Homotopy Theory of Topological Insulators

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Kurzzusammenfassung

In dieser Arbeit wird die Klassifizierung ungeordneter Fermionen auf die Beschreibung translations-invarianter Grundzustände übertragen. Anknüpfend an die Arbeit von Kitaev vervollständigen wir die Umwandlung von Symmetrien in Pseudo-Symmetrien, die eine Clifford-Algebra bilden. Dieser mathematische Rahmen wird genutzt, um einen homotopietheoretischen Beweis für die Einträge im “Periodensystem topologischer Isolatoren und Supraleiter” in der verallgemeinerten Version, die die Anwesenheit eines Gitterdefekts erlaubt, zu formulieren. Wir erweitern diese Klassifizierung, indem wir die Einschränkung einer Mindestanzahl von Valenz- und Leitungsbändern aufheben. Hierdurch erfassen wir den Hopf-Isolator, sowie eine hier erstmals identifizierte topologische Phase, den Hopf-Supraleiter. Im verallgemeinerten Rahmen zeigen wir, dass die Konzepte von “starken” und “schwachen” topologischen Phasen neu definiert werden müssen, um zu vermeiden, dass starke topologische Phasen durch das Stapeln topologischer Phasen niedrigerer Dimension realisiert werden können.

Abstract

We transfer the classification results for disordered free fermions to the setting of translation-invariant ground states and complete the framework developed by Kitaev in which true symmetries are encoded as pseudo-symmetries satisfying Clifford algebra relations. In this mathematical setting, we give a homotopy theoretic proof of the Periodic Table for topological insulators and superconductors in its generalized form allowing for the presence of a defect. Permitting arbitrary numbers of valence and conduction bands, we extend the homotopy classification to include the Hopf insulator and a newly identified topological phase we call the Hopf superconductor. In this general setting, we show that the distinction between strong and weak topological phases needs to be altered in order to prevent strong phases from being realized by stacking lower dimensional phases.

Contents

1. Introduction

Soon after the foundations of quantum mechanics were laid, the theory was applied to the problem of electrons moving in the periodic potential of a crystalline solid. The resulting dispersion relation (energy bands) of a single particle was the starting point for band theory. This theory assumes non-interacting fermions filling the energy bands according to the Fermi-Dirac distribution. Surprisingly, the assumption of independent particles turned out to be quite general if "particle" is replaced by "quasi-particle". This is the content of Fermi liquid theory: For the majority of crystals, interactions between particles can be neglected at the cost of renormalizing properties like their mass.

In recent years, the old band theory resurfaced at the forefront of condensed matter research, triggered by the discovery [vKDP80] of the quantum Hall effect in two-dimensional materials penetrated by a strong magnetic field. This presented the first example of a topological phase called a topological insulator, being characterized by an insulating interior with currents along its boundary and a quantized conductivity. It was soon recognized [Hal88] that this new topological state can be realized on a lattice, resulting in the concept of the "Chern insulator". Some years later, it was shown that by introducing symmetries [KM05, BHZ06, FK07] a whole zoo of new topological phases could be realized. Starting with the time-reversal invariant analog of the Chern insulator in two dimensions [KM05, BHZ06, KWB$^+$07], a similar topological phase was predicted [FKM07] and subsequently realized [HQW$^+$08, XQH$^+$09] in three dimensional materials. In both cases, theory preceded experiments since it was in materials suggested by theorists that the existence of these topological phases was confirmed. In contrast, the experimental discovery of superconductivity preceded the microscopic theory by a good 50 years.

In this new and very active field of research, the search for topological phases soon matured into attempts at classifying all of them. After enumerating the possible invariants that could be found for a given dimension and symmetry [SRFL08], a beautiful pattern between these invariants was revealed in [Kit09] using algebraic tools in connection with K-theory. These results were later confirmed by an analysis of possible topological terms in non-linear sigma models describing Anderson delocalization on the surface of a topological insulator [RSFL10]. An open question which we address in this thesis is the classification of topological insulators outside the range where the K-theory framework applies, capturing for instance the Hopf insulator [MRW08].

A further generalization was the introduction of lattice defects, augmenting the classification by position-like (rather than momentum-like) coordinates for surfaces enclosing a defect [TK10]. Recently, a first step towards further generalization was taken in [ZK14] by considering phase differences in Josephson junctions between topological superconductors (viewed as topological insulators of Bogoliubov quasi-particles), which are neither position- nor momentum-like.

The point of view of homotopy theory is often adopted as a starting point, but results are then derived by more indirect means either through algebraic constructions as in [Kit09, SCR11, FM13] or the calculation of homotopy invariants.

In the present work, a homotopy theoretic derivation of the classification of topological insulators is developed in the general setting of [TK10]. We rediscover the known results entirely from this natural perspective and extend them beyond the stable K-theory regime while giving the exact conditions under which the previously derived results hold. Furthermore, we investigate how concepts like the distinction between "strong" and "weak" topological insulators can be generalized to the extended setting.

This thesis is organized as follows: In Chapter 2, the setting of independent quasi-particles and the description of ground states is introduced. After reviewing the relevant tools of homotopy theory in Chapter 3, we introduce the concept of topological phases by defining an equivalence relation between ground states, contrasting some alternative approaches taken in the current literature. In Chapter 4, we determine all topological phases in the stable regime, reproducing as a corollary the K-theory classification of [Kit09] as well as the stable classification involving defects in [TK10]. The exact conditions of applicability of these stable results are derived in Chapter 5. All cases where these conditions are violated are investigated in Chapter 6 in order to complete the classification of topological phases. The fruit of this labor is the discovery of a new topological phase which we call the Hopf superconductor in symmetry class C, a close cousin to the Hopf insulator of [MRW08]. For the generalized setting encompassing both the stable and unstable regime, we revisit the notions of strong and weak topological phases in Chapter 7, giving a modified definition of these terms which we show is consistent in general. The physical implications of non-trivial topological phases are considered in Chapter 8, cumulating in a discussion of interactions and disorder.

2. Quasi-particle ground states

The goal of this chapter is to introduce the concept of a quasi-particle ground state for a translation invariant Hamiltonian. We start by describing the single particle setting, which will then be used to define the many-body ground states of an extended class of Hamiltonians given by quadratic operators on Fock space. This will be followed by the introduction of symmetries and the relations they impose on ground states. As a final result, we systematically construct symmetry groups representing each of the ten possible symmetry classes by successively adding symmetries, a procedure that was started, but left incomplete, in [Kit09] (and hence will be referred as as the "Kitaev sequence").

2.1. Single particle setting

We assume that there is a d-dimensional lattice $\mathbb{Z}^d$ with minimal distance normalized to 1, describing, for example, the positions in a lattice of atoms or, more generally, unit cells. To every unit cell in $\mathbb{Z}^d$, we associate a Hilbert space $\mathbb{C}^n$ for the electrons (or more generally fermions) in the crystal. The n complex degrees of freedom introduced in this manner can have many physical interpretations, including the spin of an electron, the orbitals associated to the underlying atoms, the number of atoms per unit cell or any combination of these. The single particle Hilbert space $\mathcal{H}$ will therefore be defined as

$$\mathcal{H} := \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n. \tag{2.1}$$

Another name for this setting is the tight binding representation.

It is useful to fix a basis $\{|\mathbf{x}\rangle \otimes |i\rangle\} \equiv \{|\mathbf{x}, i\rangle\}$ of $\mathcal{H}$, where $|\mathbf{x}\rangle \in \ell^2(\mathbb{Z}^d)$ stands for the series on $\mathbb{Z}^d$ with value 1 at $\mathbf{x}$ and 0 everywhere else and $|i\rangle$ with $i = 1, \dots, n$ is some orthonormal basis of $\mathbb{C}^n$ (for instance the basis of orbitals).

A Hermitian scalar product is defined on the basis states as

$$\langle \mathbf{x}, i|\mathbf{y}, j\rangle := \delta_{\mathbf{x}\mathbf{y}}\delta_{ij}. \tag{2.2}$$

With respect to this scalar product, a translation $t_{\mathbf{a}}$ by $\mathbf{a} \in \mathbb{Z}^d$ is defined as a unitary operator

$$t_{\mathbf{a}}|\mathbf{x}, i\rangle := |\mathbf{x} + \mathbf{a}, i\rangle \tag{2.3}$$

and a single particle Hamiltonian $H = H^\dagger : \mathcal{H} \to \mathcal{H}$ is translation invariant if

$$t_{\mathbf{a}} H = H t_{\mathbf{a}} \tag{2.4}$$

for all $\mathbf{a} \in \mathbb{Z}^d$. In that case, it has the general form

$$H|\mathbf{x}, i\rangle = \sum_{\mathbf{y};j} h_{ji}(\mathbf{y})|\mathbf{x}+\mathbf{y}, j\rangle, \tag{2.5}$$

where $h(\mathbf{y}) : \mathbb{C}^n \to \mathbb{C}^n$ are the hopping matrices which satisfy $h(\mathbf{y}) = h(-\mathbf{y})^\dagger$ to ensure hermiticity of H. For a local Hamiltonian, the magnitude of these terms decreases exponentially with $|\mathbf{y}| := \max(|y_i|)_{i=1,\dots,d}$. In common models (called tight binding models) only terms with $|\mathbf{y}| \leq 1$ (nearest neighbor hopping) or $|\mathbf{y}| \leq 2$ (next-nearest neighbor hopping) are non-vanishing.

The translation invariance of H allows for a further simplification: A simultaneous eigenbasis of all operators $t_{\mathbf{a}}$ can be defined using the discrete Fourier transform

$$|\mathbf{k}, i\rangle := \frac{1}{\sqrt{V}} \sum_{\mathbf{x}} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}, i\rangle, \tag{2.6}$$

where $\mathbf{k}$ is an element of the d-dimensional torus T^d (the dual of $\mathbb{Z}^d$) and V is the volume of the system, which is introduced as a regularization to render the set $\{|\mathbf{k}, i\rangle\}$ an orthonormal basis of $\mathcal{H}$ (with the goal of sending $V \to \infty$).[1] Indeed, applying $t_{\mathbf{a}}$ yields

$$t_{\mathbf{a}}|\mathbf{k}, i\rangle = e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}}|\mathbf{k}, i\rangle. \tag{2.7}$$

Since eq. (2.4) implies that H must leave the eigenspaces of all translations $t_{\mathbf{a}}$ invariant, H acts block diagonally as

$$\begin{aligned} H|\mathbf{k}, i\rangle &= \frac{1}{\sqrt{V}} \sum_{\mathbf{x};j} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} h_{ji}(\mathbf{y})|\mathbf{x}+\mathbf{y}, j\rangle \\ &= \frac{1}{\sqrt{V}} \sum_{\mathbf{x}';j} e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{y})} h_{ji}(\mathbf{y})|\mathbf{x}', j\rangle \\ &= \sum_j H_{ji}(\mathbf{k})|\mathbf{k}, j\rangle, \end{aligned} \tag{2.8}$$

where we have defined the *Bloch Hamiltonian* as

$$H(\mathbf{k}) := \sum_{\mathbf{y}} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}} h(\mathbf{y}). \tag{2.9}$$

[1] The vectors $|\mathbf{k}, i\rangle$ are not well defined for $V \to \infty$, but a proper regularization takes care of this problem. For instance, one may consider an arbitrarily large, but finite subset of $\mathbb{Z}^d$ with volume V and periodic boundary conditions in order to enable the implementation of translation invariance.

In other words, the Hilbert space $\mathcal{H}$ decomposes as an orthogonal sum

$$\mathcal{H} = \bigoplus_{\mathbf{k}} \mathcal{H}_{\mathbf{k}} \tag{2.10}$$

of eigenspaces $\mathcal{H}_{\mathbf{k}}$ corresponding to the eigenvalue $e^{-i\mathbf{k}\cdot\mathbf{a}}$ under translations. The Bloch Hamiltonian is simply given by the restriction of the Hamiltonian H to one of these eigenspaces and due to (2.4), its image is contained in the same component:

$$H(\mathbf{k}) = H|_{\mathcal{H}_{\mathbf{k}}} : \mathcal{H}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{k}}. \tag{2.11}$$

2.2. Fock space and many-body ground states

Up to this point, the setting was that of a single particle on a lattice with a Hamiltonian that is invariant under lattice translations. In order to describe many-body states, the first step is to specify the exchange statistics. In our case, we will be interested in fermions (usually electrons), so the proper many-body Hilbert space is given by the *Fock space* $\mathcal{F}$, which is the exterior algebra

$$\mathcal{F} := \wedge(\mathcal{H}) = \bigoplus_{m} \wedge^m(\mathcal{H}). \tag{2.12}$$

Here $\wedge^m(\mathcal{H})$ is the subspace of m-particle states $|\phi_1\rangle \wedge \cdots \wedge |\phi_m\rangle$ with $|\phi_i\rangle \in \mathcal{H}$. The orthonormal basis $\{|\mathbf{k}, i\rangle\}$ of $\mathcal{H}$ induces a basis of $\wedge^m(\mathcal{H})$ given by the set

$$\{|\mathbf{k}_1, i_1\rangle \wedge \cdots \wedge |\mathbf{k}_m, i_m\rangle\} \tag{2.13}$$

and the union over the number of fermions m yields a basis for all of $\mathcal{F}$. In this basis, a Hermitian scalar product can be defined by

$$\begin{aligned}\Big\langle |\mathbf{k}_1, i_1\rangle \wedge \cdots \wedge |\mathbf{k}_m, i_m\rangle, |\mathbf{k}'_1, i'_1\rangle \wedge \cdots \wedge |\mathbf{k}'_m, i'_m\rangle \Big\rangle& \\ := \delta_{\mathbf{k}_1,\mathbf{k}'_1} \cdots \delta_{\mathbf{k}_m,\mathbf{k}'_m} \delta_{i_1,i'_1} \cdots \delta_{i_m,i'_m}&.\end{aligned} \tag{2.14}$$

For two states with different particle numbers, it is defined to be 0.

We denote by $c_i^\dagger(\mathbf{k})$ the operator which creates a particle in the state $|\mathbf{k}, i\rangle$, realized in $\mathcal{F}$ through exterior multiplication by $|\mathbf{k}, i\rangle$. Its Hermitian conjugate with respect to the scalar product defined above will be denoted by $c_i(\mathbf{k})$. This operation annihilates the particle which is in the state $|\mathbf{k}, i\rangle$ and it is realized in $\mathcal{F}$ by contraction with the form $\langle \mathbf{k}, i| \in \mathcal{H}^*$. These operators fulfill the canonical anti-commutation relations (CAR):

$$\begin{aligned} c_i^\dagger(\mathbf{k})c_{i'}^\dagger(\mathbf{k}') + c_{i'}^\dagger(\mathbf{k}')c_i^\dagger(\mathbf{k}) &= 0 \\ c_i(\mathbf{k})c_{i'}(\mathbf{k}') + c_{i'}(\mathbf{k}')c_i(\mathbf{k}) &= 0 \\ c_i^\dagger(\mathbf{k})c_{i'}(\mathbf{k}') + c_{i'}(\mathbf{k}')c_i^\dagger(\mathbf{k}) &= \delta_{ii'}\delta_{\mathbf{k},\mathbf{k}'}. \end{aligned} \tag{2.15}$$

We call these creation and annihilation operators *bare*, as opposed to linear combinations of them which will appear later.

Single particle operators $O : \mathcal{H} \to \mathcal{H}$ are extended to operators $\hat{O} : \mathcal{F} \to \mathcal{F}$ by

$$\hat{O}(|\mathbf{k}_1, i_1\rangle \wedge \cdots \wedge |\mathbf{k}_m, i_m\rangle) := (O|\mathbf{k}_1, i_1\rangle) \wedge \cdots \wedge (O|\mathbf{k}_m, i_m\rangle) \tag{2.16}$$

and linear extension thereof for general elements in $\mathcal{F}$. Alternatively, all single particle operators may be expressed succinctly through particle creation and annihilation operators as

$$\hat{O} = \sum_{\mathbf{k},\mathbf{k}';i,j} O_{ij}(\mathbf{k}, \mathbf{k}') c_i^\dagger(\mathbf{k}) c_j(\mathbf{k}'). \tag{2.17}$$

In particular, the translation operator $t_\mathbf{a}$ has the many-body form

$$\hat{t}_\mathbf{a} = \sum_{\mathbf{k};i} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} c_i^\dagger(\mathbf{k}) c_i(\mathbf{k}), \tag{2.18}$$

while the Hamiltonian H is turned into

$$\hat{H} = \sum_{\mathbf{k};i,j} H_{ij}(\mathbf{k}) c_i^\dagger(\mathbf{k}) c_j(\mathbf{k}), \tag{2.19}$$

with the Bloch Hamiltonian $H(\mathbf{k})$ as introduced in eq. (2.9). Of course, the analog of relation (2.4) still holds true, so $\hat{H}$ is translation invariant:

$$\hat{t}_\mathbf{a} \hat{H} = \hat{H} \hat{t}_\mathbf{a}. \tag{2.20}$$

In an eigenbasis of $H(\mathbf{k})$ with creation and annihilation operators $\tilde{c}_i^\dagger(\mathbf{k})$ and $\tilde{c}_i(\mathbf{k})$ corresponding to creating (and respectively, annihilating) a particle with energy $E_i(\mathbf{k})$ in the eigenstate $|\psi_i(\mathbf{k})\rangle$, it has the form

$$\hat{H} = \sum_{\mathbf{k};i} E_i(\mathbf{k}) \tilde{c}_i^\dagger(\mathbf{k}) \tilde{c}_i(\mathbf{k}). \tag{2.21}$$

We choose to order the energies according to $E_i(\mathbf{k}) \leq E_j(\mathbf{k})$ for $i < j$. The many-body ground state $|\text{g.s.}\rangle \in \mathcal{F}$ is obtained by filling the energy eigenstates from the one with least energy upwards. The energies $E_i(\mathbf{k})$ are continuous[2] functions of the momentum $\mathbf{k}$ forming the i-th *energy band* and if the associated eigenstates are filled for all $\mathbf{k} \in \mathrm{T}^d$, this band is called an occupied (or valence) band. We will exclusively consider the case of insulators (as opposed to conductors), where the first p bands are occupied and the remaining $n - p$ bands, called conduction bands, are empty, with an

[2] The eigenstates $|\psi_i(\mathbf{k})\rangle$, however, need *not* be continuous functions of $\mathbf{k}$ in general. In fact, in many cases of topological insulators there *cannot* be a continuous choice.

energy gap $E_p(\mathbf{k}) < E_{p+1}(\mathbf{k})$ for all $\mathbf{k} \in \mathrm{T}^d$ as illustrated in Figure 2.1. In that case, the ground state of $\hat{H}$ is an element in $\mathcal{F}$ given by

$$|\text{g.s.}\rangle = \prod_{\mathbf{k}\in\mathrm{T}^d} \tilde{c}_1^\dagger(\mathbf{k})\tilde{c}_2^\dagger(\mathbf{k})\cdots\tilde{c}_p^\dagger(\mathbf{k})|0\rangle, \tag{2.22}$$

where $|0\rangle \in \wedge^0(\mathcal{H}) = \mathbb{C}$ is the vacuum state.

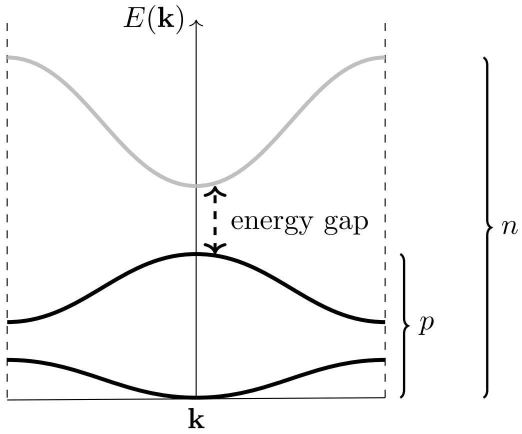


Figure 2.1.: Schematic illustration of energy bands in dimension $d = 1$ with $n = 3$ bands of which $p = 2$ bands are occupied and $n - p = 1$ band is empty. Periodic boundary conditions due to $\mathbf{k} \in \mathrm{T}^1 = \mathrm{S}^1$ are indicated by the dashed lines on the left and on the right.

Notice that in order to specify the ground state, any set of p linearly independent operators in the vector space

$$C(\mathbf{k}) := \mathrm{span}_{\mathbb{C}}\{\tilde{c}_1^\dagger(\mathbf{k}), \tilde{c}_2^\dagger(\mathbf{k}), \ldots, \tilde{c}_p^\dagger(\mathbf{k})\} \tag{2.23}$$

applied to $|0\rangle$ at every momentum $\mathbf{k}$ would yield a state proportional to $|\text{g.s.}\rangle$. Indeed, if a new set of operators is constructed from the one introduced above through an invertible matrix $S_\mathbf{k} : C(\mathbf{k}) \to C(\mathbf{k})$ (not required to be continuous in $\mathbf{k}$), then eq. (2.22) with $\tilde{c}_i^\dagger(\mathbf{k})$ replaced by $S_\mathbf{k}(\tilde{c}_i^\dagger(\mathbf{k}))$ would be identical up to a factor $\prod_\mathbf{k} \det(S_\mathbf{k}) \neq 0$. For all physical observables, only the ray $\mathbb{C}\cdot|\text{g.s.}\rangle$ is relevant, so the result is physically identical.

Furthermore, unlike the eigenstates $|\psi_i(\mathbf{k})\rangle$, the subspace $C(\mathbf{k})$ varies continuously with $\mathbf{k} \in \mathrm{T}^d$: Modeling the torus T^d as the quotient space $\mathbb{R}^d/2\pi\mathbb{Z}^d$, non-degenerate eigenstates $|\psi_i(\mathbf{k})\rangle$ merely have to satisfy the condition that both $|\psi_i(\mathbf{k})\rangle$ and $|\psi_i(\mathbf{k} + \mathbf{G})\rangle$ be eigenstates of H with the same eigenvalue for all reciprocal lattice vectors $\mathbf{G} \in 2\pi\mathbb{Z}^d$, so that

$$|\psi_i(\mathbf{k}+\mathbf{G})\rangle = \lambda_{\mathbf{G}}|\psi_i(\mathbf{k})\rangle, \tag{2.24}$$

with non-zero $\lambda_{\mathbf{G}} \in \mathbb{C}$, the phase of which can be interpreted as a *Berry phase* [Ber84]. In the degenerate case, the eigenspaces at $\mathbf{k}$ and $\mathbf{k}+\mathbf{G}$ are related by some invertible complex matrix. In either case, we have

$$C(\mathbf{k}+\mathbf{G}) = C(\mathbf{k}). \tag{2.25}$$

An alternative view of $C(\mathbf{k})$, which will be useful for the generalization of the current setting in the next section, presents itself by introducing generalized annihilation operators

$$\alpha_i(\mathbf{k}) := \begin{cases} \tilde{c}_i(\mathbf{k}) & \text{for } i > p \\ \tilde{c}_i^\dagger(-\mathbf{k}) & \text{for } i \leq p. \end{cases} \tag{2.26}$$

These operators have the property that they all annihilate the ground state $|\text{g.s.}\rangle$ and, taken together with their Hermitian conjugates, they fulfill the canonical anti-commutation relations (2.15). Shifting the energies by a constant (the chemical potential) such that, for all $\mathbf{k} \in \mathrm{T}^d$, $E_i(\mathbf{k}) < 0$ for $i \leq p$ and $E_i(\mathbf{k}) > 0$ for $i > p$, the Hamiltonian expressed in terms of the new operators reads

$$\hat{H} = \sum_{\mathbf{k};i} |E_i(\mathbf{k})| \alpha_i^\dagger(\mathbf{k})\alpha_i(\mathbf{k}) + \text{const.} \tag{2.27}$$

This expression makes manifest that a state in $\mathcal{F}$ is the ground state of $\hat{H}$ if and only if it is annihilated by all $\alpha_i(\mathbf{k})$, which confirms that $|\text{g.s.}\rangle$ is indeed the ground state. Quasi-particle excitations are given by $\alpha_i^\dagger(\mathbf{k})|\text{g.s.}\rangle$ and correspond to the creation of particles ($i > p$) or holes ($i \leq p$).

We now formalize the role of the new set of operators by introducing the $2n$-dimensional vector space $\mathcal{W}_{\mathbf{k}}$ of all linear combinations of creation and annihilation operators that decrease the momentum by $\mathbf{k}$,

$$\begin{aligned} \mathcal{W}_{\mathbf{k}} &:= \operatorname{span}_{\mathbb{C}}\{\alpha_1(\mathbf{k}), \dots, \alpha_n(\mathbf{k}), \alpha_1^\dagger(-\mathbf{k}), \dots, \alpha_n^\dagger(-\mathbf{k})\} & (2.28)\\ &= \operatorname{span}_{\mathbb{C}}\{\tilde{c}_1(\mathbf{k}), \dots, \tilde{c}_n(\mathbf{k}), \tilde{c}_1^\dagger(-\mathbf{k}), \dots, \tilde{c}_n^\dagger(-\mathbf{k})\} & (2.29)\\ &= \operatorname{span}_{\mathbb{C}}\{c_1(\mathbf{k}), \dots, c_n(\mathbf{k}), c_1^\dagger(-\mathbf{k}), \dots, c_n^\dagger(-\mathbf{k})\}. & (2.30) \end{aligned}$$

This space splits as $\mathcal{W}_{\mathbf{k}} = \mathcal{H}_{\mathbf{k}}^* \oplus \mathcal{H}_{-\mathbf{k}}$ and corresponds to a component of what is known as *Nambu space*

$$\mathcal{H}^* \oplus \mathcal{H} = \bigoplus_{\mathbf{k}} (\mathcal{H}_{\mathbf{k}}^* \oplus \mathcal{H}_{-\mathbf{k}}) = \bigoplus_{\mathbf{k}} \mathcal{W}_{\mathbf{k}}. \tag{2.31}$$

In this decomposition, we have identified the space of bare annihilation operators with $\mathcal{H}^*$ by restricting annihilators to maps $\wedge^1(\mathcal{H}) \to \wedge^0(\mathcal{H})$, where we can identify the domain with $\mathcal{H}$ and the codomain with $\mathbb{C}$. Under this identification, the component $\mathcal{H}^*_{\mathbf{k}}$ corresponds to bare annihilators reducing the momentum by $\mathbf{k}$. Similarly, the bare creation operators restrict to $\wedge^0(\mathcal{H}) \to \wedge^1(\mathcal{H})$ and we can identify them with their image of $|0\rangle$ to obtain elements in $\mathcal{H}$. From this point of view, the component $\mathcal{H}_{-\mathbf{k}}$ contains bare creation operators also decreasing the momentum by $\mathbf{k}$.

There is a canonical bijection $\mathcal{H} \to \mathcal{H}^*$ which assigns to a vector $v \in \mathcal{H}$ the function $\langle v, \cdot\rangle \in \mathcal{H}^*$. Using this bijection on the subspace $\mathcal{H} \subset \mathcal{H}^* \oplus \mathcal{H}$ and its inverse on $\mathcal{H}^* \subset \mathcal{H}^* \oplus \mathcal{H}$ defines an anti-linear map $\gamma : \mathcal{H}^* \oplus \mathcal{H} \to \mathcal{H}^* \oplus \mathcal{H}$ with $\gamma^2 = 1$. In the interpretation of $\mathcal{H}^* \oplus \mathcal{H}$ as the space of all linear combinations of creation and annihilation operators, γ is simply given by Hermitian conjugation. Its restriction to $\mathcal{W}_{\mathbf{k}} \subset \mathcal{H}^* \oplus \mathcal{H}$ can be written explicitly as

$$\begin{aligned} \gamma : \mathcal{W}_{\mathbf{k}} &\to \mathcal{W}_{-\mathbf{k}} \\ \sum_i u_i c_i(\mathbf{k}) + v_i c_i^\dagger(-\mathbf{k}) &\mapsto \sum_i \bar{u}_i c_i^\dagger(\mathbf{k}) + \bar{v}_i c_i(-\mathbf{k}). \end{aligned} \tag{2.32}$$

Another structure on $\mathcal{H}^* \oplus \mathcal{H}$ is the pairing given by the anti-commutator

$$\{\cdot,\cdot\} : (\mathcal{H}^* \oplus \mathcal{H}) \otimes (\mathcal{H}^* \oplus \mathcal{H}) \to \mathbb{C}, \tag{2.33}$$

which can only be non-zero for pairs taken from components with opposite momentum and therefore descends to a pairing

$$\{\cdot,\cdot\} : \mathcal{W}_{\mathbf{k}} \otimes \mathcal{W}_{-\mathbf{k}} \to \mathbb{C}. \tag{2.34}$$

Using the anti-commutator above in conjunction with the map γ, we can define a natural Hermitian scalar product for $w, w' \in \mathcal{W}_{\mathbf{k}}$:

$$\langle w, w'\rangle := \{\gamma w, w'\}. \tag{2.35}$$

This definition gives the standard scalar product on $\mathbb{C}^{2n}$ in a basis of $\mathcal{W}_{\mathbf{k}}$ consisting of operators obeying the CAR (2.15):

$$\langle c_i(\mathbf{k}), c_j^\dagger(-\mathbf{k})\rangle = \ 0 \ = \langle c_i^\dagger(-\mathbf{k}), c_j(\mathbf{k})\rangle \tag{2.36}$$

$$\langle c_i(\mathbf{k}), c_j(\mathbf{k})\rangle = \delta_{ij} = \langle c_i^\dagger(-\mathbf{k}), c_j^\dagger(-\mathbf{k})\rangle. \tag{2.37}$$

For general elements in $\mathcal{W}_{\mathbf{k}}$, we extend anti-linearly in the left and linearly in the right argument as usual.

With respect to this scalar product, the map γ is seen to be anti-unitary, since, for all $w, w' \in \mathcal{W}_{\mathbf{k}}$,

$$\begin{aligned}\langle \gamma w, \gamma w' \rangle &= \{\gamma^2 w, \gamma w'\} \\ &= \{w, \gamma w'\} \\ &= \{\gamma w', w\} \\ &= \langle w', w \rangle. \end{aligned} \tag{2.38}$$

We now return to the connection between the ground state and its annihilators: The continuous map assigning to every $\mathbf{k} \in \mathrm{T}^d$ the n-dimensional subspace

$$A(\mathbf{k}) := \mathrm{span}_{\mathbb{C}}\{\alpha_1(\mathbf{k}), \alpha_2(\mathbf{k}), \ldots, \alpha_n(\mathbf{k})\} \subset \mathcal{W}_{\mathbf{k}}, \tag{2.39}$$

subject to the constraint

$$\{A(\mathbf{k}), A(-\mathbf{k})\} = 0 \tag{2.40}$$

for all $\mathbf{k} \in \mathrm{T}^d$ due to the CAR (2.15), uniquely determines the ground state $|\mathrm{g.s.}\rangle$. The reason is that, with respect to the scalar product defined in eq. (2.35), $A(\mathbf{k})$ splits into an orthogonal sum

$$A(\mathbf{k}) = A^{\mathrm{p}}(\mathbf{k}) \oplus A^{\mathrm{h}}(\mathbf{k}), \tag{2.41}$$

where $A^{\mathrm{p}}(\mathbf{k}) := \mathrm{span}_{\mathbb{C}}\{\alpha_{p+1}(\mathbf{k}), \alpha_{p+2}(\mathbf{k}), \ldots, \alpha_n(\mathbf{k})\}$ and $A^{\mathrm{h}}(\mathbf{k}) = C(-\mathbf{k})$ from eq. (2.23). The superscripts p and h stand for *particle* and *hole*, since $A^{\mathrm{p}}(\mathbf{k}) \subset \mathcal{H}^*_{\mathbf{k}}$ annihilates particles, while $A^{\mathrm{h}}(\mathbf{k}) \subset \mathcal{H}_{-\mathbf{k}}$ annihilates holes (= creates particles). Thus, specifying either one of $A^{\mathrm{h}}(\mathbf{k})$ or $A^{\mathrm{p}}(\mathbf{k})$ determines the other as its orthogonal complement and therefore suffices to determine $A(\mathbf{k})$. Notice that the constraint of eq. (2.40) is automatically fulfilled here.

In the present setting, the framework introduced above is equivalent to specifying the map $\mathbf{k} \mapsto C(\mathbf{k})$ from before. However, it will be necessary for the generalized setting of the next section, where $A(\mathbf{k})$ is still well-defined in contrast to $C(\mathbf{k})$.

2.3. Superconductors

We now wish to expand the framework introduced in the previous section by generalizing the Hamiltonian $\hat{H}$ of eq. (2.19) to

$$\hat{H} = \sum_{\mathbf{k};i,j} H_{ij}(\mathbf{k}) c_i^\dagger(\mathbf{k}) c_j(\mathbf{k}) + \Delta_{ij}(\mathbf{k}) c_i^\dagger(\mathbf{k}) c_j^\dagger(-\mathbf{k}) + \overline{\Delta_{ij}(\mathbf{k})} c_j(-\mathbf{k}) c_i(\mathbf{k}). \tag{2.42}$$

The additional terms allow for a mean-field description of superconductors by including the creation and annihilation of Cooper pairs. This Hamiltonian is still translation invariant, since translations act on creation and annihilation operators as

$$\hat{t}_{\mathbf{a}} c_i^\dagger(\mathbf{k}) \hat{t}_{\mathbf{a}}^{-1} = e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} c_i^\dagger(\mathbf{k}) \tag{2.43}$$

$$\hat{t}_{\mathbf{a}} c_i(\mathbf{k}) \hat{t}_{\mathbf{a}}^{-1} = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}} c_i(\mathbf{k}). \tag{2.44}$$

Hence, for a translation invariant Hamiltonian, pairs have to be created and annihilated with opposite momenta. This is reasonable physically as translation invariance leads to momentum conservation and the only way to achieve this whilst creating or annihilating a pair of particles is to assign opposite momenta to each constituent.

Repeating the analysis of the previous section, we require a new set of operators $\tilde{c}_i^\dagger(\mathbf{k})$ and $\tilde{c}_i(\mathbf{k})$ such that

$$\hat{H} = \sum_{\mathbf{k};i} |E_i(\mathbf{k})| \alpha_i^\dagger(\mathbf{k}) \alpha_i(\mathbf{k}) + \text{const.}, \tag{2.45}$$

In contrast to before, if the coefficients $\Delta_{ij}(\mathbf{k})$ in eq. (2.42) are non-vanishing, the new operators are required to be linear combinations containing *both* types $c_i^\dagger(\mathbf{k})$ and $c_i(-\mathbf{k})$. The generalized setting introduced in the previous section applies to this situation: The ground state is the state annihilated by all $\alpha_i(\mathbf{k})$ and specifying it is equivalent to specifying the space of these annihilators in the form of a continuous map $\mathbf{k} \mapsto A(\mathbf{k}) \subset \mathcal{W}_{\mathbf{k}}$ subject to the constraint (2.40).

More formally, the vector spaces $\mathcal{W}_{\mathbf{k}}$ are, by construction, isomorphic to $(\mathbb{C}^n)^* \oplus \mathbb{C}^n \simeq \mathbb{C}^{2n}$, independent of $\mathbf{k}$. In the language of vector bundles, we therefore have a trivial bundle $\{\mathcal{W}_{\mathbf{k}}\}_{\mathbf{k}\in \mathrm{T}^d} \simeq \mathrm{T}^d \times \mathbb{C}^{2n}$. Thus, we can identify all fibers and often write $\mathcal{W}_{\mathbf{k}} \equiv \mathcal{W} \equiv \mathbb{C}^{2n}$ for simplicity. There may be situations where the vector bundle is non-trivial, for instance in effective low energy theories which discard some bands and only focus on the ones closest to the Fermi energy. In any case, the assignment $\mathbf{k} \mapsto A(\mathbf{k})$ defines a sub-vector bundle of $\{\mathcal{W}_{\mathbf{k}}\}_{\mathbf{k}\in \mathrm{T}^d}$. Focusing on the case where $\{\mathcal{W}_{\mathbf{k}}\}_{\mathbf{k}\in \mathrm{T}^d}$ is trivial, we are now in a position to give a formal definition of what kinds of ground states we will examine in this work:

Definition 2.1. *By an* IQPV *(insulator quasi-particle vacuum) we mean a complex sub-vector bundle* $\mathcal{A} \xrightarrow{\rho} \mathrm{T}^d$ *with fibers* $\rho^{-1}(\mathbf{k}) \equiv A(\mathbf{k}) \subset \mathcal{W} = \mathbb{C}^{2n}$ *of dimension* n *such that all pairs of fibers* $A(\mathbf{k})$ *and* $A(-\mathbf{k})$ *annihilate one another with respect to the CAR pairing:*

$$\forall \mathbf{k} \in \mathrm{T}^d : \quad \{A(\mathbf{k}), A(-\mathbf{k})\} = 0. \tag{2.46}$$

There is an alternative, yet equivalent description which will be adopted throughout the later parts of this thesis. It formalizes the notion of the map $\mathbf{k} \mapsto A(\mathbf{k}) \subset \mathbb{C}^{2n}$ by defining its codomain to be the *Grassmannian*

$$\mathrm{Gr}_n(\mathbb{C}^{2n}) := \{\text{Subvector spaces } A \subset \mathbb{C}^{2n} \text{ with } \dim(A) = n\}. \tag{2.47}$$

The CAR constraint (2.40) can be realized on $\mathcal{W}_{\mathbf{k}} \simeq \mathbb{C}^{2n}$ by defining an involution

$$\begin{aligned} \tau_0 : \mathrm{Gr}_n(\mathbb{C}^{2n}) &\to \mathrm{Gr}_n(\mathbb{C}^{2n}) \\ A &\mapsto A^{\perp}, \end{aligned} \tag{2.48}$$

where

$$A^{\perp} := \{w \in \mathbb{C}^{2n} : \{w, w'\} = 0 \text{ for all } w' \in A\}. \tag{2.49}$$

The alternative definition can now be given as

Definition 2.2. *By an* IQPV *(insulator quasi-particle vacuum) we mean a continuous map*

$$\begin{aligned} A : \mathrm{T}^d &\to \mathrm{Gr}_n(\mathbb{C}^{2n}) \\ \mathbf{k} &\mapsto A(\mathbf{k}), \end{aligned} \tag{2.50}$$

subject to the condition

$$A(-\mathbf{k}) = \tau_0(A(\mathbf{k})). \tag{2.51}$$

It will turn out to be useful to denote by $\tau : \mathrm{T}^d \to \mathrm{T}^d$ the involution $\tau(\mathbf{k}) = -\mathbf{k}$ such that the constraint in the definition above may be rephrased as an equivariance condition

$$A \circ \tau = \tau_0 \circ A. \tag{2.52}$$

Remark 2.3. The Hamiltonian $\hat{H}$ given in eq. (2.42) can be associated with an endomorphism[3] $H_{\mathrm{BdG}}(\mathbf{k}) : \mathcal{W}_{\mathbf{k}} \to \mathcal{W}_{\mathbf{k}}$ in analogy to the Bloch Hamiltonian $H(\mathbf{k}) : \mathcal{H}_{\mathbf{k}} \to \mathcal{H}_{\mathbf{k}}$ defined in eq. (2.11). Writing $(c_1(\mathbf{k}), \ldots, c_n(\mathbf{k}), c_1^{\dagger}(-\mathbf{k}), \ldots, c_n^{\dagger}(-\mathbf{k}))^t \equiv (\mathbf{c}(-\mathbf{k}), \mathbf{c}^{\dagger}(\mathbf{k}))^t$, it is given by

$$\hat{H} = \sum_{\mathbf{k}} \begin{pmatrix} \mathbf{c}^{\dagger}(\mathbf{k}) & \mathbf{c}(-\mathbf{k}) \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1}{2}H(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta^{\dagger}(\mathbf{k}) & -\frac{1}{2}H(-\mathbf{k})^T \end{pmatrix}}_{H_{\mathrm{BdG}}(\mathbf{k})} \begin{pmatrix} \mathbf{c}(\mathbf{k}) \\ \mathbf{c}^{\dagger}(-\mathbf{k}) \end{pmatrix} + \text{const.} \tag{2.53}$$

[3]The subscript "BdG" is short for Bogoliubov-de Gennes and $H_{\mathrm{BdG}}(\mathbf{k})$ is often referred to as the BdG- or Bogoliubov-de Gennes Hamiltonian.

In the basis $\{c_1(\mathbf{k}),\dots,c_n(\mathbf{k}),c_1^\dagger(-\mathbf{k}),\dots,c_n^\dagger(-\mathbf{k})\}$, the matrix $H_{\mathrm{BdG}}(\mathbf{k})$ is an endomorphism of $\mathcal{W}_\mathbf{k}$ and due to the CAR (2.15), it is restricted by the relation

$$\Delta(\mathbf{k}) = -\Delta(-\mathbf{k})^T. \tag{2.54}$$

Finding the set of annihilators $\{\alpha_i(\mathbf{k})\}$ annihilating the ground state amounts to finding a transformation diagonalizing $H_{\mathrm{BdG}}(\mathbf{k})$ such that $\hat{H}$ can be written as

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix}\alpha^\dagger(\mathbf{k}) & \alpha(-\mathbf{k})\end{pmatrix} \begin{pmatrix}\mathrm{diag}\,(|E_i(\mathbf{k})|) & 0 \\ 0 & \mathrm{diag}\,(-|E_i(-\mathbf{k})|)\end{pmatrix} \begin{pmatrix}\alpha(\mathbf{k}) \\ \alpha^\dagger(-\mathbf{k})\end{pmatrix} + \text{const.} \tag{2.55}$$

It follows that the space spanned by the eigenstates of $H_{\mathrm{BdG}}(\mathbf{k})$ with negative eigenvalue is equivalent to the space of annihilators at $\mathbf{k}$, while the space spanned by those with positive eigenvalue corresponds to the space of creators at $-\mathbf{k}$.

2.4. Symmetries

Symmetries are introduced into our framework through a symmetry group $\mathcal{G}$ which is represented by unitary or anti-unitary operators on the single particle Hilbert space $\mathcal{H}$ that commute with the Hamiltonian. We assume that translations form a normal Abelian subgroup $\Pi \subset \mathcal{G}$ and that all other symmetries commute with elements in this subgroup. Therefore, on $\mathcal{H}$, we have the relation

$$t_\mathbf{a} g = g t_\mathbf{a} \tag{2.56}$$

for all translations $t_\mathbf{a} \in \Pi$ and $g \in \mathcal{G}$ (note that we use the same notation for elements of the abstract group $\mathcal{G}$ and the corresponding operators on $\mathcal{H}$).

Unitary and anti-unitary representations on $\mathcal{H}$ have a natural extension to

$$\mathcal{H}^* \oplus \mathcal{H} = \bigoplus_{\mathbf{k}} (\mathcal{H}^*_\mathbf{k} \oplus \mathcal{H}_{-\mathbf{k}}) = \bigoplus_{\mathbf{k}} \mathcal{W}_\mathbf{k} \tag{2.57}$$

by assigning to an operator $g : \mathcal{H} \to \mathcal{H}$ the operator

$$(g^{-1})^T \oplus g : \mathcal{H}^* \oplus \mathcal{H} \to \mathcal{H}^* \oplus \mathcal{H}. \tag{2.58}$$

Given an eigenvector of translations $|\psi\rangle \in \mathcal{H}_\mathbf{k}$ with $t_\mathbf{a}|\psi\rangle = e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}}|\psi\rangle$, applying an element $g \in \mathcal{G}$ yields another eigenvector with

$$t_\mathbf{a} g|\psi\rangle \overset{(2.56)}{=} g t_\mathbf{a}|\psi\rangle = g e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}}|\psi\rangle = \begin{cases} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}} g|\psi\rangle & \text{for unitary } g \\ e^{\mathrm{i}\mathbf{k}\cdot\mathbf{a}} g|\psi\rangle & \text{for anti-unitary } g. \end{cases} \tag{2.59}$$

Therefore,

$$g|_{\mathcal{W}_{\mathbf{k}}} : \mathcal{W}_{\mathbf{k}} \to \mathcal{W}_{\mathbf{k}} \quad \text{for unitary } g, \tag{2.60}$$

whereas

$$g|_{\mathcal{W}_{\mathbf{k}}} : \mathcal{W}_{\mathbf{k}} \to \mathcal{W}_{-\mathbf{k}} \text{ for anti-unitary } g. \tag{2.61}$$

We are now in a position to introduce the concept of a $\mathcal{G}$-symmetric IQPV:

Definition 2.4. *An IQPV has a symmetry group $\mathcal{G}$ with the described properties if, for all $\mathbf{k} \in \mathrm{T}^d$,*

$$gA(\mathbf{k}) = A(\mathbf{k}) \tag{2.62}$$

for all unitary $g \in \mathcal{G}/\Pi$ and

$$gA(\mathbf{k}) = A(-\mathbf{k}) \tag{2.63}$$

for all anti-unitary $g \in \mathcal{G}/\Pi$.

It is sufficient to consider the reduced symmetry group $\mathcal{G}/\Pi$, since all translations act as scalars on $A(\mathbf{k})$ by construction:

$$t_{\mathbf{a}}A(\mathbf{k}) = e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{a}}A(\mathbf{k}) = A(\mathbf{k}). \tag{2.64}$$

The type of IQPVs introduced in section 2.3 are recovered by setting $\mathcal{G} = \Pi$, so that $\mathcal{G}/\Pi$ is the trivial group and only the CAR constraint (2.40) needs to be satisfied. On the other hand, the setting of section 2.2 is recovered by setting $\mathcal{G} = \Pi \times \mathrm{U}_1$, so $\mathcal{G}/\Pi = \mathrm{U}_1$. This symmetry implements particle number conservation and is intact if $\Delta(\mathbf{k}) = 0$ in eq. (2.42) (no pair creation or annihilation). For later use, we write elements in U_1 as $e^{\mathrm{i}\theta Q}$ for some $\theta \in [0, 2\pi]$ and generator Q which acts as -1 on $\mathcal{H}^*_{\mathbf{k}}$ and as $+1$ on $\mathcal{H}_{-\mathbf{k}}$. We will often make use of Q rather than the exponentiated $e^{\mathrm{i}\theta Q}$ by exploiting the fact that

$$e^{\mathrm{i}\theta Q}A(\mathbf{k}) = A(\mathbf{k}) \iff QA(\mathbf{k}) = A(\mathbf{k}). \tag{2.65}$$

2.5. Kitaev sequence

We will now construct ten examples of reduced symmetry groups $\mathcal{G}/\Pi$ by systematically adding symmetries to the setting of Section 2.3. In the end, we will show that these ten cases already give all possible settings for the kind of symmetry groups introduced in the preceding section. They will be split into two sets, one containing two classes known as the *complex* symmetry classes and another containing eight classes known as the *real* symmetry classes.

The accumulation of symmetries can be described systematically and succinctly by turning the symmetries into a set of *pseudo*-symmetries, which are defined as follows:

Definition 2.5. *An IQPV* $\mathbf{k} \mapsto A(\mathbf{k})$ *has* s *pseudo-symmetries if there is a set of* $\mathbf{k}$*-independent, orthogonal and unitary operators* $J_1, \ldots, J_s : \mathcal{W}_\mathbf{k} \to \mathcal{W}_\mathbf{k}$ *satisfying the Clifford relations*

$$J_l J_m + J_m J_l = -2\delta_{lm} \quad (l, m = 1, \ldots, s) \tag{2.66}$$

and, for all $\mathbf{k} \in \mathrm{T}^d$,

$$\langle A(\mathbf{k}), J_1 A(\mathbf{k}) \rangle = \cdots = \langle A(\mathbf{k}), J_s A(\mathbf{k}) \rangle = 0. \tag{2.67}$$

Remark 2.6. An orthogonal unitary transformation J of $\mathcal{W}_\mathbf{k}$ is a $\mathbb{C}$-linear operator with the properties

$$\langle Jw, Jw' \rangle = \langle w, w' \rangle \quad \text{and} \quad \{Jw, Jw'\} = \{w, w'\} \tag{2.68}$$

for all $w, w' \in \mathcal{W}_\mathbf{k}$.

The condition $\langle A(\mathbf{k}), JA(\mathbf{k}) \rangle = 0$ can equivalently be written as $JA(\mathbf{k}) = A(\mathbf{k})^\mathrm{c}$, where $A(\mathbf{k})^\mathrm{c}$ denotes the orthogonal complement of $A(\mathbf{k})$ in $\mathcal{W}_\mathbf{k}$. This makes the difference to true unitary symmetries apparent: The space $A(\mathbf{k})$ is not conserved, but rather mapped to its orthogonal complement.

2.5.1. Complex symmetry classes

The complex sequence starts with the setting of section 2.2, which fits into the framework of IQPVs by imposing a reduced symmetry group $\mathcal{G}/\Pi = \mathrm{U}_1$.

Definition 2.7. *By an* IQPV of complex class s *with* $s = 0, 1, 2, \ldots$ *we mean an IQPV with reduced symmetry group* $\mathcal{G}/\Pi = \mathrm{U}_1$ *and* s *pseudo-symmetries as defined in Definition 2.5.*

The following table summarizes the two complex classes and the symmetries imposed:

class	symmetries	s	pseudo-symmetries
A	Q	0	none
$A\mathrm{III}$	Q, C (twisted particle-hole)	1	$J_1 = \mathrm{i}\gamma CQ$

Complex class $s = 0$ (alias class A)

For $s = 0$, the setting corresponds precisely to the one in section 2.2: Due to the U_1 symmetry, the ground states are given by an orthogonal sum $A(\mathbf{k}) = A^\mathrm{p}(\mathbf{k}) \oplus A^\mathrm{h}(\mathbf{k})$, which renders the CAR constraint (2.40) superfluous. Therefore, no additional restrictions are imposed.

Complex class $s = 1$ (alias class *AIII*)

We now add the symmetry of twisted particle hole conjugation

$$C := \gamma S = S\gamma : \mathcal{W}_{\mathbf{k}} \to \mathcal{W}_{-\mathbf{k}}. \tag{2.69}$$

For $S = 1$, this map is the operation γ of Hermitian conjugation as introduced in eq. (2.32), but in general we allow for a twisting in the form of a $\mathbf{k}$-independent, unitary and orthogonal map

$$S : \mathcal{W}_{\mathbf{k}} \to \mathcal{W}_{\mathbf{k}}, \tag{2.70}$$

which is block diagonal with respect to the decomposition $\mathcal{W}_{\mathbf{k}} = \mathcal{H}_{\mathbf{k}}^* \oplus \mathcal{H}_{-\mathbf{k}}$ and fulfills $S^2 = 1$.

Since γ is anti-unitary and S is unitary, their composition C is anti-unitary. Therefore, according to eq. (2.63) in Definition 2.4, an IQPV with this symmetry needs to satisfy

$$CA(\mathbf{k}) = A(-\mathbf{k}) \tag{2.71}$$

for all $\mathbf{k} \in \mathrm{T}^d$.

We define the pseudo-symmetry J_1 to be the composition

$$J_1 := \mathrm{i}\gamma CQ = \mathrm{i}SQ = \mathrm{i}QS, \tag{2.72}$$

where we have used $\gamma^2 = 1$ and $SQ = QS$ (since S is block-diagonal and Q is proportional to the identity on each block). The map S is unitary and orthogonal by definition and so is $\mathrm{i}Q$, as the replacements

$$c_j(\mathbf{k}) \to -\mathrm{i}c_j(\mathbf{k}) \tag{2.73}$$

$$c_j^\dagger(-\mathbf{k}) \to \mathrm{i}c_j^\dagger(-\mathbf{k}) \tag{2.74}$$

leave both $\{\cdot,\cdot\}$ and $\langle\cdot,\cdot\rangle$ invariant. Thus, J_1 is unitary and orthogonal. Since it also squares to -1 (as Q and S commute and square to $+1$), it remains to inspect its action on the subspaces of annihilators:

$$J_1 A(\mathbf{k}) = \gamma CQA(\mathbf{k}) = \gamma CA(\mathbf{k}) = \gamma A(-\mathbf{k}) = \gamma A(\mathbf{k})^\perp = A(\mathbf{k})^{\mathrm{c}}, \tag{2.75}$$

where the last step follows from the definition of γ in eq. (2.35) relating $\{\cdot,\cdot\}$ with $\langle\cdot,\cdot\rangle$. The calculation above shows that J_1 indeed qualifies as a pseudo-symmetry according to Definition 2.5.

Remark 2.8. In the physics literature, the operator J_1 is often called the *chiral operator* or a *chiral symmetry*. We emphasize here that it is not a true symmetry, but a *pseudo*-symmetry.

We will argue at the end of this chapter that there is no new setting to be gained by adding further pseudo-symmetries. Thus, we have already completed the description of the complex symmetry classes and can now proceed to the more involved sequence of the eight real ones.

