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STUDY ON PROPERTIES OF AN ANALYTIC FUNCTION INVOLVING A GENERALISED DERIVATIVE OPERATOR

(Kajian Mengenai Sifat-sifat Fungsi Analisis Melibatkan suatu Pengoperasi Terbitan Teritlak)

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ABSTRACT

In this paper, we study a new subclass of analytic functions defined by a derivative operator. Coefficient bounds, extreme point and integral transform are investigated.

Keywords: extreme point; convolution; starlikeness; coefficient estimates

ABSTRAK

Dalam makalah ini, dikaji suatu subkelas baharu fungsi analisis yang ditakrif oleh pengoperasi terbitan. Batas pekali, titik ekstrim dan jelmaan kamiran diselidiki.

Kata kunci: titik ekstrim; konvolusi; kebakbintangan; anggaran pekali

1. Introduction

Let A denote a class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalised by $f(0) = f'(0) - 1 = 0$. A function $f(z) \in A$ is said to be starlike of order α if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in U.$$

This class is denoted by $S^*(\alpha)$. Similarly, a function $f(z) \in A$ is said to be convex of order α if satisfy

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in U.$$

Definition 1.1. A function $f(z) \in A$ is said to be γ–spirallike of order $\alpha (0 \le \alpha < 1)$, if

$$\mathrm{Re}\left\{\frac{e^{i\gamma} zf'(z)}{f(z)}\right\} > \alpha \cos\gamma, z \in U$$

for some real $\gamma\left(|\gamma|<\frac{\pi}{2}\right)$.

This class of function is denoted by $S_p^\alpha(\gamma)$.

Definition 1.2. If $f(z)=\sum_{n=0}^{\infty}a_n z^n$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n$ are analytic in U, then their Hadamard product $f*g$ defined by the power series is given by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_n b_n z^n. \tag{2}$$

Note that the convolution (2) is also analytic in U. The advent of operators in the field brought an ease and understanding to the study of analytic univalent functions. Most of these operators are expressed as convolution of analytic functions, see Al-Oboudi (2004), Bansal and Raina (2010), Carlson and Shaffer (1984), Prajapath and Raina (1984), Ruscheweyh (1975) and Zahid *et al*. (2012). Next, we shall introduce a derivative operator of certain analytic function which will be used throughout the article.

Given the operator $A_\lambda^{l,m}(n,a,c)f(z)$ involving multiplier transformations as the following:

$$A_\lambda^{l,m}(n,a,c)f(z)=z+\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\left[1+\lambda(k-1)\right]^l\left[\frac{(a)_{k-1}}{(c)_{k-1}}\right]^m a_k z^k \quad (\lambda>-1),$$

where $l,m,n\in N_\circ=N\cup\{0\}$. Here $A_\lambda^{0,0}(0,a,c)f(z)=f(z)$ and $(a)_{k-1}$ is the familiar Pochammer symbol.

By choosing suitable values, the operator reduces to several known operators in the literature. Few to mention here as follows:

$A_0^{0,0}(n,a,c)f(z)$ is the Salagean operator (1983),
$A_\lambda^{l,m}(0,a,c)f(z)$ is the class defined and studied by Bansal and Raina (2010),
$A_0^{l,0}(0,a,c)f(z)$ is the Al-Oboudi Operator (2004) and
$A_\lambda^{l,0}(n,a,a)f(z)$ is the class studied by Uralegadi *et al.* (1992).

Definition 1.3. For $f\in A$, the operator $D^{l,m,n}f(z)$ defined by $D^{l,m,n}f(z):A\to A$,

$$D^{l,m,n}f(z)=A_\lambda^{l,m}(n,a,c)f(z)*D^n f(z), z\in U \tag{3}$$

where $l,m,n\in N_\circ=N\cup\{0\}$, $D^n f(z)$ is the familiar Ruscheweyh operator [9]

$$D^{l,m,n,r}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n\left[1+\lambda(k-1)\right]^t\left[\frac{(a)_{k-1}}{(c)_{k-1}}\right]^m C(r,k)a_k z^k \tag{4}$$

where $(a)_{k-1}$ is the familiar Pochammer symbol.

We shall use the following definition and notations, with our style in the manner of Bansal and Raina (2010); Owa *et al*. (2002), to introduce a class of analytic function containing a linear operator defined in (4).

