# On the Optimal Stochastic Control of Dividend and Penalty Payments in an Insurance Company

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## Abstract

In this thesis we consider the surplus of a non-life insurance company and assume that it follows either the classical Cramér–Lundberg model or its diffusion approximation. That is, we consider a continuous time model, where premiums are cashed at a constant rate and claims occur randomly with random sizes modelled by a compound Poisson process.

In actuarial mathematics the risk of an insurance company is traditionally measured by the probability of ruin, where the time of ruin is defined as the first time when the surplus becomes negative. Using the ruin probability as a risk measure has been criticised because the time value of money is neglected and it is unrealistic to assume that an insurance company is ruined as soon as the surplus becomes negative. As an extension one can consider the probability of bankruptcy, where negative surplus is allowed and bankruptcy is the event of going out of business. In this approach, the insurance company goes bankrupt randomly for negative surplus levels at some bankruptcy rate. Another measure considers the expected discounted dividend payments which are paid to the shareholders until ruin. In this thesis, we use a similar measure, but as distinguished from classical models, we assume that the insurer is not ruined although the surplus becomes negative and that bankruptcy does not occur. In order to avoid bankruptcy, penalty payments occur, depending on the level of the surplus. For example, penalty payments occur if the insurance company needs to borrow money. As a risk measure we consider the difference between the expected discounted dividend and penalty payments.

In the first part of this thesis we consider the diffusion approximation to the Cramér–Lundberg model and we aim to determine a dividend strategy

that maximises the difference between the expected discounted dividend and penalty payments, where penalty payments are either modelled by an exponential, linear or quadratic function. We show that the optimal strategy is a barrier strategy and calculate the optimal barrier. Using this strategy, all surplus above the barrier is paid as dividends and whenever the surplus is below the barrier, no dividends are paid.

The second part studies the analogous problem where the surplus process of an insurance company is given by a Cramér–Lundberg model. We show that the optimal strategy is also a barrier strategy and consider exponentially distributed claim sizes with exponential, linear and quadratic penalty functions as examples.

In conclusion, we consider the problem where we have to determine an optimal investment and reinsurance strategy and the surplus follows the diffusion approximation to the Cramér–Lundberg model. The insurance company can invest in several risky assets and reduce the insurance risk either by excess of loss or proportional reinsurance. The aim is to find a strategy which minimises the penalty payments that are necessary to avoid bankruptcy. Various penalty functions are considered and closed form solutions are derived.

# Zusammenfassung

In dieser Dissertation betrachten wir den Überschuss eines Sachversicherungsunternehmens, der entweder durch das klassische Cramér–Lundberg–Modell modelliert ist oder der Diffusionsapproximation zu diesem Modell. Wir betrachten demzufolge ein Modell in stetiger Zeit, in dem die Prämienzahlungen durch eine konstante Rate gegeben sind und Schäden zufällig auftreten. Dabei werden die Schadenhöhen durch einen zusammengesetzen Poisson Prozess modelliert.

Das Risiko eines Versicherungsunternehmens wird in der Versicherungsmathematik in der Regel durch die Ruinwahrscheinlichkeit gemessen, wobei der Zeitpunkt des Ruins als der erste Zeitpunkt definiert ist an dem der Überschuss negativ wird. Die Verwendung der Ruinwahrscheinlichkeit als Risikomaß wird kritisiert, da der Zeitwert des Geldes vernachlässigt wird und es nicht realistisch ist anzunehmen, dass ein Versicherungsunternehmen ruiniert ist, sobald der Überschuss negativ wird. Als eine Erweiterung kann auch die Wahrscheinlichkeit des Bankrotts betrachtet werden, wobei negativer Überschuss zulässig ist und Bankrott das Ereignis bezeichnet, dass der Geschäftsbetrieb eingestellt wird. Bei diesem Ansatz tritt Bankrott zufällig ein, sobald der Überschuss negativ wird. Ein weiteres Maß betrachtet die erwarteten diskontierten Dividendenzahlungen, welche bis zum Ruin an die Aktionäre gezahlt werden. In dieser Arbeit verwenden wir ein ähnliches Maß. Abweichend von klassischen Modellen, nehmen wir jedoch an, dass das Versicherungsunternehmen nicht ruiniert ist, wenn der Überschuss negativ wird und dass Bankrott nicht eintritt. Um den Bankrott zu verhindern muss das Versicherungsunternehmen jedoch Strafzahlungen leisten, deren Höhe vom Niveau des Überschusses abhängt.

Strafzahlungen entstehen beispielsweise durch die Aufnahme von Fremdkapital. Als Risikomaß betrachten wir nun die Differenz zwischen den erwarteten diskontierten Dividenden- und Strafzahlungen.

Im ersten Teil dieser Arbeit betrachten wir die Diffusionsapproximation des Cramér–Lundberg–Modells und zielen darauf ab eine Dividendenstrategie zu bestimmen, die die Differenz zwischen den erwarteten diskontierten Dividenden- und Strafzahlungen maximiert, wobei Strafzahlungen entweder durch eine exponentielle, lineare oder quadratische Funktion modelliert werden. Wir zeigen, dass die optimale Dividendenstrategie eine Barrierenstrategie ist und bestimmen die optimale Barriere. Unter Anwendung dieser Strategie wird der Anteil des Überschusses, der die Barriere überschreitet als Dividende ausgezahlt. Sobald der Überschuss sich unterhalb der Barriere befindet, erfolgen keine weiteren Dividendenzahlungen.

Im zweiten Teil betrachten wir das analoge Problem, wobei der Überschuss hier durch das Cramér-Lundberg–Modell beschrieben ist. Wir zeigen, dass die optimale Strategie ebenfalls eine Barrierenstrategie ist und betrachten exponentialverteilte Schadenshöhen und exponentielle, lineare oder quadratische Strafzahlungen als Beispiele.

Abschließend betrachten wir ein Problem, in dem eine optimale Kapitalanlage- und Rückversicherungsstrategie zu bestimmen ist und der Überschuss durch die Diffusionsapproximation des Cramér–Lundberg–Modells gegeben ist. Das Versicherungsunternehmen hat die Möglichkeit in mehrere korrelierte Aktien zu investieren und entweder XL-Rückversicherung oder proportionale Rückversicherung zu kaufen. Das Ziel ist es eine Strategie zu ermitteln, die die erwarteten diskontierten Strafzahlungen, welche notwendig sind um Bankrott zu vermeiden, minimiert. Es werden unterschiedliche Strafkostenfunktionen betrachtet und geschlossene Lösungen bestimmt.

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# Preface

Considering an insurance company, many think at first of traffic accidents, cost of medical treatments, damage to property, etc. Indeed, we are reassured about being protected by an insurance company if such events occur. Paying a periodic premium, the insurance company promises to cover an uncertain loss. The insurance policy determines the amount of premium and for which claims the insurance company is committed to pay. Thus, the individual relies on insurance companies in order to hegde against unpredictable events. Concluding an insurance policy is somehow a form of risk management of the individual.

However, the risk management of an insurance company is far more important for the purpose of staying solvent. For example, persons who are not covered by health insurance may face unbearable costs of medical treatments in case of illness but this does not directly affect others. On the other hand, assume that the insurer becomes insolvent because of poor risk management. Then, not only one person is concerned but rather all policyholders are left without coverage. This also deteriorates the economic situation. The larger the insurance company, the larger is the effect on the economic situation. In particular during the financial crisis 2007-2008, it became evident which dramatic effects occur if a major financial player becomes insolvent. Besides several banks, for example Bradford & Bingley, Dexia, Lehman Brothers and

Hypo Real Estate, the insurance company AIG was also concerned of the financial crisis and the solvency was only ensured by interventions of the regulator. Therefore, the solvency of an insurance company is commonly of great interest and of course there are regulatory requirements for insurance companies, for instance, the regulations of Solvency II. Nevertheless, since the regulations of Solvency II primarily concern the capital adequacy requirements for insurance companies, it is necessary to apply additional measures to reduce the risk of an insurance company.

Over the last years, the theory of optimal stochastic control has become more popular in actuarial mathematics, especially to put the risk management onto a sound theoretical foundation. In concrete, the surplus of the insurance company is described by a process $S = S_t$. The first time when $S$ becomes negative defines the time of ruin. Thus, $S$ reflects the solvency of the insurance company. The insurer's strategy to reduce the risk is modelled by a control strategy $U = U_t$ that influences $S$. While $S$ is most often defined as a Cramer–Lundberg model or its diffusion approximation, various control strategies (for example, $U$ can describe a reinsurance or an investement strategy) have been proposed. The problem is to determine a strategy $U_t^*$ maximising (or minimising) a specified gain functional. This is generally achieved through solving the so-called Hamilton–Jacobi–Bellman (HJB) equation, giving the value function of the optimal control process. The most important control problems in actuarial mathematics are listed in the following. Traditionally, the risk of an insurance company is measured by the probability of ruin. For optimal decisions, the probability of ruin is minimised – for example by reinsurance and/or investments – in order to increase the solvency of an insurance company. This problem was considered for examle in [11, 35, 60, 61, 62], where further references can be found. The disadvantage of the ruin probability approach is that the time value of money is neglected and it is unrealistic to suppose that the surplus tends to infinity. A second approach distributes dividends to the shareholders. Here, the goal is to maximise the expected discounted dividends until ruin. The formulation of the dividend problem in a discrete time framework goes back to de Finetti [20]. Thereafter, Gerber [31] considered the problem in the Cramér–Lundberg model. In a more recent paper, Gerber and Shiu [33] analysed the dividend approach in a diffusion model. Li [46] considered the distribution of the dividend payments in the Cramér–Lundberg model perturbed by a Wiener process. Mishura and Schmidli [52] studied dividend strategies in a renewal risk model with generalized Erlang interarrival times. Moreover, dividend problems were considered in a Markov-modulated risk model (cf. [47, 48, 69]). In many models it was shown that the optimal dividend strategy is a barrier strategy. Here, all surplus above a specified barrier $b \geq 0$ is paid as dividend and whenever the surplus is below the barrier there are no dividend payments.

Asmussen et al. [7] also considered the dividend problem in a diffusion framework, where the insurer can buy excess of loss (XL) and proportional reinsurance. They showed that the optimal dividend strategy is a barrier strategy and that excess of loss reinsurance is always better than the proportional one. An overview of optimisiation techniques in the context of dividend payments and reinsurance, where the surplus is given by a diffusion process, can be found in [66]. Højgaard and Taksar [39] additionally assumed that the insurer may invest in a risk free and a risky asset. Here, an optimal strategy exists only if the discounting factor is larger than the yield of the stock and the risk free interest rate. If this is fulfilled, the optimal dividend strategy is also a barrier strategy and the optimal investment and reinsurance strategies depend on the market price of risk. Azcue and Muler [9, 10] considered the dividend problem where the surplus process evolves as a Cramér–Lundberg process. They showed that the optimal value function is the smallest viscosity solution to the associated HJB equation. Avanzi [8] gave an overview on the actuarial research that followed de Finetti's original paper. The disadvantage of the dividend approach is that, under the optimal strategy, generally ruin occurs almost surely. Therefore, the idea of capital injections rises.

In an approach with capital injections the shareholders should have the opportunity to inject capital whenever the surplus becomes negative in order to avoid ruin. Eisenberg and Schmidli [24, 25, 26, 27] considered an approach where the expected discounted capital injections are minimised. As proposed in [21], Kulenko and Schmidli [43] combined the approach of dividends and capital injections. They showed that the optimal strategy exists and is of barrier type. In a diffusion model an analogous problem was considered by Shreve et al. [65]. They also showed that the optimal strategy - if it exists - is a barrier strategy.

In [36, 37] the discounted average of the future surplus of an insurance company, which can buy cheap and non-cheap reinsurance, is optimised for diffusion models. Taksar and Hunderup [67] extended this approach by a penalty term for bankruptcy. A similar approach maximises the expected utility of terminal wealth. For example, this was considered in [12, 70, 72]. An overview on the application of optimal stochastic control in actuarial mathematics can be found in [11, 34, 62].

All of the approaches above have one thing in common: If the surplus becomes negative, the insurer either has to inject capital or ruin occurs. However, in practice, it can be observed that some companies continue doing business although they had large losses for a long period. Often, the regulator intervenes in order to avoid that a company goes out of business. As already mentioned, several banks and insurance companies were rescued by the regulator during the financial crisis of 2007–2008. Therefore, Albrecher et al. [4] introduced a more general bankruptcy concept by distinguishing between bankruptcy and ruin. They still define ruin as the event of the surplus becoming negative and bankruptcy as the event of going out of business. Unlike the above approaches, they assume that the insurance company can continue doing business until bankruptcy, where the probability of bankruptcy is a function of the level of negative surplus. In this framework they assume that the surplus of an insurance company follows a Brownian motion and they consider the expectation of discounted dividends until bankruptcy. Albrecher and Lautscham [5] studied the probability of bankruptcy in the Cramér–Lundberg model. Another possibility to allow negative suplusses is to observe the surplus only at discrete observation times. Such a model has been studied in [1], [2] and [3]. Nevertheless, in practice bankruptcy does not occur randomly but rather depends on the capital resources. Moreover, it is often very hard to obtain explicit solutions in an approach with a bankruptcy function.

In this thesis, we assume that bankruptcy does not occur, but whenever the surplus is negative, additional costs arise. Therefore, we introduce penalty payments. These payments reflect all costs which are necessary to prevent bankruptcy. For example, penalty payments can occur if the insurer needs to borrow money, generate additional equity or if additional administrative measures have to be taken (like reporting to the authorities). These costs may also be extended to positive surplus to penalise small surplus. Interest payments for negative surplus were also considered by Gerber [32], Embrechts and Schmidli [28] and Schmidli [59]. Note that in our modelling, the penalty payments are neither subtracted from the surplus nor be paid directly by the shareholders. The penalty payments are rather technical in order to avoid that the surplus becomes small or even negative. For a surplus level of $x$, we model the penalty payments to apply at rate $\phi(x)$, where $\phi$ is an appropriate penalty function. In particular, $\phi$ should be positive and decreasing because we assume that interest or other penalty payments are always positive and that the penalty payments increase whenever the economic situation is getting worse. In this framework we consider two stochastic control problems:

In the first problem, dividends may be paid. The value of the controlled surplus process $\{S_t^D\}$ with accumulated dividends $\{D_t\}$ is then

$$\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(S_t^D)\ \mathrm{d}t\Big|S_0^D = x]\ ,$$

where $\delta$ is a preference parameter. Dividends today are preferred to dividends tomorrow, and costs tomorrow are preferred to costs today. Thus, we assume that $\delta > 0$. Our goal will be to maximise the expected value above by choosing an optimal dividend policy.

The second problem aims to minimise the expected discounted penalty payments by investments and reinsurance, where the insurer can invest in $n$ risky assets and reduce the insurance risk either by excess of loss or proportional reinsurance. Let $R_t$ a reinsurance strategy, where $R_t$ describes the retention level at time $t$ and $\theta_t = (\theta_t^1, \theta_t^2, ..., \theta_t^n)^T$ an investment strategy, where $\theta_t^i$ describes the amount being invested into the $i$th asset at time t. Then, we aim to minimise the value

$$\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\,\phi(S_t^{(R,\theta)})\ \mathrm{d}t\ \Big|\ S_0^{(R,\theta)} = x\Big]\ ,$$

by choosing an optimal investment and reinsurance strategy.

# General Notation

| | |
|---|---|
| $\mathbb{N}$ | The natural numbers |
| $\mathbb{R}$ | The real numbers |
| $x_1 \wedge x_2$ | $\min(x_1, x_2)$ for $x_1, x_2 \in \mathbb{R}$ |
| $\mathbb{P}$ | Probability measure |
| $\Omega$ | Set of all possible outcomes |
| $\mathbb{F}$ | Set of all possible events |
| $\mathcal{F} = \{F_t\}_{t\geq 0}$ | Filtration of $\sigma$-algebras |
| $(\Omega, \mathbb{F}, \mathbb{P})$ | Probability space |
| $\mathbb{E}$ | Expected value |
| $S = \{S_t\}_{t\geq 0}$ | Surplus process of an insurance company |
| $L = \{L_t\}_{t\geq 0}$ | Cramér–Lundberg process |
| $x$ | Initial capital |
| $c$ | Premium rate |
| $N = \{N_t\}_{t\geq 0}$ | Poisson process describing the amount of claims |
| $\lambda$ | Intensity of the Poisson process |
| $\{Y_i\}_{i=1,2,\ldots}$ | Sequence of iid random variables modelling the claim sizes |
| $Y$ | Generic random variable with the same distribution as $Y_i$ |
| $F$ | Distribution function of the claim sizes |

| | |
|---|---|
| $m_1$ | Expected value of the random variable $Y_i$ |
| $m_2$ | Second moment of the random variable $Y_i$ |
| $0 = T_0 < T_1 < T_2 < ...$ | Sequence of iid random variables modelling the the claim times |
| $\eta$ | Safety loading of the insurer |
| $X = \{X_t\}_{t \geq 0}$ | Diffusion approximation to the Cramér–Lundberg model |
| $\mu, \sigma$ | Drift and diffusion parameter in the diffusion approximation |
| $W = \{W_t\}_{t \geq 0}$ | Wiener process (standard Brownian motion) |
| $D = \{D_t\}_{t \geq 0}$ | Dividend strategy |
| $b \in \mathbb{R}$ | Barrier of a dividend strategy |
| $S^D = \{S_t^D\}_{t \geq 0}$ | Surplus of an insurance company controlled by a dividend strategy $D$ |
| $Z = \{Z_t\}_{t \geq 0}$ | Stock price evolution in the Black–Scholes model |
| $a_1 > 0$ | Stock return in the Black–Scholes model |
| $v_1 > 0$ | Stock volatility in the Black-Scholes model |
| $Z^i = \{Z_t^i\}_{t \geq 0}$ | Stock price evolution of the $i$-th stock in an extension of the Black–Scholes model |
| $a \in \mathbb{R}^n$ | Stock return vector, where $a = (a_1, a_2, ..., a_n)$ |
| $v \in \mathbb{R}^{n \times n}$ | Volatility matrix of the stock prices, where $v = (v_{ij})_{i,j=1,2,...,n}$ |
| $\Sigma \in \mathbb{R}^{n \times n}$ | Covariance matrix of the stock prices |
| $B^i$, $j = 1, 2, ..., n$ | Independent Wiener processes |
| $\theta = \{\theta_t\}_{t \geq 0}$ | Investment strategy, where $\theta = (\theta^1, \theta^2, ..., \theta^n)$ |
| $S^\theta = \{S_t^\theta\}_{t \geq 0}$ | Surplus of an insurance company controlled by an investment strategy $\theta$ |
| $R = \{R_t\}_{t \geq 0}$ | Reinsurance strategy |

| | |
|---|---|
| $\rho$ | Safety loading of the reinsurer |
| $r$ | Retention level of a reinsurance policy |
| $s(r, Y)$ | Self-insurance function for a rentention level of $r$ and some insurance risk $Y$ |
| $S^R = \{S_t^R\}_{t \geq 0}$ | Surplus of an insurance company controlled by a reinsurance strategy $R$ |
| $S^U = \{S_t^U\}_{t \geq 0}$ | Surplus of an insurance company controlled by a reinsurance and investment strategy $U = (R, \theta^T)^T$ |
| $\tau_r$ | Time of ruin |
| $\omega(x)$ | Bankruptcy function |
| $\tau$ | Time of bankruptcy |
| $\phi(x)$ | Penalty function |
| $V(x)$ | (Optimal) Value function |
| $V^D(x), V^U(x)$ | Value of the strategy $D$ and $U$, respectively |
| $\delta > 0$ | Preference parameter |
| $\mathcal{D}$,$\mathcal{U}$ | Set of all admissible dividend / reinsurance and investment strategies |

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# Chapter 1

# Preliminaries

We start with an introduction to the most important models and formulate the general settings in this thesis. Throughout this thesis all stochastic objects are defined on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Moreover, $\{\mathcal{F}_t\}$ describes a complete filtration.

In actuarial mathematics the surplus of an insurance company is classically represented by a stochastic process and the insurer has the possibility to control the surplus by a number of variables. In the following chapter we assume that the uncontrolled surplus process $S = \{S_t\}_{t\geq 0}$ of an insurance company is described either by the Cramér–Lundberg model or by a diffusion approximation, i.e., we consider a continuous time framework. We start with a rough introduction to these models.

## 1.1 The Cramér–Lundberg Model and Premium Principles

A common model to describe the surplus of an insurance company is the Cramér–Lundberg model (classical risk model or compound Poisson risk model), that goes back to Cramér [18] and Lundberg [49].

Starting with an initial capital $x$ and considering a constant premium rate $c > 0$, the surplus process in the Cramér–Lundberg model is given by

$$L_t = x + ct - \sum_{i=1}^{N_t} Y_i \ , \tag{1.1}$$

where $N = \{N_t\}_{t \geq 0}$ is a Poisson process with intensity $\lambda$ and $\{Y_i\}_{i=1,2,\ldots}$ a sequence of positive, independent and identically distributed random variables with mean $m_1$, second moment $m_2$ and distribution $F$. Moreover, $\{Y_i\}_{i=1,2,\ldots}$ are independent of $N$. The number of claims arriving until time $t$ and the claim size of the $i$–th claim are denoted by $N_t$ and $Y_i$, respectively. Claims occur at random times $0 = T_0 < T_1 < T_2 < \ldots$, where the interarrival times $T_i - T_{i-1}$ are independent and exponentially distributed with mean $1/\lambda$. Furthermore, because $N$ and $\{Y_i\}_{i=1,2,\ldots}$ are independent, we have

$$\mathbb{E}[L_t - x] = ct - \mathbb{E}\Big[\mathbb{E}\Big[\sum_{i=1}^{N_t} Y_i \ \Big| \ N_t\Big]\Big] = ct - \mathbb{E}[N_t] m_1 = t(c - \lambda m_1) \ .$$

Therefore, we assume that the so-called net profit condition

$$c > \lambda m_1 \tag{1.2}$$

holds.

There are numerous premium calculation principles, most importantly is the net value principle. Here, the premium for a single claim $Y$ is calculated by

$$p = (1 + \eta)\mathbb{E}[Y] \ ,$$

where $\eta > 0$ denotes the safety loading of the insurer. In order to have a higher sensibility against large insurance risks the variance principle and the standard deviation principle are commonly used. Here, we have

$$p = \mathbb{E}[Y] + \kappa \operatorname{Var}[Y]$$

and

$$p = \mathbb{E}[Y] + \kappa\sqrt{\mathrm{Var}[Y]} \ ,$$

respectively, for some $\kappa > 0$. The variance principle is criticised because a change of the monetary unit also causes a change of the security loading. This problem is fixed by the modified variance principle, where

$$p = \mathbb{E}[Y] + \kappa \frac{\mathrm{Var}[Y]}{\mathbb{E}[Y]} \ .$$

An extension to the net value principle ist the adjusted risk principle with

$$p = \int_0^\infty (1 - F(x))^\kappa \, \mathrm{d}x \ ,$$

where $F$ denotes the distriubtion function of $Y$ and $\kappa \in (0,1)$. The net value principle is obtained as the special case $\kappa = 1$. If the insurance company aims to weight high losses stronger than small losses, they may also apply the principle of zero utility. Here, the premium $p$ is the unique solution to the equation

$$v(w) = \mathbb{E}[v(w + p - Y)] \ ,$$

where $w$ denotes the initial wealth of the insurer and $v$ is a strictly increasing and concave function with $v(0) = 0$. A well known special case is $v(y) = -\,\mathrm{e}^{-\kappa y}$, where $\kappa > 0$, i.e. the exponential premium principle.

We consider the net value principle, because the premium is easy to calculate. In case of the Cramér–Lundberg model we have

$$c = (1 + \eta)\lambda m_1 \ .$$

## 1.2 A Diffusion Approximation to the Cramér–Lundberg Model

According to Schmidli [62] it is often difficult to calculate characteristics in the Cramér–Lundberg model. Therefore one tries to find an appropriate approximation of the Cramér–Lundberg model. Let $X^n$ be a sequence of Cramér–Lundberg models. We say that $X^n$ converges weakly to a stochastic process $X$ if

$$\lim_{n\to\infty} \mathbb{E}(\psi(X^n)) = \mathbb{E}(\psi(X))$$

for every bounded continuous functional $\psi$. Our goal is to find a diffusion process

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \ , \tag{1.3}$$

where $W$ denotes a standard Wiener process, such that $X^n$ converges weakly to $X$ from equation (1.3). In [62] it is also mentioned that in case of a diffusion approximation the limiting process should be a diffusion process with stationary and independent increments, e.g.

$$X_t = x + \mu t + \sigma W_t \ . \tag{1.4}$$

Let

$$X_t^n = x_n + c_n t - \sum_{i=1}^{N_t^n} Y_i^n$$

where $N^n$ defines a sequence of Poisson processes with intensity $\lambda_n = n\lambda$, $Y_i^n = Y_i/\sqrt{n}$, $x_n = x$ and $c_n = c + \lambda m_1(\sqrt{n} - 1)$. Then, $X^n$ is a sequence of Cramér–Lundberg models and $X^1$ describes the process in (1.1). Schmidli shows in [58] that $X^n$ converges weakly to $X$, where $\mu = c - \lambda m_1$ and $\sigma = \sqrt{\lambda m_2}$. Considering the sequence $X^n$, the number of claims increases and the claim sizes decrease if $n$ increases. Therefore, the approximation is only meaningful for large portfolios.

## 1.3 Dividend Payments

In the first part of this thesis we assume that the insurer has the possibility to pay dividends to the shareholders. In this section we introduce the idea of measuring the risk of an insurance company by dividend payments and we present the most popular dividend strategies in the literature.

As already mentionend, the risk of an insurance company is classically measured by the probability of ruin. Let $L_t$ define the surplus process of an insurance company, where $L_t$ is defined as in equation (1.1). Then, the time of ruin $\tau_r$ is defined as the first time when the surplus becomes negative, i.e.

$$\tau_r = \inf\{t \geq 0 \colon L_t < 0\} \ .$$

Using the ruin probability as a risk measure, one point of criticism was that the surplus generally tends to infinity under this approach. A possibility to prevent that the surplus tends to infinity is to distribute some of the surplus to the shareholders as dividends. Then - as proposed by de Finetti [20] - the risk of an insurance company can be measured by the expected discounted dividend payments which are paid to the shareholders until ruin. A dividend strategy determines when and which amount should be paid to the shareholders. We model a dividend strategy by a stochastic process $D_t$, where $D_t$ denotes the accumulated dividend payments up to time $t$. The controlled surplus process now is given by

$$L_t^D = L_t - D_t \ .$$

We call a divdend strategy $D$ a band strategy if the state space of the surplus process is separated into three sets $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and dividends are distributed as follows: If $x \in \mathfrak{A}$, the incoming premium is paid as dividend until the next claim arrives. If $x \in \mathfrak{B}$, a dividend is paid such that the process is immediately brought back to the first set. If $x \in \mathfrak{C} = (\mathfrak{A} \cup \mathfrak{B})^{\mathrm{C}}$, there is no dividend payment.

A barrier strategy $D$ is a special type of band strategy that is characterised by a barrier $b$. Whenever the surplus is below $b$, there is no dividend payment. As soon as the surplus exceeds $b$, the difference between the surplus and $b$ is paid as dividend. Thus, $\mathfrak{A} = \{b\}$, $\mathfrak{B} = (b, \infty)$ and $\mathfrak{C} = [0, b)$. Moreover, we have $D_0 = (x - b)^+$ and

$$\mathrm{d}D_t = \int_0^t c\mathbb{1}_{\{L_s^D = b\}} \mathrm{d}s$$

for $t > 0$. This means that a barrier strategy separates the state space into two intervals ("bands") $[0, b)$ and $(b, \infty)$. Figure 1.1 illustrates a sample path of a surplus process controlled by a barrier strategy in the Cramér–Lundberg model.

Now, we assume that the surplus follows the model in (1.4). Then, we call $D$ a barrier strategy with a barrier $b$ if $D_t = (M_t - b)^+$, where

$$M_t = \sup_{0 \le s \le t} X_t \; .$$

Figure 1.2 illustrates a sample path of a surplus process controlled by a barrier strategy in a diffusion approximation.

For an overview on dividend strategies see [8].

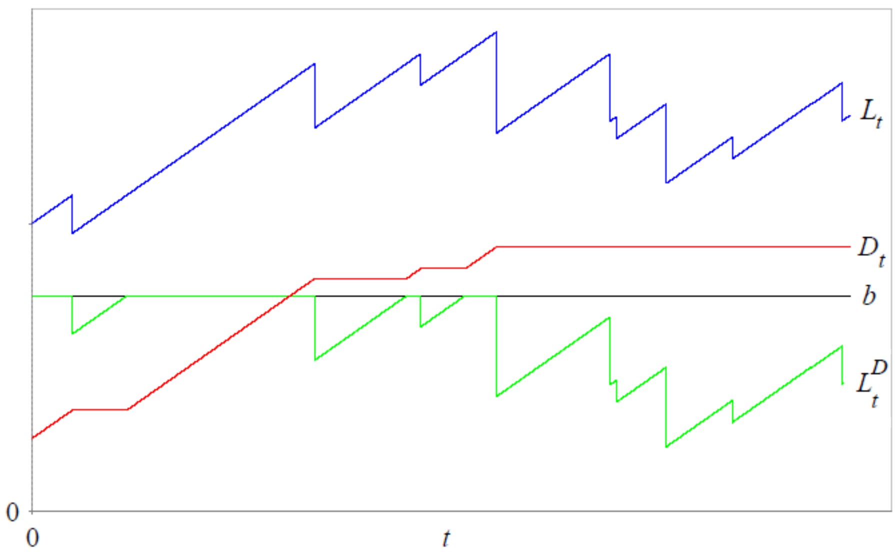


Figure 1.1: Sample path of a surplus process controlled by a barrier strategy in the Cramér–Lundberg model.

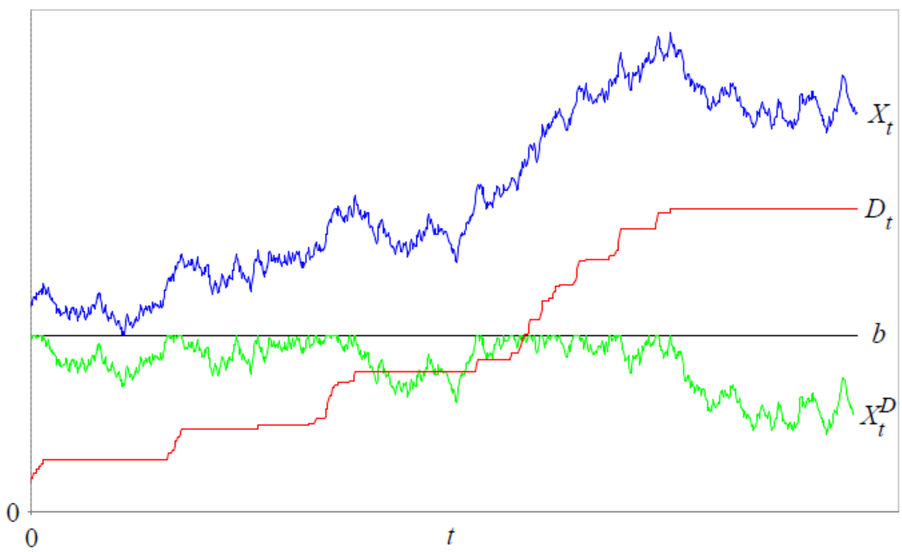


Figure 1.2: Sample path of a surplus process controlled by a barrier strategy in a diffusion approximation.