2.5.2. Real symmetry classes

Unlike in the complex symmetry classes, we start the real ones without the U_1-symmetry and define

Definition 2.9. *By an* IQPV of real symmetry class s *with* $s = 0, 1, 2, \ldots$ *we mean an IQPV with s pseudo-symmetries as defined in Definition 2.5.*

The following table summarizes the symmetries to be introduced, as well as the corresponding pseudo-symmetries formed from them:

class	symmetries	s	pseudo-symmetries
D	none	0	CAR constraint
DIII	T (time reversal)	1	$J_1 = \gamma T$
AII	T, Q (charge)	2	$J_2 = \mathrm{i}\gamma TQ$
CII	T, Q, C (twisted particle-hole)	3	$J_3 = \mathrm{i}\gamma CQ$
C	S_1, S_2, S_3 (spin rotations)	4	see text
CI	S_1, S_2, S_3, T	5	
AI	S_1, S_2, S_3, T, Q	6	
BDI	S_1, S_2, S_3, T, Q, C	7	

Real symmetry class $s = 0$ (alias class D)

This class is described in Definitions 2.1 and 2.2 and is realized here by a symmetry group $\mathcal{G} = \Pi$ consisting exclusively of translations.

Real symmetry class $s = 1$ (alias class DIII)

The first symmetry to be imposed is the operation of time-reversal, which is represented on $\mathcal{H}$ by an anti-unitary operator T with $T^2 = -1$. This operator commutes with translations and therefore, as prescribed in eq. (2.61), it maps $\mathcal{W}_{\mathbf{k}}$ to $\mathcal{W}_{-\mathbf{k}}$. Originating from an operator on $\mathcal{H}$, it is block diagonal with respect to the decomposition $\mathcal{W}_{\mathbf{k}} = \mathcal{H}_{\mathbf{k}}^* \oplus \mathcal{H}_{-\mathbf{k}}$. Using T, we can define the first pseudo-symmetry as

$$J_1 := \gamma T = T\gamma. \tag{2.76}$$

Note that this is a different J_1 than the one introduced in eq. (2.72) for the complex symmetry classes. Since $\gamma^2 = 1$ and $T^2 = -1$, the map J_1 squares to -1. As a composition of two anti-unitary maps γ and T, it is unitary, while orthogonality follows from the following calculation:

$$\begin{aligned}\{J_1 w, J_1 w'\} &= \{\gamma T w, \gamma T w'\} \\ &= \langle T w, \gamma T w'\rangle \\ &= \langle T w, T \gamma w'\rangle \\ &= \langle \gamma w', w\rangle \\ &= \{w', w\} \\ &= \{w, w'\} \end{aligned} \tag{2.77}$$

for all $w \in \mathcal{W}_{\mathbf{k}}$ and $w' \in \mathcal{W}_{-\mathbf{k}}$.

Moreover, it acts on the annihilator spaces $A(\mathbf{k})$ of an IQPV as

$$J_1 A(\mathbf{k}) = \gamma T A(\mathbf{k}) = \gamma A(-\mathbf{k}) = \gamma A(\mathbf{k})^{\perp} = A(\mathbf{k})^{\mathrm{c}}, \tag{2.78}$$

where the second equality holds since T is a true symmetry.

Real symmetry class $s = 2$ (alias class $A\mathrm{II}$)

In this class, the additional symmetry we impose is the U_1-symmetry of particle number conservation. The present setting is therefore equivalent to that of complex class $s = 0$ (class A) with the addition of time-reversal symmetry. Accordingly, we have the familiar decomposition of the annihilator space as $A(\mathbf{k}) = A^{\mathrm{p}}(\mathbf{k}) \oplus A^{\mathrm{h}}(\mathbf{k})$, albeit with the restriction

$$T A(\mathbf{k}) = A(-\mathbf{k}). \tag{2.79}$$

Remark 2.10. Since T is block diagonal with respect to $\mathcal{W}_{\mathbf{k}} = \mathcal{H}_{\mathbf{k}}^{*} \oplus \mathcal{H}_{-\mathbf{k}}$ and since $A^{\mathrm{p}}(\mathbf{k}) \subset \mathcal{H}_{\mathbf{k}}^{*}$ and $A^{\mathrm{h}}(\mathbf{k}) \subset \mathcal{H}_{-\mathbf{k}}$, relation (2.79) can be reduced to the valence bands only:

$$T A^{\mathrm{h}}(\mathbf{k}) = A^{\mathrm{h}}(-\mathbf{k}). \tag{2.80}$$

Using the perspective of Definition 2.1, an IQPV in the present class is given by a complex sub-vector bundle subject to (2.80). This kind of bundle is called *quaternionic* in [Dup69, DNG14a]. Note that the term quaternionic does not refer to the field underlying the vector spaces involved (these are always complex), but rather the presence of a quaternionic structure mapping fibers at $\mathbf{k}$ to fibers at $\tau(\mathbf{k}) = -\mathbf{k}$. A quaternionic structure is defined as a map which is anti-linear map and squares to -1, both criteria being fulfilled by T.

The second pseudo-symmetry we define as

$$J_2 := \mathrm{i} J_1 Q. \tag{2.81}$$

Since both Q and J_1 are unitary and orthogonal, so is their composition J_2. The Clifford algebra relations (2.66) are fulfilled, as the following calculations demonstrate:

$$J_2^2 = \mathrm{i}J_1Q\mathrm{i}J_1Q = \mathrm{i}\gamma TQ\mathrm{i}\gamma TQ = -\mathrm{i}^2\gamma^2T^2Q^2 = -1 \tag{2.82}$$

$$J_1J_2 = \mathrm{i}J_1QJ_1 = -\mathrm{i}J_1J_1Q = -J_2J_1. \tag{2.83}$$

In the first line we have used the fact that all involved maps commute except for γ and Q, which anti-commute. The minus sign in the second line appears for the same reason.

Recalling that the U_1-symmetry implies $QA(\mathbf{k}) = A(\mathbf{k})$, we conclude

$$J_2A(\mathbf{k}) = \mathrm{i}J_1QA(\mathbf{k}) = J_1A(\mathbf{k}) = A(\mathbf{k})^{\mathrm{c}}. \tag{2.84}$$

Real symmetry class $s = 3$ (alias class CII)

We now augment the symmetry group by twisted particle-hole conjugation C as introduced in eq. (2.69). For the third pseudo-symmetry it turns out that we can reuse the first one of the complex sequence:

$$J_3 := \mathrm{i}\gamma CQ = \mathrm{i}SQ = \mathrm{i}QS, \tag{2.85}$$

This does not come as a surprise since the setting of real class $s = 2$ (class AII) resembles that of the complex class $s = 0$ (class A), the only difference being the addition of time-reversal symmetry. Thus, using the calculations in class A, we can already conclude that J_3 is unitary and orthogonal, squares to -1 and fulfills

$$J_3A(\mathbf{k}) = A(\mathbf{k})^{\mathrm{c}}. \tag{2.86}$$

It remains to verify that it anti-commutes with J_1 and J_2, which were not present in the treatment of class A. We find that

$$J_3J_1 = \mathrm{i}QS\gamma T = \gamma\mathrm{i}QST = -\gamma T\mathrm{i}QS = -J_1J_3 \tag{2.87}$$

and

$$J_3J_2 = J_3\mathrm{i}J_1Q = -\mathrm{i}J_1J_3Q = -\mathrm{i}J_1QJ_3 = -J_2J_3. \tag{2.88}$$

In the second line, we used the fact that J_3 commutes with Q. In conclusion, J_3 is a valid member among the three pseudo-symmetries in this class.

In order to proceed to the remaining four real symmetry classes $s = 4, 5, 6$ and 7, an interlude introducing what is known as the $(1,1)$-isomorphism is required. This isomorphism will play an important role in the homotopy classification of IQPVs and will therefore be introduced in a sufficiently general manner.

2.5.3. The $(1,1)$-isomorphism

Denoting by $\mathrm{Cl}(\mathbb{R}^{p,q})$ the real Clifford algebra with p generators squaring to -1 and q generators squaring to $+1$, there is an algebra isomorphism

$$\mathrm{Cl}(\mathbb{R}^{p+1,q+1}) \simeq \mathrm{Cl}(\mathbb{R}^{p,q}) \otimes \mathrm{Cl}(\mathbb{R}^{1,1}) \tag{2.89}$$

$$\simeq \mathrm{Cl}(\mathbb{R}^{p,q}) \otimes \mathbb{R}(2). \tag{2.90}$$

In the second line we have used the fact that $\mathrm{Cl}(\mathbb{R}^{1,1})$ is isomorphic to the algebra $\mathbb{R}(2)$ of real 2-by-2 matrices. One possible realization of this isomorphism is given by assigning to the positive generator of $\mathrm{Cl}(\mathbb{R}^{1,1})$ the Pauli matrix σ_3 and to the negative generator the matrix $\mathrm{i}\sigma_2$. Using the fundamental representation on the factor $\mathbb{R}(2)$, there is a one-to-one correspondence of real representations of $\mathrm{Cl}(\mathbb{R}^{p+1,q+1})$ and those of $\mathrm{Cl}(\mathbb{R}^{p,q})$. We will use a variation of this fact in the following.

We start with the familiar space $\mathcal{W}_\mathbf{k} \equiv \mathcal{W}$, but with double the dimension as before. Hence, it is a $4n$-dimensional Hilbert space which is equipped with a non-degenerate symmetric bilinear form $\{\cdot,\cdot\}$.

Let there be $q \geq 2$ Clifford generators $J_1, \ldots, J_q$ realized as unitary operators on $\mathcal{W}$. In contrast to the pseudo-symmetries introduced in def. 2.5, we require only the first $q-1$ of them to be orthogonal, while the last one obeys

$$\{J_q w, J_q w'\} = -\{w, w'\} \tag{2.91}$$

for all $w, w' \in \mathcal{W}$. A Clifford generator with this property will be dubbed "imaginary", while the standard, orthogonal ones will be called "real".

Due to their special role in the following (they are the analogs of the additional positive and negative generator of $\mathrm{Cl}(\mathbb{R}^{p+1,q+1})$ as compared to $\mathrm{Cl}(\mathbb{R}^{p,q})$, we rename the last two generators:

$$I := J_{q-1} \tag{2.92}$$

$$K := J_q. \tag{2.93}$$

Since $K^2 = -1$, K has eigenvalues $\pm\mathrm{i}$ with corresponding eigenspaces $\mathcal{W}_\pm$. These give an orthogonal decomposition

$$\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_-. \tag{2.94}$$

Since $J_1, \ldots J_{q-2}$ and I anti-commute with K, they exchange these eigenspaces, which implies that $\dim(\mathcal{W}_+) = \dim(\mathcal{W}_-)$.

The idea of this section is to reduce all structure to the subspace $\mathcal{W}_+ \subset \mathcal{W}$. The first step is to restrict the non-degenerate symmetric bilinear form $\{\cdot,\cdot\}$ from $\mathcal{W} \times \mathcal{W}$ to $\mathcal{W}_+ \times \mathcal{W}_+$. This procedure immediately yields another symmetric bilinear form, which is also non-degenerate since, for all $w_+ \in \mathcal{W}_+$ and $w_- \in \mathcal{W}_-$,

$$\{w_+, w_-\} = \{\mathrm{i}w_+, -\mathrm{i}w_-\} = \{Kw_+, Kw_-\} = -\{w_+, w_-\} = 0. \tag{2.95}$$

Similarly, the Hermitian scalar product on $\mathcal{W}$ restricts to one on $\mathcal{W}_+$, where the non-degeneracy of the restricted pairing again follows from that of the unrestricted one. Indeed, for all $w_+ \in \mathcal{W}_+$ and $w_- \in \mathcal{W}_-$,

$$\langle w_+, w_- \rangle = -\langle \mathrm{i}w_+, -\mathrm{i}w_- \rangle = -\langle Kw_+, Kw_- \rangle = -\langle w_+, w_- \rangle = 0. \tag{2.96}$$

Therefore, also $\gamma : \mathcal{W} \to \mathcal{W}$ restricts to an anti-unitary operator $\gamma : \mathcal{W}_+ \to \mathcal{W}_+$.

Let $A \subset \mathcal{W}$ be an n-dimensional subvector space obeying the orthogonality conditions

$$J_1 A = \cdots = J_{p-2} A = IA = KA = A^{\mathrm{c}}. \tag{2.97}$$

The last two conditions imply that A is invariant under the operator $L := \mathrm{i}IK$. Since $L^2 = 1$, it has eigenvalues ± 1 with associated eigenspaces $E_{\pm 1}(L)$ and A splits into an orthogonal sum

$$A = \big(A \cap E_{+1}(L)\big) \oplus \big(A \cap E_{-1}(L)\big), \tag{2.98}$$

Let $P_\pm := \frac{1}{2}(1 \pm \mathrm{i}K)$ be the projectors onto $\mathcal{W}_\pm$ and

$$A_\pm := P_+\big(A \cap E_{\pm 1}(L)\big) \subset \mathcal{W}_+. \tag{2.99}$$

As part of the reduction to $\mathcal{W}_+$, we would like to show that $A \subset \mathcal{W}$ can be reduced to $A_+ \subset \mathcal{W}_+$ with relations (2.97) replaced by

$$j_1 A_+ = \cdots = j_{p-2} A_+ = A_+^{\mathrm{c}}, \tag{2.100}$$

where we define $j_l := LJ_l|_{\mathcal{W}_+}$ and A_+^{c} is the orthogonal complement of A_+ in $\mathcal{W}_+$. As a first step, we prove the following:

Lemma 2.11. *The space A is completely determined by A_+. More precisely, the projection map P_+ restricted to $A \cap E_{\pm 1}(L)$ gives isomorphisms*

$$A \cap E_{\pm 1}(L) \simeq A_\pm \tag{2.101}$$

and, within $\mathcal{W}_+$,

$$A_+^{\mathrm{c}} = A_-. \tag{2.102}$$

Furthermore, $\mathcal{W}$ admits an orthogonal decomposition into the following four subspaces:

$$\begin{aligned} A \cap E_{+1}(L) &= \{w + Lw \mid w \in A^{(+)}\}, \quad & A^{\mathrm{c}} \cap E_{+1}(L) &= \{w + Lw \mid w \in A^{(-)}\}, \\ A \cap E_{-1}(L) &= \{w - Lw \mid w \in A^{(-)}\}, \quad & A^{\mathrm{c}} \cap E_{-1}(L) &= \{w - Lw \mid w \in A^{(+)}\}. \end{aligned} \tag{2.103}$$

Proof. Any $v \in E_{\pm 1}(L)$ can be written

$$\begin{aligned} v &= P_+ v + P_- v \\ &= P_+ v \pm P_- L v \\ &= P_+ v \mp L P_+ v. \end{aligned} \tag{2.104}$$

Therefore, $P_+ v = 0$ implies that $v = 0$, so P_+ is injective. By definition it is also surjective and hence an isomorphism. Therefore, all $w \in A_+$ can be written $w = P_+ v$ with $v \in A \cap E_{+1}(L)$ and $w + Lw = v$. Similarly, all $w' \in A_-$ can be written $w' = P_+ v'$ with $v' \in A \cap E_{-1}(L)$ and $w' - Lw' = v'$. On the other hand, we have $w' + Lw' \in A^{\mathrm{c}} \cap E_{+1}(L)$ since

$$w' + Lw' = -\mathrm{i} K(w' - Lw') \in KA = A^{\mathrm{c}}. \tag{2.105}$$

Since $\langle A, A^{\mathrm{c}} \rangle = 0$, it follows that $0 = \langle w + Lw, w' + Lw' \rangle = 2\langle w, w' \rangle$, so A_+ is orthogonal to A_-. Furthermore,

$$\dim A^{(+)} + \dim A^{(-)} = \dim A \cap E_{+1}(L) + \dim A \cap E_{-1}(L) = \dim A = \dim \mathcal{W}_+, \tag{2.106}$$

implying that they are indeed orthogonal complements of each other in $\mathcal{W}_+$.

The last statement follows from the calculations above. □

The remaining ingredient in the reduction to $\mathcal{W}_+$ is the reduction of $J_1, \ldots, J_{p-2}$. Since the operators J_l and L commute for all $l = 1, \ldots, q-2$, the relations (2.97) can be refined to

$$L J_l \big(A \cap E_{\pm 1}(L) \big) = J_l \big(A \cap E_{\pm 1}(L) \big) = A^{\mathrm{c}} \cap E_{\pm 1}(L). \tag{2.107}$$

The operators $L J_l$ commute with K and hence also with the projections $P_\pm$. Applying P_+ to the equation above yields

$$j_l A_+ = L J_l A_+ = A_- = A_+^{\mathrm{c}} \tag{2.108}$$

for all $l = 1, \ldots, q-2$.

The operators j_l obey the relations

$$j_l j_m + j_m j_l = -2\delta_{lm} \tag{2.109}$$

for $l, m = 1, \ldots, q-2$.

2.5.4. Real classes $s \geq 4$

We now apply the reduction procedure to the case where s pseudo-symmetries are present. For this purpose, we define

$$K := \mathrm{i}J_1 J_2 J_3, \tag{2.110}$$

$$I := J_4. \tag{2.111}$$

The pseudo-symmetries $J_5, \dots, J_s$ correspond to $J_1 \dots J_{q-2}$ in the previous section with an index shift of 4 and $q = s - 2$. The crucial difference is the presence of three additional operators J_1, J_2 and J_3, which *commute* with K and therefore leave $\mathcal{W}_+$ invariant. Accordingly, we define $j_l := J_l\big|_{\mathcal{W}_+}$ for $l = 1, 2, 3$ and $j_l := LJ_l\big|_{\mathcal{W}_+}$ for $l \geq 5$ as before, where $L = \mathrm{i}IK = J_1 J_2 J_3 J_4$. This set of reduced operators obeys the following algebraic relations:

$$\begin{aligned} &j_l j_m + j_m j_l = -2\delta_{lm}\mathrm{Id}_{\mathcal{W}_+} && (1 \leq l, m \leq 2), \\ &j_l j_m - j_m j_l = 0 && (1 \leq l \leq 2;\; 5 \leq m \leq s), \\ &j_l j_m + j_m j_l = -2\delta_{lm}\mathrm{Id}_{\mathcal{W}_+} && (5 \leq l, m \leq s). \end{aligned} \tag{2.112}$$

The pseudo-symmetry conditions for J_1, J_2 and J_3 can be refined to

$$J_l\big(A \cap E_{\pm 1}(L)\big) = A^{\mathrm{c}} \cap E_{\mp 1}(L), \tag{2.113}$$

since J_1, J_2 and J_3 anti-commute with L and therefore exchange its eigenspaces. On the other hand, they commute with K and therefore also with $P_\pm$. Applying P_+ to eq. (2.113) and using Lemma 2.11 then yields

$$j_l A_+ = A_+, \tag{2.114}$$

for $l = 1, 2, 3$. Only two of these restrictions are independent, since $j_1 j_2 j_3$ is the identity on $\mathcal{W}_+$. We settle on the arbitrary choice of choosing $l = 1, 2$ and disregarding $l = 3$.

In contrast, we know from the previous section that, for $l \geq 5$,

$$j_l A_+ = A_+^{\mathrm{c}}. \tag{2.115}$$

We summarize the reduced setting in the following definition:

Definition 2.12. *A* reduced IQPV *of real symmetry class $s \geq 4$ is an* IQPV $\mathbf{k} \mapsto A_+(\mathbf{k}) \subset \mathcal{W}_+ = \mathbb{C}^{2n}$ *constrained by*

$$\begin{aligned} &j_l A_+(\mathbf{k}) = A_+(\mathbf{k}) && (1 \leq l \leq 2), \\ &j_l A_+(\mathbf{k}) = A_+(\mathbf{k})^{\mathrm{c}} && (5 \leq l \leq s), \\ &A_+(\mathbf{k})^\perp = A_+(-\mathbf{k}), \end{aligned} \tag{2.116}$$

for all $\mathbf{k} \in \mathrm{T}^d$. The j_l are unitary and orthogonal operators satisfying the relations (2.112).

It remains to be shown that the reduction procedure yields an equivalent description. Thus we prove that IQPVs of real symmetry class s can be reconstructed from their reduced versions:

Lemma 2.13. *Fix a decomposition $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_- \equiv E_{+1}(J_1J_2J_3) \oplus E_{-1}(J_1J_2J_3)$ and an isomorphism $J_1J_2J_3J_4 \equiv L : \mathcal{W}_\pm \to \mathcal{W}_\mp$. Then there exists a one-to-one correspondence between the* IQPV*s of symmetry class $s \geq 4$ and the reduced* IQPV*s of the same class s.*

Proof. We have already shown how to obtain the reduced IQPV from the original one. Thus, we prove the converse: Given a reduced IQPV $\mathbf{k} \mapsto A_+(\mathbf{k}) \subset \mathcal{W}_+$, we construct the original IQPV in the same symmetry class. For this purpose, we fix an isomorphism $L_\downarrow : \mathcal{W}_+ \to \mathcal{W}_-$ with inverse $L_\uparrow : \mathcal{W}_- \to \mathcal{W}_+$ in order to obtain $L = L_\downarrow + L_\uparrow : \mathcal{W} \to \mathcal{W}$. Setting $K = \mathrm{i}(\mathrm{Id}_{\mathcal{W}_+} - \mathrm{Id}_{\mathcal{W}_-})$, the extended pseudo-symmetries are reconstructed as

$$J_1 := j_1 - L_\downarrow j_1 L_\uparrow, \tag{2.117}$$
$$J_2 := j_1 - L_\downarrow j_2 L_\uparrow, \tag{2.118}$$
$$J_3 := \mathrm{i}KJ_1J_2, \tag{2.119}$$
$$J_4 := \mathrm{i}LK, \tag{2.120}$$
$$J_{l\geq 5} := L_\downarrow j_l + j_l L_\uparrow. \tag{2.121}$$

These operators are orthogonal, unitary and satisfy the Clifford relations (2.66).

The original IQPV $\mathbf{k} \mapsto A(\mathbf{k})$ is recovered from $\mathbf{k} \mapsto A_+(\mathbf{k})$ by defining

$$A(\mathbf{k}) := \{w + w' + L_\downarrow(w - w') \mid w \in A_+(\mathbf{k}), w' \in A_+(\mathbf{k})^{\mathrm{c}}\}. \tag{2.122}$$

By construction, the relations (2.116) translate back to the pseudo-symmetry conditions (2.67). Moreover, since $L_\downarrow$ is orthogonal and $A_+(\mathbf{k})^\perp = A_+(-\mathbf{k})$, we conclude that $A(\mathbf{k})^\perp = A(-\mathbf{k})$. □

In the following, we use the notion of reduced IQPVs in order to introduce the same sequence of symmetries as for $s = 0, 1, 2, 3$ with the addition of spin rotation symmetry SU_2. Nambu space will be denoted by $\mathcal{W}_+$ (without $\mathbf{k}$-dependence) to emphasize that we start in the reduced setting before doubling the space in order to incorporate all pseudo-symmetries.

Real symmetry class $s = 4$ (alias class C)

The setting in this class is that of real symmetry class $s = 0$ (class D) with an additional spin-1/2 degree of freedom and a corresponding SU_2 spin rotation symmetry. Possible physical realizations of this class include superconductors with spin-singlet pairing.

The Nambu space of creation and annihilation operators reducing the momentum by $\mathbf{k}$ is given by $\mathcal{W}_+ = \mathbb{C}^{n/2} \otimes (\mathbb{C}^2)_{\text{spin}}$, which emphasizes the spin degree of freedom. The group SU_2 is represented on $\mathcal{W}_+$ by unitary operators, implying that its three generators j_1, j_2 and j_3 are anti-Hermitian (for $n = 2$, they are given by $j_l = \mathrm{i}\sigma_l$). Since $j_1^2 = j_2^2 = j_3^2 = -1$, these three operators have the additional property of being unitary. Moreover, since the representation of SU_2 on $\mathcal{W}_+$ is derived from a representation on the single particle Hilbert space, j_1, j_2 and j_3 commute with γ and are therefore also orthogonal.

In the present symmetry class, spin rotations constitute the only symmetries besides translations. Thus, the reduced symmetry group is given by $\mathcal{G}/\Pi = \text{SU}_2$ and IQPVs $\mathbf{k} \mapsto A_+(\mathbf{k})$ with this symmetry group satisfy

$$j_1 A_+(\mathbf{k}) = j_2 A_+(\mathbf{k}) = j_3 A_+(\mathbf{k}) = A_+(\mathbf{k}). \tag{2.123}$$

Due to the relation $j_3 = j_2 j_1$, only two of these conditions are independent and we focus on the leftmost ones involving j_1 and j_2. The setting here is now precisely that of a reduced IQPV of real symmetry class $s = 4$. Hence, after doubling the space to $\mathcal{W}_+ \oplus \mathcal{W}_-$ with a unitary and orthogonal map $L_\downarrow : \mathcal{W}_+ \to \mathcal{W}_-$, we may use Lemma 2.13 to construct pseudo-symmetries J_1, J_2, J_3 and J_4.

Real symmetry class $s = 5$ (alias class CI)

As announced previously, the treatment of the remaining real symmetry classes will parallel that of the first four with the addition of spin-rotation symmetry. Just like real symmetry class $s = 4$ was the analog of real symmetry class $s = 0$, the present real symmetry class $s = 5$ is analogous to $s = 1$. Hence, the reduced symmetry group $\mathcal{G}/\Pi = \text{SU}_2$ is enhanced by the introduction of time-reversal symmetry T. Being represented by an anti-unitary operator, this new symmetry merits the additional requirement that

$$T A_+(\mathbf{k}) = A_+(-\mathbf{k}). \tag{2.124}$$

Similarly to $s = 1$, this leads to a pseudo-symmetry

$$j_5 := \gamma T = T\gamma, \tag{2.125}$$

which has the same properties as J_1 in real class $s = 1$ as defined in eq. (2.76). It therefore squares to -1, is unitary, orthogonal and leads to the pseudo-symmetry condition

$$j_5 A_+(\mathbf{k}) = A_+(\mathbf{k})^{\mathrm{c}}. \tag{2.126}$$

Being unitary and orthogonal, j_1 and j_2 commute with γ. On physical grounds, time reversal T inverts spin (analogous to T inverting angular momentum) and since j_1 and

j_2 are spin operators multiplied by i, they *commute* with T due to its anti-unitarity. Therefore, j_1 and j_2 commute with j_5 and the IQPV at hand is a reduced one of real symmetry class $s = 5$. Alternatively, we can reformulate it as an unreduced IQPV with five pseudo-symmetries using Lemma 2.13.

Real symmetry class $s = 6$ (alias class AI)

Continuing in the same fashion, we introduce the U_1-symmetry of particle number (or charge) conservation with generator Q, which is a unitary and orthogonal operator on $\mathcal{W}_+$. In other words, we require

$$QA_+(\mathbf{k}) = A_+(\mathbf{k}). \tag{2.127}$$

Similarly to $s = 2$, this leads to an additional pseudo-symmetry

$$j_6 := \mathrm{i} j_5 Q. \tag{2.128}$$

This operator is the analog of J_2 in eq. (2.81) and has the same properties. Hence, it squares to -1, is unitary as well as orthogonal and fulfills the pseudo-symmetry condition

$$j_6 A_+(\mathbf{k}) = A_+(\mathbf{k})^{\mathrm{c}}, \tag{2.129}$$

in addition to the one imposed by j_5. Moreover, j_6 anti-commutes with j_5 (for the same reasons that J_1 anti-commutes with J_2, see eq. (2.83)) and commutes with j_1 and j_2, so we arrive at the setting of a reduced IQPV of real symmetry class $s = 6$. Again, using Lemma 2.13, we can switch perspectives and reformulate the data above with six pseudo-symmetries.

Real symmetry class $s = 7$ (alias class BDI)

The final real symmetry class we consider is obtained as an analog of real symmetry class $s = 3$, but combined here with spin-rotation invariance. Accordingly, we assume that twisted particle-hole conjugation C is a symmetry. Since this symmetry is anti-unitary, it follows that

$$CA_+(\mathbf{k}) = A_+(-\mathbf{k}). \tag{2.130}$$

In complete analogy to real symmetry class $s = 3$, we form the pseudo-symmetry

$$j_7 := \mathrm{i}\gamma C Q = \mathrm{i} S Q = \mathrm{i} Q S. \tag{2.131}$$

The set $\{j_5, j_6, j_7\}$ corresponds to the set $\{J_1, J_2, J_3\}$ in the real symmetry class $s = 3$ and shares all of its properties, among which are unitarity, orthogonality, the Clifford algebra relations and the pseudo-symmetry properties

$$j_5 A_+(\mathbf{k}) = j_6 A_+(\mathbf{k}) = j_7 A_+(\mathbf{k}) = A_+(\mathbf{k})^{\mathrm{c}}. \tag{2.132}$$

Additionally, j_7 commutes with j_1 and j_2 since we require C to do so. The setting is therefore that of a reduced IQPV in the real symmetry class $s = 7$. Once again, we are free to convert to the setting with seven pseudo-symmetries according to Lemma 2.13.

2.6. Classifying spaces

In the previous section we have introduced a physical realization for IQPVs with any number $s = 0, \ldots, 7$ of pseudo-symmetries. This gives a well defined mathematical setting, which we describe in more detail in the present section. Given a set of s pseudo-symmetries $J_1 \ldots, J_s$, we define

$$C_s(n) := \{A \subset \mathbb{C}^{2n} \mid J_1 A = \cdots = J_s A = A^{\mathrm{c}}\} \tag{2.133}$$

$$= \cup_{p=0}^{2n} \mathrm{Gr}_p(\mathbb{C}^{2n}). \tag{2.134}$$

To allow for more generality, we include more components than $\mathrm{Gr}_n(\mathbb{C}^{2n})$ as in eq. (2.47) by removing the restriction on the dimensionality of subspaces A.

Recall from eq. (2.51) the map

$$\begin{aligned} \tau_0 : C_0(n) &\to C_0(n) \\ A &\mapsto A^{\perp}. \end{aligned} \tag{2.135}$$

Since $C_s(n)$ is a subset of $C_0(n)$ and since $J_i A = A^{\mathrm{c}}$ implies that $J_i A^{\perp} = (A^{\perp})^{\mathrm{c}}$, the map τ_0 restricts to maps

$$\tau_s := \tau_0\big|_{C_s(n)} : C_s(n) \to C_s(n). \tag{2.136}$$

We introduce the following notation for the fixed point sets of these maps:

$$R_s(n) := \{A \in C_s(n) \mid \tau_s(A) = A\}. \tag{2.137}$$

An IQPV in the real symmetry class s can therefore be described as an equivariant map

$$\begin{aligned} \psi : \mathrm{T}^d &\to C_s(n), \\ \psi \circ \tau &= \tau_s \circ \psi. \end{aligned} \tag{2.138}$$

There is an alternative picture, which will help us make a connection to the existing literature. To each element $A \in C_0(n)$ we may assign an operator

$$J(A) := \mathrm{i}(P_A - P_{A^c}), \tag{2.139}$$

where P_A and P_{A^c} are the orthogonal projectors onto the space A and its complement A^c respectively. This operator is unitary and satisfies $J(A)^2 = -1$. Since it is anti-Hermitian, we can form the Hermitian operator (reinstating the dependence on $\mathbf{k} \in \mathrm{T}^d$)

$$H(\mathbf{k}) := \mathrm{i}J(A(\mathbf{k})), \tag{2.140}$$

which is known as the flattened, or flat-band Hamiltonian. Indeed, it can be obtained from the original Hamiltonian defining the IQPV as its ground state if its eigenvalues $E_i(\mathbf{k})$ are set to -1 for all annihilation operators and $+1$ for all creation operators (see eq. (2.53)).

Using the transpose g^T of an operator g with respect to $\{\cdot,\cdot\}$, i.e.

$$\{w, gw'\} = \{g^T w, w'\} \tag{2.141}$$

for all $w, w' \in \mathcal{W}$, we obtain the relation $P_{\tau_0(A)} = (P_{A^c})^T$ and therefore

$$(J \circ \tau_0)(A) = -J(A)^T = J(A)^{-1T}. \tag{2.142}$$

It follows that the involution τ_0 on the level of subspaces $A \subset \mathcal{W}$ translates to an involution on unitary operators

$$\begin{aligned} \tau_{\mathrm{CAR}} : \mathrm{U}(\mathcal{W}) &\to \mathrm{U}(\mathcal{W}) \\ g &\mapsto (g^{-1})^T \equiv g^{-1T}. \end{aligned} \tag{2.143}$$

We use the subscript CAR to indicate that the origin of this involution is the CAR restriction of eq. (2.40). The fixed points of τ_{CAR} are the orthogonal operators $\mathrm{O}(\mathcal{W}) \subset \mathrm{U}(\mathcal{W})$.

In the presence of s pseudo-symmetries $J_1, \ldots, J_s$, the operator $J(A)$ fulfills the relations

$$J_i J(A) = -J(A) J_i, \tag{2.144}$$

for $i = 1, \ldots, s$, owing to $J_i A = A^c$. Thus, if $J(A) \in \mathrm{Fix}(\tau_{\mathrm{CAR}})$, or equivalently if $A = \tau_0(A)$, then $J(A)$ presents a choice of another pseudo-symmetry J_{s+1} extending the original set. Since the assignment $A \mapsto J(A)$ is a bijection, we can give an alternative view of the spaces $C_s(n)$ and $R_s(n)$ in terms of unitary operators:

$$C_s(n) = \{J \in \mathrm{U}(\mathcal{W}) \mid J^2 = -1 \text{ and } J_i J = -J J_i \text{ for } i = 1 \ldots, s\} \tag{2.145}$$

$$R_s(n) = \{J \in C_s(n) \mid \tau_{\mathrm{CAR}}(J) = J\} \tag{2.146}$$

These spaces are well known: $C_s(n)$ is the space of all extensions of a unitary Clifford algebra representation and $R_s(n)$ is the space of all extensions of an orthogonal Clifford algebra representation. They are used in the seminal work [Kit09] and have been determined in [Mil63] and more recently in [SCR11] with the result displayed in Table 2.1. The Clifford algebra isomorphisms $\mathrm{Cl}(\mathbb{C}^{s+2}) \simeq \mathrm{Cl}(\mathbb{C}^s) \otimes \mathbb{C}(2)$ of complex Clifford algebras and $\mathrm{Cl}(\mathbb{R}^{s+8,0}) \simeq \mathrm{Cl}(\mathbb{R}^{s,0}) \otimes \mathbb{R}(16)$ of real Clifford algebras (see [ABS64, LM89]) yield a periodicity $C_{s+2}(2n) = C_s(n)$ and $R_{s+8}(16n) = R_s(n)$ [SCR11]. This is the reason we stopped the sequence of introducing additional pseudo-symmetries at $s = 1$ for the complex symmetry classes and at $s = 7$ for the real symmetry classes: Further pseudo-symmetries would not produce any new settings.

In order to obtain the symmetric spaces displayed as quotient spaces of Lie groups in Table 2.1, we need to fix a basis of $\mathcal{W}$. Any orthonormal basis will do for identifying $\mathrm{U}(\mathcal{W})$ with U_{2n}. If we construct this orthonormal basis solely using elements fixed under γ, then we obtain a basis known as a *Majorana basis* [Kit09] and we can additionally identify $\mathrm{O}(\mathcal{W})$ with O_{2n}. It is shown in [SCR11] that the spaces $C_s(n)$ can be obtained as a union of orbits of the group

$$G_s^{\mathbb{C}}(n) := \{g \in \mathrm{U}(\mathcal{W}) \mid J_i g = g J_i \text{ for } i = 1 \ldots, s\} \tag{2.147}$$

on appropriate elements in $C_s(n)$. For instance, $C_0(n)$ is the union of orbits gJg^{-1} of $g \in \mathrm{U}_n \equiv G_0^{\mathbb{C}}(n)$ on $2n+1$ elements $J \in C_0(n)$ that have p eigenvalues $+\mathrm{i}$ and q eigenvalues $-\mathrm{i}$ for all combinations of p and q. The stabilizer for each of these orbits is the product $\mathrm{U}_p \times \mathrm{U}_q$. Since $\mathrm{U}_{2n}/\mathrm{U}_p \times \mathrm{U}_q$ is none other than the Grassmannian $\mathrm{Gr}_p(\mathbb{C}^{2n})$, the identification in eq. (2.134) follows. The next space $C_1(n)$ is given by an orbit of $G_1^{\mathbb{C}}(n) = \mathrm{U}_n \times \mathrm{U}_n$ on $J_2 \in C_1(n)$ with stabilizer the diagonal subgroup $\mathrm{U}_n \subset \mathrm{U}_n \times \mathrm{U}_n$, producing the quotient listed in Table 2.1. Due to the 2-fold periodicity $C_{s+2}(n) = C_s(n/2)$, all other spaces $C_s(n)$ can be obtained from $C_0(n)$ and $C_1(n)$.

A similar, but more involved analysis can be applied to the spaces $R_s(n)$, which can be realized as (unions of) orbits of

$$\begin{aligned} G_s(n) &:= \{g \in \mathrm{O}(\mathcal{W}) \mid J_i g = g J_i \text{ for } i = 1 \ldots, s\} \\ &= \{g \in G_s^{\mathbb{C}}(n) \mid \tau_{\mathrm{CAR}}(g) = g\}. \end{aligned} \tag{2.148}$$

In this case, all spaces $R_s(n)$ are generated by a single orbit except for $s = 2$ and $s = 6$, where $R_s(n)$ is a union of quaternionic and real Grassmannians respectively.

2.6.1. General symmetry groups

We now argue that every IQPV with general symmetry group $\mathcal{G}$ containing translations as a central subgroup can be constructed from the ten classes we have introduced. In order to accomplish this, we use the classification result of [HHZ05] for Hamiltonians acting on Nambu space $\mathcal{H}^* \oplus \mathcal{H}$. Denoting by $\mathcal{G}_0$ the unitary symmetries containing

s	$C_s(8r)$	$R_s(8r)$
0	$\cup_{p+q=16r}\mathrm{U}_{16r}/(\mathrm{U}_p\times\mathrm{U}_q)$	$\mathrm{O}_{16r}/\mathrm{U}_{8r}$
1	$(\mathrm{U}_{8r}\times\mathrm{U}_{8r})/\mathrm{U}_{8r}$	$\mathrm{U}_{8r}/\mathrm{Sp}_{8r}$
2	$\cup_{p+q=8r}\mathrm{U}_{8r}/(\mathrm{U}_p\times\mathrm{U}_q)$	$\cup_{p+q=4r}\mathrm{Sp}_{8r}/(\mathrm{Sp}_{2p}\times\mathrm{Sp}_{2q})$
3	$(\mathrm{U}_{4r}\times\mathrm{U}_{4r})/\mathrm{U}_{4r}$	$(\mathrm{Sp}_{4r}\times\mathrm{Sp}_{4r})/\mathrm{Sp}_{4r}$
4	$\cup_{p+q=4r}\mathrm{U}_{4r}/(\mathrm{U}_p\times\mathrm{U}_q)$	$\mathrm{Sp}_{4r}/\mathrm{U}_{2r}$
5	$(\mathrm{U}_{2r}\times\mathrm{U}_{2r})/\mathrm{U}_{2r}$	$\mathrm{U}_{2r}/\mathrm{O}_{2r}$
6	$\cup_{p+q=2r}\mathrm{U}_{2r}/(\mathrm{U}_p\times\mathrm{U}_q)$	$\cup_{p+q=2r}\mathrm{O}_{2r}/(\mathrm{O}_p\times\mathrm{O}_q)$
7	$(\mathrm{U}_r\times\mathrm{U}_r)/\mathrm{U}_r$	$(\mathrm{O}_r\times\mathrm{O}_r)/\mathrm{O}_r$

Table 2.1.: Realization of C_s and $R_s = \mathrm{Fix}(\tau_s)$ as homogeneous spaces.

the translations Π as a central subgroup, the most general symmetry group in our setting is given by a subgroup

$$\mathcal{G} \subset \mathcal{G}_0 \cup T\mathcal{G}_0 \cup C\mathcal{G}_0 \cup CT\mathcal{G}_0. \tag{2.149}$$

The examples we have given correspond to $\mathcal{G}_0 = \Pi$ (all complex classes and the real classes $s = 0, 1, 2, 3$) and $\mathcal{G}_0 = \Pi \times \mathrm{SU}_2$ (real classes $s = 4, 5, 6, 7$). The result of [HHZ05] states that for any reductive group $\mathcal{G}_0$, the Hamiltonian is given by a direct sum of blocks each of which is restricted to be an element of the tangent space associated to one of ten types of symmetric spaces. In the setting with $\Pi \subset \mathcal{G}_0$, the first part of this reduction is the decomposition of Nambu space $\mathcal{H}^* \oplus \mathcal{H}$ into blocks $\mathcal{W}_{\mathbf{k}}$. This reduces the unitary symmetries to the quotient group $\mathcal{G}_0/\Pi$ and all further unitary symmetries in this quotient lead to an orthogonal decomposition of $\mathcal{W}_{\mathbf{k}} = \bigoplus_i \mathcal{W}^i_{\mathbf{k}}$ with the Hamiltonians acting block-diagonally. Thus, in order to find the building blocks for the general situation, we may restrict the discussion to a single block $\mathcal{W}^i_{\mathbf{k}}$. Since the CAR constraint as well as all subgroups of the form $T\mathcal{G}_0$ and $C\mathcal{G}_0$ map the sector $\mathcal{W}^i_{\mathbf{k}}$ to $\mathcal{W}^i_{\tau(\mathbf{k})}$, the setting for $\tau(\mathbf{k}) \neq \mathbf{k}$ is that of the symmetry classes A and AIII in [Zir10]. Therefore, all Hamiltonians are elements of the tangent space to either a unitary group or a Grassmannian. For $\tau(\mathbf{k}) = \mathbf{k}$, the full classification of [HHZ05] applies and all symmetries in eq. (2.149) are relevant if they are present. In this case, there are ten possible symmetric spaces whose tangent space contains the Hamiltonians, all of which are listed in Table 2.1.

We have shown that there is a one-to-one correspondence between IQPVs and the flattened version of their defining Hamiltonian in eq. (2.140). In fact, imposing the condition of flat spectra on the space of Hamiltonians tangent to a symmetric space $C_s(n)$ in the complex classes and $R_s(n)$ in the real classes, gives symmetric spaces $C_{s+1}(n)$ and $R_{s+1}(n)$ respectively [SCR11]. For example, the tangent space to $C_1(n) = \mathrm{U}_n$ (class AIII) is given by its Lie algebra $\mathfrak{u}_n$ containing (i times) the

Hamiltonians of complex class $s = 2 \equiv 0$ (class A). Imposing a flat spectrum leads to the union of Grassmannians displayed in Table 2.1.

Thus, we can apply the classification result of [HHZ05] to arrive at the statement that an IQPV with arbitrary symmetry group containing and centralizing translations is described by a collection of IQPVs, one for each index i in $\mathcal{W}_{\mathbf{k}} = \bigoplus_i \mathcal{W}_{\mathbf{k}}^i$, each being in one of the ten complex or real symmetry classes s.

3. Tools of homotopy theory

In this chapter, we introduce a collection of tools which are tailored for the determination of topological phases as pursued in the remainder of this work. Starting with the definition of the notion of homotopy, we will introduce homotopy groups, their relative versions and various different realizations thereof, accompanied by some useful tools for their computation. This will be followed by a generalization to equivariant homotopy theory with an introduction of G-CW complexes and the G-Whitehead theorem, both of which are a vital ingredient in the homotopy theoretic derivation of the Periodic Table for topological insulators. We will finish with some facts about loop spaces and suspensions that will help formalize the notion of adding position-like and momentum-like dimensions to the configuration space of an IQPV.

Throughout this work and in particular throughout this chapter, we will use the category of topological spaces with morphisms being continuous maps. This being understood, we will omit the attributes "topological" when talking about spaces and the term "continuous" when referring to maps.

3.1. Homotopy

The backbone of homotopy theory is, as the name suggests, the notion of *homotopy*:

Definition 3.1. *Two maps $f_0, f_1 : X \to Y$ are called* homotopic *(written $f_0 \simeq f_1$) if and only if there exists a continuous interpolation, or* homotopy*, $f_t : X \to Y$ with $t \in [0, 1]$.*

The property of being homotopic is an equivalence relation on the set of all maps $X \to Y$, which therefore organize into equivalence classes called *homotopy classes*. The set of these classes will be denoted by $[X, Y]$. There is a corresponding equivalence relation on spaces: Two spaces X and Y are said to be *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$, called *homotopy equivalences*, such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. This is a coarser equivalence relation than that of *homeomorphism*, where the stronger statements $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$ are required. Thus, homeomorphisms are examples of homotopy equivalences, but not the other way around.

It is convenient to introduce the following notation: Given sequences of subsets

$X_n \subset \cdots \subset X_1 \subset X$ and $Y_n \subset \cdots \subset Y_1 \subset Y$, we denote by

$$f : (X, X_1, \dots, X_n) \to (Y, Y_1, \dots, Y_n) \tag{3.1}$$

a map $f : X \to Y$ with $f(X_i) \subset Y_i$ for all $i = 1, \dots, n$. We say that two such maps are homotopic if there exists a homotopy respecting these restrictions. The corresponding set of homotopy classes we denote by

$$[(X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n)]. \tag{3.2}$$

A common situation is that of $X_n = \{x_0\}$ and $Y_n = \{y_0\}$, where $x_0 \in X$ and $y_0 \in Y$ are distinguished points referred to as *base points*. In this case, we simplify the notation:

$$[(X, X_1, \dots, X_{n-1}, \{x_0\}), (Y, Y_1, \dots, Y_{n-1}, \{y_0\})] \tag{3.3}$$
$$\equiv [(X, X_1, \dots X_{n-1}, x_0), (Y, Y_1, \dots, Y_{n-1}, y_0)] \tag{3.4}$$
$$\equiv [(X, X_1, \dots X_{n-1}), (Y, Y_1, \dots, Y_{n-1})]_*. \tag{3.5}$$

In this case, homotopies are called base point preserving, since $f(x_0) = y_0$ stays fixed throughout. A construction central to many results in this thesis is of this kind: For $X = \mathrm{S}^d$ the d-dimensional sphere ($d \geq 0$), we define the d-th *homotopy group*

$$\pi_d(Y, y_0) := [(\mathrm{S}^d, s_0), (Y, y_0)]. \tag{3.6}$$

We often drop the base point y_0 from the notation and simply write $\pi_d(Y)$ with the base point preserving property being understood. There are two alternative definitions of the d-th homotopy group due to the fact that S^d is homeomorphic to the quotient $\mathrm{D}^d/\partial\mathrm{D}^d$ of the d-dimensional disk D^d by its boundary $\partial\mathrm{D}^d$ and, similarly, to the quotient $\mathrm{I}^d/\partial\mathrm{I}^d$ of the d-cube I^d by its boundary $\partial\mathrm{I}^d$ (see Appendix A.1 for details). Thus,

$$\pi_d(Y, y_0) = [(\mathrm{D}^d, \partial\mathrm{D}^d), (Y, y_0)] \tag{3.7}$$
$$= [(\mathrm{I}^d, \partial\mathrm{I}^d), (Y, y_0)]. \tag{3.8}$$

The realization using the d-dimensional cube $\mathrm{I}^d := [-\pi, \pi]^d$ lends itself for the definition of a group structure on these sets of homotopy classes. Given two representatives $f, g : (\mathrm{I}^d, \partial\mathrm{I}^d) \to (Y, y_0)$ with $d > 0$, we form their product as the concatenation along the first coordinate (any other choice of coordinate would lead to the same group structure, see Lemma 3.2 below):

$$(f * g)(k_1, k_2, \dots, k_d) := \begin{cases} f(2k_1 + \pi, k_2 \dots, k_d) & \text{for } -\pi \leq k_1 \leq 0 \\ g(2k_1 - \pi, k_2 \dots, k_d) & \text{for } 0 < k_1 \leq \pi. \end{cases} \tag{3.9}$$

Though formulated for representatives, this definition descends to the level of homotopy classes to give a multiplication on $\pi_d(Y)$. It can be shown [Hat02] that this multiplication is associative, has a neutral element (represented by the constant map to the base point $y_0 \in Y$) and inverses can be constructed by inverting the sign of the first coordinate ($k_1 \to -k_1$). Note that $\pi_0(Y)$ is not equipped with a group structure in general.

Viewing D^d as the unit ball in $\mathbb{R}^d$, it inherits a Euclidean structure. The action of the orthogonal group O_d on $\mathbb{R}^d$ restricts to D^d, so we can formulate the following useful lemma generalizing the construction of inverses:

Lemma 3.2. *Let Y be a space with base point $y_0 \in Y$ and $f : (\mathrm{D}^d, \partial\mathrm{D}^d) \to (Y, y_0)$ a representative of the class $[f] \in \pi_d(Y)$ with $d \geq 1$. Then the concatenation with an orthogonal transformation $g \in \mathrm{O}_d$ yields*

$$[f \circ g] = \begin{cases} [f] & \textit{for } \det(g) = 1, \\ [f]^{-1} & \textit{for } \det(g) = -1. \end{cases} \tag{3.10}$$

Proof. The group O_d has two connected components distinguished by the value of the determinant. Therefore, given $g \in \mathrm{O}_d$ with $\det(g) = 1$, there is a continuous path to any other orthogonal matrix with determinant 1. In particular, there is a path $\gamma(t)$ with $\gamma(0) = g$ and $\gamma(1) = \mathrm{Id}$. This yields a homotopy

$$F_t := f \circ \gamma(t) \tag{3.11}$$

with $F_0 = f \circ g$ and $F_1 = f$. Since $\partial\mathrm{D}^d$ is invariant under orthogonal transformations, all maps in this homotopy are base point preserving.

For the case of $\det(g) = -1$, we use a path $\gamma(t)$ with $\gamma(0) = g$ and $\gamma(1) = \mathrm{diag}(-1, 1, \ldots, 1)$. In this case, the homotopy F_t interpolates between $f \circ g$ and $f \circ \mathrm{diag}(-1, 1, \ldots, 1)$. Under the homeomorphism u^{-1} described in Appendix A.1, the latter maps to a representative of the class $[f]^{-1}$, which finishes the proof. □

Two important consequences of this Lemma 3.2 are the following:

Corollary 3.3.

(i) A permutation σ of the coordinates in I^d maps $[f] \in \pi_d(Y)$ to $[f \circ \sigma] = [f]$ if $sgn(\sigma) = 1$ and to $[f \circ \sigma] = [f]^{-1}$ if $sgn(\sigma) = -1$.

(ii) A representative of the inverse class $[f]^{-1}$ can be obtained from f by inverting the sign of any odd number of coordinates in I^d. Inverting the sign of any even number of coordinates leaves the class $[f]$ invariant.

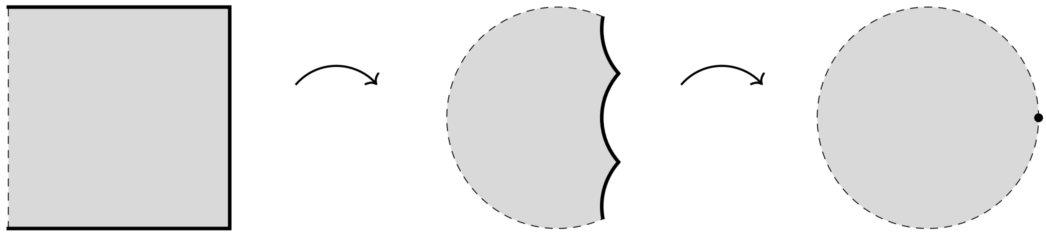


Figure 3.1.: Homotopy equivalence $(\mathrm{I}^2, \partial\mathrm{I}^2, J^1) \to (\mathrm{D}^2, \mathrm{S}^1, s_0)$: J^1 (the boundary with one side removed) is contracted to the point s_0. To relate the left picture to the definitions, note that the first coordinate runs vertically and the second horizontally.

Proof. The homeomorphism u defined in Appendix A.1 commutes with the operations of inverting the signs and permutation of coordinates, so we can use Lemma 3.2. Statement (i) is obtained by using the subgroup of O_d consisting of permutation matrices. Similarly, statement (ii) is obtained from the diagonal subgroup of O_d. □

A generalization of homotopy groups is given by the *relative homotopy groups*

$$\pi_d(Y, Y_1, y_0) := [(\mathrm{D}^d, \partial\mathrm{D}^d, s_0), (Y, Y_1, y_0)] \tag{3.12}$$

$$= [(\mathrm{D}^d, \mathrm{S}^{d-1}, s_0), (Y, Y_1, y_0)] \tag{3.13}$$

$$= [(\mathrm{I}^d, \partial\mathrm{I}^d, J^{d-1}), (Y, Y_1, y_0)]. \tag{3.14}$$

Here we have defined $J^{d-1} := \overline{\partial\mathrm{I}^d \backslash (\mathrm{I}^{d-1} \times \{-\pi\})}$ to be the boundary with one side removed (the one with last coordinate equal to $-\pi$). Figure 3.1 illustrates the two definitions in the case $d = 2$ as well as the homotopy equivalence between the respective domains. Similarly to the homotopy groups, the set $\pi_d(Y, Y_1, y_0)$ is equipped with a group structure by concatenation in the first coordinate of I^d. However, since the last coordinate is assigned a special role, this group structure is only defined for $d \geq 2$ in general. We often suppress the base point and write $\pi_d(Y, Y_1) \equiv \pi_d(Y, Y_1, y_0)$.

