A HIGHER-ORDER ENERGY EXPANSION TO TWO-DIMENSIONAL SINGULARLY PERTURBED NEUMANN PROBLEMS

JUNCHENG WEI, MATTHIAS WINTER, AND WAI-KONG YEUNG

Abstract. Of concern is the following singularly perturbed semilinear elliptic problem

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega \\ u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant and $1 < p < \left(\frac{N+2}{N-2}\right)_+$. Associated with the above problem is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right) dx$$

for $u \in H^1(\Omega)$, where $F(u) = \int_0^u s^p ds$. Ni and Takagi ([28], [29]) proved that for a single boundary spike solution u_ϵ, the following asymptotic expansion holds:

$$(1) \qquad J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + o(\epsilon) \right],$$

where $I[w]$ is the energy of the ground state, $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ and $H(P_\epsilon)$ is the boundary mean curvature function at $P_\epsilon \in \partial\Omega$. Later, Wei and Winter ([42], [43]) improved the result and obtained a higher-order expansion of $J_\epsilon[u_\epsilon]$:

$$(2) \qquad J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[\omega] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right],$$

where c_2 and $c_3 > 0$ are generic constants and $R(P_\epsilon)$ is the scalar curvature at P_ϵ. However, if $N = 2$, the scalar curvature is always zero. The expansion (2) is no longer sufficient to distinguish spike locations with same mean curvature. In this paper, we consider this case and assume that $2 \le p < +\infty$. Without loss of generality, we may assume that the boundary near $P \in \partial\Omega$ is represented by the graph $\{x_2 = \rho_P(x_1)\}$. Then we have the following higher order expansion of $J_\epsilon[u_\epsilon]$:

$$(3) \qquad J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + c_2 \epsilon^2 (H(P_\epsilon))^2 \right] + \epsilon^3 [P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^3) \Big],$$

where $H(P_\epsilon) = \rho''_{P_\epsilon}(0)$ is the curvature, $P(t) = A_1 t + A_2 t^2 + A_3 t^3$ is a polynomial, c_1, c_2, c_3 and A_1, A_2,A_3 are generic real constants and $S(P_\epsilon) = \rho^{(4)}_{P_\epsilon}(0)$. In particular $c_3 < 0$. Some applications of this expansion are given.

1. Introduction

We consider the following singularly perturbed semilinear elliptic problem

$$\begin{cases} \epsilon^2 \Delta u - bu + f(u) = 0 \text{ in } \Omega \\ u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_N^2}$ denotes the Laplace operator in $\mathbf{R}^N$, ν stands for the unit outer normal to $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ for the normal derivative, $b > 0$ is a positive constant and $f(t)$ is a function in $C^{1+\sigma}(R)$ such that $f(0) = f'(0) = 0$. Typical examples of the function $-bu + f(u)$ are

$$-bu + f(u) = -u + u_+^p \text{ with } u_+ = \max(0, u), \quad b = 1, \tag{1.2}$$

$$-bu + f(u) = u(u-a)(1-u) \text{ with } 0 < a < \frac{1}{2}, \quad b = a, \tag{1.3}$$

where

$$1 < p < \left(\frac{N+2}{N-2}\right)_+ \left(= \frac{N+2}{N-2} \text{ when } N \geq 3; = +\infty \text{ when } N = 1, 2\right). \tag{1.4}$$

Equation (1.1) with (1.2) or (1.3) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in biological pattern formation ([16], [33], [39]) or of parabolic equations in chemotaxis, population dynamics and phase transitions ([5], [6],[27], [31]).

Without loss of generality, we may assume that $b = 1$.

Associated with (1.1) is the energy functional J_ϵ defined by

$$J_\epsilon[u] := \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - F(u)\right) dx \text{ for } u \in H^1(\Omega), \tag{1.5}$$

where $F(u) = \int_0^u f(s)ds$. It is well-known that any solution of (1.1) is a critical point of J_ϵ and vice versa. In this paper, we restrict ourselves to families of solutions $\{u_\epsilon\}_{0<\epsilon<\epsilon_0}$ of (1.1) with finite energy, i.e.

$$\epsilon^{-N} J_\epsilon[u_\epsilon] < +\infty \text{ for } 0 < \epsilon < \epsilon_0. \tag{1.6}$$

It can be proved that for ϵ sufficiently small, any family of solutions of (1.1) satisfying (1.6) can have at most a finite number of local maximum points (see [28]). Let the local maximum points be $\{P_1^\epsilon, ..., P_K^\epsilon\} \subset \bar{\Omega}$. If $P_j^\epsilon \in \partial\Omega, j = 1, ..., K$, we call u_ϵ a $K-$boundary spike solution. If $K = 1$, we call u_ϵ a single boundary spike solution.

In the pioneering papers [27], [28] and [29], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ϵ sufficiently small the least-energy solution is a single boundary spike solution and has only one local maximum point P_ϵ with $P_\epsilon \in \partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of $\partial\Omega$ at P.

Since then many works have been devoted to finding solutions with multiple spikes for the Neumann problem as well as the Dirichlet problem. See [1], [2], [3], [4], [5], [6], [7], [9], [10], [11], [12], [13], [14], [15], [18], [20], [21], [22], [23], [25], [26], [28], [29], [30], [31], [32], [35], [36], [40], [41], and the references therein. Recent surveys can be found in [33], [39].

A common tool for proving the existence of spike solutions is the energy expansion: In [28] and [29], Ni and Takagi proved, among others, that for a single boundary spike solution u_ϵ the following asymptotic expansion for $J_\epsilon[u_\epsilon]$ holds

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + o(\epsilon)\right], \tag{1.7}$$

where $c_1 > 0$ is a generic constant, P_ϵ is the unique local maximum point of u_ϵ, $H(P_\epsilon)$ is the mean curvature function at $P_\epsilon \in \partial\Omega$, w is the unique solution of the following ground-state problem:

$$\begin{cases} \Delta w - w + f(w) = 0, \; w > 0 \text{ in } \mathbf{R^N} \\ w(0) = \max_{y \in \mathbf{R^N}} w(y), \; \lim_{|y| \to +\infty} w(y) = 0 \end{cases} \tag{1.8}$$

and $I[w]$ is the ground-state energy

$$I[w] = \frac{1}{2} \int_{\mathbf{R^N}} |\nabla w|^2 dy + \frac{1}{2} \int_{\mathbf{R^N}} w^2 dy - \int_{\mathbf{R^N}} F(w) dy. \tag{1.9}$$

(Note that Ni and Takagi proved (1.7) for least-energy solutions. But it is easy to see that it also holds for any single boundary spike solution.)

Based on (1.7), Ni and Takagi showed that the least energy solution must concentrate at a maximum point of the mean curvature function. However, if $H(P)$ has more than one maximum point on $\partial\Omega$, the asymptotic expansion (1.7) is no longer sufficient to derive the spike location. In the light of this, Wei and Winter ([42], [43]) obtained a higher-order expansion of $J_\epsilon[u_\epsilon]$:

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + \epsilon^2 [c_2 (H(P_\epsilon^2))^2 + c_3 R(P_\epsilon)] + o(\epsilon^2) \right], \tag{1.10}$$

where c_2, c_3 are generic constants and $R(P_\epsilon)$ is the scalar curvature at $P_\epsilon \in \partial\Omega$. In particular $c_3 > 0$. Based on this expansion, they showed that a least energy solution concentrates at a minimum point of the scalar curvature function among all maximum points of the mean curvature.

However, in the two-dimensional case, the scalar curvature is always zero. Thus the expansion (1.10) is no longer sufficient to locate the spike if there are several maximum points of the mean curvature and the next order term in (1.10) becomes important. This is exactly the motivation of this paper.

Before stating our main results, we introduce some notations.

First, we give some conditions on the function $f(t)$:

(f1) $f \in C^2(\mathbf{R})$, $f(0) = 0$, $f'(0) = 0$ and $f(t) \equiv 0$ for $t \leq 0$.

(f2) The problem (1.8) in the whole space has a unique solution w, which is nondegenerate, i.e.

$$\text{Kernel}(\Delta - 1 + f'(w)) = \text{span} \left\{ \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2} \right\}. \tag{1.11}$$

By the well-known result of Gidas, Ni and Nirenberg, [17], w is radially symmetric: $w(y) = w(|y|)$ and strictly decreasing: $w'(r) < 0$ for $r > 0$, $r = |y|$. Moreover, we have the following asymptotic behavior of w:

$$w(r) = A_0 r^{-\frac{1}{2}} e^{-r} \left(1 + O(\frac{1}{r}) \right), \tag{1.12}$$

$$w'(r) = -A_0 r^{-\frac{1}{2}} e^{-r} \left(1 + O(\frac{1}{r}) \right), \tag{1.13}$$

as $r \to \infty$, where $A_0 > 0$ is generic constant.

The uniqueness of w is proved in [24] for the case $f(u) = u^p$. For a general nonlinearity, see [8]. For $f(u)$ defined by (1.3), the uniqueness of the entire solution was proved by Peletier and Serrin [34].

In what follows, we always assume that $f(t)$ satisfies (f1) and (f2).

Remark: We have required $f(u)$ to be C^2. We believe that this is just a technical condition. This condition can be further weakened to $f \in C^{1+\sigma}$, where $\sigma > \frac{1}{2}$.

Next, we introduce boundary deformations.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary. (We need $\partial\Omega \in C^5$.) For any boundary point $P = (P_1, P_2)$, we define a diffeomorphism straightening the boundary in a neighborhood of it. After rotation and translation of the coordinate system, we may assume the inward normal to $\partial\Omega$ at P points in the direction of positive x_2-axis and that P is the origin.

We denote that

$$\begin{aligned} &B^{'}(\delta) = (-\delta, \delta), \quad && B(P,\delta) = \left\{x = (x_1, x_2) \in \mathbf{R}^2 : |x - P| < \delta\right\}, \\ &\Omega_1 = \Omega \cap B(P,\delta), && \omega_1 = \partial\Omega \cap B(P,\delta). \end{aligned} \tag{1.14}$$

Since $\partial\Omega \in C^5$, we can find a positive constant δ such that $\partial\Omega \cap B(P,\delta)$ can be represented by the graph of a smooth function $\rho_P : (-\delta, \delta) \to \mathbf{R}$ with $\rho_P(0) = \rho_P'(0) = 0$ and

$$\Omega_1 = \{(x_1, x_2) \in B(P,\delta) : x_2 - P_2 > \rho_P(x_1 - P_1)\}. \tag{1.15}$$

¿From now on, we fix a boundary point P and simply denote ρ_P by ρ if this can be done without causing confusion. From Taylor expansion, we have

$$\rho(x_1 - P_1) = \frac{1}{2}\rho''(0)(x_1 - P_1)^2 + \frac{1}{6}\rho'''(0)(x_1 - P_1)^3 + \frac{1}{24}\rho^{(4)}(0)(x_1 - P_1)^4 + O(|x|^5). \tag{1.16}$$

Here, $H(P) = \rho''(0)$ is the mean curvatures at P. We define

$$S(P) = \rho^{(4)}(0). \tag{1.17}$$

Throughout this paper, we use the following notation:

$$y = (y_1, y_2) \in \mathbf{R}^2, \quad \mathbf{R}^2_+ = \{y \in \mathbf{R}^2 : y_2 > 0\}. \tag{1.18}$$

Now, we can state the main theorem of this paper.

Theorem 1.1. *Let u_ϵ be a single boundary spike solution of* (1.1) *with local maximum point $P_\epsilon \in \partial\Omega$. Assume that $N = 2$ and that f satisfies (f1) and (f2). Then, for ϵ sufficiently small, we have*

$$J_\epsilon[u_\epsilon] = \epsilon^2\left[\frac{1}{2}I[w] - c_1\epsilon H(P_\epsilon) + c_2\epsilon^2(H(P_\epsilon))^2 + \epsilon^3[P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^3)\right], \tag{1.19}$$

where

$$P(H(P_\epsilon)) = A_1 H(P_\epsilon) + A_2(H(P_\epsilon))^2 + A_3(H(P_\epsilon))^3,$$

c_1, c_2, c_3 and A_1, A_2,A_3 are generic constants to be defined later. Moreover, we have $c_1 > 0$ and $c_3 < 0$.

As in [43], we can also obtain a similar asymptotic expansion for multiple boundary spike solutions.

