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ON FEKETE-SZEGÖ PROBLEMS FOR A SUBCLASS OF ANALYTIC FUNCTIONS

(Berkenaan Permasalahan Fekete-Szegö bagi Subkelas Fungsi Analisis)

ENTISAR EL-YAGUBI & MASLINA DARUS

ABSTRACT

The aim of this paper is to determine the Fekete-Szegö inequalities for a normalised analytic function $f(z)$ defined on the open unit disc for which $z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z))' / (\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)), \delta, m, b \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$ lies in a region starlike with respect to 1 and it is symmetric with respect to the real axis by using the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$ given recently by the authors. As a special case of this result, Fekete-Szegö inequality for a class of functions defined by fractional derivatives is also obtained.

Keywords: analytic function; starlike function; subordination; Fekete-Szegö inequality; derivative operator

ABSTRAK

Matlamat makalah ini adalah untuk menentukan ketaksamaan Fekete-Szegö bagi fungsi analisis ternormal yang ditakrif pada cakera unit terbuka dengan $z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z))' / (\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z))$, $\lambda_2 \geq \lambda_1 \geq 0$, $\delta, m, b \in \mathbb{N}_0$ terletak pada rantau bak bintang terhadap 1 dan simetri terhadap paksi nyata dengan menggunakan pengoperasi $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$ yang diberi penulis baru-baru ini. Sebagai kes khas bagi hasil ini, ketaksamaan Fekete-Szegö bagi kelas fungsi yang ditakrif oleh terbitan pecahan juga diperoleh.

Kata kunci: fungsi analisis; fungsi bak bintang; subordinasi; ketaksamaan Fekete-Szegö; pengoperasi terbitan

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; (z \in U), \tag{1}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions, which are univalent in U. Let $\phi \in P$, where $\phi(z)$ is an analytic function with positive real part on $\mathcal{A}$ with $\phi(0) = 1, \phi'(0) > 0$, and let $\mathcal{S}^*(\phi)$ be the class of functions in $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \; (z \in U), \tag{2}$$

and $\mathcal{C}(\phi)$ be the class of functions in $f \in \mathcal{A}$ for which

$$1+\frac{zf''(z)}{f'(z)} \prec \phi(z), \ (z \in U). \tag{3}$$

where $\prec$ denotes to the subordination between two analytic functions.

Let a_n be a complex number and $0 \le \mu \le 1$. A classical theorem of Fekete and Szegö (1933) states that for $f \in \mathcal{S}$ and given by (1),

$$|a_3 - \mu a_2^2| \le 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right).$$

The inequality is sharp.

For a brief history of the Fekete-Szegö problem for the class of starlike functions S^*, convex functions C and close-to-convex functions K, see the papers by Mohammed and Darus (2010), Srivastava *et al.* (2001), Darus (2002), Al-Abbadi and Darus (2011), Ravichandran *et al.* (2004) and Al-Shaqsi and Darus (2008). In particular, for $f \in K$ and given by (1), Keogh and Merkes (1969) showed that

$$|a_3 - \mu a_2^2| \le \begin{cases} 3-4\mu & \text{if} \quad 0 \le \mu \le \frac{1}{3}, \\ \frac{1}{3}+\frac{4}{9\mu} & \text{if} \quad \frac{1}{3} \le \mu \le \frac{2}{3}, \\ 1 & \text{if} \quad \frac{2}{3} \le \mu \le 1, \\ 4\mu - 3 & \text{if} \quad \mu \ge 1, \end{cases}$$

and for each μ there is a function in K for which equality holds.

Definition 1.1 (El-Yagubi & Darus 2013) Let f be in the class $\mathcal{A}$. For $\delta, m, b \in \mathbb{N}_0$ and $\lambda_2 \ge \lambda_1 \ge 0,$ we define the differential operator as follows:

$$\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) = z + \sum_{n=2}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m \mathcal{C}(\delta,n)a_n z^n, \tag{4}$$

where $C(\delta,n) = \binom{\delta+n-1}{\delta} = (\delta+1)_{n-1}/(n-1)!$ and $(\delta)_n$ denotes the Pochhammer symbol defined by

$$(\delta)_n = \begin{cases} 1 & n = 0, \\ \delta(\delta+1)(\delta+2)\ldots(\delta+n-1), & \delta \in \mathbb{C}, n \in \mathbb{N}. \end{cases}$$

Using the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$, we define the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$ as follows:

Definition 1.2. Let $\phi \in P$ be a univalent starlike function with respect to 1, which maps the unit disc U onto a region in the right half plane and symmetric with respect to the real axis, $\phi(0)=1$ and $\phi'(0)>0$. A function $f \in \mathcal{A}$ is in the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$ if

$$\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)} \prec \phi(z), \tag{5}$$

where $\delta, m, b \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$ and $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z)$ denotes the differential operator (4).