## 1.4 Investments and Reinsurance

In chapter 4, the insurer has the possibility to invest part of the surplus into risky assets and to buy reinsurance. Therefore, we now give an overview of some common investment and reinsurance models, in particular of those being applied in this thesis.

The most famous model in financial mathematics is the Black–Scholes model which goes back to Black, Scholes [15] and Merton [51]. The Black–Scholes model assumes that the financial market only consists of one risky asset (stock) and a riskfree asset (bond). The stock is modelled as

$$\mathrm{d}Z_t = a_1 Z_t \,\mathrm{d}t + v_1 Z_t \,\mathrm{d}B_t^1 \ ,$$

where $a_1 > 0$ describes the return of the stock, $v_1 > 0$ the volatility of the stock and $B^1$ denotes a Wiener process. Using Itô's formula one obtains

$$Z_t = Z_0 \exp\Big[v_1 B_t + (a_1 - \tfrac{1}{2}v_1^2)t\Big] \ .$$

The bond (sometimes called cash or money market) is modelled by

$$\mathrm{d}Z_t^m = m Z_t^m \,\mathrm{d}t \ ,$$

where $m > 0$ denotes the riskfree interest rate. Again, Itô's formula yields

$$Z_t^m = Z_0^m \,\mathrm{e}^{mt} \ .$$

In this framework Black, Scholes [15] and Merton [51] derived a closed form formula for evaluating the value of a European option (offers the buyer the right, but not the obligation, to buy (call) or sell (put) a stock or other financial assets at the maturity of the contract).

In this thesis we consider an extension to the Black–Scholes model. We assume that the insurance company has the possibility to invest in $n$ risky assets, modelled by

$$\mathrm{d}Z_t^i = a_i Z_t^i \ \mathrm{d}t + Z_t^i \sum_{j=1}^{n} v_{ij} \ \mathrm{d}B_t^j, \quad S_0^i = 1$$

for $i = 1, 2, ..., n$. Here, $B^1, B^2, ..., B^n$ are independent Wiener processes and $a_i, v_{ij} \geq 0$, $i, j = 1, 2, ..., n$. The insurer can choose an investment strategy $\theta_t = (\theta_t^1, \theta_t^2, ..., \theta_t^n)^T$, where $\theta_t^i$ describes the amount being invested into the $i$th asset at time $t$. Considering a strategy $\theta$ the controlled surplus of the insurer is given by

$$\mathrm{d}S_t^\theta = \ \mathrm{d}S_t + \sum_{i=1}^{n} a_i \theta_t^i \ \mathrm{d}t + \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_t^i v_{ij} \ \mathrm{d}B_t^j \ .$$

Buying Reinsurance is another important possibility to control the risk of an insurance company. A so called reinsurance company and the insurer (cedent) agree to share part of the claims incurred by the cedent. In return, the cedent pays a reinsurance premium to the reinsurance company. Generally, it is distinguished between facultative reinsurance, where each claim is reinsured separately and treaty reinsurance, where the cedent and reinsurer negotiate to share a part of all insurance policies which are specified in the contract. In the following we only consider facultative reinsurance. In order to model a reinsurance policy we introduce the so-called self-insurance function $0 \leq s(r, Y) \leq Y$ for a retention level $r$, where $s(r, Y)$ denotes the part of a claim $Y$ which is still covered by the insurer. The most common types of reinsurance are proportional reinsurance and excess of loss reinsurance. In case of proportional reinsurance the reinsurer covers a stated ratio of the claim. Thus, we have

$$s(r, Y) = rY \ ,$$

where $0 \leq r \leq 1$. Applying excess of loss reinsurance, the reinsurer only covers the part exceeding a specified amount and therefore

$$s(r, Y) = \min(r, Y) \ ,$$

where $0 \leq r \leq \infty$. Another possibility to buy reinsurance is proportional reinsurance in a layer. Here, we have a multidimensional retention level $(r_1, r_2, r_3)$ and

$$s((r_1, r_2, r_3), Y) = \min(r_1, Y) + (Y - r_1 - r_3)^+ + r_2 \min(r_3, (Y - r_1)^+) \ .$$

In this thesis we only consider proportional reinsurance and excess of loss reinsurance. Let $\rho$ denote the safety loading of the reinsurer. Considering a single claim and a retention level of $r$, the premium rate remaining for the insurer is given by

$$(1 + \eta)m_1 - (1 + \rho)\mathbb{E}[Y - s(r, Y)] = (1 + \rho)\mathbb{E}[s(r, Y)] - (\rho - \eta)m_1 \ .$$

The insurer can choose the retention level at any time $t$. Thus, a reinsurance strategy is an adapted process $0 \leq R_t \leq \infty$. Then, under a reinsurance strategy $R$ the surplus in the Cramér–Lundberg model is given by

$$L_t^R = \lambda(1 + \rho) \int_0^t \mathbb{E}[s(R_s, Y)] \,\mathrm{d}s - \lambda(\rho - \eta)m_1 t - \sum_{i=1}^{N_t} s(R_{T_i}, Y_i) \ .$$

## 1.5 From Ruin to Bankruptcy

At the beginning of the twenty–first century optimisation problems in acturial mathematics have extensively been studied. Mostly, the surplus process $S_t$ of an insurance company has been considered until ruin occurs. For example one has tried to maximise the expected discounted dividends payments

$$\mathbb{E}\Big(\int_0^{\tau_r} \mathrm{e}^{-\delta t} \,\mathrm{d}D_t\Big)$$

until ruin, see Section 1.3.

In the preface we pointed out that at the very latest since the beginning of the financial crisis in 2007, it can be observed that some companies, in particular banks, can still do business even though they had large losses. In order to maintain systemic stability, public money was used to bail out banks. Of course, it also could be possible that an insurance company can continue doing business despite the surplus becomes negative. Therefore we have to distinguish between the event of going out of business and the event of negative surplus. This idea was first introduced by Albrecher et al. [4]. We still define ruin as the event of negative surplus. In addition, we define bankruptcy as the event of going out of business.

In order to model the event of bankruptcy Albrecher et al. [4] introduced a bankruptcy rate function $\omega(x)$ with $\omega(x) \geq 0$, $x \leq 0$ and $\omega(x) = 0$, $x > 0$. Whenever the surplus becomes negative, bankruptcy occurs at rate $\omega(x)$. This means that

$$\mathbb{P}(\tau \leq h|\mathcal{F}_h) = 1 - \exp\Big(-\int_0^h \omega(S_t)\mathrm{d}t\Big)$$

and

$$\mathbb{P}(\tau \leq h) = \mathbb{E}\Big[\mathbb{P}(\tau \leq h|\mathcal{F}_h)\Big] = \mathbb{E}\Big[1 - \exp\Big(-\int_0^h \omega(S_t)\mathrm{d}t\Big)\Big]$$

respectively, where $\mathcal{F}_h = \sigma(S_t, 0 \leq t \leq h)$. In particular, the time of bankruptcy is given by

$$\tau = \inf\Big\{h > 0\colon \int_0^h \omega(S_t)\mathrm{d}t > E\Big\}\ ,$$

where $E \sim \mathrm{Exp}(1)$. It is assumed that $\omega$ is decreasing, i.e., the probability of bankruptcy increases if the surplus becomes more negative. If there is a $\tilde{x} < 0$ such that $\omega(x) = \infty$ for $x \leq \tilde{x}$ and $\omega(x) \geq 0$ for $\tilde{x} < x \leq 0$, bankruptcy occurs at the latest when the surplus falls below $\tilde{x}$.

Suppose that at time $h$ we have $S_h = x$ for some negative surplus. Then, the bounded convergence theorem implies

$$
\begin{aligned}
\lim_{s\downarrow 0}\frac{\mathbb{P}(h<\tau\le h+s|\tau>h)}{s} &= \lim_{s\downarrow 0}\frac{1}{s}\mathbb{E}\Big[\frac{1-\mathrm{e}^{-\int_0^{h+s}\omega(S_t)\mathrm{d}t}-\Big(1-\mathrm{e}^{-\int_0^{s}\omega(S_t)\mathrm{d}t}\Big)}{\mathrm{e}^{-\int_0^{s}\omega(L_t)\mathrm{d}t}}\Big] \\
&= \lim_{s\downarrow 0}\frac{1}{s}\mathbb{E}\Big[1-\mathrm{e}^{-\int_s^{s+h}\omega(S_t)\mathrm{d}t}\Big] \\
&= \mathbb{E}(\omega(S_h))=\omega(x)\ .
\end{aligned}
$$

Thus, it is said that bankruptcy occurs at a bankruptcy rate function $\omega$.

Considering a constant bankruptcy function $\omega(x)=\lambda$, the following is obtained: The concept of bankruptcy due to [4] coincides with the framework of randomised observation periods in [1], [2] and [3] if the time lengths between the observations are exponentially distributed with parameter $\lambda$. In [1], [2] and [3] bankruptcy occurs the first time when the surplus is negative at one of the observation times. Between the observation times it could be possible that the surplus becomes negative.

## 1.6 Introduction of Penalty Payments

The main advantage of the bankrupcty concept due to Albrecher et al. [4] is that, in contrast to classical risk models, the insurance company can continue doing business even if the surplus is negative. Despite this positive aspect, it is very hard to obtain explicit solutions and it is assumed that bankruptcy occurs randomly. In practice the solvency of an insurance company rather depends on the capital resources. In particular, an insurance company is able to raise outside funds (e.g. by borrowing money) or to conduct a capital increase if the economic situation deteriorates. Moreover, there are many other measures by the regulator and the European insurance authority (EIOPA) intended to prevent an insurance company from being insolvent. Therefore we assume that bankruptcy does not occur, but the insurance company has to pay penalty payments in order to prevent bankrupcty. These penalty payments include all costs for raising external capital or for conducting a capital increase as well as all administrative costs which can occur because of additional measures by the authorities. For a surplus level of $x$, we assume that the penalty payments occur at a penalty rate $\phi(x)$. If the economic situation of the insurer deteriorates, the penalty payments and the growth of the penalty payments increase. Moreover, the penalty payments are always positive and vanish as the surplus tends to infinity. Thus, $\phi$ should be a decreasing, convex and positive function with $\phi(x) \to 0$, $x \to \infty$. Since the expected discounted penalty payments should be bounded we assume that

$$\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\, \phi(S_t)\, \mathrm{d}t\Big] < \infty \,. \tag{1.5}$$

**Example 1.1.** *In this example we assume that for a negative surplus of $x$ the insurer has to borrow an amount of $-x$ at rate $\alpha$ and that no other penalty payments occur. This means that $\phi(x) = -\alpha x \mathbb{1}_{x>0}$ and the expected discounted penalty payments are given by*

$$-\alpha \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\, S_t^-\, \mathrm{d}t\Big] \,.$$

## 1.7 Formulation of the Problems in this Thesis

After introducing all relevant models, we are now in the position to formulate the stochastic control problems being considered in this thesis. In the previous sections we pointed out, that in classical risk models the insurance company's solvency situation is often not appropriately modelled. Therefore, we consider two optimisation problems which aim to augment classical models by penalty payments.

### 1.7.1 Maximisation of Dividends with Penalty Payments

The first problem which we consider is the dividend problem that was introduced by De Finetti [20]. In the classical framework, the aim is to maximise the expected discounted dividend payments which are distributed to the shareholders until ruin. The accumulated dividend payments are given by an increasing and adapted process $D$. We consider the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the surplus process $S_t$. As extension of the classical model, we assume that neither ruin nor bankruptcy occurs, because penalty payments more appropriately model the solvency situation of an insurance company. For a surplus level of $x$ the penalty payments occur at rate $\phi(x)$ as introduced in the previous section. The controlled surplus process is given by

$$S_t^D = S_t - D_t \; .$$

We allow all increasing càdlàg processes $D$. The value of a strategy $D$ is defined by

$$V^D(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \;\mathrm{d}D_t - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(S_t^D)\;\mathrm{d}t \;\Big|\; S_0^D = x\Big] \; , \tag{1.6}$$

where $\delta > 0$ is a preference parameter. The preference parameter expresses the investment preferences of the company holders. $\delta > 0$ implies that investing tomorrow is preferred to investing today. The set of all adapted strategies is denoted by $\mathcal{D}$ and the (optimal) value function is defined by

$$V(x) = \sup_{D\in\mathcal{D}} V^D(x) \; .$$

We aim to find a strategy $D^*$ such that

$$V^{D^*}(x) = V(x) \; .$$

In order that it is not optimal to pay an infinite amount of dividends, we have to assume that

$$\phi(x) - \phi(y) > \delta(y - x) \tag{1.7}$$

for $x < y < x_0$ and some $x_0 \in \mathbb{R}$.

### 1.7.2 Minimisation of Penalty Payments

The second problem is an investment and reinsurance optimisation problem. The aim is to minimise the expected discounted penalty payments by investments and reinsurance, where the insurer can invest in $n$ risky assets and either buy excess of loss or proportional reinsurance. Penalty payments occur at rate $\phi(x)$ for a surplus level of $x$ and the value of an investment and reinsurance strategy $U = (R, \theta^T)^T$ is given by

$$V^U(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\,\phi(S_t^U)\,\mathrm{d}t \;\Big|\; S_0^U = x\Big]\,. \tag{1.8}$$

As above, $\delta > 0$. Now, we have a control problem of the form

$$V(x) = \inf_{U \in \mathcal{U}} V^U(x)\,,$$

where $\mathcal{U}$ is the set of all admissible strategies.

## 1.8 The Dynamic Programming Approach

In this section we introduce some optimisation techniques which will help us to solve our stochastic control problems. There are many textbooks on stochastic control theory in continuous time, for example see [19, 30, 44, 45, 54, 56, 64, 68, 71]. We refer to Schmidli [62] in the following. Note, that all steps in this section are heuristic and aim to give an idea of the techniques we will use in this thesis.

The key to the solution of a stochastic control problem is the dynamic programming principle which has its origin in a discrete-time framework, see Bellman [13, 14]. For a better understanding we first introduce the approach in discrete time. The idea is to break down the control problem into easier subproblems and then determine the optimal solution recursively. Let us consider a discrete-time control system

$$X_0 = x,\ X_{n+1} = f(X_n, U_n, Y_{n+1})\ ,$$

where $n \in \mathbb{N}$ and $f$ is a measurable function. In this system $X$ describes the state of the system, $Y$ is a stochastic influence and $U$ a control strategy, which should be adapted to the natural Filtration $\mathcal{F}_n = \sigma(Y_1, Y_2, ..., Y_n)$, because we have no future information. We start at a state $x$ and have, at each time step $n$, a stochastic influence $y$ as well as a control variable $u \in \mathcal{U}$, where $\mathcal{U}$ is an arbitrary control space. In a discrete-time stochastic control problem we aim to find a stragey $U$ such that a specified value function is optimised on a finite or infinite time horizon $T$. Often, the value function has the following form

$$V_T^U(x) = \mathbb{E}\Big[\sum_{n=0}^{T} g(X_n, U_n)\,\mathrm{e}^{-\delta n}\Big]\ ,$$

where $\delta > 0$ is a discount factor and $g(X_n, U_n)$ describes the gains or costs (dependent on the current state and control variable) of the system in period $n$. We just consider the case in which we have to maximise the value function because we get the analogous minimisation problem if we maximise $-V_T^U(x)$. The optimal value function is denoted by

$$V_T(x) = \sup_U V_T^U(x)$$

and a control process $U^*$ is optimal if

$$V_T(x) = V_T^{U^*}(x)\ .$$

The idea of Bellman [13, 14] is that the optimal strategy maximises the present gains plus the future gains at each time step. In this way one can recursively determine the optimal strategy. Concretely, the optimal value function should fulfil the so-called Bellman equation

$$V_T(x) = \sup_{u\in\mathcal{U}}\Big\{g(x,u) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}(f(x,u,Y))]\Big\}\,. \tag{1.9}$$

This equation can be proven in two steps. We just give a rough summary of the proof. Firstly, let $U$ be an arbitrary strategy and $\tilde{U}_n = U_{n+1}$. Then,

$$\begin{aligned}
V_T^U(x) &= g(x,U_0) + \mathrm{e}^{-\delta}\,\mathbb{E}\Big[\sum_{n=0}^{T-1} g(X_{n+1},U_{n+1})\,\mathrm{e}^{-\delta n}\Big] \\
&= g(x,U_0) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}^{\tilde{U}}(X_1)] \\
&= g(x,U_0) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}^{\tilde{U}}(f(x,U_0,Y_1))] \\
&\le g(x,U_0) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}(f(x,U_0,Y_1))] \\
&\le \sup_{u\in\mathcal{U}}\Big\{g(x,u) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}(f(x,u,Y))]\Big\}\,.
\end{aligned}$$

On the other hand, let $u\in\mathcal{U}$ arbitrary and $U^\varepsilon$ be a strategy such that conditioned on $X_1 = f(x,u,Y_1)$ it holds

$$V_{T-1}(X_1) < V_{T-1}^{U^\varepsilon}(X_1) + \varepsilon$$

for any $\varepsilon > 0$. Moreover, define the strategy $U_n = U_{n-1}^\varepsilon$ with $U_0 = u$. Then,

$$\begin{aligned}
V_T(x) &\ge V_T^{U^\varepsilon}(x) \\
&= g(x,u) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}^U(X_1)] \\
&> g(x,u) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}(X_1)] - \varepsilon \\
&= g(x,u) + \mathrm{e}^{-\delta}\,\mathbb{E}[V_{T-1}(f(x,u,Y))] - \varepsilon
\end{aligned}$$

As $\varepsilon$ and $u$ are arbitrary, equality holds.

Now let us consider the continuous time framework. Here we have a value function of the form

$$V^U(t,x) = \mathbb{E}\Big[\int_t^T \mathrm{e}^{-\delta(s-t)}\,g(X_s^U,U_s)\,\mathrm{d}s + \mathrm{e}^{-\delta(T-t)}\,g_T(X_T^U)\;\Big|\;X_t = x\Big]\,,$$

where either $T$ is a stopping time or $T = \infty$. For simplicity, we set $t = 0$ in order to avoid partial derivatives with respect to $t$. Let $U$ be an arbitrary strategy on $[0, T \wedge t]$ and $U^\varepsilon$ a strategy from time $T \wedge t$ such that

$$V(X^U_{T\wedge t}) < V^{U^\varepsilon}(X^U_{T\wedge t}) + \varepsilon \; .$$

Similarly as above one can show that

$$V(x) > \mathbb{E}\Big[\int_0^{T\wedge t} \mathrm{e}^{-\delta s}\, g(X^U_s, U_s)\, \mathrm{d}s + \mathrm{e}^{-\delta(T\wedge t)}\, V(X^U_{T\wedge t})\Big] - \varepsilon \; .$$

Since $\varepsilon$ is arbitrary, the weak inequality must hold for $\varepsilon = 0$. Then, taking the supremum over all strategies $U$, we obtain

$$V(x) \geq \sup_U \mathbb{E}\Big[\int_0^{T\wedge t} \mathrm{e}^{-\delta s}\, g(X^U_s, U_s)\, \mathrm{d}s + \mathrm{e}^{-\delta(T\wedge t)}\, V(X^U_{T\wedge t})\Big] \; .$$

On the other hand, considering the strategy $\tilde{U}_s = U_{t+s}$, we also obtain as above

$$V(x) \leq \sup_U \mathbb{E}\Big[\int_0^{T\wedge t} \mathrm{e}^{-\delta s}\, g(X^U_s, U_s)\, \mathrm{d}s + \mathrm{e}^{-\delta(T\wedge t)}\, V(X^U_{T\wedge t})\Big] \; .$$

This implies the following dynamic programming principle

$$V(x) \quad = \quad \sup_U \mathbb{E}\Big[\int_0^{T\wedge t} \mathrm{e}^{-\delta s}\, g(X^U_s, U_s)\, \mathrm{d}s + \mathrm{e}^{-\delta(T\wedge t)}\, V(X^U_{T\wedge t})\Big] \; . \tag{1.10}$$

Rearranging the terms and dividing by $t$ yields

$$\begin{aligned} \sup_U \mathbb{E}\Big[\frac{1}{t}\int_0^{T\wedge t} \mathrm{e}^{-\delta s}\, g(X^U_s, U_s)\, \mathrm{d}s + \mathrm{e}^{-\delta(T\wedge t)}\, \frac{V(X^U_{T\wedge t}) - V(x)}{t} \qquad (1.11) \\ - \frac{1 - \mathrm{e}^{-\delta(T\wedge t)}}{t} V(x)\Big] = 0 \; . \end{aligned}$$

Letting $t \downarrow 0$ and assuming that we can interchange the limit, supremum and integration we obtain the Hamilton–Jacobi–Bellman (HJB) equation

$$\sup_{u\in\mathcal{U}}\Big[g(x,u) + \mathcal{A}_u V(x) - \delta V(x)\Big] = 0 \; , \tag{1.12}$$

where $\mathcal{A}_u$ denotes the infinitesimal generator of the process $X^u$ being controlled by the constant strategy $U_t = u$. Appendix B gives an introduction to the infinitesimal generator of a Markov process. It is also possible to motivate the HJB equation by the use of martingale techniques, see Schmidli [62]. Schmidli also states that the optimal strategy should be of the form $u^*(X_t)$, where $u^*(x)$ maximises the left-hand side of (1.12).

Now, let us consider the case, where $V$ is twice continuously differentiable and $X_t^U$ a diffusion process of the form

$$\mathrm{d}X_t^U = \mu(X_t, U_t)\ \mathrm{d}t + \sigma(X_t, U_t)\ \mathrm{d}W_t\ ,$$

where $\mu, \sigma$ functions such that $X_t$ is a continuous process. Then, Itô's formula implies

$$\begin{aligned} V(X_t^U) &= V(x) + \int_0^t V'(X_s^U)\mu(X_s, U_s)\ \mathrm{d}s + \int_0^t V'(X_s^U)\sigma(X_s, U_s)\ \mathrm{d}W_s \\ &\quad + \frac{1}{2}\int_0^t V''(X_s^U)\sigma^2(X_s, U_s)\ \mathrm{d}s\ . \end{aligned}$$

Assuming that the stochastic integral is a martingale, we obtain

$$\mathcal{A}_u V(x) = \tfrac{1}{2}\sigma^2(x,u)V''(x) + \mu(x,u)V'(x)\ .$$

In this case the HJB equation is just an ordinary differential equation. However, we also consider jump processes in this thesis and $V$ is not always twice continuously differentiable. Moreover, we made further assumptions which do not hold in general. That is why it is not enough just to solve the HJB equation in order to get the solution to a stochastic control problem. Albrecher and Thonhauser [6] state that there are generally two ways to obtain a solution for the optimisation problem based on the HJB equation:

1) It is possible to prove that there exists a unique solution to the HJB equation. Ideally, it is also possible to construct an explicit solution. In this

case a so-called verification theorem is needed that states that the unique solution dominates all other values achieved by admissible strategies. This gives the optimality. We will follow those steps in the case where we model the surplus of an insurance company by a diffusion process.

2) It is possible to show that there exist solutions of the HJB equation, but uniqueness is doubtful. Then a precise characterisation of the value function is needed and it has to be proven that the value function indeed fulfils the HJB equation by verifying that all steps in the derivation of the HJB equation are actually justified. We will follow this procedure in the case where we model the surplus of an insurance company by the Cramér–Lundberg model.

Another common approach, described in [6], is the following:
Maximise a certain value function over a (small) restricted class of admissible strategies. Then, in some cases it is possible to verify by comparison that the – within the restricted class – optimal strategy is also optimal within the bigger class of general admissible strategies.

# Chapter 2

# Maximisation of Dividends with Penalty Payments in a Diffusion Model

## 2.1 Introduction

In this chapter we consider the dividend problem described in Section 1.7.1 and we assume that the surplus of the insurance company follows a diffusion approximation

$$X_t = x + \mu t + \sigma W_t \ , \quad t \geq 0 \ , \tag{2.1}$$

where $x \in \mathbb{R}$ denotes the initial capital, $W_t$ a Wiener process and $\mu, \sigma > 0$. The information is given by the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of the Wiener process. Let $D_t$ be adapted and denote the accumulated dividend payments until time $t$. Then, the controlled surplus process is given by

$$X_t^D = X_t - D_t \ .$$

We allow all increasing càdlàg processes $D$. The value of a strategy $D$ is defined by

$$V^D(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(X_t^D)\ \mathrm{d}t \ \Big|\ X_0^D = x\Big]\ . \tag{2.2}$$

The decreasing function $\phi$ is the penalty function fulfilling $\phi(x) \to 0$ as $x \to \infty$. We further assume that $\phi$ is convex. The set of adapted and increasing strategies is denoted by $\mathcal{D}$ and the (optimal) value function is defined by

$$V(x) = \sup_{D \in \mathcal{D}} V^D(x)\ .$$

We aim to find a strategy $D^*$ such that

$$V^{D^*}(x) = V(x)\ .$$

The penalty payments are bounded by the payments obtained if no dividends are paid. We therefore have to assume

$$\int_0^\infty \mathrm{e}^{-\delta t}\,\mathbb{E}[\phi(X_t)]\ \mathrm{d}t < \infty\ .$$

Otherwise, the value function would be minus infinite. Moreover, we assume that

$$\phi(x) - \phi(y) > \delta(y - x) \tag{2.3}$$

for $x < y < x_0$ and some $x_0 \in \mathbb{R}$ in order that it is not optimal to pay an infinite amount of dividends. Since $\phi$ is assumed to be convex, this means that there is an $x_0$ such that $\phi'(x-) \le \phi'(x+) \le -\delta$ for $x < x_0$, where $\phi'(x+)$ denotes the derivative from the right and $\phi'(x-)$ the derivative from the left.

This chapter is organised as follows. In the second section we characterise the optimal strategy and we motivate the HJB equation. In Section 2.3 we prove the verification theorem. Section 2.4 considers the dividend problem with an exponential penalty function $\phi(x) = \alpha\,\mathrm{e}^{-\beta x}$, where $\alpha, \beta > 0$. Here, the value function exists only if $r_2 < -\beta$, where $r_2$ is the negative solution to the equation $\sigma^2 r^2 + 2\mu r - 2\delta = 0$. If $r_2 \geq -\beta$ no optimal strategy exists. For $r_2 < -\beta$, we show that the optimal strategy is a barrier strategy and determine the optimal barrier. Section 2.5 studies a linear penalty function $\phi(x) = -\alpha x$ for some $\alpha > 0$ if $x < 0$ and $\phi(x) = 0$ if $x \geq 0$. An optimal strategy does only exist if $\delta < \alpha$, where $\delta$ denotes the discounting factor. In this case the optimal strategy is a barrier strategy and the optimal barrier is given by $b^* = 1/r_2 \log(\delta/\alpha)$. If $\delta \geq \alpha$, the preference parameter is larger than the slope of the penalty function and it is optimal to pay an infinite amount of dividends. In the last section of this chapter we consider quadratic penalty payments, described by $\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0}$. An optimal strategy does always exist and is also a barrier strategy, but we have to distinguish between a negative and positive dividend barrier. In both cases we determine the optimal barrier.

## 2.2 Characterisation of the Optimal Strategy and the HJB Equation

It is well-known that the optimal dividend strategy in the model without penalty payments is a barrier strategy. A barrier strategy $D$ is characterised by a barrier $b$, where all surplus above $b$ is paid as dividends and whenever the surplus is below $b$, no dividends are paid. This means that

$$D_t = \max\Big(\sup_{0\leq s\leq t} X_s - b, 0\Big) .$$

We expect that in our problem the optimal strategy is also a barrier strategy. Then, $V(x) = V(b) + x - b$ for $x \geq b$. If $x < b$ let $\tau^b = \inf\{t > 0 : X_t > b\}$. We find

$$V(x) = \mathbb{E}\Big[\mathrm{e}^{-\delta(\tau^b\wedge h)}\, V(X^{D^*}_{\tau^b\wedge h}) - \int_0^{\tau^b\wedge h} \mathrm{e}^{-\delta t}\, \phi(X^{D^*}_t)\, \mathrm{d}t\Big] .$$

Assuming that $V$ is twice continuously differentiable, Itô's formula yields

$$\begin{aligned}\mathrm{e}^{-\delta(\tau^b\wedge h)}\,V(X^{D^*}_{\tau^b\wedge h}) &= V(x)+\sigma\int_0^{\tau^b\wedge h}\mathrm{e}^{-\delta t}\,V'(X^{D^*}_t)\,\mathrm{d}W_t\\ &\quad+\int_0^{\tau^b\wedge h}\mathrm{e}^{-\delta t}(\mu V'(X^{D^*}_t)+\tfrac{1}{2}\sigma V''(X^{D^*}_t)-\delta V(X^{D^*}_t))\,\mathrm{d}t\;.\end{aligned}$$

If the stochastic Integral is a martingale, we get

$$\begin{aligned}V(x) &= \mathbb{E}\Big[\mathrm{e}^{-\delta(\tau^b\wedge h)}\,V(X^{D^*}_{\tau^b\wedge h})\\ &\quad-\int_0^{\tau^b\wedge h}\mathrm{e}^{-\delta t}(\mu V'(X^{D^*}_t)+\tfrac{1}{2}\sigma V''(X^{D^*}_t)-\delta V(X^{D^*}_t))\,\mathrm{d}t\Big]\;.\end{aligned}$$

Thus,

$$\mathbb{E}\Big[\int_0^{\tau^b\wedge h}\mathrm{e}^{-\delta t}\Big(\mu V'(X^{D^*}_t)+\frac{1}{2}\sigma V''(X^{D^*}_t)-\delta V(X^{D^*}_t)-\phi(X^{D^*}_t)\Big)\,\mathrm{d}t\Big]=0\;.$$

Dividing by $h$ and letting $h\to 0$ implies

$$\tfrac{1}{2}\sigma^2 V''(x)+\mu V'(x)-\delta V(x)-\phi(x)=0\;. \tag{2.4}$$

We will see below that $V(x)$ is concave. Moreover, $V'(x)=1$ if $x>b$. This motivates the HJB equation

$$\max\Big(\tfrac{1}{2}\sigma^2 V''(x)+\mu V'(x)-\delta V(x)-\phi(x),1-V'(x)\Big)=0\;. \tag{2.5}$$

The concavity of $V(x)$ implies that the optimal strategy is a barrier strategy. If $V'(x)>1$ no dividends are paid. If $V'(x)=1$, a dividend is paid such that the process reaches a point where $V'(z)=1$ and $V'(z-h)>1$ for any $h>0$. Such a boundary point cannot be crossed. Suppose there is $x<x_0$ such that the process is reflected at $x$ and $x<y\le x_0$. Let $D_t$ be a dividend strategy for initial capital $x$. Starting with initial capital $y$, we compare the two strategies $\{D_t\}$ or $\{\tilde{D}_t\}$ where $\tilde{D}_t=y-x+D_t$. That is, we pay $y-x$ at time zero or not. Then $X^D_t-X^{\tilde{D}}_t=y-x$. So by our assumption on $\phi$

$$\mathbb{E}\Big[\int_0^\infty\mathrm{e}^{-\delta t}(\phi(X^{\tilde{D}}_t)-\phi(X^D_t))\,\mathrm{d}t\Big]>\mathbb{E}\Big[\int_0^\infty\mathrm{e}^{-\delta t}\,\delta(y-x)\,\mathrm{d}t\Big]=y-x\;.$$

This shows that it is not optimal to pay the dividend $y - x$ at time zero. We conclude that it cannot be optimal to pay dividends for $X_t < x_0$. In particular, the function $V(x)$ is bounded from above.