Theorem 1.2. *Let u_ϵ be a K-boundary spike solution of (1.1) with local maximum point $P_1^\epsilon, ..., P_K^\epsilon \in \partial\Omega$. Let $P_j^\epsilon \to P_j^0 \in \partial\Omega$. Suppose that $P_i^0 \neq P_j^0$ for $i \neq j$. Assume that*

$N=2$ and that f satisfies (f1) and (f2). Then, for ϵ sufficiently small, we have

$$J_\epsilon[u_\epsilon] = \epsilon^N \left[\frac{K}{2} I[w] - c_1 \epsilon \sum_{j=1}^K H(P_j^\epsilon) + c_2 \epsilon^2 \sum_{j=1}^K (H(P_j^\epsilon))^2 + \epsilon^3 \sum_{j=1}^K [P(H(P_j^\epsilon)) + c_3 S(P_j^\epsilon)] + o(\epsilon^3) \right]. \tag{1.20}$$

¿From Theorem 1.1, we can give a refinement of the results of [28] and [29] in the case of $N=2$. To this end, we assume that f satisfies (f1) and

(f3) For $t \geq 0$, f admits the following decomposition in $C^2(\mathbf{R})$:

$$f(t) = f_1(t) - f_2(t), \tag{1.21}$$

where (i) $f_1(t) \leq 0$ and $f_2(t) \geq 0$ with $f_1(0) = f_1'(0) = 0$, whence it follows that $f_2(0) = f_2'(0) = 0$ by (f1); and (ii) there is a $q \geq 1$ such that $\frac{f_1(t)}{t^q}$ is nondecreasing in $t > 0$, whereas $\frac{f_2(t)}{t^q}$ is nonincreasing in $t > 0$, and in case $q = 1$, we require further that the above monotonicity condition for $\frac{f_1(t)}{t}$ is strict.

(f4) $f(t) = O(t^p)$ as $t \to +\infty$, where $2 \leq p < \infty$

(f5) There exists a constant $\theta \in (0, \frac{1}{2})$ such that $F(t) \leq t\theta t f(t)$ for $t \geq 0$.

By taking a function $e(x) \equiv k$ for some constant in Ω, and choosing k large enough, we have $J_\epsilon[e] < 0$ for all $\epsilon \in (0,1)$.Then for each $\epsilon \in (0,1)$, we can define the so-called mountain-pass value:

$$c_\epsilon = \inf_{h \in \Gamma} \max_{0 \leq t \leq 1} J_\epsilon[h(t)], \tag{1.22}$$

where $\Gamma = \{h : [0,1] \to H^1(\Omega) | h(t) \text{ is continuous}, h(0) = 0, h(1) = e\}$.

In [28] and [29], it is proved that there exists a mountain-pass solution u_ϵ which is also a least energy solution. Moreover, as $\epsilon \to 0$, u_ϵ develops a spike layer behavior near a maximum point of the mean curvature function. Now we have

Corollary 1.1. *Suppose that $N = 2$ and $f(u)$ satisfies (f1), (f3), (f4) and (f5). Let u_ϵ be a least energy solution of (1.1) and let P_ϵ be the unique maximum point of u_ϵ. Then, for ϵ sufficiently small, we have*

$$H(P_\epsilon) \to \max_{P \in \partial\Omega} H(P), \quad S(P_\epsilon) \to \max_{Q \in \partial\Omega, H(Q) = \max\limits_{P \in \partial\Omega} H(P)} S(Q) \tag{1.23}$$

The proof of Theorem 1.1 is divided into three steps:

Step 1: We choose a good approximate function, concentrating at a boundary point P and called $\tilde{w}_{\epsilon,P}$, such that

$$\epsilon^2 \Delta \tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}) = O(\epsilon^2). \tag{1.24}$$

This is done in Section 3.

Step 2: Our key observation is that in order to obtain the term of order ϵ^{N+3} in the asymptotic expansion of $J_\epsilon[u_\epsilon]$, we need not expand u_ϵ up to the order $O(\epsilon^3)$. In fact, it is enough to have

$$u_\epsilon = \tilde{w}_{\epsilon,P} + O(\epsilon^\tau) \tag{1.25}$$

for some $\tau > \frac{3}{2}$. We do not even need to know the term of order ϵ^τ in the asymptotic expansion of u_ϵ. From (1.25) we derive that

$$J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P}] + o(\epsilon^{N+3}). \tag{1.26}$$

This is proved in Section 6.

Step 3: It then remains to compute the energy of $\tilde{w}_{\epsilon,P}$. A higher-order energy expansion is derived in Section 4 and in Section 5 it is shown that $c_1 < 0$ and $c_3 < 0$.

Finally, the proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.1 are contained in Section 7.

In three appendices, the technical proofs of Proposition 2.1, Proposition 3.1, and Lemma 4.1 are provided.

Throughout the paper, we use C to denote various constants independent of ϵ small.

Acknowledgments: This research of the first author is partially supported by an Earmarked Grant (CUHK4238/01P) from RGC of Hong Kong. The second author thanks the Department of Mathematics at CUHK for their kind hospitality.

2. Some Preliminaries

In this section, we introduce some preliminary analysis.

For $\mathbf{x} \in \partial\Omega$, let $\nu(\mathbf{x})$ denote the unit outward normal at x and $\frac{\partial}{\partial\nu}$ the normal derivative. In our coordinate system, for $x \in \omega_1$, we have

$$\nu(x) = \frac{1}{\sqrt{1+\rho'(x_1)^2}}(\rho'(x_1), -1), \tag{2.1}$$

$$\frac{\partial}{\partial\nu(x)} = \frac{1}{\sqrt{1+(\rho'(x_1))^2}}(\rho'(x_1)\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})|_{x_2-P_2=\rho(x_1-P_1)}. \tag{2.2}$$

For $x \in \Omega_1$, we set

$$\epsilon y_1 = x_1 - P_1, \quad \epsilon y_2 = x_2 - P_2 - \rho(x_1 - P_1). \tag{2.3}$$

We denote the corresponding transformation by T_ϵ, i.e.

$$T_{\epsilon,1}(x_1, x_2) = \frac{1}{\epsilon}x_1, \quad T_{\epsilon,2}(x_1, x_2) = \frac{1}{\epsilon}\left[x_2 - P_2 - \rho(x_1 - P_1)\right]. \tag{2.4}$$

Then, $y = T_\epsilon(x)$, where the Jacobian of T_ϵ is $\frac{1}{\epsilon^2}$. Its inverse is called $x = T_\epsilon^{-1}(y)$. It then holds that

$$x_1 = P_1 + \epsilon y_1, \quad x_2 = P_2 + \epsilon y_2 + \rho(\epsilon y_1). \tag{2.5}$$

Under the transformation T_ϵ, $\frac{|x-P|}{\epsilon}$ can be expanded

$$\begin{aligned} (\frac{|x-P|}{\epsilon})^2 &= \frac{1}{\epsilon^2}\{\epsilon^2 y_1^2 + (\epsilon y_2 + \rho(\epsilon y_1))^2\} \\ &= |y|^2 + \epsilon\rho''(0)y_1^2 y_2 + \epsilon^2[\frac{1}{3}\rho'''(0)y_1^3 y_2 + \frac{1}{4}(\rho''(0))^2 y_1^4] \\ &\quad + \epsilon^3[\frac{1}{12}\rho^{(4)}(0)y_1^4 y_2 + \frac{1}{6}\rho''(0)\rho'''(0)y_1^5] + O(\epsilon^4 e^{-a|y|}). \end{aligned} \tag{2.6}$$

It is easy to see that for $x \in \Omega_1$

$$\epsilon^2\Delta_x = \Delta_y + |\rho'(\epsilon y_1)|^2\frac{\partial^2}{\partial y_2^2} - 2\rho'(\epsilon y_1)\frac{\partial^2}{\partial y_1\partial y_2} - \epsilon\rho''(\epsilon y_1)\frac{\partial}{\partial y_2} \tag{2.7}$$

and for $x \in \omega_1$

$$\sqrt{1+(\rho'(x_1))^2}\frac{\partial}{\partial\nu} = \frac{1}{\epsilon}\{\rho'(\epsilon y_1)\frac{\partial}{\partial y_1} - (1+(\rho'(\epsilon y_1))^2)\frac{\partial}{\partial y_2}\}. \tag{2.8}$$

Let

$$\Omega_{\epsilon,P} = \{y \in \mathbf{R}^2 : \epsilon y + P \in \Omega\} \tag{2.9}$$

and let $w_{\epsilon,P}$ be the unique solution of the following problem

$$\begin{cases} \Delta_y w_{\epsilon,P} - w_{\epsilon,P} + f(w(y)) = 0 & \text{in } \Omega_{\epsilon,P}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_{\epsilon,P}. \end{cases} \tag{2.10}$$

Set $h_{\epsilon,P}(x) = w(\frac{x-P}{\epsilon}) - w_{\epsilon,P}(\frac{x-P}{\epsilon})$. Then $h_{\epsilon,P}(x)$ satisfies the following equation

$$\begin{cases} \epsilon^2 \Delta v - v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} w(\frac{x-P}{\epsilon}) & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Note that by (2.7)

$$\epsilon^2 \Delta_x h - h = \Delta_y h + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 h}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 h}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial h}{\partial y_2} - h. \tag{2.12}$$

We need to analyze the behavior of $h_{\epsilon,P}$ up to $O(\epsilon^4)$. To this end, we have to introduce five functions v_1,v_2,v_3,v_4 and v_5: v_1 is the unique solution of

$$\begin{cases} \Delta v_1 - v_1 = 0 & \text{in } \mathbf{R}^2_+ \\ \frac{\partial v_1}{\partial y_2} = -\frac{w'(|y|)}{|y|} \frac{1}{2} \rho''(0) y_1^2 & \text{on } \partial \mathbf{R}^2_+, \end{cases} \tag{2.13}$$

v_2 is the unique solution of

$$\begin{cases} \Delta v_2 - v_2 - 2\rho''(0) y_1 \frac{\partial^2 v_1}{\partial y_1 \partial y_2} - \rho''(0) \frac{\partial v_1}{\partial y_2} = 0 & \text{in } \mathbf{R}^2_+ \\ \frac{\partial v_2}{\partial y_2} = \rho''(0) y_1 \frac{\partial v_1}{\partial y_1} & \text{on } \partial \mathbf{R}^2_+, \end{cases} \tag{2.14}$$

v_3 is the unique solution of

$$\begin{cases} \Delta v_3 - v_3 = 0 & \text{in } \mathbf{R}^2_+ \\ \frac{\partial v_3}{\partial y_2} = -\frac{w'(|y|)}{|y|} \frac{1}{3} \rho'''(0) y_1^3 & \text{on } \partial \mathbf{R}^2_+, \end{cases} \tag{2.15}$$

v_4 is the unique solution of

$$\begin{cases} \Delta v_4 - v_4 - 2\rho''(0) y_1 \frac{\partial^2 v_2}{\partial y_1 \partial y_2} - \rho''(0) \frac{\partial v_2}{\partial y_2} + (\rho''(0))^2 y_1^2 \frac{\partial^2 v_1}{\partial y_2^2} = 0 & \text{in } \mathbf{R}^2_+ \\ \frac{\partial v_4}{\partial y_2} = \frac{w'(|y|)}{|y|} y_1^4 (\frac{1}{2}(\rho''(0))^3 - \frac{1}{8}\rho^{(4)}(0)) + \rho''(0) y_1 \frac{\partial v_2}{\partial y_1} - \frac{1}{16} \left(\frac{w'(|y|)}{|y|}\right)' \frac{y_1^6}{|y|} (\rho''(0))^3 & \text{on } \partial \mathbf{R}^2_+, \end{cases} \tag{2.16}$$

and v_5 is the unique solution of

$$\begin{cases} \Delta v_5 - v_5 - \rho''(0) \frac{\partial v_3}{\partial y_2} - 2\rho''(0) y_1 \frac{\partial v_3}{\partial y_1 \partial y_2} - \rho'''(0) \left[y_1 \frac{\partial v_1}{\partial y_2} + y_1^2 \frac{\partial^2 v_1}{\partial y_1 \partial y_2} \right] = 0 & \text{in } \mathbf{R}^2_+ \\ \frac{\partial v_5}{\partial y_2} = \rho'''(0) \frac{1}{2} y_1^2 \frac{\partial v_1}{\partial y_1} + \rho''(0) y_1 \frac{\partial v_3}{\partial y_1} \text{ on } \partial \mathbf{R}^2_+. \end{cases} \tag{2.17}$$

Note that v_1,v_2 and v_4 are even functions in y_1 and v_3,v_5 are odd functions in y_1, (i.e. $v_1(y_1, y_2) = v_1(-y_1, y_2)$).

Moreover, it is easy to see that $|v_1|, |v_2|, |v_3|, |v_4|, |v_5| \le Ce^{-a|y|}$ for some positive constant a.

