fundamental circuit of B with respect to 1 is $C(B; 1) = \{1, 2, 4, 5\}$. The latter equation and the fact that 2 is internally active in B' tells us that $B' \not\lessdot B$. Moreover, note that the h-vector $(1, 3, 6, 10)$ of $\mathcal{M}$ can be read off by counting the number of bases at each height. $\square$

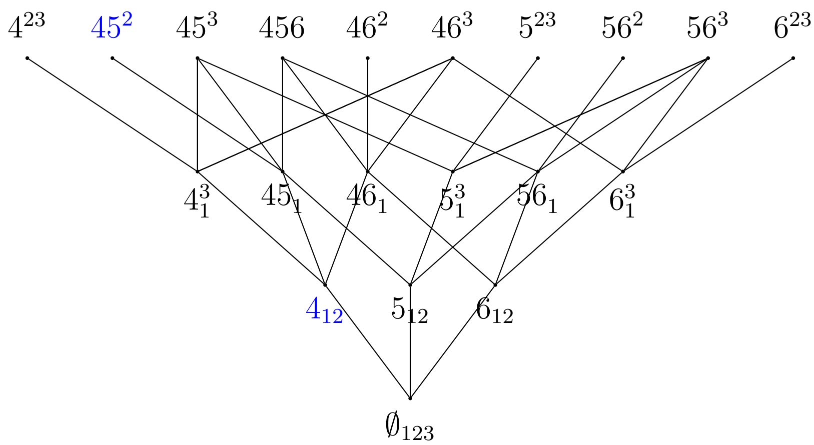


Figure 4.2: internal order of $\mathcal{U}_{3,6}$

With a view toward the construction of pure order ideals in the next section, let us pause here to see how the remarks preceding Proposition 4.3 compel a vector labeling on those bases with either 0 or 1 internally passive element. A natural construction is to allow the basis $B_0 = [r]$ to correspond to the zero vector $\mathbf{0}$ of the monoid

$$\bigoplus_{\substack{f \in [n]-[r] \\ f \text{ a nonloop}}} \mathbb{N}\mathbf{e}_f \cong \mathbb{N}^{h_1},$$

and the height-1 basis in $\mathcal{P}(\mathcal{M})$ given by $B_0 - e_f \cup f$ will correspond to the generator $\mathbf{e}_f$. The next proposition is crucial to our ability to extend such a labeling to all bases of a perfect ordered matroid in such a way that the structure of the internal order ensures that the vector labeling is in fact a pure $\mathcal{O}$-sequence.

Proposition 4.4 ([37] Lemma 3.3 dual version). *Let B and B' be bases of $\mathcal{M}$ such that $B' = B - b \cup b'$ and $B' \preccurlyeq_{\text{int}} B$ and suppose there exists an a in $C(B'; b) \cap A(B')$ such that $a > b'$. Then the set $B'' = B - a \cup b'$ is a basis and $B' \preccurlyeq_{\text{int}} B'' \preccurlyeq_{\text{int}} B$.*

Let $B = (S, T, A)$ be a basis of $\mathcal{M}$. We use Proposition 4.4 to define certain subsets of T induced by elements of S as follows. Fix an $f \in S$. As f is internally passive in B, there exists an element $a_f \in [n] \setminus B$ such that $a_f = \min C^*(B; f)$. Note that a_f depends on both f and B. Moreover, as $a_f \in C^*(B; f)$, the set $B_f := B - f \cup a_f$ is a basis that is less than B in the internal order. Now consider the (possibly empty) set $T(B; f) \subseteq T$ defined by

$$T(B; f) := \{t \in T \mid t \in C(B_f; f) \cap A(B_f) \text{ and } t > a_f\}.$$

By Proposition 4.3, the set $T(B; f)$ is empty if and only if B_f is covered by B. In this case the set of internally passive element of B_f is $\text{IP}(B) - f$, and in particular we have $T = T(B_f)$.

Now assume $T(B; f)$ is non-empty. Then Proposition 4.4 tells us that, for each $t \in T(B; f)$, the set $B_t := B - t \cup a_f$ is a basis such that

$$B_{a_f} \preccurlyeq_{\text{int}} B_t \preccurlyeq_{\text{int}} B.$$

It then follows that the set of internally active elements of B_t consist of the internally active elements of B together with the set

$$\{a_f\} \cup \{t' \in T(B; f) \mid t' < t\}.$$

If we write $T(B; f) = (t_1, t_2, \ldots, t_k)$ where $t_i < t_j$ whenever $i < j$, the repeated application of Proposition 4.4 shows that the internal order contains the following chain of bases

$$B_{a_f} \prec B_{t_k} \prec B_{t_{k-1}} \prec \cdots \prec b_{t_1} \prec B.$$

One then uses Proposition 4.3 to show that each of these relations is a cover relation. In particular we have that, when $T(B;f) \neq \emptyset$, the basis B_{t_1} is covered by B. Thus we have shown the following lemma.

