Sains Malaysiana 43(7)(2014): 1101–1104

Interval Symmetric Single-step Procedure ISS2-5D for Polynomial Zeros

(Prosedur Selang Bersimetri Langkah-tunggal ISS2-5D untuk Punca Polinomial)

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ABSTRACT

We analyzed the rate of convergence of a new modified interval symmetric single-step procedure ISS2-5D which is an extension from the previous procedure ISS2. The algorithm of ISS2-5D includes the introduction of reusable correctors $\delta_i^{(k)}$ *(*$i = 1, \ldots, n$*) for* $k \geq 0$*. Furthermore, this procedure was tested on five test polynomials and the results were obtained using MATLAB 2007 software in association with IntLab V5.5 toolbox to record the CPU times and the number of iterations.*

Keywords: Interval procedure; polynomial zeros; rate of convergence; simultaneous inclusion; symmetric single-step

ABSTRAK

Satu analisis dilakukan terhadap kadar penumpuan bagi prosedur terubahsuai selang bersimetri langkah-tunggal ISS2-5D baru yang merupakan lanjutan daripada prosedur ISS2 sebelumnya. Algoritma ISS2-5D termasuk pengenalan pembetulan yang boleh diguna semula $\delta_i^{(k)}$ *(*$i = 1, \ldots, n$*) untuk* $k \geq 0$*. Prosedur ini diuji ke atas lima jenis polinomial dan keputusan diperoleh menggunakan perisian MATLAB 2007 dan peralatan IntLab V5.5 untuk merekod masa CPU dan bilangan lelaran.*

Kata kunci: Kadar penumpuan; kemasukan serentak; prosedur selang; punca polinomial; selang langkah tunggal bersimetri

INTRODUCTION

Interval iterative procedure for simultaneous inclusion of simple polynomial zeros were discussed in Aberth (1973), Alefeld and Herzberger (1983), Gargantini and Henrici (1972), Iliev and Kyurkchiev (2010), Kyurkchiev (1998), Kyurkchiev and Markov (1983a, 1983b), Markov and Kyurkchiev (1989), Monsi and Wolfe (1988), Petkovic (1989) and Petkovic and Stefanovic (1986). In this paper, we consider the procedures developed by Bakar et al. (2012), Jamaluddin et al. (2013a, 2013b), Milovanovic and Petkovic (1983), Monsi et al. (2012), Nourien (1977), Salim et al. (2011) and Sham et al. (2013a, 2013b) in order to describe the algorithm of the interval symmetric single-step procedure ISS2-5D. This procedure needs some pre-conditions (Theorem 1) for initial intervals $X_i^{(0)}$ ($i = 1, \ldots, n$) to converge to the zeros x_i^* ($i = 1, \ldots, n$), respectively, starting with some disjoint intervals $X_i^{(0)}$ ($i = 1, \ldots, n$) each of which contains a polynomial zero. It will produce bounded closed intervals which will trap the required zero within a certain tolerance value.

The forward step by Salim et al. (2011) is modified by adding a $\delta = \delta_i^{(k)}$ ($i = 1, \ldots, n$) ($k \geq 0$) (1(c)) on the second part of the summation of the denominator (1(d)). The backward step of this procedure comes from Monsi and Wolfe (1988). The interval analysis is very straight forward compared to the analysis of the point procedures Milovanovic and Petkovic (1983) and Nourien (1977). The programming language used is Matlab 2007a with the Intlab V5.5 toolbox by Rump (1999). The effectiveness of our procedure is measured numerically using CPU time and the number of iterations.

METHODS

THE INTERVAL SYMMETRIC SINGLE-STEP PROCEDURE ISS2-5D

The interval symmetric single-step procedure ISS2-5D is an extension of the interval single-step procedure ISS2 by Salim et al. (2011) based on Aitken (1950), Alefeld and Herzberger (1983), Milovanovic and Petkovic (1983), Monsi and Wolfe (1988), Nourien (1977) and Ortega and Rheinboldt (1970). The sequences $X_i^{(k)}$ ($i = 1, \ldots, n$) are generated as follows.

Step 1: $X_i^{(1,0)} = X_i^{(0)}$ (Initial intervals). (1a)

Step 2: For $k \geq 0, x_i^{(k)} = mid\left(X_i^{(k)}\right),\ (i = 1, \ldots, n)$. (1b)

Step 3: Let $\delta_i^{(k)} = \dfrac{p\left(x_i^{(k)}\right)}{p'\left(x_i^{(k)}\right)}$ $(i = 1, \ldots, n)$. (1c)

Step 4:

$$X_i^{(k,1)} = \left\{ x_i^{(k)} + \frac{\delta_i^{(k)}}{1 + \delta_i^{(k)} \left(\sum_{j=1}^{i-1} \frac{1}{x_i^{(k)} - X_j^{(k,1)}} + \sum_{j=i+1}^{n} \frac{1}{x_i^{(k)} - X_j^{(k)} - 5\delta_j^{(k)}} \right)} \right\} \cap X_i^{(k)}.$$