Let $\chi(x)$ be a smooth cut-off function such that $\chi(x) = 1$ for $x \in B\left(0, \frac{\delta}{2}\right)$ and $\chi(x) = 0$ for x outside $B(0, \delta)$. We set

$$\begin{aligned} h_{\epsilon,P}(x) &= \epsilon v_1(y)\chi(x-P) + \epsilon^2 \left[v_2(y)\chi(x-P) + v_3(y)\chi(x-P)\right] \\ &\quad + \epsilon^3 \left[v_4(y)\chi(x-P) + v_5(y)\chi(x-P)\right] + \epsilon^4 \Psi_{\epsilon,P}(x), \end{aligned}$$

where $y = T_\epsilon(x)$ is given by (2.5).

Then, we have the following asymptotic expansion

Proposition 2.1. *For ϵ sufficiently small,*

$$\epsilon^{-2}\int_\Omega (\epsilon^2|\nabla\Psi_{\epsilon,P}|^2 + |\Psi_{\epsilon,P}|^2)dx \le C.$$

The proof of Proposition 2.1 is technical. We present it in Appendix A.

3. Approximate Function $\tilde{w}_{\epsilon,P}$

In this section, we introduce the important approximate function $\tilde{w}_{\epsilon,P}$.

We begin with the study of the properties of the following linear operator

$$L_0 := \Delta - 1 + f'(w) : H^2(\mathbf{R}^2) \longmapsto L^2(\mathbf{R}^2)$$

By our assumption (f2),

$$\text{Kernel}(L_0) = \text{span}\left\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\right\}. \tag{3.1}$$

If we restrict L_0 to

$$H^2_\nu(\mathbf{R}^2_+) = H^2(\mathbf{R}^2_+) \cap \left\{\tfrac{\partial u}{\partial y_2} = 0 \text{ on } \partial\mathbf{R}^2_+\right\} \tag{3.2}$$

then we have

$$\text{Kernel}(L_0) \cap H^2_\nu(\mathbf{R}^2_+) = \text{span}\left\{\frac{\partial w}{\partial y_1}\right\}. \tag{3.3}$$

Since $v_1(y)$ is even in y_1, there exists a unique solution to

$$\begin{cases} \Delta\Phi_0 - \Phi_0 + f'(w)\Phi_0 - f'(w)v_1 = 0 \quad \text{in } \mathbf{R}^2_+, \\ \frac{\partial\Phi_0}{\partial y_2} = 0 \text{ on } \partial\mathbf{R}^2_+, \ \Phi_0 \text{ is even in } y_1. \end{cases} \tag{3.4}$$

We call this solution Φ_0. In [43], Wei and Winter modified Φ_0 to a new function $\Phi_{\epsilon,P}$ which satisfies the Neumann boundary condition. To this end, they introduced a function $\phi_{\epsilon,P}$ which is the solution of

$$\begin{cases} \epsilon^2\Delta\phi_{\epsilon,P} - \phi_{\epsilon,P} = 0 & \text{in } \Omega, \\ \frac{\partial\phi_{\epsilon,P}}{\partial\nu} = \frac{\partial(\Phi_0(T_\epsilon(x))\chi(x-P))}{\partial\nu} & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

and set

$$\Phi_{\epsilon,P} = \Phi_0(T_\epsilon(x))\chi(x-P) - \phi_{\epsilon,P}. \tag{3.6}$$

It is easy to see that $\Phi_{\epsilon,P}$ satisfies the Neumann boundary condition, $\Phi_{\epsilon,P}(T_\epsilon^{-1}(y)) = \Phi_0(y) + O(\epsilon e^{-a|y|})$ and $|\Phi_{\epsilon,P}(T_\epsilon^{-1}(y))| \le Ce^{-a|y|}$ for some $a > 0$. Then they introduced the approximating function

$$\tilde{w}_{\epsilon,P} = w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}$$

and show that $\tilde{w}_{\epsilon,P}$ solves the problem up to the order $O(\epsilon^{1+\sigma})$.

In our problem, we need to expand $\phi_{\epsilon,P}$ up to the order $O(\epsilon^2)$. To this end, we introduce a new function Φ_1 which is the solution of

$$\begin{cases} \Delta\Phi_1 - \Phi_1 = 0 & \text{in } \mathbf{R}^2_+, \\ \frac{\partial\Phi_1}{\partial y_2} = -\rho''(0)y_1\frac{\partial\Phi_0}{\partial y_1} & \text{on } \partial\mathbf{R}^2_+ \end{cases} \tag{3.7}$$

and set

$$\phi_{\epsilon,P}(x) = \epsilon\Phi_1(T_\epsilon(x))\chi(x-P) + \epsilon^2\tilde{\phi}_{\epsilon,P}(x). \tag{3.8}$$

It is easy to see that Φ_1 is even in y_1 and $|\Phi_1(T_\epsilon^{-1}(y))| \leq Ce^{-a|y|}$ for some constant $a > 0$. Then, similar to the proof of Proposition 2.1 in Section 2, we have the following asymptotic expansion, whose proof will be given in Appendix B.

Proposition 3.1. *For ϵ sufficiently small,*

$$\tilde{w}_{\epsilon,P}(x) = w_{\epsilon,P}(x) + \epsilon\Phi_0(T_\epsilon(x))\chi(x-P) - \epsilon^2\Phi_1(T_\epsilon(x))\chi(x-P) - \epsilon^3\tilde{\phi}_{\epsilon,P}, \tag{3.9}$$

where

$$\epsilon^{-2}\int_\Omega (\epsilon^2|\nabla\tilde{\phi}_{\epsilon,P}|^2 + |\tilde{\phi}_{\epsilon,P}|^2)dx \leq C \tag{3.10}$$

$$|\tilde{\phi}_{\epsilon,P}(T_\epsilon^{-1}(y))| \leq Ce^{-a|y|} \tag{3.11}$$

for some constant $a > 0$.

The following lemma was proved in [43]. For the sake of completeness, we include the proof here.

Lemma 3.1. *Let*

$$S_\epsilon[\tilde{w}_{\epsilon,P}] := \epsilon^2\Delta\tilde{w}_{\epsilon,P} - \tilde{w}_{\epsilon,P} + f(\tilde{w}_{\epsilon,P}), \tag{3.12}$$

Then, for ϵ sufficiently small, we have

$$|S_\epsilon[\tilde{w}_{\epsilon,P}]| \leq C\epsilon^2 e^{-a|y|} \tag{3.13}$$

for some positive constant a.

Proof: Recall that

$$\begin{aligned}\tilde{w}_{\epsilon,P}(x) &= w_{\epsilon,P}(x) + \epsilon\Phi_0(T_\epsilon(x))\chi(x-P) - \epsilon^2\Phi_1(T_\epsilon(x))\chi(x-P) - \epsilon^3\tilde{\phi}_{\epsilon,P} \\ &= w_{\epsilon,P}(x) + \epsilon\Phi_{\epsilon,P}.\end{aligned}$$

We expand $S_\epsilon[\tilde{w}_{\epsilon,P}]$:

$$\begin{aligned}S_\epsilon[\tilde{w}_{\epsilon,P}] &= S_\epsilon[w_{\epsilon,P}] + \epsilon[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f'(w_{\epsilon,P})\Phi_{\epsilon,P}] \\ &\quad + [f(w_{\epsilon,P} + \epsilon\Phi_{\epsilon,P}) - f(w_{\epsilon,P}) - \epsilon f'(w_{\epsilon,P})\Phi_{\epsilon,P}] = S_1 + S_2 + S_3,\end{aligned}$$

where S_1,S_2 and S_3 are defined by the last equality.

Using (2.10), we get

$$\begin{aligned}S_1 + S_2 &= f(w_{\epsilon,P}) - f(w(\frac{x-P}{\epsilon})) + \epsilon\left[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f'(w_{\epsilon,P})\Phi_{\epsilon,P}\right] \\ &= \left[f(w_{\epsilon,P}) - f(w(\frac{x-P}{\epsilon})) + \epsilon v_1\chi f'(w(\frac{x-P}{\epsilon}))\right] \\ &\quad + \epsilon\left[\epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f'(w_{\epsilon,P})\Phi_{\epsilon,P} - v_1\chi f'(w(\frac{x-P}{\epsilon}))\right].\end{aligned}$$

Note that

$$\begin{aligned}
& \epsilon^2\Delta\Phi_{\epsilon,P} - \Phi_{\epsilon,P} + f'(w_{\epsilon,P})\Phi_{\epsilon,P} - v_1\chi f'(w(\frac{x-P}{\epsilon})) \\
= & \left[\epsilon^2\Delta\Phi_0 - \Phi_0 + f'(w(\frac{x-P}{\epsilon}))\Phi_0 - v_1 f'(w(\frac{x-P}{\epsilon}))\right]\chi \\
& +\{f'(w_{\epsilon,P}) - f'(w(\frac{x-P}{\epsilon}))\}\Phi_0\chi - f'(w_{\epsilon,P})\phi_{\epsilon,P} + E_\epsilon(\chi) \\
= & \left[|\rho'(\epsilon y_1)|^2\frac{\partial\Phi_0}{\partial y_1} - 2\rho'(\epsilon y_1)\frac{\partial^2\Phi_0}{\partial y_1\partial y_2} - \epsilon\rho''(\epsilon y_1)\frac{\partial\Phi_0}{\partial y_2}\right]\chi \\
& +\{f'(w_{\epsilon,P}) - f'(w(\frac{x-P}{\epsilon}))\}\Phi_0\chi - f'(w_{\epsilon,P})\phi_{\epsilon,P} + E_\epsilon(\chi).
\end{aligned}$$

Thus, by Proposition 2.1, we get that $S_1+S_2 = O(\epsilon^2 e^{-a|y|})$. On the other hand, it follows by the mean-value theorem that

$$|f(a+b) - f(a) - f'(a)b| \le C|a||b|^2 \tag{3.14}$$

for any a, b such that $|b| \le 2|a| \le C$. Thus,

$$S_3 = O(\epsilon^2|w_{\epsilon,P}||\Phi_{\epsilon,P}|^2) = O(\epsilon^2 e^{-a|y|}).$$

This proves the lemma. □

4. The Computation Of $J_\epsilon[\tilde{w}_{\epsilon,P}]$

In this section, we compute the energy of the approximating function $\tilde{w}_{\epsilon,P}$. In Section 6, we will show that $\tilde{w}_{\epsilon,P}$ contributes the energy expansion up to the order $o(\epsilon^{N+3})$.

Note that

$$\begin{aligned}
\tilde{w}_{\epsilon,P} &= w_{\epsilon,P} + \epsilon\Phi_0\chi - \epsilon^2\Phi_1\chi - \epsilon^3\tilde{\phi}_{\epsilon,P} \\
&= w_{\epsilon,P} + \epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi},
\end{aligned}$$

where $\widetilde{\Phi}_0$, $\widetilde{\Phi}_1$ and $\widetilde{\phi}$ are defined by the last equality. Hence

$$\begin{aligned}
J_\epsilon[\tilde{w}_{\epsilon,P}] &= J_\epsilon[w_{\epsilon,P} + \epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi}] = J_\epsilon[w_{\epsilon,P}] \\
&\quad +\epsilon\int_\Omega[\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\Phi}_0 + w_{\epsilon,P}\widetilde{\Phi}_0 - \widetilde{\Phi}_0 f(w_{\epsilon,P})]dx \\
&\quad +\frac{\epsilon^2}{2}\int_\Omega[\epsilon^2|\nabla\widetilde{\Phi}_0|^2 + |\widetilde{\Phi}_0|^2 - |\widetilde{\Phi}_0|^2 f'(w_{\epsilon,P})]dx \\
&\quad -\epsilon^2\int_\Omega[\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\Phi}_1 + w_{\epsilon,P}\widetilde{\Phi}_1 - \widetilde{\Phi}_1 f(w_{\epsilon,P})]dx \\
&\quad -\epsilon^3\int_\Omega[\epsilon^2\nabla\widetilde{\Phi}_0\nabla\widetilde{\Phi}_1 + \widetilde{\Phi}_0\widetilde{\Phi}_1 - \widetilde{\Phi}_0\widetilde{\Phi}_1 f'(w_{\epsilon,P})]dx \\
&\quad -\epsilon^3\int_\Omega[\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\phi} + w_{\epsilon,P}\widetilde{\phi} - \widetilde{\phi} f(w_{\epsilon,P})]dx \\
&\quad -\frac{\epsilon^3}{6}\int_\Omega \widetilde{\Phi}_0^3 f''(w_{\epsilon,P})dx \\
&\quad -\int_\Omega\left[F(\tilde{w}_{\epsilon,P}) - F(w_{\epsilon,P}) - (\epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi})f(w_{\epsilon,P}) - \frac{1}{2}(\epsilon^2\widetilde{\Phi}_0^2 - 2\epsilon^3\widetilde{\Phi}_0\widetilde{\Phi}_1)f'(w_{\epsilon,P}) - \frac{1}{6}\epsilon^3\widetilde{\Phi}_0^3 f''(w_{\epsilon,P})\right]dx \\
&= J_\epsilon[w_{\epsilon,P}] + J_1 + J_2 - J_3 - J_4 - J_5 - J_6 - J_7,
\end{aligned}$$

where $J_1, ..., J_7$ are defined at the last equality.