## 2.3 The Verification Theorem

We first show some basic properties of the value function.

**Lemma 2.1.** *$V$ is increasing and concave with $V(y) - V(x) \geq y - x$ for $x \leq y$. Moreover,*

$$V(x) \geq -\int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(X_t)]\, \mathrm{d}t\ .$$

*Proof.* Let $D$ be an admissible strategy for initial capital $x$. In addition, we consider the strategy $\tilde{D}_t = D_t + y - x$ for initial capital $y$. Then, we obtain

$$V(x) \geq V^{\tilde{D}}(x) = V^D(y) + y - x\ .$$

Since $D$ is arbitrary, we get $V(x) \geq V(y) + y - x$. Hence, $V$ is increasing.

Now, let $x, y \in \mathbb{R}$ and $z = kx + (1-k)y$, where $k \in [0,1]$. Moreover, we consider the strategies $D^x$ and $D^y$ for the inital capital $x$ and $y$, respectively. Then, we define $D_t = kD_t^x + (1-k)D_t^y$ for the initial capital $z$. Since $-\phi$ is concave and

$$\begin{aligned} X_t^D &= kx + (1-k)y + (k + 1 - k)(\mu t + \sigma W_t) - kD_t^x - (1-k)D_t^y \\ &= kX_t^{D^x} + (1-k)X_t^{D^y} \end{aligned}$$

we obtain

$$\begin{aligned} V(kx+(1-k)y) &= V(z) \ge V^D(z) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}(k\ \mathrm{d}D_t^x + (1-k)\ \mathrm{d}D_t^y) \\ &\quad - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(X_t^D)\ \mathrm{d}t\Big] \\ &\ge k\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \ \mathrm{d}D_t^x - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(X_t^{D_x})\ \mathrm{d}t\Big] \\ &\quad + (1-k)\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \ \mathrm{d}D_t^y - \int_0^\infty \mathrm{e}^{-\delta t}\,\phi(X_t^{D_y})\ \mathrm{d}t\Big] \\ &= kV^{D^x}(x) + (1-k)V^{D^y}(y)\ . \end{aligned}$$

Taking the supremum over all strategies $D^x$ and $D^y$, we get

$$V(kx+(1-k)y) \ge kV(x) + (1-k)V(y)\ .$$

Hence the concavity.

In conclusion, let $V^0$ be the value of the strategy where no dividends are paid. Then, Fubini's theorem implies

$$V(x) \ge V^0(x) = -\int_0^\infty \mathrm{e}^{-\delta t}\,\mathbb{E}[\phi(X_t)]\ \mathrm{d}t\ .$$

□

Now, we prove the verification theorem.

**Theorem 2.1.** *Let $f$ be a concave and twice continuously differentiable solution to* (2.5)*. Suppose that there is a $b^*$, such that $f(x) = f(b^*) + x - b^*$ for all $x > b^*$ and $f(x) < f(b^*) - (b^* - x)$ for all $x < b^*$. Moreover, we define*

$$D_t^* = \max\Big(\sup_{0\le s\le t} X_s - b^*, 0\Big)\ .$$

*If $\lim_{t\to\infty} \mathrm{e}^{-\delta t}\,\mathbb{E}[f(X_t^{D^*})] = 0$, we obtain $f(x) = V^{D^*}(x) = V(x)$ and $D^*$ is an optimal strategy.*

*Proof.* Let $D$ be an arbitrary strategy, $\tau_n = \inf(t > 0 : |X_t^D| > n)$ and $h > 0$. Then, Itô's formula implies

$$\begin{aligned}
\mathrm{e}^{-\delta h} f(X_{\tau_n \wedge h}^D) &= f(x) + \int_0^{\tau_n \wedge h} \mathrm{e}^{-\delta t}\Big(\mu f'(X_t^D) + \tfrac{1}{2}\sigma^2 f''(X_t^D) - \delta f(X_t^D)\Big)\,\mathrm{d}t \\
&\quad + \sigma \int_0^{\tau_n \wedge h} \mathrm{e}^{-\delta t} f'(X_t^D)\,\mathrm{d}W_t - \int_0^{\tau_n \wedge h} \mathrm{e}^{-\delta t} f'(X_{t-}^D)\,\mathrm{d}D_t \\
&\quad + \sum_{0<t\le\tau_n\wedge h} \mathrm{e}^{-\delta t}\Big(f(X_t^D) - f(X_{t-}^D) - f'(X_{t-}^D)(X_t^D - X_{t-}^D)\Big)\ .
\end{aligned}$$

The concavity of $f$ implies that $f$ lies below of its tangents. This means that for all $y, z$ it holds

$$f(y) \le f(z) + f'(z)(y - z)\ .$$

Thus,

$$\sum_{0<t\le\tau_n\wedge h} \mathrm{e}^{-\delta s}\Big(f(X_t^D) - f(X_{t-}^D) - f'(X_{t-}^D)(X_t^D - X_{t-}^D)\Big) \le 0\ .$$

Note that $f'(x)$ is bounded on $[-n, n]$. Thus, the stochastic integral is a martingale with mean zero. Since $f$ fulfils (2.5) and $f'(x) \ge 1$, we obtain

$$f(x) \ge \mathbb{E}\Big[\mathrm{e}^{-\delta\tau_n\wedge h} f(X_{\tau_n\wedge h}^D) + \int_0^{\tau_n\wedge h} \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^{\tau_n\wedge h} \mathrm{e}^{-\delta t}\,\phi(X_t^D)\,\mathrm{d}t\Big]\ .$$

By bounded and monotone convergence, respectively, we get

$$f(x) \ge \mathbb{E}\Big[\mathrm{e}^{-\delta\tau_n} f(X_{\tau_n}^D) + \int_0^{\tau_n} \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^{\tau_n} \mathrm{e}^{-\delta t}\,\phi(X_t^D)\,\mathrm{d}t\Big]\ ,$$

where we interpret the first term as zero if $\tau_n = \infty$. Since $f$ is increasing, we have $\mathrm{e}^{-\delta\tau_n} f(X_{\tau_n}^D) \le \mathrm{e}^{-\delta\tau_n} f(X_{\tau_n})$. The expected value of the latter tends to zero as $n \to \infty$, provided $\tau_n \to \infty$. If $\tau_n \to \infty$ as $n \to \infty$ we get $f(x) \ge V^D(x)$. Since paying dividends if $X_t^D < x_0$ is not optimal, we can find a strategy $\tilde{D}$ such that $V^{\tilde{D}}(x) \ge V^D(x)$ and $\tilde{\tau}_n \to \infty$. Thus also in this case $f(x) \ge V^{\tilde{D}}(x) \ge V^D(x)$. Since $D$ was arbitrary, we have $f(x) \ge V(x)$. Using the strategy $D_t^*$, all inequalities are replaced by equalities. Thus $f(x) = V^{D^*}(x) \le V(x)$. This proves the result. □

Having a candidate solution fulfilling (2.4) on $(-\infty, b^*]$ and $f(x) = f(b^*) + x - b^*$ on $(b^*, \infty)$ we will have to verify that (2.5) is satisfied. The following lemma shows that this holds.

**Lemma 2.2.** *Suppose that $f$ is twice continuously differentiable and concave, and solves* (2.4) *on $(-\infty, b^*]$ with $f'(b^*) = 1$ and $f''(b^*) = 0$. If $f(x) = f(b^*) + x - b^*$ on $(b^*, \infty)$, then $f$ solves* (2.5)*.*

*Proof.* Since $0 = \frac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) - \phi(x) \le \mu f'(x) - \delta f(x) - \phi(x)$ for $x \le b^*$ with equality in $b^*$, we must have $0 \ge \mu f''(b^*) - \delta f'(b^*) - \phi'(b^*-) = -\delta - \phi'(b^*)$. Here $\phi'(b^*-)$ denotes the derivative from the left. Thus by the convexity of $\phi$, $\phi(x) \ge \phi(b^*) - \delta(x - b^*)$. This implies for $x \ge b^*$ that

$$\mu - \delta(f(b^*) + x - b^*) - \phi(x) \le \mu - \delta f(b^*) - \phi(b^*) = 0 \,.$$

and therefore the assertion. □

In the following examples we will show that a solution fulfilling the conditions of the verification theorem can be found.

## 2.4 Exponential Penalty Payments

In this section we consider an exponential penalty function

$$\phi(x) = \alpha\, \mathrm{e}^{-\beta x}$$

with $\alpha, \beta > 0$. Obviously, (3.4) is fulfiled for

$$x < y < x_0 = -\beta^{-1} \max\{\log \delta - \log(\alpha\beta), 0\} \,.$$

The function

$$f(x) = C_1\, \mathrm{e}^{\xi_1 x} - C_2\, \mathrm{e}^{\xi_2 x} - A\, \mathrm{e}^{-\beta x}$$

solves equation (2.4). Here, $\xi_2 < 0 < \xi_1$ are the roots of the equation

$$\sigma^2\xi^2 + 2\mu\xi - 2\delta = 0 \;,$$

$$A = -\frac{2\alpha}{\sigma^2\beta^2 - 2\mu\beta - 2\delta}$$

and $C_1, C_2$ are some constants. Since $\mathbb{E}[\mathrm{e}^{-\beta X_t - \delta t}] = \exp\{(\frac{1}{2}\sigma^2\beta^2 - \beta\mu - \delta)t\}$, we see that $V(x) = -\infty$ if $\beta \ge -\xi_2$. Therefore we assume $0 < \beta < -\xi_2$. In particular, this means that $A > 0$.

Not paying dividends, we find

$$\begin{aligned} V(x) \;\; &\ge \;\; -\alpha \int_0^\infty \mathrm{e}^{-\delta t}\,\mathbb{E}[e^{-\beta X_t}]\,\mathrm{d}t \\ &= \;\; -\alpha \int_0^\infty e^{-\delta t - \beta(x+\mu t) + \beta^2\sigma^2 t/2}\,\mathrm{d}t = -A\,\mathrm{e}^{-\beta x} \;. \end{aligned}$$

Now, $f$ is increasing for small $x$ only if $C_2 \ge 0$. Because $C_2\,\mathrm{e}^{\xi_2 x} > A\,\mathrm{e}^{-\beta x}$ and $C_1\,\mathrm{e}^{\xi_1 x} < A\,\mathrm{e}^{-\beta x}$ for $x$ small enough, our solution has to fulfil $C_2 = 0$. We look for constants $b^*$ and $C_1$, such that $f'(b^*) = 1$ and $f''(b^*) = 0$, that is

$$C_1\xi_1\,\mathrm{e}^{\xi_1 b^*} + A\beta\,\mathrm{e}^{-\beta b^*} = 1 \;, \qquad C_1\xi_1^2\,\mathrm{e}^{\xi_1 b^*} - A\beta^2\,\mathrm{e}^{-\beta b^*} = 0 \;.$$

The solution is

$$b^* = -\frac{1}{\beta}\log\Bigl(\frac{\xi_1}{\beta A(\xi_1+\beta)}\Bigr)$$

and

$$C_1 = \frac{A\beta^2}{\xi_1^2}\,\mathrm{e}^{-(\beta+\xi_1)b^*} > 0 \;.$$

Our candidate solution becomes now

$$f(x) = \begin{cases} C_1\,\mathrm{e}^{\xi_1 x} - A\,\mathrm{e}^{-\beta x} \;, & \text{if } x \le b^* \\ C_1\,\mathrm{e}^{\xi_1 b^*} - A\,\mathrm{e}^{-\beta b^*} + x - b^*, & \text{if } x > b^* \end{cases} \;.$$

This candidate solution is a twice continuously differentiable solution. Note that $b^*$ may become negative for $\alpha$ close to zero. We further observe that $f$ is concave with $f'(x) \ge f'(b^*) = 1$ and on $[b^*, \infty)$ we have

$$\tfrac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) + \phi(x) \le 0$$

by Lemma 2.2.

From the next result it will follow that $f(x) = V(x)$.

**Lemma 2.3.** *We have*

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-\delta t} f(X_t^{D^*})\Big] = 0\ .$$

*Proof.* By Fatou's lemma, it suffices to show that $\mathrm{e}^{-\delta t}\,\mathrm{e}^{-\beta X_t^{D^*}}$ tends to zero because $C_1\,\mathrm{e}^{\xi_1 x}$ is bounded for $x \le b^*$.

The process $Y_t = b^* - X_t^{D^*}$ is a Brownian motion reflected in zero. From queueing theory it is known that the stationary distribution is exponential. Thus $X_t^{D^*}/t$ tends to zero, and $t(\delta + X_t^{D^*}/t)$ tends to infinity. This proves the result. □

Theorem 2.1 shows that $D^*$ is optimal and $V(x) = f(x)$. Figure 2.1 shows the value function for $\mu = \sigma = 1$, $\delta = 0.05$ and $\alpha = \beta = 0.1$. The dividend barrier is at $b^* = -15.59398$. The solid line gives the optimal value, the dotted line gives the value without dividend payments.

## 2.5 Linear Penalty Payments

Now, we set

$$\phi(x) = -\alpha x \mathbb{1}_{x<0}$$

for some $\alpha > 0$. Then, for $x < 0$ equation (2.5) is solved by

$$f_1(x) = C_1\,\mathrm{e}^{\xi_1 x} + C_2\,\mathrm{e}^{\xi_2 x} + \frac{\alpha(\mu + \delta x)}{\delta^2}$$

and for $x \ge 0$ by

$$f_2(x) = C_3 e^{\xi_1 x} + C_4 e^{\xi_2 x}\ ,$$

where $C_1, C_2, C_3, C_4$ are constants and $\xi_1, \xi_2$ as above. The next lemma shows that the value function exists only if $\delta \le \alpha$. In this case the value function is linearly bounded.

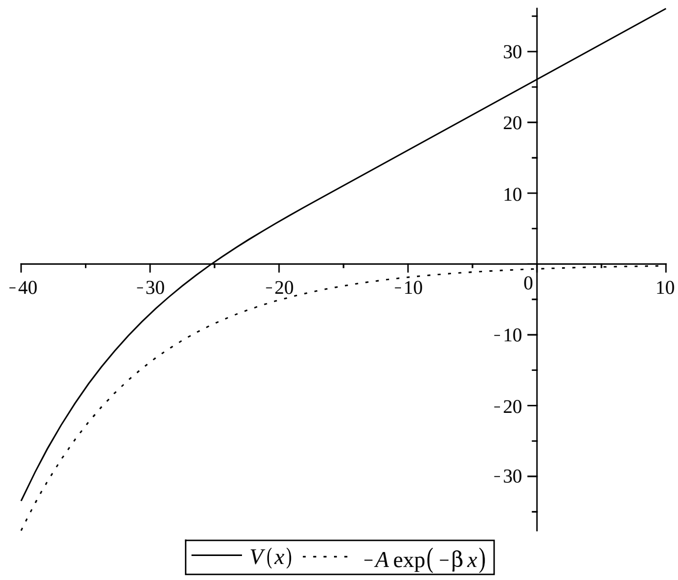


Figure 2.1: Value function for $\mu = \sigma = 1$, $\delta = 0.05$ and $\alpha = \beta = 0.1$.

**Lemma 2.4.** *i) If $\delta > \alpha$, an optimal strategy does not exist and $V(x) = \infty$.*

*ii) For $\delta < \alpha$ it holds*

$$V(x) \leq \frac{\alpha(\delta x + \mu)}{\delta^2} \ .$$

*Moreover,*

$$V(x) \geq \frac{\alpha(\delta x + \mu)}{\delta^2} + C$$

*for some $C < 0$ if $x \leq 0$.*

*iii) Let $\delta = \alpha$, then*

$$V(x) = \frac{\alpha(\delta x + \mu)}{\delta^2} \ .$$

*Proof.* $i$) Let $D^0$ be a barrier strategy with the barrier $b = 0$. Then, we define the strategy $D_t^{(0,c)} = D_t^0 + ct$ for some $c > 0$. Now, $X_t^{D^{(0,c)}} \leq 0$ and

$$\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \ \mathrm{d}D_t^{(0,c)}\Big] = \delta \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \, D_t^{(0,c)} \, \mathrm{d}t\Big] \ .$$

Thus, $\delta > \alpha$ implies

$$\begin{aligned}
V(x) &\geq V^{D^{(0,c)}}(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t^{(0,c)} + \alpha \int_0^\infty \mathrm{e}^{-\delta t}\, X_t^{D^{(0,c)}}\ \mathrm{d}t\Big] \\
&= \mathbb{E}\Big[(\delta-\alpha)\int_0^\infty \mathrm{e}^{-\delta t}\, D_t^{(0,c)}\ \mathrm{d}t + \alpha \int_0^\infty \mathrm{e}^{-\delta t}\, X_t\ \mathrm{d}t\Big] \\
&\geq \frac{c(\delta-\alpha)}{\delta^2} + \frac{\alpha(\delta x+\mu)}{\delta^2}\ .
\end{aligned}$$

Letting $c \to \infty$ implies the assertion.

*ii*) Let $D$ be an arbitrary strategy. W.l.o.g. we assume that

$$\lim_{t\to\infty} \mathbb{E}\Big[\mathrm{e}^{-\delta t}\, D_t\Big] = 0\ .$$

Otherwise $D$ cannot be optimal since it is not optimal to pay dividends if $X_t < 0$. Then,

$$\begin{aligned}
V^D(x) &\leq \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t + \alpha \int_0^\infty \mathrm{e}^{-\delta t}\, X_t^D\ \mathrm{d}t\Big] \\
&= \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \alpha \int_0^\infty \mathrm{e}^{-\delta t}\, D_t\ \mathrm{d}t\Big] + \frac{\alpha(\delta x+\mu)}{\delta^2} \\
&\leq \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \delta \int_0^\infty \mathrm{e}^{-\delta t}\, D_t\ \mathrm{d}t\Big] + \frac{\alpha(\delta x+\mu)}{\delta^2} \\
&= \frac{\alpha(\delta x+\mu)}{\delta^2}\ .
\end{aligned}$$

Since $D$ is arbitrary the first inequality follows. Now, let $x \leq 0$. Here,

$$\begin{aligned}
V(x) &\geq \alpha\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \min(X_t,0)\ \mathrm{d}t\Big] = \alpha\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\, \tfrac{1}{2}(X_t - |X_t|)\ \mathrm{d}t\Big] \\
&= \tfrac{1}{2}\Big(\frac{\alpha(\delta x+\mu)}{\delta^2} - \alpha\int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[|X_t|]\ \mathrm{d}t\Big) \\
&\geq \tfrac{1}{2}\Big(\frac{\alpha(\delta x+\mu)}{\delta^2} - \alpha\int_0^\infty \mathrm{e}^{-\delta t}\, |x+\mu t|\ \mathrm{d}t - \alpha\sigma\int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[|W_t|]\ \mathrm{d}t\Big) \\
&= \frac{\alpha(\delta x+\mu)}{\delta^2} - \frac{\alpha\mu}{\delta^2}\, \mathrm{e}^{\frac{\delta x}{\mu}} - \frac{\alpha\sigma}{2}\int_0^\infty \mathrm{e}^{-\delta t}\sqrt{2t/\pi}\ \mathrm{d}t\ .
\end{aligned}$$

We set

$$C = -\frac{\alpha\mu}{\delta^2} - \frac{\alpha\sigma}{2}\int_0^\infty \mathrm{e}^{-\delta t}\sqrt{2t/\pi}\,\mathrm{d}t\ .$$

Then, for $x \le 0$ we have

$$V(x) \ge \frac{\alpha(\delta x + \mu)}{\delta^2} + C\ .$$

*iii*) Consider the same strategy as in *i*). Now, $\delta = \alpha$ implies

$$V(x) \ge \frac{\alpha(\delta x + \mu)}{\delta^2}\ .$$

On the other hand *ii*) yields

$$V(x) \le \frac{\alpha(\delta x + \mu)}{\delta^2}\ .$$

□

In the following we assume that

$$\delta < \alpha\ . \tag{2.6}$$

This means that the preference parameter is smaller than the slope of the penalty function. Note, that this is consistent with assumption (3.4). Moreover, the dividend barrier $b^*$ must be positive.

Now, if $C_2 \neq 0$, we obtain for any $C < 0$ that for $x$ small enough either $f_1'(x) < 0$ or $f_1(x) < \frac{\alpha(\delta x+\mu)}{\delta^2} + C$. Thus, we let $C_2 = 0$. Note that the continuity of $\phi$ in $x = 0$ together with $f_1(0) = f_2(0)$ and $f_1'(0) = f_2'(0)$ implies $f_1''(0) = f_2''(0)$. At the dividend barrier we must have $f_2'(b^*) = 1$ and $f_2''(b^*) = 0$. Then,

$$C_4 = -\frac{\xi_1^2}{\xi_2^2}\,\mathrm{e}^{(\xi_1-\xi_2)b^*}\,C_3\ ,$$

$$C_3 = -\frac{\xi_2\,\mathrm{e}^{-\xi_1 b^*}}{\xi_1(\xi_1-\xi_2)}\ , \qquad C_4 = \frac{\xi_1\,\mathrm{e}^{-\xi_2 b^*}}{\xi_2(\xi_1-\xi_2)}\ ,$$

$$C_1 = \frac{\xi_1 \,\mathrm{e}^{-\xi_2 b^*} - \xi_2 \,\mathrm{e}^{-\xi_1 b^*}}{\xi_1(\xi_1 - \xi_2)} - \frac{\alpha}{\delta \xi_1} \;,$$

and

$$b^* = \frac{1}{-\xi_2} \log\Big(\frac{\alpha}{\delta}\Big) > 0 \;.$$

We obtain

$$C_3 = \frac{-\xi_2 (\alpha/\delta)^{\xi_1/\xi_2}}{\xi_1(\xi_1 - \xi_2)} > 0 \;,$$

$$C_4 = \frac{\xi_1 \alpha}{\delta \xi_2 (\xi_1 - \xi_2)} < 0 \;,$$

and

$$C_1 = \frac{\xi_1(\alpha/\delta) - \xi_2 (\alpha/\delta)^{\xi_1/\xi_2}}{\xi_1(\xi_1 - \xi_2)} - \frac{\alpha}{\delta \xi_1} = \frac{\alpha \xi_2}{\delta \xi_1 (\xi_1 - \xi_2)} (1 - (\alpha/\delta)^{\xi_1/\xi_2 - 1}) < 0 \;.$$

The candidate for the solution

$$f(x) = \begin{cases} f_1(x), & x \le 0 \\ f_2(x), & 0 < x \le b^* \\ f_2(b^*) + x - b^*, & x > b^* \end{cases}$$

is twice continuously differentiable.

Now,

$$f_2'''(x) = \xi_1^3 C_3 \,\mathrm{e}^{\xi_1 x} + \xi_2^3 C_4 \,\mathrm{e}^{\xi_2 x} > 0 \;.$$

Consequently, $f_2''(x) \le f_2''(b^*) = 0$ and $f_2'(x) \ge f_2'(b^*) = 1$ if $x \le b^*$. Furthermore,

$$f_1''(x) = \xi_1^2 C_1 \,\mathrm{e}^{\xi_1 x} < 0$$

Therefore $f_1'(x) \ge f_1'(0) = f_2'(0) \ge f_2'(b^*) = 1$ if $x \le 0$. In particular, $f$ is concave and $f'(x) > 1$ for all $x < b^*$. Alltogether, we obtain that $f$ is an increasing, concave and twice continuously differentiable function with $f'(x) \ge 1$ and by Lemma 2.2 we get

$$\tfrac{1}{2}\sigma^2 f''(x) + \mu f'(x) - \delta f(x) + \phi(x) \le 0$$

for $x \geq b^*$.

Since $f$ is linearly bounded, the following obviously holds.

**Lemma 2.5.** *We have*

$$\mathbb{E}\Big[\mathrm{e}^{-\delta t} f(X_t^{D^*})\Big] \to 0, t \to \infty \ .$$

As in Section 2.4 we obtain that $D^*$ is optimal and $V(x) = f(x) = V^{D^*}(x)$. Figure 2.2 shows the value function for $\mu = \sigma = 1$, $\delta = 0.05$ and $\alpha = 0.15$. The solid line gives the optimal value, the dotted line gives the value without dividend payments. The dividend barrier is at $b^* = 0.53622$.

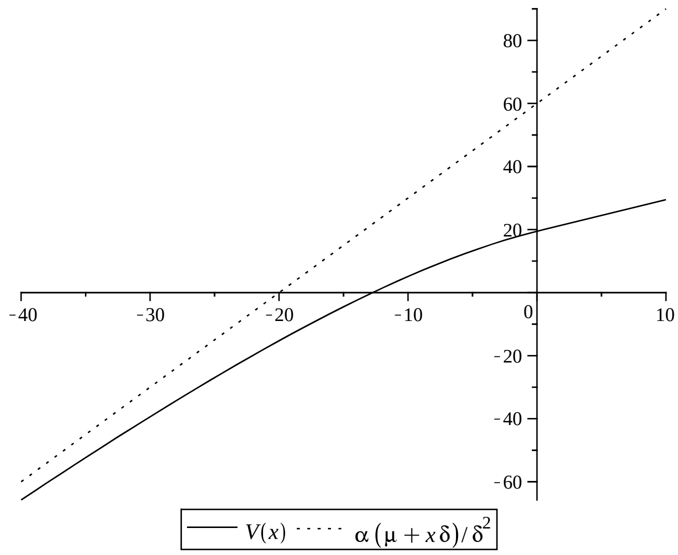


Figure 2.2: Value function for $\mu = \sigma = 1$, $\delta = 0.05$ and $\alpha = 0.15$ .

## 2.6 Quadratic Penalty Payments

In this section we let

$$\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0} \ ,$$

where $\alpha_1, \alpha_2 > 0$. Here, we have

$$x_0 = -\frac{1}{2}\frac{\alpha_1 + \delta}{\alpha_2} \; .$$

For $x < 0$ the HJB equation is solved by

$$f_1(x) = C_1 \,\mathrm{e}^{\xi_1 x} + C_2 \,\mathrm{e}^{\xi_2 x} - \frac{\alpha_2}{\delta}x^2 + \frac{\alpha_1\delta - 2\mu\alpha_2}{\delta^2}x + \frac{\mu\alpha_1\delta - 2\mu^2\alpha_2 - \sigma^2\alpha_2\delta}{\delta^3}$$

and for $x \geq 0$ by

$$f_2(x) = C_3 e^{\xi_1 x} + C_4 e^{\xi_2 x} \; ,$$

where $C_1, C_2, C_3, C_4$ are constants and $\xi_1, \xi_2$ as above. Now, the value function is quadratically bounded. Thus, again $C_2 = 0$ must hold. Note, that in this section it is possible to derive a solution with a negative optimal dividend barrier. In this case, we do not need to consider equation (2.4) for $x > 0$. Therefore, we have to distinguish between a negative and a positive dividend barrier. Let us start with the easier case, where the optimal dividend barrier is negative, i.e. $b^* = b^- \leq 0$. Then, it must hold that $f_1'(b^-) = 1$ and $f_1''(b^-) = 0$. This is fulfilled for

$$C_1 = C_1^- = \frac{2\alpha_2}{\delta\xi_1^2}\,\mathrm{e}^{-\xi_1 b^-}$$

and

$$b^- = \frac{\alpha_2\xi_1\sigma^2 + \alpha_1\delta - \delta^2}{2\delta\alpha_2} \; .$$

Thus, a necessary condition for a negative optimal dividend barrier is that the following inequality holds

$$\alpha_2\xi_1\sigma^2 + \alpha_1\delta \leq \delta^2 \; . \tag{2.7}$$

Note that,

$$b^- - x_0 = \frac{1}{2}\frac{\alpha_2\xi_1\sigma^2 + 2\alpha_1\delta}{\delta\alpha_2} > 0 \; .$$

Define

$$f_1^-(x) = C_1^- \,\mathrm{e}^{\xi_1 x} - \frac{\alpha_2}{\delta}x^2 + \frac{\alpha_1\delta - 2\mu\alpha_2}{\delta^2}x + \frac{\mu\alpha_1\delta - 2\mu^2\alpha_2 - \sigma^2\alpha_2\delta}{\delta^3} \; .$$

Obviously, $C_1^- > 0$ and therefore

$$(f_1^-)'''(x) = \xi_1^3 C_1^- \,\mathrm{e}^{\xi_1 x} > 0 \ .$$

Thus, we have for $x \le b^-$ that $(f_1^-)''(x) \le (f_1^-)''(b^-) = 0$. In particular, $f_1^-$ is concave on $(-\infty, b^-]$. This implies for $x \le b^-$ that $(f_1^-)'(x) \ge (f_1^-)'(b^-) = 1$. Now, by Lemma 2.2, we obtain that

$$f^-(x) = \begin{cases} f_1^-(x), & x \le b^- \\ f_1^-(b^-) + x - b^-, & x > b^- \end{cases}$$

fulfils the HJB equation. Furthermore, the following lemma obviously holds, because $f^-$ is quadratically bounded.

**Lemma 2.6.** *We have*

$$\mathbb{E}\Big[\mathrm{e}^{-\delta t} f^-(X_t^{D^*})\Big] \to 0, t \to \infty \ .$$

Together with the verification theorem, we obtain that the optimal dividend barrier is given by $b^-$ and $f^-(x) = V(x)$ if (2.7) is fulfilled. Figure 2.3 shows the value function for $\mu = 0.1$, $\sigma = 0.4$, $\delta = 0.05$ and $\alpha_1 = \alpha_2 = 0.01$. The dividend barrier is at $b^- = -1.38755$.