Homotopy groups together with their relative versions fit into a long exact sequence ($d \geq 0$)

$$\cdots \longrightarrow \pi_d(Y_1) \xrightarrow{i_d} \pi_d(Y) \xrightarrow{j_d} \pi_d(Y, Y_1) \xrightarrow{\partial_d} \pi_{d-1}(Y_1) \xrightarrow{i_{d-1}} \pi_{d-1}(Y) \longrightarrow \cdots$$

The map i_d is induced by the inclusion $Y_1 \hookrightarrow Y$ and j_d by the inclusion $(Y, \{y_0\}) \hookrightarrow (Y, Y_1)$. Given a representative $f : (\mathrm{D}^d, \mathrm{S}^{d-1}) \to (Y, Y_1)$ of a homotopy class $[f] \in \pi_d(Y, Y_1)$, the map ∂_d is defined by

$$\partial_d[f] := [f\big|_{\mathrm{S}^{d-1}}] \in \pi_{d-1}(Y_1). \tag{3.15}$$

The end of the exact sequence reads

$$\cdots \longrightarrow \pi_1(Y_1) \xrightarrow{i_1} \pi_1(Y) \xrightarrow{j_1} \pi_1(Y,Y_1) \xrightarrow{\partial_1} \pi_0(Y_1) \xrightarrow{i_0} \pi_0(Y).$$

It takes on a special role as the three sets on the very right do not form groups in general. In particular, among the four maps only i_1 is guaranteed to be a homomorphism. Note that all of these sets have a distinguished element represented by the constant map. We can therefore still speak about the kernel of a map as the preimage of this distinguished element. Hence, also the notion of exactness is still well defined.

The exact sequence above is closely related to the exact sequence of a fibration $Y_1 \hookrightarrow Y \xrightarrow{p} B$. In fact, the projection p induces an isomorphism (see [Hat02], p. 376)

$$p_* : \pi_d(Y,Y_1) \to \pi_d(B), \tag{3.16}$$

for all $d \geq 1$. Defining $\delta_d := \partial_d \circ (p_*)^{-1}$, there is an exact sequence

$$\cdots \longrightarrow \pi_d(Y_1) \xrightarrow{i_d} \pi_d(Y) \xrightarrow{p_*} \pi_d(B) \xrightarrow{\delta_d} \pi_{d-1}(Y_1) \xrightarrow{i_{d-1}} \pi_{d-1}(Y) \longrightarrow \cdots$$

3.2. Equivariant homotopy

In the real symmetry classes, IQPVs are given by *equivariant* maps and accordingly, we extend the notion of homotopy to this equivariant setting. Thus, we generalize the notion of topological spaces to include the action by a group G (which will be finite for all applications) and introduce

Definition 3.4. *Given two G-spaces X and Y, an* equivariant homotopy *between two equivariant maps $f_0, f_1 : X \to Y$ is a continuous family $f_t : X \to Y$ of equivariant maps.*

The property of being equivariantly homotopic is an equivalence relation on the set of all equivariant maps $X \to Y$ and we denote by $[X,Y]^G$ the corresponding set of equivalence classes. We use notation analogous to the one introduced in Section 3.1, in particular $[X,Y]^G_*$ denotes the set of base point preserving G-equivariant homotopy classes. The base point of a space X is always chosen to lie within the set of fixed points X^G of the G-action. We often use the language *free homotopy classes* for $[X,Y]^G$ as opposed to *based homotopy classes* for $[X,Y]^G_*$. Note that in the real symmetry classes of Section 2.5.2, the group action is given by the special case $G = \mathbb{Z}_2$.

Up to this point, we have considered the case $X = \mathrm{T}^d$ with $\mathbb{Z}_2$ acting through the involution $\tau : \mathrm{T}^d \to \mathrm{T}^d$ defined as $\tau(\mathbf{k}) = -\mathbf{k}$. For a more general configuration space, we make the following definition:

Definition 3.5. *The set of topological phases of the real symmetry class s with configuration space X is given by*

$$[X, C_s(n)]^{\mathbb{Z}_2} \quad or \quad [X, C_s(n)]^{\mathbb{Z}_2}_*, \tag{3.17}$$

depending on whether or not the space of annihilators is fixed for some point in X. In the complex symmetry class s, the set of topological phases with configuration space X is given by

$$[X, C_s(n)] \quad or \quad [X, C_s(n)]_*. \tag{3.18}$$

An example of a physical setting in which base point preserving homotopy classes are relevant is given in the presence of a compactified momentum space $X = \mathbb{R}^d \cup \{\infty\} = \mathrm{S}^d$. In this case we require IQPVs to map to the same point in $C_s(n)$ for infinite momentum and that this property is preserved under homotopies.

The physically relevant configuration spaces X that we will encounter in this thesis are products of spheres and as such can be described as maps from cubes with appropriate boundary conditions. If the domain is a product $\mathrm{S}^{d_1} \times \cdots \times \mathrm{S}^{d_m}$, then the corresponding cube is the product $\mathrm{I}^{d_1} \times \cdots \times \mathrm{I}^{d_m} = \mathrm{I}^{d_1+\cdots+d_m}$. Maps from this cube satisfy the property that the image of points with components on the boundary $\partial\mathrm{I}^{d_j}$ of one of the factors is invariant if these components are changed within $\partial\mathrm{I}^{d_j}$. For instance, with $d_1 = \cdots = d_m = 1$ we have the torus T^m realized on I^m with periodic boundary conditions, whereas for $m = 1$ we have S^{d_1} realized on I^{d_1} with the property that the entire boundary $\partial\mathrm{I}^{d_1}$ is mapped to a single point.

The possible $\mathbb{Z}_2$-actions on these kinds of configuration spaces can therefore be reduced to $\mathbb{Z}_2$-actions on a cube. Since a cube is simply a product of intervals $[-\pi, \pi]$, we need to choose a representation of $\mathbb{Z}_2$ on every one of the intervals. There are only two choices: Either the non-trivial element acts as the identity or it inverts the interval coordinate. We call this coordinate trivial or non-trivial respectively. If there are d_x trivial coordinates and d_k non-trivial coordinates, then we will always order them so that the trivial ones come first and the non-trivial ones last and denote the corresponding $d_x + d_k$-dimensional cube by I^{d_x,d_k}.

The most important examples of spaces realized by imposing boundary conditions on cubes are

- $X = \mathrm{T}^d$ realized as I^d with d non-trivial coordinates,
- $X = \mathrm{S}^{d_x,d_k} \equiv \mathrm{S}^{d_x+d_k}$ realized as $\mathrm{I}^{d_x+d_k}$ with the first d_x coordinates trivial and the last d_k coordinates non-trivial,
- $X = \mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ realized as $\mathrm{I}^{d_x+d_k}$ with the first d_x coordinates trivial and the last d_k ones non-trivial.

The domain $X = \mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ was previously used in [TK10] for the purpose of classifying topological phases in the presence of a defect. If a defect has codimension $d_x + 1$, it can be enclosed by a large sphere S^{d_x}, and at every point of this sphere, we can use the approximation of having translation invariance as before. Thus, the domain is enhanced to $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$. In Chapter 7, we prove that one may replace $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ by S^{d_x,d_k} at the expense of losing "weak" invariants.

The based maps $\mathrm{S}^{d_x,d_k} \to C_s(n)$ and the corresponding sets of (equivariant) homotopy classes according to Definition 3.5 will be studied extensively in Chapter 4. In this context, another important space with $\mathbb{Z}_2$-action will appear:

Definition 3.6. *Given a $\mathbb{Z}_2$-space Y, the* equivariant loop space $\Omega^{d_x,d_k} Y$ *is the space of based maps $f : \mathrm{S}^{d_x,d_k} \to Y$ equipped with the $\mathbb{Z}_2$-action $f \mapsto g \cdot f \cdot g^{-1}$. The set of $\mathbb{Z}_2$ fixed points $(\Omega^{d_x,d_k} Y)^{\mathbb{Z}_2}$ is the subset of based equivariant maps. The base point of $\Omega^{d_x,d_k} Y$ is the constant map.*

This definition enables us to reformulate the set of (based) topological phases with configuration space $X = \mathrm{S}^{d_x,d_k}$ as

$$\begin{aligned} [\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} &\simeq \pi_0((\Omega^{d_x,d_k} C_s(n))^{\mathbb{Z}_2}) \\ &\simeq \pi_{d_x}((\Omega^{0,d_k} C_s(n))^{\mathbb{Z}_2}) \\ &\equiv \pi_{d_x}(M^s_{d_k}). \end{aligned} \tag{3.19}$$

In the last line we have introduced the abbreviation $M^s_{d_k} := (\Omega^{0,d_k} C_s(n))^{\mathbb{Z}_2}$. For the frequently occurring loop spaces with one momentum-like or position-like coordinate, we will often use abbreviations $\Omega^{0,1} \equiv \bar{\Omega}$ and $\Omega^{1,0} \equiv \Omega$.

We can now prove a useful connection to the previously introduced concept of relative homotopy groups [TZMV12]:

Lemma 3.7. *For all $d_x \geq 0$, the set of topological phases in the real symmetry class s with configuration space $X = \mathrm{S}^{d_x,d_k}$ can be expressed by the relative homotopy group*

$$[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} = \pi_{d_x+1}(\Omega^{0,d_k-1} C_s(n), M^s_{d_k-1}). \tag{3.20}$$

Proof. Throughout this proof, we adopt the formulation in terms of cubes as domains. Thus, the space S^{d_x,d_k} is treated as $\mathrm{I}^{d_x+d_k}$ with coordinates $(x_1, \ldots, x_{d_x}, k_1, \ldots, k_{d_k})$. Given a map $f : \mathrm{I}^{d_x+d_k} \to C_s(n)$ representing a class in $[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2}$, we may view it as a map $\mathrm{I}^{d_x} \to M^s_{d_k}$ as in the identification (3.19). The crucial construction is shown in Figure 3.2: The domain of maps $(\psi : \mathrm{I}^{d_k} \to C_s(n)) \in M^s_{d_k}$ is cut in half at the $(d_k - 1)$-plane $(0, k_2, \ldots, k_{d_k}) \in \mathrm{I}^{d_k}$ and only points with coordinate $k_1 \geq 0$ are kept since the $\mathbb{Z}_2$-equivariance condition $\psi(-\mathbf{k}) = \tau_s(\psi(\mathbf{k}))$ determines the value of all other points. We assign an equivalence class $[\tilde{f}] \in \pi_{d_x+1}(\Omega^{0,d_k-1} C_s(n), M^s_{d_k-1})$

according to

$$\tilde{f} : (\mathrm{I}^{d_x+1}, \partial\mathrm{I}^{d_x+1}, J^{d_x}) \to (\Omega^{0,d_k-1}C_s(n), M^s_{d_k-1}, \text{const.})$$
$$(x_1, \dots, x_{d_x}, k_1) \mapsto f(x_1, \dots, x_{d_x})(k_1, \cdot, \dots, \cdot), \tag{3.21}$$

where const. denotes the constant map to the base point $A_* \in R_s(n) \subset C_s(n)$ and the last coordinate k_1 now runs from 0 to π rather than $-\pi$ to π due to the cut. Hence, the definition of J^{d_x} is changed to $J^{d_x} := \overline{\partial\mathrm{I}^{d_x+1}\backslash(\mathrm{I}^{d_x} \times \{0\})}$.

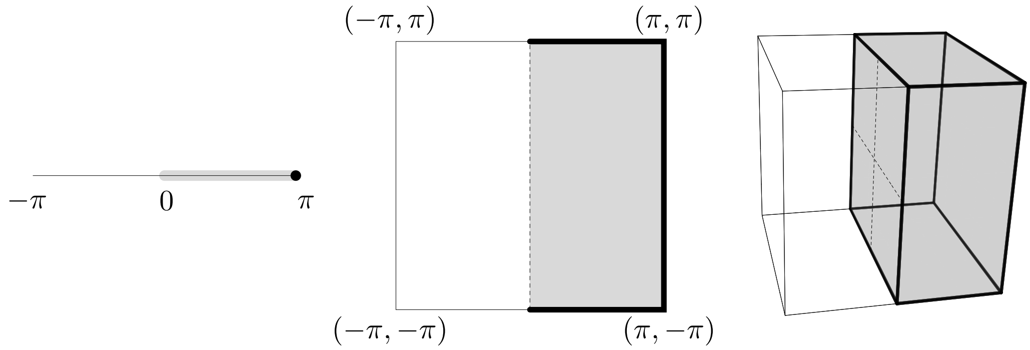


Figure 3.2.: Illustrating the cutting procedure for $d_x = 0$: Domains of elements in (from left to right) M^s_1, M^s_2 and M^s_3 are shown, with a cut along the points with first coordinate equal to zero. This shows the restriction of M^s_d to M^s_{d-1} along the cut: On the left, restriction to 0 results in maps to $R_s = M^s_0$, the fixed point set of the involution τ_s. In the middle, restriction to the dashed line results in a map in M^s_1 and finally, on the right, the restriction of a map in M^s_3 to the plane spanned by the dashed lines gives an element in M^s_2. The gray region contains all information about the mappings, since all points in the white regions are determined by the $\mathbb{Z}_2$-equivariance condition.

For arbitrary $x_1, \dots, x_{d_x}$ and k_1, $\tilde{f}(x_1, \dots, x_{d_x}, k_1)$ is a map $\mathrm{I}^{d_k-1} \to C_s(n)$, which maps the boundary $\partial\mathrm{I}^{d_k-1}$ to the base point of $C_s(n)$ (because maps in $M^s_{d_k}$ do so). Therefore, $\tilde{f}(x_1, \dots, x_{d_x}, k_1) \in \Omega^{0,d_k-1}C_s(n)$.

The subset $J^{d_x} \subset \partial \mathrm{I}^{d_x+1}$ corresponds to one of x_1 to x_{d_x} being ± 1 or $k_1 = 1$. Since $f(\partial \mathrm{I}^{d_x}) = \text{const.}$ as well as $f(\mathrm{I}^{d_x})(\partial \mathrm{I}^{d_k}) = A_*$, it follows that $\tilde{f}(J^{d_x})$ is the constant map to A_*.

The remaining face in $\partial \mathrm{I}^{d_x+1}$ is the one along the cut, namely the plane with $k_1 = 0$. With this coordinate removed, the $\mathbb{Z}_2$-equivariance condition of $M^s_{d_k}$ reduces to the one of $M^s_{d_k-1}$, so $\tilde{f}(\partial \mathrm{I}^{d_x}) \subset M^s_{d_k-1}$.

Since the assignment $f \mapsto \tilde{f}$ is merely a reinterpretation of f on half of its domain, while the other half is determined by the $\mathbb{Z}_2$-equivariance relation, it is clear that this map is well defined at the level of homotopy classes and $f \simeq g \Leftrightarrow \tilde{f} \simeq \tilde{g}$ for all $f, g \in [\mathrm{S}^{d_x,d_k}, C_s(n)]^{\mathbb{Z}_2}_*$. □

Using Lemma 3.7 to translate the set $[\mathrm{S}^{d_x,d_k}, C_s(n)]^{\mathbb{Z}_2}_*$ to a relative homotopy group $\pi_{d_x+1}(\Omega^{0,d_k-1}C_s(n), M^s_{d_k-1})$, we can determine its elements through the exact sequence associated to relative homotopy groups, provided the homotopy groups as well as the maps in the sequence are known. In fact, for $d_x \geq 1$, we can use the group structure of the relative homotopy group and the fact that the maps in the exact sequence are homomorphisms to arrive at the general result

Lemma 3.8. *For $d_x \geq 1$, all preimages under ∂_{d_x+1} of elements in $\pi_{d_x}(Y_1)$ are in bijection with* $\mathrm{im}(j_{d_x+1}) \subset \pi_{d_x+1}(Y, Y_1)$*. Therefore, as a set,*

$$\pi_{d_x+1}(Y, Y_1) = \mathrm{im}(j_{d_x+1}) \times \mathrm{im}(\partial_{d_x+1}). \tag{3.22}$$

Proof. From the long exact sequence associated to the relative homotopy groups, we take the map

$$\partial_{d_x+1} : \pi_{d_x+1}(Y, Y_1) \to \pi_{d_x}(Y_1). \tag{3.23}$$

As a set, $\pi_{d_x+1}(Y, Y_1)$ is the disjoint union of preimages of ∂_{d_x+1}. All of these preimages contain the same number of elements: Choosing two elements $\beta_1 \in \partial^{-1}_{d_x+1}(\delta_1)$ and $\beta_2 \in \partial^{-1}_{d_x+1}(\delta_2)$, a bijection is given by

$$\begin{aligned} \partial^{-1}_{d_x+1}(\delta_1) &\to \partial^{-1}_{d_x+1}(\delta_2) \\ \alpha &\mapsto \alpha\beta_1^{-1}\beta_2, \end{aligned} \tag{3.24}$$

with inverse

$$\begin{aligned} \partial^{-1}_{d_x+1}(\delta_2) &\to \partial^{-1}_{d_x+1}(\delta_1) \\ \alpha &\mapsto \alpha\beta_2^{-1}\beta_1. \end{aligned} \tag{3.25}$$

Notice that this construction makes use of the fact that, for $d_x \geq 1$, the map ∂_{d_x+1} is a group homomorphism. With the bijection above, we can identify all preimages with the preimage of the neutral element $1 \in \pi_{d_x}(Y_1)$:

$$\partial^{-1}_{d_x+1}(1) = \ker(\partial_{d_x+1}) = \mathrm{im}(j_{d_x+1}), \tag{3.26}$$

where we have used exactness in the second equality. This completes the proof. □

The case $d_x = 0$ requires separate treatment since $\pi_1(Y, Y_1)$ is not a group in general and therefore ∂_1 cannot be a homomorphism. However, even in this case there is some structure in the form of a right action of the group $\pi_1(Y)$ on $\pi_1(Y, Y_1)$ (see [tD08, p. 129]). Making the base point $y_0 \in Y_1 \subset Y$ explicit, this action is defined by assigning to a representative path $\alpha : [0, \pi] \to Y$ with $\alpha(\pi) = y_0$ and $\alpha(0) \in Y_1$ the concatenation $[\gamma] \cdot [\alpha] := [\alpha * \gamma] \in \pi_1(Y, Y_1)$ with $[\gamma] \in \pi_1(Y)$. It enables us to formulate the following statements analogous to those in Lemma 3.8:

Lemma 3.9. *The orbit of the right action of $\pi_1(Y)$ on an element $[\alpha] \in \partial_1^{-1}([y]) \subset \pi_1(Y, Y_1)$ generates all of $\partial_1^{-1}([y])$. The isotropy group of $[\alpha]$ is isomorphic to the image of $\pi_1(Y_1, y)$ in $\pi_1(Y)$ under the map $f_\alpha[\gamma] := [\bar{\alpha} * \gamma * \alpha]$, where $\bar{\alpha}$ is the inverse path of α. In particular, the union of all orbits is in bijection with the entire preimage $\partial_1^{-1}(\pi_0(Y_1))$.*

Proof. Since the action is defined through representatives, we first check that it is well defined on the level of homotopy classes. If two maps α_0 and α_1 represent the same class $[\alpha_0] = [\alpha_1] \in \pi_1(Y, Y_1)$, then there exists a homotopy $\alpha_t : [0, \pi] \to Y$ interpolating between the two, with $\alpha_t(\pi) = y_0$ and $\alpha_t(0) \in Y_1$. This yields a homotopy $\alpha_t * \gamma$ implying that $[\alpha_0 * \gamma] = [\alpha_1 * \gamma]$ in $\pi_1(Y, Y_1)$. Similarly, a homotopy between two loops γ_0 and γ_1 gives a homotopy $\alpha * \gamma_t$, so that $[\alpha * \gamma_0] = [\alpha * \gamma_1] \in \pi_1(Y, Y_1)$ and the action is indeed well defined.

The map ∂_1 maps every orbit to a single connected component of Y_1 since

$$\partial_1[\alpha] = [\alpha(0)] = \partial_1[\alpha * \gamma] \tag{3.27}$$

for all $[\gamma] \in \pi_1(Y)$. Conversely, if two elements $[\alpha], [\beta] \in \pi_1(Y, Y_1)$ satisfy $\partial_1[\alpha] = \partial_1[\beta]$, the points $\alpha(0)$ and $\beta(0)$ lie in the same connected component of Y_1. Therefore, we can find a homotopy of, e.g., α to another representative $\tilde{\alpha}$ which satisfies $\tilde{\alpha}(0) = \beta(0)$. The concatenation of the two paths gives a class of loops $[\beta^{-1} * \tilde{\alpha}] \in \pi_1(Y)$ and its action on β yields

$$[\beta^{-1} * \tilde{\alpha}] \cdot [\beta] = [\beta * \beta^{-1} * \tilde{\alpha}] = [\tilde{\alpha}] = [\alpha]. \tag{3.28}$$

Thus, $[\alpha]$ and $[\beta]$ lie in the same orbit.

For the remainder of the proof, we need to make base points explicit. The stabilizer of $[\alpha]$, where α is a path from y_0 to $y \in Y_1 \subset Y$, is given by elements $[\gamma] \in \pi_1(Y, y_0)$ with $[\gamma] \cdot [\alpha] = [\alpha]$, i.e. $\alpha * \gamma \simeq \alpha$. This property implies that

$$[\alpha * \gamma * \bar{\alpha}] = [\alpha * \bar{\alpha}] = [\text{const.}] \in \pi_1(Y, Y_1, y). \tag{3.29}$$

Using the exact sequence associated to $\pi_1(Y, Y_1, y)$ (note the change to the base point y rather than y_0), it follows that $[\alpha * \gamma * \bar{\alpha}] \in \ker(j_1) = i_1(\pi_1(Y_1, y))$. The map f_α is an isomorphism between $\pi_1(Y, y)$ and $\pi_1(Y, y_0)$ and we identify

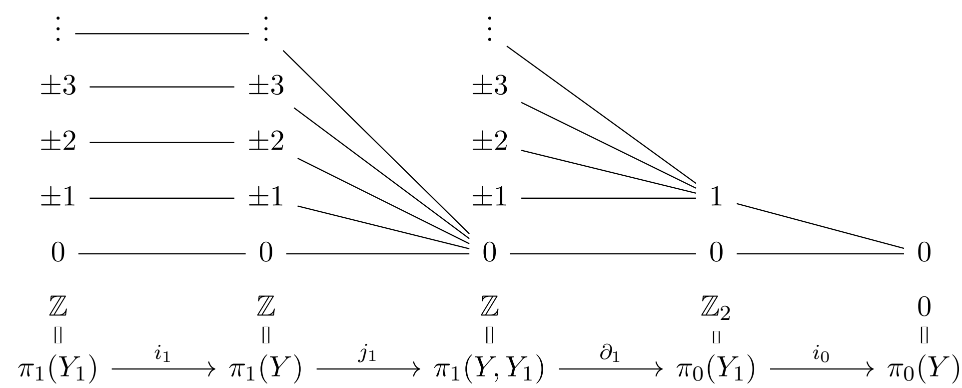
$$f_\alpha[\alpha * \gamma * \bar{\alpha}] = [\bar{\alpha} * \alpha * \gamma * \bar{\alpha} * \alpha] = [\gamma] \in \pi_1(Y, y_0). \tag{3.30}$$

Conversely, any $[\omega] \in \pi_1(Y_1, y)$ is mapped under $j_1 \circ i_1$ to the trivial element in $\pi_1(Y, Y_1, y)$ and therefore

$$\big(f_\alpha[\omega]\big) \cdot [\alpha] = [\alpha * \bar{\alpha} * \omega * \alpha] = [\omega * \alpha] = [\alpha] \in \pi_1(Y, Y_1, y_0). \tag{3.31}$$

□

An example where the preimages under ∂_1 are not in bijection is illustrated in Figure 3.3, where we take the example of $Y_1 \subset Y \subset \mathbb{R}^2$. In this example, we have $\pi_1(Y_1) \simeq \pi_1(Y) \simeq \mathbb{Z}$, where the homotopy class $n \in \mathbb{Z}$ corresponds to a winding number n around the hole in Y (white region in Figure 3.3). On the other hand, $\pi_0(Y) = 0$ (Y is connected) and $\pi_0(Y_1) = \mathbb{Z}_2$ as a set (Y_1 has two connected components). This gives the following exact sequence:



Due to exactness, ∂_1 has to be surjective. In other words, there is only one connected component of Y, so all connected components of Y_1 can be reached by paths. Since i_1 is a bijection and in particular surjective, exactness implies that $\ker(j_1) = \pi_1(Y) = \mathbb{Z}$. It follows that $\mathrm{im}(j_1) = \ker(\partial_1)$ contains only one element represented by the constant map. If ∂_1 were a homomorphism, a trivial kernel would imply that it is injective and therefore $\pi_1(Y, Y_1)$ would contain only two elements. However, due to the lack of group structure, a trivial kernel does *not* imply injectivity as illustrated in the diagram above.

From the perspective of the previous two lemmas, we can also inspect preimages under ∂_1. The preimage of the connected component of Y_1 containing the base point

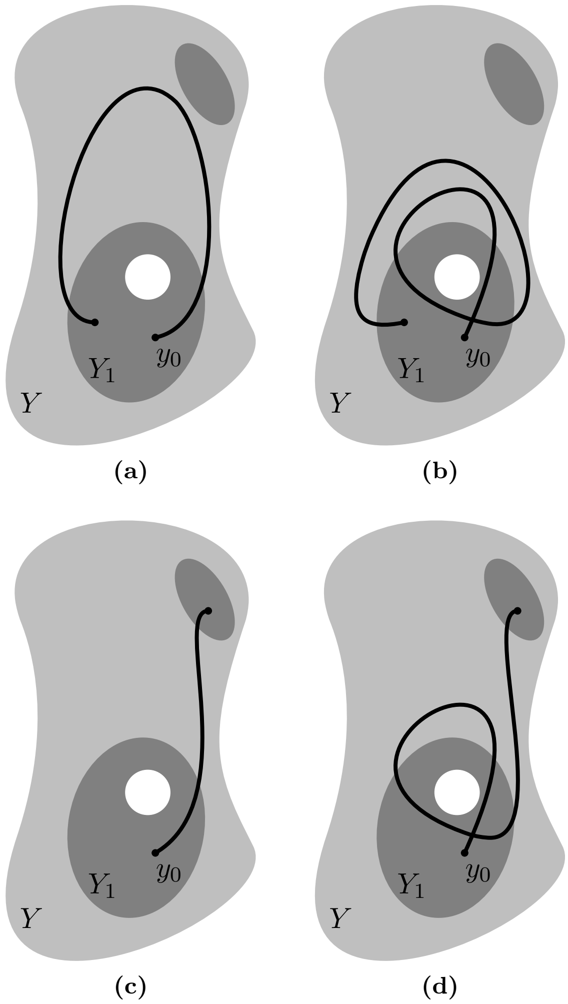


Figure 3.3.: Example illustrating non-bijective preimages under ∂_1 with Y_1 (dark gray) a subset of $Y \subset \mathbb{R}^2$ (light gray). There are two preimages since Y_1 has two connected components ($\pi_0(Y_1) = \mathbb{Z}_2$ as a set). The preimage of the connected component containing the base point y_0 includes the two paths shown in (a) and (b). Both are homotopic to the constant map (in fact, the preimage contains only one homotopy class). In contrast, (c) and (d) are not homotopic (the preimage contains infinitely many homotopy classes).

y_0 contains only one element represented by the constant map to y_0. In other words, the action of $\pi_1(Y_1)$ is trivial (no matter how many loops are added, they can all be retraced within Y_1). On the other hand, acting by $\pi_1(Y_1)$ on the path in Figure 3.3(c) yields infinitely many non-homotopic paths, one of them being 3.3(d). Thus, the preimage of the other component of Y_1 under ∂_1 contains infinitely many elements.

Sometimes it is possible to avoid using relative homotopy groups in order to obtain statements about equivariant homotopy classes. The following lemma is useful whenever a (non-equivariant) homotopy group has a generator which is represented by an equivariant map.

Lemma 3.10. *Let $\alpha : \mathrm{S}^{d_x,d_k} \to Y$ be a based and equivariant map, where we denote by τ the involution on S^{d_x,d_k} and by τ_Y the involution on Y. If $[\alpha]$ is the generator of $\pi_{d_x+d_k}(Y) = \mathbb{Z}$, then every (non-equivariant) homotopy class in $\pi_{d_x+d_k}(Y)$ has an equivariant representative.*

Proof. Let $[\alpha] = 1 \in \mathbb{Z}$. Then the map $\bar{\alpha}$ obtained by inverting the sign of first coordinate of α represents the class $[\bar{\alpha}] = [\alpha]^{-1} = -1 \in \mathbb{Z}$. Any other homotopy class $n \in \mathbb{Z}$ is represented by α^n for $n \geq 1$ or $\bar{\alpha}^n$ for $n \leq -1$ (the constant map represents the neutral element $0 \in \mathbb{Z}$), where we define

$$\begin{aligned} \alpha^n(x_1, \dots, x_{d_x+d_k}) :=& \alpha(nx_1 + (n-2m-1)\pi, x_2, \dots, x_{d_x+d_k}), \\ & \text{for } x_1 \in \left[-\pi + m\tfrac{2\pi}{n}, -\pi + (m+1)\tfrac{2\pi}{n}\right], \\ & \text{with } m = 0, \dots, n-1. \end{aligned} \tag{3.32}$$

This definition is illustrated in Figure 3.4: The domain of α^n is divided into n parts along the x_1 direction and to each part the map α is applied. For $n = 2$, this construction reduces to the product defined in eq. (3.9) and for larger n it is homotopic to n-fold iteration of eq. (3.9) in the sense of $\alpha * (\alpha * \cdots * (\alpha * \alpha) \cdots)$, which does not split the first coordinate into equal parts.

The equivariance of α implies the equivariance of $\bar{\alpha}$: If τ_1 denotes the map which inverts the first coordinate in S^{d_x,d_k}, then

$$\bar{\alpha} \circ \tau = \alpha \circ \tau_1 \circ \tau = \alpha \circ \tau \circ \tau_1 = \tau_Y \circ \alpha \circ \tau_1 = \tau_Y \circ \bar{\alpha}. \tag{3.33}$$

Inspecting eq. (3.32), we see that

$$\tau_Y \circ \alpha^n = (\tau_Y \circ \alpha)^n = (\alpha \circ \tau)^n = \alpha^n \circ \tau, \tag{3.34}$$

and the same for $\bar{\alpha}$. These relations are obvious for $d_x \geq 1$ since the first coordinate is unchanged under τ in that case. For $d_x = 0$, observe that the m-th interval in (3.32) is exchanged with the $(n-m)$-th interval and both are inverted. Since all intervals contain the same map α, the last equation follows.

□

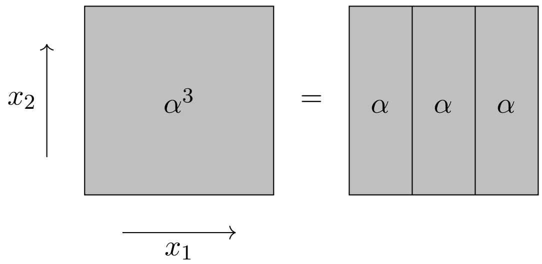


Figure 3.4.: Visualization of the domain of α^n as defined in eq. (3.32) for $n = 3$ and $d_x + d_k = 2$.

3.3. G-CW complexes

The construction of homotopy groups and their relative versions require the use of disks and their boundary spheres (or homotopy equivalent spaces) as domains. In this section, we introduce a class of spaces called CW complexes, as well as their equivariant generalization, G-CW complexes. These spaces are built by successively attaching disks of various dimensions along their boundary spheres and, not surprisingly, there are intimate ties to the homotopy groups. This fact will be exploited in the formulation of the Whitehead theorem and its equivariant generalization, the G-Whitehead theorem.

Definition 3.11. *A finite CW complex is defined inductively: Starting with a finite set of points X_0 called the 0-skeleton, we form the 1-skeleton X_1 by attaching n_1 cells I^1 (intervals) along their boundary $\partial\mathrm{I}^1$ according to maps*

$$\phi_i^1 : \partial\mathrm{I}^1 \to X_0, \tag{3.35}$$

with $i = 1, \dots, n_1$.

Having constructed the $(m-1)$-skeleton X_{m-1}, we similarly form the m-skeleton by attaching n_m cells I^m (m-cubes) along their boundary $\partial\mathrm{I}^m$ as prescribed by the attaching maps

$$\phi_i^m : \partial\mathrm{I}^m \to X_m, \tag{3.36}$$

with $i = 1, \dots, n_m$.

We stop this procedure after d steps to arrive at the finite CW complex $X = X_d$ of dimension d.

Example 3.12 (Sphere S^d)**.** The sphere S^d can be viewed as a CW complex with a single point in $X_0 = X_1 = \cdots = X_{d-1}$, and X_d obtained by attaching the boundary of a single cell I^d to this point. However, this construction is not unique: An alternative CW structure, which generalizes to the equivariant case introduced later, is given by $X_m = \mathrm{S}^m$. Thus, we start with two points ($X_0 = \mathrm{S}^0$) and form X_1 by attaching two intervals I^1 as the two hemispheres of $X_1 = \mathrm{S}^1$. This in turn is the equator of $X_2 = \mathrm{S}^2$, which is constructed by attaching the two hemispheres I^2, etc.

Proposition 3.13. *Products of finite CW complexes are again finite CW complexes.*

Proof. Given two CW complexes X and Y with skeleta X_i and Y_j, where $i \leq d_1$ and $j \leq d_2$, the product $X \times Y$ is a finite CW complex with skeleta $X_m \times Y_m$ for $m \leq \max(d_1, d_2)$. The attaching maps are given by the products of the individual ones for X and Y. This construction can be iterated to arrive at a statement about arbitrary products. □

In physical applications, it is the configuration space which will be a CW complex. In order to incorporate the action of $\mathbb{Z}_2$ on this space, we introduce a generalization in the form of G-CW complexes for a finite group G, which includes the special case $G = \mathbb{Z}_2$.

Definition 3.14. *A finite G-CW complex X is constructed inductively: Starting with a set of points X_0 with trivial G-action, we construct the m-skeleton X_m from the $(m-1)$-skeleton X_{m-1} by attaching n_m cells of the form $\mathrm{I}^m \times G/H_i^m$ with subgroups $H_i^m \subset G$ through equivariant attaching maps*

$$\phi_i^m : \partial \mathrm{I}^m \times G/H_i^m \to X_m, \tag{3.37}$$

with $i = 1, \ldots, n_m$. The group G acts as the identity on I^m (and thus on $\partial\mathrm{I}^m$) and by left multiplication on G/H_i^m. The largest value d for which $n_d \neq 0$ is called the dimension of the G-CW complex $X = X_d$.

Remark 3.15. Taking $H_i^m = G$ for all m and i, we have $\mathrm{I}^m \times G/H_i^m = \mathrm{I}^m$ and the definition reduces to that of an ordinary CW complex with non-equivariant attaching maps ϕ_i^m.

Example 3.16 (Sphere S^{d_x,d_k})**.** The sphere S^{d_x,d_k} with d_x trivial coordinates and d_k non-trivial coordinates can be equipped with the structure of a $\mathbb{Z}_2$-CW complex. We start by constructing the part with trivial involution $X_{d_x} = \mathrm{S}^{d_x}$ as an ordinary CW complex according to Example 3.12. The $(d_x + 1)$-skeleton is then formed by equivariantly attaching the pair $\mathrm{I}^{d_x+1} \times \mathbb{Z}_2$ as the two hemispheres of $X_{d_x+1} = \mathrm{S}^{d_x,1}$, leaving $X_{d_x} = \mathrm{S}^{d_x}$ as its equator which is fixed under the $\mathbb{Z}_2$-action. We iterate this process until we arrive at the full $\mathbb{Z}_2$-CW complex $X_{d_x+d_k} = \mathrm{S}^{d_x,d_k}$. This construction is illustrated in Figure 3.5 for $(d_x, d_k) = (0, 1)$.

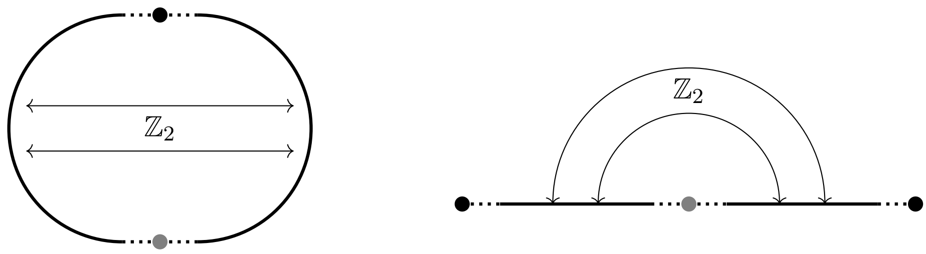


Figure 3.5.: Construction of $\mathrm{S}^{0,1}$ visualized as a circle (left) and as an interval with boundary (two black points) identified (right). Starting with X_0 consisting of two points (one gray, one black), the cell $I \times \mathbb{Z}_2$ comprising two parts related by the $\mathbb{Z}_2$-action is attached equivariantly.

Proposition 3.17. *Products of finite $\mathbb{Z}_2$-CW complexes are again finite $\mathbb{Z}_2$-CW complexes. In particular, since S^{d_x,d_k} is a $\mathbb{Z}_2$-CW complex for all d_x and d_k, so are the following spaces:*

$$\mathrm{T}^d = \prod_{i=1}^{d} \mathrm{S}^{0,i} \text{ (Brillouin zone)} \tag{3.38}$$

$$\mathrm{S}^{d_x} \times \mathrm{T}^{d_k} = \mathrm{S}^{d_x,0} \times \prod_{i=1}^{d} \mathrm{S}^{0,i} \text{ (Brillouin zone with defect)} \tag{3.39}$$

Proof. The product of two cells I^{m_1} and I^{m_2} is the cell $\mathrm{I}^{m_1+m_2}$ with trivial $\mathbb{Z}_2$-action. Similarly, the product of $\mathrm{I}^{m_1} \times \mathbb{Z}_2$ and I^{m_2} is $\mathrm{I}^{m_1+m_2} \times \mathbb{Z}_2$ with $\mathbb{Z}_2$-action on the factor $\mathbb{Z}_2$. The only difficulty arises in the product of $\mathrm{I}^{m_1} \times \mathbb{Z}_2$ and $\mathrm{I}^{m_2} \times \mathbb{Z}_2$ which is given by $\mathrm{I}^{m_1+m_2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and does not immediatly fit into Definition 3.14 of a $\mathbb{Z}_2$-CW complex. However, the action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$ leaves invariant the two subsets $\{(0,0),(1,1)\}$ and $\{(0,1),(1,0)\}$, so we have a splitting

$$\begin{aligned}(\mathrm{I}^{m_1} \times \mathbb{Z}_2) \times (\mathrm{I}^{m_2} \times \mathbb{Z}_2) &= \mathrm{I}^{m_1+m_2} \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \\ &= \mathrm{I}^{m_1+m_2} \times (\mathbb{Z}_2 \sqcup \mathbb{Z}_2) \\ &= (\mathrm{I}^{m_1+m_2} \times \mathbb{Z}_2) \sqcup (\mathrm{I}^{m_1+m_2} \times \mathbb{Z}_2).\end{aligned} \tag{3.40}$$

Note the factors $\mathbb{Z}_2$ in the above should be considered not as groups but rather as sets with two elements. The non-trivial element of the *group* $\mathbb{Z}_2$ acts by exchanging these two elements.

Given two $\mathbb{Z}_2$-CW complexes X and Y, their product $X \times Y$ can now be equipped with a $\mathbb{Z}_2$-CW structure. Given the m-skeleta X_m and Y_m of X and Y respectively, the m-skeleton of $X \times Y$ is given by the union of all sets $X_{m_1} \times Y_{m_2}$ with $m_1 + m_2 = m$. Denoting the attaching maps of X and Y by ϕ_i^m for X and θ_i^m for Y respectively,

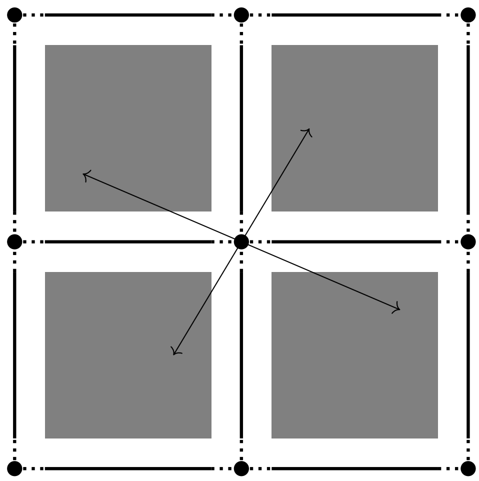


Figure 3.6.: $\mathbb{Z}_2$-CW structure of the Brillouin zone torus $\mathrm{T}^2 = \mathrm{S}^{0,1} \times \mathrm{S}^{0,1}$ as described in Example 3.18.

the attaching maps of $X \times Y$ are given by all products $\phi_i^{m_1} \times \theta_j^{m_2}$, where we include $m_1 = 0$ and $m_2 = 0$ by defining ϕ_i^0 to be the constant map to the i-th element in X_0 (and similarly for θ_j^0). If two attaching maps both have a domain with non-trivial $\mathbb{Z}_2$-action, we use eq. (3.40) to obtain two separate attaching maps in place of their product. □

Example 3.18 (Brillouin torus T^2)**.** The Brillouin zone torus $\mathrm{T}^2 = \mathrm{S}^{0,1} \times \mathrm{S}^{0,1}$ has the following $\mathbb{Z}_2$-CW complex structure shown in Figure 3.6: Opposing sides are identified, so the 0-skeleton consists of 4 points, to which 4 products $\mathrm{I}^1 \times \mathbb{Z}_2$ (a total of 8 intervals) are attached to form the 1-skeleton. The 2-skeleton is formed by attaching the product of the two 1-cells of the two circles $\mathrm{S}^{0,1}$, which splits as $\mathrm{I}^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = (\mathrm{I}^2 \times \mathbb{Z}_2) \sqcup (\mathrm{I}^2 \times \mathbb{Z}_2)$, corresponding to the 4 gray squares in the diagram. The resulting action of the non-trivial element of $\mathbb{Z}_2$ is indicated by the arrows.

3.3.1. The *G*-Whitehead Theorem

Computing homotopy classes $[X, Y]$ and especially their equivariant generalization $[X, Y]^G$ is a hard problem in general. For instance, even the seemingly innocent sets $[\mathrm{S}^m, \mathrm{S}^n]$ are unknown to a large extent. However, with knowledge about the homotopy groups of Y, there are substantial simplifications if X is a G-CW complex. In order to formalize this statement, we introduce the concept of maps being connected:

Definition 3.19. *A map $f : Y \to Z$ is said to be m-*connected *if the induced map*

$$\begin{aligned} f_* : \pi_d(Y) &\to \pi_d(Z) \\ [g] &\mapsto [f \circ g] \end{aligned}$$

is an isomorphism for all $d < m$ and surjective for $d = m$.

More generally, let Y and Z be G-spaces and $f : Y \to Z$ an equivariant map. If we denote by Y^H and Z^H the fixed point sets under a subgroup $H \subset G$, we have a more general notion:

Definition 3.20. *If G is a group, let m denote an integer-valued function $H \mapsto m(H)$ defined on all subgroups H of G. Then a G-equivariant map $f : Y \to Z$ is called m-*connected *if for any subgroup $H \subset G$ the restriction $f^H : Y^H \to Z^H$ is $m(H)$-connected.*

We are now in a position to formulate the G-Whitehead theorem, which formalizes the statement that knowledge about homotopy groups can be used to infer knowledge about the sets of homotopy classes involving a G-CW complex as a domain.

Theorem 3.21 (G-Whitehead Theorem)**.** *If X is a G-CW complex and the base-point preserving and G-equivariant map $f : Y \to Z$ is m-connected, then the induced maps*

$$\begin{aligned} f_* : [X, Y]_*^G \to [X, Z]_*^G, &\quad [g] \mapsto [f \circ g], \\ f_* : [X, Y]^G \to [X, Z]^G, &\quad [g] \mapsto [f \circ g], \end{aligned}$$

are bijective if $dim(X^H) < m(H)$ for all subgroups H of G. They are surjective if $dim(X^H) \leq m(H)$ for all subgroups H of G.

We refer to the many references for the proof of this theorem. The base-point preserving statement can be found in [MG95, Ada84] and the statement about free homotopy classes is found in [tD87, Wan80, Mat71].

3.4. Relating based and free homotopy classes

The G-Whitehead Theorem 3.21 offers statements about both base-point preserving and free homotopy classes. In this section, we state the relation between the two. More explicitly, we formulate a relation between the set $[X, Y]_*^{\mathbb{Z}_2}$ of based $\mathbb{Z}_2$-equivariant homotopy classes and the set $[X, Y]^{\mathbb{Z}_2}$ of free $\mathbb{Z}_2$-equivariant homotopy classes in the case of a $\mathbb{Z}_2$-CW complex X and a G-space Y. For if X is a $\mathbb{Z}_2$-CW complex, we can use the $\mathbb{Z}_2$-homotopy extension property [tD87] in order to define a right action of $\pi_1(Y^{\mathbb{Z}_2})$ on the set $[X, Y]_*^{\mathbb{Z}_2}$ as follows. Given a class $[\gamma] \in \pi_1(Y^{\mathbb{Z}_2})$, we can interpret any of its representatives γ as a homotopy of $\mathbb{Z}_2$-equivariant maps $f_t : \{x_0\} \to Y$ with $t \in [0,1]$ and base point $x_0 \in X^{\mathbb{Z}_2} \subset X$. Given some based $\mathbb{Z}_2$-equivariant map $F_0 : X \to Y$, we use the equivariant homotopy extension property of X to extend the homotopy f_t from $\{x_0\}$ to all of X to yield a homotopy $F_t : X \to Y$ through equivariant maps. At $t = 0$ and $t = 1$ this construction gives two based and equivariant maps F_0 and F_1 and the assignment

$$\begin{aligned} [X,Y]_*^{\mathbb{Z}_2} \times \pi_1(Y^{\mathbb{Z}_2}) &\to [X,Y]_*^{\mathbb{Z}_2} \\ [F_0] \times [\gamma] &\mapsto [F_1] \end{aligned} \tag{3.41}$$

defines a right action of $\pi_1(Y^{\mathbb{Z}_2})$ on $[X,Y]_*^{\mathbb{Z}_2}$. Denoting by $[X,Y]_*^{\mathbb{Z}_2}/\pi_1(Y^{\mathbb{Z}_2})$ the set of orbits under this action, we can formulate the following result:

Lemma 3.22. *For a $\mathbb{Z}_2$-CW complex X and a G-space Y with path-connected fixed point set $Y^{\mathbb{Z}_2}$, there is a bijection*

$$[X,Y]_*^{\mathbb{Z}_2}/\pi_1(Y^{\mathbb{Z}_2}) \simeq [X,Y]^{\mathbb{Z}_2}.$$

The proof of this statement can be found in [Whi78, p. 101]. A more detailed and elementary exposition in the case of trivial $\mathbb{Z}_2$-actions is presented in [Hat02, p. 421]. The main idea is the following: During a free homotopy between two based equivariant maps F_0 and F_1, the base point traces out a loop in $Y^{\mathbb{Z}_2}$. Therefore, even though there may be no based homotopy between F_0 and F_1, they lie within the same orbit under the $\pi_1(Y^{\mathbb{Z}_2})$-action, see Figure 3.7.

3.5. Path spaces and suspensions

Given a configuration space X, we would like to formalize the notion of adding position-like and momentum-like coordinates. For example, we would like to construct S^{d_x+1,d_k} and S^{d_x,d_k+1} given $X = \mathrm{S}^{d_x,d_k}$. The following construction accomplishes this goal:

Definition 3.23. *The position-like suspension $\mathcal{S}X$ and momentum-like suspension $\bar{\mathcal{S}}X$ of a $\mathbb{Z}_2$-space X are both given by the quotient*

$$X \times [0,1]/X \times \{0\} \cup X \times \{1\}, \tag{3.42}$$

which is a $\mathbb{Z}_2$-space where the non-trivial element of $\mathbb{Z}_2$ acts on the suspension coordinate $t \in [0,1]$ as

$$\begin{aligned} t &\mapsto t \quad \text{for } \mathcal{S}X \\ \text{and } t &\mapsto 1-t \text{ for } \bar{\mathcal{S}}X. \end{aligned}$$

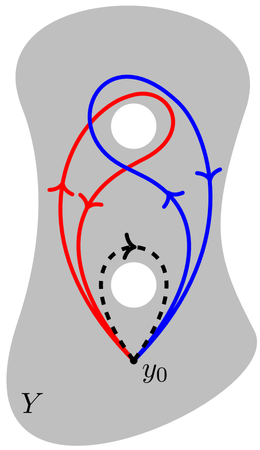


Figure 3.7.: Example of the action of $\pi_1(Y^{\mathbb{Z}_2})$ on $[X, Y]_*^{\mathbb{Z}_2}$ with $X = \mathrm{S}^1$, $Y \subset \mathbb{R}^2$ (gray with two holes), trivial $\mathbb{Z}_2$-actions and base point $y_0 \in Y^{\mathbb{Z}_2} = Y$. The red and blue loops represent distinct elements in $[X, Y]_*^{\mathbb{Z}_2}$, but share the same orbit under the action of $\pi_1(Y^{\mathbb{Z}_2})$, a representative of which is indicated by the dotted loop. Indeed, they are freely homotopic by a homotopy tracing out the dotted loop.

Example 3.24. For $X = \mathrm{S}^{d_x,d_k}$, we obtain

$$\mathcal{S}X = \mathcal{S}\mathrm{S}^{d_x,d_k} = \mathrm{S}^{d_x+1,d_k} \tag{3.43}$$
$$\bar{\mathcal{S}}X = \bar{\mathcal{S}}\mathrm{S}^{d_x,d_k} = \mathrm{S}^{d_x,d_k+1} \tag{3.44}$$

Another space of interest which is closely related to the suspension construction is the space of paths in a $\mathbb{Z}_2$-space:

Definition 3.25. *The position-like path space $\Omega(X, x_1, x_2)$ and the momentum-like path space $\bar{\Omega}(X, x_1, x_2)$ of a $\mathbb{Z}_2$-space X both consist of all paths in X starting in $x_1 \in X$ and ending in $x_2 \in X$. They are $\mathbb{Z}_2$-spaces with the non-trivial element of $\mathbb{Z}_2$ acting point-wise on points on the path and as*

$$\begin{aligned} t &\mapsto t \qquad \textit{for } \Omega(X, x_1, x_2) \\ \textit{and } t &\mapsto 1-t \ \textit{for } \bar{\Omega}(X, x_1, x_2) \end{aligned}$$

on the path coordinate $t \in [0, 1]$.

Note that in order for the path space to be different from the empty set, the points x_1 and x_2 need to lie within the same connected component of X.

Given a base point $x_* \in X^{\mathbb{Z}_2} \subset X$, there are natural base points for its suspension and path space: For both $\mathcal{S}X$ and $\bar{\mathcal{S}}X$, we choose the point $(x_*, 1/2)$, which is fixed by either $\mathbb{Z}_2$-action. In the case of the path spaces $\Omega(X, x_0, x_1)$ and $\bar{\Omega}(X, x_0, x_1)$, we take some fixed $\mathbb{Z}_2$-equivariant path from x_0 to x_1.