We estimate J_7 first. Since

$$F(\tilde{w}_{\epsilon,P}) = F(w_{\epsilon,P}) + (\epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi})f(w_{\epsilon,P}) + \frac{1}{2}(\epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi})^2 f'(w_{\epsilon,P}) + \frac{1}{6}(\epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi})^3 f''(w_{\epsilon,P}) + O(\epsilon^4),$$

the last integral J_7 is of the order $O(\epsilon^{N+4})$.

Next we estimate J_1. Since $w_{\epsilon,P}$ satisfies the equation (2.10), we get that

$$\begin{aligned}
&\int_\Omega [\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\Phi}_0 + w_{\epsilon,P}\widetilde{\Phi}_0 - \widetilde{\Phi}_0 f(w_{\epsilon,P})]dx \\
= &\int_\Omega [f(w(\frac{x-P}{\epsilon})) - f(w_{\epsilon,P})]\widetilde{\Phi}_0 dx \\
= &\int_\Omega [\epsilon v_1\chi f'(w(\frac{x-P}{\epsilon})) + \epsilon^2(v_2+v_3)\chi f'(w(\frac{x-P}{\epsilon})) - \frac{1}{2}\epsilon^2 v_1^2\chi^2 f''(w(\frac{x-P}{\epsilon}))]\widetilde{\Phi}_0 dx \\
&+O(\epsilon^{N+3}) \\
= &\ \epsilon^N \Bigg[\int_{\mathbf{R}^2_+} \epsilon v_1 f'(w)\Phi_0 dy + \int_{\mathbf{R}^2_+} \epsilon^2 v_1 \frac{\rho''(0)}{2}\frac{f''(w)w'}{|y|} y_1^2 y_2 \Phi_0 dy \\
&+ \int_{\mathbf{R}^2_+} \epsilon^2 v_2 f'(w)\Phi_0 dy - \frac{\epsilon^2}{2}\int_{\mathbf{R}^2_+} v_1^2 f''(w)\Phi_0 dy\Bigg] + O(\epsilon^{N+3}) \\
= &\ \epsilon^{N+1}\int_{\mathbf{R}^2_+} f'(w)v_1\Phi_0 dy + \epsilon^{N+2}\int_{\mathbf{R}^2_+}[f'(w)v_2 - \frac{1}{2}f''(w)v_1^2 + \frac{\rho''(0)}{2}v_1\frac{f''(w)w'}{|y|}y_1^2 y_2]\Phi_0 dy + O(\epsilon^{N+3}),
\end{aligned}$$

where we have used the following facts: $w_{\epsilon,P} = w(\frac{x-P}{\epsilon}) - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi + O(\epsilon^3)$ and v_3 is odd in y_1.

Similarly for J_3, J_4 and J_5, we can get

$$\int_\Omega [\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\Phi}_1 + w_{\epsilon,P}\widetilde{\Phi}_1 - \widetilde{\Phi}_1 f(w_{\epsilon,P})]dx = \epsilon^{N+1}\int_{\mathbf{R}^2_+} f'(w)v_1\Phi_1 dy + O(\epsilon^{N+2}), \tag{4.1}$$

$$\int_\Omega [\epsilon^2\nabla w_{\epsilon,P}\nabla\widetilde{\phi} + w_{\epsilon,P}\widetilde{\phi} - \widetilde{\phi} f(w_{\epsilon,P})]dx = O(\epsilon^{N+1}), \tag{4.2}$$

$$\epsilon^3\int_\Omega [\epsilon^2\nabla\widetilde{\Phi}_0\nabla\widetilde{\Phi}_1 + \widetilde{\Phi}_0\widetilde{\Phi}_1 - \widetilde{\Phi}_0\widetilde{\Phi}_1 f'(w_{\epsilon,P})]dx = -\epsilon^N\int_{\mathbf{R}^2_+} f'(w)v_1\Phi_1 dy + O(\epsilon^{N+1}). \tag{4.3}$$

For J_2, we have

$$\begin{aligned}
&\int_\Omega [\epsilon^2|\nabla\widetilde{\Phi}_0|^2 + |\widetilde{\Phi}_0|^2 - |\widetilde{\Phi}_0|^2 f'(w_{\epsilon,P})]dx \\
= &\int_\Omega [\epsilon^2|\nabla\widetilde{\Phi}_0|^2 + |\widetilde{\Phi}_0|^2 - |\widetilde{\Phi}_0|^2 f'(w(\frac{x-P}{\epsilon}))]dx - \int_\Omega [f'(w_{\epsilon,P}) - f'(w(\frac{x-P}{\epsilon}))]|\widetilde{\Phi}_0|^2 dx \\
= &-\epsilon^N\int_{\mathbf{R}^2_+} f'(w)v_1\Phi_0 dy + \epsilon^{N+1}\int_{\mathbf{R}^2_+} f''(w)v_1|\Phi_0|^2 dy \\
&-2\epsilon^{N+1}\int_{\mathbf{R}^2_+}\rho''(0)y_1\frac{\partial\Phi_0}{\partial y_1}\frac{\partial\Phi_0}{\partial y_2}dy - \frac{\epsilon^{N+1}}{2}\rho''(0)\int_{\mathbf{R}^2_+}\frac{f''(w)w'}{|y|}y_1^2 y_2|\Phi_0|^2 dy + O(\epsilon^{N+2}).
\end{aligned}$$

(Here, we have used the fact that $\frac{\partial\Phi_0}{\partial y_2} = 0$ on $\partial\mathbf{R}^2_+$.)

Combining the estimates for $J_1, ..., J_7$ together, we conclude

$$
\begin{aligned}
J_\epsilon[\tilde{w}_{\epsilon,P}] &= J_\epsilon[w_{\epsilon,P} + \epsilon\widetilde{\Phi}_0 - \epsilon^2\widetilde{\Phi}_1 - \epsilon^3\widetilde{\phi}] \\
&= J_\epsilon[w_{\epsilon,P}] + \frac{1}{2}\epsilon^{N+2}\int_{\mathbf{R}^2_+} f'(w)v_1\Phi_0 dy \\
&\quad +\epsilon^{N+3}\int_{\mathbf{R}^2_+}[f'(w)v_2\Phi_0 - \frac{1}{2}f''(w)v_1^2\Phi_0 + \frac{1}{2}f''(w)v_1\Phi_0^2 - \frac{1}{6}\Phi_0^3 f''(w)]dy \\
&\quad +\epsilon^{N+3}\int_{\mathbf{R}^2_+}[\frac{\rho''(0)}{2}v_1\frac{f''(w)w'}{|y|}y_1^2y_2\Phi_0 - \rho''(0)y_1\frac{\partial\Phi_0}{\partial y_1}\frac{\partial\Phi_0}{\partial y_2} - \frac{\rho''(0)}{4}\frac{f''(w)w'}{|y|}y_1^2y_2|\Phi_0|^2]dy \\
&\quad +O(\epsilon^{N+4}). \qquad (4.4)
\end{aligned}
$$

It remains to compute $J_\epsilon[w_{\epsilon,P}]$ up to the order $o(\epsilon^{N+3})$.

The computation of $J_\epsilon[w_{\epsilon,P}]$ is quite long. We begin with

$$
\begin{aligned}
& J_\epsilon[w_{\epsilon,P}] \\
= & \frac{\epsilon^2}{2}\int_\Omega |\nabla w_{\epsilon,P}|^2 dx + \frac{1}{2}\int_\Omega w_{\epsilon,P}^2 dx - \int_\Omega F(w_{\epsilon,P})dx \\
= & \frac{1}{2}\int_\Omega f(w)w_{\epsilon,P}dx - \int_\Omega F(w_{\epsilon,P})dx \\
= & \frac{1}{2}\int_\Omega f(w)(w - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi)dx \\
& -\int_\Omega F(w - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi)dx + o(\epsilon^{N+3}) \\
= & \int_\Omega \frac{1}{2}f(w)w - F(w)dx - \frac{\epsilon}{2}\int_\Omega f(w)v_1\chi dx - \frac{\epsilon^2}{2}\int_\Omega f(w)v_2\chi dx - \frac{\epsilon^3}{2}\int_\Omega f(w)v_4\chi dx \\
& +\int_\Omega [F(w) - F(w - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi)]dx + o(\epsilon^{N+3}).
\end{aligned}
$$

We see that

$$
\begin{aligned}
& F(w - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi) \\
= & F(w) - f(w)(\epsilon v_1\chi + \epsilon^2(v_2+v_3)\chi + \epsilon^3(v_4+v_5)\chi) \\
& +\frac{1}{2}f'(w)(\epsilon v_1\chi + \epsilon^2(v_2+v_3)\chi + \epsilon^3(v_4+v_5)\chi)^2 \\
& -\frac{1}{6}f''(w)(\epsilon v_1\chi + \epsilon^2(v_2+v_3)\chi + \epsilon^3(v_4+v_5)\chi)^3 + o(\epsilon^3).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \int_\Omega F(w) - F(w - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi)dx \\
= & \int_\Omega f(w)(\epsilon v_1 + \epsilon^2(v_2+v_3) + \epsilon^3(v_4+v_5))\chi dx \\
& -\int_\Omega \frac{1}{2}f'(w)(\epsilon^2 v_1^2 + 2\epsilon^3(v_1v_2 + v_1v_3))\chi^2 dx + \int_\Omega \frac{1}{6}f''(w)\epsilon^3 v_1^3\chi^3 dx + o(\epsilon^{N+3}) \\
= & \epsilon\int_\Omega f(w)v_1\chi dx + \epsilon^2\int_\Omega f(w)v_2\chi - \frac{1}{2}f'(w)v_1^2\chi^2 dx \\
& +\epsilon^3\int_\Omega f(w)v_4\chi - f'(w)v_1v_2\chi^2 + \frac{1}{6}f''(w)v_1^3\chi^3 dx + o(\epsilon^{N+3}).
\end{aligned}
$$

Here we have used the facts that v_3 and v_5 are odd in y_1 and hence $\int_{\mathbf{R}^2_+} f(w)v_3dy = \int_{\mathbf{R}^2_+} f(w)v_3dy = 0$. Thus,

$$
\begin{aligned}
J_\epsilon[w_{\epsilon,P}] &= \int_\Omega \frac{1}{2}f(w)w - F(w)dx + \frac{\epsilon}{2}\int_\Omega f(w)v_1\chi dx + \frac{\epsilon^2}{2}\int_\Omega f(w)v_2\chi - f'(w)v_1^2\chi^2 dx \\
&\quad + \epsilon^3 \int_\Omega \frac{1}{2}f(w)v_4\chi - f'(w)v_1v_2\chi^2 + \frac{1}{6}f''(w)v_1^3\chi dx + o(\epsilon^{N+3}).
\end{aligned}
$$

¿From now on, we omit the factor χ in the integrals for simplicity.

Let

$$
\begin{aligned}
I_{1,1} &= \int_\Omega \frac{1}{2}f(w)w - F(w)dx \\
I_{1,2} &= \frac{\epsilon}{2}\int_\Omega f(w)v_1 dx \\
I_{1,3} &= \frac{\epsilon^2}{2}\int_\Omega \left(f(w)v_2 - f'(w)v_1^2\right)dx \\
I_{1,4} &= \epsilon^3\int_\Omega \left(\frac{1}{2}f(w)v_4 - f'(w)v_1v_2 + \frac{1}{6}f''(w)v_1^3\right)dx
\end{aligned}
$$

We compute these terms up to the order $o(\epsilon^{N+3})$. We state the following useful lemma, whose proof is delayed to Appendix C.