The motivation of this paper is to generalise the Fekete-Szegö inequalities proved by Srivastava and Mishra (2000) for functions in the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$. We also give some applications of our results for certain functions defined by fractional derivatives.

To prove our main results, the following lemma is required.

Lemma 1.1 (Ma & Minda 1994). *If* $p_1(z)=1+c_1(z)+c_2z^2+\ldots$ *is an analytic function with positive real part in* U, *then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2 & \text{if} \quad v \leq 0, \\ 2 & \text{if} \quad 0 \leq v \leq 1, \\ 4v+2 & \text{if} \quad v \geq 1. \end{cases}$$

When $v<0$ *or* $v>1$, *the equality holds if and only if* $p_1(z)$ *is* $\frac{1+z}{1-z}$ *or one of its rotations. If* $0<v<1$, *then equality holds if and only if* $p_1(z)$ *is* $\frac{1+z^2}{1-z^2}$ *or one of its rotations. If* $v=0$, *the equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z}, \ (0 \leq \gamma \leq 1),$$

or one of its rotations. If $v=1$, *the equality holds if and only if* $p_1(z)$ *is the reciprocal of one of the functions such that the equality holds in the case of* $v=0$. *Also the above upper bound is sharp, it can be improved as follows when* $0<v<1$:

$$|c_2 - vc_1^2| + v|c_1^2| \leq 2, \quad (0 < v \leq \frac{1}{2})$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1^2| \leq 2, \quad (\frac{1}{2} < v \leq 1).$$

2. Main Results

Our first result is contained in the following theorem.

Theorem 2.1. *Let* $\phi(z)$ *be an analytic function with positive real part on* $\mathcal{A}$ *and* $\phi(z)=1+\mathcal{B}_1 z+\mathcal{B}_2 z^2+\ldots$. *If* $f(z)$ *is given by* (1) *and belongs to* $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}\phi(z)$, *then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}-\dfrac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}}\\ +\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m} & \text{if}\quad \mu\le\sigma_1,\\ \dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m} & \text{if}\quad \sigma_1\le\mu\le\sigma_2,\\ -\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}+\dfrac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}}\\ -\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m} & \text{if}\quad \mu\ge\sigma_2,\end{cases} \tag{6}$$

where

$$\sigma_1:=\frac{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2-\mathcal{B}_1)+\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mathcal{B}_1^2} \tag{7}$$

and

$$\sigma_2:=\frac{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2+\mathcal{B}_1)+\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mathcal{B}_1^2}. \tag{8}$$

Proof. For $f\in\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}\phi(z)$, let

$$p(z)=\frac{z(\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta}f(z))'}{\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta}f(z)}=1+b_1z+b_2z^2+\ldots. \tag{9}$$

Substituting (4) in (9) and comparing the coefficients of z^2 and z^3 on both sides in equation (9), we have

$$\left[\frac{1+\lambda_1+\lambda_2+b}{1+\lambda_2+b}\right]^m(\delta+1)a_2=b_1$$

and

$$\left[\frac{1+2(\lambda_1+\lambda_2)+b}{1+2\lambda_2+b}\right]^m(\delta+2)(\delta+1)a_3=\left[\frac{1+\lambda_1+\lambda_2+b}{1+\lambda_2+b}\right]^{2m}(\delta+1)^2a_2^2+b_2. \tag{10}$$

Now, we want to find out the values for b_1 and b_2. Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \dots \tag{11}$$

is analytic and has a positive real part in U. Thus, we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

From the equations (9) and (11), we obtain

$$1 + b_1 z + b_2 z^2 + \dots = \phi\left(\frac{c_1 z + c_2 z^2 + \dots}{2 + c_1 z + c_2 z^2 + \dots}\right)$$

$$= \phi\left[\frac{1}{2}c_1 z + \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \dots\right]$$

$$= 1 + \mathcal{B}_1 \frac{1}{2}c_1 z + \mathcal{B}_1 \frac{1}{2}(c_2 - \frac{1}{2}c_1^2)z^2 + \mathcal{B}_2 \frac{1}{4}c_1^2 z^2 + \dots, \tag{12}$$

and this implies

$$b_1 = \frac{1}{2}\mathcal{B}_1 c_1 \text{ and } b_2 = \frac{1}{2}\mathcal{B}_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}\mathcal{B}_2 c_1^2.$$

