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Analytical Solution for Cauchy Reaction-Diffusion Problems by Homotopy Perturbation Method

(Penyelesaian Beranalisis Bagi Masalah Tindak Balas-resapan Cauchy dengan Kaedah Usikan Homotopi)

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ABSTRACT

In this paper, the homotopy-perturbation method (HPM) is applied to obtain approximate analytical solutions for the Cauchy reaction-diffusion problems. HPM yields solutions in convergent series forms with easily computable terms. The HPM is tested for several examples. Comparisons of the results obtained by the HPM with that obtained by the Adomian decomposition method (ADM), homotopy analysis method (HAM) and the exact solutions show the efficiency of HPM.

Keywords: Cauchy problems; Homotopy-perturbation method; reaction-diffusion equation

ABSTRAK

Dalam makalah ini, kaedah usikan homotopi (KUH) diaplikasikan bagi memperoleh penyelesaian hampiran beranalisis untuk masalah tindak balas-resapan. KUH menghasilkan penyelesaian dalam bentuk siri yang menumpu dengan sebutan mudah dihitung. KUH diuji terhadap beberapa contoh masalah. Perbandingan keputusan yang diperoleh menerusi KUH dengan kaedah penguraian Adomian (KPA), kaedah homotopi analisis (KHA) dan penyelesaian tepat menunjukkan keefisienan KUH.

Kata kunci: Kaedah homotopi usikan; masalah Cauchy; persamaan tindak balas-resapan

INTRODUCTION

Cauchy reaction-diffusion equations model many problems in mathematical physics, astrophysics, engineering and science. The one-dimensional, time-dependent reaction diffusion equation can be written as (Dehghan & Shakeri 2008; Lesnic 2007),

$$\frac{\partial w}{\partial y}(x,t) - A\frac{\partial^2 w}{\partial x^2}(x,t) - q(x,t)w(x,t) = 0, \quad (x,t)\in\Omega\subset\Re^2. \tag{1}$$

where w is the concentration, q is the reaction parameter and $A > 0$ is the diffusion coefficient, subject to the initial or boundary conditions

$$w(x,0) = g(x), \quad x\in\Re, \tag{2}$$

$$w(0,t) = f_0(t), \quad \frac{\partial w}{\partial x}(0,t) = f_1(t), \quad t\in\Re. \tag{3}$$

The problem given by Equation (1) and (2) is called the characteristic Cauchy problem in the domain $\Omega = \Re \times \Re_+$, and the problem given by Equation (1) and (3) is called the non-characteristic Cauchy problem in the domain $\Omega = \Re_+ \times \Re$, . Very recently, Dehghan and Shakeri (2008) employed the variational iteration method (VIM) and Lesnic (2007) applied the Adomian decomposition (ADM) and Sami Bataineh et al. (2008) applied the homotopy analysis method (HAM) to solve the Cauchy reaction-diffusion problems.

In recent years, much attention has been given to the study of the homotopy-perturbation method (HPM) (He 1999, 2000, 2003, 2005a, 2005b, 2006a, 2006b, 2006c) for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations. HPM deforms a difficult problem into an infinite set of problems which are easier to solve without any need to transform non linear terms. The HAM, different from perturbation methods, can be categorized into a generalized Taylor expansion method. The HPM applies the homotopy parameter p, as an expanding parameter. The applications of HPM in non linear problems have been demonstrated by many researchers. For examples, the HPM was employed to solve variational problems (Abdulaziz et al. 2008a), fractional initial value problems (Abdulaziz et al. 2008b), systems of fractional differential equations (Abdulaziz et al. 2008c), singular second-order differential equations (Chowdhury & Hashim 2007a), time-dependent Emden-Fowler type equations (Chowdhury & Hashim 2007b), Klein-Gordon and sine-Gordon equations (Chowdhury & Hashim 2007c), n th-order IVPs directly (Chowdhury & Hashim 2007d), non linear population dynamics models (Chowdhury et al. 2007a), chaotic Lorenz system (Chowdhury et al. 2007b), squeezing flow of a Newtonian fluid (Ghori et al. 2007), non linear partial differential equations of fractional order (Momani & Odibat 2007), quadratic Riccati differential equation of fractional order (Odibat & Momani 2008), an inverse problem of diffusion equation (Shakeri & Dehghan 2007) and Couette and Poiseuille flows for non-Newtonian fluids (Siddiqui et al. 2006).

In this paper, we shall obtain analytical solutions to (1)-(3) by the HPM. The HPM yields convergent series solutions with easily computable terms. Test examples demonstrate the efficiency of the HPM.

SOLUTION APPROACH BY HPM

Since the HPM has now become standard and for brevity, the reader is referred to (He 1999, 2000, 2003, 2005a, 2005b, 2006a 2006b, 2006c) for the basic ideas of HPM. In this section, we shall demonstrate the application of HPM to solve Equation (1)-(3).