Lemma 4.5. *Let $B = (S, T, A)$ be a basis of an ordered matroid $\mathcal{M}$. Fix an element $f \in S$ and let $a_f := \min C^*(B;f)$ as above. Then there is a unique $b \in T(B;f) \cup f$ such that the basis $B - b \cup a_f$ is covered by B in the internal order. Moreover, $b = f$ if and only if $T(B;f) = \emptyset$ and, otherwise, we have $b = \min T(B;f)$.*

The basis $B - b \cup a_f$ from the previous lemma will be denoted $B(f)$ in the sequel.

Now let us examine to what extent the union $\widetilde{T} := \bigcup_{f \in S} T(B;f)$ covers T. In general there are three possibilities:

1. $\widetilde{T}$ is proper subset of T;
2. $\widetilde{T} = T$ and for some $f, g \in S$ the sets $T(B;f)$ and $T(B,g)$ intersect; or
3. T is the disjoint union of the $T(B,f)$ as f ranges over S.

For example, from Figure 4.2 we see that the basis 45^2 of $\mathcal{U}_{3,6}$ is of the first type, while the bases 45^3 and 456 are of the second and third type, respectively.

The bases of the third type will be the central focus of the next section, and so we make the following definitions.

Definition 1. A basis $B = (S, T, A)$ of an ordered matroid is ***internally perfect*** (or ***perfect***, for short) if $T = \bigsqcup_{f \in S} T(B;f)$, i.e., T is the disjoint union of the $T(B;f)$. An ordered matroid is ***internally perfect*** if every basis of $\mathcal{M}$ is. An unordered matroid $\mathcal{M} = (E, \mathcal{B})$ on n elements is ***internally perfect*** if there is an ordering $<$ of its ground set such that the ordered matroid $\mathcal{M}(E, <, \mathcal{B})$ is internally perfect.

In the sequel, we typically abbreviate by referring to an internally perfect matroid as perfect.

Example 4. We have already seen that $\mathcal{U}_{3,6}$ is not perfect. Indeed, using Figure 4.2 one can see that a basis of $\mathcal{U}_{3,6}$ is perfect (with respect to the ordering of the ground set of the matroid) if and only if it is not of the form (S, T, A) where $A = \emptyset$, $S \subset \{4, 5, 6\}$ is a subset of size 2, and $T \subset \{1, 2\}$ is a singleton.

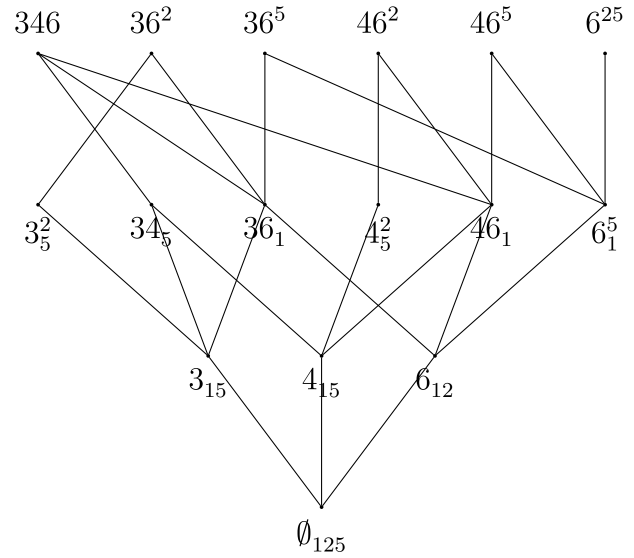


Figure 4.3: internal poset of a perfect rank-3 matroid

Now consider the set of bases $\mathcal{B}(\mathcal{U}_{3,6}) \setminus \{123, 124, 134, 234\}$. One can show that this is the set of bases of a matroid $\mathcal{M}$ and that the h-vector of $\mathcal{M}$ is $(1, 3, 6, 8)$. By inspection of the internal order of $\mathcal{M}$ given in Figure 4.3 one can conclude that the matroid is perfect with respect to the ordering $(1, 2, 5, 3, 4, 6)$. Moreover, one can show that $\mathcal{M}$ is the perfect simple rank-3 matroid on 6 elements with the greatest number of bases. $\square$