Notations: The following notions will be used in this paper without loss of generality.

$$N = \{1,2,3,...\},$$

$$N_{\circ} = \{0\},$$

$$N_{-1} = \{u \in \mathbb{R} : u > -1\}$$

and

$$\mathbb{R}^{\circ}_{-1} = \mathbb{R}_{-1} \setminus \{0\}.$$

Definition 1.4. Let $f \in A$ and $u \in \mathbb{R}_{-1}$, we redefined the stated operator (4) as follows:

$$D^u f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda} \right)^n \left[1+\lambda(k-1)\right]^t \left[\frac{(a)_{k-1}}{(c)_{k-1}} \right]^m C(u,k) a_k z^k. \tag{5}$$

Definition 1.5. Let $P(z) = \dfrac{D^u f(z)}{D^v f(z)}$, a function $f \in S$ is said to be in the class $B(u,v,\gamma)$ if it satisfy the inequality

$$\left| \frac{1}{P(z)} - \frac{1}{2\gamma} \right| < \frac{1}{\gamma} \tag{6}$$

for $0 < \gamma < 1$.

We shall characterise $B(u,v,\gamma)$ by investigating coefficient bounds, extreme point and integral operator. We shall make use of the methods and techniques in Aouf and Cho (1998), Cho and Kim (2003), Owa *et al*. (2002) and Salagean (1983) to establish our results.

2. Conditions for Functions in the Class $B(u,v,\gamma)$ and Coefficient Inequality

Theorem 2.1. *$f \in B(u,v,\gamma)$ iff* $\operatorname{Re}\left(\dfrac{D^u f(z)}{D^v f(z)} \right) > \gamma,$ *for some $0 < \gamma < 1$, $u \in \mathbb{R}^{\circ}_{-1}$ and $v \in \mathbb{R}_{-1}$.*

Proof. Let $P(z) = \dfrac{D^u f(z)}{D^v f(z)}$, for $f \in B(u,v,\gamma)$, we can write

$$\left|\frac{2\gamma - P(z)}{2\gamma P(z)}\right| < \frac{1}{2\gamma}. \tag{7}$$

Squaring both side of (7) and simplifying, we have

$$(2\gamma - P(z))\overline{(2\gamma - P(z))} < \overline{P(z)}P(z),$$

$$4\gamma^2 - 2\gamma(\overline{P(z)} + P(z)) < 0,$$

$$2\gamma - 2\gamma\, \mathrm{Re}(P(z)) < 0,$$

$$\mathrm{Re}(P(z)) > \gamma,$$

therefore,

$$\mathrm{Re}\left(\frac{D^u f(z)}{D^v f(z)}\right) > \gamma. \square$$

Theorem 2.2. *If* $f \in A$ *satisfies*

$$\sum_{k=2}^{\infty}\left\{\begin{array}{l}\{B(\delta(u,\lambda))C(u,k) + |2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)|\} \\ |a_k| \le 1 - |1-2\gamma|\end{array}\right. \tag{8}$$

where

$$B(\delta(u,\lambda))C(u,k) = \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n [1+\lambda(k-1)]^t \left[\frac{(a)_{k-1}}{(c)_{k-1}}\right]^m C(r,k),$$

then $f \in B(u,v,\gamma)$.

Proof. It suffices to show that

$$\left|\frac{2\gamma D^v f(z) - D^u(z)}{D^u f(z)}\right| < 1,$$

$$\left|\frac{2\gamma D^v f(z) - D^u F(z)}{D^u f(z)}\right| = \left|\frac{2\gamma z + \sum B(\delta(v,\lambda))C(v,k)a_k z^k - z - \sum B(\delta(u,\lambda))C(u,k)a_k z^k}{z + \sum B(\delta(u,\lambda))C(u,k)a_k z^k}\right|,$$

$$= \left|\frac{z(1-2\gamma) + \sum\{2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)\}\, a_k z^k}{z + \sum B(\delta(u,\lambda))C(u,k)a_k z^k}\right|,$$

$$\le \frac{|(1-2\gamma)| + \sum \left|\{2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)\}|a_k||z^{k-1}|\right|}{1 - \sum |B(\delta(u,\lambda))C(u,k)||a_k|}, \tag{9}$$

$$< \frac{|(1-2\gamma)| + \sum \left|\{2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)\}|a_k||z^{k-1}|\right|}{1 - \sum |B(\delta(u,\lambda))C(u,k)||a_k|}.$$