CHARACTERISTIC SOLUTION (PARTIAL t -SOLUTION) OF CAUCHY PROBLEM

According to HPM, we construct a homotopy into Equation (1) which satisfies the following relation:

$$w_t-(y_0)_t+p\left[(y_0)_t-Aw_{xx}-qw\right]=0, \tag{4}$$

where $p\in[0,1]$ is an embedding parameter and y_0 is an initial approximation which generally satisfies the initial conditions. Let us choose the initial approximations as

$$y_0(x,t)=u_0(x,t)=w(x,0)=g(x), \tag{5}$$

and

$$w(x,t)=u_0(x,t)+pu_1(x,t)+p^2u_2(x,t)+\cdots, \tag{6}$$

where $u_j (j = 1,2,3,...)$ are functions yet to be determined. Substituting (5)-(6) into (4) and collecting terms of the same powers of p, we have

$$(u_1)_t+(y_0)_t-A(u_0)_{xx}-qu_0=0, \quad u_1(x,0)=0, \tag{7}$$

$$(u_2)_t-A(u_1)_{xx}-qu_1=0, \quad u_2(x,0)=0, \tag{8}$$

$$(u_3)_t-A(u_2)_{xx}-qu_2=0, \quad u_3(x,0)=0, \tag{9}$$

etc. Now we can easily solve the above equations for u_1, u_2 and u_3 etc. using the Maple package.

NON-CHARACTERISTIC SOLUTION (PARTIAL x-SOLUTION) OF CAUCHY PROBLEM

Now we construct a homotopy into Equation (1) as follows:

$$w_{xx}-(y_0)_{xx}+p\left[(y_0)_{xx}-\frac{1}{A}w_t+\frac{1}{A}q(x,t)w(x,t)\right]=0, \tag{10}$$

where $p\in[0,1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. Let us choose the initial approximations as

$$y_0(x,t)=u_0(x,t)=w(0,t)+xw_x(0,t)=f_0(t)+xf_1(t). \tag{11}$$

where $u_j (j = 1,2,3,...)$ are functions yet to be determined. Substituting (11) and (6) into (10) and collecting terms of the same powers of p, we have

$$(u_1)_{xx}+(y_0)_t-\frac{1}{A}(u_0)_t+\frac{1}{A}qu_0=0, \quad u_1(0,t)=0, \quad (u_1)_x(0,t)=0, \tag{12}$$

$$(u_2)_{xx}-\frac{1}{A}(u_1)_t+\frac{1}{A}qu_1=0, \quad u_2(0,t)=0, \quad (u_2)_x(0,t)=0, \tag{13}$$

$$(u_3)_{xx}-\frac{1}{A}(u_2)_t+\frac{1}{A}qu_2=0, \quad u_3(0,t)=0, \quad (u_3)_x(0,t)=0, \tag{14}$$

etc. Finally, the series solution can be written as

$$w; u_0+u_1+u_2+u_3+\ldots. \tag{15}$$

The convergence of series (15) has been proven by He in his papers (1999, 2000).

APPLICATIONS OF HPM

The efficiency and accuracy of HPM to (1)-(3) through four examples demonstrated. The HPM algorithm is coded in the computer algebra package Maple.

Example 1. Case: q = constant

Setting and, Equation (1) becomes the Kolmogorov-Petrovsky-Piskunov (KPP) equation

$$\frac{\partial w}{\partial t}(x,t)-\frac{\partial^2 w}{\partial x^2}(x,t)+w(x,t)=0, \quad (x,t)\in\Omega, \tag{16}$$

subject to the initial and boundary conditions

$$w(x,0)=e^{-x}+x=g(x), \quad x\in\Re, \tag{17}$$

$$w(0,t)=1=f_0(t), \quad \frac{\partial w}{\partial x}(0,t)=e^{-t}-1=f_1(t), \quad t\in\Re. \tag{18}$$

CHARACTERISTIC SOLUTION (PARTIAL t -SOLUTION)

According to the HPM, we construct a homotopy into Eq. (16) which satisfies the following relation:

$$w_t-(y_0)_t+p[(y_0)_t-w_{xx}+w]=0, \tag{19}$$

where $p\in[0,1]$ is an embedding parameter. Let us choose the initial approximations as

$$y_0(x,t)=u_0(x,t)=w(x,0)=g(x)=e^{-x}+x. \tag{20}$$

where u_j (j = 1,2,3,...) are functions yet to be determined. Substituting (6) and (20) into (19) and collecting terms of the same powers of p, we have

$$(u_1)_t + (y_0)_t - (u_0)_{xx} + u_0 = 0, \quad u_1(x,0) = 0, \tag{21}$$

$$(u_2)_t - (u_1)_{xx} + u_1 = 0, \quad u_2(x,0) = 0, \tag{22}$$

$$(u_3)_t - (u_2)_{xx} + u_2 = 0, \quad u_3(x,0) = 0, \tag{23}$$

$$(u_4)_t - (u_3)_{xx} + u_3 = 0, \quad u_4(x,0) = 0, \tag{24}$$

etc. Solving the differential equations (21)-(24) we obtain,

$$u_1(x,t) = -xt, \quad u_2(x,t) = \frac{1}{2}xt^2, \quad u_3(x,t) = -\frac{1}{6}xt^3, \quad u_4(x,t) = \frac{1}{24}xt^4,$$

etc.