Now, we try to determine a solution with a positive optimal dividend barrier $b^+$. As in the section with linear penalty payments we have to solve the equations $f_1(0) = f_2(0)$ and $f_1'(0) = f_2'(0)$ in order to obtain, together with the continuity of $\phi$ in $x = 0$, a twice continuously differentiable candidate for the value function. At the dividend barrier we must have $f_2'(b^+) = 1$ and $f_2''(b^+) = 0$. This is fulfilled for

$$C_3 = C_3^+ = -\frac{\xi_2 \,\mathrm{e}^{-\xi_1 b^+}}{\xi_1(\xi_1 - \xi_2)} \ , \qquad C_4 = C_4^+ = \frac{\xi_1 \,\mathrm{e}^{-\xi_2 b^+}}{\xi_2(\xi_1 - \xi_2)} \ ,$$

$$C_1 = C_1^+ = \frac{\xi_1 \,\mathrm{e}^{-\xi_2 b^+} - \xi_2 \,\mathrm{e}^{-\xi_1 b^+}}{\xi_1(\xi_1 - \xi_2)} - \frac{a_1}{\xi_1} \ ,$$

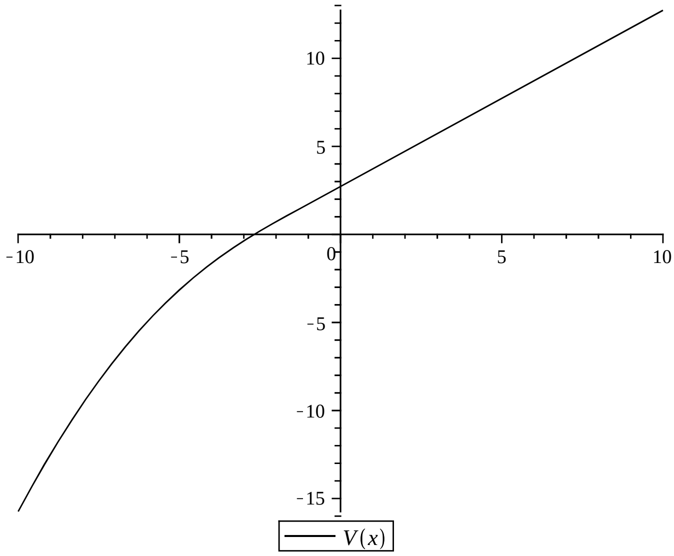


Figure 2.3: Value function for $\mu = 0.1$, $\sigma = 0.4$, $\delta = 0.05$ and $\alpha_1 = \alpha_2 = 0.01$.

where

$$a_1 = \frac{\alpha_1\delta - 2\mu\alpha_2}{\delta^2}$$

and

$$b^+ = \frac{1}{-\xi_2}\log\Big(\frac{\alpha_2\xi_1\sigma^2 + \alpha_1\delta}{\delta^2}\Big)\,.$$

Thus, a necessary condition for a positive optimal dividend barrier is that (2.7) does not hold. Moreover,

$$C_3^+ = \frac{-\xi_2(\alpha/\delta)^{\xi_1/\xi_2}}{\xi_1(\xi_1 - \xi_2)} > 0$$

and

$$C_4^+ = \frac{\xi_1\alpha}{\delta\xi_2(\xi_1 - \xi_2)} < 0$$

Define

$$f_1^+(x) = C_1^+\,\mathrm{e}^{\xi_1 x} - \frac{\alpha_2}{\delta}x^2 + \frac{\alpha_1\delta - 2\mu\alpha_2}{\delta^2}x + \frac{\mu\alpha_1\delta - 2\mu^2\alpha_2 - \sigma^2\alpha_2\delta}{\delta^3}\,.$$

and

$$f_2^+(x) = C_3^+ e^{\xi_1 x} + C_4^+ e^{\xi_2 x} \, .$$

The candidate for the solution

$$f^+(x) = \begin{cases} f_1^+(x), & x \le 0 \\ f_2^+(x), & 0 < x \le b^+ \\ f_2^+(b^+) + x - b^+, & x > b^+ \end{cases}$$

is twice continuously differentiable. Moreover,

$$\begin{aligned} (f_1^+)''(0) &= C_1^+ \xi_1^2 - 2\frac{\alpha_2}{\delta} \\ &= -2\frac{\alpha_2 \xi_1 \sigma^2 + \alpha_1 \delta}{\delta^2}\Big[1 - \Big(\frac{\alpha_2 \xi_1 \sigma^2 + \alpha_1 \delta}{\delta^2}\Big)^{\xi_1/\xi_2 - 1}\Big] < 0 \, . \end{aligned}$$

Now,

$$(f_2^+)'''(x) = \xi_1^3 C_3^+ \,\mathrm{e}^{\xi_1 x} + \xi_2^3 C_4^+ \,\mathrm{e}^{\xi_2 x} > 0 \, .$$

Consequently, $(f_2^+)''(x) \le (f_2^+)''(b^+) = 0$ and $(f_2^+)'(x) \ge f_2'(b^+) = 1$ if $x \le b^+$. Furthermore, if $C_1^+ \le 0$ it holds

$$(f_1^+)''(x) = \xi_1^2 C_1^+ \,\mathrm{e}^{\xi_1 x} - 2\frac{\alpha_2}{\delta} < 0 \, .$$

On the other hand, if $C_1^+ > 0$ we have for $x \le 0$ that

$$(f_1^+)''(x) = \xi_1^2 C_1^+ \,\mathrm{e}^{\xi_1 x} - 2\frac{\alpha_2}{\delta} \le C_1 \xi_1^2 - 2\frac{\alpha_2}{\delta} = (f_1^+)''(0) < 0 \, .$$

Therefore $(f_1^+)'(x) \ge (f_1^+)'(0) = (f_2^+)'(0) \ge (f_2^+)'(b^+) = 1$ if $x \le 0$. In particular, $f^+$ is concave and $(f^+)'(x) > 1$ for all $x < b^+$. Altogether, we obtain that $f^+$ is an increasing, concave and twice continuously differentiable function with $(f^+)'(x) \ge 1$ and by Lemma 2.2 we get

$$\tfrac{1}{2}\sigma^2 (f^+)''(x) + \mu (f^+)'(x) - \delta f^+(x) + \phi(x) \le 0$$

for $x \ge b^+$. As above the following lemma holds.

**Lemma 2.7.** *We have*

$$\mathbb{E}\Big[\mathrm{e}^{-\delta t} f^+(X_t^{D^*})\Big] \to 0, t \to \infty\ .$$

In sum we obtain that the optimal dividend barrier is given by $b^+$ and $f^+(x) = V(x)$ if (2.7) is not fulfilled. Figure 2.4 shows the value function for $\mu = 0.08$, $\sigma = 0.4$, $\delta = 0.05$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.01$. The dividend barrier is at $b^+ = 1.62327$.

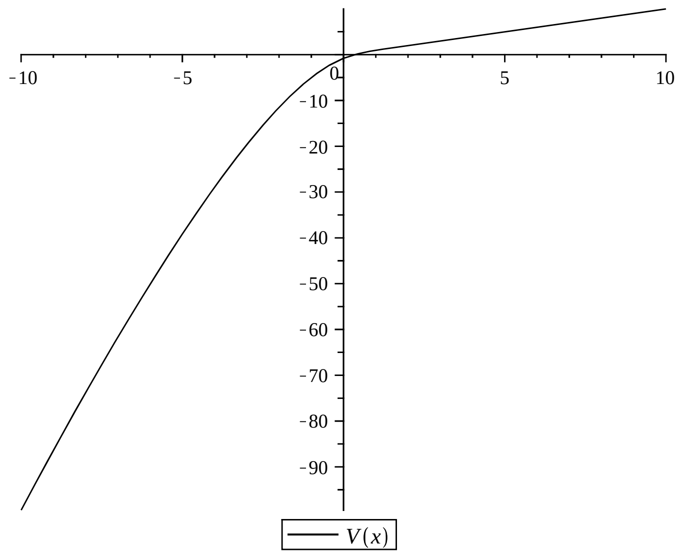


Figure 2.4: Value function for $\mu = 0.08$, $\sigma = 0.4$, $\delta = 0.05$, $\alpha_1 = 0.5$ and $\alpha_2 = 0.01$.

# Chapter 3

# Maximisation of Dividends with Penalty Payments in the Cramér–Lundberg Model

## 3.1 Introduction

Now, we consider the dividend problem in the Cramér–Lundberg model. That is, the surplus is given by

$$L_t = x + ct - \sum_{i=1}^{N_t} Y_i \ , \tag{3.1}$$

where $x$ denotes the initial capital and $c > 0$ a constant premium rate. The amount of claims arriving until time $t$ is given by the Poisson process $N = \{N_t\}_{t \geq 0}$ with intensity $\lambda$ and the claim size of the $i$-th claim is denoted by $Y_i$, where $\{Y_i\}_{i=1,2,...}$ is a sequence of positive, independent and identically distributed random variables with mean $m_1$, second moment $m_2$ and a continuous distribution function $F$. Moreover, $\{Y_i\}_{i=1,2,...}$ are independent of $N$. Claims occur at random times $0 = T_0 < T_1 < T_2 < ...$ and we consider independent and exponentially distributed interarrival times with mean $1/\lambda$. In addition, since $E(L_t - x) = t(c - \lambda m_1)$, we assume that the so-called net profit condition $c > \lambda m_1$ holds.

The information is given by the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of the aggregate claim process. Let $D_t$ be adapted and denote the accumulated dividend payments until time $t$. Then, the controlled surplus process is given by

$$L_t^D = L_t - D_t \ .$$

We allow all increasing càdlàg processes $D$. The value of a strategy $D$ is defined by

$$V^D(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t} \ \mathrm{d}D_t - \int_0^\infty \mathrm{e}^{-\delta t} \, \phi(L_t^D) \ \mathrm{d}t \ \Big| \ L_0^D = x\Big] \ , \tag{3.2}$$

where $\delta > 0$ denotes a preference parameter and the continuous, decreasing, positive and convex function $\phi$ models the penalty payments fulfilling $\phi(x) \to 0$ as $x \to \infty$. The set of admissible strategies is denoted by $\mathcal{D}$ and the (optimal) value function is defined by

$$V(x) = \sup_{D \in \mathcal{D}} V^D(x) \ .$$

We aim to find a strategy $D^*$ such that

$$V^{D^*}(x) = V(x) \ .$$

As in the chapter above, we assume that

$$\int_0^\infty \mathrm{e}^{-\delta t} \, \mathbb{E}[\phi(L_t)] \ \mathrm{d}t < \infty \ . \tag{3.3}$$

and that

$$\phi(x) - \phi(y) > \delta(y - x) \tag{3.4}$$

for $x < y < x_0$ and some $x_0 \in \mathbb{R}$.

This chapter is organised as follows. In the second section we show that $V$ is continuous, increasing and concave. Moreover, we derive some bounds of the value function and show that it solves the Hamilton–Jacobi–Bellman (HJB) equation

$$\max\Big\{cV'(x) + \lambda \int_0^\infty V(x-y)\,\mathrm{d}F(y) - (\lambda+\delta)V(x) - \phi(x),\\ 1 - V'(x)\Big\} = 0\ . \tag{3.5}$$

In Section 3 we prove that the optimal strategy is a barrier strategy. Section 4 studies an exponential penalty function $\phi(x) = \alpha\,\mathrm{e}^{-\beta x}$ for some $\alpha, \beta > 0$, a linear penalty function $\phi(x) = -\alpha x \mathbb{1}_{x<0}$ for some $\alpha > 0$ and a quadratic penalty function $\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0}$ for some $\alpha_1, \alpha_2 > 0$.

## 3.2 First Properties and the HJB Equation

We start with some basic properties of the value function that will help us to prove the HJB equation.

The first lemma states that the value function is concave. The concavity is crucially important to prove our main results.

**Lemma 3.1.** *The function $V(x)$ is concave.*

*Proof.* The proof is analogous to the proof of Lemma 2.1 in Chapter 2. □

**Remark 3.1.** *The concavity implies that $V$ is differentiable from the left and from the right and $V'(x-) \geq V'(x+) \geq V'(y-) \geq V'(y+)$ for $x < y$. In particular, $V$ is differentiable almost everywhere. Moreover, the concavity implies that $V$ is continuous.*

The next result gives some useful bounds of the value function.

**Lemma 3.2.** *$V(x)$ increasing with $V(y) - V(x) \geq y - x$ if $x \leq y$ and*

$$-\int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\, \mathrm{d}t \leq V(x) \leq (x - x_0)^+ + \frac{c}{\delta}\, . \tag{3.6}$$

*Proof.* Let $D$ be an admissible strategy for initial capital $x$. In addition, we consider the strategy $\tilde{D}_t = D_t + y - x$ for initial capital $y$. Then, we obtain

$$V(x) \geq V^{\tilde{D}}(x) = V^D(y) + y - x\, .$$

Since $D$ is arbitrary, we get $V(x) \geq V(y) + y - x$.

As in Chapter 2 we can show that a strategy that pays dividends if the surplus is below $x_0$ is dominated by a strategy where no dividends are paid for a surplus below $x_0$. Then, consider the pseudo-strategy $D$ where $(x - x_0)^+$ is immediately paid as dividends and thereafter dividends paid at rate $c$ and no penalty payments occur. Obviously,

$$V(x) \leq V^D(x) = (x - x_0)^+ + \frac{c}{\delta}\, .$$

Considering the strategy where no dividends are paid, the lower bound is obtained by the application of Fubini's theorem. □

Using that $V$ is locally bounded, we obtain the following

**Lemma 3.3.** *The function $V$ is locally Lipschitz continuous.*

*Proof.* Let $h > 0$ and $\tilde{D}$ be a strategy with initial capital $x + ch$. Then, for the strategy

$$D_t = \begin{cases} 0, & t \leq h \text{ or } T_1 \leq h, \\ \tilde{D}_{t-h}, & T_1 \wedge t > h, \end{cases}$$

with initial capital $x$, we obtain

$$\begin{aligned}
V(x) &\geq V^D(x) = \mathbb{P}(T_1 > h)\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^\infty \phi(L_t^D)\ \mathrm{d}t \,\Big|\, T_1 > h\Big] \\
&\quad + \mathbb{P}(T_1 \leq h)\mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}D_t - \int_0^\infty \phi(L_t^D)\ \mathrm{d}t \,\Big|\, T_1 \leq h\Big] \\
&= \mathrm{e}^{-(\lambda+\delta)h}\, V^{\tilde{D}}(x+ch) - \int_0^h \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t \\
&\quad - (1-\mathrm{e}^{-\lambda h}) \int_h^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t\ .
\end{aligned}$$

Since $\tilde{D}$ is arbitrary, we get

$$V(x) \geq \mathrm{e}^{-(\lambda+\delta)h}\, V(x+ch) - \int_0^h \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t - (1-\mathrm{e}^{-\lambda h}) \int_h^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t\ .$$

Thus,

$$\begin{aligned}
0 &\leq V(x+ch) - V(x) \leq V(x+ch)(1-\mathrm{e}^{-(\lambda+\delta)h}) \\
&\quad + \int_0^h \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t + (1-\mathrm{e}^{-\lambda h}) \int_h^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t\ . \qquad (3.7)
\end{aligned}$$

Note that $V$ is locally bounded and

$$\begin{aligned}
&\frac{1}{h}\Big[V(x+ch)(1-\mathrm{e}^{-(\lambda+\delta)h}) + \int_0^h \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t \\
&+ (1-\mathrm{e}^{-\lambda h}) \int_h^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t\Big] \\
&\quad \to (\lambda+\delta)V(x) + \phi(x) + \lambda \int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(L_t)]\ \mathrm{d}t,\ \ h \to 0\ .
\end{aligned}$$

Dividing inequality (3.7) by $h$ and letting $h \to 0$, we obtain that the derivatives from the right are locally bounded. Similarly one can show that the derivatives from the left are locally bounded. Thus, $V$ is locally Lipschitz continuous. □

We can now derive the HJB equation and prove that the value function is a solution to this equation.

**Theorem 3.1.** *The function $V(x)$ is differentiable and fulfils equation* (3.5)*. Moreover, there exists a $b^* \in \mathbb{R}$ such that $V'(b^*) = 1$ and $V(x) = V(b^*) + x - b^*$ for $x \geq b^*$.*

*Proof.* Let $h > 0$ and $d \geq 0$. Since $V$ is locally Lipschitz continuous and it cannot be optimal to pay dividends if $x < x_0$, we can choose in a measurable way a strategy $D^\varepsilon$ such that $V^{D^\varepsilon}(x') > V(x') - \varepsilon$ for $x' \in (-\infty, x + (c-d)h]$ and for a fixed $\varepsilon > 0$. Then, we define the strategy

$$D_t = \begin{cases} dt, & 0 \leq t < T_1 \wedge h, \\ D^\varepsilon_{t - T_1 \wedge h}, & t \geq T_1 \wedge h. \end{cases}$$

For this strategy we obtain

$$\begin{aligned}
V(x) &\geq V^D(x) \\
&= \mathbb{E}\Big[\int_0^{T_1\wedge h} \mathrm{e}^{-\delta s}(d-\phi(L_s^D))\,\mathrm{d}s + \mathrm{e}^{-\delta(T_1\wedge h)}\,V^{D^\varepsilon}(L^D_{T_1\wedge h})\Big] \\
&> \mathbb{E}\Big[\int_0^{T_1\wedge h} \mathrm{e}^{-\delta s}(d-\phi(L_s^D))\,\mathrm{d}s + \mathrm{e}^{-\delta(T_1\wedge h)}\,V(L^D_{T_1\wedge h})\Big] - \varepsilon \\
&= \mathbb{E}\Big[\Big(\int_0^{T_1\wedge h} \mathrm{e}^{-\delta s}(d-\phi(L_s^D))\,\mathrm{d}s \\
&\qquad + \mathrm{e}^{-\delta(T_1\wedge h)}\,V(L^D_{T_1\wedge h})\Big)(\mathbb{1}_{T_1>h}+\mathbb{1}_{T_1\leq h})\Big] - \varepsilon \\
&= \mathbb{P}(T_1>h)\Big(\int_0^h \mathrm{e}^{-\delta s}(d-\phi(x+(c-d)s))\,\mathrm{d}s \\
&\qquad + \mathrm{e}^{-\delta h}\,V(x+(c-d)h)\Big) + \mathbb{E}\Big[\Big(\int_0^{T_1} \mathrm{e}^{-\delta s}(d-\phi(L_s^D))\,\mathrm{d}s \\
&\qquad + \mathrm{e}^{-\delta T_1}\,V(x+(c-d)T_1-Y_1)\Big)\mathbb{1}_{T_1\leq h}\Big] \\
&= \mathrm{e}^{-\lambda h}\Big(\int_0^h \mathrm{e}^{-\delta s}(d-\phi(x+(c-d)s))\,\mathrm{d}s + \mathrm{e}^{-\delta h}\,V(x+(c-d)h)\Big) \\
&\qquad + \int_0^h \lambda\,\mathrm{e}^{-\lambda t}\Big[\int_0^t (d-\phi(x+(c-d)s))\,\mathrm{d}s \\
&\qquad + \mathrm{e}^{-\delta t}\int_0^\infty V(x+(c-d)t-y)\,\mathrm{d}F(y)\Big]\,\mathrm{d}t - \varepsilon \\
&\qquad + V(x+(c-d)h) - V(x+(c-d)h)\,.
\end{aligned}$$

Since $\varepsilon$ is arbitrary we can let it tend to zero and obtain the weak inequality.

Rearranging the terms and dividing by $h$ implies

$$\begin{aligned}0 \quad \geq \quad & \frac{V(x+(c-d)h)-V(x)}{h} - \frac{1-\mathrm{e}^{-h(\delta+\lambda)}}{h} V(x+(c-d)h) \\ & + \frac{\mathrm{e}^{-\lambda h}}{h} \int_0^h \mathrm{e}^{-\delta s}(d-\phi(x+(c-d)s))\,\mathrm{d}s \\ & + \frac{1}{h}\int_0^h \lambda\,\mathrm{e}^{-\lambda t}\Big[\int_0^t (d-\phi(x+(c-d)s))\,\mathrm{d}s \\ & + \mathrm{e}^{-\delta t}\int_0^\infty V(x+(c-d)t-y)\,\mathrm{d}F(y)\Big]\,\mathrm{d}t\,. \qquad (3.8)\end{aligned}$$

Since $V$ is concave, the derivatives from the left or from the right exist and $V$ is differentiable almost everywhere. Thus, the first term in the equation above converges to $V'(x+)(c-d)$ if $c>d$ and to $V'(x-)(c-d)$ if $c<d$ and vice versa if we start with an initial capital of $x-(c-d)h$. For simplicity of notation we just write $V'(x)$ for the derivative from the left and from the right. We will soon see that $V'(x+)=V'(x-)$. Letting $h\downarrow 0$, we get

$$(c-d)V'(x)-(\lambda+\delta)V(x)+d-\phi(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}F(y)\leq 0\,.$$

Since $d$ is arbitrary, we obtain

$$\begin{aligned}\sup_{d\geq 0}\Big[&(c-d)V'(x)-(\lambda+\delta)V(x)+d-\phi(x)+ \\ & \lambda\int_0^\infty V(x-y)\,\mathrm{d}F(y)\Big]\leq 0\,. \qquad (3.9)\end{aligned}$$

This implies that $V'(x)\geq 1$. Otherwise

$$(c-d)V'(x)-(\lambda+\delta)V(x)+d-\phi(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}F(y)$$

would be positive for $d$ large enough. In addition we obtain for $d=0$ that

$$cV'(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}F(y)-(\lambda+\delta)V(x)-\phi(x)\leq 0\,.$$

Thus, (3.9) can also be written as

$$\max\Big\{cV'(x)+\lambda\int_0^\infty V(x-y)\,\mathrm{d}F(y)-(\lambda+\delta)V(x)-\phi(x),1-V'(x)\Big\}\leq 0\,.$$

Analogously, one can show that " $\geq$ " holds. For example, see [62, Section 2.4.1]. Now, consider the value $b^* = \inf\{x\colon V'(x-) = 1\}$. Assume that $b^* = \infty$. Then, we have $V'(x) > 1$ for all $x$ and therefore the HJB equation implies

$$\begin{aligned} cV'(x) &= (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}F(y) + \phi(x) \\ &> (\lambda+\delta)V(x) - \lambda V(x) \int_0^\infty 1\,\mathrm{d}F(y) = \delta V(x)\,. \end{aligned}$$

But this yields that $V(x) > \mathrm{e}^{\delta x/c}$. Since $V$ is linearly bounded from above, we obtain $b^* < \infty$. Obviously, $V'(x) = 1$ for $x > b^*$ because of the concavity. Consequently, $V$ is differentiable on $(b^*, \infty)$ and $V(x) = V(b^*) + x - b^*$ for $x > b^*$. For $x < b^*$ we have $V'(x-) \geq V'(x+) > 1$ and therefore the HJB equation implies that $V'(x-)$ and $V'(x+)$ fulfil

$$cV'(x) = (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}F(y) + \phi(x)\,. \tag{3.10}$$

Since $F, V$, and $\phi$ are continuous, we obtain

$$cV'(x+) = (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y)\,\mathrm{d}F(y) + \phi(x) = cV'(x-)\,.$$

Thus, $V$ is also differentiable on $(-\infty, b^*)$. In conclusion, we show that $V'(b^*) = 1$. As in [57, Section 3.2.2] we can show that $V(b^*)$ is characterised through

$$V(b^*) = \frac{c - \phi(b^*)}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta} \int_0^\infty V(b^* - y)\,\mathrm{d}F(y)\,.$$

Plugging $V(b^*)$ into (3.10), we obtain $V'(b^*-) = 1$. Again, the concavity implies $1 = V'(b^*-) \geq V'(b^*+)$. Thus, either $V$ is differentiable at $b^*$ or $1 > V'(b^*+)$. Since the latter is impossible, we obtain that $V$ is differentiable at $b^*$ with $V'(b^*) = 1$.

$\square$

## 3.3 The Optimal Strategy and Characterisation of the Value Function

In this section we show that the optimal strategy $D^*$ is a barrier strategy with the barrier $b^*$. That is $D_t^* = D_t^{b^*}$, where

$$D_t^{b^*} = \max(x - b^*, 0) + c \int_0^t \mathbb{1}_{\{L_s^{b^*} = b\}} \, \mathrm{d}s$$

and $L_s^{b^*} = L_s^{D^{b^*}}$. We first state a useful lemma.

**Lemma 3.4.** *Let $N_t$ be an $\mathcal{F}_t$-adapted Poisson process with intensity $\lambda$ and $Z_t$ an $\mathcal{F}_t$-predictable process with*

$$\mathbb{E}\Big[\int_0^t |Z_s| \, \mathrm{d}s\Big] < \infty$$

*for all $t \geq 0$. Then,*

$$\int_0^t Z_s \, \mathrm{d}N_s - \lambda \int_0^t Z_s \, \mathrm{d}s$$

*is a $\mathcal{F}_t$-adapted martingale.*

*Proof.* See Brémaud [16, Page 27]. □

Now, we prove the main result in this chapter.

**Theorem 3.2.** *The strategy $D^{b^*}$ is optimal, that is, $V(x) = V^{D^{b^*}}(x)$.*

*Proof.* Firstly, we consider the telescoping sum

$$\begin{aligned}
\mathrm{e}^{-\delta t} V(L_t^{b^*}) &= V(x) + \sum_{i=1}^{N_t} \Big[ \mathrm{e}^{-\delta T_i} V(L_{T_i}^{b^*}) - \mathrm{e}^{-\delta T_{i-1}} V(L_{T_i-1}^{b^*}) \Big] \\
&\quad + \mathrm{e}^{-\delta t} V(L_t^{b^*}) - \mathrm{e}^{-\delta T_{N_t}} V(L_{T_{N_t}}^{b^*}) \\
&= V(x) + \sum_{i=1}^{N_t} \Big[ \mathrm{e}^{-\delta T_i} V(L_{T_i}^{b^*}) - \mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*}) \Big] \\
&\quad + \sum_{i=1}^{N_t} \Big[ \mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*}) - \mathrm{e}^{-\delta T_{i-1}} V(L_{T_i-1}^{b^*}) \Big] \\
&\quad + \mathrm{e}^{-\delta t} V(L_t^{b^*}) - \mathrm{e}^{-\delta T_{N_t}} V(L_{T_{N_t}}^{b^*}) \\
&= V(x) + \sum_{i=1}^{N_t} \Big[ \mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*} - Y_i) - \mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*}) \Big] \\
&\quad + \sum_{i=1}^{N_t} \Big[ \mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*}) - \mathrm{e}^{-\delta T_{i-1}} V(L_{T_i-1}^{b^*}) \Big] \\
&\quad + \mathrm{e}^{-\delta t} V(L_t^{b^*}) - \mathrm{e}^{-\delta T_{N_t}} V(L_{T_{N_t}}^{b^*}) \,.
\end{aligned}$$

Now, let $Y$ be a generic random variable with the same distribution as $Y_i$ and define the process

$$Z_t = \mathrm{e}^{-\delta t} \Big[ V(L_{t-}^{b^*} - Y) - V(L_{t-}^{b^*}) \Big] \,.$$

The lemma above implies that

$$\int_0^t Z_s \,\mathrm{d}N_s - \lambda \int_0^t Z_s \,\mathrm{d}s$$

is a martingale with mean zero. Moreover,

$$\begin{aligned}
\mathrm{e}^{-\delta T_i} V(L_{T_i-}^{b^*}) - \mathrm{e}^{-\delta T_{i-1}} V(L_{T_{i-1}}^{b^*}) &= \int_{T_{i-1}}^{T_i-} \Big[ \mathrm{e}^{-\delta s} V(L_s^{b^*}) \Big]' (\mathbb{1}_{\{L_s^{b^*} < b^*\}} + \mathbb{1}_{\{L_s^{b^*} = b^*\}}) \,\mathrm{d}s \\
&= \int_{T_{i-1}}^{T_i-} \mathrm{e}^{-\delta s} \Big[ cV'(L_s^{b^*}) - \delta V(L_s^{b^*}) \Big] \mathbb{1}_{\{L_s^{b^*} < b^*\}} \,\mathrm{d}s \\
&\quad - \int_{T_{i-1}}^{T_i-} \delta\, \mathrm{e}^{-\delta s} V(L_s^{b^*}) \mathbb{1}_{\{L_s^{b^*} = b^*\}} \,\mathrm{d}s
\end{aligned}$$

and

$$\int_0^t \mathrm{e}^{-\delta s}\, Z_s\, \mathrm{d}N_s = \sum_{i=1}^{N_t} \mathrm{e}^{-\delta T_i}\, Z_{T_i}\ .$$

Taking together, we obtain

$$\begin{aligned}
0 \;&=\; \mathbb{E}\Big\{\int_0^t Z_s\, \mathrm{d}N_s - \lambda \int_0^t Z_s\, \mathrm{d}s + 0\Big\} = \mathbb{E}\Big\{\mathrm{e}^{-\delta t}\, V(L_t^{b^*}) - V(x) \\
&\quad - \int_0^t \mathrm{e}^{-\delta s}\Big[cV'(L_s^{b^*}) + \lambda \int_0^\infty V(L_s^{b^*} - y)\, \mathrm{d}F(y) - (\lambda+\delta)V(L_s^{b^*})\Big]\mathbb{1}_{\{L_s^{b^*} < b^*\}}\, \mathrm{d}s \\
&\quad - \int_0^t \mathrm{e}^{-\delta s}\Big[\lambda \int_0^\infty V(L_s^{b^*} - y)\, \mathrm{d}F(y) - (\lambda+\delta)V(L_s^{b^*})\Big]\mathbb{1}_{\{L_s^{b^*} = b^*\}}\, \mathrm{d}s\Big\}\ .
\end{aligned}$$

On $\{L_s^{b^*} < b^*\}$ we have $V'(L_s^{b^*}) > 1$ and therefore the HJB equation implies

$$cV'(L_s^{b^*}) + \lambda \int_0^\infty V(L_s^{b^*} - y)\, \mathrm{d}F(y) - (\lambda+\delta)V(L_s^{b^*}) = \phi(L_s^{b^*})\ .$$

Similarly,

$$\lambda \int_0^\infty V(L_s^{b^*} - y)\, \mathrm{d}F(y) - (\lambda+\delta)V(L_s^{b^*}) = \phi(L_s^{b^*}) - c$$

on $\{L_s^{b^*} = b\}$. Thus,

$$0 = \mathbb{E}\Big[\mathrm{e}^{-\delta t}\, V(L_t^{b^*}) - V(x) + c \int_0^t \mathbb{1}_{\{L_s^{b^*} = b\}}\, \mathrm{d}s - \int_0^t \mathrm{e}^{-\delta s}\, \phi(L_s^{b^*})\, \mathrm{d}s\Big]\ .$$

Letting $t \to \infty$, we get by the bounded convergence theorem and by (3.6) that $V(x) = V^{D^{b^*}}(x)$. □

The next theorem characterises the value function as the minimal solution to the HJB equation.