There is a useful relation connecting suspension and path spaces, which is stated as follows:

Proposition 3.26. *Given a $\mathbb{Z}_2$-CW complex X and a $\mathbb{Z}_2$-space Y, there are bijections*

$$[\mathcal{S}X, Y]_*^{\mathbb{Z}_2} \simeq [X, \Omega(Y, y_0, y_1)]_*^{\mathbb{Z}_2} \tag{3.45}$$
$$[\bar{\mathcal{S}}X, Y]_*^{\mathbb{Z}_2} \simeq [X, \bar{\Omega}(Y, y_0, y_1)]_*^{\mathbb{Z}_2} \tag{3.46}$$

Proof. Put simply, the correspondence is established by reinterpreting the suspension coordinate as a path coordinate and vice versa. However, there is a mismatch when it comes to homotopies fixing base points: While in the suspension only a single point is fixed, in the path space the entire path constituting the base point is fixed. This discrepancy is remedied by considering a version of the suspension called the reduced suspension ΣX or $\bar{\Sigma}X$ obtained by additionally collapsing the subspace $\{x_*\} \times I$ in $\mathcal{S}X$ or $\bar{\mathcal{S}}X$ to a point. In the case of a $\mathbb{Z}_2$-CW complex X, there are $\mathbb{Z}_2$-homotopy equivalences $\Sigma X \simeq \mathcal{S}X$ and $\bar{\Sigma}X \simeq \bar{\mathcal{S}}X$ (see [Ada84, p. 491]). Analogously, we modify the path spaces $\Omega(Y, y_0, y_1)$ and $\bar{\Omega}(Y, y_0, y_1)$ of a $\mathbb{Z}_2$-space Y to be loop spaces ΩY and $\bar{\Omega}Y$ by taking $y_1 = y_0$ and choosing the base point to be the constant loop to

$y_0 \in Y^{\mathbb{Z}_2} \subset Y$. Again, there are $\mathbb{Z}_2$- homotopy equivalences $\Omega(Y, y_1, y_0) \simeq \Omega Y$ and $\bar{\Omega}(Y, y_0, y_1) \simeq \bar{\Omega} Y$. The reduced suspension and the based loop space are adjoints of one another in the sense that there are bijections

$$[\Sigma X, Y]_*^{\mathbb{Z}_2} \simeq [X, \Omega Y]_*^{\mathbb{Z}_2} \tag{3.47}$$

$$[\bar{\Sigma} X, Y]_*^{\mathbb{Z}_2} \simeq [X, \bar{\Omega} Y]_*^{\mathbb{Z}_2}. \tag{3.48}$$

In both cases, the suspension coordinate on the left hand side is reinterpreted as the loop coordinate on the right hand side and there are no longer issues regarding base points. Due to the homotopy equivalences between reduced and unreduced suspensions as well as loop spaces and path spaces, the original statement follows. □

4. Homotopy classification

The goal of this chapter is to determine the set $[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2}$ with $n \gg d_x, d_k$ for all d_x, d_k and complex and real symmetry classes s. The results of this endeavor are displayed in Table 4.1. In the absence of defects ($d_x = 0$), this table is known as the Periodic Table for topological insulators and superconductors [Kit09] and has been generalized in [TK10] to all d_x. On a historical note, the entries of the Periodic Table were initially not presented as in Table 4.1, but were rather calculated case by case for low dimensions. Indeed, there is no a priori reason for expecting the diagonal pattern that only exhibits itself if the symmetry classes are arranged in the presented order. The latter was first realized by Kitaev [Kit09] by noticing that there is a deep relation to a mathematical result known as Bott periodicity [Bot59]. The plan of this chapter is to make use of Bott periodicity in order to prove that there are bijections $[X, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\bar{\mathcal{S}}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2}$ and $[X, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\mathcal{S}X, C_{s-1}(n)]_*^{\mathbb{Z}_2}$ for path-connected $\mathbb{Z}_2$-CW complexes X with $\dim X \ll n$ ($\bar{\mathcal{S}}X$ and $\mathcal{S}X$ denote the suspension of X which adds a momentum-like or position-like dimension to X respectively, see Section 3.5). Specializing to $X = \mathrm{S}^{d_x,d_k}$, the result shown in Table 4.1 follows.

Before we proceed with the homotopy classification, we point out crucial differences to other choices of equivalence relations that are used widely in the literature.

4.1. Alternative equivalence relations

There are two mathematical languages in which IQPVs may be viewed, as introduced in Definitions 2.1 and 2.2 in the setting without symmetries. Both capture the fact that the spaces of annihilators should vary continuously with some parameter in a configuration space X (e.g. with momentum $\mathbf{k} \in X = \mathrm{T}^d$). We can either encode this feature in the form of a continuous map from X to a classifying space or by viewing the collection of annihilator spaces as a sub-vector bundle over X.

More formally, and in the presence of symmetries, we may view it as a rank-n complex sub-vector bundle $\mathcal{A} \xrightarrow{\rho} X$ with fibers $\rho^{-1}(k) = A(\mathbf{k}) \subset \mathcal{W}_{\mathbf{k}} = \mathbb{C}^{2n}$ subject to $A(\mathbf{k})^{\perp} = A(\tau(\mathbf{k}))$ and the pseudo-symmetry conditions (2.67). On the other hand, we may describe it by a classifying map $A : X \to C_s(n)$ subject to the $\mathbb{Z}_2$-equivariance condition $\tau_s \circ A = A \circ \tau$.

We used the description in terms of classifying maps with the natural equivalence relation of being homotopic in order to define topological phases of IQPVs. In this

	index	symmetry	$d_k - d_x$			
	s	label	0	1	2	3
complex classes	0	A	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	1	$A\mathrm{III}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
real classes	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	1	$D\mathrm{III}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	2	$A\mathrm{II}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	3	$C\mathrm{II}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	4	C	0	0	$\mathbb{Z}$	0
	5	$C\mathrm{I}$	0	0	0	$\mathbb{Z}$
	6	$A\mathrm{I}$	$\mathbb{Z}$	0	0	0
	7	$BD\mathrm{I}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0

Table 4.1.: The sets $[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2}$ for $1 \leq d_x + d_k \ll n$, also known as the Periodic Table for topological insulators and superconductors. The complex symmetry classes are included with trivial $\mathbb{Z}_2$-actions. The entries 0, $\mathbb{Z}_2$ and $\mathbb{Z}$ mean sets with one, two and (countably) infinitely many elements, respectively. They are groups only when $d_x \geq 1$. For $d_x = d_k = 0$, the three entries of $\mathbb{Z}$ change to $\mathbb{Z}_{2n+1}$ (class A), $\mathbb{Z}_{n/2+1}$ (class $A\mathrm{II}$) and $\mathbb{Z}_{n/4+1}$ (class $A\mathrm{I}$), corresponding to the connected components of $C_0(n)$ (class A), $R_2(n)$ (class $A\mathrm{II}$) and $R_6(n)$ (class A); see Table 2.1.

section, we point out two alternative equivalence relations based on the vector bundle description, both of which will turn out to give, in general, a coarser classification than using homotopy. These differences can already be illustrated for the complex class $s = 0$ (class A), where IQPVs can be viewed either as classifying maps $X \to \mathrm{Gr}_p(\mathbb{C}^n)$ or complex p-dimensional sub-vector bundles of $X \times \mathbb{C}^n$. We assume here that X is path-connected and thus focus on only one connected component of $C_0(n)$. In other words, the dimension of the fibers will be constant.

Two such sub-vector bundles $\mathcal{A}_0$ and $\mathcal{A}_1$ with p-dimensional fibers and projections $\rho_0 : \mathcal{A}_0 \to X$ and $\rho_1 : \mathcal{A}_1 \to X$ are said to be isomorphic if there exists a homeomorphism $h : \mathcal{A}_0 \to \mathcal{A}_1$ which maps fibers to fibers ($\rho_1 \circ h = \rho_0$) by linear isomorphisms. This notion defines an equivalence relation and its equivalence classes, called isomorphism classes, will be denoted by $\mathrm{Vect}_p^{\mathbb{C}}(X)$.

There is a bijection [Hus66]

$$\mathrm{Vect}_p^{\mathbb{C}}(X) \simeq [X, \mathrm{Gr}_p(\mathbb{C}^n)] \text{ if } 2(n-p) \geq \dim X, \tag{4.1}$$

so the two equivalence relations lead to the same equivalence classes if the condition on the dimensionality of X is met. However, if it is violated, it may occur that two IQPVs are isomorphic as sub-vector bundles but not homotopic. A concrete example is provided by the "Hopf insulator" [MRW08] for $X = \mathrm{S}^3$ with $n = 2$ and $p = 1$, where $2(n-p) = 2 < 3 = \dim \mathrm{S}^3$. Indeed, while all complex line bundles over S^3 are isomorphic to the trivial one ($\mathrm{Vect}_1^{\mathbb{C}}(\mathrm{S}^3) = 0$), such vector bundles, viewed as subbundles of $\mathrm{S}^3 \times \mathbb{C}^2$, organize into distinct homotopy classes:

$$[\mathrm{S}^3, \mathrm{Gr}_1(\mathbb{C}^2)] = \pi_3(\mathrm{S}^2) = \mathbb{Z}. \tag{4.2}$$

These homotopy classes are distinguished by what is called the Hopf invariant (see Section 6.3 for details).

A standard approach used in the literature is to work with a further reduction of the topological information contained in isomorphism classes, by adopting the equivalence relation of *stable equivalence* between vector bundles. Two vector bundles $\mathcal{A}_0 \to X$ and $\mathcal{A}_1 \to X$ are stably equivalent if they are isomorphic after adding trivial bundles (meaning trivial valence bands in physics language), i.e. if there exist $m_1, m_2 \in \mathbb{N}$ such that

$$\mathcal{A}_0 \oplus (X \times \mathbb{C}^{m_1}) \simeq \mathcal{A}_1 \oplus (X \times \mathbb{C}^{m_2}). \tag{4.3}$$

Under the direct-sum operation, the stable equivalence classes constitute a group called the reduced complex K-group of X, denoted as $\widetilde{K}_{\mathbb{C}}(X)$. Inverses in this group are given by the fact that for compact X, all complex vector bundles $\mathcal{A}$ have a partner $\mathcal{A}'$ such that $\mathcal{A} \oplus \mathcal{A}' \simeq X \times \mathbb{C}^n$ for some $n \in \mathbb{N}$, where the right-hand side represents the neutral element. In the limit of a large number of valence and conduction bands, called the *stable regime*, the elements of the reduced K-group are in bijection with the homotopy classes of maps into the classifying space [Hus66]:

$$\widetilde{K}_{\mathbb{C}}(X) \simeq [X, \mathrm{Gr}_p(\mathbb{C}^n)] \text{ if } 2(n-p) \geq \dim X \text{ and } 2p \geq \dim X. \tag{4.4}$$

Outside the stable regime, stably equivalent vector bundles need not be isomorphic, much less homotopic. A class of examples demonstrating the differences between all three equivalence relations is provided by imposing, on top of the U_1-symmetry of the complex symmetry class A, the combined operation $T \circ I$ of time-reversal and inversion as a symmetry with $T^2 = 1$ and $I^2 = 1$ as well as $T \circ I = I \circ T$. Although fundamentally time-reversal squares to -1 for fermions, the property $T^2 = 1$ can be realized by the reduction of an SU_2 symmetry as carried out in the real class $s = 6$ (class AI) in Section 2.5.4. Since both T and I invert the momentum ($\mathbf{k} \mapsto -\mathbf{k}$), their combination fixes it and therefore the spaces of annihilators are restricted by the condition $(T \circ I)A(\mathbf{k}) = A(\mathbf{k})$. The anti-linear map $T \circ I$ acts as a real structure on $\mathcal{W}_{\mathbf{k}} = \mathbb{C}^n$ and due to the condition on the spaces of annihilators, we may restrict our attention to the $+1$ eigenspace of this real structure, defining a real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. Accordingly, the IQPVs in this setting are given by classifying maps $X \to \mathrm{Gr}_p(\mathbb{R}^n)$ or p-dimensional real sub-vector bundles of $X \times \mathbb{R}^n$, where X is momentum space. Note that this realization falls outside of the complex and real symmetry classes introduced in Section 2.4 since inversion I does not commute with translations. To stay within the scope of the symmetry classes introduced earlier, we could instead impose only time-reversal T with $T^2 = 1$ and choose X to exclusively have position-like coordinates. This would lead to the same description and correspond to the real class $s = 6$ (class AI).

We will focus on the choice $X = \mathrm{S}^d$ which is to be understood as $\mathrm{S}^{0,d}$ if X is momentum space and $\mathrm{S}^{d,0}$ if all dimensions of X are position-like. The latter can be interpreted as a measuring surface around a defect of codimension $d+1$. In any case, the analogs of equations (4.1) and (4.4) read

$$\mathrm{Vect}_p^{\mathbb{R}}(X) \simeq [X, \mathrm{Gr}_p(\mathbb{R}^n)] \text{ if } n-p-1 \geq \dim X \tag{4.5}$$

and

$$\widetilde{K}_{\mathbb{R}}(X) \simeq [X, \mathrm{Gr}_p(\mathbb{R}^n)] \text{ if } n-p-1 \geq \dim X \text{ and } p-1 \geq \dim X. \tag{4.6}$$

Setting $p = 1$ and $X = \mathrm{S}^1$, we have a bijection $\mathrm{Vect}_1^{\mathbb{R}}(\mathrm{S}^1) \simeq [\mathrm{S}^1, \mathrm{Gr}_1(\mathbb{R}^n)] \simeq \mathbb{Z}_2$ for $n \geq 3$, where the non-trivial element is represented by the Moebius bundle [Hat03]. For $n = 2$ we see the difference between homotopy classes and isomorphism classes: Since $\mathrm{Gr}_1(\mathbb{R}^2) = \mathrm{S}^1$, it is clear that $[\mathrm{S}^1, \mathrm{Gr}_1(\mathbb{R}^2)] = [\mathrm{S}^1, \mathrm{S}^1] \simeq \mathbb{Z}$, where the homotopy classes are distinguished by their winding number. However, the bundle which twists by 2π when traversing the base space S^1 and thus has a non-trivial winding number 2 is isomorphic to the trivial bundle.

Taking now $X = \mathrm{S}^2$, we can combine equations (4.5) and (4.6) to obtain the statement that there is a bijection $\widetilde{K}_{\mathbb{R}}(\mathrm{S}^2) \simeq \mathrm{Vect}_p^{\mathbb{R}}(\mathrm{S}^2)$ for $p \geq 3$. Indeed, violating the latter requirement with $p = 2$, we have $\mathrm{Vect}_2^{\mathbb{R}}(\mathrm{S}^2) \simeq \mathbb{N}_0$ while $\widetilde{K}_{\mathbb{R}}(\mathrm{S}^2) = \mathbb{Z}_2$. A representative of the non-trivial class $2 \in \mathbb{N}_0 = \mathrm{Vect}_2^{\mathbb{R}}(\mathrm{S}^2)$ is the tangent bundle to S^2 denoted by $T\mathrm{S}^2$. By regarding S^2 as the unit sphere in $\mathbb{R}^3$, we can also construct the normal bundle $N\mathrm{S}^2 \simeq \mathrm{S}^2 \times \mathbb{R}$. The direct sum of $T\mathrm{S}^2$ and $N\mathrm{S}^2$ is $\mathrm{S}^2 \times \mathbb{R}^3$ and therefore $T\mathrm{S}^2$ is stably equivalent to the trivial bundle. Yet, the isomorphism class of $T\mathrm{S}^2$ differs from that of the trivial bundle. The result $\mathrm{Vect}_2^{\mathbb{R}}(\mathrm{S}^2) = \mathbb{N}_0$ can be found in [Hat03] and, in the context of classifying topological phases, in Table A.1 of [DNG14b].

The notion of isomorphism (and stable equivalence) classes of vector bundles can be extended to the two real symmetry classes $s = 2$ (class AII) and $s = 6$ (class AI). In these symmetry classes, there is an additional time-reversal operator T acting on the total space of a bundle $\mathcal{A} \xrightarrow{\rho} X$ with $T^2 = -1$ for class AII and $T^2 = +1$ for class AI. This action covers the involution τ on the base space, i.e. $T\rho^{-1}(x) = \rho^{-1}(\tau(x))$. These bundles are called Real vector bundles [Ati66] or Quaternionic vector bundles [Dup69] (with capital R and Q in order to distinguish them from vector bundles over the real and quaternionic numbers). An isomorphism of two Real or Quaternionic vector bundles is an isomorphisms of the underlying complex vector bundles with the additional property that it commutes with T. The corresponding reduced K-groups are written $\widetilde{KR}(X)$ [Ati66, DNG14b] and $\widetilde{KQ}(X)$ [Dup69, DNG14a].

The K-theory groups for the other symmetry classes can be inferred indirectly by an algebraic construction using Clifford modules as in [Kit09, FM13]. In all cases, the K-theory groups of momentum space X are in bijection with $[X, C_s(n)]^{\mathbb{Z}_2}$ as a set, in the limit of large n (as well as large p where applicable, see Table 2.1).

To sum up, the natural equivalence relation for us to use is that of homotopy. It is a finer tool than stable equivalence (as considered in [Kit09]) and even ordinary isomorphy of vector bundles (as considered in [DNG14a, DNG14b] for $s = 2, 6$), and is therefore adopted as our topological classification principle. In Chapter 5 we give the precise bounds on the number of conduction and valence bands for all complex and real symmetry classes beyond which the three equivalence relations differ, including equations (4.1), (4.4), (4.5) and (4.6) as special cases.

4.2. The diagonal map

In this section we prove the bijection $[X, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\bar{S}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2}$ for any path-connected $\mathbb{Z}_2$-CW complex X. For this purpose we introduce the “diagonal map” increasing the momentum-like dimension as well as the symmetry index by one.

The spaces $C_s(n)$ and their τ_s-fixed point sets $R_s(n)$ are defined through pseudo-symmetries satisfying Clifford algebra relations (see Section 2.6). To formulate the diagonal map, we exploit this close connection to Clifford algebras by using the counterpart of the algebra isomorphisms $\mathrm{Cl}(\mathbb{C}^2) \otimes \mathrm{Cl}(\mathbb{C}^s) \simeq \mathrm{Cl}(\mathbb{C}^{s+2})$ as well as $\mathrm{Cl}(\mathbb{R}^{1,1}) \otimes \mathrm{Cl}(\mathbb{R}^{s,0}) \simeq \mathrm{Cl}(\mathbb{R}^{s+1,1})$. Recall that $\mathrm{Cl}(\mathbb{C}^m)$ is the complex Clifford algebra with m generators and $\mathrm{Cl}(\mathbb{R}^{p,q})$ is the real Clifford algebra with p negative and q positive generators.

The following treatment is analogous to the one in Section 2.5.4. Let there be s real pseudo-symmetries $j_1 \ldots, j_s$ on $\mathcal{W}_+ = \mathbb{C}^{2n}$ forming the space $C_s(n)$ with τ_s-fixed point set $R_s(n)$ as defined in Section 2.6. We now choose to view $\mathcal{W}_+$ as part of a space $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_- = \mathbb{C}^{4n}$ on which an imaginary generator K acts with eigenspaces $\mathcal{W}_\pm$ for its eigenvalues $\pm\mathrm{i}$. We fix an isomorphism $L_\downarrow : \mathcal{W}_+ \to \mathcal{W}_-$ with inverse $L_\uparrow : \mathcal{W}_- \to \mathcal{W}_+$ and set $L := L_\downarrow + L_\uparrow$. A new set of $s+2$ generators can now be defined on W as

$$J_l := L_\downarrow j_l + j_l L_\uparrow \quad (l = 1, \ldots, s) \tag{4.7}$$

$$J_{s+1} := \mathrm{i}LK \tag{4.8}$$

$$J_{s+2} := K. \tag{4.9}$$

We interpret this set of operators as a set of pseudo-symmetries on the doubled space $\mathcal{W}$. The pseudo-symmetries $J_1, \ldots, J_{s+1}$ are real, while J_{s+2} is imaginary by construction. This enhanced set of pseudo-symmetries defines the space $C_{s+2}(2n)$ with fixed point set $R_{s+1,1}(2n)$ (we use a double subscript to indicate that the number of real and imaginary pseudo-symmetries). In Section 2.5.4, we have constructed a map $C_{s+2}(2n) \to C_s(n)$ which restricts under τ_{s+2} to a map $R_{s+1,1}(2n) \to R_s(n)$. The following assignment constitutes the inverse of this map:

$$\begin{aligned} f : C_s(n) &\to C_{s+2}(2n) \\ A_+ &\mapsto \left\{ w + w' + L_\downarrow(w - w') \mid w \in A_+, w' \in A_+^{\mathrm{c}} \right\}. \end{aligned} \tag{4.10}$$

This map is well defined since

$$J_1 f(A_+) = \cdots = J_{s+2} f(A_+) = f(A_+)^{\mathrm{c}}. \tag{4.11}$$

Furthermore, if $A = A^\perp$ then $f(A) = f(A)^\perp$, so f restricts to a map $f' : R_s(n) \to R_{s+1,1}(2n)$.

In order to define the diagonal map adding a momentum-like coordinate, recall from eq. (2.139) that we can associate to a subspace $A \subset \mathcal{W}$ the anti-Hermitian operator

$$J(A) = \mathrm{i}(P_A - P_{A^{\mathrm{c}}}). \tag{4.12}$$

with the properties $J(A)^2 = -1$ and $\tau_{\mathrm{CAR}}(J(A)) = J(A^\perp)$.

Given $A \in C_{s+2}(2n)$ and the eigenspace $E_{+\mathrm{i}}(K)$ associated to the eigenvalue $+\mathrm{i}$ of K, we can define the heart of the diagonal map as the one-parameter family

$$\beta_t(A) := \mathrm{e}^{(t\pi/2)KJ(A)} \cdot E_{+\mathrm{i}}(K). \tag{4.13}$$

The following lemma summarizes the key features of this map.

Lemma 4.1. *The assignment $[0,1] \ni t \mapsto \beta_t(A)$ for $A \in C_{s+2}(2n)$ is a curve in $C_{s+1}(2n)$ with initial point $\beta_0(A) = E_{+\mathrm{i}}(K)$, final point $\beta_1(A) = E_{-\mathrm{i}}(K)$, and midpoint $\beta_{1/2}(A) = A$. It is $\mathbb{Z}_2$-equivariant in the sense that $\beta_t(A)^\perp = \beta_{1-t}(A^\perp)$.*

Proof. Since the Clifford generators $J_1, \ldots, J_s$, and I anti-commute with K, they exchange the two eigenspaces $E_{+\mathrm{i}}(K)$ and $E_{-\mathrm{i}}(K) = E_{+\mathrm{i}}(K)^{\mathrm{c}}$, so $E_{\pm\mathrm{i}}(K) \in C_{s+1}(2n)$. Similarly, $J(A)$ anti-commutes with all generators $J_1, \ldots, J_s, I$, implying that the latter commute with the product $KJ(A)$ as well as the unitary operator $\mathrm{e}^{(t\pi/2)KJ(A)}$. Therefore, since $E_{+\mathrm{i}}(K)$ lies in $C_{s+1}(2n)$, so does $\mathrm{e}^{(t\pi/2)KJ(A)} \cdot E_{+\mathrm{i}}(K) = \beta_t(A)$. In other words, $\beta_t(A)$ satisfies the pseudo-symmetry relations

$$J_1\beta_t(A) = \ldots = J_s\beta_t(A) = I\beta_t(A) = \beta_t(A)^{\mathrm{c}}. \tag{4.14}$$

To see that the curve ends at $E_{-\mathrm{i}}(K)$, we recall that $K^2 = J(A)^2 = -1$ and $KJ(A) = -J(A)K$ (due to $KA = A^{\mathrm{c}}$). These relations imply that $(KJ(A))^2 = -1$ and

$$\begin{aligned} \beta_1(A) &= \mathrm{e}^{(\pi/2)KJ(A)} \cdot E_{+\mathrm{i}}(K) \\ &= \sin(\pi/2)KJ(A) \cdot E_{+\mathrm{i}}(K) \\ &= J(A) \cdot E_{+\mathrm{i}}(K) \\ &= E_{-\mathrm{i}}(K), \end{aligned} \tag{4.15}$$

since $J(A)$ swaps the eigenspaces of K.

The property that the midpoint of the curve evaluates as $\beta_{1/2}(A) = A$ can be deduced by computing

$$\begin{aligned} \mathrm{e}^{(\pi/4)KJ(A)} &= \cos(\pi/4)\mathrm{Id}_{\mathcal{W}} + \sin(\pi/4)KJ(A) \\ &= (\mathrm{Id}_W + KJ(A))/\sqrt{2}. \end{aligned} \tag{4.16}$$

Applying this to any $w \in E_{+\mathrm{i}}(K)$ we get

$$\begin{aligned} (\mathrm{Id}_W + KJ(A))w &= w - \mathrm{i}J(A)w \\ &= -\mathrm{i}J(A)(w - \mathrm{i}J(A)w) \\ &\in E_{+\mathrm{i}}(J(A)) = A. \end{aligned} \tag{4.17}$$

The linear transformation $\mathrm{e}^{(\pi/4)KJ(A)} : E_{+\mathrm{i}}(K) \to A$, $w \mapsto w - \mathrm{i}J(A)w$, is an isomorphism because $J(A) \cdot E_{+\mathrm{i}}(K) = E_{-\mathrm{i}}(K)$. Hence

$$\beta_{1/2}(A) = \mathrm{e}^{(\pi/4)KJ(A)} \cdot E_{+\mathrm{i}}(K) = A. \tag{4.18}$$

Turning to the property stated last, we use $\tau_{\mathrm{CAR}}(J(A)) = J(A^{\perp})$ and $\tau_{\mathrm{CAR}}(K) = -K$ (since K is imaginary) as well as $E_{+\mathrm{i}}(K)^{\perp} = E_{-\mathrm{i}}(K)$ to obtain

$$\begin{aligned}\beta_t(A)^{\perp} &= \tau_{\mathrm{CAR}}\big(\mathrm{e}^{(t\pi/2)KJ(A)}\big) \cdot E_{+\mathrm{i}}(K)^{\perp} \\ &= \mathrm{e}^{(-t\pi/2)KJ(A^{\perp})} \cdot E_{-\mathrm{i}}(K) \\ &= \beta_{1-t}(A^{\perp}),\end{aligned} \tag{4.19}$$

where we have additionally used the identity $(g\cdot A)^{\perp} = \tau_{\mathrm{CAR}}(g)\cdot A^{\perp}$ for all $g \in \mathrm{GL}(\mathcal{W})$. Thus $t \mapsto \beta_t(A)$ is $\mathbb{Z}_2$-equivariant in the stated sense. □

Let the notation for the space of paths in $C_{s+1}(2n)$ from $E_{+\mathrm{i}}(K)$ to $E_{-\mathrm{i}}(K)$ be abbreviated to

$$\bar{\Omega}(C_{s+1}(2n), E_{+\mathrm{i}}(K), E_{-\mathrm{i}}(K)) \equiv \bar{\Omega}_K C_{s+1}(2n). \tag{4.20}$$

Interpreting the parameter t in the definition of β_t as a path parameter, we obtain an equivariant map

$$\begin{aligned}\beta : C_s(n) &\to \bar{\Omega}_K C_{s+1}(2n) \\ A_+ &\mapsto \{t \mapsto \beta_t(f(A_+))\}.\end{aligned} \tag{4.21}$$

Due to its $\mathbb{Z}_2$-equivariance, it restricts to a map

$$\beta' : C_s(n)^{\mathbb{Z}_2} = R_s(n) \to (\bar{\Omega}_K C_{s+1}(2n))^{\mathbb{Z}_2}, \tag{4.22}$$

where the $\mathbb{Z}_2$-action on the path space is the one introduced in Section 3.5.

Let an IQPV in the real symmetry class s with configuration space X be described by the map $\psi : X \to C_s(n)$. Using β, we can form $\beta \circ \psi : X \to \bar{\Omega}_K C_{s+1}(2n)$ and interpret this map as a map $\bar{\mathcal{S}}X \to C_{s+1}(2n)$ describing an IQPV in the real symmetry class $s+1$ with double the number of bands and an additional momentum-like coordinate in its configuration space (see Section 3.5). In the following, we demonstrate these features on an example.

Example 4.2 (From $(d, s) = (0, 0)$ to $(1, 1)$)**.** Starting with a superconductor ground state in the real symmetry class $s = 0$ (class D) in zero dimensions ($X = \mathrm{S}^{0,0}$) and with $\mathcal{W}_+ = \mathbb{C}^2$ ($n = 1$), applying the construction above produces the ground state of a time-reversal invariant superconductor in class DIII in one dimension ($\bar{\mathcal{S}}X = \mathrm{S}^{0,1}$). Recall that both points of $X = \mathrm{S}^{0,0}$ are fixed under the involution τ and therefore the image of a map describing an IQPV is restricted to lie within $R_0(1) \subset C_0(1)$, which consists of only two points:

$$R_0(1) = \{\mathbb{C} \cdot c, \mathbb{C} \cdot c^{\dagger}\}, \tag{4.23}$$

which correspond to the empty and occupied state $|0\rangle$ and $|1\rangle$ respectively. Choosing the base point to be $\mathbb{C}|0\rangle$, there are two based maps $\mathrm{S}^0 \to R_0(1)$: the constant map and the map ψ assigning to the point in S^0 which is not the base point the image $|1\rangle$. The procedure of doubling the number of bands amounts to forming the tensor product with the two-dimensional spinor space $(\mathbb{C}^2)_{\text{spin}}$ to obtain the space $\mathcal{W} = \mathcal{W}_+ \oplus \mathcal{W}_- = \mathbb{C}^4 = (\mathbb{C}^2)_{\text{BdG}} \otimes (\mathbb{C}^2)_{\text{spin}}$, where we use the subscript "BdG" as for the Bogoliubov-de Gennes Hamiltonian of eq. (2.53). We set

$$K := \mathrm{i}(\sigma_1)_{\text{BdG}} \otimes (\sigma_1)_{\text{spin}}, \tag{4.24}$$

so the image of $A = \mathbb{C} \cdot c^\dagger \in R_0(1)$ under f as defined in eq. (4.10) is given by

$$f(A) = \operatorname{span}_{\mathbb{C}}\{c_\uparrow^\dagger, c_\downarrow^\dagger\}, \tag{4.25}$$

while the base point $A_* = \mathbb{C} \cdot c \in R_0(1)$ is mapped to

$$f(A_*) = \operatorname{span}_{\mathbb{C}}\{c_\uparrow, c_\downarrow\}. \tag{4.26}$$

The operator I is chosen to be the pseudo-symmetry J_1 introduced in Section 2.5.2 as part of the Kitaev sequence:

$$I \equiv J_1 = \gamma T = (\sigma_1)_{\text{BdG}} \otimes (\mathrm{i}\sigma_2)_{\text{spin}}. \tag{4.27}$$

We can now apply β_t with $t \in [0,1]$ to obtain a one-dimensional IQPV in the real symmetry class $s = 1$ (class $D\mathrm{III}$). Since the parameter t will play the role of the momentum coordinate, we use the parametrization $k := \pi(t - 1/2)$ to obtain

$$\begin{aligned} A(k) &= \beta_{t/\pi+1/2}(f(A)) \\ &= e^{(k/2-1/2)KJ(A)} \cdot E_{+\mathrm{i}}(K) \\ &= e^{(k/2)KJ(A)} \cdot A \\ &= \operatorname{span}_{\mathbb{C}} \left\{ c_\sigma^\dagger(-k)\cos(k/2) - c_{-\sigma}(k)\sin(k/2) \right\}_{\sigma=\uparrow,\downarrow}. \end{aligned} \tag{4.28}$$

Similarly, the base point $f(A_*) \in R_{1,1}(2)$ maps to

$$A_*(k) = \operatorname{span}_{\mathbb{C}} \left\{ c_\sigma(k)\cos(k/2) - c_{-\sigma}^\dagger(-k)\sin(k/2) \right\}_{\sigma=\uparrow,\downarrow}. \tag{4.29}$$

Since $A(\pm\pi/2) = A_*(\pm\pi/2)$, the two parts $A(k)$ and $A_*(k)$ fit together to produce a single $\mathbb{Z}_2$-equivariant map $\bar{\mathcal{S}}\mathrm{S}^{0,0} = \mathrm{S}^{0,1} \to C_1(2)$, as shown in Figure 4.1. In fact, the part $A_*(k)$ can be absorbed into the part $A(k)$ by extending the range of k from $[-\pi/2, \pi/2]$ to $[-\pi, \pi]$.

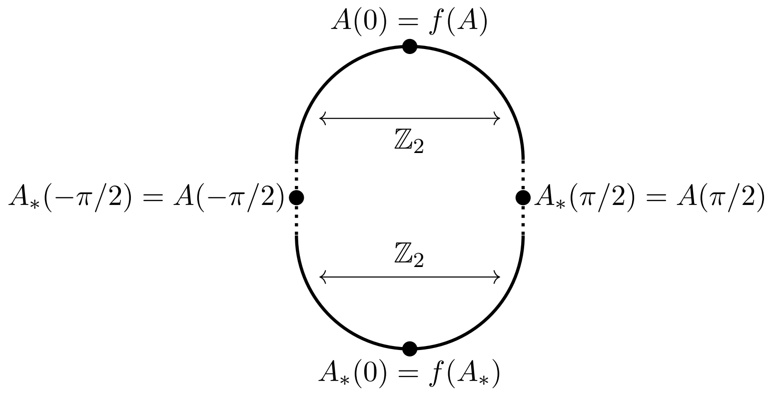


Figure 4.1.: The additional coordinate introduced may be viewed as a suspension coordinate since $A_*(k)$ (lower arc) and $A(k)$ (upper arc) agree at $k = \pm\pi/2$, producing a domain $\mathrm{S}^{0,1}$. The $\mathbb{Z}_2$-action on $\mathrm{S}^{0,1}$ is indicated by the arrows.

The many-body ground state which is annihilated by all elements in $A(k)$ for all $k \in [-\pi, \pi]$ can be written as

$$|\text{g.s.}\rangle = \exp\left(\textstyle\sum_k \cot(k/2)P(k)\right)|\text{vac}\rangle, \tag{4.30}$$

with

$$P(k) := c_\uparrow^\dagger(k)c_\downarrow^\dagger(-k). \tag{4.31}$$

There are other choices of imaginary generator K, for instance the family $K(\alpha) = \mathrm{i}(\sigma_1)_{\mathrm{BdG}} \otimes (\sigma_1 \cos\alpha + \sigma_3 \sin\alpha)_{\mathrm{spin}}$. With respect to an arbitrary choice of α, the generalized Cooper pair operator $P_\alpha(k)$ reads

$$P_\alpha(k) = c_\uparrow^\dagger(k)c_\downarrow^\dagger(-k)\cos\alpha + \left(c_\uparrow^\dagger(k)c_\uparrow^\dagger(-k) - c_\downarrow^\dagger(k)c_\downarrow^\dagger(-k)\right)\sin\alpha, \tag{4.32}$$

manifesting the spin-triplet pairing of the superconductor at hand.

Example 4.3 (From $(d, s) = (1, 1)$ to $(2, 2)$)**.** Starting from the result of the previous example, we now apply the diagonal map once more to arrive at a two-dimensional system in the real symmetry class $s = 2$ (class *A*II). The result of this exercise will be a representative of the topological phase known as the quantum spin Hall effect. Having already introduced spin, doubling the dimension of $\mathcal{W}$ has the physical interpretation of introducing two bands, which we label by p and h. Applying the $(1, 1)$-isomorphism f of eq. (4.10) to the outcome of the previous example yields

$$f(A(k_1)) = \mathrm{span}_{\mathbb{C}}\left\{a_{\uparrow,+}(k_1), a_{\downarrow,-}(k_1), b_{\downarrow,-}^\dagger(-k_1), b_{\uparrow,+}^\dagger(-k_1)\right\}, \tag{4.33}$$

with

$$a_{\sigma,\varepsilon}(k_1) = c_{\sigma,\mathrm{p}}(k_1)\cos(k_1/2) + \mathrm{i}\varepsilon c_{-\sigma,\mathrm{h}}(k_1)\sin(k_1/2), \tag{4.34}$$

$$b_{\sigma,\varepsilon}(k_1) = c_{\sigma,\mathrm{h}}(k_1)\cos(k_1/2) - \mathrm{i}\varepsilon c_{-\sigma,\mathrm{p}}(k_1)\sin(k_1/2). \tag{4.35}$$

Here we have used a convenient basis in order to avoid linear combinations of creation and annihilation operators, in anticipation of the particle number (or charge) conservation to be introduced. The following operators present a set of pseudo-symmetries:

$$J_1 = \gamma T = (\sigma_1)_{\mathrm{BdG}} \otimes (\mathrm{i}\sigma_2)_{\mathrm{spin}} \otimes \mathrm{Id}_{\mathrm{ph}}, \tag{4.36}$$

$$I = J_2 = \mathrm{i}QJ_1 = (\sigma_2)_{\mathrm{BdG}} \otimes (\mathrm{i}\sigma_2)_{\mathrm{spin}} \otimes \mathrm{Id}_{\mathrm{ph}}, \tag{4.37}$$

$$K = \mathrm{i}\mathrm{Id}_{\mathrm{BdG}} \otimes (\sigma_1)_{\mathrm{spin}} \otimes (\sigma_1)_{\mathrm{ph}}. \tag{4.38}$$

We now translate the subspace $f(A(k_1)) \subset \mathbb{C}^8$ to the operator $J(f(A(k_1)))$ according to eq. (2.139):

$$J(f(A(k_1))) = \mathrm{i}(\sigma_3)_{\mathrm{BdG}} \otimes (\mathrm{Id}_{\mathrm{spin}} \otimes (\sigma_3)_{\mathrm{ph}}\cos(k_1) + (\sigma_2)_{\mathrm{spin}} \otimes (\sigma_1)_{\mathrm{ph}}\sin(k_1)). \tag{4.39}$$

The diagonal map can now be evaluated as

$$\begin{aligned} A(\mathbf{k}) &= \mathrm{e}^{(k_2/2)KJ(A(k_1))} \cdot A(k_1) \\ &= \mathrm{span}_{\mathbb{C}} \left\{ \widetilde{a}_{\uparrow,+}(\mathbf{k}), \widetilde{a}_{\downarrow,-}(\mathbf{k}), \widetilde{b}^{\dagger}_{\downarrow,-}(-\mathbf{k}), \widetilde{b}^{\dagger}_{\uparrow,+}(-\mathbf{k}) \right\}, \end{aligned} \tag{4.40}$$

where $\mathbf{k} = (k_1, k_2)$ and

$$\begin{aligned} \widetilde{a}_{\sigma,\varepsilon}(\mathbf{k}) &= (c_{\sigma,\mathrm{p}}(\mathbf{k})\cos(k_1/2) + \mathrm{i}\varepsilon c_{-\sigma,\mathrm{h}}(\mathbf{k})\sin(k_1/2))\cos(k_2/2) \\ &\quad - (c_{-\sigma,\mathrm{h}}(\mathbf{k})\cos(k_1/2) + \mathrm{i}\varepsilon c_{\sigma,\mathrm{p}}(\mathbf{k})\sin(k_1/2))\sin(k_2/2), \end{aligned} \tag{4.41}$$

$$\begin{aligned} \widetilde{b}_{\sigma,\varepsilon}(\mathbf{k}) &= (c_{\sigma,\mathrm{h}}(\mathbf{k})\cos(k_1/2) - \mathrm{i}\varepsilon c_{-\sigma,\mathrm{p}}(\mathbf{k})\sin(k_1/2))\cos(k_2/2) \\ &\quad - (c_{-\sigma,\mathrm{p}}(\mathbf{k})\cos(k_1/2) - \mathrm{i}\varepsilon c_{\sigma,\mathrm{h}}(\mathbf{k})\sin(k_1/2))\sin(k_2/2). \end{aligned} \tag{4.42}$$

By construction, the space $A(\mathbf{k})$ is k_1-independent for $k_2 = \pm\pi/2$, so the momentum space can be viewed as $\bar{\mathcal{S}}\mathrm{S}^{0,1} = \mathrm{S}^{0,2}$. In order to verify that the present IQPV represents the non-trivial phase called the quantum spin Hall phase, we follow [KM05] and consider the bilinear form assigning to $w, w' \in A_{\mathbf{k}}$ the complex number

$$\theta_{\mathbf{k}}(w, w') := \langle Tw, w' \rangle = \{J_1 w, w'\}, \tag{4.43}$$

where we identify $\mathcal{W}_{\mathbf{k}} \equiv \mathcal{W}_{-\mathbf{k}} = \mathbb{C}^8$. A short computation using the facts that J_1 is orthogonal with respect to $\{\cdot,\cdot\}$ and $J_1^2 = -1$ reveals that $\theta_{\mathbf{k}}$ is skew:

$$\begin{aligned} \theta_{\mathbf{k}}(w, w') &= \{J_1 w, w'\} \\ &= \{J_1^2 w, J_1 w'\} \\ &= -\{w, J_1 w'\} \\ &= -\{J_1 w', w\} \\ &= -\theta_{\mathbf{k}}(w', w). \end{aligned} \tag{4.44}$$

The Kane-Melé Pfaffian [KM05] is defined to be the Pfaffian of the skew bilinear form $\theta_{\mathbf{k}}$ and evaluates in the present example as

$$\mathrm{Pf}(\theta_{\mathbf{k}}) \propto \cos^2(k_2). \tag{4.45}$$

This expression vanishes only for the two points with $k_2 = \pm\pi/2$. The fact that the zeros occur in a pair at $\mathbf{k}$ and $-\mathbf{k}$ is guaranteed due to $\overline{\theta_{\mathbf{k}}(w, w')} = \theta_{\mathbf{k}}(Tw, Tw')$ and $TA(\mathbf{k}) = A(-\mathbf{k})$. Being zeros of a complex-valued function, all zeros of the Pfaffian carry a vorticity and homotopies of IQPVs can only create zeros in pairs with opposite vorticities. Furthermore, at the special momenta $\mathbf{k} = -\mathbf{k}$ the form $\theta_{\mathbf{k}}$ is non-degenerate so the Pfaffian cannot vanish and pairs of zeros cannot be created or annihilated there. Thus, the property of having an even or odd number of pairs of zeros is an invariant and since the trivial topological phase is represented by a constant map, all of its representatives belong to the even sector. On the other hand, the result of applying our diagonal map yields a Kane-Melé Pfaffian with a single pair of zeros and therefore represents the quantum spin Hall phase.

4.3. Homotopy theory of the diagonal map

Having constructed the equivariant map β that maps an IQPV $X \to C_s(n)$ to an IQPV $\bar{S}X \to C_{s+1}(2n)$, we now investigate its induced map on equivariant homotopy classes (= topological phases):

$$\beta_*^{\mathbb{Z}_2} : [X, C_s(n)]_*^{\mathbb{Z}_2} \to [\bar{S}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2}. \tag{4.46}$$

We wish to apply the $\mathbb{Z}_2$-Whitehead theorem (Theorem 3.21) in order to show that, under certain circumstances, $\beta_*^{\mathbb{Z}_2}$ is a bijection and therefore leads to the diagonal pattern in the Periodic Table 4.1. First, we identify $[\bar{S}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2} = [X, \bar{\Omega} C_{s+1}(2n)]_*^{\mathbb{Z}_2}$ according to Proposition 3.26. The problem now fits the format given in Theorem 3.21 with $Y = C_s(n)$ and $Z = \bar{\Omega}_K C_{s+1}(2n)$. Therefore, in order for $\beta_*^{\mathbb{Z}_2}$ to be a bijection for $d \ll n$, both the map β (forgetting its equivariance) and its restriction β' to $\mathbb{Z}_2$-fixed points (see eq. (4.22)) need to induce bijections on the level of all homotopy groups π_d for $d \ll n$.

The first of these statements follows immediately from a result known as complex Bott periodicity [Bot59], since our map β reduces to the complex Bott map when the $\mathbb{Z}_2$-action is ignored:

Proposition 4.4. *The induced map*

$$\beta_* : \pi_d(C_s(n)) \to \pi_d(\bar{\Omega}_K C_{s+1}(2n)) \tag{4.47}$$

is an isomorphism for all s and $1 \leq d \ll n$.

We exclude the case $d = 0$ since, for instance, $\pi_0(C_0(n)) = \pi_0\big(\cup_{q=0}^{2n} \operatorname{Gr}_q(\mathbb{C}^{2n})\big) = \mathbb{Z}_{2n+1}$ but $\pi_0(\bar{\Omega}_K C_1(2n)) = \pi_1(\mathrm{U}_n) = \mathbb{Z}$. Often this discrepancy is evaded in the literature by the ad hoc definition $C_0(n) = \mathbb{Z} \times \operatorname{Gr}_n(\mathbb{C}^{2n})$. However, in the physical setting there is no justification for this adjustment and with some care, our proofs will work without it for $d \geq 1$, leaving $d = 0$ to be treated separately.

The second statement, which concerns the connectivity of the map β' defined in eq. (4.22), is more intricate and we will devote the remainder of this section to it. As a first step, we know from Section 3.5 that we can identify the equivariant path space with the equivariant loop space. Using Lemma 3.7, we know that loops are already determined by half of their length, so we can conclude

$$\pi_d((\bar{\Omega}_K C_{s+1}(2n))^{\mathbb{Z}_2}) \simeq \pi_d((\bar{\Omega} C_{s+1}(2n))^{\mathbb{Z}_2}) \simeq \pi_{d+1}(C_{s+1}, R_{s+1}) \tag{4.48}$$

for all $d \geq 1$. Thus, showing that β' is highly connected amounts to showing that

$$\beta'_* : \pi_d(R_s(n)) \to \pi_{d+1}(C_{s+1}, R_{s+1}) \tag{4.49}$$

is an isomorphism for $1 \leq d \ll n$.

In the next subsection, we prove the above statement for the two real symmetry classes $s = 2$ and $s = 6$. The other classes will be handled by a more indirect proof based on this result.

4.3.1. Bijection for $s \in \{2, 6\}$

In order to show that β'_* is highly connected, we will make use of the fact that there is, for $s = 2$ or $s = 6$, a fibration (actually, even a fiber bundle)

$$R_{s+1}(2n) \hookrightarrow C_{s+1}(2n) \xrightarrow{p} \widetilde{R}_{s,1}(2n), \tag{4.50}$$

for a base space $\widetilde{R}_{s,1}(2n) \simeq R_{s,1}(2n)$ to be introduced. Inspecting the spaces in Table 2.1, these two fiber bundles correspond to

$$\mathrm{Sp}_n \hookrightarrow \mathrm{U}_n \xrightarrow{p} \mathrm{U}_n/\mathrm{Sp}_n \qquad (s = 2), \tag{4.51}$$

$$\mathrm{O}_{n/4} \hookrightarrow \mathrm{U}_{n/4} \xrightarrow{p} \mathrm{U}_{n/4}/\mathrm{O}_{n/4} \qquad (s = 6), \tag{4.52}$$

where n needs to be a multiple of 2 for the first one and a multiple of 4 for the second one.

The projection p induces an isomorphism

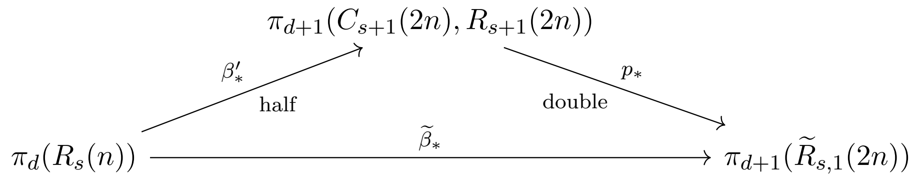
$$p_* : \pi_{d+1}(C_{s+1}(2n), R_{s+1}(2n)) \to \pi_{d+1}(\widetilde{R}_{s,1}(2n)) \tag{4.53}$$

for all d by basic principles (see [Hat02, p. 376]). On the other hand, we will show that the map β' can be interpreted as a map $\widetilde{\beta}_*$ into $\Omega_K \widetilde{R}_{s,1}$ rather than $(\bar{\Omega}_K C_{s+1})^{\mathbb{Z}_2}$, yielding the real analog of Proposition 4.4 in the form of isomorphisms

$$\widetilde{\beta}_* : \pi_d(R_s(n)) \to \pi_{d+1}(\widetilde{R}_{s,1}(2n)). \tag{4.54}$$

Ultimately, we will show that $\beta' \circ p = \widetilde{\beta}$ which proves that β'_* also induces isomorphisms on homotopy groups.

The outlined strategy is summarized in the diagram below: Since the diagram is commutative and two out of three maps are isomorphisms (p_* and $\widetilde{\beta}_*$), the third map β'_* has to be an isomorphism as well. In fact, it will turn out that the step of halving the interval of the path coordinate in the map β' to $t \in [0, \frac{1}{2}]$ in order to arrive at the relative homotopy group will be reversed under the projection p which doubles the interval to $t \in [0, 1]$ again.



Changing the CAR involution

Recall that the CAR pairing of $\mathcal{W}$ is given by the anti-commutator bracket $\{\cdot, \cdot\}$ for fermionic operators. Using the two pseudo-symmetries I and K in the definition of $C_{s+2}(2n)$, we can form the operator

$$u_0 := \frac{1}{\sqrt{2}}(1 - IK). \tag{4.55}$$

This operator is unitary since IK is anti-Hermitian and $(IK)^2 = -1$:

$$u_0 u_0^\dagger = \tfrac{1}{2}(1 - IK)(1 + IK) = \tfrac{1}{2}(1 - (-1)) = 1. \tag{4.56}$$

Using u_0, we define another bracket

$$\widetilde{\{w, w'\}} = \{u_0 w, u_0 w'\}, \tag{4.57}$$

for $w, w' \in \mathcal{W}$.

Since I is a real and K an imaginary pseudo-symmetry, I preserves the bracket $\{\cdot, \cdot\}$, while K reverses its sign. Therefore,

$$\widetilde{\{w, w'\}} = \tfrac{1}{2}\{w - IKw, w' - IKw'\} = \{-IKw, w'\} = \{(IK)^{-1}w, w'\}. \tag{4.58}$$

This implies that if $\{w, w'\} = 0$, then $\widetilde{\{IKw, w'\}} = 0$, so the modified bracket results in a modified involution $\widetilde{\tau}_{s+1} : C_{s+1}(2n) \to C_{s+1}(2n)$ related to the original involution by

$$\widetilde{\tau}_{s+1} = IK \circ \tau_{s+1}. \tag{4.59}$$

Consequently, also the involution τ_{CAR} on operators is modified to

$$\widetilde{\tau}_{\mathrm{CAR}} = IK \circ \tau_{\mathrm{CAR}} \circ (IK)^{-1}, \tag{4.60}$$

and in particular

$$\widetilde{\tau}_{\mathrm{CAR}}(I) = IK\tau_{\mathrm{CAR}}(I)(IK)^{-1} = KIK^{-1} = -I, \tag{4.61}$$
$$\widetilde{\tau}_{\mathrm{CAR}}(K) = IK\tau_{\mathrm{CAR}}(K)(IK)^{-1} = I(-K)I^{-1} = +K. \tag{4.62}$$

Thus, the roles of I and K are reversed under the modified bracket: I becomes an imaginary pseudo-symmetry, while K is turned into a positive one. All remaining pseudo-symmetries commute with the product IK and therefore

$$\widetilde{\tau}_{\mathrm{CAR}}(J_l) = \tau_{\mathrm{CAR}}(J_l) = J_l \tag{4.63}$$

for $l = 1, \ldots, s$. These s pseudo-symmetries define the space $C_s(2n)$ as before and we now have two options of extending the set of pseudo-symmetries by an imaginary one to obtain $C_{s+1}(2n)$: Either we take K with the usual involution τ_{CAR} leading to the fixed point set

$$R_{s,1}(2n) = \{A \in C_s(2n) \mid KA^{\mathrm{c}} = A = \tau_{s+1}(A)\}, \tag{4.64}$$

or we choose I with the modified involution $\widetilde{\tau}_{\mathrm{CAR}}$, which results in a different fixed point set

$$\widetilde{R}_{s,1}(2n) = \{A \in C_s(2n) \mid IA^{\mathrm{c}} = A = \widetilde{\tau}_{s+1}(A)\}. \tag{4.65}$$

These two spaces are in bijection ($R_{s,1}(2n) \simeq \widetilde{R}_{s,1}(2n)$), since they are related by the invertible transformation u_0.

Connection with real Bott periodicity

We recall from eq. (4.21) the definition of the map $\beta : C_{s+2}(2n) \to \bar{\Omega}_K(C_{s+1}(2n))$ adding a momentum-like coordinate:

$$\beta_t(A) = \mathrm{e}^{(t\pi/2)KJ(A)} \cdot E_{+\mathrm{i}}(K).$$

Lemma 4.5. *For $A \in R_{s+1,1}(2n)$ the curve $t \mapsto \beta_t(A)$ lies entirely within $\widetilde{R}_{s,1}(2n)$.*

Proof. By inspecting the definitions (4.65) and (2.137) one sees that

$$R_{s+1,1}(2n) = \widetilde{R}_{s,1}(2n) \cap R_{s+1}(2n). \tag{4.66}$$

Indeed, the two spaces on the right-hand side have the same pseudo-symmetries $J_1, \ldots, J_s$ and I, but the points of the second space are fixed with respect to τ_{s+1}

while the first space is the fixed-point set of $\widetilde{\tau}_{s+1}$. As a consequence, all elements $A \in \widetilde{R}_{s,1}(2n) \cap R_{s+1}(2n)$ are fixed under the map IK:

$$IKA = IK\tau_{s+1}(A) = \widetilde{\tau}_{s+1}(A) = A. \tag{4.67}$$

Since I is a pseudo-symmetry, it follows that

$$KA = KIKA = IA = A^{\mathrm{c}}. \tag{4.68}$$

Therefore the intersection on the right-hand side of Eq. (4.66) does give the space on the left-hand side.