Hence, the characteristic (partial t -solution) series solution is

$$w(x,t) \cong e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots\right), \tag{25}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = e^{-x} + xe^{-t}, \tag{26}$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

NON-CHARACTERISTIC SOLUTION (PARTIAL x -SOLUTION)

Again, we construct a homotopy into Equation (16) as

$$w_{xx} - (y_0)_{xx} + p\,[(y_0)_{xx} - w_t - w] = 0, \tag{27}$$

and choose the initial approximations are

$$y_0(x,t) = u_0(x,t) = w(0,t) + xw_x(0,t) = 1 + xe^{-t} - x. \tag{28}$$

Substituting (6) and (28) into (27) and collecting terms of the same powers of p, we have

$$(u_1)_{xx} + (y_0)_{xx} - (u_0)_t - u_0 = 0, \; u_1(0,t) = 0, \; (u_1)_x(0,t) = 0, \tag{29}$$

$$(u_2)_{xx} - (u_1)_t - u_1 = 0, \; u_2(0,t) = 0, \; (u_2)_x(0,t) = 0, \tag{30}$$

$$(u_3)_{xx} - (u_2)_t - u_2 = 0, \; u_3(0,t) = 0, \; (u_3)_x(0,t) = 0, \tag{31}$$

$$(u_4)_{xx} - (u_3)_t - u_3 = 0, \; u_4(0,t) = 0, \; (u_4)_x(0,t) = 0, \tag{32}$$

etc. Solving the differential equations (29)-(32) we obtain,

$$u_1(x,t) = -\frac{1}{6}x^3 + \frac{1}{2}x^2, \quad u_2(x,t) = -\frac{1}{120}x^5 + \frac{1}{24}x^4,$$

$$u_3(x,t) = -\frac{1}{5040}x^7 + \frac{1}{720}x^6, \quad u_4(x,t) = -\frac{1}{362880}x^9 + \frac{1}{40320}x^8,$$

etc.

Hence, the non-characteristic (partial x -solution) series solution is

$$w(x,t) \cong xe^{-t} + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right), \tag{33}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = xe^{-t} + e^{-x}, \tag{34}$$

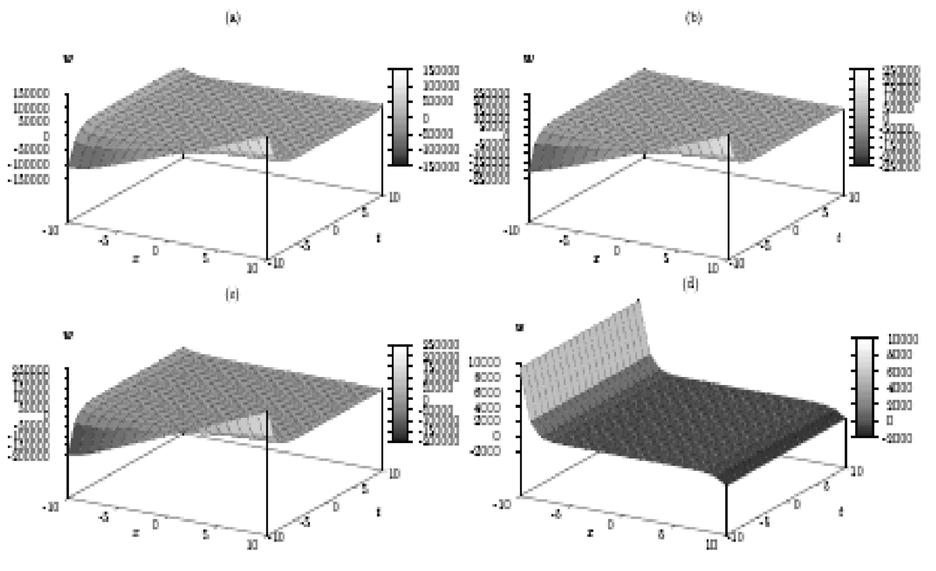

FIGURE 1. The numerical results for $w(x, t)$ for Example 1: (a) 10-term HPM partial t -solution, (b) 10-term HPM partial x -solution, (c) exact and (d) error between exact and 10-term HPM partial x -solution

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

From Table 1, it is observed that the 10-term HPM solutions agree with exact solutions for , while the 5-term HPM solutions are only valid for $-2 < x < 2$. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms or further components of $w(x,t)$ when the HPM is used.