Lemma 4.1. *Suppose that $A(|y|)$ is a radially symmetric function such that*

$$|A'(|y|)| + |A''(|y|)| + |A'''(|y|)| + |A^{(4)}(|y|)| \le Ce^{-a|y|}$$

for some $a > 0$. Then, for ϵ sufficiently small, we have

$$
\begin{aligned}
A(\frac{x-P}{\epsilon}) &= A(y) + \epsilon\left[\frac{1}{2}\frac{A'(|y|)}{|y|}\rho''(0)y_1^2y_2\right] \\
&\quad + \epsilon^2\left[\frac{1}{2}\frac{A'(|y|)}{|y|}(\frac{1}{3}\rho'''(0)y_1^3y_2 + \frac{1}{4}(\rho''(0))^2y_1^4)\right] \\
&\quad + \epsilon^2\left[\frac{1}{8}(\frac{A''(|y|)}{|y|^2} - \frac{A'(|y|)}{|y|^3})(\rho''(0))^2y_1^4y_2^2\right] \\
&\quad + \epsilon^3\left[\frac{1}{2}\frac{A'(|y|)}{|y|}(\frac{1}{12}\rho^{(4)}(0)y_1^4y_2 + \frac{1}{6}\rho''(0)\rho'''(0)y_1^5)\right] \\
&\quad + \epsilon^3\left[\frac{1}{8}(\frac{A''(|y|)}{|y|^2} - \frac{A'(|y|)}{|y|^3})(\frac{2}{3}\rho''(0)\rho'''(0)y_1^5y_2^2 + \frac{1}{2}(\rho''(0))^3y_1^6y_2)\right]
\end{aligned}
$$

$$+\epsilon^3\left[\frac{1}{48}(\frac{A'''(|y|)}{|y|^3} - 3\frac{A''(|y|)}{|y|^4} + 3\frac{A'(|y|)}{|y|^5})(\rho''(0))^3y_1^6y_2^3\right] + O(\epsilon^4e^{-a|y|}) \tag{4.5}$$

and

$$
\begin{aligned}
\int_\Omega A(\frac{x-P}{\epsilon})dx &= \epsilon^N\int_{\mathbf{R}^2_+} A(y)dy - \frac{1}{2}\epsilon^{N+1}\rho''(0)\int_{\partial\mathbf{R}^2_+} A(y)y_1^2dy_1 \qquad (4.6) \\
&\quad - \frac{1}{24}\epsilon^{N+3}\int_{\partial\mathbf{R}^2_+}[\rho^{(4)}(0)A(y)y_1^4 + \frac{1}{2}(\rho''(0))^3A'(|y|)|y_1|^5]dy_1 + O(\epsilon^{N+4}).
\end{aligned}
$$

¿From Lemma 4.1, we obtain

$$I_{1,1} = \epsilon^N \int_{\mathbf{R}^2_+} \frac{1}{2}[wf(w)-F(w)]dy - \frac{1}{2}\epsilon^{N+1}\rho''(0)\int_{\mathbf{R}}(\frac{1}{2}wf(w)-F(w))y_1^2dy_1$$
$$-\frac{\epsilon^{N+3}}{24}\rho^{(4)}(0)\int_{\mathbf{R}}[\frac{1}{2}wf(w)-F(w)]y_1^4dy_1 - \frac{\epsilon^{N+3}}{96}(\rho''(0))^3\int_{\mathbf{R}}[wf'(w)-f(w)]w'|y_1|^5dy_1.$$

This finishes the computation for $I_{1,1}$.

For $I_{1,2}$, we need to expand $\int_\Omega f(w)v_1dx$ up to the order $O(\epsilon^{N+2})$. Using Lemma 4.1 again, we have

$$\int_\Omega f(w)v_1dy = \epsilon^N\int_{\mathbf{R}^2_+} f(w)v_1dy + \epsilon^{N+1}\rho''(0)\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{2|y|}y_1^2y_2v_1dy$$
$$+\frac{\epsilon^{N+2}}{6}\rho'''(0)\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{|y|}y_1^3y_2v_1dy + \frac{\epsilon^{N+2}}{8}(\rho''(0))^2\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{|y|}y_1^4v_1dy$$
$$+\frac{\epsilon^{N+2}}{8}(\rho''(0))^2\int_{\mathbf{R}^2_+}[\frac{f''(w)(w')^2+f'(w)w''}{|y|^2}-\frac{f'(w)w'}{|y|^3}]y_1^4y_2^2v_1dy + O(\epsilon^{N+3})$$

which implies

$$I_{1,2} = \frac{\epsilon}{2}\int_\Omega f(w)v_1dy$$
$$= \epsilon^{N+1}\int_{\mathbf{R}^2_+}\frac{1}{2}f(w)v_1dy + \epsilon^{N+2}\rho''(0)\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{4|y|}y_1^2y_2v_1dy$$
$$+\frac{\epsilon^{N+3}}{16}(\rho''(0))^2\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{|y|}y_1^4v_1dy$$
$$+\frac{\epsilon^{N+3}}{16}(\rho''(0))^2\int_{\mathbf{R}^2_+}[\frac{f''(w)(w')^2+f'(w)w''}{|y|^2}-\frac{f'(w)w'}{|y|^3}]y_1^4y_2^2v_1dy + O(\epsilon^{N+4}).$$

Next we compute $I_{1,3}$. Observe that

$$f'(w(\frac{x-P}{\epsilon})) = f'(w(|y|)) + \epsilon\rho''(0)\frac{f''(w)w'}{2|y|}y_1^2y_2 + O(\epsilon^2).$$

Hence,

$$\int_\Omega f(w)v_2dx - \int_\Omega f'(w)v_1^2dx = \epsilon^N\int_{\mathbf{R}^2_+}[f(w)v_2 - f'(w)v_1^2]dy$$
$$+\epsilon^{N+1}\rho''(0)\int_{\mathbf{R}^2_+}[\frac{f'(w)w'}{2|y|}y_1^2y_2v_2 - \frac{f'(w)w'}{2|y|}y_1^2y_2v_1^2]dy + O(\epsilon^{N+2}).$$

Thus,

$$I_{1,3} = \frac{\epsilon^{N+2}}{2}\int_{\mathbf{R}^2_+}[f(w)v_2 - f'(w)v_1^2]dy$$
$$+\frac{\epsilon^{N+3}}{4}\rho''(0)\int_{\mathbf{R}^2_+}\frac{f'(w)w'}{|y|}y_1^2y_2v_2 - \frac{f''(w)w'}{|y|}y_1^2y_2v_1^2dy + O(\epsilon^{N+4}).$$

Finally we estimate $I_{1,4}$:

$$I_{1,4} = \epsilon^{N+3} \int_{\mathbf{R}^2_+} [\frac{1}{2} f(w) v_4 - f'(w) v_1 v_2 + \frac{1}{6} f''(w) v_1^3] dy + O(\epsilon^{N+4}). \tag{4.7}$$

Combining the estimates for $I_{1,1}, I_{1,2}, I_{1,3}$ and $I_{1,4}$, we conclude that

$$J_\epsilon[w_{\epsilon,P}] = \epsilon^N \frac{1}{2} I[w] + B_1 \epsilon^{N+1} + \widetilde{B_2} \epsilon^{N+2} + \widetilde{B_3} \epsilon^{N+3} + O(\epsilon^{N+4}),$$

where

$$\begin{aligned}
B_1 &= \int_{\mathbf{R}^2_+} \frac{1}{2} f(w) v_1 dy - \frac{1}{2} \rho''(0) \int_{\partial \mathbf{R}^2_+} [\frac{1}{2} w f(w) - f(w)] y_1^2 dy_1 \\
\widetilde{B_2} &= \frac{1}{2} \int_{\mathbf{R}^2_+} [f(w) v_2 - f'(w) v_1^2] dy + \rho''(0) \int_{\mathbf{R}^2_+} \frac{f'(w) w'}{4|y|} y_1^2 y_2 v_1 dy \\
\widetilde{B_3} &= \int_{\mathbf{R}^2_+} [\frac{1}{2} f(w) v_4 - f'(w) v_1 v_2 + \frac{1}{6} f''(w) v_1^3] dy \\
&\quad + (\rho''(0))^2 \{ \int_{\mathbf{R}^2_+} \frac{f'(w) w'}{16|y|} y_1^4 v_1 dy + \int_{\mathbf{R}^2_+} \frac{1}{16} [\frac{f''(w)(w')^2 + f'(w) w''}{|y|^2} - \frac{f'(w) w'}{|y|^3}] y_1^4 y_2^2 v_1 dy \} \\
&\quad + \rho''(0) [\frac{1}{4} \int_{\mathbf{R}^2_+} \frac{f'(w) w'}{|y|} y_1^2 y_2 v_2 - \frac{f'(w) w'}{|y|} y_1^2 y_2 v_1^2] - \frac{1}{96} \rho'''(0) \int_{\mathbf{R}} [w f'(w) - f(w)] w' |y_1|^5 dy_1 \\
&\quad - \rho^{(4)}(0) \frac{1}{24} \int_{\mathbf{R}} (\frac{1}{2} w f(w) - F(w)) y_1^4 dy_1.
\end{aligned}$$

Recalling (4.4), we conclude that

$$J_\epsilon[\tilde{w}_{\epsilon,P}] = \frac{1}{2} I[w] \epsilon^N + B_1 \epsilon^{N+1} + B_2 \epsilon^{N+2} + B_3 \epsilon^{N+3} + O(\epsilon^{N+4}),$$

where

$$\begin{aligned}
B_2 &= \int_{\mathbf{R}^2_+} \frac{1}{2} f'(w) v_1 \Phi_0 dy + \widetilde{B_2} \\
B_3 &= \int_{\mathbf{R}^2_+} [f'(w) v_2 \Phi_0 - \frac{1}{2} f''(w) v_1^2 \Phi_0 + \frac{1}{2} f''(w) v_1 \Phi_0^2 - \frac{1}{6} \Phi_0^3 f''(w)] dy \\
&\quad + \int_{\mathbf{R}^2_+} [\frac{\rho''(0)}{2} v_1 \frac{f''(w) w'}{|y|} y_1^2 y_2 \Phi_0 - \rho''(0) y_1 \frac{\partial \Phi_0}{\partial y_1} \frac{\partial \Phi_0}{\partial y_2} - \frac{\rho''(0)}{4} \frac{f''(w) w'}{|y|} y_1^2 y_2 |\Phi_0|^2] dy + \widetilde{B_3}.
\end{aligned}$$

Since we are interested in the contributions of $\rho^{(4)}(0)\epsilon^{N+3}$, we only consider those coefficients of ϵ^{N+3} involving $\rho^{(4)}(0)$. It turns out that we only have to study the terms $\int_{\mathbf{R}^2_+} f(w) v_4$ and $-\frac{1}{24} \int_{\mathbf{R}} (\frac{1}{2} w f(w) - F(w) y_1^4 dy_1$. Note that

$$\begin{aligned}
\int_{\mathbf{R}^2_+} f(w) v_4 dy &= - \int_{\mathbf{R}^2_+} (\Delta w - w) v_4 dy = - \int_{\partial \mathbf{R}^2_+} w \frac{\partial v_4}{\partial y_2} \\
&= - \int_{\partial \mathbf{R}^2_+} w [\frac{w^{'}(|y|)}{|y|} y_1^4 (\frac{1}{2} (\rho^{''}(0))^3 - \frac{1}{8} \rho^{(4)}(0)) + \rho^{''}(0) y_1 \frac{\partial v_2}{\partial y_1}] dy_1.
\end{aligned}$$

Hence, we conclude that the coefficient of $\rho^{(4)}(0)$ is

$$\begin{aligned} c_3 &= \frac{1}{2}\int_{\mathbf{R}} \frac{ww'}{8|y|} y_1^4 dy_1 - \frac{1}{24}\int_{\mathbf{R}} (\frac{1}{2}wf(w) - F(w)) y_1^4 dy_1 \\ &= \frac{1}{48}\int_{\mathbf{R}} [3\frac{ww'}{|y|} - wf(w) + 2F(w)] y_1^4 dy_1. \end{aligned} \tag{4.8}$$

Furthermore, we can also simplify the coefficient $-c_1$ of $\rho''(0)\epsilon^{N+1}$ in the same way and we get

$$c_1 = \frac{1}{4}\int_{\mathbf{R}} [\frac{ww'}{|y|} - wf(w) + 2F(w)] y_1^2 dy_1. \tag{4.9}$$

¿From the Lemma 3.2 in [43], we know that B_2 can be simplified as follows:

$$B_2 = \frac{1}{8}(\rho''(0))^2 \int_{\partial \mathbf{R}^2_+} \Psi \frac{\partial \Psi}{\partial y_2} dy_1 = c_2 (H(P))^2, \tag{4.10}$$

where c_2 is defined by the last equality and Ψ is the unique solution of the following problem:

$$\begin{cases} \Delta \Psi - \Psi + f'(w)\Psi = 0 & \text{in } \mathbf{R}^2_+, \\ \frac{\partial \Psi}{\partial y_2} = \frac{w'(|y|)}{|y|} y_1^2 & \text{on } \partial \mathbf{R}^2_+. \end{cases} \tag{4.11}$$

Finally, due to integration by parts, the coefficient of ϵ^{N+3} can be written as

$$A_1(\rho''(0)) + A_2(\rho''(0))^2 + A_3(\rho''(0))^3 + c_3\rho^{(4)}(0),$$

where A_1, A_2 and A_3 are generic constants.