4.3 Perfect Matroids and Stanley's Conjecture

In this section, we begin by proving that every (nonempty) ordered matroid contains some internally perfect and that every rank-2 matroid is perfect. We then introduce a map $\mu = \mu(\mathcal{B})$ that sends the bases $\mathcal{B}$ of an order matroid $\mathcal{M}$ to a collection of integer vectors and prove that $\mu(\mathcal{B})$ is a pure order ideal whose $\mathcal{O}$-sequence coincides with the h-vector of $\mathcal{M}$, whenever $\mathcal{M}$ is internally perfect and μ is injective. In this case the matroid $\mathcal{M}$ satisfies Stanley's conjecture. First we show that an arbitrary ordered matroid always contains perfect bases.

Proposition 4.6. *Let $\mathcal{M}$ be a rank r ordered matroid on $[n]$ and consider the basis $B = (S, T, A) \in \mathcal{B}(\mathcal{M})$ be a basis. Then B is internally perfect if either T is empty or $|S| \leq 1$.*

Proof. When $T = \emptyset$ the result is trivial. Moreover, when $S = \emptyset$, then $T = \emptyset$ and so we are done. If $S = \{f\}$, then

$$\begin{aligned}\bigsqcup_{f\in S} \widetilde{T}(B; f) &= \widetilde{T}(B; f)\\ &= T(B)\end{aligned}$$

and the result follows. □

From this simple fact it is easy to deduce the following corollary.

Corollary 4.7. *Every rank-2 ordered matroid is internally perfect.*

Proof. Let $B = (S, T, A)$ be a basis of a rank 2 ordered matroid. Then the cardinality of $|S| \in \{0, 1, 2\}$ and if $|S| = 2$ then $|T| = 0$. So B is perfect by Proposition 4.6. □

We now define a map that takes a basis B of a fixed matroid $\mathcal{M}$ to an element in the semigroup $\mathcal{S} := \bigoplus_{f\in D} \mathbb{N}\mathbf{e}_f \cong \mathbb{N}^{h_1}$, where $h_1 = h_1(\mathcal{M})$. Consider the map $\mu : \mathcal{B}(\mathcal{M}) \to \mathcal{S}$ defined by

$$B \quad\mapsto\quad \sum_{f\in S} \mathbf{e}_f + \sum_{f\in S} |T(B; f)|\, \mathbf{e}_f. \tag{4.1}$$

First we show that for a perfect basis $B = (S, T, A)$, the vector $\mu(B)$ has coordinate sum $|S \cup T|$ equal to the number of internally passive elements of B.

Proposition 4.8. *Let $\mathcal{M}$ be a matroid and let $B \in \mathcal{B}(\mathcal{M})$ be a basis. Then B is internally perfect if and only if the number of internally passive elements of B is exactly the coordinate sum $\sum_{f\in S}(\mu(B))_f$ of $\mu(B)$.*

Proof. The basis B is perfect if and only if $\sum_{f\in S} \big|T(B; f)\big| = \big|\bigsqcup_S T(B; f)\big|$ which is equivalent to

$$\begin{aligned}\sum_{f\in S}(\mu(B))_f &= |S| + \Big|\bigsqcup_{f\in S} T(B; f)\Big|\\ &= |S| + |T|,\end{aligned}$$

and the result follows. □

It now follows directly from Propositions 4.2 and 4.8 that if the map μ is injective on the set of bases $\mathcal{B}$ of a perfect matroid $\mathcal{M}$ then the image of $\mathcal{B}$ under μ is a collection of elements in $\mathcal{S}$ such that the number of elements with coordinate sum i coincides with the i^{th} entry of the h-vector of $\mathcal{M}$. We record this in the following corollary.

Corollary 4.9. *Let $\mathcal{M}$ be an internally perfect matroid and suppose the map $\mu : \mathcal{B}(\mathcal{M}) \to \mathcal{S}$ is injective. Then the number of vectors $\mu(B)$ with coordinate sum i is exactly $h_i(\mathcal{M})$.*

The next step toward proving that internally perfect matroids satisfy Stanley's conjecture whenever the map μ is injective is to show that μ gives an order ideal when all bases are perfect. Toward this end we first give a necessary condition for a basis to be perfect in terms of cover relations in the internal order $\mathcal{P}(\mathcal{M})$.