Example 2. Case: $q = q(t)$

Considering $A = 1$ and $q = 2t$, Equation (1) recasts as

$$\frac{\partial w}{\partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) - 2tw(x,t) = 0, \quad (x,t)\in\Omega, \qquad (35)$$

subject to the initial and boundary conditions

$$w(x,0) = \mathrm{e}^x = g(x), \quad x\in\Re, \qquad (36)$$

$$w(0,t) = \mathrm{e}^{t+t^2} = f_0(t), \quad \frac{\partial w}{\partial x}(0,t) = \mathrm{e}^{t+t^2} = f_1(t), \quad t\in\Re. \qquad (37)$$

CHARACTERISTIC SOLUTION (PARTIAL t-SOLUTION)

According to HPM, the homotopy equation is

$$w_t - (y_0)_t + p[(y_0)_t - w_{xx} - 2tw] = 0. \qquad (38)$$

and take the initial approximations as

$$y_0(x,t) = u_0(x,t) = w(x,0) = g(x) = e^x. \qquad (39)$$

Substituting (6) and (39) into (38), gives

$$(u_1)_t + (y_0)_t - (u_0)_{xx} - 2tu_0 = 0, \; u_1(x,0) = 0, \qquad (40)$$

$$(u_2)_t - (u_1)_{xx} - 2tu_1 = 0, \; u_2(x,0) = 0, \qquad (41)$$

$$(u_3)_t - (u_2)_{xx} - 2tu_2 = 0, \; u_3(x,0) = 0, \qquad (42)$$

etc. Solving the differential equation (40)-(42) , we obtain

$$u_1(x,t) = \mathrm{e}^x t + \mathrm{e}^x t^2, \quad u_2(x,t) = \frac{1}{2}\mathrm{e}^x t^4 + \mathrm{e}^x t^3 + \frac{1}{2}\mathrm{e}^x t^2,$$

etc.

Hence, the characteristic (partial t -solution) series solution is

$$w(x,t) ; \; \mathrm{e}^x\left(1 + t + \frac{3t^2}{2!} + \frac{7t^3}{3!} + \frac{25t^4}{4!} + \cdots\right), \qquad (43)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = ex + t + t_2, \qquad (44)$$

TABLE 1. Absolute errors between the exact and 5-term HPM and 10-term HPM solutions for Ex-1 for $t = 1$

x	*Exact – HPM* 5	*Exact – HPM* 10
-10	1.194E+04	76.09
-9	3343	8.556
-8	844.7	0.754
-7	185.9	0.04869
-6	33.86	0.00209
-5	4.724	5.120E-05
-4	0.444	5.400E-07
-3	0.02214	1.000E-08
-2	0.0003436	0
-1	3.010E-07	2.000E-09
0	0	0
1	2.524E-07	0
2	0.0002383	1.000E-10
3	0.01273	1.000E-09
4	0.2106	3.790E-07
5	1.834	3.160E-05
6	10.66	0.001166
7	46.91	0.02453
8	168.5	0.3419
9	518.6	3.483
10	1413	27.71

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

NON-CHARACTERISTIC SOLUTION (PARTIAL x-SOLUTION)

Again, we construct a homotopy in Equation (35) as follows:

$$w_{xx} - (y_0)_{xx} + p[(y_0)_{xx} - w_t = 2tw] = 0, \qquad (45)$$

and take the initial approximations as

$$y_0(x,t) = u_0(x,t) = w(0,t) + xw_x(0,t) = \mathrm{e}^{t+t^2} + x\mathrm{e}^{t+t^2}. \qquad (46)$$

Substituting (6) and (46) into (45), gives

$$(u_1)_{xx} + (y_0)_{xx} - (u_0)_t + 2tu_0 = 0, \quad u_1(0,t) = 0,$$
$$(u_1)_x(0,t) = 0, \qquad (47)$$

$$(u_2)_{xx} - (u_1)_t + 2tu_1 = 0, \quad u_2(0,t) = 0, \quad (u_2)_x(0,t) = 0, \qquad (48)$$

$$(u_3)_{xx} - (u_2)_t + 2tu_2 = 0, \quad u_3(0,t) = 0, \quad (u_3)_x(0,t) = 0, \qquad (49)$$

etc. Solving the equations (47)-(49) we obtain,

$$u_1(x,t) = \frac{1}{2}\mathrm{e}^{t+t^2}x^2 + \frac{1}{6}\mathrm{e}^{t+t^2}x^3, \quad u_2(x,t) = \frac{1}{24}\mathrm{e}^{t+t^2}x^4 + \frac{1}{120}\mathrm{e}^{t+t^2}x^5,$$
$$u_3(x,t) = \frac{1}{720}\mathrm{e}^{t+t^2}x^6 + \frac{1}{5040}\mathrm{e}^{t+t^2}x^7,$$

etc.