$(i = 1, \ldots, n)$ (1d)

Step 5:

$$X_i^{(k,2)} = \left\{ x_i^{(k)} + \frac{\delta_i^{(k)}}{1+\delta_i^{(k)}\left(\sum_{j=1}^{i-1}\frac{1}{x_i^{(k)}-X_j^{(k,1)}} + \sum_{j=i+1}^{n}\frac{1}{x_i^{(k)}-X_j^{(k,2)}-5\delta_i^{(k,2)}}\right)} \right\} \cap X_i^{(k,1)}.$$

$(i = 1, \dots, n)$ (1e)

Step 6: $X_i^{(k)} \to x_i^* \ (k \to \infty)\ (i = 1, \dots, n)$. (1f)

Step 7: If $w\left(X_i^{k+1}\right) < \varepsilon$, then stop. Else set $k = k + 1$ and go to Step 2. (1g)

Step 4 is from Milovanovic and Petkovic (1983) and pointed out without δ by Nourien (1977), while Step 5 is from Monsi and Wolfe (1988).

The procedure ISS2-5D has the following attractive features:

The use of $5\delta_j$ instead of $\delta_i^{(k)}$ as in Milovanovic and Petkovic (1983); the values $\delta_i^{(k)}$ computed for use in Step 4 are reused in Step 5; the summations $\sum_{j=1}^{i-1}\frac{1}{x_i^{(k)}-x_j^{(k,1)}}\left(i = 1,\dots n\right)$ used in Step 4 are reused in Step 5 and $x_n^{(k,1)} = x_n^{(k,2)}\left(k \ge 0\right)$ so that $x_n^{(k,2)}$ need not be computed.

THE RATE OF CONVERGENCE OF ISS2-5D

Now we have additional description of the Algorithm ISS2-5D regarding the conditions, inclusion, convergent and the rate of convergence.

Theorem 1: Let p defined by $p(x) = \sum_{i=0}^{n} a_i x^i$ $(a_n \ne 0)$. If (i) p has n distinct zeros x_i^* $(i = 1, \dots, n)$, $x_i^* \in X_i^{(0)}$ and $X_i^{(0)} \cap X_j^{(0)} = \varnothing$, $(i, j = 1, \dots, n;\ i \ne j)$ hold; (ii) $0 \notin D_i \in I(R)$ $(D_i = [d_{iI}, d_{iS}])$ is such that $p'(x) \in D_i$ $(\forall x \in D_i(\forall x \in X_i^{(0)})$, $(i = 1, \dots, n)$ and

$$w\left(X_i^{(k+1)}\right) \le \frac{1}{2}\left(1 - \frac{d_{iI}}{d_{iS}}\right) w\left(X_i^{(k)}\right),$$

holds (where $w(X_i^{(k)}) \le w\left(\left[x_{iI}^{(k)}, x_{iS}^{(k)}\right]\right) = x_{iS}^{(k)} - x_{iI}^{(k)}$); (iii) the sequence $\left\{X_i^{(k)}\right\}$ $(i = 1,\dots,n)$ are generated from (1), then (iv) $(\forall k \ge 0)$ $x_i^* \in X^{(k,i)} \subseteq X^{(k)} (i = 1,\dots,n)$; (v) $X^{(0,i)} \supset X^{(0,1)} \supset X^{(0,2)} \supset \dots$ with $\lim_{k\to\infty}{}^{(k)} = x_i^*$, $X_i^{(k)} \to x_i^*$ $(k \to \infty)$ $(i = 1,\dots,n)$, and (vi) $O_R\,(ISS2-5D, x_i^*) \ge 6$ for $(i = 1,\dots,n)$.

The proofs of (iv) and (v) are available in Aitken (1950). Now the proof of (vi) is as follows.

Proof

By Step 4 and Step 5, $\exists\alpha > 0$ such that $(\forall k \ge 0)$,

$$w_i^{(k,1)} \le \beta\left(w_i^{(k,0)}\right)^2 \left\{\sum_{j=1}^{i-1} w_j^{(k,1)} + \sum\left(w_j^{(k,0)}\right)^2\right\}$$

$(i = 1, \dots, n)$, (2)

and

$$w_i^{(k,2)} \le \beta\left(w_i^{(k,0)}\right)^2 \left\{\sum_{j=1}^{i-1} w_j^{(k,1)} + \sum_{j=i+1}^{n} w_j^{(k,2)}\right\} \quad (i = n,\dots,1), \quad (3)$$

where

$$w_i^{(k,s)} = (n-1)\,\alpha w\,(X_i^{(k,s)} \quad (s = 0,1,2), \quad (4)$$

and

$$\beta = \frac{1}{n-1}. \quad (5)$$

Let

$$u_i^{(1,1)} = \begin{cases} 4 & \left(i = 1,\dots,n-1\right) \\ 6 & \left(i = n\right) \end{cases}, \quad (6)$$

$$u_i^{(1,2)} = \begin{cases} 8 & \left(i = 1\right) \\ 6 & \left(i = 2,\dots,n-1\right) \end{cases}, \quad (7)$$

and for $(r = 1,2)$, let

$$u_i^{(k+1,r)} = \begin{cases} 6u_i^{(k,r)} + 2 & \left(i = 1\right) \\ 6u_i^{(k,r)} & \left(i = 2,\dots,n\right) \end{cases}. \quad (8)$$