Lemma 4.10. *Let $\mathcal{M}$ be an ordered matroid and let $B = (S, T, A) \in \mathcal{B}(\mathcal{M})$ be a basis. If B is internally perfect, then B covers exactly $|S|$ bases in $\mathcal{P}(\mathcal{M})$.*

Proof. Let $B = (S, T, A)$ be a perfect basis of $\mathcal{M}$. By Lemma 4.5, for every f in S there is a unique $b \in T(B; f) \cup \{f\}$ such that the basis

$$B(f) = B - b \cup a_f$$

is covered by B in the internal order. We now prove the lemma in two steps. First we show that B covers at least $|S|$ bases by proving that, for distinct $f, g \in S$ we have $B(f) \neq B(g)$. Consider the basis $B(f) = B - b_f \cup a_f$ and $B(g) = B - b_g \cup a_g$. If $a_f \neq a_g$, then evidently $B(f) \neq B(g)$. On the other hand, if $a_f = a_g$, then the bases coincide if and only if $b_f = b_g$. But this can happen if and only if the intersection $T(B; g) \cap T(B; f)$ is nonempty, which is impossible as B is a perfect basis. This implies that $b_f \neq b_g$ and hence that $B(f) \neq B(g)$.

Next we prove that B covers exactly $|S|$ bases by showing that, if B' is a basis covered by B in the internal order, then $B' = B(f)$ for some $f \in S$. To see this let B' be a basis such that $B' \lessdot B$. By Proposition 4.3, the basis B' is of the form $B' = B - b \cup a$ where b is internally passive in B and a is simultaneously the minimal element of $C^*(B'; a)$ and the maximal element of $C(B'; b) \cap A(B')$. Since B is perfect, there is a unique $f \in S$ such that b is in $\{f\} \cup T(B; f)$. For this f we have

$$\begin{aligned} a &= \min C^*(B'; a) \\ &= \min C^*(B; b) \\ &= \min C^*(B; f). \end{aligned}$$

It follows that $a = a_f$ and hence that $B' = B(f)$ by Lemma 4.5 as desired. $\square$

We are now ready to prove Stanley's conjecture for internally perfect matroids under the assumption that the map $\mu : \mathcal{B}(\mathcal{M}) \to \mathcal{S}$ defined by

$$B \quad \mapsto \quad \sum_{f \in S} \mathbf{e}_f + \sum_{f \in S} |T(B; f)|\, \mathbf{e}_f$$

is injective.

Theorem 4.11. *For an internally perfect matroid $\mathcal{M}$ with bases $\mathcal{B}$, the image $\mathcal{O}$ of $\mu : \mathcal{B} \to \mathcal{S}$ is a collection of vectors that form a pure order ideal whenever μ is injective. Thus, any such internally perfect matroid satisfies Stanley's conjecture.*

Proof. Let $\mathbf{m} = \mu(B) \in \mathcal{O}$ with $B = (S, T, A)$. We need to check that for each $f \in \operatorname{supp}(\mathbf{m})$ the vector $\mathbf{m}(f) = \mathbf{m} - \mathbf{e}_f$ equals $\mu(B')$ for some $B' \in \mathcal{B}$. By Lemma 4.10, for each $f \in S$ the basis B is covered by a unique basis of the form $B_f = B - b_f \cup a_f$ corresponding to f, where either $b_f = f$ or f is an element of $T(B, f)$. In either case we have $\mu(B_f) = \mu(B) - \mathbf{e}_f$. Therefore $\mathcal{O}$ is an order ideal.

By Corollary 4.9, we know that the $\mathcal{O}$-sequence of $\mu(\mathcal{B})$ is the h-vector of $\mathcal{M}$, so all that remains to prove is that $\mu(\mathcal{B})$ is pure. To see this notice that we have now shown that the divisibility order on $\mathcal{O}$ is isomorphic to the internal order of $\mathcal{M}$ with the top element $\hat{1}$ removed. Thus the purity of $\mathcal{O}$ follows from the fact that $\mathcal{P}(\mathcal{M})$ is a graded lattice and hence $\mathcal{P}(\mathcal{M}) \setminus \hat{1}$ is pure. $\square$

We have now shown that an internally perfect matroid satisfies Stanley's conjecture whenever the map μ is injective. The following conjecture states that this is always the case.