Finally, the non-characteristic (partial x-solution) series solution is

$$w(x,t) \,;\; \mathrm{e}^{t+t^2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right),$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = \mathrm{e}^{x+t+t^2} \qquad (51)$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for (Sami Bataineh et al. 2008).

From Table 2, it is observed that the 10-term HPM solutions agree with exact solutions for $-6 < x < 6$, while the 5-term HPM solutions are only valid for $-2 < x < 2$.

Example 3. Case: $q = q(x)$

Taking A =1 and $q = -(1 + 4x^2)$, Equation (1) reduces to

$$\frac{\partial w}{\partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) + (1+4x^2)w(x,t) = 0, \quad (x,t) \in \Omega, \qquad (52)$$

subject to the initial and boundary conditions

$$w(x,0) = \mathrm{e}^{x2} = \mathrm{g}(x), \; x \in \Re, \qquad (53)$$

$$w(0,t) = \mathrm{e}^t = f_0(t), \quad \frac{\partial w}{\partial x}(0,t) = 0 = f_1(t), \quad t \in \Re. \qquad (54)$$

CHARACTERISTIC SOLUTION (PARTIAL t-SOLUTION)

We construct a homotopy in Equation (52) which satisfies the following relation:

$$w_t - (y_0)_t + p\left[(y_0)_t - w_{xx} + (1+4x^2)w\right] = 0. \qquad (55)$$

Let us choose the initial approximations as

$$y_0(x,t) = u_0(x,t) = w(x,0) = g(x) = \mathrm{e}^{x^2}. \qquad (56)$$

Substituting (6) and (56) into (55) and collecting terms of the same powers of p, we have

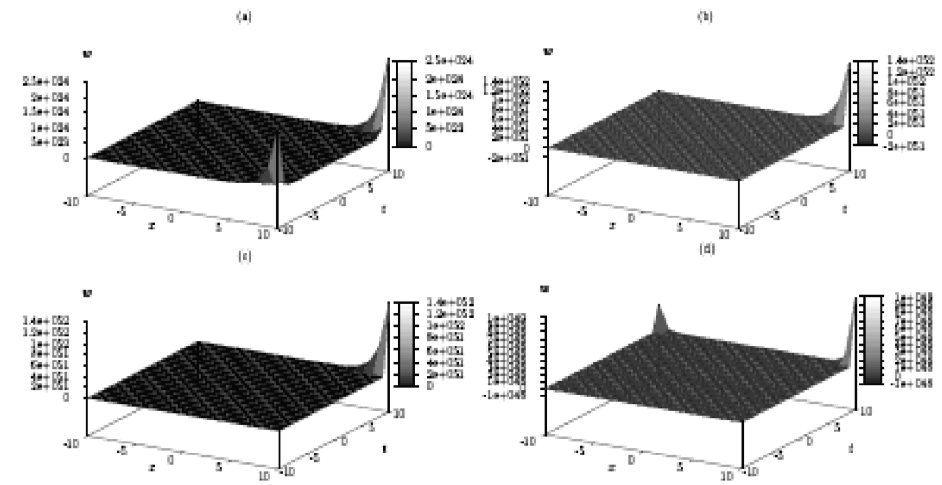

FIGURE 2. The numerical results for $w(w, \mathrm{t})$ for Example 2: (a) 10-term HPM partial t -solution, (b) 10-term HPM partial x-solution, (c) exact and (d) error between exact and 10-term HPM partial x -solution

TABLE 2. Absolute errors between the exact and 5-term HPM and 10-term HPM solutions for Ex-2 for $t = 1$

x	*Exact* – HPM 5	*Exact* – HPM 10
-10	1.044E+ 04	204.7
-9	3832	25.74
-8	1245	2.526
-7	346.6	0.1812
-6	78.76	0.008617
-5	13.55	0.0002335
-4	1.556	2.803E-06
-3	0.09409	8.900E-09
-2	0.001761	0
-1	1.864E-06	1.000E-09
0	0	0
1	2.220E-06	1.000E-08
2	0.002539	0
3	0.1636	1.000E-07
4	3.281	4.000E-06
5	34.9	0.000378
6	250.2	0.01544
7	1374	0.3598
8	6242	5.571
9	2.470E+04	63.22
10	8.822E+04	562.2

$$(u_1)_t + (y_0)_t - (u_0)_{xx} + (1+4x^2)u_0 = 0, \quad u_1(x,0) = 0, \tag{57}$$

$$(u_2)_t - (u_1)_{xx} + (1+4x^2)u_1 = 0, \quad u_2(x,0) = 0, \tag{58}$$

$$(u_3)_t - (u_2)_{xx} + (1+4x^2)u_2 = 0, \quad u_3(x,0) = 0, \tag{59}$$

etc. Solving the above equations (57)-(59), we obtain

$$u_1(x,t) = \mathrm{e}^{x^2} t, \quad u_2(x,t) = \frac{1}{2}\mathrm{e}^{x^2} t^2, \quad u_3(x,t) = \frac{1}{6}\mathrm{e}^{x^2} t^3,$$

etc.