Owing to (4.66) all points A of $R_{s+1,1}(2n)$ lie in both $R_{s+1}(2n)$ and $\widetilde{R}_{s,1}(2n)$. Also, the product $KJ(A)$ commutes with all generators $J_1, \ldots, J_s$ and I. It follows that the one-parameter group of unitary operators $\mathrm{e}^{(t\pi/2)KJ(A)}$ preserves the pseudo-symmetry relations of $\widetilde{R}_{s,1}(2n)$. Moreover, $\mathrm{e}^{(t\pi/2)KJ(A)}$ is real with respect to the $\widetilde{\mathrm{CAR}}$ structure since $\widetilde{\tau}_{\mathrm{CAR}}(K) = +K$ and

$$\widetilde{\tau}_{\mathrm{CAR}}(J(A)) = J(\widetilde{\tau}_{s+1}(A)) = J(A).$$

Hence $\beta_t(A) \in \widetilde{R}_{s,1}(2n)$ for all $t \in [0,1]$ as claimed. □

As a consequence, the map $\beta' : R_s(n) \to (\bar{\Omega}_K C_{s+1}(2n))$ may be reinterpreted as a map

$$\widetilde{\beta} : R_s(n) \to \Omega_K \widetilde{R}_{s,1}(2n) \tag{4.69}$$

by using the modified involution $\widetilde{\tau}_{\mathrm{CAR}}$ on $C_{s+1}(2n)$. After identifying $\widetilde{R}_{s,1}(2n) \simeq R_{s,1}(2n) \simeq R_{s-1}(n)$, this map corresponds to the well known real Bott map [Bot59] and we therefore have the result

Proposition 4.6. *The induced map*

$$\widetilde{\beta}_* : \pi_d(R_s(n)) \to \pi_{d+1}(\widetilde{R}_{s,1}(2n))$$

is an isomorphism for $1 \le d \ll n$.

Thus, the only ingredient remaining is the projection p.

The projection p

For the remainder of this section, we adopt the simplified notation

$$C_{s+1} \equiv C_{s+1}(2n), \tag{4.70}$$
$$R_{s+1} \equiv R_{s+1}(2n), \tag{4.71}$$
$$R_{s+1,1} \equiv R_{s+1,1}(2n), \tag{4.72}$$
$$\widetilde{R}_{s,1} \equiv \widetilde{R}_{s,1}(2n). \tag{4.73}$$

We reiterate the following relations between these spaces:

$$R_{s+1} \subset C_{s+1}, \tag{4.74}$$

$$\widetilde{R}_{s,1} \subset C_{s+1}, \tag{4.75}$$

$$R_{s+1,1} = R_{s+1} \cap \widetilde{R}_{s,1}. \tag{4.76}$$

Recall that all of the pseudo-symmetries $J_1, \ldots, J_s$ and I anti-commute with the operator K, so they map the eigenspace $E_{+\mathrm{i}}(K)$ to its orthogonal complement $E_{-\mathrm{i}}(K)$. Thus, $E_{\mathrm{i}}(K)$ is an element in C_{s+1} and we may realize the latter as an orbit

$$C_{s+1} = U \cdot E_{+\mathrm{i}}(K), \tag{4.77}$$

by the group

$$U := \{u \in \mathrm{U}(\mathcal{W}) \mid u = J_1 u J_1^{-1} = \ldots = J_s u J_s^{-1} = I u I^{-1}\} \tag{4.78}$$

with stabilizer

$$U_K := \{u \in U \mid u = K u K^{-1}\}. \tag{4.79}$$

Note that we met U in Section 2.6 as $U = G_{s+1}^{\mathbb{C}}(2n)$. In that section, we pointed out that the spaces C_{s+1} can all be realized by orbits for even s and as unions of orbits for odd s. We focus on the case with even s and thus a single orbit, since our goal will be to apply the machinery developed here to the case $s \in \{2, 6\}$. For odd s, some details would have to be changed, including the replacement of C_{s+1} by one of its connected components.

Since all $u \in U$ commute with I, the stabilizer U_K can be realized alternatively as the fixed point set of the Cartan involution

$$\theta(u) := IKu(IK)^{-1}, \tag{4.80}$$

rendering C_{s+1} a symmetric space. Indeed, for all odd s it is a unitary group, see Table 2.1. The involution θ has the useful property that it relates the two involutions $\widetilde{\tau}_{\mathrm{CAR}}$ and τ_{CAR} on C_{s+1} by the formula

$$\widetilde{\tau}_{\mathrm{CAR}} = \theta \circ \tau_{\mathrm{CAR}}, \tag{4.81}$$

see eq. (4.60). In fact, all three involutions commute with one another.

The two subgroups of U fixed by the involutions $\widetilde{\tau}_{\mathrm{CAR}}$ and τ_{CAR} will play an important role in this section:

$$G := \mathrm{Fix}(\tau_{\mathrm{CAR}}) \subset U, \tag{4.82}$$

$$\widetilde{G} := \mathrm{Fix}(\widetilde{\tau}_{\mathrm{CAR}}) \subset U. \tag{4.83}$$

Similar to the realization of C_{s+1} as an orbit of U, we can realize R_{s+1} and $\widetilde{R}_{s,1}$ as orbits of G and $\widetilde{G}$ respectively:

$$R_{s+1} = G \cdot A_*, \tag{4.84}$$
$$\widetilde{R}_{s,1} = \widetilde{G} \cdot E_{+\mathrm{i}}(K) = \widetilde{G}/H, \tag{4.85}$$

where we have chosen a base point $A_* \in R_{s+1} \cap \widetilde{R}_{s,1} = R_{s+1,1}$ and defined the stabilizer subgroup

$$H := U_K \cap \widetilde{G} = U_K \cap G = G \cap \widetilde{G}. \tag{4.86}$$

The situation is illustrated in Figure 4.2: The groups G and $\widetilde{G}$ generate R_{s+1} (blue) and $\widetilde{R}_{s,1}$ (green), which intersect in the space $R_{s+1,1}$ (red circle) containing the base point A_*.

The different realizations of H follow from eq. (4.81). Since the Cartan involution θ restricts to $\widetilde{G}$ and has H as its fixed points within $\widetilde{G}$, the space $\widetilde{R}_{s,1}$ is also a symmetric space. A construction which will be used in the proof of the next lemma is the Cartan embedding $\widetilde{G}/H \equiv U(\widetilde{G}/H) \subset \widetilde{G}$ defined by the bijection

$$\widetilde{G}/H \to U(\widetilde{G}/H) \tag{4.87}$$
$$\tilde{g}H \mapsto \tilde{g}\theta(\tilde{g})^{-1}. \tag{4.88}$$

We omit an analogous discussion for R_{s+1} since it will not be required for the following.

Lemma 4.7. *Suppose that the principal bundle $U \to U/U_K = C_{s+1}$ admits a global section, i.e. a map $\sigma : C_{s+1} \to U$ with $\sigma(A) \cdot E_{+\mathrm{i}}(K) = A$ for all $A \in C_{s+1}$. Suppose further that*

(i) for all $A \in C_{s+1}$, the group element $\sigma(A)$ commutes with its images under θ and $\widetilde{\tau}_{\mathrm{CAR}}$, and

(ii) for all $A \in \widetilde{R}_{s,1}$ the relation $\tau_{\mathrm{CAR}}(\sigma(A)) = \sigma(A)^{-1}$ holds.

Then the mapping $p : C_{s+1} \to C_{s+1}$ defined by

$$p(A) := \tau_{\mathrm{CAR}}(\sigma(A))^{-1} \cdot A \tag{4.89}$$

has the following properties:

1. *p is onto $\widetilde{R}_{s,1}$.*
2. *$p(\beta_t(A)) = \beta_{2t}(A)$ for all $A \in R_{s+1,1}$.*
3. *$p(R_{s+1}) = E_{-\mathrm{i}}(K)$.*

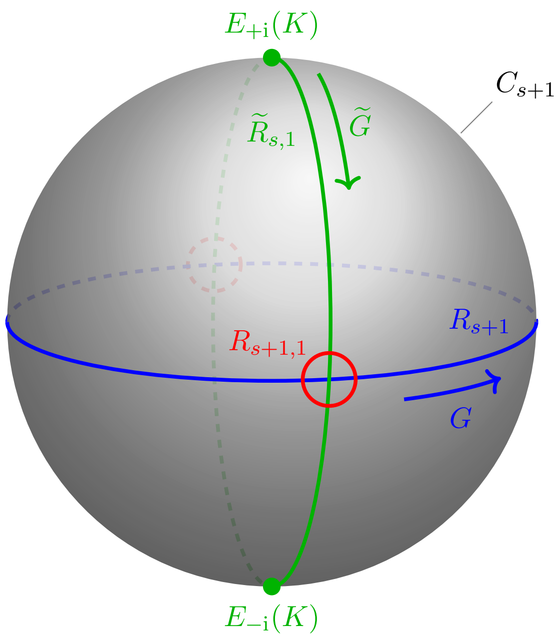


Figure 4.2.: Schematic visualization of the setting in Lemma 4.7: The orbits under the groups $\widetilde{G}$ and G are the spaces $\widetilde{R}_{s,1}$ and R_{s+1} (green and blue) respectively. Their intersection (red) is the space $R_{s+1,1}$. The projection p squares ("doubles") in the green direction (property 1 and 2) and thereby sends the blue part to the south pole $E_{-\mathrm{i}}(K)$ (property 3).

Proof. Before proving that p is onto $\widetilde{R}_{s,1}$, we show that the image of p is contained in $\widetilde{R}_{s,1}$. For this purpose, we write $p(A) = \Sigma(A) \cdot E_{+\mathrm{i}}(K)$ with $\Sigma(A) = \tau_{\mathrm{CAR}}(\sigma(A))^{-1}\sigma(A)$ and send $p(A)$ to its image under the Cartan embedding:

$$p(A) \mapsto \Sigma(A)\theta(\Sigma(A))^{-1} \equiv \tilde{g}.$$

Using the fact that $\tau_{\mathrm{CAR}}(\Sigma(A)) = \Sigma(A)^{-1}$ as well as assumption (i), applying $\widetilde{\tau}_{\mathrm{CAR}}$ to $\tilde{g}$ evaluates to

$$\begin{aligned}
\widetilde{\tau}_{\mathrm{CAR}}(\tilde{g}) &= \widetilde{\tau}_{\mathrm{CAR}}\left(\Sigma(A)\theta(\Sigma(A))^{-1}\right) \\
&= (\theta \circ \tau_{\mathrm{CAR}})(\Sigma(A))\tau_{\mathrm{CAR}}(\Sigma(A))^{-1} \\
&= \theta(\Sigma(A))^{-1}\Sigma(A) \\
&\overset{\text{(i)}}{=} \Sigma(A)\theta(\Sigma(A))^{-1} \\
&= \tilde{g}.
\end{aligned}$$

This implies that $\theta(\tilde{g}) = \tau_{\mathrm{CAR}}(\tilde{g}) = \tilde{g}^{-1}$ and therefore $\tilde{g}$ lies in the Cartan embedding $\mathrm{U}(\widetilde{G}/H)$. This in turn implies that $p(A) \in \widetilde{G} \cdot E_{+\mathrm{i}}(K)$ and therefore p maps into $\widetilde{R}_{s,1}$. To see that it is in fact surjective, let $A = \sigma(A) \cdot E_{+\mathrm{i}}(K) \in \widetilde{R}_{s,1} \subset C_{s+1}$. By assumption (ii), the expression for $p(A)$ in this case takes the form

$$p(A) = \tau_{\mathrm{CAR}}(\sigma(A))^{-1} \cdot A = \sigma(A)^2 \cdot E_{+\mathrm{i}}(K). \tag{4.90}$$

Thus $p : \widetilde{R}_{s,1} \to \widetilde{R}_{s,1}$ is the operation of squaring (or doubling the geodesic distance) from the point $E_{+\mathrm{i}}(K)$: in normal coordinates by the exponential mapping with respect to $E_{+\mathrm{i}}(K)$ it is the map

$$p(A) = p(\exp(X) \cdot E_{+\mathrm{i}}(K)) = \exp(2X) \cdot E_{+\mathrm{i}}(K). \tag{4.91}$$

Since the squaring map is surjective, it follows that $p : C_{s+1} \to \widetilde{R}_{s,1}$ is onto. In Figure 4.2, the property of squaring can be visualized as “stretching” by a factor of two into the green direction.

Now for $A \in R_{s+1,1} \subset \widetilde{R}_{s,1}$ we recall that $\beta_t(A) = \mathrm{e}^{(t\pi/2)KJ(A)} \cdot E_{+\mathrm{i}}(K)$. The second stated property is then an immediate consequence of the squaring property in eq. (4.90):

$$p(\beta_t(A)) = \left(\mathrm{e}^{(t\pi/2)KJ(A)}\right)^2 \cdot E_{+\mathrm{i}}(K) = \beta_{2t}(A).$$

Turning to the third property, we observe that σ as a section of $U \to U/U_K$ satisfies, for all $u \in U$,

$$\sigma(u \cdot A) = u\sigma(A)h(u, A), \tag{4.92}$$

for some $h(u,A)$ taking values in the isotropy group U_K of $E_{+\mathrm{i}}(K)$. By specializing this to $A = g \cdot A_* \in R_{s+1}$ for $u = g \in G$ and using $g = \tau_{\mathrm{CAR}}(g)$ we obtain

$$\begin{aligned} p(A) &= \tau_{\mathrm{CAR}}(\sigma(A))^{-1}\sigma(A) \cdot E_{+\mathrm{i}}(K) && (4.93)\\ &= \tau_{\mathrm{CAR}}(h)^{-1}\tau_{\mathrm{CAR}}(\sigma(A_*))^{-1}\sigma(A_*) \cdot E_{+\mathrm{i}}(K) && (4.94)\\ &= \tau_{\mathrm{CAR}}(h)^{-1}p(A_*) && (4.95)\\ &= \tau_{\mathrm{CAR}}(h)^{-1}E_{-\mathrm{i}}(K) && (4.96)\\ &= E_{-\mathrm{i}}(K). && (4.97) \end{aligned}$$

In the second to last line we have used the second property of p in the form $p(A_*) = p(\beta_{1/2}(A_*)) = \beta_1(A_*) = E_{-\mathrm{i}}(K)$ and in the last line we have used the fact that $\tau_{\mathrm{CAR}}(h)^{-1} \in U_K$ since the subgroup U_K of θ-fixed points is stable under τ_{CAR} (as θ and τ_{CAR} commute). In the schematic picture presented in Figure 4.2, this property corresponds to p sending the entirety of the blue subset $R_{s+1} \subset C_{s+1}$ to the south pole $E_{-\mathrm{i}}(K)$. □

Remark 4.8. The section σ with the stated properties, whose existence is a necessary condition for the statement of Lemma 4.7 to hold, exists if and only if $s \in \{2, 6\}$.

Proposition 4.9. *The map β'_* of eq.* (4.49) *is an isomorphism for $s \in \{2, 6\}$ and $1 \le d \ll n$.*

Proof. Let $s = 2$ for definiteness. Then $U = \mathrm{U}_n \times \mathrm{U}_n$ and the Cartan involution θ has the effect of exchanging the two factors of $U = \mathrm{U}_n \times \mathrm{U}_n$, so the subgroup $\mathrm{Fix}(\theta) = U_K$ is the diagonal subgroup $\mathrm{U}_n \subset \mathrm{U}_n \times \mathrm{U}_n$. The involution τ_{CAR} acts by τ_{Sp} in each factor, where we define $\tau_{\mathrm{Sp}} : \mathrm{U}_n \to \mathrm{U}_n$ to be the involution with $\mathrm{Fix}(\tau_{\mathrm{Sp}}) = \mathrm{Sp}_n$. Hence $G = \mathrm{Fix}(\tau_{\mathrm{CAR}}) = \mathrm{Sp}_n \times \mathrm{Sp}_n$ and $\widetilde{G} = \mathrm{Fix}(\widetilde{\tau}_{\mathrm{CAR}}) = \mathrm{U}_n$, with intersection $H = G \cap \widetilde{G} = \mathrm{Sp}_n$. The orbit of $\widetilde{G}$ on $E_{+\mathrm{i}}(K)$ is $\widetilde{R}_{2,1} = \widetilde{G}/H = \mathrm{U}_n/\mathrm{Sp}_n$.

The principal bundle $U \to U/U_K = C_{s+1}$ is the projection $\mathrm{U}_n \times \mathrm{U}_n \to \mathrm{U}_n \times \mathrm{U}_n/\mathrm{U}_n$ and is trivial. We may take σ to be of the form $\sigma(A) = (u, 1)$, with the second factor being the identity. The involution τ_{CAR} does not mix the two factors, implying that the second factor of $\tau_{\mathrm{CAR}}(\sigma(A))$ is still the identity. Because the Cartan involution θ exchanges factors and thus moves the identity map to the first factor, $\theta(\sigma(A))$ commutes with $\sigma(A)$ and $\tau_{\mathrm{CAR}}(\sigma(A))$, as is required in order for the first condition of Lemma 4.7 to be met. Moreover, an element $A \in \widetilde{R}_{2,1}$ lifts to $\sigma(A) = (u\tau_{\mathrm{Sp}}(u)^{-1}, \mathrm{Id})$ for some $u \in \mathrm{U}_n$. In this case one has $\tau_{\mathrm{CAR}}(\sigma(A)) = (\tau_{\mathrm{Sp}}(u)u^{-1}, 1) = \sigma(A)^{-1}$, which means that also the second condition of Lemma 4.7 is satisfied. The case of $s = 6$ proceeds along the same lines with the substitutions $n \to n/4$ and $\mathrm{Sp} \to \mathrm{O}$.

Thus Lemma 4.7 applies, and from the properties stated there it follows that for $s \in \{2, 6\}$ we have a short exact sequence of spaces

$$R_{s+1} \hookrightarrow C_{s+1} \xrightarrow{p} \widetilde{R}_{s,1}, \qquad (4.98)$$

where the first map is simply the inclusion of $R_{s+1} = p^{-1}(E_{-\mathrm{i}}(K))$ into C_{s+1}. The second map, $p : C_{s+1} \to \widetilde{R}_{s,1}$, has the so-called homotopy lifting property: for any mapping $f : X \times [0,1] \to \widetilde{R}_{s,1}$ there exists a mapping $F := \sigma \circ f : X \times [0,1] \to C_{s+1}$, which is a lift of f in the sense that $p \circ \widetilde{f} = f$. This means that the short exact sequence (4.98) is a fibration.

It is a standard result of homotopy theory (see [Hat02, Thm. 4.41,p. 376]) that the mapping p_* induced by the projection p of a fibration induces isomorphisms

$$p_* : \pi_{d+1}(C_{s+1}, R_{s+1}, A_*) \to \pi_{d+1}(\widetilde{R}_{s,1}, E_{-\mathrm{i}}(K))$$

for all d (for clarity, we make the base points explicit here). By composing p_* with the mapping β'_* of eq. (4.49)), we arrive at the map

$$p_* \circ \beta'_* : \pi_d(R_s(n), A_*) \to \pi_{d+1}(\widetilde{R}_{s,1}(2n), E_{-\mathrm{i}}(K)). \tag{4.99}$$

By the second property of p stated in Lemma 4.7, we have

$$p_* \circ \beta'_* = \widetilde{\beta}_* \tag{4.100}$$

and since, in addition to p_*, the induced map $\widetilde{\beta}_*$ is an isomorphism by Proposition 4.6, so is β'_* for all $1 \leq d \ll n$. □

Remark 4.10. To draw the same conclusion for all real classes s, one would need eight fibrations of the following type:

$$\begin{aligned}
\mathrm{U}/\mathrm{Sp} &\hookrightarrow (\mathrm{U}\times\mathrm{U})/\mathrm{U} \longrightarrow (\mathrm{O}\times\mathrm{O})/\mathrm{O},\\
\mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp}) &\hookrightarrow \mathrm{U}/(\mathrm{U}\times\mathrm{U}) \longrightarrow \mathrm{O}/\mathrm{U},\\
(\mathrm{Sp}\times\mathrm{Sp})/\mathrm{Sp} &\hookrightarrow (\mathrm{U}\times\mathrm{U})/\mathrm{U} \longrightarrow \mathrm{U}/\mathrm{Sp},\\
\mathrm{Sp}/\mathrm{U} &\hookrightarrow \mathrm{U}/(\mathrm{U}\times\mathrm{U}) \longrightarrow \mathrm{Sp}/(\mathrm{Sp}\times\mathrm{Sp}),\\
\mathrm{U}/\mathrm{O} &\hookrightarrow (\mathrm{U}\times\mathrm{U})/\mathrm{U} \longrightarrow (\mathrm{Sp}\times\mathrm{Sp})/\mathrm{Sp},\\
\mathrm{O}/(\mathrm{O}\times\mathrm{O}) &\hookrightarrow \mathrm{U}/(\mathrm{U}\times\mathrm{U}) \longrightarrow \mathrm{Sp}/\mathrm{U},\\
(\mathrm{O}\times\mathrm{O})/\mathrm{O} &\hookrightarrow (\mathrm{U}\times\mathrm{U})/\mathrm{U} \longrightarrow \mathrm{U}/\mathrm{O},\\
\mathrm{O}/\mathrm{U} &\hookrightarrow \mathrm{U}/(\mathrm{U}\times\mathrm{U}) \longrightarrow \mathrm{O}/(\mathrm{O}\times\mathrm{O}).
\end{aligned}$$

The third ($s = 2$) and seventh ($s = 6$) of these are the fibrations discussed in the proof of Proposition 4.9. While the others are available [Gif96] in the K-theory limit of infinitely many bands ($n \to \infty$), they do not seem to exist at finite n.

We are now in a position to use the $\mathbb{Z}_2$-Whitehead Theorem 3.21 in order to prove the following:

Proposition 4.11. *Let X be a path-connected $\mathbb{Z}_2$-CW complex, and let $s = 2$ or $s = 6$. Then the map* (4.46) *between homotopy classes of $\mathbb{Z}_2$-equivariant maps,*

$$\beta_*^{\mathbb{Z}_2} : [X, C_s(n)]_*^{\mathbb{Z}_2} \to [\bar{\mathcal{S}}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2},$$

which increases the symmetry index and the momentum-space dimension of a topological phase by one, is bijective for $\dim X \ll n$.

Proof. After the identification $[\bar{\mathcal{S}}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2} = [X, \bar{\Omega}_K C_{s+1}(2n)]_*^{\mathbb{Z}_2}$ given by Proposition 3.26, our statement is an immediate consequence of the $\mathbb{Z}_2$-Whitehead Theorem (Theorem 3.21). Recall that in order for that theorem to apply in the case of a $\mathbb{Z}_2$-equivariant map $\beta : Y \to Z$, one has to show that $\beta^H : Y^H \to Z^H$ is highly connected for all subgroups H of $\mathbb{Z}_2$. We have done so (with the identifications $Y = C_s(n)$ and $Z = \bar{\Omega}_K C_{s+1}(2n)$) for $H = \{e\}$ (by Proposition 4.4) and $H = \mathbb{Z}_2$ (for $s = 2$ and $s = 6$ by Proposition 4.9). In both cases, the fact that (for $s = 2, 6$) there is no bijection between $\pi_0(C_s(n))$ and $\pi_0(\bar{\Omega}_K C_{s+1}(2n))$ (resp. between $\pi_0(R_s(n))$ and $\pi_0\big((\bar{\Omega}_K C_{s+1}(2n))^{\mathbb{Z}_2}\big)$) is remedied by the assumption that X is path-connected. Indeed, under that condition the image of the base-point preserving map β (resp. $\beta^{\mathbb{Z}_2}$) lies entirely within the connected component of $\Omega_K C_{s+1}(2n)$ (resp. $(\Omega_K C_{s+1}(2n))^{\mathbb{Z}_2}$) containing the base point and we may simply restrict to that single connected component. With this detail in mind, the $\mathbb{Z}_2$-Whitehead Theorem indeed applies to give the stated result. □

4.4. Classification for all s

In this section we extend the statement of Proposition 4.11 to all real symmetry classes s. In order to do so, we construct a mapping which increases the position-like dimension of the configuration space X by one (rather than the momentum-like dimension as before) while decreasing (as opposed to increasing) the symmetry index. As a corollary, choosing the configuration space $X = \mathrm{S}^{d_x,d_k}$ as introduced in Section 3.2 will allow us to recover the generalized Periodic Table for topological phases (Table 4.1) including the presence of a single defect with codimension $d_x + 1$ as put forward in [TK10]. The connection to the physical configuration spaces given by the Brillouin zone T^{d_k} without defect and the product $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ in the presence of a defect will be made in Chapter 7: There we prove that, in the stable regime, the sets of topological phases $[\mathrm{T}^{d_k}, C_s(n)]_*^{\mathbb{Z}_2}$ and $[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s(n)]_*^{\mathbb{Z}_2}$ decompose into a product with factors exclusively of the form $[\mathrm{S}^{d_x,r}, C_s(n)]_*^{\mathbb{Z}_2}$, $0 \le r \le d_k$, all of which are determined here.

4.4.1. Additional position-like dimensions

Recall from Definition 4.13 that the map β is given by

$$\beta_t(A) = \mathrm{e}^{(t\pi/2)KJ(A)} \cdot A.$$

In the following, we use the same definition, for $A \in C_s(n)$ (rather than the previous $A \in C_{s+2}(2n)$) and with $\tau_{\text{CAR}}(K) = K$ (rather than $\tau_{\text{CAR}}(K) = -K$). Thus, all pseudo-symmetries $J_1, \ldots, J_{s-1}, K$ are assumed to be of the real type. The change of the formerly imaginary generator K to a real one has an important consequence: the second property listed in Lemma 4.1 changes from $\beta_t(A)^\perp = \beta_{1-t}(A^\perp)$ to $\beta_t(A)^\perp = \beta_t(A^\perp)$. Hence, the additional coordinate t is now position-like rather than momentum-like. This means that the modified curve $t \mapsto \beta_t(A)$ agrees with the original Bott map [Bot59, Mil63]: all $\mathbb{Z}_2$-fixed points $A \in R_s(n) \subset C_s(n)$ are now mapped to $\mathbb{Z}_2$-fixed points $\beta_t(A) \in R_{s-1}(n) \subset C_{s-1}(n)$ for all t. A treatise on the relationships between complex and real Bott periodicity can be found in [MQ12]. The alternative use of β described here leads to the following result:

Theorem 4.12. *For a path-connected $\mathbb{Z}_2$-CW complex X with $\dim X \ll n$, the original Bott map β induces a bijection*

$$[X, C_s(n)]_*^{\mathbb{Z}_2} \xrightarrow{\sim} [\mathcal{S}X, C_{s-1}(n)]_*^{\mathbb{Z}_2}.$$

Proof. We use Proposition 3.26 to identify $[\mathcal{S}X, C_{s-1}(n)]_*^{\mathbb{Z}_2} = [X, \Omega_K C_{s-1}(n)]_*^{\mathbb{Z}_2}$ and in order to be able to apply the $\mathbb{Z}_2$-Whitehead Theorem (Theorem 3.21). For the trivial subgroup $\{e\} \subset \mathbb{Z}_2$, the map $\beta : C_s(n) \to \Omega_K C_{s-1}(n)$ is the complex Bott map and therefore highly connected. Similarly, for the full group $\mathbb{Z}_2$, the map β restricts to the real Bott map $R_s(n) \to \Omega_K R_{s-1}(n)$, which is also highly connected. The obstruction that there may be a mismatch between π_0 for $C_s(n)$ resp. $R_s(n)$ and $\Omega_K C_{s-1}(n)$ resp. $\Omega_K R_{s-1}(n)$, is avoided by the reasoning described in the proof of Proposition 4.11. □

By specializing the result above to the case of $X = \mathrm{S}^{d_x,d_k}$ (which is path-connected unless $d_x = d_k = 0$) and using $\mathcal{S}X = \mathcal{S}(\mathrm{S}^{d_x,d_k}) = \mathrm{S}^{d_x+1,d_k}$ we immediately get the following:

Corollary 4.13. *There exists a bijection*

$$[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} \xrightarrow{\sim} [\mathrm{S}^{d_x+1,d_k}, C_{s-1}(n)]_*^{\mathbb{Z}_2}$$

for $1 \leq d_x + d_k \ll n$.

4.4.2. Additional momentum-like dimensions

We now state and prove for all real symmetry classes s an analog of Theorem 4.12 for an increase in the momentum-like dimension:

Theorem 4.14. *For a path-connected $\mathbb{Z}_2$-CW complex X with $\dim X \ll n$ there is, for any real symmetry class s, a bijection*

$$[X, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\bar{\mathcal{S}}X, C_{s+1}(2n)]_*^{\mathbb{Z}_2}.$$

Proof. The idea of the proof is to first apply Theorem 4.12 repeatedly in order to adjust the symmetry index s to be either 2 or 6 (for concreteness, we settle on the arbitrary choice of 2 here), then use the statement of Proposition 4.11 to increase the momentum-like dimension of X by one, and finally go to the symmetry index $s+1$ by retracing the initial steps.

To spell out the details, let $s = 2 + r$ with $r \geq 0$ (the cases $s = 0$ and $s = 1$ are included as $s = 8$ and $s = 9$ respectively by making use of the eightfold periodicity $C_s(n) = C_{s+8}(n/16)$ and $R_s(n) = R_{s+8}(n/16)$). Then Theorem 4.12 implies that there is a bijection

$$[X, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\mathcal{S}^r X, C_2(n)]_*^{\mathbb{Z}_2},$$

where $\mathcal{S}^r X$ is the r-fold suspension of X. Here we made use of the fact that if X is path-connected, then so is its suspension. We next apply Proposition 4.11 to obtain a bijection

$$[\mathcal{S}^r X, C_2(n)]_*^{\mathbb{Z}_2} \simeq [\bar{\mathcal{S}}\mathcal{S}^r X, C_3(2n)]_*^{\mathbb{Z}_2}.$$

Finally, we observe that $\bar{\mathcal{S}}\mathcal{S}^r X = \mathcal{S}^r \bar{\mathcal{S}} X$ and carry out r applications of Theorem 4.12 in reverse:

$$[\mathcal{S}^r \bar{\mathcal{S}} X, C_3(2n)]_*^{\mathbb{Z}_2} \simeq [\bar{\mathcal{S}} X, C_{s+1}(2n)]_*^{\mathbb{Z}_2},$$

which completes the proof. □

Specializing once more to $X = \mathrm{S}^{d_x, d_k}$ we have

Corollary 4.15. *For $1 \leq d_x + d_k \ll n$, there is a bijection*

$$[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} \simeq [\mathrm{S}^{d_x,d_k+1}, C_{s+1}(2n)]_*^{\mathbb{Z}_2}.$$

Proof. Although this result follows directly from the more general one in Theorem 4.14, it may be instructive to repeat the proof in order to show our chain of reasoning for a special case of importance in physics:

$$\begin{aligned}
[\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} &\simeq [\mathrm{S}^{d_x+s-2,d_k}, C_2(n)]_*^{\mathbb{Z}_2} \\
&\simeq [\mathrm{S}^{d_x+s-2,d_k+1}, C_3(2n)]_*^{\mathbb{Z}_2} \\
&\simeq [\mathrm{S}^{d_x,d_k+1}, C_{s+1}(2n)]_*^{\mathbb{Z}_2}.
\end{aligned}$$

□

From the combination of the Corollaries 4.13 and 4.15, the entries of Table 4.1 are determined by just specifying one column of entries for variable symmetry index s but fixed values for the dimensions d_x and d_k, subject to $d_x + d_k \geq 1$. For example, one may take $(d_x, d_k) = (1, 0)$, in which case $[\mathrm{S}^{1,0}, C_s(n)]_*^{\mathbb{Z}_2}$ is none other than the well-known fundamental group $\pi_1(R_s(n))$ for the real symmetry classes and $\pi_1(C_s(n))$ for the complex ones.

5. Beyond the Periodic Table

In stating our Theorems 4.12 and 4.14, we simply posed the qualitative condition $d = \dim X \ll n$, leaving their range of validity unspecified. In this chapter, we fill this quantitative void and formulate precise conditions on d (as a function of n) in order for the theorems to apply. This sets up the investigation of changes to the homotopy classification displayed in Table 4.1 to be carried out in Chapter 6.

5.1. Connectivity of inclusions

In the definition of the space $C_s(n)$ with involution τ_s fixing the subspace $R_s(n)$, the dimension n takes values in $m_s\mathbb{N}$ for a minimal integer $m_s \geq 1$ which depends on the symmetry class s. This restriction $n \in m_s\mathbb{N}$ stems from the requirement that $\mathcal{W} = \mathbb{C}^{2n}$ must carry a representation of the Clifford algebra generated by the pseudo-symmetries $J_1, \ldots, J_s$. The numbers m_s can be obtained by choosing the minimal parameters in Table 2.1 and are related to the ones found in [ABS64, Table 2] and [SCR11, Table V]. The result is shown in the following list, which can be continued beyond $s = 8$ by the relation $m_{s+8} = m_s/16$:

s	0	1	2	3	4	5	6	7	8
m_s	1	2	2	4	4	4	4	8	16

Let the Clifford generators in the definition of $C_s(n)$ be denoted by J_l and those of $C_s(m_s)$ by J'_l ($l = 1, \ldots, s$). For any symmetry class s, let a fixed element $A_0 \in R_s(m_s) \subset C_s(m_s)$ be given. We then have a natural inclusion

$$i_s : C_s(n) \hookrightarrow C_s(n + m_s), \quad A \mapsto A \oplus A_0, \tag{5.1}$$

where $C_s(n + m_s)$ is defined with Clifford generators $J_l \oplus J'_l$ (for $l = 1, \ldots, s$). The map i_s has the property of being equivariant with respect to the $\mathbb{Z}_2$-action on its image and domain:

$$i_s(A)^\perp = A^\perp \oplus A_0^\perp = A^\perp \oplus A_0 = i_s(A^\perp). \tag{5.2}$$

In particular, its restriction $i_s^{\mathbb{Z}_2}$ to the fixed point set $C_s(n)^{\mathbb{Z}_2} = R_s(n)$ has image in $C_s(n + m_s)^{\mathbb{Z}_2} = R_s(n + m_s)$.

The goal of this chapter is to prove the following theorem:

Theorem 5.1. *Given a path-connected $\mathbb{Z}_2$-CW complex X and a number (of bands) $n = m_s r$ for some integer $r \in \mathbb{N}$, the induced map*

$$(i_s)_* : [X, C_s(n)]_*^{\mathbb{Z}_2} \to [X, C_s(n + m_s)]_*^{\mathbb{Z}_2}$$

is bijective if $\dim X < d_1$ and $\dim X^{\mathbb{Z}_2} < d_2$, and remains surjective under the weakened conditions $\dim X \leq d_1$ and $\dim X^{\mathbb{Z}_2} \leq d_2$. The values of d_1 and d_2 are given in the following table (the complex symmetry classes are included by replacing the $\mathbb{Z}_2$-actions on $X, C_s(n)$ and $C_s(n + m_s)$ by the trivial one and neglecting the conditions on $X^{\mathbb{Z}_2}$):

s	$C_s(m_s r)_0$	–	d_1	*Case*
even	$\mathrm{U}_{p+q}/\mathrm{U}_p \times \mathrm{U}_q$	–	$\min(2p+1, 2q+1)$	*(iv)*
odd	U_r	–	$2r$	*(i)*

s	$C_s(m_s r)_0$	$C_s(m_s r)_0^{\mathbb{Z}_2}$	d_2	*Case*
0	$\mathrm{U}_{2r}/\mathrm{U}_r \times \mathrm{U}_r$	$\mathrm{O}_{2r}/\mathrm{U}_r$	$2r - 1$	*(ii)*
1	U_{2r}	$\mathrm{U}_{2r}/\mathrm{Sp}_{2r}$	$4r$	*(ii)*
2	$\mathrm{U}_{2p+2q}/\mathrm{U}_{2p} \times \mathrm{U}_{2q}$	$\mathrm{Sp}_{2p+2q}/\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q}$	$\min(4p+3, 4q+3)$	*(iv)*
3	U_{2r}	Sp_{2r}	$4r + 2$	*(i)*
4	$\mathrm{U}_{2r}/\mathrm{U}_r \times \mathrm{U}_r$	$\mathrm{Sp}_{2r}/\mathrm{U}_r$	$2r + 1$	*(iii)*
5	U_r	$\mathrm{U}_r/\mathrm{O}_r$	r	*(iii)*
6	$\mathrm{U}_{p+q}/\mathrm{U}_p \times \mathrm{U}_q$	$\mathrm{O}_{p+q}/\mathrm{O}_p \times \mathrm{O}_q$	$\min(p, q)$	*(iv)*
7	U_r	O_r	$r - 1$	*(i)*

For the complex symmetry classes with even s (class A) as well as the real classes $s = 2$ (class AII) and $s = 6$ (class AI), the single parameter r is refined to $r = p + q$ in order to accommodate the possibility of the base point lying in different connected components of $C_s(m_s r)$.

Remark 5.2. The choice of p and q in the refinement $r = p + q$ amounts to choosing a chemical potential and thus declaring the number of valence bands to be p and the number of conduction bands to be q (or vice versa).

Proof. Since X is path-connected and all maps are base-point preserving, we may replace $C_s(n) = C_s(m_s r)$ by its connected component (denoted by $C_s(m_s r)_0$ in the table) containing the base point $A_* \in R_s(n) \subset C_s(n)$. Then, by applying the $\mathbb{Z}_2$-Whitehead Theorem, we obtain the desired statements provided that i_s is d_1-connected and $i_s^{\mathbb{Z}_2}$ is d_2-connected, with numbers d_1 and d_2 that are yet to be determined. The latter is done in the remainder of the proof, where we distinguish between four cases.

Case (i)

We start with the three rows attributed to case (i) in the tables. These enjoy the property of having Lie groups for their target spaces and we can make use of the following three fiber bundles:

$$\begin{aligned} \mathrm{O}_r \hookrightarrow \mathrm{O}_{r+1} \to \mathrm{O}_{r+1}/\mathrm{O}_r &= \mathrm{S}^r, \\ \mathrm{U}_r \hookrightarrow \mathrm{U}_{r+1} \to \mathrm{U}_{r+1}/\mathrm{U}_r &= \mathrm{S}^{2r+1}, \\ \mathrm{Sp}_{2r} \hookrightarrow \mathrm{Sp}_{2r+2} \to \mathrm{Sp}_{2r+2}/\mathrm{Sp}_{2r} &= \mathrm{S}^{4r+3}, \end{aligned}$$

each of which gives rise to a long exact sequence in homotopy. By using $\pi_l(\mathrm{S}^d) = 0$ for $l < d$, we infer from these sequences the following values of d_1 and d_2:

$$\begin{aligned} d_2 &= r-1 && \text{for} \quad \mathrm{O}_r \hookrightarrow \mathrm{O}_{r+1}, \\ d_1 &= 2r && \text{for} \quad \mathrm{U}_r \hookrightarrow \mathrm{U}_{r+1}, \\ d_2 &= 4r+2 && \text{for} \quad \mathrm{Sp}_{2r} \hookrightarrow \mathrm{Sp}_{2r+2}. \end{aligned}$$

For the next two cases, (ii) and (iii), the target spaces are quotients G_r/H_r with G_r and H_r being either an orthogonal, a unitary or a symplectic group. The strategy in the following will be to apply the result of case (i) to the exact sequence associated to the fiber bundle

$$H_r \hookrightarrow G_r \to G_r/H_r.$$

We distinguish between case (ii) where the inclusion $G_r \hookrightarrow G_{r+1}$ is at most as connected as the inclusion $H_r \hookrightarrow H_{r+1}$, and case (iii) where it is more connected.

Case (ii)

Let $G_r \hookrightarrow G_{r+1}$ be m-connected, where m is less than or equal to the connectivity of $H_r \hookrightarrow H_{r+1}$. Then for all $j \in \mathbb{N}$ with $1 \leq j \leq m-1$ there is the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_j(H_r) & \longrightarrow & \pi_j(G_r) & \longrightarrow & \pi_j(G_r/H_r) & \longrightarrow & \pi_{j-1}(H_r) & \longrightarrow & \pi_{j-1}(G_r) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow (i_s^{\mathbb{Z}_2})_* & & \downarrow \simeq & & \downarrow \simeq \\ \pi_j(H_{r+1}) & \to & \pi_j(G_{r+1}) & \to & \pi_j(G_{r+1}/H_{r+1}) & \to & \pi_{j-1}(H_{r+1}) & \to & \pi_{j-1}(G_{r+1}) \end{array}$$

The Five-Lemma (for $j \geq 2$) and the Special Five-Lemma (for $j = 1$) of Appendix A.2 imply that $(i_s^{\mathbb{Z}_2})_*$ is an isomorphism for all j with $1 \leq j \leq m-1$. The map $(i_s^{\mathbb{Z}_2})_* : \pi_0(G_r/H_r) \to \pi_0(G_{r+1}/H_{r+1})$ needs to be investigated separately. This task is facilitated by the fact that domain or codomain only contain more than one element for $G_r/H_r = \mathrm{O}_{2r}/\mathrm{U}_r$. In that case, $\pi_0(\mathrm{O}_{2r}/\mathrm{U}_r) = \mathbb{Z}_2 = \pi_0(\mathrm{O}_{2r+2}/\mathrm{U}_{r+1})$. In the realization of $\mathrm{O}_{2r}/\mathrm{U}_\mathrm{r}$ as an orbit, all elements $A \in \mathrm{O}_{2r}/\mathrm{U}_r$ may be written $A = gA_*$ for a fixed $A_* \in \mathrm{O}_{2r}/\mathrm{U}_r$ and $g \in \mathrm{O}_{2r}$. The two connected components are distinguished by $\det(g) = \pm 1$ and we compute

$$\begin{aligned} i^{\mathbb{Z}_2}(A) &= i^{\mathbb{Z}_2}(gA_*) \\ &= gA_* \oplus A_0 \\ &= (g \oplus \mathrm{Id})(A_* \oplus A_0). \end{aligned} \tag{5.3}$$

Since $\det(g \oplus \mathrm{Id}) = \det(g)$, it follows that the map $(i^{\mathbb{Z}_2})_*$ is a bijection on π_0.

By considering the part further left in the long exact sequences, we obtain the commutative diagram

$$\begin{array}{ccccccc} \pi_m(G_r) & \longrightarrow & \pi_m(G_r/H_r) & \longrightarrow & \pi_{m-1}(H_r) & \longrightarrow & \pi_{m-1}(G_r) \\ \downarrow{\scriptstyle \text{surjective}} & & \downarrow{\scriptstyle (i_s^{\mathbb{Z}_2})_*} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \pi_m(G_{r+1}) & \longrightarrow & \pi_m(G_{r+1}/H_{r+1}) & \longrightarrow & \pi_{m-1}(H_{r+1}) & \longrightarrow & \pi_{m-1}(G_{r+1}) \end{array}$$

Here, the second Four-Lemma (see Lemma A.2 of Appendix A.2) implies that $(i_s^{\mathbb{Z}_2})_*$ is surjective. Combining all results, it follows that the inclusion $i_s^{\mathbb{Z}_2}$ is m-connected, so $d_2 = m$.

Case (iii)

Consider now the complementary case, where $H_r \hookrightarrow H_{r+1}$ is m-connected with m less than the connectivity of $G_r \hookrightarrow G_{r+1}$. We again use parts of the long exact sequence associated to the bundle $H_r \hookrightarrow G_r \to G_r/H_r$ in order to determine the connectivity of the inclusion $i_s^{\mathbb{Z}_2}$. Similar to the previous case, consider the following commutative diagram for $1 \leq j \leq m$:

$$\begin{array}{ccccccccc} \pi_j(H_r) & \longrightarrow & \pi_j(G_r) & \longrightarrow & \pi_j(G_r/H_r) & \longrightarrow & \pi_{j-1}(H_r) & \longrightarrow & \pi_{j-1}(G_r) \\ \downarrow{\scriptstyle \text{surjective}} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle (i_s^{\mathbb{Z}_2})_*} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \pi_j(H_{r+1}) & \longrightarrow & \pi_j(G_{r+1}) & \longrightarrow & \pi_j(G_{r+1}/H_{r+1}) & \longrightarrow & \pi_{j-1}(H_{r+1}) & \longrightarrow & \pi_{j-1}(G_{r+1}) \end{array}$$

Again, the Five-Lemma and the Special Five-Lemma of Appendix A.2 imply that $(i_s^{\mathbb{Z}_2})_*$ is an isomorphism for all j with $1 \leq j \leq m$. Notice that a difference to the previous case is the fact that the leftmost vertical map is only surjective. The extension to $j = 0$, where the diagram above is not defined, is trivial here since all spaces in involved are path-connected. Further to the left in the exact sequence, we find the commutative diagram

$$\begin{array}{ccccccc}
\pi_{m+1}(G_r) & \longrightarrow & \pi_{m+1}(G_r/H_r) & \longrightarrow & \pi_m(H_r) & \longrightarrow & \pi_m(G_r) \\
\downarrow \simeq & & \downarrow (i_s^{\mathbb{Z}_2})_* & & \downarrow \text{surjective} & & \downarrow \simeq \\
\pi_{m+1}(G_{r+1}) & \longrightarrow & \pi_{m+1}(G_{r+1}/H_{r+1}) & \longrightarrow & \pi_m(H_{r+1}) & \longrightarrow & \pi_m(G_{r+1})
\end{array}$$

The second Four-Lemma A.2 again implies that $(i_s^{\mathbb{Z}_2})_*$ is surjective. Therefore, in this case, $i_s^{\mathbb{Z}_2}$ is $(m+1)$-connected, so that $d_2 = m+1$.

Case (iv)

In the remaining three rows of the table, the target space has the form of a quotient $G_{p+q}/G_p \times G_q$. For the product of any two spaces Y and Z, one has a natural isomorphism [Hat02]

$$\pi_j(Y \times Z) \simeq \pi_j(Y) \times \pi_j(Z) \tag{5.4}$$

for all $j \geq 0$. Setting $Y = G_p$ and $Z = G_q$, it is compatible with the inclusions $G_p \hookrightarrow G_{p+1}$ and $G_q \hookrightarrow G_{q+1}$ giving a commutative diagram

$$\begin{array}{ccc}
\pi_j(G_p \times G_q) & \longrightarrow & \pi_j(G_{p+1} \times G_{q+1}) \\
\downarrow \simeq & & \downarrow \simeq \\
\pi_j(G_p) \times \pi_j(G_q) & \longrightarrow & \pi_j(G_{p+1}) \times \pi_j(G_{q+1})
\end{array}$$

Hence, if $G_p \hookrightarrow G_{p+1}$ is m-connected and $G_q \hookrightarrow G_{q+1}$ m'-connected, then $G_p \times G_q \hookrightarrow G_{p+1} \times G_{q+1}$ is $\min(m, m')$-connected. In particular, excluding the trivial case where $p = 0$ or $q = 0$, the inclusion $G_p \times G_q \hookrightarrow G_{p+1} \times G_{q+1}$ is always less connected than $G_{p+q} \hookrightarrow G_{p+q+2}$ and we can follow the steps of case (iii) with H_r replaced by $G_p \times G_q$. As a result, $d_1 = \min(m, m') + 1 = \min(m+1, m'+1)$ (and the same for d_2). This completes the determination of d_1 and d_2 and, hence, the proof of the theorem. □

Specializing to the physically most relevant case of $X = \mathrm{S}^{d_x, d_k}$, we obtain

Corollary 5.3. *The induced map*

$$(i_s)_* : [\mathrm{S}^{d_x,d_k}, C_s(n)]_*^{\mathbb{Z}_2} \to [\mathrm{S}^{d_x,d_k}, C_s(n+m_s)]_*^{\mathbb{Z}_2}$$

is bijective if $1 \leq d_x + d_k < d_1$ *and* $d_x < d_2$ *and surjective if* $1 \leq d_x + d_k \leq d_1$ *and* $d_x \leq d_2$.

Once the conditions for $(i_s)_*$ to be bijective are met, we are in what is called the *stable regime*. In that case, given some path-connected finite $\mathbb{Z}_2$-CW complex X, Corollary 5.3 can be applied repeatedly to give a bijection

$$(i_s)_* : [X, C_s(n)]_*^{\mathbb{Z}_2} \to [X, C_s(\infty)]_*^{\mathbb{Z}_2}, \tag{5.5}$$

where $C_s(\infty)$ is the direct limit under i_s. This is the limit where K-theory applies for arbitrary $\mathbb{Z}_2$-CW complexes X of finite dimension. For example, taking complex class A (even s and trivial $\mathbb{Z}_2$-actions), the right hand side is often written $[X, \mathrm{BU}]_*$ and is in bijection with $\tilde{K}_{\mathbb{C}}(X)$.

Returning to the case of a fixed configuration space X, Theorem 5.1 gives the exact boundary to the stable regime of K-theory. However, as discussed in Section 4.1, on the unstable side there is a further distinction in some symmetry classes between homotopy classes and isomorphism classes of vector bundles. This is the case for the real symmetry classes $s = 2$ (class $A\mathrm{II}$) and $s = 6$ (class $A\mathrm{I}$) as well as the complex symmetry class with even s (class A), all three of which have been handled in case (iv) in the proof of Theorem 5.1. In these symmetry classes, there is a U_1-symmetry leading to a decomposition of the fibers $A(x) \in C_s(n)$ $(x \in X)$ as $A(x) = A^{\mathrm{p}}(x) \oplus A^{\mathrm{h}}(x)$, where p stands for particles or conduction bands and h for holes or valence bands. Recall from Section 2.2 that $A(x)$ is already determined by $A^{\mathrm{h}}(x)$. The bundle with fiber $A^{\mathrm{h}}(x)$ over $x \in X$ is a Quaternionic vector bundle in the sense of [Dup69] (class $A\mathrm{II}$), a Real vector bundle in the sense of [Ati66] (class $A\mathrm{I}$) or an ordinary complex vector bundle (class A) over X. In [DNG14b] and [DNG14a], these vector bundles have been classified up to isomorphism for $X = \mathrm{S}^{d_x,d_k}$ with $d_k \leq 4$ and $d_x \leq 1$. However, as was emphasized in Section 4.1, in the situation at hand, where we have *sub*vector bundles, isomorphism classes agree with homotopy classes only when $\dim A^{\mathrm{p}}(x)$ is large compared to $\dim X$ and $\dim X^{\mathbb{Z}_2}$. It is the goal of the following to specify precisely what is meant by "large" in the three symmetry classes respectively.

The inclusion i_s adds dimensions to both A^{p} and A^{h}, corresponding to the addition of conduction bands *and* valence bands. This increases q to $q+1$ and p to $p+1$, as was considered in case (iv) of Theorem 5.1 above. This inclusion can be refined by two separate inclusions: Given a fixed $A_0 = A_0^{\mathrm{p}} \oplus A_0^{\mathrm{h}} \in C_s(m_s)$, one may add additional valence bands,

$$i_s^{\mathrm{h}} : C_s(n) \hookrightarrow C_s(n + m_s/2), \quad A \mapsto A \oplus A_0^{\mathrm{h}}, \tag{5.6}$$

or additional conduction bands,

$$i_s^{\mathrm{p}} : C_s(n) \hookrightarrow C_s(n + m_s/2), \quad A \mapsto A \oplus A_0^{\mathrm{p}}. \tag{5.7}$$

Since the situation is entirely symmetric, we will focus on i_s^{p} for the remainder of this section. In the realization of $C_s(n)$ and $R_s(n)$ as (unions of) homogeneous spaces, we have (restricting to one connected component as in Theorem 5.1)

$$\begin{aligned} i_2^{\mathrm{p}} &: \mathrm{U}_{2p+2q}/\mathrm{U}_{2p} \times \mathrm{U}_{2q} \hookrightarrow \mathrm{U}_{2p+2q+2}/\mathrm{U}_{2p} \times \mathrm{U}_{2q+2}, \\ (i_2^{\mathrm{p}})^{\mathbb{Z}_2} &: \mathrm{Sp}_{2p+2q}/\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q} \hookrightarrow \mathrm{Sp}_{2p+2q+2}/\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q+2}, \\ i_6^{\mathrm{p}} &: \mathrm{U}_{p+q}/\mathrm{U}_p \times \mathrm{U}_q \hookrightarrow \mathrm{U}_{p+q+1}/\mathrm{U}_p \times \mathrm{U}_{q+1}, \\ (i_6^{\mathrm{p}})^{\mathbb{Z}_2} &: \mathrm{O}_{p+q}/\mathrm{O}_p \times \mathrm{O}_q \hookrightarrow \mathrm{O}_{p+q+1}/\mathrm{O}_p \times \mathrm{O}_{q+1}. \end{aligned} \tag{5.8}$$

Note that the complex symmetry class A may be included in this treatment by taking the inclusion i_6^{p} with $\mathbb{Z}_2$-action ignored.