Then by (6) - (8), for $(\forall k \ge 0)$

$$u_i^{(k,1)} = \begin{cases} \frac{22}{5}\left(6^{k-1}\right) - \frac{2}{5} & \left(i = 1\right) \\ 4\left(6^{(k-1)}\right) & \left(i = 2,\dots,n-1\right), \\ 6\left(6^{(k-1)}\right) & \left(i = n\right) \end{cases} \quad (9)$$

and

$$u_i^{(k,2)} = \begin{cases} \frac{42}{5}\left(6^{(k-1)}\right) - \frac{2}{5} & \left(i = 1\right) \\ 6\left(6^{k-1}\right) & \left(i = 2,\dots,n\right) \end{cases}. \quad (10)$$

Suppose, without any loss of generality, that

$$w_i^{(0,0)} \le h < 1 \quad (i = 1, \dots, n). \quad (11)$$

Then by inductive argument it follows from (2) - (10) that for $(i = 1, \dots, n)$ $(k \ge 0)$

$$w_i^{(k,1)} \le h^{u_i^{(k+1,1)}}, \text{ and } w_i^{(k,2)} \le h^{u_i^{(k+1,2)}},$$

whence by (1f) and (9),

$$w_i^{(k+1)} \le h^{6^{(k+1)}} \quad (i = 1,\dots,n). \quad (12)$$

So, $(\forall k \ge 0)$, by (4) and (12),

$$w\left(X_i^{(k)}\right) \le \left(\frac{\beta}{\alpha}\right) h^{6^{(k)}} \quad (i = 1, \dots, n), \quad \alpha > 0. \quad (13)$$

TABLE 1. Number of Iterations and CPU Times

Polynomial	Degree n	ISS2		ISS2-5D	
		No. of iterations	CPU time	No. of iterations	CPU time
1	5	2	0.086719	1	0.040234
2	9	2	0.124219	2	0.123438
3	9	1	0.089844	1	0.084766
4	6	2	0.099609	1	0.055859
5	10	2	0.134375	2	0.130078

Let

$$w^{(k)} = \max_{1 \le i \le n} \left\{ w\left(X_i^{(k)} \right) \right\}. \tag{14}$$

Then by (13) and (14),

$$w^{(k)} \le \left(\frac{\beta}{\alpha} \right) h^{6^k} \quad \left(\forall k \ge 0 \right).$$

So, by the definition of R-factor in Monsi et al. (2012), we have

$$R\left(w^{(k)}\right) = \lim_{k\to\infty} \sup \left\{ \left(w^{(k)} \right)^{1/(6^k)} \right\} = \lim_{k\to\infty} \left\{ \left(\frac{\beta}{\alpha} \right)^{1/(6^k)} h \right\} = h < 1.$$

Therefore, it is proven (as defined in Aitken (1950), Gargantini and Henrici (1972) and Monsi and Wolf (1988) that the order of convergence of ISS2-5D is at least 6 or $O_R(ISS2-5D, x_i^*) \ge 6$, ($i = 1, \dots, n$).

DISCUSSION AND NUMERICAL RESULTS

We used the Intlab V5.5 toolbox by Rump (1999) for MATLAB R2007 to get the following results below as computed by Jamaludin et al. (2013a). The algorithms ISS2 and ISS2-5D are run on five test polynomials where the stopping criterion used is $w^{(k)} \le 10^{10}$. Test Polynomial 1 was from Alefeld and Herzberger (1983), Test Polynomial 2 was from Salim et al. (2011), Test Polynomial 3 was from Monsi and Wolfe (1988), Test Polynomial 4 and Test Polynomial 5 were from Monsi and Wolfe (1988).

Table 1 as computed by Jamaludin et al. (2013a) shows that the procedure ISS2-5D required less CPU times than the procedure ISS2 for all five test polynomials, and required less number of iterations meaning ISS2-5D converges faster than ISS2. However, for test polynomials 2, 3 and 5, the number of iterations for both procedures is the same, but the time consumed for procedure ISS2-5D is still less than the ISS2 procedure.

CONCLUSION

The above results have shown analytically in Section 3 that ISS2-5D has faster rate of convergence of at least 6, whereas the R-order of convergence of ISS2 Salim et al. (2011) is at least 5. Thus, we have this relationship $Q_R(ISS2-5D, x^*) > O_R\ (ISS2,\ x^*)$. The attractive features of our procedure mentioned in Section 2 contribute to less CPU times and number of iterations.

ACKNOWLEDGEMENTS

We are indebted to Universiti Kebangsaan Malaysia for funding this research under the grant PMT-1.

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Received: 24 June 2013
Accepted: 14 October 2013