Finally, the characteristic (partial t -solution) series solution is

$$w(x,t) \cong \mathrm{e}^{x^2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right), \tag{60}$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = \mathrm{e}^{x^2 + t}, \tag{61}$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

NON-CHARACTERISTIC SOLUTION (PARTIAL x -SOLUTION)

Now we construct a homotopy in Equation (52) which satisfies the following relation:

$$w_{xx} - (y_0)_{xx} + p\left[(y_0)_{xx} - w_t - (1+4x^2)w\right] = 0, \tag{62}$$

where $p \in [0, 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. Let us choose the initial approximations as

$$y_0(x,t) = u_0(x,t) = w(0,t) + x w_x(0,t) = \mathrm{e}^t. \tag{63}$$

where $u_j (j = 1,2,3,...)$ are functions yet to be determined. Substituting (6) and (63) into (62) and collecting terms of the same powers of p, we have

$$(u_1)_{xx} + (y_0)_{xx} - (u_0)_t - (1+4x^2)u_0 = 0,$$
$$u_1(0,t) = 0, \quad (u_1)_x(0,t) = 0, \tag{64}$$

$$(u_2)_{xx} - (u_1)_t - (1+4x^2)u_1 = 0, \quad u_2(0,t) = 0,$$
$$(u_2)_x(0,t) = 0, \tag{65}$$

$$(u_3)_{xx} - (u_2)_t - (1+4x^2)u_2 = 0, \quad u_3(0,t) = 0,$$
$$(u_3)_x(0,t) = 0, \tag{66}$$

etc. Solving the differential equation (64)-(66) we obtain,

$$u_1(x,t) = \mathrm{e}^t x^2 + \frac{1}{3}\mathrm{e}^t x^4, \quad u_2(x,t) = \frac{1}{6}\mathrm{e}^t x^4 + \frac{7}{45}\mathrm{e}^t x^6 + \frac{1}{42}\mathrm{e}^t x^8,$$

$$u_3(x,t) = \frac{1}{90}\mathrm{e}^t x^6 + \frac{11}{630}\mathrm{e}^t x^8 + \frac{211}{28350}\mathrm{e}^t x^{10} + \frac{1}{1386}\mathrm{e}^t x^{12},$$

etc.

Finally, the approximate non-characteristic solution (partial x-solution) in a series form is

$$w(x,t) \approx e^{t}\left(1+x^{2}+\frac{x^{4}}{2!}+\frac{x^{6}}{3!}+\cdots\right), \qquad (67)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = e^{t+x^2}, \qquad (68)$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

From Table 3, it is observed that the 20-term HPM solutions agree with exact solutions for $-6 < t < 6$, , while the 10-term HPM solutions are only valid for $-2 < t < 2$.

Example 4. Case: $q = q(x, t)$

Taking $A = 1$ and $q = -(4x^2 - 2t + 2)$, Equation (1) reduce to

$$\frac{\partial w}{\partial t}(x,t)-\frac{\partial^{2} w}{\partial x^{2}}(x,t)+(4x^{2}-2t+2)w(x,t)=0, \quad (x,t)\in\Omega, \qquad (69)$$

subject to the initial and boundary conditions

$$w(x,0)=e^{x^{2}}=g(x), \quad x\in\Re, \qquad (70)$$

$$w(0,t)=e^{t^{2}}=f_{0}(t), \quad \frac{\partial w}{\partial x}(0,t)=0=f_{1}(t), \quad t\in\Re. \qquad (71)$$

CHARACTERISTIC SOLUTION (PARTIAL t-SOLUTION)

We construct a homotopy in Equation (69) which satisfies the following relation:

$$w_{t}-(y_{0})_{t}+p\left[(y_{0})_{t}-w_{xx}+(4x^{2}-2t+2)w\right]=0, \qquad (72)$$

and take the initial approximations as

$$y_{0}(x,t)=u_{0}(x,t)=w(x,0)=g(x)=e^{x^{2}}. \qquad (73)$$

Substituting (6) and (73) into (72), gives

$$(u_{1})_{t}+(y_{0})_{t}-(u_{0})_{xx}+(4x^{2}-2t+2)u_{0}=0, \quad u_{1}(x,0)=0, \qquad (74)$$

$$(u_{2})_{t}-(u_{1})_{xx}+(4x^{2}-2t+2)u_{1}=0, \quad u_{2}(x,0)=0, \qquad (75)$$

$$(u_{3})_{t}-(u_{2})_{xx}+(4x^{2}-2t+2)u_{2}=0, \quad u_{3}(x,0)=0, \qquad (76)$$

etc. Solving the differential equation (74)-(76), we obtain

$$u_{1}(x,t)=e^{x^{2}}t^{2}, \quad u_{2}(x,t)=\frac{1}{2}e^{x^{2}}t^{4}, \quad u_{3}(x,t)=\frac{1}{6}e^{x^{2}}t^{6},$$

etc.