**Theorem 3.3.** *Let $f$ be a solution to* (3.5) *with* $\mathbb{E}[\mathrm{e}^{-\delta t}\, f(L_t^{b^*})] \to 0$ *as* $t \to \infty$*. Then, we have* $f(x) \geq V(x)$*.*

*Proof.* As in the proof above, one can show that

$$\begin{aligned}
0 &= \mathbb{E}\Big\{\mathrm{e}^{-\delta t} f(L_t^{b^*}) - f(x) \\
&\quad - \int_0^t \mathrm{e}^{-\delta s}\Big[cf'(L_s^{b^*}) + \lambda\int_0^\infty f(L_s^{b^*} - y)\,\mathrm{d}F(y) - (\lambda+\delta)f(L_s^{b^*})\Big]\mathbb{1}_{\{L_s^{b^*}<b^*\}}\,\mathrm{d}s \\
&\quad - \int_0^t \mathrm{e}^{-\delta s}\Big[\lambda\int_0^\infty f(L_s^{b^*} - y)\,\mathrm{d}F(y) - (\lambda+\delta)f(L_s^{b^*})\Big]\mathbb{1}_{\{L_s^{b^*}=b^*\}}\,\mathrm{d}s\Big\}\,.
\end{aligned}$$

Moreover, we obtain from (3.5) that

$$cf'(L_s^{b^*}) + \lambda\int_0^\infty f(L_s^{b^*} - y)\,\mathrm{d}F(y) - (\lambda+\delta)f(L_s^{b^*}) \le \phi(L_s^{b^*})$$

and

$$\lambda\int_0^\infty f(L_s^{b^*} - y)\,\mathrm{d}F(y) - (\lambda+\delta)f(L_s^{b^*}) \le \phi(L_s^{b^*}) - cf'(L_s^{b^*}) \le \phi(L_s^{b^*}) - c\,.$$

Thus,

$$0 \ge \mathbb{E}\Big[\mathrm{e}^{-\delta t} f(L_t^{b^*}) - f(x) + c\int_0^t \mathbb{1}_{\{L_s^{b^*}=b\}}\,\mathrm{d}s - \int_0^t \mathrm{e}^{-\delta s}\,\phi(L_s^{b^*})\,\mathrm{d}s\Big]$$

and the the assertion follows for $t \to \infty$. □

**Remark 3.2.** *In order to solve the HJB equation explicitly, we need an initial condition. Kulenko and Schmidli [43] proposed to determine the value $V(0)$ by comparing the barrier strategies with a barrier $b \ge 0$. Note that in our model it is possible that $b$ becomes negative. Let $V^b$ be the value of a barrier strategy with the barrier $b$. Then, we have similarly as in [43] that*

$$V(0) = \sup_{b\in\mathbb{R}} V^b(0)\,. \tag{3.11}$$

*As in Bühlmann [17] we can show that $V^b$ fulfils*

$$c(V^b)'(x) + \lambda\int_0^\infty V^b(x-y)\,\mathrm{d}F(y) - (\lambda+\delta)V^b(x) - \phi(x) = 0$$

*on $(-\infty, b]$ with $(V^b)'(b) = 1$ and $V^b(x) = V^b(b) + x - b$ on $(b,\infty)$.*

## 3.4 Examples

In our examples we assume that the claim sizes are exponentially distributed. That is $F(y) = (1 - \mathrm{e}^{\gamma y})\mathbb{1}_{y \geq 0}$ for some $\gamma > 0$. Then, $m_1 = 1/\gamma$. Before we consider the examples we have to prove analogous to Lemma 2.2 the following.

**Lemma 3.5.** *Suppose that $f$ is continuously differentiable and concave, and solves*

$$cf'(x) = (\lambda + \delta)f(x) - \lambda \int_0^\infty f(x-y)\,\mathrm{d}F(y) + \phi(x) \tag{3.12}$$

*on $(-\infty, b^*]$ with $f'(b^*) = 1$ and $f''(b^*) = 0$. Moreover, assume that $\phi$ is differentiable for $x \geq b^*$. If $f(x) = f(b^*) + x - b^*$ on $(b^*, \infty)$, then $f$ solves* (3.5).

*Proof.* Note that (3.12) is equivalent to

$$cf'(x) = (\lambda + \delta)f(x) - \gamma\lambda\,\mathrm{e}^{-\gamma x} \int_{-\infty}^x f(z)\,\mathrm{e}^{\gamma z}\ \mathrm{d}z + \phi(x)\ . \tag{3.13}$$

If $\phi(x)$ is differentiable at $x$, the right-hand side of (3.13) is also differentiable at $x$ with

$$cf''(x) = (\lambda + \delta)f'(x) + \gamma^2\lambda\,\mathrm{e}^{-\gamma x} \int_{-\infty}^x f(z)\,\mathrm{e}^{\gamma z}\ \mathrm{d}z - \gamma\lambda f(x) + \phi'(x)\ , \tag{3.14}$$

Plugging (3.13) into (3.14) yields

$$cf''(x) = (\lambda + \delta - \gamma c)f'(x) + \gamma\delta f(x) + \phi'(x) + \gamma\phi(x)\ . \tag{3.15}$$

Thus,

$$0 = cf''(b^*) = \lambda + \delta - \gamma c + \gamma\delta f(b^*) + \phi'(b^*) + \gamma\phi(b^*)\ .$$

Now, we set

$$\begin{aligned} g(x) &= cf'(x) + \lambda \int_0^\infty f(x-y)\,\mathrm{d}F(y) - (\lambda+\delta)f(x) - \phi(x) \\ &= cf'(x) + \gamma\lambda\,\mathrm{e}^{-\gamma x} \int_{-\infty}^x f(z)\,\mathrm{e}^{\gamma z}\ \mathrm{d}z - (\lambda+\delta)f(x) - \phi(x)\ . \end{aligned} \tag{3.16}$$

Then, if $\phi(x)$ is differentiable at $x$, we obtain

$$\begin{aligned} g'(x) &= cf''(x) - \gamma^2\lambda\,\mathrm{e}^{-\gamma x}\int_{-\infty}^{x} f(z)\,\mathrm{e}^{\gamma z}\;\mathrm{d}z + \gamma\lambda f(x) \\ &\quad - (\lambda+\delta)f'(x) + \phi'(x)\;. \end{aligned} \tag{3.17}$$

As above, plugging (3.16) into (3.17) yields

$$g'(x) = -\gamma g(x) + cf''(x) + (\gamma c - \lambda - \delta)f'(x) - \gamma\delta f(x) - \phi'(x) - \gamma\phi(x)\;.$$

In the following we let $x \geq b^*$. Then,

$$g'(x) = -\gamma g(x) + (\gamma c - \lambda - \delta) - \gamma\delta(f(b^*) + x - b^*) - \gamma\phi(x) - \phi'(x)\;.$$

As in Lemma 2.2 it holds $\phi(x) \geq \phi(b^*) - \delta(x - b^*)$. Moreover, by the convexity of $\phi$, we obtain $-\phi'(x) \leq -\phi'(b^*)$. Taking together, we get

$$g'(x) \leq -\gamma g(x) + \gamma c - \lambda - \delta - \gamma\delta f(b^*) - \gamma\phi(b^*) - \phi'(b^*) = -\gamma g(x)\;.$$

In conclusion, assume $g(x) \geq 0$. Then, $g'(x) \leq 0$ and therefore $0 = g(b^*) \geq g(x)$. Thus, $g(x) \leq 0$. □

### 3.4.1 Exponential Penalty Payments

In this section we consider the function $\phi(x) = \alpha\,\mathrm{e}^{-\beta x}$ with $\alpha, \beta > 0$. Note that (3.4) is fulfiled for

$$x < y < x_0 = -\beta^{-1}\max\{\log\delta - \log(\alpha\beta), 0\}\;.$$

Let $M_Y(r) = \mathbb{E}[\mathrm{e}^{rY}]$ denote the moment-generating function of the claim sizes. Then,

$$\mathbb{E}[\mathrm{e}^{-\beta L_t - \delta t}] = \exp\Big[-\beta(x + ct) + \lambda t(M_Y(\beta) - 1) - \delta t\Big]\;.$$

If the claim sizes are exponentially distributed, $M_Y(\beta)$ only exists if $\beta < \gamma$. In this case we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\beta L_t - \delta t}] &= \exp\Big[-\beta(x+ct) + \lambda t \frac{\beta}{\gamma-\beta} - \delta t\Big] \\
&= \exp\Big[-\beta x + t \frac{c\beta^2 + (\lambda+\delta-\gamma c)\beta - \gamma\delta}{\gamma - \beta}\Big] .
\end{aligned}$$

Thus, (4.4) is fulfilled if

$$\beta < \max(\gamma, -\xi_2) , \tag{3.18}$$

where $\xi_2 < 0 < \xi_1$ are the roots of the equation

$$c\xi^2 - (\lambda + \delta - \gamma c)\xi - \gamma\delta = 0 .$$

Then,

$$V(x) \geq -\alpha \int_0^\infty \mathbb{E}[\mathrm{e}^{-\beta L_t - \delta t}] \, \mathrm{d}t = -A \, \mathrm{e}^{-\beta x} ,$$

where

$$A = -\frac{\alpha(\gamma - \beta)}{c\beta^2 + (\lambda + \delta - \gamma c)\beta - \gamma\delta} .$$

If (3.18) is not fulfilled we have $V(x) = \infty$.

For $x > b^*$ we have $V(x) = V(b^*) + x - b^*$. On $(-\infty, b^*]$ the value function fulfils

$$\begin{aligned}
cV'(x) &= (\lambda+\delta)V(x) - \lambda \int_0^\infty V(x-y) \, \mathrm{d}F(y) + \alpha \, \mathrm{e}^{-\beta x} \\
&= (\lambda+\delta)V(x) - \gamma\lambda \, \mathrm{e}^{-\gamma x} \int_{-\infty}^x V(z) \, \mathrm{e}^{\gamma z} \, \mathrm{d}z + \alpha \, \mathrm{e}^{-\beta x} . \qquad (3.19)
\end{aligned}$$

Obviously, the right-hand side is differentiable and therefore

$$cV''(x) = (\lambda+\delta)V'(x) + \gamma^2\lambda \, \mathrm{e}^{-\gamma x} \int_{-\infty}^x V(z) \, \mathrm{e}^{\gamma z} \, \mathrm{d}z - \gamma\lambda V(x) - \beta\alpha \, \mathrm{e}^{-\beta x} .$$

Using (3.19), we obtain

$$cV''(x) = (\lambda + \delta - \gamma c)V'(x) + \gamma\delta V(x) + \alpha(\gamma - \beta) \, \mathrm{e}^{-\beta x} .$$

This equation is solved by

$$V(x) = C_1 \,\mathrm{e}^{\xi_1 x} + C_2 \,\mathrm{e}^{\xi_2 x} - A \,\mathrm{e}^{-\beta x} \ ,$$

where $\xi_1, \xi_2$ are defined above and $C_1, C_2$ are some constants. Note that this solution solves equation (3.19) even though it was derived by differentiation of it. Now, since $\xi_1 > 0 > -\beta > \xi_2$, we obtain that $V(x)$ is only increasing for $x$ small enough if $C_2 \leq 0$. Furthermore, if $C_2 < 0$ we have $V(x) < -A \,\mathrm{e}^{-\beta x}$ for $x$ small enough. Thus, it must hold that $C_2 = 0$. By $V'(b^*) = 1$ we obtain

$$C_1 = \frac{1 - \beta A \,\mathrm{e}^{-\beta b^*}}{\xi_1 \,\mathrm{e}^{\xi_1 b^*}} \ .$$

The optimal barrier $b^*$ is calculated through (3.11). That is, $b^*$ maximises the function

$$g(b) = \frac{1 - \beta A \,\mathrm{e}^{-\beta b}}{\xi_1 \,\mathrm{e}^{\xi_1 b}} - A \ .$$

Solving

$$0 = g'(b) = \frac{(\beta + \xi_1)\beta A \,\mathrm{e}^{-(\beta+\xi_1) b} - \xi_1 \,\mathrm{e}^{-\xi_1 b}}{\xi_1}$$

we obtain

$$b^* = -\frac{1}{\beta} \log\Big( \frac{\xi_1}{(\beta + \xi_1)\beta A} \Big) \ .$$

Since

$$\begin{aligned} g''(b^*) &= \frac{\xi_1^2 \,\mathrm{e}^{-\xi_1 b^*} - (\beta + \xi_1)^2 \beta A \,\mathrm{e}^{-(\beta+\xi_1) b^*}}{\xi_1} \\ &= \frac{\xi_1^2 \,\mathrm{e}^{-\xi_1 b^*} - (\beta + \xi_1)\xi_1 \,\mathrm{e}^{-\xi_1 b^*}}{\xi_1} = -\beta \,\mathrm{e}^{-\xi_1 b^*} < 0 \ , \end{aligned}$$

we obtain that $b^*$ is a maximum of $g$. Note that

$$V''(b^*) = \xi_1 - \xi_1 \beta A \,\mathrm{e}^{-\beta b^*} - A\beta^2 \,\mathrm{e}^{-\beta b^*} = 0 \ .$$

Thus, Theorem 3.2 and Lemma 3.5 yield that $D^*$ is optimal with the barrier $b^*$ and

$$V(x) = \begin{cases} C_1 \mathrm{e}^{\xi_1 x} - A \mathrm{e}^{-\beta x}, & x \leq b^*, \\ C_1 \mathrm{e}^{\xi_1 b^*} - A \mathrm{e}^{-\beta b^*} + x - b^*, & x > b^*. \end{cases}$$

In Figure 3.1 the value function is shown for $c = \gamma = \lambda = 1$, $\alpha = 0.3$ and $\beta = \delta = 0.1$. In this case we have $b^* = -8.47049$.

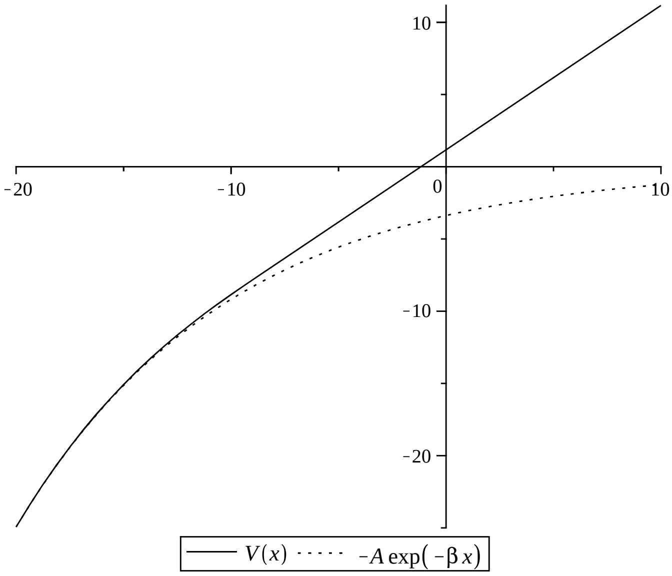


Figure 3.1: Value function for $c = \gamma = \lambda = 1$, $\alpha = 0.3$ and $\beta = \delta = 0.1$.

### 3.4.2 Linear Penalty Payments

Now, we let $\phi(x) = -\alpha x \mathbb{1}_{x<0}$ for some $\alpha > 0$. That is, for a negative surplus of $x$ the insurer has to borrow an amount of $-x$ at rate $\alpha$ in order to avoid bankruptcy. Obviously, (4.4) holds. Moreover, (3.4) is also fulfilled if $\alpha > \delta$, where $x_0 = 0$. The following can be proved analogously as in the proof of Lemma 2.4.

**Lemma 3.6.** *i) If $\alpha < \delta$, an optimal strategy does not exist and $V(x) = \infty$.*

*ii) Let*

$$f(x) = \frac{\alpha(\delta x + c - \lambda m_1)}{\delta^2} \; .$$

*Then, for $\alpha > \delta$, it holds $V(x) \leq f(x)$. Moreover, $V(x) \geq f(x) + C$ for some $C < 0$ if $x \leq 0$.*

*iii) Let $\delta = \alpha$, then $V(x) = f(x)$.*

In the following we assume that

$$\alpha > \delta \; . \tag{3.20}$$

Then, the dividend barrier $b^*$ must be positive or equal to zero, because it cannot be optimal to pay dividends if the surplus is negative. On $(-\infty, 0]$ we have

$$cV'(x) = (\lambda + \delta)V(x) - \gamma\lambda \, \mathrm{e}^{-\gamma x} \int_{-\infty}^{x} V(z) \, \mathrm{e}^{\gamma z} \; \mathrm{d}z - \alpha x \; .$$

As above we get by differentiation

$$cV''(x) = (\lambda + \delta - \gamma c)V'(x) + \gamma\delta V(x) - \alpha(\gamma x + 1) \; .$$

Here, a solution is given by

$$V_1(x) = C_1 \, \mathrm{e}^{\xi_1 x} + C_2 \, \mathrm{e}^{\xi_2 x} + \frac{\alpha(\delta x + c - \lambda m_1)}{\delta^2} \; ,$$

where $C_1, C_2$ some constants and $\xi_1, \xi_2$ as above. Since $V$ is linearly bounded, $C_2 = 0$ must hold. On $(0, b^*]$ the HJB equation is solved by

$$V_2(x) = C_3 \, \mathrm{e}^{\xi_1 x} + C_4 \, \mathrm{e}^{\xi_2 x}$$

for some constants $C_3, C_4$. If $b^* = 0$ we do not have to consider $V_2(x)$.

In order to determine the optimal dividend barrier we need to calculate $V^b(0)$, where $V^b$ denotes the value of a barrier strategy with the barrier $b$. For $b = 0$ we have

$$V^0(x) = \begin{cases} C_1^0 \, \mathrm{e}^{\xi_1 x} + \frac{\alpha(\delta x + c - \lambda m_1)}{\delta^2}, & x \leq 0, \\ C_1^0 + \frac{\alpha(c - \lambda m_1)}{\delta^2} + x, & x > 0, \end{cases}$$

where

$$C_1^0 = \frac{\delta - \alpha}{\delta \xi_1} \ .$$

If $b > 0$ it holds

$$V^b(x) = V_+^b(x) = \begin{cases} C_1^+ \, \mathrm{e}^{\xi_1 x} + \frac{\alpha(\delta x + c - \lambda m_1)}{\delta^2}, & x \leq 0, \\ C_3^+ \, \mathrm{e}^{\xi_1 x} + C_4^+ \, \mathrm{e}^{\xi_2 x}, & 0 < x \leq b, \\ C_3^+ \, \mathrm{e}^{\xi_1 b} + C_4^+ \, \mathrm{e}^{\xi_2 b} + x - b, & x > b, \end{cases}$$

where $C_1^+$, $C_3^+$ and $C_4^+$ determined such that $V_+^b(0-) = V_+^b(0+)$, $(V_+^b)'(0-) = (V_+^b)'(0+)$ and $(V_+^b)'(b) = 1$. That is,

$$C_1^+ = C_3^+ + C_4^+ + \frac{\lambda m_1 - c}{\delta^2} \ ,$$

$$C_3^+ = \frac{1 - \xi_2 C_4^+ \, \mathrm{e}^{\xi_2 b}}{\xi_1 \, \mathrm{e}^{\xi_1 b}}$$

and

$$C_4^+ = \frac{\alpha(\lambda \xi_1 + \gamma \delta - \gamma c \xi_1)}{\gamma \delta^2 (\xi_2 - \xi_1)} \ .$$

Now, $b^*$ is the maximum of

$$g(b) = \begin{cases} V^0(0), & b = 0, \\ V_+^b(0), & b > 0 \end{cases} \ .$$

Note that

$$g(0-) = C_1^0 + \frac{\alpha(c - \lambda m_1)}{\delta^2} = \frac{1 - \xi_2 C_4}{\xi_1} + C_4 = g(0+) \ .$$

Thus, $g$ is continuous at $b = 0$. Moreover, since

$$\lambda \xi_1 + \gamma \delta - \gamma c \xi_1 = -\frac{\gamma \delta \xi_1}{\xi_2} - \delta \xi_1 \ ,$$

we obtain for $b > 0$ that

$$\begin{aligned} g'(b) &= -\frac{\xi_1 \, \mathrm{e}^{-\xi_1 b} + (\xi_2 - \xi_1) \xi_2 C_4^+ \, \mathrm{e}^{(\xi_2 - \xi_1) b}}{\xi_1} \\ &= -\mathrm{e}^{-\xi_1 b} + \zeta \, \mathrm{e}^{(\xi_2 - \xi_1) b} \\ &= \mathrm{e}^{-\xi_1 b} (\zeta \, \mathrm{e}^{\xi_2 b} - 1) \ , \end{aligned}$$

where

$$\zeta = \frac{\alpha(\gamma+\xi_2)}{\delta\gamma}\ .$$

If $\zeta \le 1$, we obtain $g'(b) \le 0$ for $b > 0$. This yields $b^* = 0$. If $\zeta > 1$, we obtain $g'(b_0) = 0$, where

$$b_0 = -\frac{1}{\xi_2}\log(\zeta)\ .$$

and

$$g''(b_0) = \xi_2 \zeta^{\frac{\xi_2}{\xi_1}} < 0\ .$$

Moreover, $(V_+^{b_0})''(b_0) = 0$ if $\zeta > 1$. In the case where $b^* = 0$ we cannot apply Lemma 3.5. Thus we have to prove the following.

**Lemma 3.7.** *If $\zeta \le 1$, we obtain that $V^0(x)$ fulfills* (3.5)*.*

*Proof.* We set

$$h(x) = cf'(x) + \lambda\int_0^\infty f(x-y)\,\mathrm{d}F(y) - (\lambda+\delta)f(x) - \phi(x)\ .$$

For $x > 0$ it holds

$$\begin{aligned}
h(x) &= c + \lambda\int_0^x (f(0)+x-y)\gamma\,\mathrm{e}^{\gamma y}\ \mathrm{d}y - (\lambda+\delta)(f(0)+x) \\
&\quad + \lambda\int_x^\infty \Big[C_1\,\mathrm{e}^{\xi(x-y)} + \frac{\alpha(\gamma\delta(x-y)+\gamma c-\lambda)}{\gamma\delta^2}\Big]\gamma\,\mathrm{e}^{\gamma y}\ \mathrm{d}y \\
&= c + \frac{\lambda(1-\gamma f(0))}{\gamma}\,\mathrm{e}^{-\gamma x} + \frac{\lambda(\gamma f(0)+\gamma x-1)}{\gamma} - (\lambda+\delta)(f(0)+x) \\
&\quad + \frac{\lambda\gamma C_1}{\gamma+\xi_1}\,\mathrm{e}^{-\gamma x} + \frac{\lambda\alpha(\gamma c-\lambda-\delta)}{\gamma\delta^2}\,\mathrm{e}^{-\gamma x} \\
&= c + \frac{\lambda}{\gamma}\Big[1 - \frac{\gamma(\delta-\alpha)}{\delta\xi_1} - \frac{\alpha(\gamma c-\lambda)}{\delta^2}\Big]\,\mathrm{e}^{-\gamma x} - \frac{\lambda}{\gamma} - \delta\Big[\frac{\delta-\alpha}{\delta\xi_1} + \frac{\alpha(\gamma c-\lambda)}{\gamma\delta^2}\Big] - \delta x \\
&\quad + \frac{\lambda\gamma(\delta-\alpha)}{(\gamma+\xi_1)\delta\xi_1}\,\mathrm{e}^{-\gamma x} + \frac{\lambda\alpha(\gamma c-\lambda-\delta)}{\gamma\delta^2}\,\mathrm{e}^{-\gamma x}
\end{aligned}$$

Note that $\gamma+\xi_2 > 0$. Thus, from $\zeta \le 1$ it follows $\alpha \le \delta\gamma/(\gamma+\xi_2)$. Moreover, $\alpha > \delta$, $\xi_1\xi_2 = -\gamma\delta/c$ and $\xi_1+\xi_2 = (\lambda+\delta-\gamma c)/c$. Therefore, for $x > 0$ we obtain

$$\begin{aligned}
h'(x) &= \frac{\lambda\xi_1(\alpha-\delta)\,\mathrm{e}^{-\gamma x} - \gamma\delta^2 - \delta^2\xi_1}{\delta(\gamma+\xi_1)} \\
&\leq \frac{\lambda\xi_1(\alpha-\delta) - \gamma\delta^2 - \delta^2\xi_1}{\delta(\gamma+\xi_1)} \\
&\leq \frac{\lambda\xi_1\delta\gamma - \lambda\xi_1\delta\gamma - \lambda\xi_1\xi_2\delta - \gamma^2\delta^2 - \gamma\delta^2\xi_2 - \gamma\delta^2\xi_1 - \delta^2\xi_1\xi_2}{(\gamma+\xi_2)\delta(\gamma+\xi_1)} \\
&= \frac{\lambda\gamma\delta^2 - \gamma^2\delta^2 c + (\gamma c - \lambda - \delta)\gamma\delta^2 + \delta^3\gamma}{c(\gamma+\xi_2)\delta(\gamma+\xi_1)} = 0\,.
\end{aligned}$$

Thus, for $x > 0$ we have $h(x) \leq h(0) = 0$. □

In sum we get by Theorem 3.2, Lemma 3.5 and Lemma 3.7 that

$$b^* = b_0 \vee 0$$

and

$$V(x) = \begin{cases} V^0(x), & \zeta \leq 1, \\ V_+^{b_0}(x), & \zeta > 1, \end{cases}$$

where $D^*$ is optimal with the barrier $b^*$. Figure 3.2 illustrates the value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$ and $\alpha = 0.2$. The optimal dividend barrier is given by $b^* = 0.33408$ and $\zeta = 1.15215$. Figure 3.3 illustrates the value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 2$ and $\alpha = 0.11$. Here, $b^* = 0$ and $\zeta = 0.50356$.

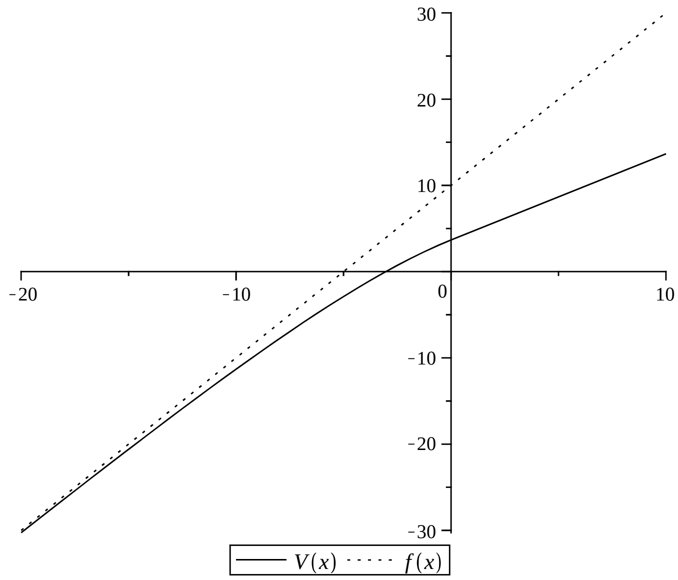


Figure 3.2: Value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$ and $\alpha = 0.2$.

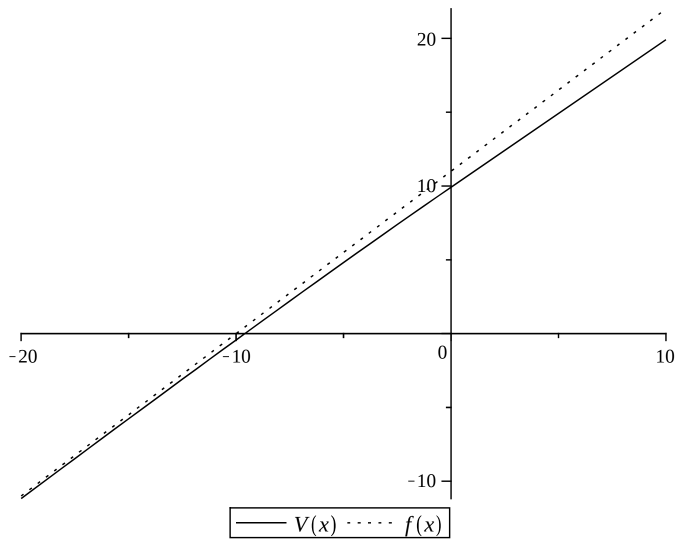


Figure 3.3: Value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 2$ and $\alpha = 0.11$.

### 3.4.3 Quadratic Penalty Payments

In this section we consider a quadratic function $\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0}$, where $\alpha_1, \alpha_2 > 0$. Here, we have

$$x_0 = -\frac{1}{2}\frac{\alpha_1 + \delta}{\alpha_2} \, .$$

Therefore, it is possible that the optimal dividend barrier is negative. If $b^*$ is positive, we have as above on $(-\infty, 0]$ that

$$cV'(x) = (\lambda + \delta)V(x) - \gamma\lambda \, \mathrm{e}^{-\gamma x} \int_{-\infty}^{x} V(z) \, \mathrm{e}^{\gamma z} \, \mathrm{d}z + \alpha_2 x^2 - \alpha_1 x$$

and by differentiation we get

$$cV''(x) = (\lambda + \delta - \gamma c)V'(x) + \gamma\delta V(x) + \gamma\alpha_2 x^2 + (2\alpha_2 - \alpha_1)x - \alpha_1 \, .$$

Here, a solution is given by

$$V_1(x) = C_1 \, \mathrm{e}^{\xi_1 x} + C_2 \, \mathrm{e}^{\xi_2 x} + h(x) \, ,$$

where $C_1, C_2$ some constants, $\xi_1, \xi_2$ as above and $h(x) = p_0 + p_1 x + p_2 x^2$ with

$$p_0 = \frac{\gamma\delta(\alpha_1\delta - 2c\alpha_2) + (\lambda + \delta - \gamma c)(2\alpha_2\delta - \alpha_1\delta - 2\alpha_2(\lambda + \delta - \gamma c))}{\delta^3\gamma^2} \, ,$$

$$p_1 = \frac{2\alpha_2\lambda + \alpha_1\delta - 2\alpha_2 c\gamma}{\gamma\delta^2}$$

and

$$p_2 = -\frac{\alpha_2}{\delta} \, .$$

Since $V$ is quadratically bounded, $C_2 = 0$ must hold. On $(0, b^*]$ the HJB equation is solved by

$$V_2(x) = C_3 \, \mathrm{e}^{\xi_1 x} + C_4 \, \mathrm{e}^{\xi_2 x} \, ,$$

where $C_3, C_4$ some constants. If $b^*$ is negative we do not have to consider $V_2(x)$.