In summary, we have derived the following proposition.

Proposition 4.1. *Let $P \in \partial\Omega$ and $\tilde{w}_{\epsilon,P}$ be defined in (3.9). Then, for ϵ sufficiently small, we have*

$$J_\epsilon[\tilde{w}_{\epsilon,P}] = \epsilon^2 \{ \frac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + c_2 \epsilon^2 (H(P_\epsilon))^2 + \epsilon^3 [P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^3) \}, \tag{4.12}$$

where

$$P(H(P_\epsilon)) = A_1 H(P_\epsilon) + A_2 (H(P_\epsilon))^2 + A_3 (H(P_\epsilon))^3,$$

c_1 is defined by (4.9), c_2 is defined by (4.10), c_3 is defined by (4.8), and A_1, A_2,A_3 are generic constants.

5. The Signs of c_1 And c_3

In this section, we are concerned with the signs of c_1 and c_3. Even though we can not compute them explicitly, we can determine their sign.

The sign of c_1 has been shown to be positive (Proposition 3.2 of [28]). So we just need to determine the sign of c_3.

By (4.8), we have

$$\begin{aligned}
96c_3 &= 2\int_0^\infty [3\frac{ww'}{r} - wf(w) + 2F(w)]r^4 dr \\
&= 2\int_0^\infty [3\frac{ww'}{r} + w(w'' + \frac{1}{r}w' - w) + 2F(w)]r^4 dr \\
&= 2\int_0^\infty [ww''r^4 + 4ww'r^3]dr - \int_0^\infty [w^2 - 2F(w)]r^4 dr \\
&= -2\int_0^\infty [(w')^2 + w^2 - 2F(w)]r^4 dr \\
&= -\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(w')^2 + w^2 - 2F(w)]r^4 \cos\theta d\theta dr \\
&= -\int_{\mathbf{R}^2_+} [|\nabla w|^2 + w^2 - 2F(w)]|y|^2 y_2 dy.
\end{aligned}$$

Now we state the following lemma

Lemma 5.1. *Let w be the ground state solution of*

$$\Delta w - w + f(w) = 0 \text{ in } \mathbf{R^2_+}. \tag{5.1}$$

Then we have

$$\int_{\mathbf{R}^2_+} [|\nabla w|^2 + w^2 - 2F(w)]|y|^2 y_2 dy = \int_{\mathbf{R}^2_+} 2(\frac{\partial w}{\partial y_2})^2 y_2 |y|^2 dy + \int_{\mathbf{R}^2_+} 2y_1 y_2^2 \frac{\partial w}{\partial y_1}\frac{\partial w}{\partial y_2} dy \tag{5.2}$$

Let us first assume that Lemma 5.1 holds. We then have

Lemma 5.2. *We have $c_3 < 0$.*

Proof: From Lemma 5.1, we have

$$-48c_3 = \int_{\mathbf{R}^2_+} 2(\frac{\partial w}{\partial y_2})^2 y_2 |y|^2 dy + \int_{\mathbf{R}^2_+} 2y_1 y_2^2 \frac{\partial w}{\partial y_1}\frac{\partial w}{\partial y_2} dy. \tag{5.3}$$

Since w is radially symmetric and $w^{'}(r) \le 0$, it is easy to see that both terms on the right hand side of (5.3) are positive. Hence $c_3 < 0$. □

We are now ready to prove Lemma 5.1.

Proof: We first multiply both sides of (5.1) by $|y|^2 y_2^2 \frac{\partial w}{\partial y_2}$ and then integrate over $\mathbf{R^2_+}$:

$$\int_{\mathbf{R}^2_+} (|y|^2 y_2^2 \frac{\partial w}{\partial y_2})\Delta w dy - \int_{\mathbf{R}^2_+} w|y|^2 y_2^2 \frac{\partial w}{\partial y_2} dy + \int_{\mathbf{R}^2_+} f(w)|y|^2 y_2^2 \frac{\partial w}{\partial y_2} dy = 0. \tag{5.4}$$

We compute the three integrals of the left-hand side of (5.4) separately:

$$\begin{aligned}
&\int_{\mathbf{R^2_+}} (|y|^2 y_2^2 \frac{\partial w}{\partial y_2}) \Delta w dy \\
= & -\int_{\mathbf{R^2_+}} \nabla w \cdot \nabla (|y|^2 y_2^2 \frac{\partial w}{\partial y_2}) dy \\
= & -\int_{\mathbf{R^2_+}} (\nabla w \cdot \nabla \frac{\partial w}{\partial y_2}) |y|^2 y_2 dy - \int_{\mathbf{R^2_+}} (\nabla w \cdot \nabla y_2^2) |y|^2 \frac{\partial w}{\partial y_2} dy - \int_{\mathbf{R^2_+}} (\nabla w \cdot \nabla |y|^2) y_2^2 \frac{\partial w}{\partial y_2}) dy \\
= & -\int_{\mathbf{R^2_+}} \frac{1}{2} \frac{\partial |\nabla w|^2}{\partial y_2} |y|^2 y_2^2 dy - \int_{\mathbf{R^2_+}} 2y_2 |y|^2 (\frac{\partial w}{\partial y_2})^2 dy - \int_{\mathbf{R^2_+}} [2y_1 \frac{\partial w}{\partial y_1} + 2y_2 \frac{\partial w}{\partial y_2}] y_2^2 \frac{\partial w}{\partial y_2} dy \\
= & \int_{\mathbf{R^2_+}} |\nabla w|^2 |y|^2 y_2 dy + \int_{\mathbf{R^2_+}} |\nabla w|^2 y_2^3 dy - \int_{\mathbf{R^2_+}} 2y_2 |y|^2 (\frac{\partial w}{\partial y_2})^2 dy \\
& -\int_{\mathbf{R^2_+}} 2y_1 y_2^2 \frac{\partial w}{\partial y_1} \frac{\partial w}{\partial y_2} dy - \int_{\mathbf{R^2_+}} 2y_2^3 (\frac{\partial w}{\partial y_2})^2 dy.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{\mathbf{R^2_+}} w |y|^2 y_2^2 \frac{\partial w}{\partial y_2} dy &= \int_{\mathbf{R^2_+}} \frac{1}{2} |y|^2 y_2^2 (\frac{\partial w^2}{\partial y_2}) dy \\
&= -\int_{\mathbf{R^2_+}} w^2 |y|^2 y_2 dy - \int_{\mathbf{R^2_+}} w^2 y_2^3 dy,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbf{R^2_+}} f(w) |y|^2 y_2^2 \frac{\partial w}{\partial y_2} dy = 0 &= \int_{\mathbf{R^2_+}} |y|^2 y_2^2 \frac{\partial F(w)}{\partial y_2} dy \\
&= -\int_{\mathbf{R^2_+}} 2F(w) |y|^2 y_2 dy - \int_{\mathbf{R^2_+}} 2F(w) y_2^3 dy.
\end{aligned}$$

Combining all together, we obtain

$$\begin{aligned}
&\int_{\mathbf{R^2_+}} 2(\frac{\partial w}{\partial y_2})^2 y_2 |y|^2 dy + \int_{\mathbf{R^2_+}} 2y_1 y_2^2 \frac{\partial w}{\partial y_1} \frac{\partial w}{\partial y_2} dy \\
= & \int_{\mathbf{R^2_+}} 2y_2 |y|^2 (\frac{\partial w}{\partial y_2})^2 dy + \int_{\mathbf{R^2_+}} 2y_1 y_2^2 \frac{\partial w}{\partial y_1} \frac{\partial w}{\partial y_2} dy + \int_{\mathbf{R^2_+}} 2y_2^3 (\frac{\partial w}{\partial y_2})^2 dy + \int_{\mathbf{R^2_+}} [2F(w) - |\nabla w|^2 - w^2] y_2^3 dy.
\end{aligned} \tag{5.5}$$

Lemma 5.2 follows from the following identity:

$$\int_{\mathbf{R^2_+}} [2F(w) - |\nabla w|^2 - w^2] y_2^3 dy = -2 \int_{\mathbf{R^2_+}} (\frac{\partial w}{\partial y_2})^2 y_2^3 dy. \tag{5.6}$$

The proof of (5.6) is similar to that of (5.5): multiplying both sides of (5.1) by $y_2^4 \frac{\partial w}{\partial y_2}$ and integrating over $\mathbf{R^2_+}$, we obtain

$$\int_{\mathbf{R^2_+}} y_2^4 \frac{\partial w}{\partial y_2} \Delta w dy - \int_{\mathbf{R^2_+}} y_2^4 \frac{\partial w}{\partial y_2} w dy + \int_{\mathbf{R^2_+}} y_2^4 \frac{\partial w}{\partial y_2} f(w) dy = 0. \tag{5.7}$$

Note that

$$\begin{aligned} LHS \text{ of } (5.7) &= -\int_{\mathbf{R}^2_+} \nabla w \cdot \nabla (y_2^4 \frac{\partial w}{\partial y_2}) dy - \int_{\mathbf{R}^2_+} \frac{1}{2} y_2^4 \frac{\partial w^2}{\partial y_2} dy + \int_{\mathbf{R}^2_+} y_2^4 \frac{\partial F(w)}{\partial y_2} dy \\ &= -\int_{\mathbf{R}^2_+} 4y_2^3 (\frac{\partial w}{\partial y_2})^2 dy - \int_{\mathbf{R}^2_+} \frac{1}{2} y_2^4 \frac{\partial |\nabla w|^2}{\partial y_2} dy + \int_{\mathbf{R}^2_+} 2w^2 y_2^3 dy - \int_{\mathbf{R}^2_+} 4F(w) y_2^3 dy \\ &= -\int_{\mathbf{R}^2_+} 4y_2^3 (\frac{\partial w}{\partial y_2})^2 dy + \int_{\mathbf{R}^2_+} 2|\nabla w|^2 y_2^3 dy + \int_{\mathbf{R}^2_+} 2w^2 y_2^3 dy - \int_{\mathbf{R}^2_+} 4F(w) y_2^3 dy \\ &= RHS \text{ of } (5.7) = 0 \end{aligned}$$

yielding (5.6). □

6. The Asymptotic Behavior of u_ϵ and $J_\epsilon[u_\epsilon]$

Let u_ϵ be a single boundary spike solution of (1.1) and P_ϵ be its local maximum point. In this section, we compute the energy of u_ϵ. The key observation is that by using $\tilde{w}_{\epsilon,P_\epsilon}$ as our approximating function, we just need to expand u_ϵ up to $O(\epsilon^\tau)$ for some $\tau > \frac{3}{2}$. Now, we choose $\frac{3}{2} < \tau < 2$.

The main result in this section is the following theorem.