Hence, the characteristic (partial t -solution) series solution is

$$w(x,t) \approx e^{x^{2}}\left(1+t^{2}+\frac{t^{4}}{2!}+\frac{t^{6}}{3!}+\frac{t^{8}}{4!}+\cdots\right), \qquad (77)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = e^{x^2+t^2}, \qquad (78)$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

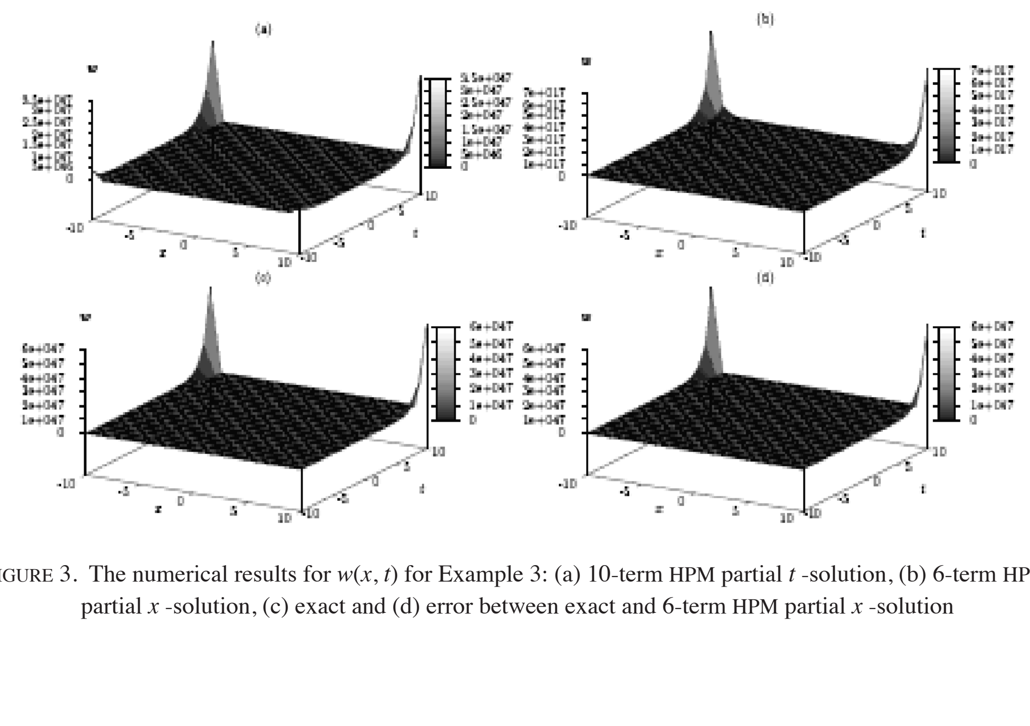

FIGURE 3. The numerical results for $w(x, t)$ for Example 3: (a) 10-term HPM partial t -solution, (b) 6-term HPM partial x -solution, (c) exact and (d) error between exact and 6-term HPM partial x -solution

TABLE 3. Absolute errors between the exact and 10-term HPM and 20-term HPM solutions for Ex-3 for $x = 1$

t	$Exact - HPM\,10$	$Exact - HPM\,20$
-10	3650	36.42
-9	1202	4.116
-8	346.3	0.3587
-7	84.08	0.02249
-6	16.32	0.0009152
-5	2.33	2.066E-05
-4	0.2131	1.972E-07
-3	0.00962	6.000E-10
-2	0.0001193	1.000E-10
-1	6.300E-08	1.000E-10
0	0	0
1	7.100E-08	3.000E-09
2	0.0001669	0
3	0.01596	4.000E-08
4	0.4215	3.000E-07
5	5.525	3.270E-05
6	46.74	0.001595
7	293.7	0.04321
8	1492	0.7614
9	6476	9.675
10	2.497E+04	95

NON-CHARACTERISTIC SOLUTION (PARTIAL x -SOLUTION)

Again construct a homotopy in Equation (69) which satisfies the following relation:

$$w_{xx} - (y_0)_{xx} + p\left[(y_0)_{xx} - w_t - (4x^2 - 2t + 2)w\right] = 0, \quad (79)$$

where $p \in [0,\ 1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. Let us choose the initial approximations as

$$y_0(x,t) = u_0(x,t) = w(0,t) + xw_x(0,t) = e^{t^2}. \quad (80)$$

Substituting (6) and (80) into (79), gives

$$(u_1)_{xx} + (y_0)_{xx} - (u_0)_t - (4x^2 - 2t + 2)u_0 = 0,$$
$$u_1(0,t) = 0, \quad (u_1)_x(0,t) = 0, \quad (81)$$

$$(u_2)_{xx} - (u_1)_t - (4x^2 - 2t + 2)u_1 = 0,$$
$$u_2(0,t) = 0, \quad (u_2)_x(0,t) = 0, \quad (82)$$

$$(u_3)_{xx} - (u_2)_t - (4x^2 - 2t + 2)u_2 = 0,$$
$$u_3(0,t) = 0, \quad (u_3)_x(0,t) = 0, \quad (83)$$

etc.