Again, we first calculate $V^b(0)$, where $V^b$ denotes the value of a barrier strategy with the barrier $b$. For $b \le 0$ we have

$$V^b(x) = V^b_-(x) = \begin{cases} C_1^- \,\mathrm{e}^{\xi_1 x} + h(x), & x \le b, \\ C_1^- \,\mathrm{e}^{\xi_1 b} + h(b) + x - b, & x > b, \end{cases}$$

where $C_1^-$ is given by $(V^b_-)'(b) = 1$, i.e.

$$C_1^- = \frac{1 - 2p_2 b - p_1}{\xi_1 \,\mathrm{e}^{\xi_1 b}} \;.$$

If $b > 0$ it holds

$$V^b(x) = V^b_+(x) = \begin{cases} C_1^+ \,\mathrm{e}^{\xi_1 x} + h(x), & x \le 0, \\ C_3^+ \,\mathrm{e}^{\xi_1 x} + C_4^+ \,\mathrm{e}^{\xi_2 x}, & 0 < x \le b, \\ C_3^+ \,\mathrm{e}^{\xi_1 b} + C_4^+ \,\mathrm{e}^{\xi_2 b} + x - b, & x > b, \end{cases}$$

where $C_1^+$, $C_3^+$ and $C_4^+$ determined such that $V^b_+(0-) = V^b_+(0+)$, $(V^b_+)'(0-) = (V^b_+)'(0+)$ and $(V^b_+)'(b) = 1$. That is,

$$C_1^+ = C_3^+ + C_4^+ - p_0 \;,$$
$$C_3^+ = \frac{1 - \xi_2 C_4^+ \,\mathrm{e}^{\xi_2 b}}{\xi_1 \,\mathrm{e}^{\xi_1 b}}$$

and

$$C_4^+ = \frac{\xi_1 p_0 - p_1}{\xi_1 - \xi_2} \;.$$

As above, $b^*$ is the maximum of

$$\begin{aligned} g(b) &= \begin{cases} V^b_-(0), & b \le 0, \\ V^b_+(0), & b > 0 \end{cases} \\ &= p_0 + \begin{cases} C_1^-, & b \le 0, \\ C_1^+, & b > 0. \end{cases} \\ &= \frac{1}{\xi_1} \begin{cases} \mathrm{e}^{-\xi_1 b}(1 - 2p_2 b - p_1) + p_0 \xi_1, & b \le 0, \\ \mathrm{e}^{-\xi_1 b} + \frac{\xi_1 p_0 - p_1}{\xi_1 - \xi_2}(\xi_1 - \xi_2 \,\mathrm{e}^{(\xi_2 - \xi_1) b}), & b > 0. \end{cases} \end{aligned}$$

and $g(0-) = \frac{p_0\xi_1+1-p_1}{\xi_1} = g(0+)$. Now, define

$$b^+ = -\frac{1}{\xi_2}\log\Big(\frac{\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma)}{\gamma\delta^2}\Big)$$

and

$$b^- = \frac{1}{2}\frac{\alpha_1\delta + 2\alpha_2\lambda - 2c\alpha_2(\xi_2+\gamma) - \gamma\delta^2}{\alpha_2\gamma\delta} .$$

Let us first assume that $b^+ > 0$ (in particular $b^- > 0$). Then, $g'(b^+) = 0$ and

$$g''(b^+) = \xi_2\Big(\frac{\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma)}{\gamma\delta^2}\Big)^{\frac{\xi_1}{\xi_2}} < 0 .$$

Moreover, for $b < 0$, we have

$$\begin{aligned} g'(b) &= \frac{(2\xi_1 p_2 b + \xi_1 p_1 - 2p_2 - \xi_1)\xi_2}{\xi_1\xi_2}\,\mathrm{e}^{-\xi_1 b} \\ &> \frac{(\xi_1 p_1 - 2p_2 - \xi_1)\xi_2}{\xi_1\xi_2}\,\mathrm{e}^{-\xi_1 b} \\ &= \frac{\alpha_1\delta + 2\alpha_2\lambda - 2c\alpha_2(\xi_2+\gamma) - \gamma\delta^2}{\gamma\delta^2}\,\mathrm{e}^{-\xi_1 b} \\ &> \frac{2\alpha_2 b^-}{\delta}\,\mathrm{e}^{-\xi_1 b} > 0 , \end{aligned}$$

where we used that $\xi_1\xi_2 = \gamma\delta/c$. Since $g$ is continuous, we obtain that $b^+$ maximises $V^b(0)$ and therefore $b^* = b^+$. Now, assume that $b^- < 0$ (in particular $b^+$ does not exists or is negative). As above $g'(b^-) = 0$ and

$$g''(b^-) = -\frac{2\alpha_2}{\delta}\,\mathrm{e}^{-\xi_1 b^-} < 0 .$$

For $b > 0$ it holds

$$g'(b) = \frac{\xi_2(\xi_1 p_0 - p_1)\,\mathrm{e}^{(\xi_2-\xi_1)b} - \xi_1\,\mathrm{e}^{-\xi_1 b}}{\xi_1} .$$

If $\xi_1 p_0 - p_1 > 0$, we obtain directly $g'(b) < 0$ for $b > 0$. Moreover, by $\xi_1(\lambda+\delta-\gamma c) + \gamma\delta = -\gamma\delta\xi_1/\xi_2$ and $\xi_1\xi_2 = \gamma\delta/c$ we obtain

$$\frac{\xi_2(\xi_1 p_0 - p_1) - \xi_1}{\xi_1} = \frac{\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma) - \gamma\delta^2}{\gamma\delta^2} .$$

Thus, if $\xi_1 p_0 - p_1 \le 0$, we have

$$\begin{aligned} g'(b) &\le \frac{\xi_2(\xi_1 p_0 - p_1) - \xi_1}{\xi_1}\,\mathrm{e}^{-\xi_1 b} \\ &= \frac{\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma) - \gamma\delta^2}{\gamma\delta^2}\,\mathrm{e}^{-\xi_1 b} \\ &= \frac{2\alpha_2 b^-}{\delta}\,\mathrm{e}^{-\xi_1 b} < 0 \end{aligned}$$

for $b < 0$. As above, we obtain $b^* = b^-$ if $b^- < 0$. Note that

$$\begin{aligned} (b^- - x_0)2\alpha_2\gamma\delta &= \alpha_1\delta + 2\alpha_2\lambda + \alpha_1\gamma\delta - 2c\alpha_2(\xi_2+\gamma) \\ &> 2\alpha_2(\lambda - c\xi_2 - c\gamma) \\ &= \frac{\lambda - \delta - \gamma c + \sqrt{(\lambda+\delta-\gamma c)^2 + 4\gamma\delta c}}{2} \\ &= \frac{\lambda - \delta - \gamma c + \sqrt{(\lambda-\delta-\gamma c)^2 + 4\lambda\delta}}{2} > 0 \end{aligned}$$

and therefore $b^- > x_0$.

In conclusion, let us consider the cases where $b^* = 0$. If $b^- > 0$, we obtain as above that $g'(b) > 0$ for $b < 0$. If $b^+$ does not exists, we have

$$\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma) \le 0\,.$$

This implies $\xi_1 p_0 - p_1 \ge 0$ and therefore $g'(b) < 0$ for $b > 0$. If $b^+$ is negative, we obtain for $b > 0$ that

$$\begin{aligned} g'(b) &\le \frac{\xi_2(\xi_1 p_0 - p_1) - \xi_1}{\xi_1}\,\mathrm{e}^{(\xi_2-\xi_1)b} \\ &= \frac{\alpha_1\delta + 2\alpha_2\lambda + \alpha_1\delta\xi_2 - 2c\alpha_2(\xi_2+\gamma) - \gamma\delta^2}{\gamma\delta^2}\,\mathrm{e}^{(\xi_2-\xi_1)b} < 0\,. \end{aligned}$$

Note that $(V_-^{b^-})''(b^-) = 0$ if $b^- < 0$ and $(V_+^{b^+})''(b^+) = 0$ if $b^+ > 0$. Moreover, similar as in Lemma 3.7 one can show that $V_-^0$ fulfils the HJB equation if $b^- \geq 0$ and $b^+ \leq 0$ or $b^+$ does not exist. In sum, we get the following. The optimal dividend barrier is given by

$$b^* = \begin{cases} b^-, & b^- < 0, \\ 0, & b^- \geq 0 \wedge (b^+ \leq 0 \vee b^+ \text{ does not exists}) \\ b^+, & b^+ > 0. \end{cases}$$

Moreover,

$$V(x) = \begin{cases} V_-^{b^-}(x), & b^- < 0, \\ V_-^0(x), & b^- \geq 0 \wedge (b^+ \leq 0 \vee b^+ \text{ does not exists}) \\ V_+^{b^+}(x), & b^+ > 0, \end{cases}$$

where $V_-^{b^-}(x), V_-^0(x)$ and $V_+^{b^+}$ as above.

In Figure 3.4 the value function is shown for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 4$, $\alpha_1 = 0.02$ and $\alpha_2 = 0.01$. In this case we have $b^* = b^- = -3.68071$ and $b^+ = -2.88519$. Figure 3.5 illustrates the value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$, $\alpha_1 = 0.05$ and $\alpha_2 = 0.02$. Here, it holds $b^- = 0.10889$ and $b^+ = -0.43500$. Thus, the optimal dividend barrier is 0. Figure 3.6, shows the value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$, $\alpha_1 = \alpha_2 = 0.1$ with a positive barrier $b^* = b^+ = 2.81196$, where $b^- = 1.35889$.

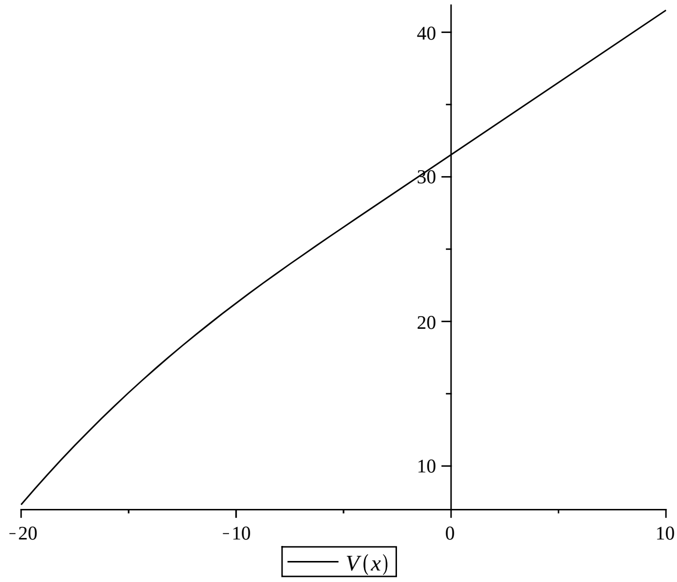


Figure 3.4: Value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 4$, $\alpha_1 = 0.02$ and $\alpha_2 = 0.01$.

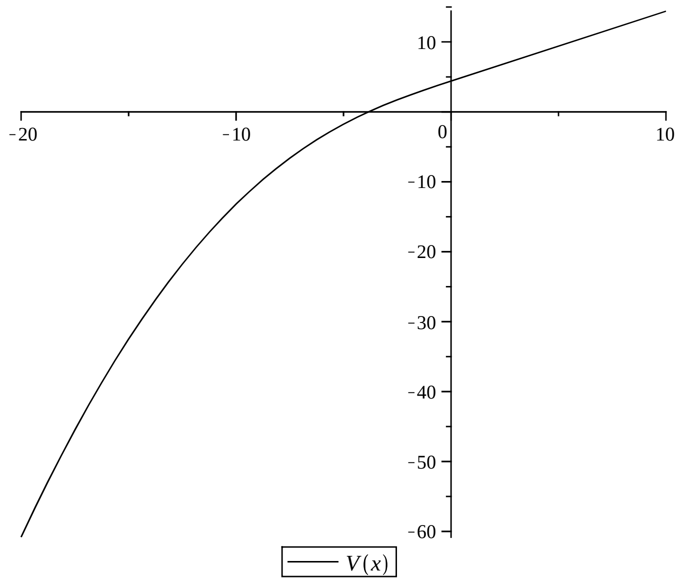


Figure 3.5: Value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$, $\alpha_1 = 0.05$ and $\alpha_2 = 0.02$.

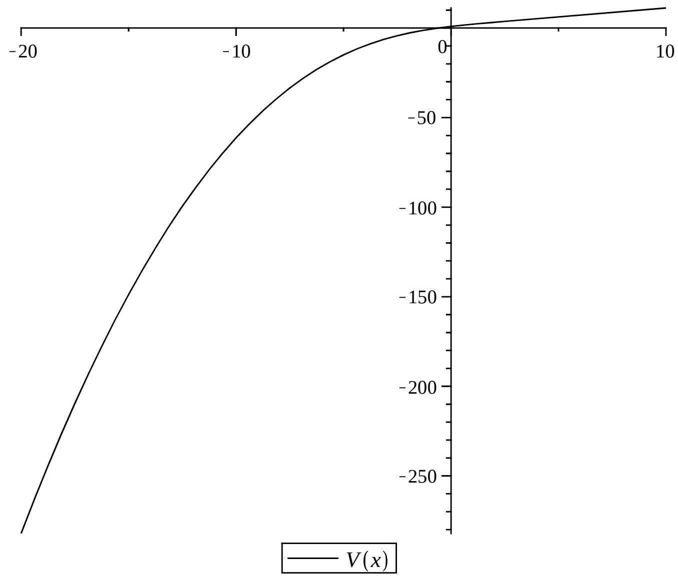


Figure 3.6: Value function for $\gamma = \lambda = 1$, $\delta = 0.1$, $c = 1.5$, $\alpha_1 = \alpha_2 = 0.1$.

# Chapter 4

# Minimisation of Penalty Payments by Investments and Reinsurance in a Diffusion Model

## 4.1 Introduction

This chapter studies the investment and reinsurance problem described in Section 1.7.2, where the surplus follows a diffusion process. In the Cramér–Lundberg model, the surplus is given by

$$L_t = x + ct - \sum_{i=1}^{N_t} Y_i \ , \tag{4.1}$$

where $x$ denotes the initial capital, $c > 0$ a constant premium rate, $N_t$ the amount of claims arriving until time $t$ and $Y_i$ the claim size of the $i$-th claim. Moreover, we consider the net value principle. That is, the premium rate is given by

$$c = (1+\eta)\lambda\mathbb{E}(Y) = (1+\eta)\lambda m_1 \ ,$$

where $\eta > 0$ denotes the safety loading of the insurer.

The insurer has the possibility to buy excess of loss or proportional reinsurance for individual claims. For a reinsurance strategy $0 \le R_t \le \infty$ the controlled surplus is given by

$$L_t^R = x + \lambda(1+\rho)\int_0^t \mathbb{E}[s(R_s, Y)]\ \mathrm{d}s - \lambda(\rho-\eta)m_1 t - \sum_{i=1}^{N_t} s(R_{T_i}, Y_i)\ .$$

In Section 1.2 we have already introduced a diffusion approximation to the uncontrolled Cramér-Lundberg process. The next lemma gives a motivation for an approximation to the Cramér–Lundberg process that is controlled by a reinsurance strategy.

**Lemma 4.1.** *Let $\mu\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and $X, X^n$, $n \in \mathbb{N}$ semi-martingales such that $X_0 = X_0^n = 0$. Further assume that $Y^n$ fulfils the equation*

$$Y_t^n = x + X_t^n + \int_0^t \mu(Y_s^n)\ \mathrm{d}s$$

*and $Y$ fulfils the equation*

$$Y_t = x + X_t^n + \int_0^t \mu(Y_s)\ \mathrm{d}s\ .$$

*Then, $Y^n$ converges weakly to $Y$ if and only if $X^n$ converges weakly to $X$.*

*Proof.* For proof see Schmidli [59]. □

Given a reinsurance strategy $R_t$ we now assume that the surplus fulfils

$$X_t^R = x + \lambda\rho\int_0^t \mathbb{E}[s(R_s, Y)]\ \mathrm{d}s - \lambda(\rho-\eta)m_1 t + \int_0^t \sqrt{\lambda\mathbb{E}[s(R_s, Y)^2]}\ \mathrm{d}W_s\ .$$

In addition, the insurance company has the possibility to invest in $n$ risky assets, modelled by

$$\mathrm{d}Z_t^i = a_i Z_t^i\ \mathrm{d}t + Z_t^i \sum_{j=1}^n v_{ij}\ \mathrm{d}B_t^j, \quad S_0^i = 1$$

for $i = 1, 2, ..., n$. Here, $B^1, B^2, ..., B^n$ and $W$ are independent Wiener processes and $a_i, v_{ij} \geq 0$, $i, j = 1, 2, ..., n$. We assume that the matrix $v = (v_{ij})_{i,j=1,2,...,n}$ of volatilities is invertible. Then, the covariance matrix $vv^T$ is positive definite. The insurer can choose an investment strategy $\theta_t = (\theta_t^1, \theta_t^2, ..., \theta_t^n)^T$, where $\theta_t^i < \infty$ describes the amount being invested into the $i$-th asset at time $t$. For a control strategy $U = (R, \theta^T)^T$ the surplus is governed by

$$\begin{aligned} \mathrm{d}X_t^U &= \lambda(\rho\mathbb{E}[s(R_t, Y)] - (\rho - \eta)m_1)\,\mathrm{d}t + \sqrt{\lambda\mathbb{E}[s(R_t, Y)^2]}\,\mathrm{d}W_t \\ &\quad + \sum_{i=1}^{n} a_i\theta_t^i\,\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_t^i v_{ij}\,\mathrm{d}B_t^j\;. \end{aligned} \tag{4.2}$$

For simplicity of notation we set $(R, \theta) = (R, \theta^T)^T$ in the following. To ensure that the differential equation (4.2) is well-defined we require that

$$\int_0^t (\theta_s^i)^2\,\mathrm{d}s < \infty$$

for $t > 0$ and $i = 1, 2, ..., n$.

In order to prevent bankruptcy, the insurer has to pay penalty payments at a rate $\phi(x)$, where $\phi(x)$ is the convex, decreasing and positive penalty function vanishing at infinity. The value of a strategy $U$ is given by

$$V^U(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\,\phi(X_t^U)\,\mathrm{d}t\;\Big|\;X_0^U = x\Big]\;, \tag{4.3}$$

where $\delta > 0$ denotes a preference parameter. The insurer aims to minimise the penalty payments. That is, we consider the control problem

$$V(x) = \inf_{U\in\mathcal{U}} V^U(x)\;.$$

Let $\mathrm{cad}(\mathcal{F})$ be the set of all càdlàg processes being adapted to $\mathcal{F}_t = \sigma(X_t, t \geq 0)$. We only consider adapted càdlàg processes and at any time it is not allowed to invest an infinite amount. Thus, $\mathcal{U} \subset \mathrm{cad}(\mathcal{F})$ and for an admissible strategy

$U \in \mathcal{U}$ it holds $\tau_k^U \to \infty$ as $k \to \infty$, where $\tau_k^U = \inf(t > 0 : |X_t^U| > k)$ and $\inf \emptyset = \infty$. As in the chapters above, $\phi$ has to fulfil

$$\int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(X_t)] < \infty \,. \tag{4.4}$$

This chapter is organised as follows. In section 2 we motivate the HJB equation and prove a verification theorem for a general penalty function $\phi$. Section 3 considers the control problem with an exponential penalty function $\phi(x) = \alpha\, \mathrm{e}^{-\beta x}$, where $\alpha, \beta > 0$. We show that the optimal investment and reinsurance strategy is constant and determine an explicite solution. Section 4 studies a linear penalty function $\phi(x) = -\alpha x \mathbb{1}_{x<0}$ for some $\alpha > 0$. Here, it is very difficult to solve the HJB equation explicitly. Thus, we only determine an optimal strategy in the case $n = 1$ without reinsurance and assume that there are additional investment constraints. Under the same restrictions we obtain an analogous result in section 5 for a quadratic penalty function $\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0}$, where $\alpha_1, \alpha_2 > 0$. Moreover, it is possible to determine an explicit solution for $n > 1$ without reinsurance and with no investment constraints if we make some restrictions on $\alpha_1$ and $\alpha_2$. In the last section we assume that the penalty payments are given by a power function $\phi(x) = \alpha(-x)^k \mathbb{1}_{x<0}$, where $\alpha > 0$ and $k > 2$. We derive a solution in the case where all claims are reinsured by so-called cheap reinsurance.

## 4.2 The HJB Equation and the Verification Theorem

We begin by stating some basic properties of the value function.

**Lemma 4.2.** *$V$ is positive, decreasing and vanishes at infinity. Moreover,*

$$V(x) \le \int_0^\infty \mathrm{e}^{-\delta t}\, \mathbb{E}[\phi(X_t)]\, \mathrm{d}t < \frac{\phi(x+k)}{\delta}$$

*for a constant $k$.*

*Proof.* Obviously, $V$ is positive, decreasing and vanishes at infinity. Moreover, the first inequality follows because $V(x) \leq V^{U^0}(x)$, where $U^0$ describes the strategy where neither reinsurance is bought nor any investments are made. Then, we prove the second inequality similar to the proof of the mean value theorem for integrals. We set $f(z) = \phi(z + x + \mu t)$ and

$$g(z) = \frac{\mathrm{e}^{\frac{-z^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}} .$$

Since $f$ is decreasing with $f \geq 0$ and $\lim_{z\to-\infty} f(z) = \infty$ as well as $\lim_{z\to\infty} f(z) = 0$, we obtain by the intermediate value theorem that for

$$y = \frac{\int_{-\infty}^{\infty} f(z)g(z)\,\mathrm{d}z}{\int_{-\infty}^{\infty} g(z)\,\mathrm{d}z} = \int_{-\infty}^{\infty} f(z)g(z)\,\mathrm{d}z > 0$$

there exists a unique $k \in \mathbb{R}$ such that $y = f(k)$. This implies

$$\begin{aligned}
\mathbb{E}[\phi(X_t)] &= \mathbb{E}[\phi(x + \mu t + \sigma W_t)] \\
&= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} \phi(u)\,\mathrm{e}^{-\frac{(u-x-\mu t)^2}{2\sigma^2 t}}\,\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^{\infty} \phi(z + x + \mu t)\,\mathrm{e}^{-\frac{z^2}{2\sigma^2 t}}\,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} f(z)g(z)\,\mathrm{d}z = \phi(k + x + \mu t) \int_{-\infty}^{\infty} g(z)\,\mathrm{d}z \\
&= \phi(k + x + \mu t) < \phi(x + k)
\end{aligned}$$

and therefore the assertion. $\square$

Now, we motivate the HJB equation heuristically. We choose $r \in [0, \infty]$, $\vartheta \in \mathbb{R}^n$ and define the strategy

$$U_t = \begin{cases} (r, \vartheta), & t \leq h, \\ U^{\varepsilon}_{t-h}, & t > h, \end{cases}$$

where $h > 0$ and $U^\varepsilon$ is a strategy such that

$$V^{U^\varepsilon}(x) < V(x) - \varepsilon$$

for all $x$ and some $\varepsilon > 0$. Note that we do not address the problem of whether we can do that in a measurable way. In the case of proportional reinsurance, $r$ should be chosen within $[0, 1]$. This yields

$$\begin{aligned} V(x) &\leq V^U(x) = \int_0^h \mathrm{e}^{-\delta t}\, \phi(X_t^U)\, \mathrm{d}t + \mathrm{e}^{-\delta h}\, V^U(X_h^U) \\ &< \int_0^h \mathrm{e}^{-\delta t}\, \phi(X_t^U)\, \mathrm{d}t + \mathrm{e}^{-\delta h}\, V(X_h^U) - \varepsilon\ . \end{aligned} \tag{4.5}$$

If $V$ is twice continuously differentiable, Itô's formula implies

$$\begin{aligned} V(X_h^U) &= V(x) + \int_0^h \Big[\{\lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1) + a^T\vartheta\}V'(X_t^U) \\ &\quad + \tfrac{1}{2}\{\lambda\mathbb{E}[s(r,Y)^2] + \vartheta^T\Sigma\vartheta\}V''(X_t^U)\Big]\, \mathrm{d}t \\ &\quad + \int_0^h \sqrt{\lambda\mathbb{E}[s(r,Y)^2]}V'(X_t^U)\, \mathrm{d}W_t + \sum_{i=1}^n\sum_{j=1}^n \int_0^h \vartheta_i v_{ij} V'(X_t^U)\, \mathrm{d}B_t^j\ , \end{aligned}$$

where $a = (a_1, a_2, ..., a_n)^T$ and $\Sigma = vv^T$. Let us assume that the stochastic integrals are martingales with mean zero. Now, taking the expected value in (4.5) and letting $\varepsilon \downarrow 0$, we get

$$\begin{aligned} 0 &\leq V(x)(\mathrm{e}^{-\delta h} - 1) + \mathbb{E}\Big(\int_0^h \mathrm{e}^{-\delta t}\, \phi(X_t^U)\, \mathrm{d}t \\ &\quad + \mathrm{e}^{-\delta h}\int_0^h \Big[\{\lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1) \\ &\quad + a^T\vartheta\}V'(X_t^U) + \tfrac{1}{2}\{\lambda\mathbb{E}[s(r,Y)^2] + \vartheta^T\Sigma\vartheta\}V''(X_t^U)\Big]\, \mathrm{d}t\Big)\ . \end{aligned}$$

Dividing by $h$ and letting $h \downarrow 0$, we obtain

$$\begin{aligned} 0 &\leq -\delta V(x) + \phi(x) + \{\lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1) + a^T\vartheta\}V'(x) \\ &\quad + \tfrac{1}{2}\{\lambda\mathbb{E}[s(r,Y)^2] + \vartheta^T\Sigma\vartheta\}V''(x)\ . \end{aligned}$$

This motivates the HJB equation

$$\inf_{(r,\vartheta)\in[0,r_0]\times I(x)} \Big[\tfrac{1}{2}\{\lambda\mathbb{E}[s(r,Y)^2] + \vartheta^T\Sigma\vartheta\}V''(x) + \{\lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1) + a^T\vartheta\}V'(x) - \delta V(x) + \phi(x)\Big] = 0\;, \tag{4.6}$$

where $I(x) \subset \mathbb{R}^n$ describes the set of all admissible control values if there are any investment constraints. In case of proportional and excess of loss reinsurance we have $r_0 = 1$ and $r_0 = \infty$, respectively. Note that $\Sigma$ is symmetric, positive definite and invertible. Moreover, we may minimise $r$ and $\vartheta$ independently. Thus, if $I(x) = \mathbb{R}^n$ and $V'(x) < 0 < V''(x)$, we obtain that

$$\vartheta^*(x) = -\frac{V'(x)}{V''(x)}\Sigma^{-1}a$$

minimises the HJB equation in $\vartheta$. Plugging this into (4.6), we obtain

$$\inf_{r\in[0,r_0]}\Big[-\gamma\frac{V'(x)^2}{V''(x)} + \tfrac{1}{2}\lambda\mathbb{E}[s(r,Y)^2]V''(x) + \lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1)V'(x) - \delta V(x) + \phi(x)\Big] = 0\;,$$

where

$$\gamma = \tfrac{1}{2}a^T\Sigma^{-1}a\;.$$

Now, we are in the position to prove the following verification theorem.

**Theorem 4.1.** *Let $f$ be a twice continuously differentiable solution to* (4.6) *with*

$$f(x) \le \frac{\phi(x+k)}{\delta} \tag{4.7}$$

*for a constant $k$ and assume that $u^*(x) = (r^*(x), \vartheta^*(x))$ minimises the left-hand side of* (4.6)*. Moreover, assume that $X_t^*$ is a continuous solution to*

$$\mathrm{d}X_t^* = \lambda(\rho\mathbb{E}[s(r^*(X_t^*),Y)] - (\rho-\eta)m_1)\,\mathrm{d}t + \sqrt{\lambda\mathbb{E}[s(r^*(X_t^*),Y)^2]}\,\mathrm{d}W_t + \sum_{i=1}^n a_i\vartheta_i^*(X_t^*)\,\mathrm{d}t + \sum_{i=1}^n\sum_{j=1}^n \vartheta_i^*(X_t^*)v_{ij}\,\mathrm{d}B_t^j$$

*and*

$$\mathbb{E}\Big[\mathrm{e}^{-\delta h}\, f(X_h^{U^*})\Big] \to 0, \quad h \to \infty\ , \tag{4.8}$$

*where* $U_t^* = u^*(X_t^*)$ *is admissible. Then, we obtain* $f(x) = V^{U^*}(x) = V(x)$ *and* $U^*$ *is an optimal strategy.*

*Proof.* Let $U = (R, \theta) \in \mathcal{U}$ be an arbitrary strategy, $h > 0$ and $\tau_k = \inf(t > 0 : |X_t^U| > k)$. Then,

$$\int_0^{\tau_k \wedge h} \sqrt{\lambda \mathbb{E}[s(R_t, Y)^2]}\, V'(X_t^U)\ \mathrm{d}W_t + \sum_{i=1}^n \sum_{j=1}^n \int_0^{\tau_k \wedge h} \theta_t^i v_{ij} V'(X_t^U)\ \mathrm{d}B_t^j$$

is a martingale with mean zero. Thus, Itô's formula implies

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\delta(\tau_k \wedge h)}\, f(X_{\tau_k \wedge h}^U)] \;\; = \;\; & f(x) + \mathbb{E}\Big\{\int_0^{\tau_k \wedge h} \mathrm{e}^{-\delta t} \Big[\{\lambda(\rho \mathbb{E}[s(R_t, Y)] - (\rho - \eta) m_1) \\ & + a^T \theta_t\} V'(X_t^U) + \tfrac{1}{2}\{\lambda \mathbb{E}[s(R_t, Y)^2] + \theta_t^T \Sigma \theta_t\} V''(X_t^U) \\ & - \delta V(X_t^U)\Big]\ \mathrm{d}t\Big\}\ .\end{aligned}$$

Since $f$ fulfils (4.6), we obtain

$$\mathbb{E}\Big[\mathrm{e}^{-\delta(\tau_k \wedge h)}\, f(X_{\tau_k \wedge h}^U) + \int_0^{\tau_k \wedge h} \mathrm{e}^{-\delta t}\ \phi(X_t^U)\ \mathrm{d}t\Big] \geq f(x)\ .$$

We can assume that

$$\mathbb{E}\Big[\mathrm{e}^{-\delta h}\, f(X_h^U)\Big] \to 0, \quad h \to \infty \tag{4.9}$$

for all strategies $U$. Indeed, if this not fulfilled for some strategy $U$, we get by (4.7) that $V^U(x) = \infty$ and therefore $U$ cannot be optimal. Moreover, since $U \in \mathcal{U}$, we get $\tau_k \to \infty$. Letting $h, k \to \infty$, we get by the bounded convergence theorem that $V^U(x) \geq f(x)$. Note that equality holds if $U = U^*$. Since $U$ is arbitrary, we get $f(x) \leq V(x)$. The optimality of the strategy $U^*$ follows because $f(x) = V^{U^*}(x) \geq V(x)$. □

In the following we study equation (4.6) for various penalty functions and aim to find a solution fulfilling the regularity conditions of the verification theorem.