All of these maps have the form

$$i_s^{\mathrm{p}} : G_{p+q}/G_p \times G_q \hookrightarrow G_{p+q+1}/G_p \times G_{q+1}. \tag{5.9}$$

Since the inclusion $G_{p+q} \hookrightarrow G_{p+q+1}$ is always more connected than the inclusion $G_q \hookrightarrow G_{q+1}$, we find ourselves in the setting of case (iii) in the proof of Theorem 5.1. Thus, if $G_q \hookrightarrow G_{q+1}$ is m-connected, then the inclusion i_s^{p} is $(m+1)$-connected, independent of the parameter p. Using the $\mathbb{Z}_2$-Whitehead Theorem once more, we can now prove the following:

Corollary 5.4. *For a path-connected $\mathbb{Z}_2$-CW complex X, the induced map adding a conduction band,*

$$(i_s^{\mathrm{p}})_* : [X, C_s(n)]_*^{\mathbb{Z}_2} \to [X, C_s(n + m_s/2)]_*^{\mathbb{Z}_2},$$

is bijective or surjective according to the following table:

	bijective	*surjective*
class A	$\dim X < 2q+1$	$\dim X \leq 2q+1$
class AI	$\dim X < 2q+1$ *and* $\dim X^{\mathbb{Z}_2} < q$	$\dim X \leq 2q+1$ *and* $\dim X^{\mathbb{Z}_2} \leq q$
class AII	$\dim X < 4q+3$	$\dim X \leq 4q+3$

Proof. The proof is analogous to that of Theorem 5.1. For class A, the fact that i_6^{p} is $(2q+1)$-connected leads to the result. Proceeding to class AI, we have a non-trivial $\mathbb{Z}_2$-action and therefore the additional requirement on $\dim X^{\mathbb{Z}_2}$ due to the fact that $(i_6^{\mathrm{p}})^{\mathbb{Z}_2}$ is q-connected. For class AII, there is a slight change in the requirement for $\dim X$ due to the factor two in the indices ($q \to 2q$, see eq. (5.8)). Furthermore, since $(i_2^{\mathrm{p}})^{\mathbb{Z}_2}$ is $(4q+3)$-connected while i_2^{p} is only $(4q+1)$-connected, the additional requirement on $\dim X^{\mathbb{Z}_2}$ is always fulfilled due to $\dim X^{\mathbb{Z}_2} \leq \dim X$. □

For $X = \mathrm{S}^{d_x, d_k}$, the table in the Corollary simplifies to the following:

	bijective	surjective
class A	$d_x + d_k < 2q+1$	$d_x + d_k < 2q+1$
class AI	$d_x + d_k < 2q+1$ and $d_x < q$	$d_x + d_k \leq 2q+1$ and $d_x \leq q$
class AII	$d_x + d_k < 4q+3$	$d_x + d_k \leq 4q+3$

Notice the difference to the result in Theorem 5.1: Rather than requiring both p and q to be large, only one of the two indices is required to be large. In fact, if the configuration space X meets the conditions for bijectivity as listed above, the set of (equivariant) homotopy classes is in bijection with the set of isomorphism classes of rank-p complex vector bundles (class A), rank-p Real vector bundles (class AI) and rank-$2p$ Quaternionic vector bundles (class $A\mathrm{II}$) with fixed fibers over the base point of X. Thus, we have derived the exact boundary, within the unstable regime, below which isomorphism classes of vector bundles may differ from homotopy classes.

Remark 5.5. The restriction of fixed fibers over the base point of X can be removed by applying the free version of the $\mathbb{Z}_2$-Whitehead Theorem (rather than the one with fixed base points, see Theorem 3.21) for a connected component of $C_s(n)$.

The following table qualitatively summarizes the relationship between the three equivalence relations in this context:

p and q large	homotopy = isomorphism = stable equivalence
p arbitrary and q large	homotopy = isomorphism $\supset$ stable equivalence
p and q arbitrary	homotopy $\supset$ isomorphism $\supset$ stable equivalence

The first line is the setting of Chapter 4, since this is the regime where Bott periodicity holds. The meaning of "large" in this case is derived in Theorem 5.1. The second line includes the regime discussed above, where the conditions of bijectivity in Corollary 5.4 are met. These conditions are allowed to be violated in the third line, which includes the regime where all three equivalence relations may be different.

We now list all potentially unstable cases violating the conditions of bijectivity in Corollary 5.3 and Corollary 5.4. There are infinitely many possibilities in general if d_x and d_k are unrestricted. However, the physically most relevant cases are those with $d_k \leq 3$ and $d_x < d_k$. The latter inequality is needed on physical grounds since the dimension of the defect is $d_k - d_x - 1 \geq 0$. Table 5.1 lists all cases which are not in the stable regime and may therefore differ from the stable classification.

In Table 5.1, the cases in which isomorphism classes of vector bundles give the same classification as homotopy classes are included. In order to leave this intermediate regime (i.e to have more homotopy classes than isomorphism classes), the conditions for q need to be met additionally by p. For instance, neither the stable classification nor the classification of complex vector bundles give any non-trivial topological phases for $d_k + d_x = 3$ in class A, but the Hopf insulator [MRW08] with $q = p = 1$ has a homotopy classification by $\mathbb{Z}$. In the next chapter, we investigate all potential changes beyond the stable regime in the part of Table 5.1 with $d_x = 0$ (no defect).

complex class s	symmetry label	$d_x = 0$			$d_x = 1$		$d_x = 2$
		$d_k = 1$	$d_k = 2$	$d_k = 3$	$d_k = 2$	$d_k = 3$	$d_k = 3$
even	A			$q = 1$	$q = 1$	$q = 1$	$q \leq 2$
odd	$A\mathrm{III}$		$r = 1$	$r = 1$	$r = 1$	$r \leq 2$	$r \leq 2$

real class s	symmetry label	$d_x = 0$			$d_x = 1$		$d_x = 2$
		$d_k = 1$	$d_k = 2$	$d_k = 3$	$d_k = 2$	$d_k = 3$	$d_k = 3$
0	D			$r = 1$	$r = 1$	$r = 1$	$r \leq 2$
1	$D\mathrm{III}$					$r = 1$	$r = 1$
2	$A\mathrm{II}$						
3	$C\mathrm{II}$					$r = 1$	$r = 1$
4	C			$r = 1$	$r = 1$	$r = 1$	$r \leq 2$
5	$C\mathrm{I}$		$r = 1$	$r = 1$	$r = 1$	$r \leq 2$	$r \leq 2$
6	$A\mathrm{I}$			$q = 1$	$q = 1$	$q = 1$	$q \leq 2$
7	$BD\mathrm{I}$	$r = 1$	$r = 1$	$r = 1$	$r \leq 2$	$r \leq 2$	$r \leq 3$

Table 5.1.: All potentially unstable cases for $d_k \leq 3$ and $d_x < d_k$.

6. Novel topological phases

In this chapter, we will go through all possible exceptions to the Periodic Table cumulating in the result displayed in Table 6.1.

6.1. One dimension ($d_k = 1$)

We begin in the lowest dimension $d_k = 1$, where the only possible exception resides in the real symmetry class BDI. In this case, we have $C_7(m_7) = \mathrm{U}_1$ with τ_7 being complex conjugation, which leads to a fixed point set $R_7(m_7) = \mathrm{O}_1$. The reason that this case violates the conditions of Theorem 5.1 is the fact that $\pi_1(\mathrm{O}_1)$ is trivial while $\pi_1(\mathrm{O}_2) = \mathbb{Z}$ and $\pi_1(\mathrm{O}_n) = \mathbb{Z}_2$ for all $n \geq 3$. The topological phases in this setting are given by the set

$$[\mathrm{S}^{0,1}, \mathrm{U}_1]_*^{\mathbb{Z}_2} = \mathbb{Z}, \tag{6.1}$$

since there is a bijection with its non-equivariant analog $[\mathrm{S}^1, \mathrm{U}_1]_* = \pi_1(\mathrm{U}_1) = \mathbb{Z}$. Thus, the topological phases here are already in bijection with the stable classification $[\mathrm{S}^{0,1}, \mathrm{U}_n]_*^{\mathbb{Z}_2} = \mathbb{Z}$ for $n \geq 2$, but since all countably infinite sets are in bijection with each other, we aim for the stronger statement that this bijection is induced by the inclusion $i : \mathrm{U}_1 \hookrightarrow \mathrm{U}_n$. If i_* were not surjective, then some topological phases would be lost for $n = 1$ and if it were not injective, then there would be some additional topological phases for $n = 1$.

The generator for $\pi_1(\mathrm{U}_1) = \mathbb{Z}$ can be chosen to be the loop $f(\mathbf{k}) = e^{i\mathbf{k}}$, which has the property of being equivariant: $f(-\mathbf{k}) = \overline{f(\mathbf{k})}$. Therefore, every non-equivariant homotopy class has an equivariant representative. Recall that the inclusions

$$\mathrm{U}_1 \hookrightarrow \mathrm{U}_2 \hookrightarrow \cdots \hookrightarrow \mathrm{U}_n$$

are equivariant (as a special case of eq. (5.2)) and induce isomorphisms on π_1 by Theorem 5.1. Therefore, for any $n \in \mathbb{N}$, the set $[\mathrm{S}^{0,1}, \mathrm{U}_n]_*^{\mathbb{Z}_2}$ is in bijection with $\pi_1(\mathrm{U}_n)$ for the same reason as the one given for $n = 1$. In fact, there is a commutative diagram

$$\begin{array}{ccc} [\mathrm{S}^{0,1}, \mathrm{U}_1]_*^{\mathbb{Z}_2} & \longrightarrow & \pi_1(\mathrm{U}_1) \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} \\ [\mathrm{S}^{0,1}, \mathrm{U}_n]_*^{\mathbb{Z}_2} & \longrightarrow & \pi_1(\mathrm{U}_n) \end{array} \tag{6.2}$$

complex class s	symmetry label	$d_x = 0$		
		$d_k = 1$	$d_k = 2$	$d_k = 3$
even	A			$0 \to \mathbb{Z}$
odd	AIII		$0 \to 0$	$\mathbb{Z} \to 0$

real class s	symmetry label	$d_x = 0$		
		$d_k = 1$	$d_k = 2$	$d_k = 3$
0	D			$0 \to 0$
1	DIII			
2	AII			
3	CII			
4	C			$0 \to \mathbb{Z}_2$
5	CI		$0 \to 0$	$\mathbb{Z} \to 0$
6	AI			$0 \to 0$
7	BDI	$\mathbb{Z} \to \mathbb{Z}$	$0 \to 0$	$0 \to 0$

Table 6.1.: Comparison between the stable classification of Table 4.1 (entries located to the left of the arrows) and the classification outside the stable regime (entries to the right of the arrows) which is neither captured by K-theory nor by isomorphism classes of vector bundles. Entries here are for the case of $r = q = 1$ and $d_x = 0$ in Table 5.1.

and thus i_* is the bijection $[\mathrm{S}^{0,1}, \mathrm{U}_1]_*^{\mathbb{Z}_2} = [\mathrm{S}^{0,1}, \mathrm{U}_n]_*^{\mathbb{Z}_2} = \mathbb{Z}$. The classification in this case is therefore identical to the stable classification. The same arguments apply when the restriction for maps to be base-point preserving is lifted, since $[\mathrm{S}^1, \mathrm{U}_n] = \pi_1(\mathrm{U}_n)$. This follows from Lemma 3.22, since the action of $\pi_1(\mathrm{U}_n)$ on itself is trivial (the action is given by conjugation in this case and $\pi_1(\mathrm{U}_n)$ is Abelian).

6.2. Two dimensions ($d_k = 2$)

For $d_k = 2$, there are three symmetry classes to consider. In all of these the stable classification leads to only one topological phase. We start with the complex symmetry class $A\mathrm{III}$, where the set of topological phases to determine is given by the non-equivariant homotopy classes

$$[\mathrm{S}^2, \mathrm{U}_1]_* = \pi_2(\mathrm{U}_1) = \pi_2(\mathrm{S}^1) = 0. \tag{6.3}$$

From this it is immediate that also the set of free homotopy classes is trivial. Thus, the fact remains that there is no non-trivial topological phase.

For the real symmetry class $s = 5$ (class CI) with $d_k = 2$, the target space is $C_5(m_5) = \mathrm{U}_1$ with τ_5 being the identity[1] and therefore $R_5(m_5) = C_5(m_5) = \mathrm{U}_1$. Using Lemma 3.7, the set of (based) topological phases in this case can be rewritten as

$$[\mathrm{S}^{0,2}, \mathrm{U}_1]_*^{\mathbb{Z}_2} = \pi_1(\Omega \mathrm{U}_1, M_1^5), \tag{6.4}$$

where M_1^5 stands for the set of IQPVs in the real class $s = 5$ and dimension $d_k = 1$. The set on the right hand side fits into an exact sequence, part of which is displayed in the following diagram:

$$\begin{array}{ccccc} \pi_1(\Omega \mathrm{U}_1) & \longrightarrow & \pi_1(\Omega \mathrm{U}_1, M_1^5) & \longrightarrow & \pi_0(M_1^5) \\ \| & & & & \| \\ 0 & & & & 0 \end{array} \tag{6.5}$$

On the right, we have used the fact that for $d_k = 1$ there is no unstable regime (see Table 5.1). Alternatively, we may use Lemma 3.7 to rewrite $\pi_0(M_1^5) = \pi_1(\mathrm{U}_1, \mathrm{U}_1)$, which is trivial ($\pi_1(Y, Y)$ is trivial for any Y since all paths can be retraced). Due to the exactness of the sequence shown in the diagram above, we conclude that

$$[\mathrm{S}^{0,2}, \mathrm{U}_1]_*^{\mathbb{Z}_2} = 0. \tag{6.6}$$

Using Lemma 3.22, this also holds without base points being preserved.

[1]For more bands, τ_5 is the operation of taking the transpose, but here we deal with scalars, on which the transpose acts as the identity.

The last case to consider for $d_k = 2$ is the real symmetry class $s = 7$ (class BDI), which we have considered for $d_k = 1$ previously. In fact, we can use the previous result in conjunction with diagram (6.5) (with $s = 7$ rather than $s = 5$ and a different $\mathbb{Z}_2$-action on U_1) to show that $[\mathrm{S}^{0,2}, \mathrm{U}_1]_*^{\mathbb{Z}_2} = 0$ (and therefore also $[\mathrm{S}^{0,2}, \mathrm{U}_1]^{\mathbb{Z}_2} = 0$ using Lemma 3.22).

6.3. Three dimensions ($d_k = 3$)

The number of possible exceptions to the stable classification increases to seven for $d_k = 3$ (see Table 5.1). We begin by investigating the three symmetry classes which we have already encountered in the previous two sections for $d_k = 1$ and $d_k = 2$. Starting with the complex symmetry class AIII, we find immediately that

$$[\mathrm{S}^3, \mathrm{U}_1]_* = \pi_3(\mathrm{U}_1) = \pi_3(\mathrm{S}^1) = 0 \tag{6.7}$$

and the same for the free homotopy classes. This marks the first change to the stable classification: For $n \geq 2$, $\pi_3(\mathrm{U}_n) = \mathbb{Z} \neq 0$, so there exist non-trivial phases which are absent for $n = 1$.

Turning to the real symmetry classes CI and BDI, we can use a diagram similar to (6.5) but for one dimension higher:

$$\begin{array}{ccccc} & & [\mathrm{S}^{0,3}, \mathrm{U}_1]_*^{\mathbb{Z}_2} & & \\ & & \| & & \\ \pi_1(\Omega^2\mathrm{U}_1) & \longrightarrow & \pi_1(\Omega^2\mathrm{U}_1, M_2^s) & \longrightarrow & \pi_0(M_2^s) \\ \| & & & & \| \\ \pi_3(\mathrm{U}_1) & & & & [\mathrm{S}^{0,2}, \mathrm{U}_1]_*^{\mathbb{Z}_2} \\ \| & & & & \| \\ 0 & & & & 0 \end{array} \tag{6.8}$$

In the right column we have used the previous results (no non-trivial phases in $d_k = 2$ for both CI and BDI) and the left column follows from the basic fact that $\pi_d(\mathrm{S}^1) = 0$ for all $d \geq 2$. The exactness of the sequence again implies that $[\mathrm{S}^{0,3}, \mathrm{U}_1]_*^{\mathbb{Z}_2}$ (and therefore also $[\mathrm{S}^{0,3}, \mathrm{U}_1]^{\mathbb{Z}_2}$) is trivial for the classes CI and BDI. In the case of class CI this marks a change from a $\mathbb{Z}$ classification to a trivial one.

The rest of this section will be devoted to the remaining four symmetry cases, for which the Hopf fibration will play a major role since the target space will be $\mathrm{Gr}_1(\mathbb{C}^2) = \mathrm{S}^2$ rather than $\mathrm{U}_1 = \mathrm{S}^1$. This treatment will reveal cases where there are more topological phases than in the stable regime, including the Hopf insulator [MRW08] and a newly identified phase we call the Hopf superconductor.

We will continue with our strategy of determining based homotopy classes first. In the complex symmetry class A, the set of (based) topological phases is the set of non-equivariant homotopy classes

$$[\mathrm{S}^3, \mathrm{Gr}_1(\mathbb{C}^2)]_* = \pi_3(\mathrm{Gr}_1(\mathbb{C}^2)) = \mathbb{Z}. \tag{6.9}$$

This group $\mathbb{Z}$ on the right hand side is generated by the celebrated Hopf map h (meaning that $[h] = 1 \in \mathbb{Z}$). It is the projection map of a fibration

$$\mathrm{S}^1 \hookrightarrow \mathrm{S}^3 \xrightarrow{h} \mathrm{Gr}_1(\mathbb{C}^2) \tag{6.10}$$

and defined as follows: We view S^3 as the unit sphere in $\mathbb{C}^2$ by assigning to an element $(x_1, x_2, x_3, x_4) \in \mathrm{S}^3$ the two complex numbers $z_1 := x_1 + ix_2$ and $z_2 := x_3 + ix_4$, where the requirement $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ translates to $|z_1|^2 + |z_2|^2 = 1$. The Hopf map h is the canonical map assigning to the point $(z_1, z_2) \in \mathbb{C}^2$ the complex line in $\mathrm{Gr}_1(\mathbb{C}^2)$ which passes through the origin and this point. The preimage of this line consists of all pairs $(\lambda z_1, \lambda z_2)$ with $\lambda \in \mathrm{U}_1 = \mathrm{S}^1$, explaining the fiber in the above sequence.

In the following, we deviate from this canonical definition by identifying both domain and codomain of h with spaces which are more suitable for computations and visualizations. First, we identify $\mathrm{Gr}_1(\mathbb{C}^2)$ with S^2 by assigning to a complex line through (z_1, z_2) its complex slope $z_1/z_2 \in \mathbb{C} \cup \{\infty\}$ and subsequently using the inverse of the stereographic projection $p_2 : \mathrm{S}^2 \to \mathbb{C} \cup \{\infty\}$ as defined in Appendix A.1. Explicitly, we obtain the expression

$$\begin{aligned}(p_2^{-1} \circ h)(z_1, z_2) &= (2\mathrm{Re}(z_1\bar{z}_2), 2\mathrm{Im}(z_1\bar{z}_2), |z_1|^2 - |z_2|^2) \qquad (6.11)\\ &= (2x_1x_3 + 2x_2x_4,\\ &\quad\; 2x_2x_3 - 2x_1x_4,\\ &\quad\; x_1^2 + x_2^2 - x_3^2 - x_4^2) \qquad (6.12)\end{aligned}$$

Furthermore, we will often use the homeomorphism $r \circ p_3 : \mathrm{S}^3 \to \mathrm{I}^3/\partial\mathrm{I}^3$ as defined in Appendix A.1 and thus replace h by the composition

$$p_2^{-1} \circ h \circ (r \circ p_3)^{-1} : \mathrm{I}^3 \to \mathrm{S}^2, \tag{6.13}$$

with the property that $\partial\mathrm{I}^3$ is mapped to a point in S^2. We take the liberty of denoting all of these variations of the Hopf map by the same symbol h, since we will be interested in its homotopy-invariant properties which are not affected by homeomorphisms.

The fact that $\pi_3(\mathrm{S}^2) = \mathbb{Z}$ and that it is generated by $[h]$ can be deduced from part of the exact sequence associated to the fibration (6.10):

$$\begin{array}{ccccccc}\pi_3(\mathrm{S}^1) & \longrightarrow & \pi_3(\mathrm{S}^3) & \xrightarrow{h_*} & \pi_3(\mathrm{S}^2) & \longrightarrow & \pi_2(\mathrm{S}^1)\\ \| & & \| & & \| & & \|\\ 0 & & \mathbb{Z} & & \mathbb{Z} & & 0\end{array}$$

Exactness implies that h_* has to be an isomorphism. Using the basic fact that $\pi_d(\mathrm{S}^d) = \mathbb{Z}$ is generated by the homotopy class of the identity map, $[\mathrm{Id}] = 1 \in \mathbb{Z} = \pi_d(\mathrm{S}^d)$ for all $d \geq 1$ (to be introduced in more detail later), the generator of $\pi_3(\mathrm{S}^2)$ is given by

$$h_*[\mathrm{Id}] = [h \circ \mathrm{Id}] = [h]. \tag{6.14}$$

Although they will not be necessary for the computations in the remainder of this chapter, we complete the discussion here by introducing integral formulas for the homotopy invariants distinguishing the homotopy classes in $\pi_2(\mathrm{S}^2) = \mathbb{Z}$ (by an invariant called the mapping degree) and $\pi_3(\mathrm{S}^2) = \mathbb{Z}$ (by the Hopf invariant). Starting with the former, let there be a differentiable representative $f : \mathrm{S}^2 \to \mathrm{S}^2$ of a homotopy class in $\pi_2(\mathrm{S}^2)$. Given the volume 2-form ω on S^2 with normalization $\int_{\mathrm{S}^2} \omega = 1$, the mapping degree n of the map f is defined to be the integer

$$n_{\mathrm{deg}}(f) := \int_{\mathrm{S}^2} f^* \omega. \tag{6.15}$$

The fact that the identity on S^2 represents the generator of $\pi_2(\mathrm{S}^2) = \mathbb{Z}$ is reflected in the fact that

$$n_{\mathrm{deg}}(\mathrm{Id}) = \int_{\mathrm{S}^2} \omega = 1. \tag{6.16}$$

For a differentiable map $g : \mathrm{S}^3 \to \mathrm{S}^2$, we again use the pullback $g^*\omega$. However, this time a 3-form is needed that can be integrated over S^3, so we form the wedge product with a 1-form α chosen as follows: The second de Rham-cohomology group is trivial on S^3 ($H^2_{\mathrm{dR}}(\mathrm{S}^3) = 0$), implying that all closed 2-forms on S^3 are exact. Therefore, since $dg^*\omega = g^* d\omega = 0$, we choose a 1-form α with $d\alpha = g^*\omega$. The Hopf invariant of g is defined to be the integral

$$n_{\mathrm{Hopf}}(g) := \int_{\mathrm{S}^3} \alpha \wedge d\alpha. \tag{6.17}$$

The Hopf invariant of the Hopf map h is

$$n_{\mathrm{Hopf}}(h) = 1. \tag{6.18}$$

Both invariants described above have canonical generalizations: The mapping degree can be generalized to maps $\mathrm{S}^d \to \mathrm{S}^d$ giving a complete invariant of $\pi_d(\mathrm{S}^d) = \mathbb{Z}$ for all $d \geq 1$ (all homotopy classes are distinguished by this invariant). Alternatively, it may be generalized to the Chern number [Nak03] of maps $\mathrm{S}^2 \to \mathrm{Gr}_m(\mathbb{C}^n)$ with general m and n. The Hopf invariant can be generalized to maps $\mathrm{S}^{2d-1} \to \mathrm{S}^d$, giving a partial invariant of $\pi_{2d-1}(\mathrm{S}^d)$ (maps with different Hopf invariants are not homotopic, but the converse is not true in general for $d \neq 2$).

The set of (based) topological phases with configuration space S^3 in class A was given in (6.9) with the Hopf map h representing the phase known as the Hopf insulator [MRW08]. All other topological phases in this case may be realized by making use of the addition in π_3 (an alternative way of obtaining representatives in these phases is described in [DWSD13]).

Continuing on to the three real symmetry classes D, C and AI, we need to understand how the Hopf map behaves under the different $\mathbb{Z}_2$-actions. Even though the ultimate goal is to determine topological phases for $d_k = 3$ and $d_x = 0$, we find it useful to keep the more general $\mathbb{Z}_2$-action on the domain, so $I^3 = I^{d_x,d_k}$ with $d_x + d_k = 3$ as introduced in Section 3.2. On the codomain S^2, there are three different $\mathbb{Z}_2$-actions with non-trivial element $\tau_s : S^2 \to S^2$, corresponding to the three real symmetry classes according to the following list:

$$\tau_s(x_1, x_2, x_3) = \begin{cases} (-x_1, -x_2, x_3) & \text{for } s = 0 \text{ (class } D) \\ (x_1, x_2, x_3) & \text{for } s = 4 \text{ (class } C) \\ (-x_1, x_2, x_3) & \text{for } s = 6 \text{ (class } A\mathrm{I}). \end{cases} \tag{6.19}$$

We take this opportunity to point out that $s = 2$ (class AII) is excluded, since the minimal number of bands in that case is at least four, with two of them occupied, ($p = q = 1$ in $\mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})$, see Table 2.1). This doubling is required in order to satisfy the conditions imposed by the pseudo-symmetries or, in the most prominent physical realization thereof, the conditions of particle number conservation and time-reversal symmetry.

Hence, the space $(S^2)^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-fixed points is given by

$$(S^2)^{\mathbb{Z}_2} = \begin{cases} S^0 = O_2/U_1 & \text{for } s = 0 \text{ (class } D) \\ S^2 = Sp_2/U_1 & \text{for } s = 4 \text{ (class } C) \\ S^1 = \mathrm{Gr}_1(\mathbb{R}^2) & \text{for } s = 6 \text{ (class } A\mathrm{I}). \end{cases} \tag{6.20}$$

Our aim for the remainder of the section is to determine the set $[S^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} = [S^{0,3}, S^2]_*^{\mathbb{Z}_2}$ for the three cases listed above. Similarly to previous calculations, we employ Lemma 3.7 to identify

$$[S^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} = \pi_0(M_3^s) \simeq \pi_1(\Omega^2 S^2, M_2^s), \tag{6.21}$$

where M_2^s denotes the space of 2-dimensional, 2-band IQPVs in class s. The relevant part of the associated exact sequence is displayed in the following diagram:

$$\begin{array}{ccccccccc} & & & & \big[\mathrm{S}^{0,3}, C_s(m_s)\big]_*^{\mathbb{Z}_2} & & & & \\ & & & & \| & & & & \\ \pi_1(M_2^s) & \xrightarrow{i_1} & \pi_1(\Omega^2\mathrm{S}^2) & \xrightarrow{j_1} & \pi_1(\Omega^2\mathrm{S}^2, M_2^s) & \xrightarrow{\partial_1} & \pi_0(M_2^s) & \xrightarrow{i_0} & \pi_0(\Omega^2\mathrm{S}^2) \\ & & \| & & & & & & \| \\ & & \pi_3(\mathrm{S}^2) & & & & & & \pi_2(\mathrm{S}^2) \\ & & \| & & & & & & \| \\ & & \mathbb{Z} & & & & & & \mathbb{Z} \end{array} \tag{6.22}$$

The entry on the right, $\pi_2(\mathrm{S}^2) = \mathbb{Z}$, is the set of homotopy classes of based maps $\mathrm{S}^2 \to \mathrm{S}^2$, which are distinguished by the mapping degree. In order to determine the set $[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2}$ using the diagram above, we will show that $\ker(i_0)$ contains only the trivial element, which must be the only element in the image of ∂_1 due to exactness. This will be established in the next proposition and will enable us to apply Lemma 3.9 to obtain a bijection

$$[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} \simeq \pi_1(\Omega^2\mathrm{S}^2)/i_1(\pi_1(M_2^s)), \tag{6.23}$$

for $s = 0, 4, 6$.

Proposition 6.1. *For the three cases $s = 0, 4, 6$ we have the following result for $\pi_0(M_2^s)$ in diagram* (6.22)*:*

$$\pi_0(M_2^s) = \begin{cases} \mathbb{Z} & \text{for } s = 0 \text{ (class D)} \\ 2\mathbb{Z} & \text{for } s = 4 \text{ (class C)} \\ 0 & \text{for } s = 6 \text{ (class AI)}, \end{cases} \tag{6.24}$$

and the invariant distinguishing all homotopy classes is the mapping degree. In particular, for all three cases $s = 0, 4, 6$, the image of the map $\partial_1 : \pi_1(\Omega^2\mathrm{S}^2, M_2^s) \to \pi_0(M_2^s)$ in diagram (6.22) *consists only of the class represented by the constant map.*

Proof. The cardinalities of these results follow from the fact that $d_x = 0$ and $d_k = 2$ are sufficiently low to be in the stable regime (c.f. Table 5.1). However, in order to prove the important statement about the mapping degree, and to be able to use details about the nature of representatives at a later stage, we need to go into more detail. Starting with the last line ($s = 6$), there is a diagram (we take the liberty of writing $A\mathrm{I}$ in the superscript in place of $s = 6$ for clarity)

$$\begin{array}{ccccccccc} & & & & \pi_0(M_2^{A\mathrm{I}}) & & & & \\ & & & & \| & & & & \\ \pi_1(M_1^{A\mathrm{I}}) & \xrightarrow{i_1} & \pi_1(\Omega \mathrm{S}^2) & \xrightarrow{j_1} & \pi_1(\Omega \mathrm{S}^2, M_1^{A\mathrm{I}}) & \xrightarrow{\partial_1} & \pi_0(M_1^{A\mathrm{I}}) & \xrightarrow{i_0} & \pi_0(\Omega \mathrm{S}^2) \\ \| & & \| & & & & \| & & \| \\ \pi_2(\mathrm{S}^2, \mathrm{S}^1) & & \pi_2(\mathrm{S}^2) & & & & \pi_1(\mathrm{S}^2, \mathrm{S}^1) & & \pi_1(\mathrm{S}^2) \\ \| & & \| & & & & \| & & \| \\ \mathbb{Z} \times \mathbb{Z} & & \mathbb{Z} & & & & 0 & & 0 \end{array} \tag{6.25}$$

The bottom row needs some explanation. The two entries on the right, $\pi_1(\mathrm{S}^2)$ and $\pi_1(\mathrm{S}^2, \mathrm{S}^1)$, are trivial since loops on S^2 as well as paths ending on the equator $\mathrm{S}^1 \subset \mathrm{S}^2$ can be contracted to a point. The leftmost result, $\pi_2(\mathrm{S}^2, \mathrm{S}^1) = \mathbb{Z} \times \mathbb{Z}$, can be deduced from the associated exact sequence

$$\begin{array}{ccccccccc} \pi_2(\mathrm{S}^1) & \xrightarrow{i_2} & \pi_2(\mathrm{S}^2) & \xrightarrow{j_2} & \pi_2(\mathrm{S}^2, \mathrm{S}^1) & \underset{\delta}{\overset{\partial_2}{\rightleftarrows}} & \pi_1(\mathrm{S}^1) & \xrightarrow{i_1} & \pi_1(\mathrm{S}^2) \\ \| & & \| & & \| & & \| & & \| \\ 0 & & \mathbb{Z} & & \mathbb{Z} \times \mathbb{Z} & & \mathbb{Z} & & 0 \end{array} \tag{6.26}$$

The fact that, as a set, $\pi_2(\mathrm{S}^2, \mathrm{S}^1) = \mathbb{Z} \times \mathbb{Z}$ follows from Lemma 3.8 using the injectivity of j_2 and the surjectivity of ∂_2, both statements being implied by the exactness of the sequence above. However, we wish to use the group structure of $\pi_2(\mathrm{S}^2, \mathrm{S}^1)$, so a more detailed analysis is required which can be found, for instance, in [Hil53, p. 41]. As indicated in diagram (6.26), there is a splitting $\delta : \pi_1(\mathrm{S}^1) \to \pi_2(\mathrm{S}^2, \mathrm{S}^1)$, defined as follows: The inclusion $i : \mathrm{S}^1 \hookrightarrow \mathrm{S}^2$ is (based) homotopic to the constant map to the base point of S^2 (see Figure 6.1). Therefore, for any loop γ representing a class $[\gamma] \in \pi_1(\mathrm{S}^1)$, the composition $i \circ \gamma$ is homotopic to the constant map. By taking the homotopy parameter as part of the domain, we obtain a map $(\mathrm{D}^2, \mathrm{S}^1) \to (\mathrm{S}^2, \mathrm{S}^1)$, which descends to the map δ on homotopy classes.

From this definition, it is clear that $\partial_2 \circ \delta = \mathrm{Id}$ on $\pi_1(\mathrm{S}^1)$. Recalling the injectivity of j_2, we conclude that every element of $\pi_2(\mathrm{S}^2, \mathrm{S}^1)$ is uniquely represented by a sum $j_2[\alpha] + \delta[\beta]$ for $[\alpha] \in \pi_2(\mathrm{S}^2)$ and $[\beta] \in \pi_1(\mathrm{S}^1)$. Since $j_2(\pi_2(\mathrm{S}^2))$ is contained in the center of $\pi_2(\mathrm{S}^2, \mathrm{S}^1)$ (see [tD08, p. 128]), the latter is isomorphic to a direct product of groups

$$\pi_2(\mathrm{S}^2, \mathrm{S}^1) \simeq j_2(\pi_2(\mathrm{S}^2)) \times \delta(\pi_1(\mathrm{S}^1)) \simeq \mathbb{Z} \times \mathbb{Z}. \tag{6.27}$$

It remains to determine the map $i_1 : \pi_1(M_1^{A\mathrm{I}}) \to \pi_2(\mathrm{S}^2)$ in diagram (6.25), which assigns to an equivariant map $\mathrm{S}^{1,1} \to \mathrm{S}^2$ its mapping degree. For this purpose, it is necessary to use the equivariance relation in order to double the domain in the isomorphism $\pi_2(\mathrm{S}^2, \mathrm{S}^1) \simeq \pi_1(M_1^{A\mathrm{I}})$. Since i_1 is a homomorphism, it is sufficient to

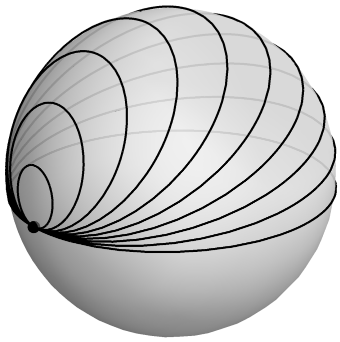


Figure 6.1.: Homotopy between $i : \mathrm{S}^1 \hookrightarrow \mathrm{S}^2$ and the constant map to the base point.

know its values on the generators of $\mathbb{Z} \times \mathbb{Z} = \pi_2(\mathrm{S}^2, \mathrm{S}^1)$. The generator $(0, 1) \in \mathbb{Z} \times \mathbb{Z}$ is given by $\delta[\mathrm{Id}]$ for $[\mathrm{Id}] = 1 \in \mathbb{Z} = \pi_1(\mathrm{S}^1)$. Since the involution on both domain and codomain is reflection about the equator, it extends to the identity map on $\mathrm{S}^{1,1}$. On the other hand, the generator $(1, 0)$, which is given by $j_2[\mathrm{Id}]$ for $[\mathrm{Id}] \in \pi_2(\mathrm{S}^2)$, extends to the concatenation $\mathrm{Id} * \mathrm{Id}$ (as defined in eq. (3.9)) on $\mathrm{S}^{1,1}$. Therefore, the map i_1 evaluates as

$$\begin{aligned} i_1 : \mathbb{Z} \times \mathbb{Z} &\to \mathbb{Z} \\ (m, n) &\mapsto 2m + n. \end{aligned} \tag{6.28}$$

In particular, it is surjective and exactness implies that $j_1 : \pi_2(\mathrm{S}^2) \to \pi_0(M_2^{\mathrm{AI}})$ has to be the constant map. At the same time, j_1 has to be surjective since $\mathrm{im}(j_1) = \ker(\partial_1) = \pi_0(M_2^{\mathrm{AI}})$. Hence, $\pi_0(M_2^{\mathrm{AI}})$ can only contain a single element and we write

$$\pi_0(M_2^{\mathrm{AI}}) = 0. \tag{6.29}$$

Turning to the case $s = 0$ (class D), we have an exact sequence

$$\begin{array}{ccccccc} & & \pi_0(M_2^D) & & & & \\ & & \| & & & & \\ \pi_1(\Omega \mathrm{S}^2) & \xrightarrow{j_1} & \pi_1(\Omega \mathrm{S}^2, M_1^D) & \xrightarrow{\partial_1} & \pi_0(M_1^D) & \xrightarrow{i_0} & \pi_0(\Omega \mathrm{S}^2) \\ \| & & & & \| & & \| \\ \pi_2(\mathrm{S}^2) & & & & \pi_1(\mathrm{S}^2, \mathrm{S}^0) & & \pi_1(\mathrm{S}^2) \\ \| & & & & \| & & \| \\ \mathbb{Z} & & & & \mathbb{Z}_2 & & 0 \end{array} \tag{6.30}$$

The only changes to diagram (6.25) used for the calculations with $s = 6$ (class AI) are the omission of the leftmost column (which will not be required here) and the fact that $(\mathrm{S}^2)^{\mathbb{Z}_2} = \mathrm{S}^0$ rather than S^1.

The entry $\pi_0(M_1^D) = \pi_1(\mathrm{S}^2, \mathrm{S}^0)$ can be computed through the following exact sequence:

$$\begin{array}{ccccccccc} \pi_1(\mathrm{S}^0) & \xrightarrow{i_1} & \pi_1(\mathrm{S}^2) & \xrightarrow{j_1} & \pi_1(\mathrm{S}^2,\mathrm{S}^0) & \xrightarrow{\partial_1} & \pi_0(\mathrm{S}^0) & \xrightarrow{i_0} & \pi_0(\mathrm{S}^2) \\ \| & & \| & & & & \| & & \| \\ 0 & & 0 & & & & \mathbb{Z}_2 & & 0 \end{array} \tag{6.31}$$

It immediately follows that the map ∂_1 is an isomorphism. The two homotopy classes of $\pi_1(\mathrm{S}^2, \mathrm{S}^0) = \mathbb{Z}_2$ are paths that start and end at the base point, represented by the constant map, and those that start at the base point and end at the other point of S^0. Physically, these two homotopy classes are the trivial and non-trivial topological phase in one-dimensional class D superconductors, both of which can be realized, for instance, in the Kitaev Majorana chain model [Kit01].

is the constant map to the trivial homotopy class (with mapping degree 0).

Returning to diagram (6.30), we see that the map ∂_1 has to be surjective due to exactness (note that the two maps named ∂_1 in diagrams (6.30) and (6.31) are different). Thus, the set $\pi_0(M_2^D) = \pi_1(\Omega\mathrm{S}^2, M_1^D)$ is the disjoint union of the two preimages under ∂_1. Using Lemma 3.9, each preimage can be realized as an orbit of $\pi_1(\Omega\mathrm{S}^2) = \mathbb{Z}$. One is the orbit on the constant path as illustrated in the upper part of Figure 6.2 and the other is the orbit on a path of loops ending in the non-trivial loop of $\pi_0(M_1^D) = \mathbb{Z}_2$ as shown in the lower part of Figure 6.2. In order to construct an element in the latter orbit, we can use the fact that the identity map $\mathrm{Id} : \mathrm{S}^2 \to \mathrm{S}^2$ is $\mathbb{Z}_2$-equivariant in symmetry class D (since the involution happens to be the same on domain and target) and restricting it to one hemisphere gives a path of loops ending in the non-trivial element of $\pi_0(M_1^D) = \mathbb{Z}_2$.

Upon doubling the domain by using the $\mathbb{Z}_2$-equivariance to undo the application of Lemma 3.7 and return from $\pi_1(\Omega\mathrm{S}^2, M_1^D)$ to $\pi_0(M_2^D)$, the action by an element of $\pi_1(\Omega\mathrm{S}^2) = \pi_2(\mathrm{S}^2)$ with mapping degree $n \in \mathbb{Z}$ effectively adds a mapping degree $2n$. The reason is that two coordinates of the extended part of the domain are inverted, which is a transformation with determinant 1 leaving the homotopy class invariant according to Lemma 3.2, while the involution on the target S^2 is homotopic to the identity. Therefore the action of $\pi_1(\Omega\mathrm{S}^2) = \pi_2(\mathrm{S}^2) = \mathbb{Z}$ on the constant map yields all even mapping degrees, while the action on the element corresponding to the identity map yields all odd mapping degrees, as illustrated in Figure 6.2. The result is

$$\pi_0(M_2^D) = \mathbb{Z}, \tag{6.32}$$

with elements distinguished by their mapping degree. Note that the stabilizers of both orbits have to be trivial since the action of any non-trivial element in $\pi_1(\Omega\mathrm{S}^2) = \mathbb{Z}$ changes the mapping degree and therefore also the homotopy class.

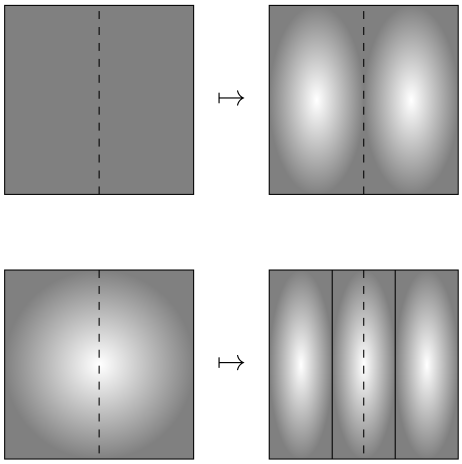


Figure 6.2.: Illustration of the action of $1 \in \mathbb{Z} = \pi_1(\Omega S^1) = \pi_2(S^2)$ on the constant path in $\pi_1(\Omega S^2, M_1^D)$ (upper row) and on a path ending in the non-trivial element of $\pi_0(M_1^D)$ (lower row), creating representatives with mapping degree 2 and 3 respectively. Depicted is the domain and different colors indicate different images. The cut of Lemma 3.7 is indicated by the dashed line. For instance, the lower left is the identity map of S^2 (which is equivariant for class D) and can be viewed as a path ending in the non-trivial element of $\pi_0(M_1^D)$ when restricted to the right half.

The last case to consider is that of symmetry class $s = 4$ (class C), where we have $(\mathrm{S}^2)^{\mathbb{Z}_2} = \mathrm{S}^2$ and the following exact sequence:

$$
\begin{array}{ccccccccc}
 & & & & \pi_0(M_2^C) & & & & \\
 & & & & \| & & & & \\
\pi_1(M_1^C) & \xrightarrow{i_1} & \pi_1(\Omega\mathrm{S}^2) & \xrightarrow{j_1} & \pi_1(\Omega\mathrm{S}^2, M_1^C) & \xrightarrow{\partial_1} & \pi_0(M_1^C) & \xrightarrow{i_0} & \pi_0(\Omega\mathrm{S}^2) \\
\| & & \| & & & & \| & & \| \\
\pi_2(\mathrm{S}^2,\mathrm{S}^2) & & \pi_2(\mathrm{S}^2) & & & & \pi_1(\mathrm{S}^2,\mathrm{S}^2) & & \pi_1(\mathrm{S}^2) \\
\| & & \| & & & & \| & & \| \\
0 & & \mathbb{Z} & & & & 0 & & 0
\end{array}
\tag{6.33}
$$

Here the situation is particularly simple: Both relative homotopy groups $\pi_2(\mathrm{S}^2, \mathrm{S}^2)$ and $\pi_1(\mathrm{S}^2, \mathrm{S}^2)$ vanish due to the general statement that $\pi_d(Y, Y) = 0$ for all spaces Y and dimensions $d \geq 1$. As a consequence, j_1 is an isomorphism and all homotopy classes in $\pi_1(\Omega\mathrm{S}^2, M_1^C)$ are represented by maps $\mathrm{S}^2 \to \mathrm{S}^2$ and classified by their mapping degree. Upon the identification $\pi_1(\Omega\mathrm{S}^2, M_1^C) = \pi_0(M_2^C)$, the domain is doubled and so is the mapping degree, since the involution on the target S^2 is the identity. We have therefore arrived at the final result

$$\pi_0(M_2^C) = 2\mathbb{Z}. \tag{6.34}$$

□

We have shown that the elements of $\pi_0(M_2^s)$ are distinguished by the mapping degree in all cases $s = 0, 4, 6$. It follows that the map i_0 in diagram (6.22) has trivial kernel for all $s = 0, 4, 6$ and therefore, due to exactness, the image of ∂_1 can only contain one element. Hence, for $s = 0, 4, 6$,

$$[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} = \pi_1(\Omega^2\mathrm{S}^2, M_2^s) = \ker(\partial_1). \tag{6.35}$$

This situation constitutes a special case of Lemma 3.9 where there is only one preimage under ∂_1 and therefore the left hand side of the equation above may be realized as a single orbit under $\pi_1(\Omega^2\mathrm{S}^2) = \pi_3(\mathrm{S}^2)$ with stabilizer $i_1(\pi_1(M_2^s))$. For later reference, we summarize this result in the following lemma:

Lemma 6.2. *For $s \in \{0, 4, 6\}$, there is a bijection*

$$[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} \simeq \pi_1(\Omega^2\mathrm{S}^2)/i_1(\pi_1(M_2^s)).$$

Before proceeding to the main result of this section, we take a moment to unravel some of the indirect arguments that were necessary to prove Lemma 6.2 above. For this purpose, we make the usual identification of domains S^{d_x,d_k} with I^{d_x,d_k}, where it is understood that the entire boundary of I^{d_x,d_k} is mapped to a point.

Any equivariant map $\psi : \mathrm{I}^{0,3} \to \mathrm{S}^2$ restricts to an equivariant map $\mathrm{I}^{0,2} \to \mathrm{S}^2$ when any of its three momentum-like coordinates is set to zero. For concreteness let (k_1, k_2, k_3) be the coordinates of $\mathrm{I}^{0,3}$ with $-\pi \leq k_i \leq \pi$, then $\psi(0, k_2, k_3)$ is said restriction (we make the arbitrary choice of setting the first coordinate to zero). It must have mapping degree zero, since ψ is continuous and a homotopy to the constant map is given by $\psi(t, k_2, k_3)$ for $t \in [0, \pi]$ (recall that the boundary of $\mathrm{I}^{0,3}$ is mapped to a single point). This homotopy is through non-equivariant maps in general, but we have proved that there always exists a homotopy through equivariant maps as well ($\ker(i_0)$ is trivial in diagram (6.22), a corollary to Proposition 6.1). Consequently, every homotopy class of equivariant maps $\mathrm{I}^{0,3} \to \mathrm{S}^2$ has a representative that is constant in the k_2, k_3-plane, which we still denote by ψ. The domain of ψ can be viewed as two 3-spheres joined in a point, one being the part with $k_1 \geq 0$ and the other the one with $k_1 \leq 0$. We can therefore assign two well defined Hopf invariants $n_+ = n_{\mathrm{Hopf}}(\psi_+)$ and $n_- = n_{\mathrm{Hopf}}(\psi_-)$, where ψ_+ and ψ_- are the map ψ restricted to $k_1 \geq 0$ and $k_1 \leq 0$ respectively. Note that these numbers are only well-defined (and in particular invariant) as long as the k_2, k_3-plane maps to a single point. They are not independent, since ψ is equivariant and therefore

$$\psi_+ = \tau_s \circ \psi_- \circ \tau. \tag{6.36}$$

Pulling back the volume 2-form ω of S^2 by the composition $\tau_s \circ g$ rather than g yields

$$(\tau_s \circ g)^* \omega = g^* \tau_s^* \omega = \pm g^* \omega, \tag{6.37}$$

where the sign is positive for the orientation preserving involutions τ_0 and τ_4 (class D and class C respectively) and it is negative for the orientation-reversing τ_6 (class AI). Thus $d\alpha$ in eq 6.17 is changed to $\pm d\alpha$. However, this sign change is compensated for since the same sign needs to be chosen for the potential 1-form α, so the form $\alpha \wedge d\alpha$ stays invariant for all three τ_s $(s = 0, 4, 6)$. As a consequence, $n_{\mathrm{Hopf}}(\tau_s \circ g) = n_{\mathrm{Hopf}}(g)$, or in other words, $(\tau_s)_* : \pi_3(\mathrm{S}^2) \to \pi_3(\mathrm{S}^2)$ is the identity. The involution τ on the domain inverts three coordinates and therefore $\det(\tau) = -1$. Lemma 3.2 then implies that $[g \circ \tau] = [g]^{-1}$ for $[g] \in \pi_3(\mathrm{S}^2)$ or $n_{\mathrm{Hopf}}(g \circ \tau) = -n_{\mathrm{Hopf}}(g)$ (this can also be seen directly in eq. (6.17)). Collecting these results, we can relate n_+ and n_-:

$$\begin{aligned} n_+ &= n_{\mathrm{Hopf}}(\psi_+) \\ &= n_{\mathrm{Hopf}}(\tau_s \circ \psi_- \circ \tau) \\ &= n_{\mathrm{Hopf}}(\psi_- \circ \tau) \\ &= -n_{\mathrm{Hopf}}(\psi_-) \\ &= -n_-. \end{aligned} \tag{6.38}$$

Hence, the total Hopf invariant of ψ is zero and the Hopf insulator in the complex symmetry class A does not have an immediate equivariant realization in the real symmetry classes. However, this does not mean that there is only the trivial topological phase. As an analogy, the topological phases in the complex symmetry class A in two dimensions are distinguished by the Chern number (and hence are often called Chern insulators [Hal88]) and this is always zero when imposing time-reversal symmetry in the real symmetry class $A\mathrm{II}$. Yet, there is a $\mathbb{Z}_2$-classification with the non-trivial phase represented by the quantum spin Hall effect [KM05] we met in Example 4.3.

In the situation at hand, we have constructed a representative ψ for an arbitrary class in $[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2}$ which maps the k_2, k_3-plane to a point, but general homotopies through equivariant maps do not respect this property. In fact, any such general homotopy ψ_t between two representatives ψ_0 and ψ_1, both of which map the k_2, k_3-plane to a point, restricts to a loop in M_2^s for $k_1 = 0$. As such, it represents an element in $\pi_1(M_2^s)$. If this element has a non-trivial Hopf invariant, then n_+ (and therefore also n_-) may be changed by this number. This is the reason for the quotient in Lemma 6.2. It will turn out that for the symmetry classes $s = 0$ and $s = 6$ (class D and class AI), all Hopf invariants are realized in $\pi_1(M_2^s)$, while for class $s = 4$ (class C) only even Hopf invariants can be realized. The former means that there are no non-trivial topological phases in classes D and AI, while the latter implies that there is one non-trivial topological phase in class C: A representative with odd n_+ can never be deformed to the constant map with $n_+ = 0$.

We summarize and prove these results in the following:

Theorem 6.3. *The topological phases of two-band IQPVs in the real symmetry classes $s = 0, 4$ and 6 in three spatial dimensions ($d_k = 3$) without defect ($d_x = 0$) are*

$$[\mathrm{S}^{0,3}, C_s(m_s)]^{\mathbb{Z}_2} = [\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} = \begin{cases} 0 & \text{for } s = 0 \text{ (class } D\text{)} \\ \mathbb{Z}_2 & \text{for } s = 4 \text{ (class } C\text{)} \\ 0 & \text{for } s = 6 \text{ (class } A\mathrm{I}\text{)}. \end{cases}$$

Proof. We use the identification $[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} \simeq \pi_1(\Omega^2\mathrm{S}^2)/i_1(\pi_1(M_2^s))$ of Lemma 6.2 and determine the subgroup $i_1(\pi_1(M_2^s)) \subset \pi_1(\Omega\mathrm{S}^2) = \pi_3(\mathrm{S}^2) = \mathbb{Z}$. This task is equivalent to determining which classes in $\pi_3(\mathrm{S}^2) = \mathbb{Z}$ can be realized by equivariant maps $\mathrm{S}^{d_x,d_k} \to \mathrm{S}^2$ with $d_x = 1$ and $d_k = 2$.