Solving the equation (81)-(83) we obtain,

$$u_1(x,t) = e^{t^2}x^2 + \frac{1}{3}e^{t^2}x^4, \quad u_2(x,t) = \frac{105}{630}e^{t^2}x^4 + \frac{98}{630}e^{t^2}x^6 + \frac{1}{42}e^{t^2}x^8,$$

$$u_3(x,t) = \frac{3465}{311850}e^{t^2}x^6 + \frac{5445}{311850}e^{t^2}x^8 + \frac{2321}{311850}e^{t^2}x^{10} + \frac{225}{311850}e^{t^2}x^{12},$$

etc.

Hence, the non-characteristic (partial x -solution) series solution is

$$w(x,t) ; \; e^{t^2}\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right), \quad (84)$$

and this will, in the limit of infinitely many terms, yield the closed-form solution,

$$w(x,t) = e^{t^2 + x^2}, \quad (85)$$

which is the same as the solutions obtained by ADM (Lesnic 2007) and HAM for $\hbar = -1$ (Sami Bataineh et al. 2008).

From Table 4, it is observed that the 10-term HPM solutions agree with exact solutions for $-2 < x < 2$, while the 5-term HPM solutions are only valid for $-1 < x < 1$.

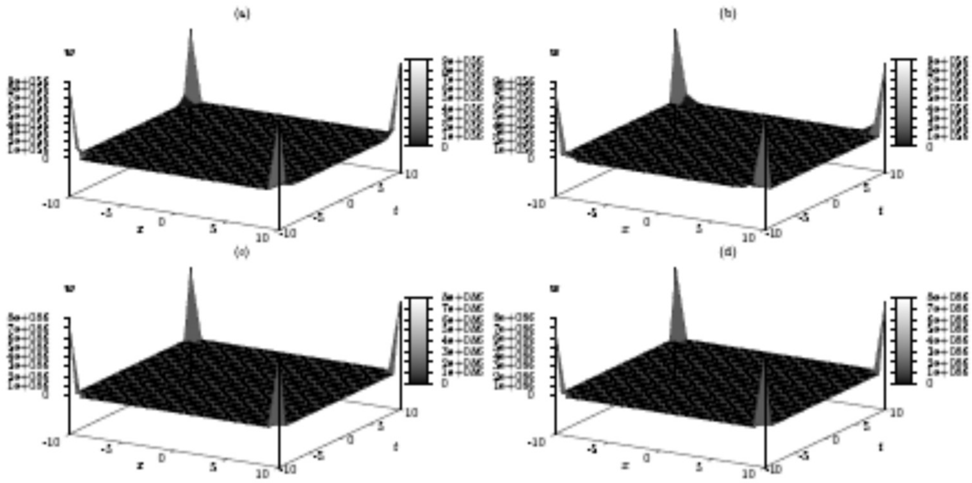

FIGURE 4. The numerical results for $w(x\ \ t\ )$ for Example 4: (a) 10-term HPM partial t-solution, (b) 6-term HPM partial x-solution, (c) exact and (d) error between exact and 6-term HPM partial x-solution

TABLE 4. Absolute errors between the exact and 5-term HPM and 10-term HPM solutions for Ex-4 for $t = 1$

x	*Exact – HPM* 5	*Exact – HPM* 10
-10	7.307E+43	7.307E+43
-9	4.094E+35	4.094E+35
-8	1.695E+28	1.695E+28
-7	5.185E+21	5.185E+21
-6	1.172E+16	1.171E+16
-5	1.957E+11	1.777E+11
-4	2.362E+07	6.224E+06
-3	1.199E+04	53.44
-2	3.171	7.600E-06
-1	8.312E-06	3.000E-09
0	0	0
1	8.312E-06	3.000E-09
2	3.171	7.600E-06
3	1.199E+04	53.44
4	2.362E+07	6.224E+06
5	1.957E+11	1.777E+11
6	1.172E+16	1.171E+16
7	5.185E+21	5.185E+21
8	1.695E+28	1.695E+28
9	4.094E+35	4.094E+35
10	7.307E+43	7.307E+43

Conclusion

In this paper, the standard homotopy-perturbation method (HPM) was applied to solve Cauchy reaction-diffusion problems. In all examples, in the limit of infinitely many terms the HPM yields the exact solution. The results of the test examples also show that the ADM and HAM results are same as the results of HPM. Comparisons with the exact solution reveals that HPM is simple, efficient and reliable. In addition, the calculations involved in HPM are very simple and straightforward. It is demonstrated that HPM is a powerful and efficient tool for PDEs.

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