## 4.3 Exponential Penalty payments

In this section we model the penalty payments by an exponential function $\phi(x) = \alpha\, \mathrm{e}^{-\beta x}$ with $\alpha, \beta > 0$. Since $\mathbb{E}[\mathrm{e}^{-\beta X_t - \delta t}] = \exp\{-\beta x + (\frac{1}{2}\sigma^2\beta^2 - \beta\mu - \delta)t\}$, we obtain that (4.4) is fulfilled if

$$\beta < \xi_1 \;, \tag{4.10}$$

where $\xi_1$ is the positive root of the equation

$$\sigma^2\xi^2 - 2\mu\xi - 2\delta = 0 \;.$$

If (4.10) holds, we get by Fubini's theorem that

$$V^0(x) = \mathbb{E}\Big[\int_0^\infty \mathrm{e}^{-\delta t}\, \phi(X_t)\, \mathrm{d}t\Big] = A\, \mathrm{e}^{-\beta x}$$

where

$$A = -\frac{2\alpha}{\sigma^2\beta^2 - 2\mu\beta - 2\delta} \;.$$

If $\beta \geq \xi_1$, we get $V^0(x) = \infty$. In particular, it follows that (4.4) is not fulfilled. For this reason we assume that (4.10) holds. Furthermore, we assume in this section that there are no investment constraints. That is $I(x) = \mathbb{R}^n$. Then, the HJB equation becomes

$$\begin{aligned}\inf_{r\in[0,r_0]}\Big[-\gamma\frac{V'(x)^2}{V''(x)} + \tfrac{1}{2}\lambda\mathbb{E}[s(r,Y)^2]V''(x) + \lambda(\rho\mathbb{E}[s(r,Y)]\\ - (\rho-\eta)m_1)V'(x) - \delta V(x) + \alpha\, \mathrm{e}^{-\beta x}\Big] = 0 \;.\end{aligned} \tag{4.11}$$

We make the ansatz $f(x) = C\,\mathrm{e}^{-\beta x}$ for some constant $C$. Plugging this into (4.11), we get

$$\inf_{r\in[0,r_0]}\Big[-2\gamma+\lambda\mathbb{E}[s(r,Y)^2]\beta^2-2\lambda(\rho\mathbb{E}[s(r,Y)]-(\rho-\eta)m_1)\beta \\ -2\delta+\frac{2\alpha}{C}\Big]=0\,. \qquad (4.12)$$

Since this expression is continuous in $r$, there exists an $r^*$ at which the minimum is attained and we have

$$-2\gamma+\lambda\mathbb{E}[s(r^*,Y)^2]\beta^2-2\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)\beta \\ -2\delta+\frac{2\alpha}{C}=0\,. \qquad (4.13)$$

Solving equation (4.13) in $C$, we obtain

$$C=\frac{2\alpha}{2\gamma+2\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)\beta-\lambda\mathbb{E}[s(r^*,Y)^2]\beta^2+2\delta}\,.$$

Obviously,

$$\begin{aligned}&\lambda\mathbb{E}[s(r^*,Y)^2]\beta^2-2\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)\beta-2\delta\\ &\le\lambda\mathbb{E}[s(r_0,Y)^2]\beta^2-2\lambda(\rho\mathbb{E}[s(r_0,Y)]-(\rho-\eta)m_1)\beta-2\delta\\ &=\sigma^2\beta^2-2\mu\beta-2\delta<0\end{aligned}$$

holds true and $\gamma > 0$ since $\Sigma^{-1}$ is positive definite. This yields $C > 0$. Moreover, we have a constant minimiser

$$u^*=\Big(r^*,\frac{1}{\beta}\Sigma^{-1}a\Big)$$

and for $U_t^* = u^*$ we obtain that

$$\begin{aligned}X_t^{U^*} &= x+\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)t+\sqrt{\lambda\mathbb{E}[s(r^*,Y)^2]}W_t\\ &\quad+\sum_{i=1}^n a_i\vartheta_i^*t+\sum_{i=1}^n\sum_{j=1}^n\vartheta_i^*v_{ij}B_t^j\end{aligned}$$

is a continuous process. Finally, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{-\delta t-\beta X_t^{U^*}}] &= \exp\Big\{-\delta t-\beta x-\beta\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)t \\
&\quad +\tfrac{1}{2}\beta^2\lambda\mathbb{E}[s(r^*,Y)^2]t-\beta a^T\vartheta^*t+\tfrac{1}{2}\beta^2\vartheta^{*T}\Sigma\vartheta^*t\Big\} \\
&= \exp\Big\{-\beta x-\frac{t}{2}\Big[2\gamma+2\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)\beta \\
&\quad -\lambda\mathbb{E}[s(r^*,Y)^2]\beta^2+2\delta\Big]\Big\}\ .
\end{aligned}$$

We have already shown that

$$2\gamma+2\lambda(\rho\mathbb{E}[s(r^*,Y)]-(\rho-\eta)m_1)\beta-\lambda\mathbb{E}[s(r^*,Y)^2]\beta^2+2\delta>0\ .$$

Thus, (4.8) is fulfilled. In sum, we obtain that $f$ is a solution to the HJB equation fulfilling the regularity conditions of the verification theorem. As a result $V(x)=f(x)=V^{U^*}(x)$ and $U^*$ is an optimal strategy.

**Example 4.1.** *In case of proportional reinsurance we have $s(r,Y)=rY$ and equation* (4.12) *becomes*

$$\inf_{r\in[0,1]}\Big[-2\gamma+\lambda r^2m_2\beta^2-2\lambda m_1(\rho r-(\rho-\eta))\beta-2\delta+\frac{2\alpha}{C}\Big]=0\ .$$

*Here, the minimum is attained at*

$$r^*=\frac{m_1\rho}{m_2\beta}\wedge 1$$

*and we get*

$$C=\frac{2\alpha}{2\gamma+2\lambda m_1(\rho r^*-(\rho-\eta))\beta-\lambda r^{*2}m_2\beta^2+2\delta}\ .$$

**Example 4.2.** *Now, we consider excess of loss reinsurance. Then, $s(r,Y)=\min(r,Y)$ and equation* (4.12) *reads*

$$\begin{aligned}
\inf_{r\in[0,\infty]}\Big[-\gamma+\lambda\beta^2\int_0^r y(1-F(y))\,\mathrm{d}y-\lambda\rho\beta\int_0^r(1-F(y))\,\mathrm{d}y \\
+\lambda m_1(\rho-\eta)\beta-\delta+\frac{\alpha}{C}\Big]=0\ . \qquad (4.14)
\end{aligned}$$

*Differentiating this expression w.r.t. $r$, we obtain*

$$\lambda\beta(1-F(r))(\beta r-\rho)$$

*Let $y_0 = \inf\{y\colon F(y) < 1\}$. If $y_0 \le \rho/\beta$, it is optimal to buy no reinsurance at all. If $y_0 > \rho/\beta$ the left-hand side of* (4.14) *is minimised at $\rho/\beta$. Thus,*

$$r^* = \frac{\rho}{\beta} \wedge y_0$$

*and*

$$C = \frac{2\alpha}{2\gamma + 2\lambda\rho\beta\int_0^{r^*}(1-F(y))\,\mathrm{d}y - 2\lambda m_1(\rho-\eta)\beta - 2\lambda\beta^2\int_0^{r^*} y(1-F(y))\,\mathrm{d}y + 2\delta} \,.$$

Figure 4.1 illustrates the value functions for proportional and excess of loss reinsurance, where the claims are exponentially distributed with $m_2 = 2m_1 = 1$ and $\lambda = 10$, $\rho = 0.1$, $\eta = 0.05$, $\delta = 0.06$, $\alpha = 1$, $\beta = 0.1$, $n = 2$, $a_1 = 0.1$, $a_2 = 0.15$ and

$$b = \begin{pmatrix} 0.5 & 0 \\ 0.5 & 0.4 \end{pmatrix} .$$

## 4.4 Linear Penalty Payments

Now, we consider $\phi(x) = -\alpha x \mathbb{1}_{x<0}$ for some $\alpha > 0$. Obviously, (4.4) is fulfilled. The HJB equation becomes

$$\inf_{(r,\vartheta)\in[0,r_0]\times I(x)} \Big[\tfrac{1}{2}\{\lambda\mathbb{E}[s(r,Y)^2] + \vartheta^T\Sigma\vartheta\}V''(x) + \{\lambda(\rho\mathbb{E}[s(r,Y)] - (\rho-\eta)m_1) + a^T\vartheta\}V'(x) - \delta V(x) - \alpha x\mathbb{1}x<0\Big] = 0\,.$$

In the general case it is very difficult to find a closed-form solution. Therefore, we have to make some restriction. At the beginning we investigate several cases and we motivate the key problems. Firstly, let $x \ge 0$. If $\eta \ge \rho$, it is optimal to apply the trivial strategy $(R_t, \theta_t) = (0, \mathbf{0})$ and $V(x) = 0$. If $\eta < \rho$

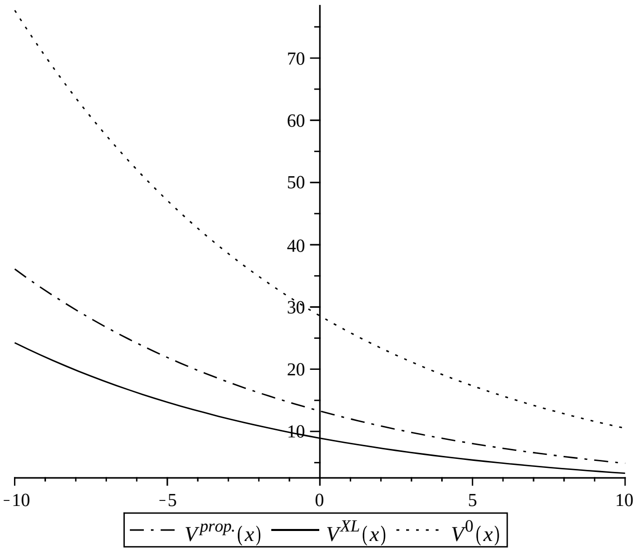


Figure 4.1: Value functions for proportional and excess of loss reinsurance from Example 4.1 and Example 4.2.

and investments are not constrained, we have the same equation as in [23, Section 2.1]. Here, a solution is given by $f(x) = C\,\mathrm{e}^{\zeta_1 x}$, where $C$ is a free constant and $\zeta_1$ is choosen such that the HJB equation is solved.

If $x < 0$ it appears difficult to solve the HJB equation explicitly. Considering proportional reinsurance and $I(x) = \mathbb{R}^n$, we obtain the equation

$$-\tilde{\gamma}\frac{V'(x)^2}{V''(x)} - \lambda m_1(\rho - \eta)V'(x) - \delta V(x) - \alpha x = 0$$

where

$$\tilde{\gamma} = \gamma + \frac{\lambda(\rho m_1)^2}{2m_2}$$

in case of

$$-\frac{\rho m_1}{m_2}\frac{V'(x)^2}{V''(x)} \leq 1 \,.$$

A similar equation was already solved in [37]. Nevertheless, the problem remains to find a solution to the HJB equation in case of

$$-\frac{\rho m_1}{m_2}\frac{V'(x)^2}{V''(x)} > 1\ .$$

Due to the complexity of reinsurance, this section is restricted to the case where the insurer does not buy reinsurance and where $n = 1$. Moreover, investment constraints are that neither short-selling nor taking money from any other sources to buy stocks is allowed. That is, the set of all admissible control values becomes

$$I(x) = \{\vartheta \in \mathbb{R} \colon 0 \le \vartheta \le \max(x, 0)\}\ .$$

Note, that it is not necessary to assume that $0 \le \vartheta$.

Now, the controlled surplus process is given by

$$\mathrm{d}X_t^\theta = (\mu + a_1\theta_t)\ \mathrm{d}t + \sigma\ \mathrm{d}W_t + v_1\theta_t\ \mathrm{d}B_t^1 \tag{4.15}$$

and the HJB equation becomes

$$\inf_{\vartheta\in I(x)}\Big[\tfrac{1}{2}(\sigma^2 + v_1^2\vartheta^2)V''(x) + (\mu + a_1\vartheta)V'(x) \\ - \delta V(x) - \alpha x\mathbb{1}x < 0\Big] = 0\ . \tag{4.16}$$

Following [39] we assume that

$$a_1 < \delta\ . \tag{4.17}$$

Otherwise the solution to the HJB equation becomes very complex. If the surplus is negative, no investments are allowed at all and the HJB equation is given by

$$\tfrac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) - \alpha x = 0\ .$$

This equation is solved by

$$f_1(x) = C_1\,\mathrm{e}^{\xi_1 x} + C_2\,\mathrm{e}^{\xi_2 x} - \alpha\frac{\delta x + \mu}{\delta^2}\ ,$$

where $\xi_2 < 0 < \xi_1$ are the roots of the equation $\sigma^2\xi^2 + 2\mu\xi - 2\delta = 0$ and $C_1, C_2$ some constants. As in Section 2.5, the value function is linearly bounded which enforces $C_2 = 0$.

For $x > 0$ we have to solve

$$\inf_{0\le\vartheta\le x}\Big[\tfrac{1}{2}(\sigma^2 + v_1^2\vartheta^2)V''(x) + (\mu + a_1\vartheta)V'(x) - \delta V(x)\Big] = 0 \; . \tag{4.18}$$

If $V'(x) < 0 < V''(x)$, we obtain the minimum at $\tilde\vartheta(x) = \bar\vartheta(x) \wedge x$, where

$$\bar\vartheta(x) = -\frac{a_1}{v_1^2}\frac{V'(x)}{V''(x)} \; .$$

If $0 \le x \le \bar\vartheta(x)$, equation (4.18) becomes

$$\tfrac{1}{2}(\sigma^2 + v_1^2x^2)V''(x) + (\mu + a_1x)V'(x) - \delta V(x) = 0 \; . \tag{4.19}$$

Paulsen and Gjessing [55] showed that, if (4.17) holds, equation (4.19) is solved by

$$f_2(x) = C_3D(x, \nu + 1) + C_4E(x, \nu + 1) \; ,$$

where

$$\begin{aligned}
\nu &= \tfrac{1}{2}\Big[\sqrt{\Big(\frac{2a_1}{v_1^2} - 1\Big)^2 + \frac{8\delta}{v_1^2}} - \Big(1 + \frac{2a_1}{v_1^2}\Big)\Big] \; , \\
D(x,\kappa) &= \int_x^\infty (t-x)^\kappa K(t)\,\mathrm{d}t, \quad -1 < \kappa < 1 + 2\nu + \frac{2a_1}{v_1^2} \; , \\
E(x,\kappa) &= \int_{-\infty}^x (x-t)^\kappa K(t)\,\mathrm{d}t, \quad -1 < \kappa < 1 + 2\nu + \frac{2a_1}{v_1^2}
\end{aligned}$$

with

$$K(t) = (v_1^2t^2 + \sigma^2)^{-(\nu+1+a_1/v_1^2)} \exp\Big[-\frac{2\mu}{\sigma v_1}\arctan\Big(\frac{v_1t}{\sigma}\Big)\Big]$$

and some constants $C_3, C_4$. Moreover, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}D(x,\kappa) = -\kappa D(x,\kappa-1), \quad \frac{\mathrm{d}}{\mathrm{d}x}E(x,\kappa) = \kappa E(x,\kappa-1)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x^2}D(x,\kappa) = \kappa(\kappa-1)D(x,\kappa-2), \quad \frac{\mathrm{d}}{\mathrm{d}x^2}E(x,\kappa) = \kappa(\kappa-1)E(x,\kappa-2) \; .$$

Notice that (4.17) implies $\nu > 0$ and therefore $f_2''(x)$ exists. If $x > \bar{\vartheta}(x)$, we obtain the equation

$$-\frac{a_1^2}{2v_1^2}\frac{V'(x)^2}{V''(x)} + \tfrac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) = 0 \; .$$

A solution to this equation is given by

$$f_3(x) = C_5 \, \mathrm{e}^{\xi_3 x} \; ,$$

where $\xi_3$ is the negative solution to the equation $\sigma^2\xi^2 + 2\mu\xi - 2\delta - a_1^2/v_1^2 = 0$ and $C_5$ is a free constant. Supposing that $V(x) = f_3(x)$ for $x \geq \bar{\vartheta}(x)$, we get

$$\bar{\vartheta}(x) = \bar{\vartheta} = -\frac{a_1}{v_1^2\xi_3} > 0 \; .$$

Now, we have to determine $C_1, C_3, C_4$ and $C_5$ such that $f_1(0) = f_2(0)$, $f_1'(0) = f_2'(0)$, $f_2(\bar{\vartheta}) = f_3(\bar{\vartheta})$ and $f_2'(\bar{\vartheta}) = f_3'(\bar{\vartheta})$. These equations are fulfilled if

$$C_1 = \frac{\alpha\mu}{\delta^2} + C_3 D(0, \nu+1) + C_4 E(0, \nu+1) \; ,$$

$$C_3 = K\alpha(\delta - \mu\xi_1)[(\nu+1)E(\bar{\vartheta}, \nu) - \xi_3 E(\bar{\vartheta}, \nu+1)] \; ,$$

$$C_4 = K\alpha(\delta - \mu\xi_1)[(\nu+1)D(\bar{\vartheta}, \nu) + \xi_3 D(\bar{\vartheta}, \nu+1)]$$

and

$$C_5 = K\alpha(\delta - \mu\xi_1)[D(\bar{\vartheta}, \nu+1)E(\bar{\vartheta}, \nu) + D(\bar{\vartheta}, \nu)E(\bar{\vartheta}, \nu+1)] \, \mathrm{e}^{-\xi_3\bar{\vartheta}},$$

where

$$\begin{aligned}
\frac{\delta^2}{K} &= \xi_1(\nu+1)[D(0, \nu+1)E(\bar{\vartheta}, \nu) + D(\bar{\vartheta}, \nu)E(0, \nu+1)] \\
&\quad + (\nu+1)^2[D(0, \nu)E(\bar{\vartheta}, \nu) - D(\bar{\vartheta}, \nu)E(0, \nu)] \\
&\quad + \xi_1\xi_3[D(\bar{\vartheta}, \nu+1)E(0, \nu+1) - D(0, \nu+1)E(\bar{\vartheta}, \nu+1)] \\
&\quad - (\nu+1)\xi_3[D(0, \nu)E(\bar{\vartheta}, \nu+1) + D(\bar{\vartheta}, \nu+1)E(0, \nu)] \; .
\end{aligned}$$

From the differential equations we get that $f_1''(0) = f_2''(0)$ and $f_2''(\bar{\vartheta}) = f_3''(\bar{\vartheta})$.

In the next step, we show that the above function is a decreasing and convex solution to the HJB equation. Obviously, $D(x,\kappa), E(x,\kappa) > 0$. Moreover, $D(x,\kappa)$ is decreasing and $E(x,\kappa)$ is increasing. This implies

$$D(x,\kappa)E(y,\kappa) - D(y,\kappa)E(x,\kappa) > 0$$

if $x < y$. Together with $\xi_1, \nu > 0 > \xi_3$, we obtain $K > 0$. Since $\delta - \mu\xi_1 = -\delta\xi_1/\xi_2 > 0$, we also obtain that $C_3$ and $C_5$ are positive. Thus, $f_3(x)$ is convex and decreasing. If $C_4 \geq 0$, we get $f_2''(x) > 0$. If $C_4 < 0$ and $x \leq \bar{\vartheta}$ we get that

$$f_2''(x) \geq f_2''(\bar{\vartheta}) = f_3''(\bar{\vartheta}) > 0 \ .$$

Thus, $f_2$ is convex at least on $(-\infty, \bar{\vartheta}]$. In particular,

$$C_1\xi_1^2 = f_1''(0) = f_2''(0) > 0 \ .$$

Therefore, $C_1 > 0$ and $f_1$ is also convex. Moreover,

$$f_2'(x) \leq f_2'(\bar{\vartheta}) = f_3'(\bar{\vartheta}) < 0$$

if $x \leq \bar{\vartheta}$ and

$$f_1'(x) \leq f_1'(0) = f_2'(0) < 0$$

if $x \leq 0$. In sum, we obtain that

$$f(x) = \begin{cases} f_1(x), & x \leq 0, \\ f_2(x), & 0 < x \leq \bar{\vartheta}, \\ f_3(x), & x > \bar{\vartheta} \end{cases}$$

is a solution to the HJB equation fulfilling the regularity conditions of the verification theorem

Now, let $\vartheta^*(x) = \tilde{\vartheta}(x) \vee 0$ and

$$\mathrm{d}X_t^{\vartheta^*} = [\mu + a_1\vartheta^*(X_t^{\vartheta^*})]\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + v_1\vartheta^*(X_t^{\vartheta^*})\,\mathrm{d}B_t^1 \ . \tag{4.20}$$

Obviously, there exists an unique and continuous solution $X_t^{\theta^*}$ of (4.20) and (4.8) is fulfilled. Taken together, we get $V(x) = f(x) = V^{\theta^*}(x)$, where $\theta_t^* = \vartheta^*(X_t^{\theta^*})$.

**Example 4.3.** *Consider a HJB equation of the form*

$$\tfrac{1}{2}p_1^2(x)V''(x) + p_2(x)V'(x) - \delta f(x) = 0 \, ,$$

*where $p_1, p_2$ are continuous functions. In the literature it has been discussed that the behaviour of the associated stochastic control problem can become very complex if the assumption*

$$p_1'(x) \leq \delta \tag{4.21}$$

*is violated. For instance, see Shreve et al. [65]. If $p_1(x) = \sqrt{\sigma^2 + v_1^2 x^2}$ and $p_2(x) = \mu + a_1 x$, we obtain equation* (4.19)*. Here, we had to assume that the strict inequality holds in order to find a solution.*

*Nevertheless, there are some nice solutions even if* (4.17) *is violated. Let $v_1^2 = 2$, $\mu = 0$ and $\sigma^2 = a_1 = \delta = 1$. Then* (4.19) *is solved by*

$$f_2(x) = C_1 x + C_2 \sqrt{1 + 2x^2} \, ,$$

*where $C_1, C_2$ some constants. For*

$$\alpha = \frac{1}{3} \frac{\sqrt{2}(11\sqrt{6} + 26)}{\sqrt{6} + 6} \, ,$$

*we get*

$$V(x) = \begin{cases} \mathrm{e}^{\sqrt{2}x} - \alpha x, & x \leq 0, \\ (\sqrt{2} - \alpha)x + \sqrt{1 + 2x^2}, & 0 < x \leq \bar{\vartheta}, \\ \frac{\sqrt{5}}{\sqrt{27} + \sqrt{32}} \, \mathrm{e}^{(1 - \sqrt{10}x)/2}, & x > \bar{\vartheta} \end{cases}$$

*and $\bar{\vartheta} = \sqrt{1/10}$.*

**Example 4.4.** *In this example, the insurer has the additional possibility to inject capital in order to avoid penalty payments. Then, the controlled surplus process becomes*

$$\mathrm{d}X_t^{(\theta,Q)} = (\mu + a_1\theta_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + v_1\theta_t\,\mathrm{d}B_t^1 + \,\mathrm{d}Q_t\ ,$$

*where $Q_t$ denotes the accumulated capital injections until time $t$. The value of a strategy is given by*

$$W^{(\theta,Q)}(x) = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-\delta t}\ \mathrm{d}Q_t + \alpha\int_0^\infty \mathrm{e}^{-\delta t}\, X_t^{(\theta,Q)^-}\ \mathrm{d}t\Big]$$

*and*

$$W(x) = \inf_{(\theta,Q)} V^{(\theta,Q)}(x)$$

*denotes the optimal value function. We allow all increasing and adapted càdlàg processes $Q_t$ with $Q_{0-} = 0$. Furthermore, we only consider the case $\alpha > \delta$ and suppose that there exists a level $q^* > 0$ such that it is optimal to inject capital as soon as the surplus drops to $-q^*$. That is, under the optimal strategy $(\theta^*, Q^*)$ it holds that*

$$X_t^{(\theta^*,Q^*)} \geq -q^*\ .$$

*This is fulfilled if*

$$Q_t^* = -\min\Big(\inf_{0\leq s\leq t} X_s^{(\theta^*,Q^*)} + q^*, 0\Big)\ .$$

*Moreover, $q^*$ is characterised by $W'(-q^*) = -1$ and $W''(-1^*) = 0$. The HJB equation becomes*

$$\min\Big\{1 + W'(x), \inf_{\vartheta\in I(x)}\Big[\tfrac{1}{2}(\sigma^2 + v_1^2\vartheta^2)W''(x) + (\mu + a_1\vartheta)W'(x) - \delta W(x) - \alpha x \mathbb{1}_{x<0}\Big]\Big\} = 0\ . \tag{4.22}$$

*We define $\sigma, \mu, \delta, a_1, v_1, \alpha$ as in example 4.3. Let*

$$f(x) = \begin{cases} f_1(-q^*) - (x + q^*), & x \leq -q^*, \\ f_1(x), & q^* < x \leq 0, \\ f_2(x), & 0 < x \leq \bar{\vartheta}, \\ f_3(x), & x > \bar{\vartheta}, \end{cases}$$

*where*

$$f_1(x) = C_1 \,\mathrm{e}^{\sqrt{2}x} + C_2 \,\mathrm{e}^{-\sqrt{2}x} \;,$$

$$f_2 = C_3 x + C_4 \sqrt{1 + 2x^2} \;,$$

$$f_3(x) = C_5 \,\mathrm{e}^{-\sqrt{10}x/2}$$

*and $C_1, C_2, C_3, C_4, C_5, q^*$ are determined such that $f_1'(-q^*) = -1$, $f_1''(-q^*) = 0$, $f_1(0) = f_2(0)$, $f_1'(0) = f_2'(0)$, $f_2(\bar{\vartheta}) = f_3(\bar{\vartheta})$ and $f_2'(\bar{\vartheta}) = f_3'(\bar{\vartheta})$. Then, $f(x)$ is a twice continuously differentiable, decreasing and convex solution to (4.22) that vanishes at infinity. Now, one can easily show that $W(x) = f(x) = W^{(\theta^*, Q^*)}(x)$, where $X_t^{(\theta^*, Q^*)}$ is the unique solution to*

$$\mathrm{d}X_t^{(\theta^*, Q^*)} = \vartheta^*(X_t^{\theta^*})\,\mathrm{d}t + \,\mathrm{d}W_t + \sqrt{2}\vartheta^*(X_t^{\theta^*})\,\mathrm{d}B_t^1 + \,\mathrm{d}Q_t^*$$

*and that $\theta_t^* = \vartheta^*(X_t^{(\theta^*, Q^*)})$. It remains unclear whether or not this result also holds in the general case.*

Figure 4.2 illustrates the value functions of Example 4.3 and Example 4.4.

## 4.5 Quadratic Penalty Payments

In this section we study a quadratic function $\phi(x) = (\alpha_2 x^2 - \alpha_1 x)\mathbb{1}_{x<0}$, where $\alpha_1, \alpha_2 > 0$. The condition (4.4) obviously holds. As in the section above it is very hard to solve the general HJB equation explicitly and we assume that the insurer does not buy reinsurance.

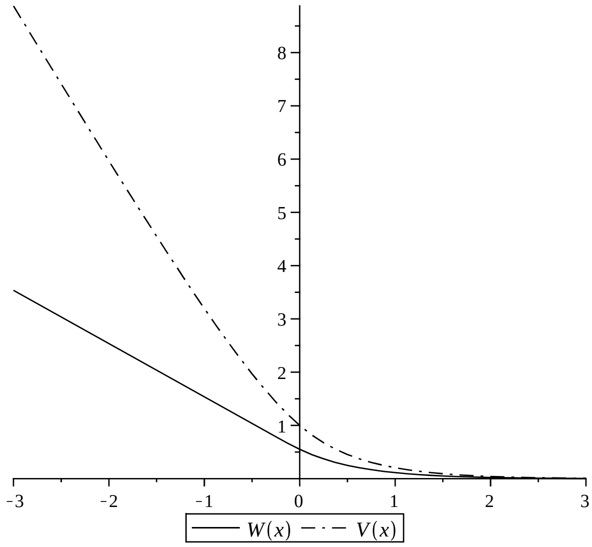


Figure 4.2: Value function with and without capital injections from Example 4.3 and Example 4.4.

If $n = 1$ and

$$I(x) = \{\vartheta \in \mathbb{R} \colon 0 \le \vartheta \le \max(x, 0)\} \ ,$$

the HJB equation becomes

$$\tfrac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) + \alpha_2 x^2 - \alpha_1 x = 0$$

when $x < 0$. This equation is solved by

$$f_1(x) = C_1 \,\mathrm{e}^{\xi_1 x} + C_2 \,\mathrm{e}^{\xi_2 x} + \frac{(\alpha_2 x^2 - \alpha_1 x)\delta^2 + [(\sigma^2 + 2x\mu)\alpha_2 - \mu\alpha_1]\delta + 2\mu^2\alpha_2}{\delta^3} \ ,$$

where $\xi_2 < 0 < \xi_1$ are the roots of the equation $\sigma^2\xi^2 + 2\mu\xi - 2\delta = 0$. For $x \ge 0$ we obtain the same equations as in section 5 and therefore we can determine the optimal strategy analogously. Thus, we only consider an example at the end of this section.

In the following we let $n > 1$ and set $I(x) = \mathbb{R}^n$. If $x \geq 0$, the HJB equation reads

$$-\gamma \frac{V'(x)^2}{V''(x)} + \tfrac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) = 0$$

This equation is solved by $h(x) = C\, \mathrm{e}^{\xi_1 x}$, where $\xi_1$ is the negative root of the equation $\sigma^2 \xi^2 + 2\mu\xi - \gamma - 2\delta = 0$. If $x < 0$, we have

$$-\gamma \frac{V'(x)^2}{V''(x)} + \tfrac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) + \alpha_2 x^2 - \alpha_1 x = 0\ . \tag{4.23}$$

We make the ansatz

$$g(x) = C + C\xi_1 x + \tfrac{1}{2} C \xi_1^2 x^2\ .$$

Obviously,

$$f(x) = \begin{cases} g(x), & x < 0, \\ h(x), & x \geq 0 \end{cases}$$

is twice continuously differentiable and vanishes at infinity. Moreover, $f$ is convex and decreasing if $C > 0$. Choosing

$$C = \frac{2\alpha_2}{\xi_1^2(\delta + 2\gamma)}$$

and plugging $g(x)$ into equation (4.23), we obtain

$$(\delta + \gamma)\xi_1 \alpha_1 + (4\gamma - 2\mu\xi_1 + 2\delta)\alpha_2 = 0\ . \tag{4.24}$$

Thus, if we choose $\alpha_1$ and $\alpha_2$ such that (4.24) holds, $f$ is a solution to the HJB equation fulfilling the regularity conditions of the verification theorem . For general $\alpha_1$ and $\alpha_2$ it is very difficult to find a closed-form solution. Nevertheless, even under the restriction in (4.24) there are still some meaningful parameters. For example, if

$$\alpha_1 = \frac{2(2\gamma - \mu\xi_1 + \delta)}{4\gamma + 2\delta - \xi_1(2\mu + \delta + 2\gamma)} \quad \alpha_2 = \frac{-\xi_1(2\gamma + \delta)}{4\gamma + 2\delta - \xi_1(2\mu + \delta + 2\gamma)}\ , \tag{4.25}$$

we have $0 < \alpha_1, \alpha_2 < 1$ and $\alpha_1 + \alpha_2 = 1$.