Therefore if

$$\sum_{k=2}^{\infty} \begin{cases} \{B(\delta(u,\lambda))C(u,k) + |2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)|\} \\ |a_k| \le 1 - |1-2\gamma| \end{cases}$$

then

$$\begin{aligned} &\sum \left|\{2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k)\}|a_k||z^{k-1}|\right| \\ &\quad \le 1 - |(1-2\gamma)| - \sum |B(\delta(u,\lambda))C(u,k)||a_k|. \end{aligned} \tag{10}$$

Relating (9) and (10), we have our desired results. □

3. Extreme Points

Theorem 3.1. *Let* $\overline{B}(u,v,\gamma)$ *be the subclasses of* $B(u,v,\gamma)$ *which consists of function*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (a_k \ge 0) \tag{11}$$

whose coefficient satisfy inequality (8). *Then* $f \in \overline{B}(u,v,\gamma)$ *iff it can be express in the form*

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z), \tag{12}$$

where $\delta_k \ge 0$ *and* $\sum_{k=1}^{\infty} \delta_k = 1.$

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z), \tag{13}$$

then

$$f(z)=\delta_1 f(z)+\sum_{k=2}^{\infty}\delta_k f_k(z)$$

$$=\delta_1 z+\sum_{k=2}^{\infty}\delta_k\left(z+\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}\right),$$

where

$$Q_k(u,v,\gamma)=\sum_{k=2}^{\infty}\left\{B(\delta(u,\lambda))C(u,k)+\left|2\gamma B(\delta(v,\lambda))C(v,k)-B(\delta(u,\lambda))C(u,k)\right|\right\}$$

$$=z+\sum_{k=2}^{\infty}\delta_k f_k(z)\delta_1 z+\sum_{k=2}^{\infty}\delta_k\left(z+\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}\right),$$

thus we have

$$\sum_{k=2}^{\infty}\delta_k\left(z+\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}\right)\cdot Q_k(u,v,\gamma)=1-|1-2\gamma|\left(1-\delta_1\right)\leq 1-|1-2\gamma|.$$

Therefore we have $f\in\overline{B}(u,v,\gamma)$.

Conversely, suppose that $f\in\overline{B}(u,v,\gamma)$, we know that

$$a_k\leq\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}$$

for $\left(k=2,3,...\right)$. Define $\delta_k=\dfrac{Q_k(u,v,\gamma)}{1-|1-2\gamma|}$ and $\delta_1=1-\sum_{k=2}^{\infty}\delta_k$.

Thus we have

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k$$

$$=\left(\sum_{k=1}^{\infty}\delta_k\right)z+\sum_{k=2}^{\infty}\delta_k\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}z^k$$

$$=\delta_1 z+\sum_{k=2}^{\infty}\delta_k\left(z+\frac{1-|1-2\gamma|}{Q_k(u,v,\gamma)}z^k\right)$$

$$= \delta_1 f_1(z) + \sum_{k=2}^{\infty} \delta_k f_k(z)$$

$$= \sum_{k=2}^{\infty} \delta_k f_k(z),$$

which was to be established. Thus we conclude that the extreme point assumed are correct.□

4. Integral Operator

Theorem 4.1. *Let* $f \in B(u,v,\gamma)$ *and let* c *be a real number such that* $c > -1$. *Then the function defined by*

$$F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \tag{14}$$

also belongs to the class $B(u,v,\gamma)$.

Proof. From the expanded representation of $F_c(z)$, we have

$$F_c(z) = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k} \right) a_k z^k, \tag{15}$$

relating this with Theorem 2.1, we can therefore write

$$\sum_{k=2}^{\infty} \left\{ B(\delta(u,\lambda))C(u,k) + \left| 2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k) \right| \right\} \left| \frac{c+1}{c+k} \right| |a_k| \tag{16}$$

$$\leq \sum_{k=2}^{\infty} \left\{ B(\delta(u,\lambda))C(u,k) + \left| 2\gamma B(\delta(v,\lambda))C(v,k) - B(\delta(u,\lambda))C(u,k) \right| \right\} |a_k|$$

$$\leq 1 - |1 - 2\gamma|.$$

Thus since $f \in B(u,v,\gamma)$, by Theorem 2.2, we have that $F_c(z) \in B(u,v,\gamma)$. This proves our theorem.□

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