As a first attempt, it is instructive to see whether the Hopf map $\tilde{h}$ as defined in (6.13) is already equivariant as it is. If it were so, the fact that no homotopy through non-equivariant maps to the constant map exists implies that in particular no homotopy through equivariant maps does. Therefore, a non-trivial class would be realized. The Hopf map is equivariant if it fulfills the condition

$$\tau_s \circ h = h \circ \tau \tag{6.39}$$

for a pair of involutions τ_s on S^2 (with $s \in \{0, 4, 6\}$) and τ on I^{d_x,d_k} (with $d_x + d_k = 3$). We would like to make use of the Hopf map as written explicitly in eq. (6.12), so we need to transfer the involution on I^{d_x,d_k} to S^{d_x,d_k} using the homeomorphism $p_3^{-1} \circ r^{-1} : \mathrm{I}^{d_x,d_k}/\partial\mathrm{I}^{d_x,d_k} \to \mathrm{S}^{d_x,d_k}$. The latter is given by

$$(p_3^{-1} \circ r^{-1})(\mathbf{x}) \mapsto f_1(|\mathbf{x}|)\big(\mathbf{x}, f_2(|\mathbf{x}|)\big) \tag{6.40}$$

for some functions f_1 and f_2 (see Appendix A.1), so the involution τ is realized on $\mathrm{S}^{d_x,d_k} \subset \mathbb{R}^4$ by acting as the identity on the first d_x coordinates and as multiplication by -1 on the next d_k coordinates. The last coordinate is always left invariant as it only depends on the absolute value of coordinates in I^{d_x,d_k}. Furthermore, permuting coordinates of I^{d_x,d_k} corresponds to permuting the first 3 coordinates of $\mathrm{S}^{d_x,d_k} \subset \mathbb{R}^4$ while leaving the last coordinate in place. Hence, we may resort to studying the original Hopf map as displayed in eq. (6.12) with x_4 fixed under τ.

Starting with $d_x = 0$ and $d_k = 3$, we have

$$\begin{aligned} h(-x_1, -x_2, -x_3, x_4) = (&2(-x_1)(-x_3) + 2(-x_2)x_4, \\ &2(-x_2)(-x_3) + 2(-x_1)x_4, \\ &(-x_1)^2 + (-x_2)^2 - (-x_3)^2 - x_4^2). \end{aligned} \tag{6.41}$$

While the third component in the image remains invariant, the sign changes for only one of the summands in the first and second component respectively. Therefore, the right hand side does not equal $\tau_s \circ h$ for any s, which comes as no surprise in view of the derivation leading to eq. (6.38), where we showed that the Hopf invariant always vanishes for equivariant maps in classes $s = 0, 4, 6$.

Turning to the case $d_x = 1$ and $d_k = 2$, which corresponds to the leftmost entry in the diagram 6.22, we have

$$\begin{aligned} (h \circ \tau)(x_1, x_2, x_3, x_4) &= h(x_1, -x_2, -x_3, x_4) \\ &= (2x_1(-x_3) - 2(-x_2)x_4, \\ &\quad\; 2(-x_2)(-x_3) + 2x_1x_4, \\ &\quad\; x_1^2 + (-x_2)^2 - (-x_3)^2 - x_4^2) \\ &= (-[2x_1x_3 - 2x_2x_4], \\ &\quad\; 2x_2x_3 + 2x_1x_4, \\ &\quad\; x_1^2 + x_2^2 - x_3^2 - x_4^2) \\ &= (\tau_6 \circ h)(x_1, x_2, x_3, x_4), \end{aligned} \tag{6.42}$$

so the Hopf map can be realized directly as an equivariant map $\mathrm{S}^{1,2} \to \mathrm{S}^2$ for $s = 6$ (class AI). Composing h with a transformation σ which permutes the first three coordinates cyclically, we compute

$$\begin{aligned}
(h \circ \sigma \circ \tau)(x_1, x_2, x_3, x_4) &= (h \circ \sigma)(x_1, -x_2, -x_3, x_4) \\
&= h(-x_2, -x_3, x_1, x_4) \\
&= (2(-x_2)x_1 - 2(-x_3)x_4, \\
&\quad\; 2(-x_3)x_1 - 2(-x_2)x_4, \\
&\quad\; (-x_2)^2 + (-x_3)^2 - x_1^2 - x_4^2) \\
&= (-[2x_2x_1 - 2x_3x_4], \\
&\quad\; -[2x_3x_1 + 2x_2x_4], \\
&\quad\; x_2^2 + x_3^2 - x_1^2 - x_4^2) \\
&= (\tau_0 \circ h)(x_2, x_3, x_1, x_4) \\
&= (\tau_0 \circ h \circ \sigma)(x_1, x_2, x_3, x_4). \qquad (6.43)
\end{aligned}$$

Hence, $h \circ \sigma$ is equivariant as a map $\mathrm{S}^{1,2} \to \mathrm{S}^2$ for $s = 0$ (class D). Recall that cyclic permutations of the coordinates leave the homotopy class invariant (see Lemma 3.2), so $[h \circ \sigma] = [h] = 1 \in \mathbb{Z}$. In fact, using Lemma 3.10, all classes $n \in \mathbb{Z} = \pi_3(\mathrm{S}^2)$ have equivariant representatives $\mathrm{S}^{1,2} \to \mathrm{S}^2$ for $s = 0$ (class D) and $s = 6$ (class AI). Therefore, for these two classes the map i_1 in diagram (6.22) is surjective and Lemma 6.2 implies that

$$[\mathrm{S}^{0,3}, C_s(m_s)]_*^{\mathbb{Z}_2} \simeq \pi_1(\Omega^2\mathrm{S}^2)/i_1(\pi_1(M_2^s)) \simeq i_1(\pi_1(M_2^s))/i_1(\pi_1(M_2^s)) = 0. \qquad (6.44)$$

It remains to investigate symmetry class $s = 4$ (class C), where the Hopf map cannot be realized equivariantly as above. Using Lemma 3.7 to identify $\pi_1(M_2^C) = \pi_2(\Omega\mathrm{S}^2, M_1^C)$, we can utilize the following exact sequence:

$$\begin{array}{ccccccccc}
 & & & & \pi_1(M_2^C) & & & & \\
 & & & & \| & & & & \\
\pi_2(M_1^C) & \xrightarrow{i_2} & \pi_2(\Omega\mathrm{S}^2) & \xrightarrow{j_2} & \pi_2(\Omega\mathrm{S}^2, M_1^C) & \xrightarrow{\partial_2} & \pi_1(M_1^C) & \xrightarrow{i_1} & \pi_1(\Omega\mathrm{S}^2) \\
\| & & \| & & & & \| & & \| \\
\pi_3(\mathrm{S}^2, \mathrm{S}^2) & & \pi_3(\mathrm{S}^2) & & & & \pi_2(\mathrm{S}^2, \mathrm{S}^2) & & \pi_2(\mathrm{S}^2) \\
\| & & \| & & & & \| & & \| \\
0 & & \mathbb{Z} & & & & 0 & & \mathbb{Z}
\end{array} \qquad (6.45)$$

The trivial entries follow again from $\pi_d(Y, Y) = 0$ for all $d \geq 1$ and the non-trivial entries $\mathbb{Z}$ are familiar from before. Due to exactness, j_2 is an isomorphism, so every map $(\mathrm{D}^2, \mathrm{S}^1) \to (\Omega\mathrm{S}^2, M_1^C)$ is homotopic to one that maps the entirety of S^1 to the base point, yielding a map $\mathrm{S}^2 \to \Omega\mathrm{S}^2$ whose homotopy class is determined by the Hopf invariant. Upon doubling the domain in order to undo the application of Lemma 3.7, two coordinates are inverted and since the involution on the target space S^2 is simply the identity, the Hopf invariant is also doubled (see Lemma 3.2). Thus, every representative in $\pi_1(M_2^C)$ has even mapping degree and we write

$$\pi_1(M_2^C) = 2\mathbb{Z}. \tag{6.46}$$

Returning to the computation of $[S^{0,3}, S^2]_*^{\mathbb{Z}_2}$ in the present symmetry class C, we apply Lemma 6.2 to obtain the final result

$$[S^{0,3}, S^2]_*^{\mathbb{Z}_2} \simeq \pi_1(\Omega^2 S^2)/i_1(\pi_1(M_2^C)) \simeq \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

For all three real symmetry classes $s = 0, 4, 6$, the topological phases with a preserved base point coincide with the free topological phases. For $s = 0$ and $s = 6$ this is obvious as there is only one based topological phase and the freedom of being able to move the base point during homotopies cannot yield more homotopy classes. More formally, we can use Lemma 3.22: If $[X, Y]_*^{\mathbb{Z}_2} = 0$, then $[X, Y]^{\mathbb{Z}_2} = [X, Y]_*^{\mathbb{Z}_2}/\pi_1(Y^{\mathbb{Z}_2}) = 0$. For symmetry class $s = 4$, the fundamental group of the fixed points of τ_4 is trivial, $\pi_1((S^2)^{\mathbb{Z}_2}) = \pi_1(S^2) = 0$, so the action is trivial and

$$[S^{0,3}, S^2]^{\mathbb{Z}_2} = [S^{0,3}, S^2]_*^{\mathbb{Z}_2}/\pi_1(S^2) = [S^{0,3}, S^2]_*^{\mathbb{Z}_2}. \tag{6.47}$$

□

In conclusion, we have identified a superconducting analog in the real symmetry class C of the Hopf insulator in complex symmetry class A, which we propose to call the Hopf superconductor.

6.3.1. Many bands

In general, the Hopf insulator and superconductor only have non-trivial topology when realized in a situation with exactly two bands. However, there is a generalization to many-band models, which we present for the Hopf insulator in complex symmetry class A. Using the homogeneous space model $\mathrm{Gr}_1(\mathbb{C}^2) = U_2/U_1 \times U_1$, an alternative view of the fact that $\pi_3(U_2/U_1 \times U_1) = \mathbb{Z}$ presents itself by considering the fiber bundle

$$U_1 \times U_1 \hookrightarrow U_2 \to U_2/U_1 \times U_1. \tag{6.48}$$

Part of the associated long exact sequence reads

$$\begin{array}{ccccccc} \pi_3(U_1 \times U_1) & \longrightarrow & \pi_3(U_2) & \longrightarrow & \pi_3(U_2/U_1 \times U_1) & \longrightarrow & \pi_2(U_1 \times U_1) \\ \parallel & & \parallel & & & & \parallel \\ 0 & & \mathbb{Z} & & & & 0 \end{array} \tag{6.49}$$

It follows immediately that $\pi_3(\mathrm{U}_2/\mathrm{U}_1 \times \mathrm{U}_1) = \mathbb{Z}$.

In the presence of more conduction or valence bands, the leftmost map becomes surjective due to $\pi_3(\mathrm{U}_m) = \mathbb{Z}$ for $m \geq 2$ and we retrieve the familiar[2] result $\pi_3(\mathrm{U}_n/\mathrm{U}_p \times \mathrm{U}_q) = 0$ for $p > 1$ or $q > 1$ (or both) from Table 4.1 (class A with $d_k = 3$ and $d_x = 0$). If we however impose the condition that no energy levels are degenerate and that this property is preserved under all homotopies, the space of annihilators turns into the flag manifold

$$\mathrm{U}_n/(\mathrm{U}_1)^n, \tag{6.50}$$

where $(\mathrm{U}_1)^n := \mathrm{U}_1 \times \cdots \times \mathrm{U}_1$ is an n-fold product with factors U_1. This is the space of all collections of n mutually orthogonal, one-dimensional subspaces of $\mathbb{C}^n$. In the physics literature, it was considered for the integer quantum Hall effect in two dimensions [ASS83], in which the assumption about separated energy levels is justified, since the Landau levels are flat with constant energy differences. In fact, using the generalized fiber bundle

$$(\mathrm{U}_1)^n \hookrightarrow \mathrm{U}_n \to \mathrm{U}_n/(\mathrm{U}_1)^n, \tag{6.51}$$

the associated exact sequence has a part

$$\begin{array}{ccccccc} \pi_2(\mathrm{U}_n) & \longrightarrow & \pi_2(\mathrm{U}_n/(\mathrm{U}_1)^n) & \longrightarrow & \pi_1((\mathrm{U}_1)^n) & \longrightarrow & \pi_1(\mathrm{U}_n) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z}^n & & \mathbb{Z} \end{array} \tag{6.52}$$

The rightmost map assigns to a set of winding numbers $(m_1, \ldots, m_n) \in \pi_1((\mathrm{U}_1)^n)$ their sum $m_1 + \cdots + m_n \in \mathbb{Z} = \pi_1(\mathrm{U}_n)$. In particular, it is surjective and therefore exactness implies that $\pi_2(\mathrm{U}_n/(\mathrm{U}_1)^n) = \mathbb{Z}^{n-1}$, the subset of $\mathbb{Z}^n$ with sum equal to zero. These invariants can be interpreted as Chern numbers of the line bundles associated to each energy band with a zero sum rule due to the fact that the n-dimensional vector bundle into which they are embedded is assumed to be trivial.

Moving to three dimensions, another part of the long exact sequence generalizes diagram (6.49):

$$\begin{array}{ccccccc} \pi_3((\mathrm{U}_1)^n) & \longrightarrow & \pi_3(\mathrm{U}_n) & \longrightarrow & \pi_3(\mathrm{U}_n/(\mathrm{U}_1)^n) & \longrightarrow & \pi_2((\mathrm{U}_1)^n) \\ \| & & \| & & & & \| \\ 0 & & \mathbb{Z} & & & & 0 \end{array} \tag{6.53}$$

Thus, the non-trivial result $\pi_3(\mathrm{U}_n/(\mathrm{U}_1)^n) = \mathbb{Z}$ remains in this generalized setting with an arbitrary number of bands.

[2] Actually, if one of $p = 1$ or $q > 1$ and vice versa, we are not yet in the stable regime, but in the intermediate regime of vector bundle isomorphism classes. However, these turn out to be trivial too.

7. Strong and weak topological phases

Up to this point we have exclusively computed (equivariant) homotopy classes for IQPVs with configuration space S^{d_x,d_k}. In this chapter, we address the problem of classifying topological phases with the important configuration spaces $X = \mathrm{T}^d$ (the Brillouin zone torus) and $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ (the Brillouin zone torus in the presence of a defect of codimension $d_x + 1$). The solution we present sheds new light on the notion of strong and weak topological phases, especially outside the stable regime.

7.1. Stable regime

We begin the exposition in the stable regime and write $C_s(n) \equiv C_s$ and $R_s(n) \equiv R_s$ for brevity throughout this section, with the understanding that n is always large enough with respect to the dimension of the configuration space in order for the conditions of bijectivity in Theorem 5.1 to be fulfilled. In the stable regime, we determined the set

$$[\mathrm{S}^{d_x,d_k}, C_s]_*^{\mathbb{Z}_2} \tag{7.1}$$

of base-point preserving equivariant homotopy classes in Chapter 4. For $d_x = 0$, the set

$$[\mathrm{S}^{0,d_k}, C_s]_*^{\mathbb{Z}_2} \tag{7.2}$$

can be interpreted physically as classifying topological phases invariant under a continuous translation group, which leads to momenta $\mathbf{k} \in \mathbb{R}^{d_k}$. Imposing the physical requirement that the image is fixed for $|\mathbf{k}| \to \infty$, momentum space compactifies to S^{0,d_k} and the point ∞ is the base point with fixed image.

However, in the setting introduced in Chapter 2 with discrete translation group, the set of topological phases in the absence of defects is

$$[\mathrm{T}^d, C_s]^{\mathbb{Z}_2}, \tag{7.3}$$

and in the presence of a defect with codimension $d_x + 1$, it is

$$[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]^{\mathbb{Z}_2}. \tag{7.4}$$

In the following, we demonstrate that our results completely determine the above sets of topological phases in the stable regime, since they decompose as a product of sets of the form given in eq (7.1). This result has been derived in [FHN+11, FM13] using K-theory - here we give an independent proof from the perspective of homotopy theory.

Theorem 7.1. *In the stable regime, there are bijections*

$$[\mathrm{T}^d, C_s]^{\mathbb{Z}_2} \simeq \prod_{r=0}^{d} \left([\mathrm{S}^{0,r}, C_s]_*^{\mathbb{Z}_2}\right)^{\binom{d}{r}},$$

$$[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]^{\mathbb{Z}_2} \simeq \prod_{r=0}^{d_k} \left([\mathrm{S}^{d_x,r}, C_s]_*^{\mathbb{Z}_2}\right)^{\binom{d_k}{r}},$$

for all real and complex symmetry classes s and dimensions d (respectively d_x and d_k), with the exception of classes A, AI and AII, where we need to replace C_s by its connected component $(C_s)_0$ containing the base point and omit the factor with $r = 0$ on the right hand sides.

Before proving these statements, we introduce a tool called the equivariant free loop fibration (for the non-equivariant version see [tD08, p. 116]; the equivariant extension is found in [tD87]): Let Y be a $\mathbb{Z}_2$-space on which the non-trivial element of $\mathbb{Z}_2$ acts by the involution τ_Y. Then the space LY of free loops $f : \mathrm{S}^{0,1} \to Y$ is equipped with the $\mathbb{Z}_2$-action $f \mapsto \tau_Y \circ f \circ \tau$ and the equivariant free loop fibration is defined by

$$(\bar{\Omega} Y)^{\mathbb{Z}_2} \hookrightarrow (LY)^{\mathbb{Z}_2} \xrightarrow{p} Y^{\mathbb{Z}_2}, \tag{7.5}$$

where p assigns to an equivariant loop $f : \mathrm{S}^{0,1} \to Y$ its value $f(s_0) \in Y^{\mathbb{Z}_2}$ at the base point $s_0 \in \mathrm{S}^{0,1}$. Thus, the fiber over a point $y \in Y^{\mathbb{Z}_2}$ is the space of equivariant loops based at y.

Importantly, this fibration is equipped with a section $q : Y^{\mathbb{Z}_2} \to (LY)^{\mathbb{Z}_2}$ given by assigning to $y \in Y^{\mathbb{Z}_2}$ the constant loop at y, which results in $p \circ q = id_{Y^{\mathbb{Z}_2}}$. Therefore, the associated long exact sequence splits into short exact sequences

$$0 \longrightarrow \pi_d((\bar{\Omega} Y)^{\mathbb{Z}_2}) \longrightarrow \pi_d((LY)^{\mathbb{Z}_2}) \underset{q_*}{\overset{p_*}{\rightleftarrows}} \pi_d(Y^{\mathbb{Z}_2}) \longrightarrow 0 \tag{7.6}$$

for all $d \geq 0$. Note that the maps in these short exact sequences are homomorphisms only for $d \geq 1$. In that case, the splitting yields an isomorphism

$$\begin{aligned} \pi_d((LY)^{\mathbb{Z}_2}) &\simeq \pi_d((\bar{\Omega} Y)^{\mathbb{Z}_2}) \rtimes \pi_d(Y^{\mathbb{Z}_2}) \\ &\overset{\text{as sets}}{\simeq} \pi_d((\bar{\Omega} Y)^{\mathbb{Z}_2}) \times \pi_d(Y^{\mathbb{Z}_2}). \end{aligned} \tag{7.7}$$

Since we are interested in the set of topological phases and not any group structures on this set, we only require the lower line stating a bijection between sets.

The prerequisite $d \geq 1$ is a crucial condition. In fact, there exists an identification

$$[\mathrm{T}^d, C_s]^{\mathbb{Z}_2} \simeq \pi_0((L^d C_s)^{\mathbb{Z}_2}), \tag{7.8}$$

where $(L^{d_k} C_s)^{\mathbb{Z}_2}$ is the d_k-fold iterated equivariant free loop space of C_s, and if eq. (7.7) were true for $d = 0$, then a product decomposition would follow immediately, in contradiction to the counter-examples that exist outside the stable regime (see eqs. (7.27) and (7.31) in the next section). In the presence of a position-like dimension $d_x \geq 1$ on the other hand, we have

$$\begin{aligned}
[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]_*^{\mathbb{Z}_2} &\simeq \pi_{d_x}((L^{d_k} C_s)^{\mathbb{Z}_2}) \\
&\simeq \pi_{d_x}((L^{d_k-1}\bar{\Omega} C_s)^{\mathbb{Z}_2}) \times \pi_{d_x}((L^{d_k-1} C_s)^{\mathbb{Z}_2}) \\
&\vdots \\
&\simeq \prod_{r=0}^{d_k} \left(\pi_{d_x}((\Omega^{0,r} C_s)^{\mathbb{Z}_2}) \right)^{\binom{d_k}{r}} \\
&\simeq \prod_{r=0}^{d_k} \left([\mathrm{S}^{d_x,r}, C_s]_*^{\mathbb{Z}_2} \right)^{\binom{d_k}{r}}.
\end{aligned} \tag{7.9}$$

This result holds independently of stability conditions. However, it assumes that a base point is preserved, for which there is no physical justification in this case.

Remark 7.2. With $d_x = 1$ and trivial $\mathbb{Z}_2$-actions, the left hand side of eq. (7.9) is known as the d_k-th torus homotopy group of C_s, a concept developed in [Fox45] (with a more detailed exposition in [Fox48]) and applied in the seminal paper [ASS83] to the homotopy theory of the quantum Hall effect.

Proof of Theorem 7.1. We now prove the statements of Theorem 7.1 without base points, including the important case without defect. Assuming first that the symmetry index s is odd in order for C_s to have only a single connected component, we use the Bott map as in Theorem 4.12 in conjunction with the $\mathbb{Z}_2$-Whitehead Theorem 3.21 in its free version to obtain bijections

$$[\mathrm{T}^d, C_s]^{\mathbb{Z}_2} \simeq [\mathrm{T}^d, \Omega C_{s-1}]^{\mathbb{Z}_2} \tag{7.10}$$

and

$$[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]^{\mathbb{Z}_2} \simeq [\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, \Omega C_{s-1}]^{\mathbb{Z}_2}, \tag{7.11}$$

for odd s. Notice that we choose to use the loop space rather than the space of geodesics (see Section 3.5) and that the loop coordinate has the trivial $\mathbb{Z}_2$-action. We use this loop coordinate to identify the sets on the right hand side with a fundamental group π_1, enabling the use of the decomposition in eq. (7.7). Without defect, we find

$$\begin{aligned}
[\mathrm{T}^d, C_s]^{\mathbb{Z}_2} &\simeq [\mathrm{T}^d, \Omega C_{s-1}]^{\mathbb{Z}_2} \\
&\simeq \pi_1((L^d C_{s-1})^{\mathbb{Z}_2}) \\
&\simeq \pi_1((L^{d-1}\bar{\Omega} C_{s-1})^{\mathbb{Z}_2}) \times \pi_1((L^{d-1} C_{s-1})^{\mathbb{Z}_2}) \\
&\vdots \\
&\simeq \prod_{r=0}^{d} \left(\pi_1((\bar{\Omega}^r C_{s-1})^{\mathbb{Z}_2})\right)^{\binom{d}{r}} \\
&\simeq \prod_{r=0}^{d} \left([\mathrm{S}^{0,r}, \Omega C_{s-1}]_*^{\mathbb{Z}_2}\right)^{\binom{d}{r}} \\
&\simeq \prod_{r=0}^{d} \left([\mathrm{S}^{0,r}, C_s]_*^{\mathbb{Z}_2}\right)^{\binom{d}{r}}. \qquad (7.12)
\end{aligned}$$

In the last equation we used the $\mathbb{Z}_2$-Whitehead Theorem in reverse (the based version) in order to readjust the symmetry index from $s-1$ to s. A similar chain of bijections is obtained for $d_x \geq 1$:

$$\begin{aligned}
[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]^{\mathbb{Z}_2} &\simeq [\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, \Omega C_{s-1}]^{\mathbb{Z}_2} \\
&\simeq \pi_{d_x}((L^{d_k} C_{s-1})^{\mathbb{Z}_2}) \\
&\simeq \pi_{d_x}((L^{d_k-1}\bar{\Omega} C_{s-1})^{\mathbb{Z}_2}) \times \pi_1((L^{d_k-1} C_{s-1})^{\mathbb{Z}_2}) \\
&\vdots \\
&\simeq \prod_{r=0}^{d_k} \left(\pi_{d_x}((\bar{\Omega}^r C_{s-1})^{\mathbb{Z}_2})\right)^{\binom{d_k}{r}} \\
&\simeq \prod_{r=0}^{d_k} \left([\mathrm{S}^{d_x,r}, \Omega C_{s-1}]_*^{\mathbb{Z}_2}\right)^{\binom{d_k}{r}} \\
&\simeq \prod_{r=0}^{d_k} \left([\mathrm{S}^{d_x,r}, C_s]_*^{\mathbb{Z}_2}\right)^{\binom{d_k}{r}}. \qquad (7.13)
\end{aligned}$$

The difference to the result of eq. (7.9) is the lack of a base point condition at the outset.

For even s, the requirements for the $\mathbb{Z}_2$-Whitehead Theorem are not met since there are only a finite number of connected components of C_s in contrast to infinitely many connected components of ΩC_{s-1} (see Table 4.1). We therefore resort to the same strategy as in the proofs of the theorems in Chapters 4 and 5 and replace C_s and ΩC_{s-1} by their connected components $(C_s)_0$ and $(\Omega C_{s-1})_0$ containing the base point (the base point of ΩC_{s-1} being the constant loop at the base point of $(C_s)_0 \subset C_{s-1}$). The $\mathbb{Z}_2$-Whitehead Theorem then gives bijections

$$[\mathrm{T}^d, (C_s)_0]^{\mathbb{Z}_2} \simeq [\mathrm{T}^d, (\Omega C_{s-1})_0]^{\mathbb{Z}_2} \tag{7.14}$$

and

$$[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, (C_s)_0]^{\mathbb{Z}_2} \simeq [\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, (\Omega C_{s-1})_0]^{\mathbb{Z}_2}. \tag{7.15}$$

In the first bijection, the right hand side is a subset of $[\mathrm{T}^d, \Omega C_{s-1}]^{\mathbb{Z}_2}$. It can be identified in the decomposition (7.12) as the subset with the factor $\pi_1(C_{s-1}^{\mathbb{Z}_2}) = \pi_1(R_{s-1})$ replaced by $\ker(i_*) \subset \pi_1(R_{s-1})$ as illustrated in Figure 7.1 for the case $d = 1$, where

$$i_* : \pi_1(R_{s-1}) \to \pi_1(C_{s-1}) \tag{7.16}$$

is the induced map of the inclusion $i : R_{s-1} \hookrightarrow C_{s-1}$.

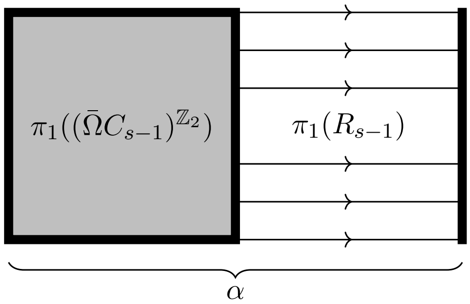


Figure 7.1.: Decomposition of $[\mathrm{S}^1, \Omega C_{s-1}]^{\mathbb{Z}_2} = \pi_1((LC_{s-1})^{\mathbb{Z}_2})$ into the product $\pi_1((\bar{\Omega}C_{s-1})^{\mathbb{Z}_2}) \times \pi_1(R_{s-1})$ as viewed from the domain. Thick black lines are mapped to the base point and lines with arrows all indicate the same loop representing an element in $\pi_1(R_{s-1})$. Elements in the subset $[\mathrm{S}^1, (\Omega C_{s-1})_0]^{\mathbb{Z}_2} \subset [\mathrm{S}^1, \Omega C_{s-1}]^{\mathbb{Z}_2}$ restrict to a loop α homotopic to the constant loop, corresponding to an element $\alpha \in \ker(i_*) \subset \pi_1(R_{s-1})$.

Similarly, the set on the right hand side of the other bijection (eq. (7.15)) is a subset of $[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, \Omega C_{s-1}]^{\mathbb{Z}_2}$, which can be identified with the subset of the result in eq. (7.13) with the factor $\pi_{d_x+1}(C_{s-1}^{\mathbb{Z}_2}) = \pi_{d_x+1}(R_{s-1})$ replaced by $\ker(i'_*) \subset \pi_{d_x+1}(R_{s-1})$, where this time

$$i'_* : \pi_{d_x+1}(R_{s-1}) \to \pi_{d_x+1}(C_{s-1}). \tag{7.17}$$

We use a slightly different notation here in order to distinguish the two maps i_* and i'_* even though both are induced by the same map i. For the real symmetry classes with even $s \neq 2, 6$, we have $\ker(i_*) = \pi_1(R_{s-1})$ and $\ker(i'_*) = \pi_{d_x+1}(R_{s-1})$. Furthermore, since $R_s \subset (C_s)_0$ in these cases and since both T^{d_k} and $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ are path-connected, it follows that

$$[\mathrm{T}^d, (C_s)_0]^{\mathbb{Z}_2} \simeq [\mathrm{T}^d, C_s]^{\mathbb{Z}_2} \tag{7.18}$$

and

$$[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, (C_s)_0]^{\mathbb{Z}_2} \simeq [\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s]^{\mathbb{Z}_2}. \tag{7.19}$$

Thus, the results for real symmetry classes with even $s \neq 2, 6$ are the same as for odd s.

For the remaining symmetry classes – complex class A and real classes $A\mathrm{II}$ $(s = 2)$ and $A\mathrm{I}$ $(s = 6)$ – the sets $\ker(i_*)$ and $\ker(i'_*)$ contain only one element. Therefore, the factor with $r = 0$ in the product decompositions (7.12) and (7.13) vanishes. Moreover, $R_s \not\subset (C_s)_0$ in these cases, so we cannot use eqs. (7.18) and (7.19). It follows that the main statements of the theorem need to be modified as announced. □

Remark 7.3. Physically, the replacement $C_s \to (C_s)_0$ amounts to choosing a chemical potential which fixes the number of valence and conduction bands. In order to fully classify all topological phases in symmetry classes A, $A\mathrm{I}$ and $A\mathrm{II}$, one needs to move the base point to every connected component of $(C_s)_0$ and apply Theorem 7.1. In doing so, one needs to be careful not to leave the stable regime, beyond which Theorem 7.1 is not valid in general.

Since we have determined in Chapter 4 all factors in the product decomposition offered by Theorem 7.1, it follows that we have determined all topological phases with configuration spaces T^d and $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$ in the stable regime. Another use of Theorem 7.1 is the option of distinguishing topological phases according to certain factors in the product decomposition. For instance, the notion of strong and weak can be defined:

Definition 7.4. *A topological phase is* strong in the stable sense *if the bijection in Theorem 7.1 maps it to a product with non-trivial element in the factor* $[\mathrm{S}^{0,d}, C_s]_*^{\mathbb{Z}_2}$ *(resp.* $[\mathrm{S}^{d_x,d_k}, C_s]_*^{\mathbb{Z}_2}$*) with domain of the largest dimension. Otherwise, it is called* weak in the stable sense*.*

The weak topological phases contain those phases that are realized simply by stacking IQPVs with momentum-like dimension lower than d into d dimensions. In the case where a defect is present, the weak phases contain those that are stacked at every point of the measuring surface S^{d_x,d_k}. The distinction between strong and weak will be revisited and in fact revised when leaving the stable regime in the next section (hence the addendum "in the stable sense"). For now however, we stay in the stable regime and give two examples of how the result in Theorem 7.1 can be applied to identify the strong and weak topological phases.

Example 7.5 (Class *AII*)**.** In the real symmetry class $s = 2$ (class *AII*) without defect and $d = 3$, we pick a connected component $(C_s)_0 = \mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})$ corresponding to $2p$ valence bands and $2q$ conduction bands. Then Theorem 7.1 implies

$$[\mathrm{T}^3, \mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})]^{\mathbb{Z}_2} \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2), \tag{7.20}$$

since we know from Table 4.1 that

$$[\mathrm{S}^{0,3}, \mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})]_*^{\mathbb{Z}_2} = \mathbb{Z}_2, \tag{7.21}$$

$$[\mathrm{S}^{0,2}, \mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})]_*^{\mathbb{Z}_2} = \mathbb{Z}_2, \tag{7.22}$$

$$[\mathrm{S}^{0,1}, \mathrm{Gr}_{2p}(\mathbb{C}^{2p+2q})]_*^{\mathbb{Z}_2} = 0. \tag{7.23}$$

This is the result given in the seminal work [FKM07] generalizing the two-dimensional quantum spin Hall effect to three dimensions and predicting the existence of a three-dimensional time-reversal invariant topological phase with no two-dimensional analog. In that work, a quartet of independent invariants $(\nu_0; \nu_1, \nu_2, \nu_3)$ is constructed with $\nu_i \in \mathbb{Z}_2$ corresponding to the four $\mathbb{Z}_2$ factors in eq. (7.20). The strong phases in this example are those with $\nu_0 = 1$ (non-trivial value), while the weak phases are the ones with $\nu_0 = 0$ (trivial value). All non-trivial weak phases have representatives constructed by piling layers of two-dimensional quantum spin Hall phases into three dimensions.

Example 7.6 (Class *D*)**.** We have already considered real symmetry class $s = 0$ (class *D*) in one dimension in Example 4.2, but with a fixed base point. Theorem 7.1 gives the result

$$[\mathrm{T}^1, C_0]^{\mathbb{Z}_2} = [\mathrm{S}^{0,1}, C_0]^{\mathbb{Z}_2} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{7.24}$$

as the results listed in Table 4.1 imply

$$[\mathrm{S}^{0,1}, C_0]_*^{\mathbb{Z}_2} = \mathbb{Z}_2, \tag{7.25}$$

$$[\mathrm{S}^{0,0}, C_0]_*^{\mathbb{Z}_2} = \mathbb{Z}_2. \tag{7.26}$$

A representative of each topological phase is shown in Figure 7.2: Two of them (displayed as blue and red dashed lines) map to only one connected component of R_s at both momenta $k = 0$ and $k = \pm\pi$. These are homotopic to constant maps, but not homotopic to one another. The remaining two representatives switch connected components at $k = 0$ and $k = \pm\pi$ (blue and red solid lines). If a base-point is preserved, one of the choices of connected component is fixed and therefore only a $\mathbb{Z}_2$ classification remains.

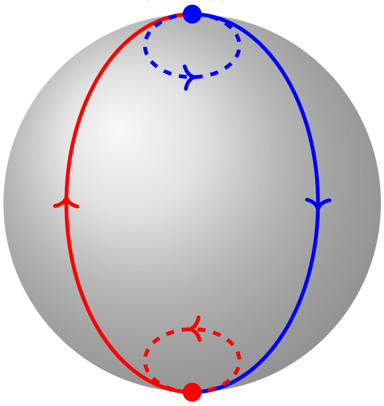


Figure 7.2.: Representatives of the four topological phases in $[\mathrm{S}^{0,1}, C_0(n)]^{\mathbb{Z}_2} = \mathbb{Z}_2 \times \mathbb{Z}_2$ of Example 7.6, for the (already stable) case $n = 1$. Shown is half of each image (the other half is determined by the $\mathbb{Z}_2$-equivariance, see Lemma 3.7). The τ_0-fixed point set $R_0(1)$ has two connected components $(R_0(1))_0$ and $(R_0(1))_1$ corresponding to the blue and red dot respectively.

7.2. Outside the stable regime

The proof of Theorem 7.1 required the use of Bott periodicity, a result applicable only in the stable regime. The next two examples demonstrate that this is not merely a shortcoming of the technique used in the proof, but rather that the product decomposition does not exist in general.

Example 7.7. In Section 6.3, we introduced the Hopf insulator [MRW08] as a nontrivial representative of the set $[\mathrm{S}^3, \mathrm{Gr}_1(\mathbb{C}^2)]$. On a lattice in three dimensions however, the set of topological phases is given by $[\mathrm{T}^3, \mathrm{Gr}_1(\mathbb{C}^2)]$. This set has been determined in [AK10]:

$$\begin{aligned}[\mathrm{T}^3, \mathrm{Gr}_1(\mathbb{C}^2)] = \{(n_0; n_1, n_2, n_3) \mid\, & n_1, n_2, n_3 \in \mathbb{Z}; \\ & n_0 \in \mathbb{Z} \text{ for } n_1 = n_2 = n_3 = 0 \text{ and} \\ & n_0 \in \mathbb{Z}_{2\cdot\gcd(n_1,n_2,n_3)} \text{ otherwise}\},\end{aligned} \tag{7.27}$$

where $\gcd(n_1, n_2, n_3)$ is the greatest common divisor of the integers n_1, n_2 and n_3.

This example demonstrates that invariants may not be independent of each other as in the stable regime. Only for $n_1 = n_2 = n_3 = 0$ do we find $n_0 \in \mathbb{Z} = [\mathrm{S}^3, \mathrm{Gr}_1(\mathbb{C}^2)] \subset [\mathrm{T}^3, \mathrm{Gr}_1(\mathbb{C}^2)]$. In all other cases, the range for n_0 is finite.

Example 7.8. The setting of this next example is the one introduced in Section 4.1. Recall that this example is set in complex symmetry class A as above, but with the additional symmetry $T \circ I$ (combination of time-reversal and inversion with $T^2 = I^2 = 1$). For this setup, Theorem 7.1 still applies by using the $\mathbb{Z}_2$-action of real symmetry class $s = 6$ (class AI) on the target space and the trivial $\mathbb{Z}_2$-action on all domains. Hence, for large values of p and q, the set of topological phases (without defect) in two dimensions is given by

$$\begin{aligned}[\mathrm{T}^2, \mathrm{Gr}_p(\mathbb{R}^{p+q})] &= \pi_2(\mathrm{Gr}_p(\mathbb{R}^{p+q})) \times \big(\pi_1(\mathrm{Gr}_p(\mathbb{R}^{p+q})) \times \pi_1(\mathrm{Gr}_p(\mathbb{R}^{p+q}))\big) \\ &= \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2).\end{aligned}$$

For dimensions greater than $\min(p, q)$, we find ourselves outside the stable regime (see the table in Theorem 5.1). This is in particular the case for two dimensions with $p = 1$ and $q = 3$, where we have

$$[\mathrm{S}^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{N}_0, \tag{7.28}$$

$$[\mathrm{S}^1, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{Z}_2. \tag{7.29}$$

Elements in the first set are classified by the absolute value of their skyrmion number, which is defined as follows: Consider the fiber bundle $\mathbb{Z}_2 \hookrightarrow \mathrm{S}^2 \to \mathrm{Gr}_1(\mathbb{R}^3)$, where the projection assigns to a point $x \in \mathrm{S}^2 \subset \mathbb{R}^3$ the line passing through the origin and x. From the associated exact sequence, it follows immediately that the projection induces an isomorphism $\pi_2(\mathrm{S}^2) \simeq \pi_2(\mathrm{Gr}_1(\mathbb{R}^3))$. Since $\pi_2(\mathrm{S}^2) = \mathbb{Z}$ (classified by the mapping degree, see eq. (6.15)), we conclude that $\pi_2(\mathrm{Gr}_1(\mathbb{R}^3)) = \mathbb{Z}$. We say a map $\mathrm{S}^2 \to \mathrm{Gr}_1(\mathbb{R}^3)$ has skyrmion number n if it represents a class in $\pi_2(\mathrm{Gr}_1(\mathbb{R}^3))$ originating from a class in $\pi_2(\mathrm{S}^2)$ with mapping degree n. The fact that only the absolute value of the skyrmion number is a homotopy invariant is explained by applying Lemma 3.22 to obtain

$$[\mathrm{S}^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \pi_2(\mathrm{Gr}_1(\mathbb{R}^3))/\pi_1(\mathrm{Gr}_1(\mathbb{R}^3)) = \mathbb{Z}/\mathbb{Z}_2 = \mathbb{N}_0. \tag{7.30}$$

The result of (7.29) has been discussed in the context of eq. (4.5): The single non-trivial class is represented by the Moebius bundle.

With momentum space T^2, the topological phases are given by [Jän87, BSH99, Che12]

$$\begin{aligned}[\mathrm{T}^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \{(n_0; n_1, n_2) \mid\ & n_1, n_2 \in \mathbb{Z}_2; \\ & n_0 \in \mathbb{N} \text{ for } n_1 = n_2 = 0 \text{ and} \\ & n_0 \in \mathbb{Z}_2 \text{ otherwise}\}.\end{aligned} \tag{7.31}$$

Again, only if the lower-dimensional invariants n_1 and n_2 vanish does the invariant n_0 have full range.

From these two examples we take the lesson that, in general, we can only hope that homotopy classes of maps from spheres will remain distinct as maps over a torus of the same dimension. In this way, the results of Chapter 6 would still be valid in the physically more relevant case with torus as a domain, with the slight drawback of giving only a partial answer. This hope turns out to be justified, as we demonstrate in the following theorem:

Theorem 7.9. *The sets of topological phases without and with defect have subsets*

$$[\mathrm{S}^{0,d}, C_s(n)]^{\mathbb{Z}_2} \subset [\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2} \tag{7.32}$$

and

$$[\mathrm{S}^{d_x,d_k}, C_s(n)]^{\mathbb{Z}_2} \subset [\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, C_s(n)]^{\mathbb{Z}_2}. \tag{7.33}$$

Both inclusions are defined by relaxing the appropriate boundary conditions for representatives $\mathrm{I}^d \to C_s(n)$ (resp. $\mathrm{I}^{d_x+d_k} \to C_s(n)$) on the left hand sides.

This theorem allows for a more general definition of the attributes strong and weak.

Definition 7.10. *A topological phase is called* strong in the general sense *if it is non-trivial and contained in the image of one of the maps in Theorem 7.9. Otherwise, it is called* weak in the general sense.

The statement of Theorem 7.9 translates to the following: If there is no homotopy between two equivariant maps $\mathrm{S}^{0,d} \to C_s(n)$ (resp. $\mathrm{S}^{d_x,d_k} \to C_s(n)$) through equivariant maps with the same domain, then there cannot be a homotopy through maps with domain T^d (resp. $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$). Put differently, allowing homotopies through maps that obey less strict boundary conditions on I^{d_x,d_k} does not result in less homotopy classes. The difficulty of proving this statement directly is illustrated in Figure 7.3.

Recall from Section 3.2 that we can model all maps from products of spheres with arbitrary $\mathbb{Z}_2$-actions as maps from a cube of the appropriate dimension (and with the appropriate $\mathbb{Z}_2$-action) and certain boundary conditions. We make use of this model in the following and use the simple notation I^d since all $\mathbb{Z}_2$-actions will be covered at once. Similarly, we use $\Omega^d Y$ and $L^d Y$ for the based and free loop spaces of Y without specifying which coordinates are equipped with a non-trivial $\mathbb{Z}_2$-action.

In order to prove Theorem 7.9, we need to investigate some properties of Lemma 3.22 in the case where the domain is a sphere. If a $\mathbb{Z}_2$-space Y has a path-connected fixed point set $Y^{\mathbb{Z}_2}$, then Lemma 3.22 gives a bijection

$$[\mathrm{S}^d, Y]^{\mathbb{Z}_2} \simeq [\mathrm{S}^d, Y]_*^{\mathbb{Z}_2} / \pi_1(Y^{\mathbb{Z}_2}). \tag{7.34}$$

If $Y^{\mathbb{Z}_2}$ has multiple connected components, we denote by $Y_0^{\mathbb{Z}_2}$ the component containing the base point. Then a modified version of the above bijection holds:

$$[(\mathrm{S}^d, s_0), (Y, Y_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \simeq [\mathrm{S}^d, Y]_*^{\mathbb{Z}_2} / \pi_1(Y^{\mathbb{Z}_2}), \tag{7.35}$$

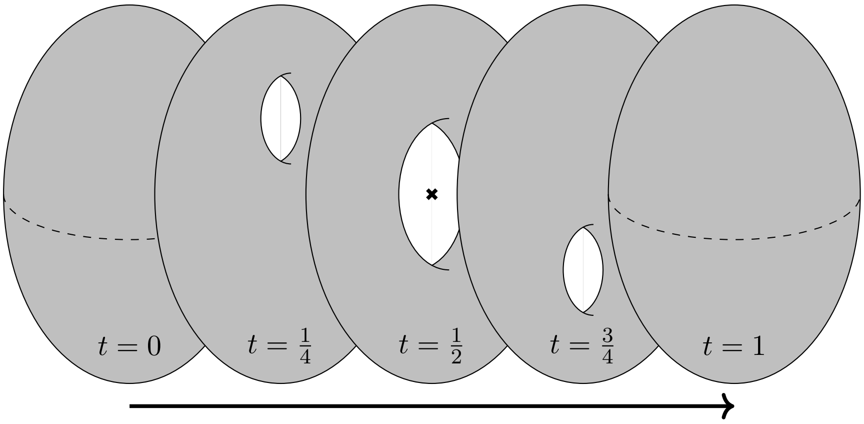


Figure 7.3.: Homotopy h_t between maps from S^2 ($t = 0$ and $t = 1$) through maps from T^2 ($t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$) in $\mathbb{R}^3$ with a point removed (black cross in the middle). The homotopy uses the hole of T^2 and it is not obvious how a homotopy through maps from S^2 can be constructed from it canonically.

where the left hand side stands for homotopy classes of equivariant maps $\mathrm{S}^d \to Y$ sending the base point into $Y_0^{\mathbb{Z}_2}$. This amounts to replacing $Y^{\mathbb{Z}_2}$ by its connected component $Y_0^{\mathbb{Z}_2}$ and therefore follows from Lemma 3.22.

The identifications (7.34) and (7.35) have a simple geometrical interpretation: Points on the boundary of $\mathrm{S}^d = [-\pi, \pi]^d$ are always fixed under the $\mathbb{Z}_2$-action and therefore have to map to $Y^{\mathbb{Z}_2}$. A loop γ representing an element in $\pi_1(Y^{\mathbb{Z}_2})$ now acts on a representative f of a class in $[\mathrm{S}^d, Y]_*^{\mathbb{Z}_2}$ by moving the image point of the boundary along γ to give a map $b_d(\gamma, f) : \mathrm{S}^d \to Y$ (see Figure 7.4). In formulas,

$$b_d(\gamma, f)(x) := \begin{cases} f(2x) & \text{for } |x| \leq \frac{\pi}{2} \\ \gamma(3\pi - 4|x|) & \text{for } |x| > \frac{\pi}{2}, \end{cases} \tag{7.36}$$

where $|x| := \max(x_i)_{i=1\ldots d}$.

Although defined on the level of representatives, eq. (7.36) yields a well-defined action on the level of homotopy classes and the orbit of this action is identified on the right hand side of (7.34). In the following special case, the map b_d simplifies considerably, which will be important for the proof of Theorem 7.9:

Lemma 7.11. *For* $[\gamma] \in \pi_1((LY)^{\mathbb{Z}_2})$ *and* $[f] \in [\mathrm{S}^d, \Omega Y]_*^{\mathbb{Z}_2}$,

$$[b_d(\gamma, f)] = [b_{d+1}(\gamma(\cdot)(0), f)] \text{ in } [\mathrm{S}^d, LY]_*^{\mathbb{Z}_2}, \tag{7.37}$$

where the first coordinate of γ *is the loop coordinate of* π_1 *and* f *is interpreted as a map* $\mathrm{S}^{d+1} \to Y$ *on the right hand side.*

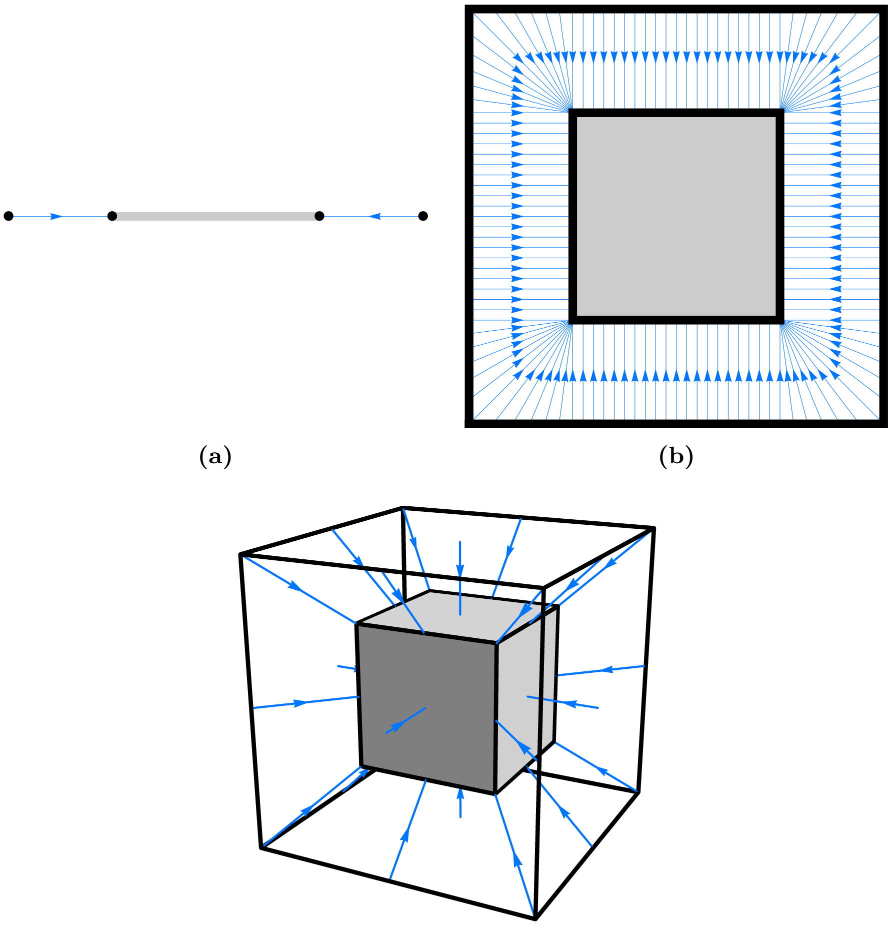


Figure 7.4.: The domain of $b_d(\gamma, f)$ for (a) $d = 1$, (b) $d = 2$ and (c) $d = 3$. The loop γ is represented in blue with an arrow indicating the direction in which it is traversed and the domain of f is depicted in gray. In (a) and (b), black points are mapped to the base point $y_0 \in Y^{\mathbb{Z}_2}$. In (c), the entire surfaces of the two cubes are mapped to y_0.

Proof. The map γ is a based loop of free loops with base point being the constant loop at $y_0 \in Y$. Alternatively, it may be viewed as a free loop of based loops by switching the two loop coordinates. The latter interpretation is shown in Figure 7.5a for $d = 1$, where lines with arrows represent based loops. The fact that this is a free loop of based loops is indicated by the color code: All these loops may be different, but there are periodic boundary conditions (the most upper based loop is the same as the lowest one, both being shown in orange). The goal of the present proof is to construct homotopies in order to arrive at the picture in Figure 7.5d which illustrates the right hand side of eq. (7.37) for $d = 1$: The argument "0" in $\gamma(\cdot)(0)$ is reflected in the fact that all loops are the same (depicted in blue) and the origin of the increased index in b_{d+1} (as opposed to b_d) is the fact that this loop surrounds the domain of f, in contrast to the initial picture in Figure 7.5a (c.f. the difference between Figure 7.4a and Figure 7.4b).

The map $b_d(\gamma, f)(\cdot, \pm\pi)$ is homotopic to $f(\cdot)(\pm\pi)$, since $f(x)(\pm\pi) = y_0$ and the action fixes the neutral element. This can be seen in Figure 7.5a for $d = 1$: The upper and lower boundaries correspond to the concatenation of the based loop $\gamma(\cdot)(\pm\pi)$ (orange), the constant loop $f(\cdot)(\pm\pi)$ (black) and the reversed version of $\gamma(\cdot)(\pm\pi)$ (orange, reversed arrow). This combination is clearly homotopic to the constant loop and this homotopy is used to arrive at Figure 7.5b.

For the next homotopies, the central part of the cube $[-\pi, \pi]^{d+1}$ which is associated with f (gray area in Figure 7.5) will remain invariant. The surrounding part is equivalent to a map $\mathrm{S}^d \to \Omega Y$, but since we will only use special homotopies that leave the part with last coordinate $x_{d+1} = 0$ in $[-\pi, \pi]^{d+1}$ invariant (the blue loops in Figure 7.5b), we will restrict to only one hemisphere of S^d, which is a disk D^d. The same homotopies will be applied to the other hemisphere.