Conjecture 4.12. *The map $\mu : \mathcal{B} \to \mathbb{N}^{h_1}$ is injective on the bases of an ordered matroid $\mathcal{M}$ whenever $\mathcal{M}$ is internally perfect. Hence, internally perfect matroids satisfy Stanley's conjecture.*

4.4 Further Directions

Having shown in the previous section that internally perfect matroids satisfy Stanley's conjecture whenever the map μ is injective, we conclude by discussing the construction of perfect matroids. While some general methods for producing new perfect matroids exist, we discuss three substantial obstacles that have hindered us in proving that there are perfect matroids not belonging to any of the classes for which Stanley's conjecture is known to hold. We close by suggesting some directions for further research inspired by computations taking advantage of the matroid database described in [39] and available online. Through out this section we call a matroid *interesting* if it satisfies Stanley's conjecture and is not in any of classes for which Stanley's conjecture is known to hold.

We begin by proving that perfect matroids are closed under 1- and 2-sums.

Proposition 4.13. *Internally perfect matroids are closed with respect to* 1*-sums. Moreover, let* $\mathcal{M}_1 = (E_1, \mathcal{B})$ *and* $\mathcal{M}_2 = (E_2, \mathcal{B})$ *be matroids such that* $E_1 \cap E_2 = \{e\}$ *where e is not a coloop of at least one of the two matroids. Then the* 2*-sum* $\mathcal{M}_1 \oplus_2 \mathcal{M}_2$ *is perfect if and only if both summands are.*

Proof. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be ordered matroids on E_1 and E_2, respectively and let $B_i \in \mathcal{B}_i$ be a basis of $\mathcal{M}_i$ for $i = 1, 2$. Recall that a basis $B = (S, T, A)$ is (internally) perfect if T is the disjoint union of the sets $T(B; f)$. Since every basis of a 1-sum is a 1-sum of bases, it follows immediately that the direct sum $B_1 \oplus B_2$ is perfect if and only if each summand is. Thus the matroid $\mathcal{M}_1 \oplus \mathcal{M}_2$ is perfect if and only if $\mathcal{M}_1$ and $\mathcal{M}_2$ are, proving the first claim.

Now suppose $E_1 \cap E_2 = \{e\}$ where e is not a coloop of $\mathcal{M}_1$, say. Recall from Section 1.4 that a set B is a basis of the 2-sum $\mathcal{M}_1 \oplus_2 \mathcal{M}_2$ if and only if $B = B_1 \bigsqcup B_2$ is the disjoint union of bases of $\mathcal{M}_1$ and $\mathcal{M}_2$. As the disjoint union of perfect bases is perfect, it follows that the 2-sum of perfect matroids is perfect. $\square$

Proposition 4.13 tells us how to construct a perfect matroid from a given set of perfect matroids. For matroids with rank less than five on no more than nine elements we are able to verify the existence of many perfect matroids using the `Posets` package of `Macaulay2` ([17, 26]). Therefore we can create new perfect matroids from these by taking 1- and 2-sums. This technique, however, will not produce new matroids satisfying Stanley's conjecture for the following reason.

For general matroids, the Tutte polynomial of $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is the product of the Tutte polynomials of the components. Thus, the h-vector of $\mathcal{M}$ is $(h_0, h_1, \ldots, h_r)$ where $r = \mathrm{rank}(\mathcal{M}_1) + \mathrm{rank}(\mathcal{M}_2)$ and

$$h_k = \sum_{i+j=k} h_i(\mathcal{M}_1) h_j(\mathcal{M}_2).$$

Now suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two matroids satisfying Stanley's Conjecture, and let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then there are pure order ideals, $\mathcal{O}_1$ and $\mathcal{O}_2$, such that the $\mathcal{O}$-sequence of $\mathcal{O}_i$ is $h(\mathcal{M}_i)$. It is then a simple exercise to check that the poset product $\mathcal{O}_1 \times \mathcal{O}_2$ is a pure order ideal with $\mathcal{O}$-sequence equal to $h(\mathcal{M})$.

In light of this fact and the fact that Stanley's conjecture is known to hold for all matroids with rank at most four ([35]), taking 1-sums of the perfect matroids of low rank discussed in the previous paragraph produces no new interesting matroids. Stanley's conjecture also holds for all matroids on at most nine elements ([22]), so the search for interesting perfect matroids begins with rank five matroids on ten elements. For rank five matroids on ten elements we confront two obstacles. The first is finding matroids that are not members of any of the classes known to satisfy Conjecture 4.1. The second is that, given such a matroid $\mathcal{M}$, one must theoretically check up to 10! orderings of the ground set of $\mathcal{M}$ to show that it is not perfect. We hope to overcome these obstacles in future work.

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