Theorem 6.1. *For ϵ sufficiently small, we have*

$$u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon, \tag{6.1}$$

where ϕ_ϵ satisfies

$$\|\phi_\epsilon\|_{L^\infty(\overline{\Omega})} + \epsilon^{-N} \int_\Omega (\epsilon^2 |\nabla \phi_\epsilon|^2 + |\phi_\epsilon|^2) dx \le C. \tag{6.2}$$

Let us first assume that Theorem 6.1 holds. We then have

Lemma 6.1. *For ϵ sufficiently small, we have*

$$J_\epsilon[u_\epsilon] = J_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] + o(\epsilon^{N+3}). \tag{6.3}$$

Proof: Note that both $\tilde{w}_{\epsilon,P_\epsilon}$ and ϕ_ϵ satisfy the Neumann boundary condition. So we have

$$\begin{aligned} J_\epsilon[u_\epsilon] &= \frac{1}{2} \int_\Omega \{\epsilon^2 |\nabla \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \nabla \phi_\epsilon|^2 + |\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon|^2\} dx - \int_\Omega F(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon) dx \\ &= J_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] + \epsilon^\tau \int_\Omega \{\epsilon^2 \nabla \tilde{w}_{\epsilon,P_\epsilon} \nabla \phi_\epsilon + \tilde{w}_{\epsilon,P_\epsilon} \phi_\epsilon - f(\tilde{w}_{\epsilon,P_\epsilon}) \phi_\epsilon\} dx \\ &\quad + \frac{\epsilon^{2\tau}}{2} \int_\Omega \{\epsilon^2 |\nabla \phi_\epsilon|^2 + |\phi_\epsilon|^2 - f'(\tilde{w}_{\epsilon,P_\epsilon}) \phi_\epsilon^2\} dx \\ &\quad - \int_\Omega \{F(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon) - F(\tilde{w}_{\epsilon,P_\epsilon}) - f(\tilde{w}_{\epsilon,P_\epsilon}) \epsilon^\tau \phi_\epsilon - \frac{1}{2} f'(\tilde{w}_{\epsilon,P_\epsilon}) \epsilon^{2\tau} \phi_\epsilon^2\} dx. \end{aligned}$$

By Theorem 6.1, the last two terms are $O(\epsilon^{N+2\tau})$. Now, we consider that

$$\begin{aligned} \epsilon^\tau \int_\Omega \{\epsilon^2 \nabla \tilde{w}_{\epsilon,P_\epsilon} \nabla \phi_\epsilon + \tilde{w}_{\epsilon,P_\epsilon} \phi_\epsilon - f(\tilde{w}_{\epsilon,P_\epsilon}) \phi_\epsilon\} dx &= \epsilon^\tau \int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] \phi_\epsilon dx \\ &\le \epsilon^\tau \int_\Omega |S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}]| dx \|\phi_\epsilon\|_{L^\infty} = O(\epsilon^{N+2+\tau}) \end{aligned}$$

which finishes the proof of Lemma 6.1. □

We are now ready to prove Theorem 6.1. The key step is the following lemma.

Lemma 6.2. *For ϵ sufficiently small, we have*

$$\|\phi_\epsilon\|_{L^\infty(\overline{\Omega})} \le C. \tag{6.4}$$

Proof: Recall

$$\begin{aligned} S_\epsilon[u] &= \epsilon^2\Delta u - u + f(u), \\ S'_\epsilon[u](\phi) &= \epsilon^2\Delta\phi - \phi + f'(u)\phi. \end{aligned}$$

Substituting $u_\epsilon = \tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau\phi_\epsilon$ into the equation

$$\epsilon^2\Delta u - u + f(u) = 0,$$

we see that ϕ_ϵ satisfies

$$\begin{cases} \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon = -\epsilon^{-\tau}S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] + N_\epsilon[\phi_\epsilon] \text{ in } \Omega, \\ \frac{\partial\phi_\epsilon}{\partial\nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.5}$$

where

$$N_\epsilon[\phi_\epsilon] = -\epsilon^{-\tau}[f(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau\phi_\epsilon) - f(\widetilde{\epsilon, P_\epsilon}) - \epsilon^\tau f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon].$$

By Lemma 3.1, $S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] = O(\epsilon^2)$, we have

$$\epsilon^{-\tau}S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}] = O(\epsilon^{2-\tau}).$$

On the other hand, by mean-value theorem, we get

$$\begin{aligned} |N_\epsilon[\phi_\epsilon]| &= \epsilon^{-\tau}|f(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau\phi_\epsilon) - f(\tilde{w}_{\epsilon,P_\epsilon}) - \epsilon^\tau f'(\tilde{w}_{\epsilon,P_\epsilon})\phi_\epsilon| \\ &\le C|\phi_\epsilon||\epsilon^\tau\phi_\epsilon|. \end{aligned}$$

Thus,

$$|N_\epsilon[\phi_\epsilon]| = o(1)|\phi_\epsilon|.$$

Now, we can prove Lemma 6.2.

Suppose not. That is, there exists a sequence $\epsilon_k \to 0$ such that $\|\phi_{\epsilon_k}\|_{L^\infty(\overline{\Omega})} \to +\infty$. For simplicity, we still denote ϵ_k as ϵ. Set

$$M_\epsilon = \|\phi_\epsilon\|_{L^\infty(\overline{\Omega})} \to +\infty.$$

Let $M_\epsilon = |\phi_\epsilon(x_\epsilon)|$, where

$x_\epsilon \in \overline{\Omega}$. Without loss of generality, we may assume that x_ϵ is a maximum point of ϕ_ϵ. We proceed in two claims.

Claim 1: $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \le C$.

In fact, suppose not. That is $\frac{|x_\epsilon - P_\epsilon|}{\epsilon} \to +\infty$. Then

$$-1 + f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)) \le -\tfrac{1}{4} \text{ for } \epsilon \text{ small}.$$

Since $\frac{\partial\phi_\epsilon}{\partial\nu} = 0$, by the Hopf boundary Lemma, it is impossible to have $x_\epsilon \in \partial\Omega$. Thus, $x_\epsilon \in \Omega$, which implies that

$$\Delta\phi_\epsilon \le 0.$$

¿From (6.5), we deduce that

$$(1 - f'(\tilde{w}_{\epsilon,P_\epsilon}(x_\epsilon)))M_\epsilon + o(1)M_\epsilon + O(\epsilon^{\tau-1}) \le 0$$

and hence M_ϵ is bounded. This gives a contradiction and the proof of Claim 1 is completed.

Let

$$\widehat{\phi}_\epsilon(y) = \frac{\widehat{\phi_\epsilon(x)}}{M_\epsilon}\chi(x - P_\epsilon),\ y = T_\epsilon(x). \tag{6.6}$$

Claim 2: $\widehat{\phi}_\epsilon(y) \to 0$ in $C^1_{loc}(\mathbf{R^2_+})$ as $\epsilon \to 0$.
In fact, from the equation for $\widehat{\phi}_\epsilon$, we see that as $\epsilon \to 0$, $\widehat{\phi}_\epsilon \to \widehat{\phi}_0$ which satisfies

$$\begin{cases} \Delta\widehat{\phi}_0 - \widehat{\phi}_0 + f'(w)\widehat{\phi}_0 = 0, \ |\widehat{\phi}_0| \le 1 \text{ in } \mathbf{R^2_+}, \\ \frac{\partial \widehat{\phi}_0}{\partial y_2} = 0 \text{ on } \partial\mathbf{R^2_+}. \end{cases} \tag{6.7}$$

By the nondegeneracy of w, there exists a constant a_1 such that

$$\widehat{\phi}_0 = a_1 \frac{\partial w}{\partial y_1}.$$

On the other hand, we know that

$$\nabla_{x_1} u_\epsilon(P_\epsilon) = 0.$$

Hence, we have

$$\begin{aligned} 0 &= \nabla_{x_1}(\tilde{w}_{\epsilon,P_\epsilon} + \epsilon^\tau \phi_\epsilon) \\ &= O(\epsilon^2) + \nabla_{x_1}(w(\frac{x-P_\epsilon}{\epsilon}) - \epsilon v_1\chi - \epsilon^2(v_2+v_3)\chi - \epsilon^3(v_4+v_5)\chi) + \epsilon^{\tau-1}M_\epsilon \nabla_{y_1}\widehat{\phi}_\epsilon(0) \\ &= O(\epsilon^2) + \epsilon^{\tau-1}M_\epsilon \nabla_{y_1}\widetilde{\phi}_\epsilon(0). \end{aligned}$$

(Note that $\nabla_{y_1} v_1(0) = \nabla_{y_1} v_2(0) = 0$.) Thus, we have $\nabla_{y_1}\widehat{\phi}_\epsilon(0) \to 0$ which shows that $\nabla_{y_1}\widehat{\phi}_\epsilon = 0$. This implies that

$$\nabla_{y_1}(a_1 \frac{\partial w}{\partial y_1})|_{y=0} = 0$$

and $a_1 = 0$. This proves Claim 2.

Lemma 6.2 now follows from Claim 1 and Claim 2: let $y_\epsilon = \frac{x_\epsilon - P_\epsilon}{\epsilon}$, then by Claim 1, we have $|y_\epsilon| \le C$. So we may assume that $y_\epsilon \to y_0$ as $\epsilon \to 0$. Since $\widehat{\phi}_\epsilon(y_\epsilon) = 1$, we have $\widehat{\phi}_0(y_0) = 1$ which contradicts Claim 2. □

Proof of Theorem 6.1: Theorem 6.1 now follows from Lemma 6.2. In fact, multiplying (6.5) by ϕ_ϵ and integrating over Ω, we obtain

$$\begin{aligned} &\epsilon^2 \int_\Omega |\nabla\phi_\epsilon|^2 dx + \int_\Omega |\phi_\epsilon|^2 dx \\ = &\int_\Omega f'(\tilde{w}_{\epsilon,P})\phi_\epsilon dx - \int_\Omega N_\epsilon[\phi_\epsilon]\phi_\epsilon dx + \epsilon^{-\tau}\int_\Omega \phi_\epsilon S_\epsilon[\tilde{w}_{\epsilon,P_\epsilon}]dx \\ \le &\ C\epsilon^N + o(1)\int_\Omega |\phi_\epsilon|^2 dx. \end{aligned}$$

This finishes the proof of Theorem 6.1. □

7. The Proofs Of Theorem 1.1 And Corollary 1.1

Theorem 1.1 follows from Lemma 6.1, Lemma 5.2 and Proposition 4.1.

To prove Theorem 1.2, we follow the proof of Theorem 1.1: first we note that

$$S_\epsilon[\sum_{j=1}^K \tilde{w}_{\epsilon,P_j^\epsilon}] = \sum_{j=1}^K S_\epsilon[\tilde{w}_{\epsilon,P_j^\epsilon}] + O(e^{-\delta/\epsilon}) \tag{7.1}$$

for some $\delta > 0$, since $\min_{i\neq j} |P_i^\epsilon - P_j^\epsilon| \geq \delta$. Then we decompose

$$u_\epsilon = \sum_{j=1}^K \tilde{w}_{\epsilon, P_j^\epsilon} + \epsilon^\tau \phi_\epsilon$$

and show that $\|\phi_\epsilon\|_{L^\infty(\bar{\Omega})} \leq C$. The rest of the proof is exactly the same.

It remains to prove Corollary 1.1.

Proof: Let u_ϵ be a least energy solution of (1.1). By Theorem 1.1, we have

$$\begin{aligned} c_\epsilon &= J_\epsilon[u_\epsilon] \\ &= \epsilon^N \left[\tfrac{1}{2} I[w] + c_1 \epsilon H(P_\epsilon) + c_2 \epsilon^2 (H(P_\epsilon))^2 + \epsilon^3 [P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^3)\right]. \end{aligned} \tag{7.2}$$

On the other hand, let

$$\beta(t) = J_\epsilon[t\tilde{w}_{\epsilon,P}], \quad t > 0 \tag{7.3}$$

By Lemma 3.1 of [28], we have

$$c_\epsilon \leq \max_{t>0} \beta(t). \tag{7.4}$$

By assumption (f3) (see(3.16) of [28]), there exists a unique $t = t_{\epsilon,P}$ such that

$$\beta'(t_{\epsilon,P}) = 0 \quad \beta(t_{\epsilon,P}) = \max\nolimits_{t>o} \beta(t).$$

Note that

$$\begin{aligned} \beta'(1) &= \int_\Omega [\epsilon^2 |\nabla \tilde{w}_{\epsilon,P}|^2 + (\tilde{w}_{\epsilon,P})^2 - f(\tilde{w}_{\epsilon,P})\tilde{w}_{\epsilon,P}] dx \\ &= \int_\Omega S_\epsilon[\tilde{w}_{\epsilon,P}]\tilde{w}_{\epsilon,P} dx = O(\epsilon^{N+2}). \end{aligned}$$

Similar to (3.16) of [28], one can show that

$$t_{\epsilon,P} = 1 + O(\epsilon^2). \tag{7.5}$$

Then

$$\begin{aligned} \beta(t_{\epsilon,P}) &= \beta(1) + \beta'(1)(t_{\epsilon,P} - 1) + O(\epsilon^N |t_{\epsilon,P} - 1|^2) \\ &= \beta(1) + O(\epsilon^{N+4}) \end{aligned}$$

which implies that

$$\begin{aligned} c_\epsilon &\leq \max\nolimits_{t>0} \beta(t) = J_\epsilon[t_{\epsilon,P}\tilde{w}_{\epsilon,P}] = J_\epsilon[\tilde{w}_{\epsilon,P}] + o(\epsilon^{N+3}) \\ &\leq \epsilon^N \{\tfrac{1}{2} I[w] - c_1 \epsilon H(P_\epsilon) + c_2 \epsilon^2 (H(P_\epsilon))^2 + \epsilon^3 [P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^3)\} \end{aligned} \tag{7.6}$$

for any $P \in \partial\Omega$.