By substituting the values of b_1 and b_2 in equation (10), we have

$$a_2 = \frac{\mathcal{B}_1 c_1 (1+\lambda_2+b)^m}{2(1+\lambda_1+\lambda_2+b)^m(\delta+1)}$$

and

$$a_3 = \frac{(\frac{1}{4}\mathcal{B}_1^2 c_1^2 + \frac{1}{2}\mathcal{B}_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}\mathcal{B}_2 c_1^2)(1+2\lambda_2+b)^m}{(1+2(\lambda_1+\lambda_2)+b)^m(\delta+2)(\delta+1)}. \tag{13}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{\mathcal{B}_1(1+2\lambda_2+b)^m}{2(1+2(\lambda_1+\lambda_2)+b)^m(\delta+2)(\delta+1)}(c_2 - c_1^2[\frac{1}{2}(1 - \frac{\mathcal{B}_2}{\mathcal{B}_1} +$$
$$\frac{(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mu - (1+\lambda_1+\lambda_2+b)^{2m}(\delta+1)^2}{(1+\lambda_1+\lambda_2+b)^{2m}(\delta+1)^2}\mathcal{B}_1)]),$$

which implies

$$a_3-\mu a_2^2=\frac{\mathcal{B}_1(1+2\lambda_2+b)^m}{2(1+2(\lambda_1+\lambda_2)+b)^m(\delta+2)(\delta+1)}[c_2-\nu c_1^2],$$

where

$$\nu=\frac{1}{2}(1-\frac{\mathcal{B}_2}{\mathcal{B}_1}$$
$$+\frac{(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mu-(1+\lambda_1+\lambda_2+b)^{2m}(\delta+1)^2}{(1+\lambda_1+\lambda_2+b)^{2m}(\delta+1)^2}\mathcal{B}_1).$$

If $\mu\leq\sigma_1$, then by Lemma 1.1 we obtain

$$|a_3-\mu a_2^2|\leq\frac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}-\frac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}}$$
$$+\frac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}.$$

If $\mu\geq\sigma_2$, then we get

$$|a_3-\mu a_2^2|\leq-\frac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}+\frac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}}$$
$$-\frac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}.$$

Similarly if $\sigma_1\leq\mu\leq\sigma_2$, we get

$$|a_3-\mu a_2^2|\leq\frac{(1+2\lambda_2+b)^m\mathcal{B}_1}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}.$$

3. Application of Fractional Derivatives

For fixed $g \in \mathcal{A}$, let $\mathcal{M}^{m,b,g}_{\lambda_1,\lambda_2,\delta}(\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in \mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$. Note that, for any two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution is defined by

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 3.1 (Owa & Srivastava 1987)**.** Let *f* be analytic in a simply connected region of the *z*-plane containing the origin. The functional derivative of *f* of order γ is defined by

$$\mathcal{D}^{\gamma}_{z} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}} d\zeta, (0 \leq \gamma < 1),$$

where the multiplicity of $(z-\zeta)^{\gamma}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

Using Definition 3.1 and the well known extension involving fractional derivatives and fractional integrals, Owa and Srivastava (1987) introduced the operator $\Omega^{\gamma} : A \to A$, which is defined by

$$\Omega^{\gamma} f(z) = \Gamma(2-\gamma) z^{\gamma} \mathcal{D}^{\gamma}_{z} f(z), (\gamma \neq 2,3,4,\ldots).$$

The class $\mathcal{M}^{m,b,\gamma}_{\lambda_1,\lambda_2,\delta}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\gamma} f \in \mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$. Note that $\mathcal{M}^{m,b,\gamma}_{\lambda_1,\lambda_2,\delta}(\phi)$ is the special case of the class $\mathcal{M}^{m,b,g}_{\lambda_1,\lambda_2,\delta}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n.$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0).$$

Since $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) \in \mathcal{M}^{m,b,g}_{\lambda_1,\lambda_2,\delta}(\phi)$ if and only if $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta} f(z) * g(z) \in \mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$, we obtain the coefficient estimate for functions in the class $\mathcal{M}^{m,b,g}_{\lambda_1,\lambda_2,\delta}(\phi)$, from the corresponding estimate for functions in the class $\mathcal{M}^{m,b}_{\lambda_1,\lambda_2,\delta}(\phi)$. Applying Theorem 2.1 for the function

$$\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta}f(z)*g(z)=z+\left[\frac{1+\lambda_1+\lambda_2+b}{1+\lambda_2+b}\right]^m(\delta+1)a_2g_2z^2+\ldots,$$

we get the following result after an obvious change of the parameter μ.