Let

$$\psi(x) = \begin{cases} -1/\xi_1 - x, & x < 0, \\ -1/\xi_1, & x \geq 0 \end{cases}$$

and

$$\vartheta^*(x) = \psi(x)\Sigma^{-1}a \; .$$

Then, the optimal strategy is given by

$$\theta_t^* = \vartheta^*(X_t^{\theta_t^*}) \; ,$$

where $X_t^{\theta_t^*}$ is the unique solution to

$$\mathrm{d}X_t^{\theta_t^*} = \mu \; \mathrm{d}t + \sigma \; \mathrm{d}W_t + \psi(X_t^{\theta_t^*})a^T\Sigma^{-1}a \; \mathrm{d}t + \psi(X_t^{\theta_t^*})(\Sigma^{-1}a)v \; \mathrm{d}B_t$$

with $B = (B^1, B^2, ..., B^n)$.

**Example 4.5.** *In this example we choose $\alpha_1, \alpha_2$ as in (4.25). Furthermore, we set $n = 3$, $\mu = 0$, $\sigma^2 = \delta = 1$, $a = (1, 0.3, 0.2)^T$ and*

$$v = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0 \end{pmatrix} .$$

*Then, the optimal value function is given by*

$$V^{n=3}(x) = C \begin{cases} 1 + \xi_1 x + \frac{1}{2}\xi_1^2 x^2, & x < 0, \\ \mathrm{e}^{\xi_1 x}, & x \geq 0, \end{cases}$$

*where $\xi_1 = -\sqrt{2(1+\gamma)}$, $\gamma = 0.26299$ and $C = 0.22975$. Moreover,*

$$\vartheta^*(x) = (0.62920 - x)(0.64667, -0.52764, 0.18799)^T \; .$$

**Example 4.6.** *As mentioned at the beginning of this section, we now consider an example where $n = 1$ and investments are constrained by*

$$I(x) = \{\vartheta \in \mathbb{R} \colon 0 \le \vartheta \le \max(x, 0)\} .$$

*Moreover, we set $\mu$, $\sigma$, $\delta$, $\alpha_1$, $\alpha_2$ as in the previous example, $a_1 = 1$ and $v_1 = \sqrt{2}$. Then, the optimal value function becomes*

$$V^{n=1}(x) = \begin{cases} C_1 \,\mathrm{e}^{\sqrt{2}} + \alpha_2 - \alpha_1 x + \alpha_2 x^2, & x < 0, \\ C_2 x + C_3\sqrt{1 + 2x^2}, & 0 \le x < \bar{\vartheta}, \\ C_4 \,\mathrm{e}^{-\sqrt{10}/2x}, & x \ge \bar{\vartheta}, \end{cases}$$

*where $\bar{\vartheta} = 1/\sqrt{10}$ and $C_1, C_2, C_3, C_4$ are determined such that $V^{n=1}$ is continuously differentiable. As in section 5 it follows from the differential equations that $V^{n=1}$ is twice continuously differentiable. Moreover, $\theta^*_t = \vartheta^*_{n=1}(X^{\theta^*}_t)$ is an optimal strategy where $\vartheta^*_{n=1}(x) = (\bar{\vartheta} \wedge x) \vee 0$ and $X^{\theta^*}_t$ is defined as in (4.20).*

Figure 4.3 and 4.4 illustrate the value and control functions from Example 4.5 and Example 4.6.

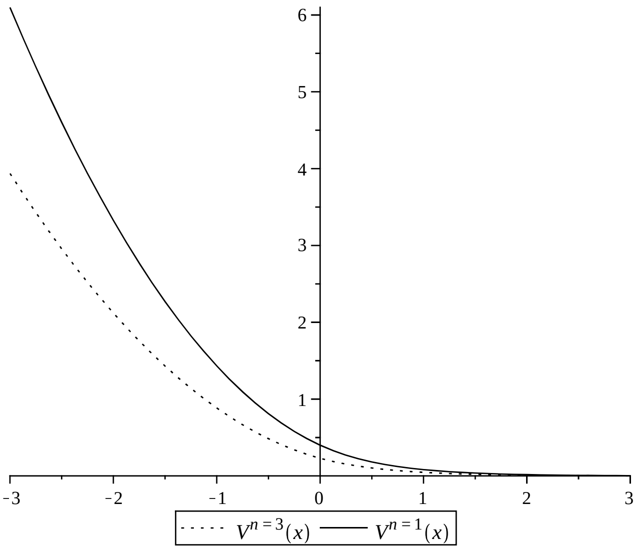


Figure 4.3: Value functions from Example 4.5 and Example 4.6.

## 4.6 Power Functions

Finally, we consider power functions $\phi(x) = \alpha(-x)^k \mathbb{1}_{x<0}$, where $\alpha > 0$ and $k > 2$. Here, (4.4) holds for all $\alpha, k$. In this section, we assume that all claims are reinsured by proportional reinsurance with $\rho = \eta$ and that there are no investment constraints. In practice it is not usual that all claims are reinsured, but note that it is optimal to apply the trivial strategy $(R_t, \theta_t) = (0, \mathbf{0})$ and $V(x) = 0$ when $x \geq 0$. The HJB equation becomes

$$-\gamma \frac{V'(x)^2}{V''(x)} - \delta V(x) + \alpha(-x)^k \mathbb{1}_{x<0} = 0 \,. \tag{4.26}$$

Plugging $h(x) = C(-x)^k \mathbb{1}_{x<0}$ into the HJB equation, we obtain

$$-\gamma \frac{kC}{k-1} - \delta C + \alpha = 0 \,.$$

Thus, $h$ solves (4.26) if

$$C = \frac{\alpha(k-1)}{\gamma + \delta(k-1)} \,.$$

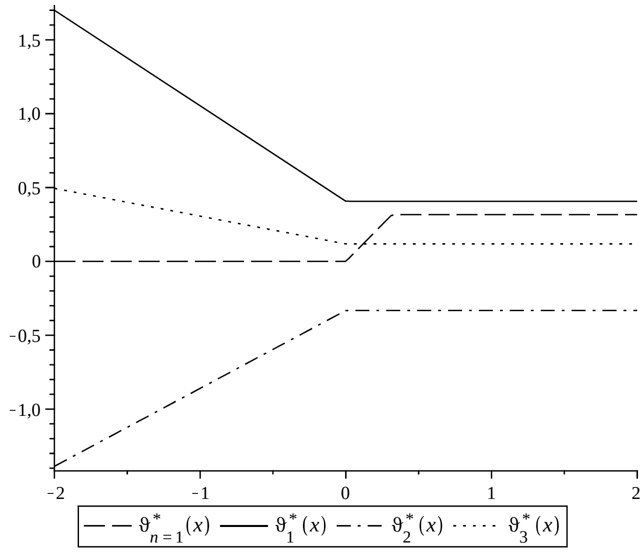


Figure 4.4: Control functions from Example 4.5 and Example 4.6.

Now,

$$f(x) = \begin{cases} h(x), & x < 0, \\ 0, & x \geq 0 \end{cases}$$

is a solution to the HJB equation that fulfils all regularity conditions. Therefore, $f(x) = V(x)$ and

$$\vartheta^*(x) = -\frac{x\mathbb{1}_{x<0}}{k-1}\Sigma^{-1}a$$

denotes the optimal control function.

# Appendix A

# Stochastic Analysis

We used several results from probability theory and stochastic calculus in this thesis. This appendix gives a short overview to this topic. For a more detailed insight see for example [22, 40, 41, 53, 63]. Note, that we do not prove the lemmas and theorems in this section, because we only mention well-known results.

## A.1 Stochastic Processes and Martingales

We assumed that the surplus of an insurance company is given by a stochastic process. Before we give the mathematical definition of a stochastic process we have to introduce the concept of almost surely (a.s.) and null sets.

**Definition A.1.** *We say that an event $A$ occurs almost surely (a.s.) or that an event holds for almost all $\omega$ if it occurs with probabilty 1, that is $\mathbb{P}(A) = 1$. We call an event $\mathbb{P}-$null set if $\mathbb{P}(A) = 0$.*

**Definition A.2.** *A stochastic process is a family $X = \{X_t\}_{t\in I}$ of $E$-valued random variables defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where $(E, \mathcal{E})$ is a measurable space and $I$ is given either by $\mathbb{N}$ or $\mathbb{R}$. If $I = \mathbb{N}$, we call $X$ a* *discrete-time stochastic process and if $I = \mathbb{R}_+$ we call $X$ a continuous-time stochastic process, respectively. For an $\omega \in \Omega$, the function $t \to X_t(\omega)$ is called path or realisation of the process. If $t \to X_t(\omega)$ is continuous (left continuous, right continuous) for almost all $\omega$, the process $X$ is called a.s. continuous (left continuous, right continuous). If $X$ is right-continuous and its left limits exist at all points, we say that $X$ is a càdlàg process.*

The most popular stochastic process is the (standard) Wiener process which is also called standard Brownian motion. We used this process to get an approximation to the Cramér–Lundberg model.

**Definition A.3.** *We call a process $W = \{W_t\}_{t \geq 0}$ (standard) Wiener process or Brownian motion if*

1. *It holds a.s. that $W_0 = 0$.*
2. *For $0 < t_0 < t_1 < t_2 < \cdots < t_n$ the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.*
3. *If $0 \leq s < t$ it holds that $W_t - W_s \sim \mathcal{N}(0, t-s)$, where $\mathcal{N}(0, t-s)$ is a normal distribution with mean 0 and variance $t - s$.*
4. *$X$ is a.s. continuous.*

The Wiener process has a number of nice properties given in the following lemma.

**Lemma A.1.** *Let $W$ be a Wiener process and $s, t \in \mathbb{R}_+$. Then*

*i) We have $\mathbb{E} W_s W_t = s \wedge t$.*

*ii) The processes $\{W_{s+t} - W_s\}_{t \geq 0}$, $\{-W_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$, with $B_t = t W_{1/t}$ if $t > 0$ and $B_0 = 0$, are Wiener processes.*

*iii) Let $s < t$ and $(\Delta_n)_{n\in\mathbb{N}}$ be a sequence of partitions $s = t_0^n < t_1^n < \cdots < t_{k_n}^n = t$ of the interval $[s,t]$. Define*

$$T_{[s,t]}^n = \sum_{i=0}^{k_n-1} [W_{t_{i+1}^n} - W_{t_i^n}]^2 \ .$$

*If $\max_{i=0,\ldots,k_n-1}(t_{i+1}^n - t_i^n)$ converges to zero, then*

$$\lim_{n\to\infty} \mathbb{P}(|T_{[s,t]}^n - (t-s)| \geq \varepsilon) = 0$$

*for all $\varepsilon > 0$. We say that $T_{[s,t]}^n$ converges to $t-s$ in probability and that the Wiener process is of bounded quadratic variation. Consequently, the Wiener process has unbounded variation, i.e. the value*

$$\sup_{\Delta_n} \sum_{i=0}^{k_n-1} [W_{t_{i+1}^n} - W_{t_i^n}]$$

*does not exist almost surely.*

*iv) It holds a.s. that*

$$\limsup_{t\to\infty} \frac{W_t}{\sqrt{2t\log\log t}} = 1$$

*and*

$$\liminf_{t\to-\infty} \frac{W_t}{\sqrt{2t\log\log t}} = -1 \ .$$

Another popular process which we used to define the Cramér–Lundberg model is the Poisson process.

**Definition A.4.** *A càdlàg process $P = \{P_t\}_{t\geq 0}$ is called Poisson process with intensity $\lambda$ if*

*i) $P_0 = 0$ almost surely.*

*ii) $P_t - P_s \sim \mathcal{P}_{\lambda(t-s)}$ for $s < t$, where $\mathcal{P}_{\lambda(t-s)}$ is a Poisson distribution with parameter $\lambda(t-s)$.*

*iii) For* $0 < t_0 < t_1 < t_2 < \cdots < t_n$ *the increments* $P_{t_1} - P_{t_0}, P_{t_2} - P_{t_1}, \ldots, P_{t_n} - P_{t_{n-1}}$ *are independent.*

Obviously, a Poisson process is a jump process and does not have continuous paths, but there are also some nice properties given in the next lemma.

**Lemma A.2.** *Let* $P$ *be a Poisson process and* $0 < s < t$. *Then the following holds.*

*i) The process* $\{P_{s+t} - P_s\}_{t\geq 0}$ *is a Poisson process.*

*ii) Let* $(\Delta_n)_{n\in\mathbb{N}}$ *be a sequence of partitions* $0 = t_0^n < t_1^n < \ldots < t_k^n = t$ *of the interval* $[0,t]$. *If* $\max_{i=0,\ldots,k-1}(t_{i+1}^n - t_i^n)$ *converges to zero, then*

$$\sum_{i=0}^{k-1}[P_{t_{i+1}^n} - P_{t_i^n}]^2 \to P_t$$

*in probability. One says that a Poisson process has quadratic variation equal to itself.*

An extension to the Poisson process is the compound Poisson process, which is used to model the claims in the Cramér–Lundberg model.

**Definition A.5.** *Let* $N$ *be a Poisson process and* $\{Y_n\}_{n=1,2,\ldots}$ *a sequence of iid random variables independent of* $N$. *The process* $\{Z_t\}_{t\geq 0}$ *with*

$$Z_t = \sum_{n=1}^{N_t} Y_n$$

*is called compound Poisson process. As a special case the Poisson process is obtained if* $Y_n = 1$ *for all* $n$.

Considering a stochastic process $\{X_t\}_{t\in I}$, we are often interested in the limiting value as $t$ tends to $\infty$. The following theorems give sufficient conditions for the interchange of the limit and the expectation.

**Theorem A.1** (Bounded convergence theorem)**.** *Let* $\{X_t\}_{t\in I}$ *be a stochastic process and it holds a.s. that* $X_t \to X$, $t \to \infty$. *Moreover,* $|X_t| \leq Y$ *for all* $t$, *where* $\mathbb{E}(Y) < \infty$. *Then,* $\lim_{t\to\infty} \mathbb{E}(X_t) = \mathbb{E}(X)$.

**Theorem A.2** (Monotone convergence theorem)**.** *Let* $\{X_t\}_{t\in I}$ *be a stochastic process and it holds a.s. that* $X_t \to X$, $t \to \infty$. *Moreover,* $X_s \leq X_t$ *a.s. for all* $s < t$. *Then,* $\lim_{t\to\infty} \mathbb{E}(X_t) = \mathbb{E}(X)$.

Now, we introduce the concept of filtrations. A filtration contains all historical information which is available about a stochastic process. Concretely, a filtration is a family of $\sigma$-algebras $\{\mathcal{F}_t\}_{t\in I}$ with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$, where $\mathcal{F}_t$ represents all information of a process until time $t$. We call a filtration complete if $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. For $I = \mathbb{R}$ we call a filtration right-continuous if

$$\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s \, .$$

A process $X$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$ if $X_t$ is $\mathcal{F}_t$-measurable for all $t$. The filtration $\mathcal{F}_t^X$ generated by the process $X$ is called natural filtration.

Further important stochastic processes are martingales, which are used to model a fair game. In this thesis, we consider martingales in continuous time.

**Definition A.6.** *Let* $X = \{X_t\}_{t\geq 0}$ *be a stochastic process in continuous time and* $\{\mathcal{F}_t\}_{t\geq 0}$ *a filtration. Then, we call* $\{(X_t, \mathcal{F}_t)\}_{t\geq 0}$ *a martingale in continuous time if for all* $t \geq 0$ *it holds*

*i)* $X_t$ *is* $\mathcal{F}_t$*-measurable,*

*ii)* $\mathbb{E}|X_t| < \infty$,

*iii)* $E[X_t|\mathcal{F}_s] = X_s$ *for* $s < t$.

**Remark A.1.** *i) Having a stochastic process $\{X_t\}_{t\geq 0}$, one typically considers integrable random variables and the natural filtration. So the characteristic property of a martingale is given by iii) in the definition above. Therefore, we generally call $\{X_t\}_{t\geq 0}$ a martingale if iii) is fulfilled.*

*ii) The process $\{X_t\}_{t\geq 0}$ is called supermartingale if for $s < t$ it holds*

$$E[X_t|\mathcal{F}_s] \leq X_s$$

*and submartingale if*

$$E[X_t|\mathcal{F}_s] \geq X_s \ ,$$

*respectively.*

**Example A.1.** *i) Let us consider a Wiener process $W$. Then, $\{W_t\}_{t\geq 0}$, $\{W_t^2 - t\}_{t\geq 0}$ and $\{W_t^4 - 6tW_t^2 + 3t^2\}_{t\geq 0}$ are martingales.*

*ii) For a Poisson process $P = \{P_t\}_{t\geq 0}$ with intensity $\lambda$ the process $\{P_t - \lambda t\}_{t\geq 0}$ is also a martingale.*

A very popular result in martingale theory is Doob's martingale convergence theorem stated in the following.

**Theorem A.3** (Martingale convergence theorems)**.** *Let $\{X_t\}_{t\in I}$ be a submartingale with $\sup_t \mathbb{E}|X_t| < \infty$ and $I = \mathbb{R}^+$ or $I = \mathbb{N}$, then there exists a random variable $X^\infty$ such that $X_t$ converges a.s. to $X^\infty$.*

In Chapter 1 we mentioned, that in classical risk models, ruin occurs the first time when the surplus process becomes negative. To describe the time of ruin we have to define a so-called stopping time.

**Definition A.7.** *Let $\{\mathcal{F}_t\}_{t\in I}$ be a filtration. A mapping $\tau\colon \Omega \to I$ is called $\mathcal{F}_t$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in I$. The set*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in I\}$$

*is called pre-$\tau$-$\sigma$-algebra.*

**Example A.2.** *Let $X_t = x + \mu t + \sigma W_t$ describe the surplus of an insurance company and $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated from the process $X_t$. Then, the first time when the surplus becomes negative $\tau = \inf\{t > 0 : X_t < 0\}$ is a $\mathcal{F}_t$-stopping time.*

Considering stopping times, the stopping theorem, mentioned below, is a very helpful result.

**Theorem A.4** (Stopping theorem)**.** *Let $\{(X_t, \mathcal{F}_t)\}_{t\in I}$ be a submartingale, $I = \mathbb{R}^+$ or $I = \mathbb{N}$ and $\tau$ a $\mathcal{F}_t$-stopping time. Then, $\{(X_{t\wedge\tau}, \mathcal{F}_{t\wedge\tau})\}_{t\in I}$ is also a submartingale.*

## A.2 Stochastic Integration

In this thesis we considered integrals of the form $\int_0^t f(X_t)\ \mathrm{d}W_s$, where $W$ is a Wiener process, $f$ a twice continuously differentiable function and $X_t$ some continuous stochastic process. Now, we give a short introduction to integrals, where the integrator is a continuous martingale $M$.

In the first step we define the stochastic integral for so-called simple processes given by

$$H_s(\omega) = \sum_{i=1}^{n} h_{i-1}(\omega)\mathbb{1}_{(t_{i-1},t_i]}(s)\ ,$$

where $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_n$ and $h_{i-1}$ is bounded and $\mathcal{F}_{t_{i-1}}$-measurable for $i = 1, 2, ..., n$. Let $\mathcal{H}$ denote the set of all simple processes. For a simple process we define

$$\int_0^t H_s\ \mathrm{d}M_s = \sum_{i=1}^{n} h_{i-1}(M_{t_i\wedge t} - M_{t_{i-1}\wedge t})\ ,$$

for $t \geq 0$.

In the next step we consider progressively measurable processes, but first we have to introduce the product $\sigma$-algebra.

**Definition A.8.** *Let $\mathcal{C}_i$ be a $\sigma$-algebra on $\Omega_i$, $i = 1, 2$. Then, we define the product $\sigma$-algebra $\mathcal{C}$ on*

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$$

*by*

$$\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_2 = \sigma(\{\pi_i^{-1}(A_i) : A_i \in \mathcal{C}_i, i = 1, 2\})$$

*with $\pi_i \colon \Omega \to \Omega_i, \pi_i(\omega) = \omega_i$, $i = 1, 2$.*

**Definition A.9.** *Considering a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, we call a process $\{X_t\}_{t\geq 0}$ progressively measurable if for all $t \geq 0$ the mapping*

$$\Omega \times [0, t] \to \mathbb{R}, (\omega, s) \to X_s(\omega)$$

*is $\mathcal{F}_t \otimes (\mathcal{B} \cap [0, t])$ measurable, where $\mathcal{B} \cap [0, t]$ is the Borel $\sigma$-algebra on the interval $[0, t]$.*

Now, we have to introduce the quadratic variation of a continuous martingale, which is specified by the next theorem.

**Theorem A.5.** *Let $\{M_t\}_{t\geq 0}$ be a continuous martingale. Then, there exists an a.s. unique, continuous, increasing and adapted process $\{[M]_t\}_{t\geq 0}$ with $[M]_0 = 0$ such that $\{M_t^2 - [M]_t\}_{t\geq 0}$ is a continuous martingale.*

**Remark A.2.** *i) The process $[M]$ is called quadratic variation of the continuous martingale $M$.*

*ii) For a Wiener process $W$, we get by Lemma A.1, iv) and Example A.1, i) that*

$$[W]_t = t = \lim_{n\to\infty} \sum_{i=0}^{k_n - 1} [W_{t_{i+1}^n} - W_{t_i^n}]^2 \ ,$$

*where* $(\Delta_n)_{n\in\mathbb{N}}$ *is a sequence of partitions* $0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t$ *of the interval* $[0,t]$ *with* $\max_{i=0,\dots,k_n-1}(t_{i+1}^n - t_i^n) \to 0$, $n \to \infty$.

*iii) Considering stochastic martingales* $M, N$ *one can also show that there exists an a.s. unique and adapted process* $\{[M,N]_t\}_{t\geq 0}$ *with* $[M,N]_0 = 0$ *such that* $\{M_t N_t - [M,N]_t\}_{t\geq 0}$ *is a continuous martingale. The process* $[M,N]$ *is called covariation of* $M$ *and* $N$ *and is given by*

$$[M,N]_t = \frac{1}{4}([M+N] - [M-N]) \ .$$

*Note that* $[M,M] = [M]$.

Since $[M]$ is an increasing process, we can define the measure $\mu_{[M]}((0,t])(\omega) = [M]_t(\omega)$. For a progressively measurable process $H$ the Lebesgue-Stieltjes integral is given by

$$\int_0^t H_s(\omega)\ \mathrm{d}[M]_s(\omega) = \int_{[0,t)} H_s(\omega)\ \mathrm{d}\mu_{[M]}(\omega) \ ,$$

where the expression of the right-hand side is the Lebesgue integral of $H$ with respect to the measure $\mu_{[M]}$. We do not introduce the Lebesgue integral because we only consider Riemann integrable functions $H$ and in case of a Wiener process we get the Riemann integral

$$\int_0^t H_s\ \mathrm{d}[B]_s = \int_0^t H_s\ \mathrm{d}s \ .$$

Now, let $\mathcal{P}^2(M)$ be the set of all progressively measurable processes $H$ with

$$\|H\|_M = \Big(\mathbb{E}\Big[\int_0^t H_s^2\ \mathrm{d}[M]_s\Big]\Big)^{1/2} < \infty \ .$$

Moreover, let $\mathcal{M}^2$ be the set of all continuous martingales with

$$\|M\| = \Big(\sup_{0\leq s\leq t} \mathbb{E}[M_t^2]\Big)^{1/2} < \infty \ .$$

One can show that $\mathcal{M}^2$ is complete, that is every Cauchy sequence of points in $\mathcal{M}^2$ has a limit in $\mathcal{M}^2$. Further, for $H \in \mathcal{P}^2(M)$ there exists a sequence of simple processes $H^n \in \mathcal{H}$ such that $\|H^n - H\|_M$ converges to zero as $n$ tends to infinity. The Itô isometry says that

$$\left\| \int_0^t H_s^n \, \mathrm{d}M_s \right\| = \|H^n\|_M \ .$$

Now, we define

$$\int_0^t H_s \, \mathrm{d}M_s$$

as the limit of the Cauchy sequence

$$\mathcal{I}_t^n = \int_0^t H_s^n \, \mathrm{d}M_s \ .$$

## A.3 Itô's Formula and Stochastic Differential Equations

In this section we state the main result of stochastic calculus, Itô's formula, and we consider stochastic differential equations. Generally, we apply Itô's formula to continuous martingales in this thesis, but it is also possible to apply Itô's formula to semimartingales. The definition of a semimartingale is given in the following.

**Definition A.10.** *i) A stochastic process $M$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is called a local martingale if there exists a sequence of stopping times $\tau_n \uparrow \infty$ such that $\{M_{\tau_n \wedge t} - M_0\}_{t\geq 0}$ is a martingale adapted to the filtration $\{\mathcal{F}_{\tau_n \wedge t}\}_{t\geq 0}$.*

*ii) A semimartingale is a stochastic process $X$ with $X_t = M_t + Y_t$, $t \geq 0$, where $M$ is a local martingale and $Y$ a process of bounded variation with $Y_0 = 0$.*

Note that the stochastic integral introduced in the previous section can be extended to semimartingales. In the next theorem we give Itô's formula.

**Theorem A.6** (Itô's formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function and $X = (X^1, X^2, ..., X^n)$ be a semimartingale. Then, it holds a.s.*

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{j=1}^{n} \int_0^t \frac{\partial f}{\partial x^j}(X_{s-})\, \mathrm{d}X_s^j \\ &\quad + \frac{1}{2} \sum_{j,k=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x^j \partial x^k}(X_{s-})\, \mathrm{d}[X^j, X^k]_s \\ &\quad + \sum_{0<s\le t} \Big[ f(X_s) - f(X_{s-}) - \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(X_{s-})(X_s^j - X_{s-}^j) \Big] \; . \end{aligned}$$

Now, we introduce stochastic differential equations. We consider differential equations of the form

$$\mathrm{d}X_t = \mu(t, X_t)\, \mathrm{d}t + \sigma(t, X_t)\, \mathrm{d}W_t \; , \tag{A.1}$$

$t \in [0, T]$, where $\{W_t\}_{t\in[0,T]}$ is a standard Wiener process and $\mu, \sigma\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ are measurable functions. Here, equation (A.1) is interpreted as the integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s)\, \mathrm{d}W_s \; .$$

**Definition A.11.** *A stochastic process $X$ with initial value $X_0$ is called a solution to* (A.1) *if for all $t \in [0, T]$ it holds*

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, \mathrm{d}s + \int_0^t \sigma(s, X_s)\, \mathrm{d}W_s \; .$$

A very popular differential equation is the following.

$$\mathrm{d}X_t = \mu X_t\, \mathrm{d}t + \sigma X_t\, \mathrm{d}W_t \; , \tag{A.2}$$

where $\mu, \sigma > 0$. Let us suppose that we have a solution of the form $X_t =$

$f(t, W_t)$. Then, Itô's formula implies

$$\begin{aligned} f(t, W_t) &= f(0, W_0) + \int_0^t \frac{\partial f}{\partial s}(s, W_s)\ \mathrm{d}s + \int_0^t \frac{\partial f}{\partial w}(s, W_s)\ \mathrm{d}W_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial w^2}(s, W_s)\ \mathrm{d}[W, W]_s \\ &= f(0, 0) + \int_0^t \Big(\frac{\partial f}{\partial s}(s, W_s) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(s, W_s)\Big)\ \mathrm{d}s \\ &\quad + \int_0^t \frac{\partial f}{\partial w}(s, W_s)\ \mathrm{d}W_s \end{aligned}$$

or in differential notation

$$\mathrm{d}X_t = \Big(\frac{\partial f}{\partial t}(t, W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t, W_t)\Big)\ \mathrm{d}t + \frac{\partial f}{\partial w}(t, W_t)\ \mathrm{d}W_t\ .$$

Solving the differential equations

$$\mu f(t, w) = \frac{\partial f}{\partial t}(t, w) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t, w)$$

and

$$\sigma f(t, w) = \frac{\partial f}{\partial w}(t, w)$$

we get a solution to (A.2). One easily can show that

$$X_t = X_0 \exp\Big[(\mu - \sigma^2/2)t + \sigma W_t\Big]\ .$$

This process is called geometric Brownian motion and is commonly used to model future prices in financial mathematics, for example to model the evolution of stock prices, FX rates and swap rates (here $\mu = 0$).

# Appendix B

# Markov Theory and Infinitesimal Generators

In this thesis we considered HJB equations of the form

$$\sup_{u \in \mathcal{U}} \Big[ g(x,u) + \mathcal{A}_u V(x) - \delta V(x) ] \Big] = 0 \ ,$$

where $\mathcal{A}_u$ denotes the infinitesimal generator of a stochastic process. Here, we give a short introduction to Markov theory and infinitesimal generators. For a more in depth study see [19, 29].

Firstly, we define a Markov process.

**Definition B.1.** *Let $(E, \mathcal{E})$ be some measurable space. An $E$-valued stochastic Process $X$ is called $\mathcal{F}$-Markov process if $X$ is adapted to $\mathcal{F}$ and*

$$\mathbb{P}[X_{t+s} \in A | \mathcal{F}_t] = \mathbb{P}[X_{t+s} \in A | X_t]$$

*for each $A \in \mathcal{E}$ and $s < t$. One says that the future of the process only depends on the present state of the process. The function*

$$P_t(s, x, A) = \mathbb{P}[X_{t+s} \in A | X_t = x]$$

*is called transition function. The process $X$ is called homogenous Markov process if the transition function does not depend on $t$. If*

$$\mathbb{P}(X_{T+s} \in A | \mathcal{F}_T) = P(s, X_T, A)$$

*for all stopping times $T$, we call $X$ a strong Markov Process.*

In the following $B(E)$ denotes the set of all measurable bounded real functions on $E$ endowed with the supremum norm $\|f\| = \sup_{x \in E} |f(x)|$. Now, we define the infinitesimal generator of a Markov process

**Definition B.2.** *Let $X$ be a Markov process. Then, define the operator*

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}[f(X_t) - f(x) | X_0 = x]$$

*if the right-hand side converges uniformly on $B(E)$. We call $\mathcal{A}$ the infinitesimal generator of the process $X$. The set $\mathcal{D}(\mathcal{A})$ of all functions $f$ for which $\mathcal{A}f(x)$ exists is called domain of the infinitesimal generator.*

Obviously, the Cramér–Lundberg process is a Markov process, because it has stationary and independent increments. The next example gives the generator of the Cramér–Lundberg process.

**Example B.1.** *Let $L_t = x + ct - \sum_{i=1}^{N_t} Y_i$, $t \geq 0$ be a Cramér–Lundberg process and $F$ be the distribution function of the claims. Then, the infinitesimal generator of $X$ is given by*

$$\mathcal{A}f(x) = cf'(x) + \lambda \int_0^\infty f(x-y) \, \mathrm{d}F(y) - \lambda f(x) \, .$$

A very popular theorem linking infinitesimal generators to martingales is the following

**Theorem B.1** (Dynkin's theorem)**.** *Let $X$ be a Markov process and $f \in \mathcal{D}(\mathcal{A})$. Then, the process $Y$ with*

$$Y_t = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s) \, \mathrm{d}s \, ,$$

$t \geq 0$, *is a martingale.*

The solution to the HJB equation is often unbounded. In the literature, the definition above is extended to a full generator (cf. [29]).

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