Abstract

A graph parameter φ is monotone if $\varphi(H) \leq \varphi(G)$ whenever H is an induced subgraph of a graph G. In this work we investigate the relationship between the maximum degree Δ of a graph and other monotone graph parameters, with a focus on the clique number ω and the chromatic number χ.

By a characterization of the minimal forbidden induced subgraphs, we describe the largest hereditary graph class $\mathcal{G}$ where the maximum degree of each graph in this class is bounded by a monotonically increasing function on a monotone graph parameter on the graph. For ω and χ, respectively, we show that if a function on one of these graph parameters obeys a certain condition, the set of minimal forbidden induced subgraphs of the thereby defined hereditary graph class is finite. We prove that the largest hereditary graph classes where the maximum degree is bounded by a function on ω are exactly those where the maximum degree is bounded by a function on χ.

For every $k \in \mathbb{N}_0$, we introduce the hereditary graph classes $\mathit{\Omega}_k$ and $\mathit{\Upsilon}_k$, that is, the largest hereditary graph classes where the maximum degree is bounded by the function $\omega + k - 1$ or by the function $\chi + k - 1$, respectively. For $\mathit{\Omega}_k$ and for $\mathit{\Upsilon}_k$, we provide the finite set of minimal forbidden induced subgraphs for every k. We compare $\mathit{\Omega}_k$ and $\mathit{\Upsilon}_k$ and subsequently characterize the graph class where the minimal forbidden induced subgraphs of $\mathit{\Omega}_k$ and $\mathit{\Upsilon}_k$ coincide for each k. Moreover, we provide some results on the chromatic number of the graphs in $\mathit{\Omega}_k$ and show that if $\mathit{\Upsilon}_k$ is restricted to claw-free graphs, then the minimal forbidden induced subgraphs can be described in terms of Turán graphs. In addition, the computational complexity of CLIQUE and K-COLORABILITY is investigated in the considered hereditary graph classes.

Given a monotone graph parameter φ, we say that a vertex in a graph is φ-critical if removing the vertex from the graph produces a graph with a lower value of φ. We detect the graphs such that every connected induced subgraph of the graph contains a vertex that is φ-critical given that φ equals Δ, ω and χ, respectively.

We suppose the existence of a vertex-minimal counterexample to a conjecture of Reed on χ, Δ and ω. We reveal some structural properties of this counterexample and derive some graph classes for which the conjecture holds. Among other results, we show that the verification of the conjecture on triangle-free graphs is equivalent to the verification on graphs that do not contain a diamond or a net as induced subgraph.