We introduce the radial coordinate $0 \leq r \leq 1$ of D^d, which corresponds to $x_1 = \cdots = x_d = 0$ at $r = 0$ and to $x_{d+1} = 0$ at $r = 1$. The result of using the homotopy of $b_d(\gamma, f)(\cdot, \pm\pi)$ to the constant map is a map $\alpha_0 : \mathrm{D}^d \to \Omega Y$ depicted for $d = 1$ in Figure 7.5b and given in general by

$$\alpha_0(r) := \begin{cases} \gamma(\pi) & \text{for } r \leq \frac{1}{2} \\ \gamma(2\pi(1-r)) & \text{for } r > \frac{1}{2}, \end{cases} \tag{7.38}$$

The next step takes the form of a homotopy

$$\alpha_t(r) := \begin{cases} \gamma(\pi) & \text{for } r \leq \frac{1-t}{2} \\ \gamma(\frac{2\pi}{1+t}(1-r)) & \text{for } r > \frac{1-t}{2}, \end{cases} \tag{7.39}$$

where $0 \leq t \leq 1$. For $d = 1$, this is the step from Figure 7.5b to Figure 7.5c: The former shows $\alpha_0 : \mathrm{D}^1 \to \Omega Y$, which maps to the orange loop at $r = 0$ and to the blue loop at $r = 1$. The homotopy α_t pushes the orange region completely to $r = 0$ while "stretching" the remainder accordingly, which results in α_1 shown in Figure 7.5c.

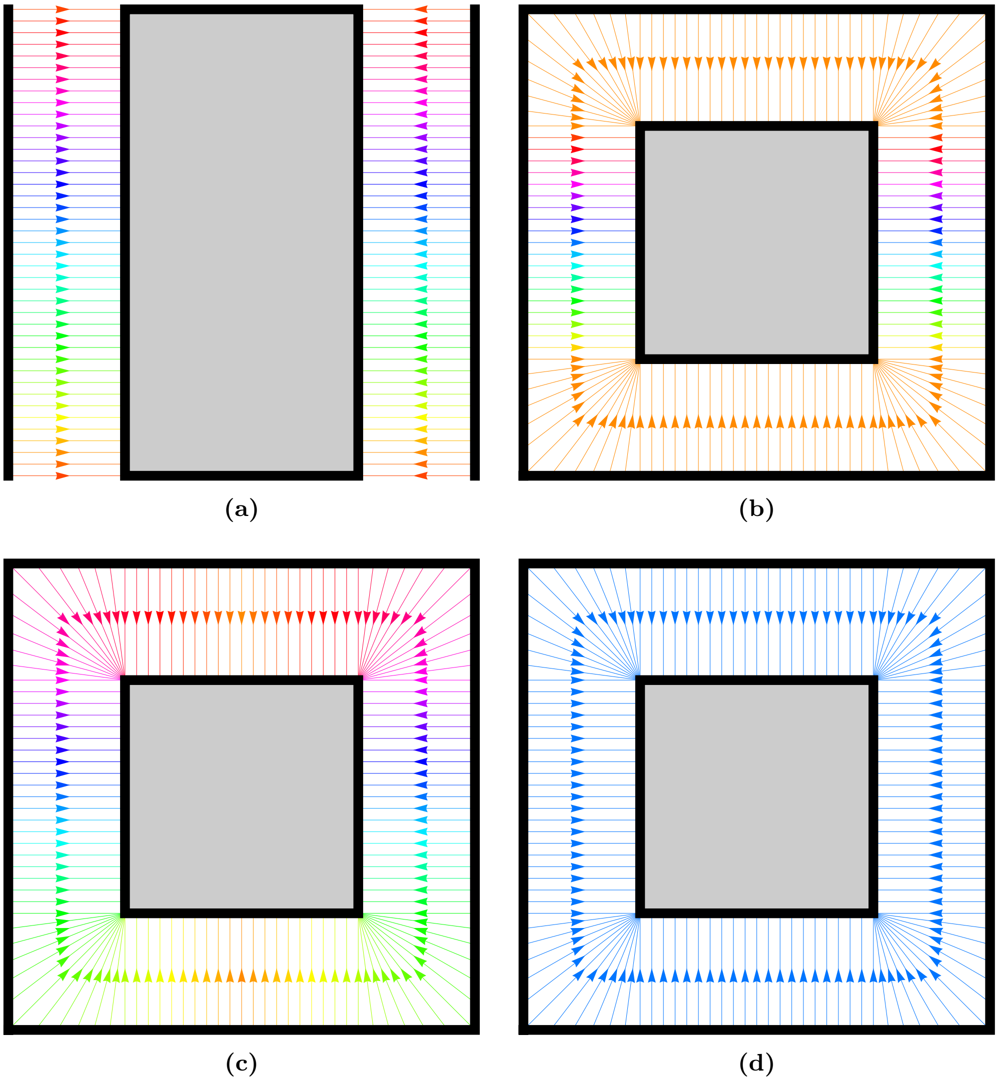


Figure 7.5.: Steps in the proof of Lemma 7.11 for $d = 1$. The gray area corresponds to the domain of $f : \mathrm{S}^1 \to \Omega Y$ interpreted as a map $\mathrm{S}^2 \to Y$. All black lines are mapped to the base point of Y. (a) shows the domain of $b_1(\gamma, f)$, in this case given by conjugation of f by $\gamma : \mathrm{S}^1 \to (LY)^{\mathbb{Z}_2}$. The latter can be viewed as a free loop of based loops (colored lines) and arrows indicate the direction in which the based loops are traversed. (b) shows the result of applying the homotopy of the upper and lower sides to the constant map, giving the configuration with α_0. The stage at $\alpha_1 = \beta_0$ is shown in (c), while (d) depicts the final configuration with β_1, which corresponds to the domain of $b_2(\gamma(\cdot)(0), f)$.

Subsequently, all other loops are also pushed to $r = 0$ and "annihilate", leaving only the blue one. In formulas, this second homotopy is given by

$$\beta_t(r) := \gamma(\pi(1-r)(1-t)), \tag{7.40}$$

where $\beta_0 = \alpha_1$. Since all $\mathbb{Z}_2$-actions introduced for $\mathrm{I}^{d+1} = [-\pi, \pi]^{d+1}$ fix the radial coordinate r and at the same time all homotopies depend only on this coordinate r, they all go through equivariant maps. □

For the next result, we use Lemma 7.11 to show that the homotopy classes of maps with periodic boundary conditions in one coordinate of $[-\pi, \pi]^d$ include the classes of maps that map to a fixed point at the edges of that interval.

Lemma 7.12.

$$[(\mathrm{S}^d, s_0), (LY, (LY)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \supset [(\mathrm{S}^{d+1}, s_0), (Y, Y_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \tag{7.41}$$

Proof.

$$\begin{aligned}
[(\mathrm{S}^d, s_0), (LY, (LY)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} &= [\mathrm{S}^d, LY]_*^{\mathbb{Z}_2} / \pi_1((LY)^{\mathbb{Z}_2}) & (7.42)\\
&= [\mathrm{S}^1, \Omega^d Y]^{\mathbb{Z}_2} / \pi_1((LY)^{\mathbb{Z}_2}) & (7.43)\\
&\supset [(\mathrm{S}^1, s_0), (\Omega^d Y, (\Omega^d Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} / \pi_1((LY)^{\mathbb{Z}_2}) & (7.44)\\
&= \Big([\mathrm{S}^1, \Omega^d Y]_*^{\mathbb{Z}_2} / \pi_1((\Omega^d Y)^{\mathbb{Z}_2})\Big) / \pi_1((LY)^{\mathbb{Z}_2}) & (7.45)\\
&= [\mathrm{S}^1, \Omega^d Y]_*^{\mathbb{Z}_2} / \pi_1(Y^{\mathbb{Z}_2}) & (7.46)\\
&= [(\mathrm{S}^{d+1}, s_0), (Y, Y_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} & (7.47)
\end{aligned}$$

This chain of equalities and inclusions needs some explanation: We first use the relation (7.35) between based and unbased homotopy classes to arrive at (7.42). Then, for eq. (7.43), the perspective is changed to viewing the (free) loop parameter of LY as the domain and the d coordinates of S^d as the domain of elements in $\Omega^d Y$. Importantly, this effects a change from based homotopy classes to unbased ones. The inclusion (7.44) is well defined on the quotient since $(\Omega^d Y)_0^{\mathbb{Z}_2}$ is fixed under conjugation by elements in $(LY)^{\mathbb{Z}_2}$. Having arrived at (7.45) by again using (7.35), we use Lemma 7.11 to find a homotopy of the action of elements in $[\mathrm{S}^1, (\Omega^d Y)^{\mathbb{Z}_2}]_*$ as well as $\pi_1((LY)^{\mathbb{Z}_2})$ to the action of some element in $\pi_1(Y^{\mathbb{Z}_2})$, yielding (7.46). In the last step, we use (7.35) again to complete the proof. □

Proof of Theorem 7.9. We have now accumulated the necessary ingredients in order to prove Theorem 7.9: For the case without defects, if $Y^{\mathbb{Z}_2}$ is connected,

$$\begin{aligned}
[\mathrm{T}^d, Y]^{\mathbb{Z}_2} &= [\mathrm{S}^1, L^{d-1}Y]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^1, s_0), (L^{d-1}Y, (L^{d-1}Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^2, s_0), (L^{d-2}Y, (L^{d-2}Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset \cdots \\
&\supset [(\mathrm{S}^{d-1}, s_0), (LY, (LY)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^d, s_0), (Y, Y_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&= [\mathrm{S}^d, Y]^{\mathbb{Z}_2}
\end{aligned} \tag{7.48}$$

If $Y^{\mathbb{Z}_2}$ has several components $Y_n^{\mathbb{Z}_2}$, we repeat the above steps for different base points $y_0 \in Y_n^{\mathbb{Z}_2}$ to obtain

$$\begin{aligned}
[\mathrm{T}^d, Y]^{\mathbb{Z}_2} &= \coprod_n [(\mathrm{T}^d, s_0), (Y, Y_n^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset \coprod_n [(\mathrm{S}^d, s_0), (Y, Y_n^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&= [\mathrm{S}^d, Y]^{\mathbb{Z}_2}
\end{aligned} \tag{7.49}$$

In the presence of defects, similar steps lead to the result of Theorem 7.9. Assuming again that $Y^{\mathbb{Z}_2}$ is connected,

$$\begin{aligned}
[\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}, Y]^{\mathbb{Z}_2} &= [\mathrm{S}^{d_x}, L^{d_k}Y]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^{d_x}, s_0), (L^{d_k}Y, (L^{d_k}Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^{d_x,1}, s_0), (L^{d_k-1}Y, (L^{d_k-1}Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^{d_x,2}, s_0), (L^{d_k-2}Y, (L^{d_k-2}Y)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset \cdots \\
&\supset [(\mathrm{S}^{d_x,d_k-1}, s_0), (LY, (LY)_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&\supset [(\mathrm{S}^{d_x,d_k}, s_0), (Y, Y_0^{\mathbb{Z}_2})]^{\mathbb{Z}_2} \\
&= [\mathrm{S}^{d_x,d_k}, Y]^{\mathbb{Z}_2}
\end{aligned} \tag{7.50}$$

By the same argument as in (7.49), the result generalizes to fixed point sets $Y^{\mathbb{Z}_2}$ with multiple connected components by repeating the above for base points in all different components. This completes the proof of Theorem 7.9. □

7.3. Stacked IQPVs

The primary goal of introducing the distinction between strong and weak topological phases is to grasp the dimensionality of a given topological phase. This is motivated by the fact that, given a d-dimensional IQPV in a non-trivial topological phase, there are infinitely many realizations of IQPVs in dimensions greater than d produced simply by stacking the original IQPV. For instance, a one-dimensional IQPV can be stacked in two linearly independent ways into two dimensions and three linearly independent ways into three dimensions, etc. Similarly, a two-dimensional IQPV can be extended to a layered three-dimensional IQPV in three linearly independent directions.

In this section, we demonstrate at the hand of two examples the following statements:

(i) Definition 7.10 gives the maximal set of strong topological phases if we require that strong topological phases cannot be realized by stacking lower-dimensional IQPVs.

(ii) There are also weak topological phases (in both stable and general sense) that cannot be realized by stacking lower-dimensional IQPVs.

We begin by formalizing the notion of stacked IQPVs. For simplicity, we start from the setting of Section 2.1 (corresponding to complex symmetry class A). Recall from eq. (2.5) that a general translation-invariant Hamiltonian in this setting acts as

$$H|\mathbf{x}, i\rangle = \sum_{\mathbf{y},j} h_{ji}(\mathbf{y})|\mathbf{x}+\mathbf{y}, j\rangle, \tag{7.51}$$

with $h_{ji}(\mathbf{y}) = \overline{h_{ij}(-\mathbf{y})}$ to ensure hermiticity, $i, j = 1, \ldots, n$ and $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$. Changing to a an eigenbasis of translations, we obtain the Bloch Hamiltonian (see eq. (2.9))

$$H(\mathbf{k}) := \sum_{\mathbf{y}} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}} h(\mathbf{y}), \tag{7.52}$$

with $\mathbf{k} \in \mathrm{T}^d$.

We now view the d-dimensional lattice $\mathbb{Z}^d$ as being embedded into another lattice $\mathbb{Z}^D$ in a higher dimension $D > d$. In eq. (7.51), a canonical embedding is given by letting $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^D$ and setting $h_{ji}(\mathbf{y}) = 0$ whenever $y_i \neq 0$ for $i = d+1, \ldots, D$. Physically, this signifies no hopping of fermions into the new $D - d$ directions or, equivalently, that the system is stacked into these directions.

To generalize the stacking direction, we introduce an invertible, integer D-by-D matrix $A \in \mathrm{GL}_D(\mathbb{Z})$ and define the stacked Hamiltonian to be given by the replacement $h_{ji}(\mathbf{y}) \mapsto h_{ji}(A^{-1}\mathbf{y})$, corresponding to changing the hopping from the $\mathbf{y}$-direction to the $A\mathbf{y}$-direction.

Defining the projection $P : T^D \to T^d$ by $P(k_1, \ldots, k_D) := (k_1, \ldots, k_d)$, the Bloch Hamiltonian of the stacked system can be expressed by the lower-dimensional Bloch

Hamiltonian:

$$\begin{aligned} H_{\text{stack}}(\mathbf{k}) &= \sum_{\mathbf{y}\in\mathbb{Z}^D} h(A^{-1}\mathbf{y})e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}} \\ &= \sum_{\mathbf{y}\in\mathbb{Z}^D} h(\mathbf{y})e^{-\mathrm{i}\mathbf{k}\cdot(A\mathbf{y})} \\ &= \sum_{\mathbf{y}\in\mathbb{Z}^D} h(\mathbf{y})e^{-\mathrm{i}(A^T\mathbf{k})\cdot\mathbf{y}} \\ &= \sum_{\mathbf{y}\in\mathbb{Z}^d} h(\mathbf{y})e^{-\mathrm{i}(PA^T\mathbf{k})\cdot\mathbf{y}} \\ &= H(PA^T\mathbf{k}). \end{aligned} \tag{7.53}$$

The change in **k**-dependence descends to the level of IQPVs. Therefore, given an IQPV $\psi : \mathrm{T}^d \to C_s(n)$, stacking it in D dimensions according to the matrix A yields an IQPV

$$\psi_{\text{stack}}(\mathbf{k}) = \psi(PA^T\mathbf{k}), \tag{7.54}$$

with $\mathbf{k} \in \mathrm{T}^D$.

An important diagnostic is the following: Since A^T is invertible and the projection P has a $(D-d)$-dimensional kernel, there are exactly $D-d$ linearly independent directions in the Brillouin zone T^D in which ψ is constant.

7.3.1. Stacked skyrmions

We begin by investigating Example 7.8 in more detail in order to explain why the product formula in Theorem 7.9 fails and to show that only the strong topological phases (in the general sense of Definition 7.10) do not have stacked representatives. The result stated in eq. (7.31) can be derived following the more general procedure outlined presently, which uses the free loop fibration introduced in the proof of Theorem 7.1, but with trivial $\mathbb{Z}_2$-actions. Denoting by $(LY)_n$ the n-th connected component of the free loop space LY, the set $[\mathrm{T}^2, Y]$ is a disjoint union of subsets labeled by the pair (n_1, n_2), which contain classes whose representatives restrict to $(LY)_{n_1}$ on $\mathrm{S}^1 \times \{s_0\} \subset \mathrm{T}^2$ and to $(LY)_{n_2}$ on $\{s_0\} \times \mathrm{S}^1 \subset \mathrm{T}^2$. Notice that the number of elements in a sector (n_1, n_2) is the same as in (n_2, n_1).

Let $p : LY \to Y$ be the evaluation map of the free loop fibration with trivial $\mathbb{Z}_2$-action. Then the number of elements in a subset (n_1, n_2) can be determined by computing $[\mathrm{S}^1, (LY)_{n_1}]$ and counting the elements that map to $(LY)_{n_2}$ under the induced map p_*. In our example with $Y = \mathrm{Gr}_1(\mathbb{R}^3)$, the free loop space LY has two connected components, since

$$[\mathrm{S}^1, \mathrm{Gr}_1(\mathbb{R}^3)] = \pi_1(\mathrm{Gr}_1(\mathbb{R}^3)) = \mathbb{Z}_2. \tag{7.55}$$

Note that Lemma 3.22 implies that based and free homotopy classes agree in this case, since π_1 is Abelian and therefore the action on itself by conjugation is trivial. We denote by $(LY)_0$ the component containing the constant map and by $(LY)_1$ the component containing all non-trivial loops. The set we will study in the following is $[\mathrm{S}^1, (LY)_1]$, which is the union of sectors of the form $(n_1, 1)$ or, equivalently, of the sectors $(1, n_2)$, with $n_1, n_2 \in \{0, 1\}$. If Theorem 7.1 were applicable in the present setting, all four sectors would satisfy $(n_1, n_2) = \mathbb{N}_0$, so in particular, any union of two sectors would have to contain infinitely many elements. We will show that $[\mathrm{S}^1, (LY)_1]$ contains only finitely many elements and thus confirm that Theorem 7.1 can only hold in the stable regime.

We will show later that $\pi_1((LY)_1)$ is Abelian and therefore Lemma 3.22 can be used to obtain

$$[\mathrm{S}^1, (LY)_1] = \pi_1((LY)_1). \tag{7.56}$$

Choosing a base point in $(LY)_1$, the long exact sequence associated to the free loop fibration contains the right hand side of the above equation and reads

$$\begin{array}{ccccccccc} \pi_2(Y) & \xrightarrow{\partial_2} & \pi_1((\Omega Y)_1) & \xrightarrow{i_*} & \pi_1((LY)_1) & \xrightarrow{p_*} & \pi_1(Y) & \xrightarrow{\partial_1} & \pi_0((\Omega Y)_1) \\ \| & & \| & & & & \| & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & \mathbb{Z}_2 & & 0 \end{array}$$

This exact sequence is *not* split like the one with a base point in $(LY)_0$ in eq. (7.6). Since the first map ∂_2 is *not* the constant map as in the split case, but rather multiplication by -2 [BSH99], exactness implies that $\pi_1((LY)_1)$ must be a group with exactly four elements. This leaves only the possibilities $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ and in either case, it is an Abelian group as previously claimed and therefore $[\mathrm{S}^1, (LY)_1]$ also contains only four elements.

The other point, that all phases not captured in the strong subgroup as defined in Theorem 7.9 have stacked representatives in the present example, is explained by the fact that $\pi_1((LY)_1) = \mathbb{Z}_4$ rather than $\mathbb{Z}_2 \times \mathbb{Z}_2$ [Jän87, BSH99]. If $\psi : \mathrm{S}^1 \to \mathrm{Gr}_1(\mathbb{R}^3)$ is a non-trivial topological insulator in one dimension, i.e. represents the non-trivial class in $\pi_1(\mathrm{Gr}_1(\mathbb{R}^3)) = [\mathrm{S}^1, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{Z}_2$, then the generator of $\pi_1((LY)_1) = \mathbb{Z}_4$ is represented by $\psi(k_1 + k_2)$, where k_1 is the coordinate associated to π_1 and k_2 is the free loop coordinate. Since the group structure in π_1 is concatenation of loops (see eq. (3.9)), all elements in $\mathbb{Z}_4$ are represented respectively by one of

$$\psi(mk_1 + k_2), \tag{7.57}$$

with $m = 0, 1, 2, 3$. These configurations are illustrated in Figs. 7.6b ($m = 0$), 7.6d ($m = 2$) and 7.6f ($m = 3$). The ones with even m belong to the sector $(1, 0)$, while the ones with odd m belong to the sector $(1, 1)$. All of these maps correspond to

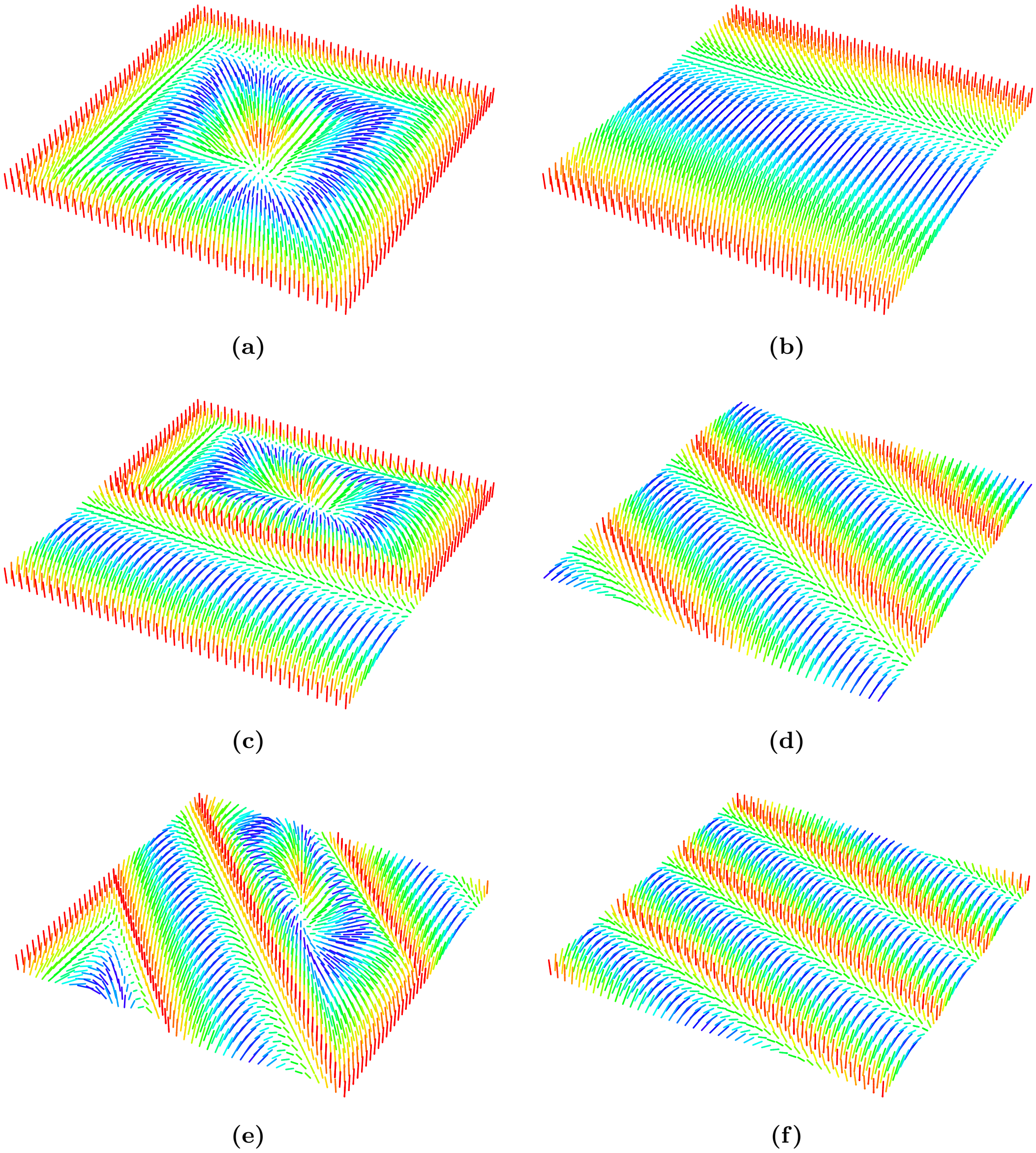


Figure 7.6.: Maps $T^2 \to \mathrm{Gr}_1(\mathbb{R}^3)$ visualized by placing the image of a point (a line in $\mathbb{R}^3$) on the point itself. T^2 is modeled here as a square with periodic boundary conditions. Colors represent the angle to the axis out of the plane. Using the notation $(n_0; n_1, n_2)$ as in eq. (7.31), (a) corresponds to $(1; 0, 0)$, (b) to $(0; 1, 0)$, (c) and (d) to $(1; 1, 0)$ and (e) and (f) to $(1; 1, 1)$. Remarkably, all except (a) are homotopic to stacked one-dimensional phases.

the one-dimensional non-trivial IQPV stacked along the $(-1, m)$-direction of the two-dimensional lattice $\mathbb{Z}^2$.

The above implies that the sectors $(1,1)$, $(1,0)$ and therefore also the sector $(0,1)$ contain two elements, all of which have stacked representatives. Together with the result of Theorem 7.9, which can be interpreted as stating that the sector $(0,0)$ is in bijection with $[\mathrm{S}^2, \mathrm{Gr}_1(\mathbb{R}^3)] = \mathbb{N}_0$, the result shown in eq. (7.31) follows. Of the sector $(0,0)$ only the class of the constant map has a stacked representative. Thus, the only topological phases which cannot be realized by stacking are the non-trivial elements in $\mathbb{N}_0 = [\mathrm{S}^2, \mathrm{Gr}_1(\mathbb{R}^3)] \subset [\mathrm{T}^2, \mathrm{Gr}_1(\mathbb{R}^3)]$.

7.3.2. Weak but not stackable

The following is an example – in the stable regime – of a weak topological insulator (in both the stable and the general sense), which cannot be constructed through stacking: In two dimensions, consider a $4n$-band model with $2n$ occupied and $2n$ empty bands in complex symmetry class *A*III (see Section 2.5.1). Let there be an additional U_1-symmetry, for example conservation of a spin component, which commutes with the single pseudo-symmetry. Then all IQPVs are given by maps

$$\psi : \mathrm{T}^2 \to \mathrm{U}_n \times \mathrm{U}_n, \tag{7.58}$$

which can be viewed as two separate IQPVs as explained in Section 2.6.1. This view carries over to the set of topological phases, since they split as

$$[\mathrm{T}^2, \mathrm{U}_n \times \mathrm{U}_n] = [\mathrm{T}^2, \mathrm{U}_n] \times [\mathrm{T}^2, \mathrm{U}_n]. \tag{7.59}$$

Therefore, we may apply Theorem 7.1 to each factor separately. We know from Table 4.1 that

$$\pi_2(\mathrm{U}_n) = 0, \tag{7.60}$$
$$\pi_1(\mathrm{U}_n) = \mathbb{Z}, \tag{7.61}$$
$$\pi_0(\mathrm{U}_n) = 0, \tag{7.62}$$

so we can conclude that

$$\begin{aligned}[\mathrm{T}^2, \mathrm{U}_n \times \mathrm{U}_n] &= [\mathrm{T}^2, \mathrm{U}_n] \times [\mathrm{T}^2, \mathrm{U}_n] \\ &= \big(\pi_1(\mathrm{U}_n) \times \pi_1(\mathrm{U}_n)\big) \times \big(\pi_1(\mathrm{U}_n) \times \pi_1(\mathrm{U}_n)\big) \\ &= (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z}). \end{aligned} \tag{7.63}$$

Writing $\psi(\mathbf{k}) = \psi(k_1, k_2) = (\psi_1(k_1, k_2), \psi_2(k_1, k_2)) \in \mathrm{U}_n \times \mathrm{U}_n$, the invariants in eq. (7.63) are given by the winding numbers n_i of $\det(\psi_i(k_1, 0))$ and m_i of $\det(\psi_i(0, k_2))$ for $i = 1, 2$, arranged according to

$$(n_1, n_2) \times (m_1, m_2) \in (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z}). \tag{7.64}$$

One-dimensional versions of this model are classified by $[\mathrm{S}^1, \mathrm{U}_n \times \mathrm{U}_n] = \pi_1(\mathrm{U}_n) \times \pi_1(\mathrm{U}_n) = \mathbb{Z} \times \mathbb{Z}$, with invariants given by the winding numbers n_i of $\det(\psi_i(k))$ with $i = 1, 2$ and $k \in \mathrm{S}^1$. Stacking a representative ϕ of the class (n_1, n_2) according to some matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ yields two-dimensional IQPV (see eq. (7.54))

$$\begin{aligned} \phi_{\text{stack}}(\mathbf{k}) &= \phi(PA^T\mathbf{k}) \\ &= \phi(A_{11}k_1 + A_{21}k_2) \\ &= (\phi_1(A_{11}k_1 + A_{21}k_2), \phi_2(A_{11}k_1 + A_{21}k_2)), \end{aligned} \tag{7.65}$$

representing the topological phase

$$(A_{11}n_1, A_{11}n_2) \times (A_{21}n_1, A_{21}n_2) \in (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z}). \tag{7.66}$$

Not all classes can be of this form, the simplest counter-example being $(1, 0) \times (0, 1)$: For the first and fourth invariant to be non-zero, both n_1 and n_2 would have to be non-zero. However, this would imply that, in order for the second and third invariant to vanish, $A_{11} = A_{21} = 0$, which would in turn lead to an invariant $(0, 0) \times (0, 0)$, giving a contradiction.

The mathematical reason is the fact that $\mathbb{Z} \times \mathbb{Z}$ is generated by two elements rather than only one. In physical terms, if the U_1-symmetry is realized by the conservation of a spin component, then the non-trivial winding for spin up takes place along a linearly independent direction from that of the non-trivial winding for spin down and therefore there is no corresponding one-dimensional system.

8. Physical implications

One of the main outcomes of this thesis is a classification of topological phases $[\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$ as defined in Definition 3.5. We now address the physical properties of IQPVs described by maps $\psi_0 : \mathrm{T}^d \to C_s(n)$ representing a non-trivial topological phase $[\psi_0] \neq [\text{const.}] \in [\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$. By Definition 3.1, each member of a family ψ_t depending continuously on a parameter $t \in [0,1]$ and satisfying the same set of symmetries resides in the same topological phase: $[\psi_t] = [\psi_0]$ for all $t \in [0,1]$. The parameter t can always be associated to a continuous family of Hamilton operators H_t with the same symmetries by assigning the flattened Hamiltonian at every t using eqs. (2.139) and (2.140). As such, it can have a multitude of physical interpretations, like the hopping amplitude or the strength of spin-orbit coupling. However, there can be homotopies of Hamiltonians that do *not* descend to homotopies of IQPVs. This is the case whenever the energy gap between the occupied and empty eigenstates vanishes for some $t_0 \in [0,1]$ and the system is no longer insulating – it becomes a metal. Given a homotopy H_t from H_0 with IQPV ψ_0 to H_1 with IQPV ψ_1, the only way to obtain distinct classes $[\psi_0] \neq [\psi_1]$ is the presence of at least one such value t_0 in order for ψ_{t_0} to be ill defined, allowing a "jump" into another topological phase. Thus, the hallmark of a topological phase transition is the closing of the energy gap.

In the next sections, we discuss two important physical manifestations of the homotopy parameter t.

8.1. Atomic limit

We start in the complex symmetry class A as introduced in Section 2.1. Recall from eq. (2.5) the action of a generic translation-invariant Hamiltonian on a basis $\{|\mathbf{x}, i\rangle\}$ of the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n$:

$$H|\mathbf{x}, i\rangle = \sum_{\mathbf{y};j} h_{ji}(\mathbf{y})|\mathbf{x}+\mathbf{y}, j\rangle,$$

with hopping matrix $h(\mathbf{y})^\dagger = h(-\mathbf{y})$. Defining a homotopy H_t by

$$H_t|\mathbf{x}, i\rangle := \sum_{j} h_{ji}(0)|\mathbf{x}, j\rangle + (1-t) \sum_{\substack{\mathbf{y};j \\ (\mathbf{y}\neq\mathbf{x})}} h_{ji}(\mathbf{y})|\mathbf{x}+\mathbf{y}, j\rangle, \tag{8.1}$$

with $t \in [0,1]$, we recover the original Hamiltonian at $t = 0$ ($H_0 = H$). The endpoint H_1 is called the *atomic limit*. During the homotopy H_t, the original hopping amplitudes of $H_0 = H$ are gradually diminished until at $t = 1$ they vanish completely. One may think of this process as a continuous increase of the inter-atomic separation from the original one at $t = 0$ to infinity at $t = 1$, so that only the atomic n-by-n Hamiltonian $h(0)$ remains for each lattice site, completely oblivious of the other sites.

Repeating the steps outlined in Section 2.1, we arrive at the Bloch Hamiltonian of H_t:

$$H_t(\mathbf{k}) = h(0) + (1-t) \sum_{\mathbf{y} \neq \mathbf{x}} e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}} h(\mathbf{y}). \tag{8.2}$$

For $t = 0$, this expression reduces to the Bloch Hamiltonian $H_0(\mathbf{k}) = H(\mathbf{k})$ of eq. (2.9) and, not surprisingly, there is no momentum dependence in the atomic limit, $H_1(\mathbf{k}) = h(0)$. The latter property descends to the IQPV associated to H_1 and therefore the topological phase $[\text{const.}] \in [\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$ is represented by the IQPV of the atomic limit. Here we fix a chemical potential in order to single out one connected component of $C_s(n)$, so that there is only one topological phase represented by a constant map. The present discussion can be generalized to the setting of Section 2.2 by using the BdG Hamiltonian in eq. (2.53) in place of the one from eq. (2.5) we used above. In the presence of additional symmetries, we require H_t to have these symmetries for all $t \in [0,1]$. This may require a different homotopy than the one given in eq. (8.2), but a homotopy always exists (recall the result of [HHZ05] stating that every Hamiltonian decomposes into blocks each taken from the tangent space of a symmetric space, which is always path-connected).

We have thus found a physical manifestation of non-trivial topological phases: Whenever $[\psi_0] \neq [\text{const.}] \in [\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$, the homotopy H_t to the atomic limit (with H_0 the defining Hamiltonian of ψ_0) must undergo a topological phase transition in the form of a gapless Hamiltonian. Conversely, if $[\psi_0] = [\text{const.}]$, there is always a homotopy to the atomic limit varying exclusively through gapped Hamiltonians. These features are used in [HPB11] to define the terms "topologically trivial" and "topologically non-trivial", agreeing with our definitions. In [HPB11], the entanglement spectrum is used as a diagnostic tool: If it exhibits spectral flow, the atomic limit cannot be reached without a topological phase transition.

8.2. Boundaries

While the process of taking the atomic limit may be assigned the status of a Gedankenexperiment, there are features of topological phases that are more accessible experimentally. Recall the reasoning for introducing the configuration space $\mathrm{S}^{d_x} \times \mathrm{T}^{d_k}$: If there is a defect of codimension $d_x + 1$, then we may enclose it by a sphere S^{d_x}.

If the radius of this sphere is sufficiently large, then we can use the approximation of translation-invariance at every point and treat the dependence on the position on S^{d_x} semi-classically as additional, independent continuous parameters. In a similar fashion, we can describe the crossover between two representatives of the same topological phase in real space if the region of crossover is large enough. This crossover can occur, in particular, to the atomic limit IQPV as shown in the lower part of Figure 8.1. Thus, we have found yet another physical manifestation of the homotopy parameter $t \in [0, 1]$ in the form of the relative position within the crossover region in real space between two IQPVs. If two IQPVs represent different topological phases, we have learned that the energy gap is bound to close (there must be a topological phase transition) if an interpolation between the corresponding Hamiltonians is created. Therefore, in the crossover region there must be at least one gapless state. In the example where one IQPV is that of the atomic limit, the crossover region can be interpreted as a (continuous) boundary of the material as shown in Figure 8.1 and the gapless state is located at this boundary. This relationship between topological phases of the bulk and boundary properties is known as the *bulk-boundary correspondence*. The experimental observation of topological phases has so far been limited to the measurement of precisely these properties. For instance, the non-trivial phase of two-dimensional systems in the real symmetry class $A\mathrm{II}$ (the quantum spin Hall phase) was discovered by transport experiments [KWB$^+$07], which confirmed that conduction only occurred along the rim of the sample. Similarly, in its three-dimensional generalization, photo-emission spectroscopy was used to show that the two-dimensional surface carried gapless states [HQW$^+$08, XQH$^+$09].

In real materials there are no large crossover regions forming continuous boundaries as displayed in the lower part of Figure 8.1, but rather sharp boundaries as shown in the upper part. For this case, more quantitative information is needed about the attribute "sufficiently large" ascribed to the crossover region. In many models, it can be verified numerically that the gapless boundary states persist for sharp boundaries. Under certain circumstances, there are rigorous proofs that this must be so: in symmetry classes A and $A\mathrm{II}$ in $d = 2$, there is a proof in [GP13] for quite general Hamiltonians and in [MS11] for Dirac Hamiltonians. Using a semi-classical approximation for the Green's function akin to the one we introduced here for IQPVs, [EG11] presents an argument for the bulk-boundary correspondence encompassing the other symmetry classes.

However, it is not true that a sharp boundary on a non-trivial IQPV will carry gapless states in general when the semi-classical approximations break down. A counterexample is given in [HPB11] in the form of a non-trivial inversion-symmetric IQPV without gapless boundary states (a slight generalization of the setting in this thesis allows the accommodation of inversion symmetry).

8.3. Interactions and disorder

Although the many-body picture of independent particles as introduced in Section 2.2 seems limited at first glance, it covers a plethora of interacting systems that are well described by non-interacting *quasi*-particles potentially differing from the elementary particles in the microscopic description. In particular, interactions are essential for the concept of superconductivity and are thus included in the mean-field description of Section 2.2. These arguments only account for interacting topological phases that are homotopic to non-interacting ones (in the sense that there is a homotopy through gapped, interacting Hamiltonians) and there are many phases beyond this "weakly interacting" regime, like the fractional quantum Hall phases [Lau99]. It was shown in [MKF13, WPS14] that the bulk-boundary correspondence discussed in the last section needs to be revisited in the realm of strong interactions, with an additional possibility of exotic gapped (rather than gapless) edge theories displaying topological order much like the fractional quantum Hall states.

Another concept seemingly disregarded in this work is the possible presence of disorder, since translation-invariance is assumed from the outset. However, at least in the stable regime of many valence and conduction bands, an argument previously used in [NTW85, LP12, QWZ06] can be made to incorporate disorder into our framework. Let a translation-invariant system with topological phases $[\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$ be given, where n is the number of complex degrees of freedom per unit cell (corresponding to the factor $\mathbb{C}^n$ in the definition of the Hilbert space in eq. (2.1)). We now choose to merge multiple unit cells in order to define a larger unit cell. For instance, we can form a new unit cell containing N^d of the original unit cells in a cube of length N. This amounts to changing n by nN^d in eq. (2.1) defining the Hilbert space. The topological phases with the enlarged unit cell are given by $[\mathrm{T}^d, C_s(nN^d)]^{\mathbb{Z}_2}$ and due to Theorem 5.1 in combination with Theorem 7.1, this set is in bijection with the original set $[\mathrm{T}^d, C_s(n)]^{\mathbb{Z}_2}$ if the latter resides in the stable regime. Thus, an arbitrary amount of disorder repeating with a period of N lattice sites in real space does not alter the topological phase (provided the energy gap remains open). Since N is arbitrary, we can take $N \to \infty$ to remove the restriction on the disorder to be periodic.

Note that the statement derived here guarantees that if, in the stable regime, the disorder is continuously increased starting from the clean system, the topological phase cannot change as long as the energy gap remains open. However, generically even an infinitesimal amount of disorder fills the energy gap (even though the density of states may be small). We can still apply the formalism developed in this thesis since the number of occupied states is the same for all $\mathbf{k} \in \mathrm{T}^d$. In fact, any amount of disorder leaving open a mobility gap rather than an energy gap will leave the topological phase invariant [NTW85, LP12]. For completely general disorder it is still guaranteed that the set of topological phases as a whole remains the same, but the disordered topological phase may not be that of the clean system [LP12].

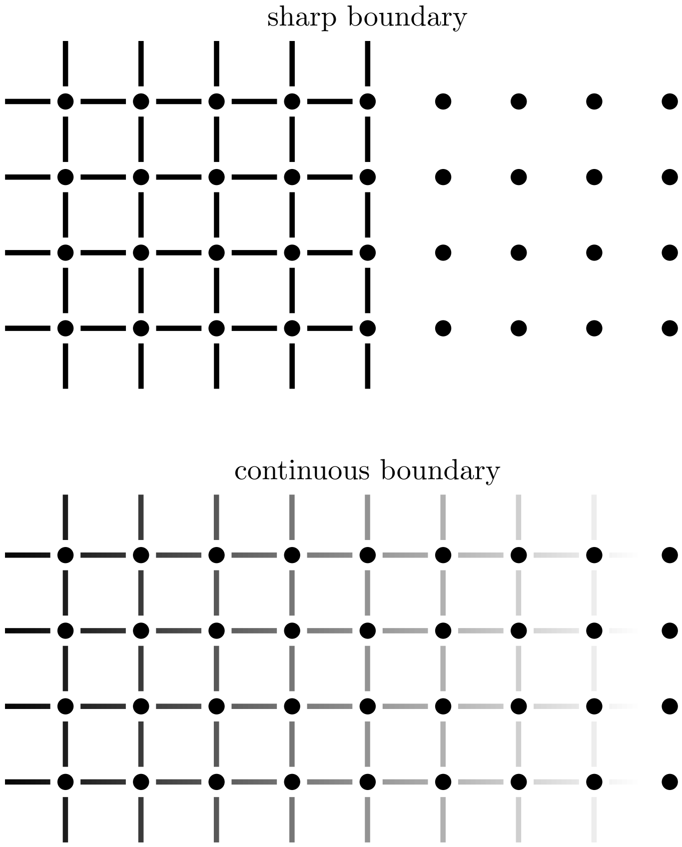


Figure 8.1.: Comparison of sharp and continuous boundaries on a lattice $\mathbb{Z}^2$ (dots) with hopping amplitudes indicated by the color of links (black: full amplitude, white: zero amplitude).

9. Conclusion

In this thesis, we employed the natural notion of homotopy as an equivalence relation defining topological phases. The task of obtaining a broad classification of these phases seems daunting at first given the many variables like dimensions, symmetries and the number of conduction and valence bands. However, if translations are symmetries and all other symmetries commute with them, we showed that the problem reduces to ten symmetry classes as in [HHZ05]. We have organized these ten symmetry classes systematically using pseudo-symmetries which satisfy Clifford algebra relations. Furthermore, we have proved that there are critical numbers of conduction and valence bands above which the set of topological phases stabilizes. For spherical configuration spaces with arbitrary numbers of momentum-like and position-like dimensions, we have classified all topological phases in this stable regime for the ten symmetry classes. While this result can be obtained by more algebraic means using K-theory, we have given an independent homotopy theoretic derivation thereof.

On top of the alternative proof of the known results, we have extended the classification beyond the stable regime. In this endeavor, we have identified the exact boundaries to the stable regime for all ten symmetry classes and determined all exceptions in the case of spherical configuration spaces with up to three exclusively momentum-like dimensions. These exceptions include the Hopf insulator of [MRW08], as well as a newly identified topological phase which we call the $\mathbb{Z}_2$-Hopf superconductor in symmetry class C. We have shown that all these results are also valid when the configuration space is the physically more relevant Brillouin zone torus (or a product of position-like sphere and this torus in the presence of a defect). In fact, in the stable regime, the set of topological phases over the torus splits into a product of topological phases over spheres, so we have given an exhaustive classification in that case. Outside the stable regime, we showed the situation to be more intricate, since there is no such product decomposition. However, we demonstrated that the results with spherical configuration spaces give at least a partial answer to the full classification problem there.

The question of sphere or torus as configuration space is intimately linked to the concept of strong and weak topological insulators. While the latter distinction can be defined in the stable regime using the product decomposition, we showed that outside this regime a modified distinction has to be made in order to avoid strong topological phases being realizable by stacking lower-dimensional systems. However, we demonstrated that independent of which definition for the distinction between strong and weak is used, there can also be weak topological insulators that cannot be stacked.

A. Appendix

A.1. Cubes, disks and spheres

In this appendix we spell out the homeomorphisms between S^d, $\mathrm{I}^d/\partial\mathrm{I}^d$ and $\mathrm{D}^d/\partial\mathrm{D}^d$ leading to the equivalent definitions of the homotopy groups introduced in Chapter 3.

First, defining $\mathrm{I}^d := [-\pi, \pi]^d$ and D^d as the unit ball in $\mathbb{R}^d$ with radius π, we have a homeomorphism

$$\begin{aligned} u : \mathrm{I}^d/\partial\mathrm{I}^d &\to \mathrm{D}^d/\partial\mathrm{D}^d \\ \mathbf{k} &\mapsto k_{\max}\frac{\mathbf{k}}{|\mathbf{k}|} \end{aligned} \tag{A.1}$$

with inverse

$$\begin{aligned} u^{-1} : \mathrm{D}^d/\partial\mathrm{D}^d &\to \mathrm{I}^d/\partial\mathrm{I}^d \\ \mathbf{k} &\mapsto |\mathbf{k}|\frac{\mathbf{k}}{k_{\max}}, \end{aligned} \tag{A.2}$$

where $k_{\max} := \max\{|k_1|, \ldots, |k_d|\}$. Since the boundaries are mapped to each other, these maps are well defined on the corresponding quotient spaces.

The second homeomorphism between will be a composite $r \circ p_d : \mathrm{S}^d \to \mathrm{I}^d/\partial\mathrm{I}^d$, where we view S^d as the unit sphere in $\mathbb{R}^{d+1}$. The first part is the stereographic projection

$$\begin{aligned} p_d : \mathrm{S}^d &\to \mathbb{R}^d \cup \{\infty\} \\ (\mathbf{x}, t) &\mapsto \frac{1}{1-t}\mathbf{x}, \end{aligned} \tag{A.3}$$

with inverse

$$\begin{aligned} p_d^{-1} : \mathbb{R}^d \cup \{\infty\} &\to \mathrm{S}^d \\ \mathbf{y} &\mapsto \frac{1}{1+|\mathbf{y}|^2}(2\mathbf{y}, |\mathbf{y}|^2 - 1). \end{aligned} \tag{A.4}$$

This is followed by a rescaling to the cube I^d:

$$\begin{aligned} r : \mathbb{R}^d \cup \{\infty\} &\to \mathrm{I}^d/\partial\mathrm{I}^d \\ \mathbf{y} &\mapsto \frac{\pi}{1+y_{\max}^2}\mathbf{y}, \end{aligned} \tag{A.5}$$

with inverse

$$\begin{aligned} r^{-1} : \mathrm{I}^d/\partial\mathrm{I}^d &\to \mathbb{R}^d \cup \{\infty\} \\ \mathbf{x} &\mapsto \frac{\pi}{1 - x_{\max}^2}\mathbf{x}. \end{aligned} \tag{A.6}$$

Note that r maps ∞ to the boundary $\partial\mathrm{I}^d$ (and vice versa for r^{-1}), so it descends to the quotient $\mathrm{I}^d/\partial\mathrm{I}^d$.

A.2. Four-Lemmas and Five-Lemma

Lemma A.1 (First Four-Lemma)**.** *Let there be groups X_i and Y_i and homomorphisms $f_i : X_i \to Y_i$ $(i = 1, 2, 3, 4)$ forming the following commutative diagram with exact rows:*

$$\begin{array}{ccccccc} X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 & \xrightarrow{h_3} & Y_4 \end{array}$$

If f_1 is surjective and if f_2 and f_4 are injective, then f_3 is injective.

Proof. Given any $x_3 \in \ker(f_3)$, commutativity implies that $f_4(g_3(x_3)) = h_3(f_3(x_3)) = h_3(1) = 1$ and therefore $g_3(x_3) = 1$ since f_4 is injective. Due to $\ker(g_3) = \operatorname{im}(g_2)$, there is $x_2 \in X_2$ with $g_2(x_2) = x_3$. Using commutativity again, we obtain $1 = f_3(x_3) = f_3(g_2(x_2)) = h_2(f_2(x_2))$ and since $\ker(h_2) = \operatorname{im}(h_1)$, there is $y_1 \in Y_1$ with $h_1(y_1) = f_2(x_2)$. Surjectivity of f_1 implies that there is an element $x_1 \in X_1$ with $f_1(x_1) = y_1$, so we can apply commutativity again to get $f_2(g_1(x_1)) = h_1(f_1(x_1)) = h_1(y_1) = f_2(x_2)$. Since f_2 is injective by assumption, $g_1(x_1) = x_2$ and exactness implies that $x_3 = g_2(x_2) = g_2(g_1(x_1)) = 1$, proving that f_3 is injective. □

Lemma A.2 (Second Four-Lemma)**.** *Let there be groups X_i and Y_i forming the following commutative diagram of homomorphisms with exact rows:*

$$\begin{array}{ccccccc} X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 & \xrightarrow{g_4} & X_5 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_2 & \xrightarrow{h_2} & Y_3 & \xrightarrow{h_3} & Y_4 & \xrightarrow{h_4} & Y_5 \end{array}$$

If f_5 is injective and if f_2 and f_4 be surjective, then f_3 is surjective.

Proof. Let $y_3 \in Y_3$ be given. Then $h_3(y_3) = f_4(x_4)$ for some $x_4 \in X_4$, since f_4 is surjective. Applying h_4 to this equation gives $1 = h_4(h_3(y_3)) = h_4(f_4(x_4)) = f_5(g_4(x_4))$, where we have used exactness and commutativity. Since f_5 is injective, this implies that $g_4(x_4) = 1$. Thus, due to exactness, $x_4 = g_3(x_3)$ for some $x_3 \in X_3$. It follows that $h_3(y_3) = f_4(x_4) = f_4(g_3(x_3)) = h_3(f_3(x_3))$ and therefore there is $y \in \ker(h_3)$ with $y_3 = y \cdot f_3(x_3)$. Since $\ker(h_3) = \operatorname{im}(h_2)$, there is an element $y_2 \in Y_2$ with $h_2(y_2) = y$. Furthermore, f_2 is surjective, so there is some $x_2 \in X_2$ with $f_2(x_2) = y_2$. Collecting these results, we obtain

$$\begin{aligned} f_3(g_2(x_2) \cdot x_3) &= f_3(g_2(x_2)) \cdot f_3(x_3) \\ &= h_2(f_2(x_2)) \cdot f_3(x_3) \\ &= h_2(y_2) \cdot f_3(x_3) \\ &= y \cdot f_3(x_3) \\ &= y_3. \end{aligned}$$

□

Lemma A.3 (Five-Lemma). *Let there be groups X_i and Y_i and homomorphisms $f_i : X_i \to Y_i$ $(i = 1, 2, 3, 4, 5)$ forming the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{g_1} & X_2 & \xrightarrow{g_2} & X_3 & \xrightarrow{g_3} & X_4 & \xrightarrow{g_4} & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 & \xrightarrow{h_3} & Y_4 & \xrightarrow{h_4} & Y_5 \end{array}$$

If f_1 is surjective, f_5 injective and if f_2 and f_4 are bijective, then f_3 is bijective.

Proof. Injectivity of f_3 follows from the first Four-Lemma A.1 and surjectivity from the second Four-Lemma A.2. □

A special version of the Five-Lemma relaxing the requirement of all entries to be groups and all maps to be homomorphisms is given in [tD08, p. 129] and we reproduce it here:

Lemma A.4 (Special Five-Lemma). *Let there be two fiber bundles $X_1 \hookrightarrow X \xrightarrow{\rho_1} A$ and $Y_1 \hookrightarrow Y \xrightarrow{\rho_2} B$ with a based map $f : X \to Y$ satisfying $f \circ \rho_1 = \rho_2 \circ f$, so that f restricts to maps $f : A \to B$ and $f : X_1 \to Y_1$. Consider the resulting commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} \pi_1(X_1) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_0(X_1) & \longrightarrow & \pi_0(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \pi_1(Y_1) & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(B) & \longrightarrow & \pi_0(Y_1) & \longrightarrow & \pi_0(Y) \end{array}$$

Additionally, suppose that all statements in the following list are true for all choices of base point $x_0 \in A \subset X$ *(and the corresponding base point* $f(x_0) \in B \subset Y$*):*

- $f_* : \pi_1(X_1) \to \pi_1(Y_1)$ *is surjective,*
- $f_* : \pi_1(X) \to \pi_1(Y)$ *is bijective,*
- $f_* : \pi_0(X_1) \to \pi_0(Y_1)$ *is bijective,*
- $f_* : \pi_0(X) \to \pi_0(Y)$ *is injective.*

Then $f_* : \pi_1(A) \to \pi_1(B)$ *is bijective for all choices of base point* $x_0 \in A$*.*

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