Theorem 3.1. *Let* $g(z)=z+\sum_{n=2}^{\infty}g_nz^n,(g_n>0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z)=1+\sum_{n=1}^{\infty}\mathcal{B}_nz^n$. *If* $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta}f(z)$ *given by* (4) *belongs to* $\mathcal{M}^{m,b,g}_{\lambda_1,\lambda_2,\delta}(\phi)$, *then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^mg_3}-\dfrac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}g_2^2}\\+\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^mg_3} & \text{if}\quad \mu\le\sigma_1,\\ \dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^mg_3} & \text{if}\quad \sigma_1\le\mu\le\sigma_2,\\ -\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^mg_3}+\dfrac{(1+2\lambda_2+b)^m\mu\mathcal{B}_1^2}{(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}g_2^2}\\-\dfrac{(1+2\lambda_2+b)^m\mathcal{B}_1^2}{(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^mg_3} & \text{if}\quad \mu\ge\sigma_2,\end{cases}$$

where

$$\sigma_1:=\frac{g_2^2(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2-\mathcal{B}_1)+\mathcal{B}_1^2}{g_3(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mathcal{B}_1^2},$$

$$\sigma_2:=\frac{g_2^2(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2+\mathcal{B}_1)+\mathcal{B}_1^2}{g_3(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m\mathcal{B}_1^2}.$$

Since

$$(\Omega^\gamma\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,\delta})f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}\left[\frac{1+(\lambda_1+\lambda_2)(n-1)+b}{1+\lambda_2(n-1)+b}\right]^m\mathcal{C}(\delta,n)a_nz^n,$$

we have

$$g_2:=\frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)}=\frac{2}{(2-\gamma)}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)}.$$

Proof. By using the same technique as in the proof of Theorem 2.1, the required result is obtained.

For g_2 and g_3 given by above equalities, Theorem 3.1 reduces to the following result.

Corollary 3.1. *Let* $g(z) = z + \sum_{n=2}^{\infty} g_n z^n, (g_n > 0)$ *and let the function* $\phi(z)$ *be given by* $\phi(z) = 1 + \sum_{n=1}^{\infty} \mathcal{B}_n z^n$. *If* $\mathcal{D}_{\lambda_1,\lambda_2,\delta}^{m,b} f(z)$ *given by* (3) *belongs to* $\mathcal{M}_{\lambda_1,\lambda_2,\delta}^{m,b,g}(\phi)$. *Then,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{(2-\gamma)(3-\gamma)(1+2\lambda_2+b)^m \mathcal{B}_2}{6(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m} - \dfrac{(2-\gamma)^2(1+2\lambda_2+b)^m \mu \mathcal{B}_1^2}{4(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}} \\ + \dfrac{(2-\gamma)(3-\gamma)(1+2\lambda_2+b)^m \mathcal{B}_1^2}{6(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}, & \text{if } \mu \le \sigma_1, \\ \dfrac{(2-\gamma)(3-\gamma)(1+2\lambda_2+b)^m \mathcal{B}_1}{6(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{(2-\gamma)(3-\gamma)(1+2\lambda_2+b)^m \mathcal{B}_2}{6(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m} + \dfrac{(2-\gamma)^2(1+2\lambda_2+b)^m \mu \mathcal{B}_1^2}{4(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}} \\ -\dfrac{(2-\gamma)(3-\gamma)(1+2\lambda_2+b)^m \mathcal{B}_1^2}{6(\delta+2)(\delta+1)(1+2(\lambda_1+\lambda_2)+b)^m}, & \text{if } \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{2(3-\gamma)(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2 - \mathcal{B}_1) + \mathcal{B}_1^2}{3(2-\gamma)(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m \mathcal{B}_1^2}$$

and

$$\sigma_2 := \frac{2(3-\gamma)(\delta+1)^2(1+\lambda_1+\lambda_2+b)^{2m}(\mathcal{B}_2 + \mathcal{B}_1) + \mathcal{B}_1^2}{3(2-\gamma)(\delta+2)(\delta+1)(1+2\lambda_2+b)^m(1+2(\lambda_1+\lambda_2)+b)^m \mathcal{B}_1^2}.$$

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School of Mathematical Sciences
Faculty of Science and Technology
University Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, MALAYSIA
*E-mail: entisar_e1980@yahoo.com, maslina@ukm.edu.my**

*Corresponding author