Now, we take $P = Q_0$ such that

$$H(Q_0) = \max_{P\in\partial\Omega} H(P), \quad S(Q_0) = \max\{S(Q) : Q \in \partial\Omega, H(Q) = \max_{P\in\partial\Omega} H(P)\}. \tag{7.7}$$

Comparing (7.6) with (7.2), we arrive at

$$\begin{aligned} & -c_1 H(Q_0) - c_2 \epsilon (H(Q_0))^2 - \epsilon^2 [P(H(Q_0)) + c_3 S(Q_0)] + o(\epsilon^2) \\ \leq & -c_1 H(P_\epsilon) - c_2 \epsilon (H(P\epsilon))^2 - \epsilon^2 [P(H(P_\epsilon)) + c_3 S(P_\epsilon)] + o(\epsilon^2). \end{aligned}$$

Since $c_1 > 0$, $c_3 < 0$,(the sign of c_2 and the $A_i's$ are not important), we conclude that

$$H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P) \quad S(P_\epsilon) \to \max_{Q\in\partial\Omega, H(Q) = \max_{P\in\partial\Omega} H(P)} S(Q)$$

as $\epsilon \to 0$.

This finishes the proof of Corollary 1.1. □

Appendix A: Proof of Proposition 2.1

To prove Proposition 2.1, we recall a lemma in [40].

Lemma A: *(Lemma 2.1 of* [40]*.) Let u be a solution of*

$$\begin{cases} \epsilon^2 \Delta u - u + f = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{7.8}$$

Assume that $\int_\Omega |f|^2 \le C\epsilon^N$ and $\int_{\partial\Omega} |g|^2 \le C\epsilon^{N-1}$, then

$$\epsilon^{-N} \int_\Omega \left(\epsilon^2 |\nabla u|^2 + |u|^2 \right) dx \le C. \tag{7.9}$$

We first compute the equation for $\Psi_{\epsilon,P}$:

$$\begin{aligned}
& -\epsilon^2 \Delta_x \Psi_{\epsilon,P} + \Psi_{\epsilon,P} \\
= \quad & \frac{1}{\epsilon^4} \left[\epsilon^2 \Delta_x \left\{ \epsilon v_1 \chi + \epsilon^2 (v_2 + v_3)\chi + \epsilon^3 (v_4 + v_5)\chi \right\} - \left\{ \epsilon v_1 \chi + \epsilon^2 (v_2 + v_3)\chi + \epsilon^3 (v_4 + v_5)\chi \right\} \right] \\
= \quad & \frac{1}{\epsilon^3} \left[\left[\Delta_y v_1 + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_1}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_1}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_1}{\partial y_2} - v_1 \right] \chi \right. \\
& + \epsilon \left[\Delta_y v_2 + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_2}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_2}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_2}{\partial y_2} - v_2 \right] \chi \\
& + \epsilon \left[\Delta_y v_3 + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_3}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_3}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_3}{\partial y_2} - v_3 \right] \chi \\
& + \epsilon^2 \left[\Delta_y v_4 + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_4}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_4}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_4}{\partial y_2} - v_4 \right] \chi \\
& \left. + \epsilon^2 \left[\Delta_y v_5 + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_5}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_5}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_5}{\partial y_2} - v_5 \right] \chi + E_\epsilon(\chi) \right] \\
= \quad & \frac{1}{\epsilon^3} \left[\left[|\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_1}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_1}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_1}{\partial y_2} \right] \chi \right. \\
& + \epsilon \left[2\rho''(0) y_1 \frac{\partial^2 v_1}{\partial y_1 \partial y_2} + \rho''(0) \frac{\partial v_1}{\partial y_2} + |\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_2}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_2}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_2}{\partial y_2} \right] \chi \\
& + \epsilon \left[|\rho'(\epsilon y_1)|^2 \frac{\partial^2 v_3}{\partial y_2^2} - 2\rho'(\epsilon y_1) \frac{\partial^2 v_3}{\partial y_1 \partial y_2} - \epsilon \rho''(\epsilon y_1) \frac{\partial v_3}{\partial y_2} \right] \chi
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^2\left[2\rho^{''}(0)y_1\frac{\partial^2 v_2}{\partial y_1\partial y_2}+\rho^{''}(0)\frac{\partial v_2}{\partial y_2}-(\rho^{''}(0))^2y_1^2\frac{\partial^2 v_1}{\partial y_2^2}\right.\\
&\left.+|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_4}{\partial y_2^2}-2\rho'(\epsilon y_1)\frac{\partial^2 v_4}{\partial y_1\partial y_2}-\epsilon\rho''(\epsilon y_1)\frac{\partial v_4}{\partial y_2}\right]\chi\\
&+\epsilon^2\left[\rho^{''}(0)\frac{\partial v_3}{\partial y_2}+2\rho^{''}(0)y_1\frac{\partial v_3}{\partial y_1\partial y_2}+\rho^{'''}(0)[y_1\frac{\partial v_1}{\partial y_2}+y_1^2\frac{\partial^2 v_1}{\partial y_1\partial y_2}]\right.\\
&\left.\left.+|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_5}{\partial y_2^2}-2\rho'(\epsilon y_1)\frac{\partial^2 v_5}{\partial y_1\partial y_2}-\epsilon\rho''(\epsilon y_1)\frac{\partial v_5}{\partial y_2}\right]\chi+E_\epsilon(\chi)\right]\\
=\ &\frac{1}{\epsilon^3}\left[\left[(|\rho'(\epsilon y_1)|^2-(\rho''(0))^2\epsilon^2y_1^2)\frac{\partial^2 v_1}{\partial y_2^2}-2(\rho'(\epsilon y_1)-\rho''(0)\epsilon y_1-\frac{1}{2}\rho'''(0)\epsilon^2y_1^2)\frac{\partial^2 v_1}{\partial y_1\partial y_2}\right.\right.\\
&\left.+\epsilon(\rho''(0)-\rho''(\epsilon y_1)+\rho'''(0)\epsilon y_1)\frac{\partial v_1}{\partial y_2}\right]\chi\\
&+\epsilon\left[|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_2}{\partial y_2^2}+2(\rho''(0)\epsilon y_1-\rho'(\epsilon y_1))\frac{\partial^2 v_2}{\partial y_1\partial y_2}+\epsilon(\rho''(0)-\rho''(\epsilon y_1))\frac{\partial v_2}{\partial y_2}\right]\chi\\
&+\epsilon\left[|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_3}{\partial y_2^2}+2(\rho''(0)\epsilon y_1-\rho'(\epsilon y_1))\frac{\partial^2 v_3}{\partial y_1\partial y_2}+\epsilon(\rho''(0)-\rho''(\epsilon y_1))\frac{\partial v_3}{\partial y_2}\right]\chi\\
&+\epsilon^2\left[|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_4}{\partial y_2^2}-2\rho'(\epsilon y_1)\frac{\partial^2 v_4}{\partial y_1\partial y_2}-\epsilon\rho''(\epsilon y_1)\frac{\partial v_4}{\partial y_2}\right]\chi\\
&\left.+\epsilon^2\left[|\rho'(\epsilon y_1)|^2\frac{\partial^2 v_5}{\partial y_2^2}-2\rho'(\epsilon y_1)\frac{\partial^2 v_5}{\partial y_1\partial y_2}-\epsilon\rho''(\epsilon y_1)\frac{\partial v_5}{\partial y_2}\right]\chi+E_\epsilon(\chi)\right]\\
=\ &f_\epsilon(x),
\end{aligned}
$$

where E_ϵ denotes all the terms involving derivatives of χ.

Since $|v_1|,|v_2|,|v_3|,|v_4|,|v_5|\le Ce^{-a|y|}$ for some positive constant a, we have $f_\epsilon\in L^2(\Omega_{\epsilon,P})$ and $\int_{\Omega_{\epsilon,P}}f_\epsilon^2\le C$. On the other hand, for $x\in\partial\Omega$, it holds that

$$
\epsilon\frac{\partial\Psi_{\epsilon,P}}{\partial v}=\frac{1}{\epsilon^3}\left[\frac{\partial h_{\epsilon,P}}{\partial\nu}-\epsilon\frac{\partial(v_1\chi)}{\partial\nu}-\epsilon^2\frac{\partial(v_2\chi)}{\partial\nu}-\epsilon^2\frac{\partial(v_3\chi)}{\partial\nu}-\epsilon^3\frac{\partial(v_4\chi)}{\partial\nu}-\epsilon^3\frac{\partial(v_5\chi)}{\partial\nu}\right].
$$

Using (2.6), we have for $x\in\omega_1$,

$$
\frac{|x-P|}{\epsilon}=|y|\left(1+\frac{\epsilon^2}{4}\left(\rho''(0)\right)^2\frac{y_1^4}{|y|^2}+O(\epsilon^3)\right)^{\frac{1}{2}}. \tag{7.10}
$$

Using (2.2) and (2.8) , we have the following

$$
\begin{aligned}
\sqrt{1+(\rho')^2}\frac{\partial h_{\epsilon,P}}{\partial \nu} &= \sqrt{1+(\rho')^2}\frac{\partial w(\frac{x-P}{\epsilon})}{\partial \nu} \\
&= w'(\frac{x-P}{\epsilon})\frac{\epsilon y_1 \rho'(\epsilon y_1)-\rho(\epsilon y_1)}{\epsilon|x-P|} \\
&= \frac{w'(|y|)}{|y|}\left[\frac{1}{2}\rho''(0)y_1^2+\frac{\epsilon}{3}\rho'''(0)y_1^3+\frac{\epsilon^2}{8}\rho^{(4)}(0)y_1^4\right] \\
&\quad +\frac{\epsilon^2}{16}\left(\rho''(0)\right)^3\left(\frac{w'(|y|)}{|y|}\right)'\frac{y_1^6}{|y|}+O(\epsilon^3 e^{-a|y|}), \\
\sqrt{1+(\rho')^2}\frac{\partial v_1}{\partial \nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial v_1}{\partial y_1}+\frac{w'(|y|)}{|y|}\frac{1}{2}\rho''(0)y_1^2+(\rho'(\epsilon y_1))^2\frac{w'(|y|)}{|y|}\frac{1}{2}\rho''(0)y_1^2\right], \\
\sqrt{1+(\rho')^2}\frac{\partial v_2}{\partial \nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial v_2}{\partial y_1}-\rho''(0)y_1\frac{\partial v_1}{\partial y_1}-(\rho'(\epsilon y_1))^2\rho''(0)y_1\frac{\partial v_1}{\partial y_1}\right], \\
\sqrt{1+(\rho')^2}\frac{\partial v_3}{\partial \nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial v_3}{\partial y_1}+\frac{1}{3}\frac{w'(|y|)}{|y|}\rho'''(0)y_1^3+(\rho'(\epsilon y_1))^2\frac{1}{3}\frac{w'(|y|)}{|y|}\rho'''(0)y_1^3\right], \\
\sqrt{1+(\rho')^2}\frac{\partial v_4}{\partial \nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial v_4}{\partial y_1}-\frac{\partial v_4}{\partial y_2}-\left(\rho'(\epsilon y_1)\right)^2\frac{\partial v_4}{\partial y_2}\right] \\
&= \frac{1}{\epsilon}\Bigg[-\frac{w'(|y|)}{|y|}y_1^4[\frac{1}{2}(\rho''(0))^3-\frac{1}{8}\rho^{(4)}(0)]-\rho''(0)y_1\frac{\partial v_2}{\partial y_1} \\
&\quad +\frac{1}{16}\left(\rho''(0)\right)^3\left(\frac{w'(|y|)}{|y|}\right)'\frac{y_1^6}{|y|}+O(\epsilon e^{-a|y|})\Bigg], \\
\sqrt{1+(\rho')^2}\frac{\partial v_5}{\partial \nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial v_5}{\partial y_1}-\frac{\partial v_5}{\partial y_2}-\left(\rho'(\epsilon y_1)\right)^2\frac{\partial v_5}{\partial y_2}\right] \\
&= \frac{1}{\epsilon}\left[-\rho''(0)y_1\frac{\partial v_3}{\partial y_1}-\frac{1}{2}\rho'''(0)y_1^2\frac{\partial v_1}{\partial y_1}+O(\epsilon e^{-a|y|})\right].
\end{aligned}
$$

Hence, we get

$$\begin{aligned}
&\epsilon\sqrt{1+(\rho'(\epsilon y_1))^2}\frac{\partial \Psi_{\epsilon,P}}{\partial \nu}\\
= &\ \frac{1}{\epsilon^3}\left[\frac{w'(|y|)}{|y|}\left[\frac{1}{2}\rho''(0)y_1^2+\frac{1}{3}\rho'''(0)\epsilon y_1^3+\frac{\epsilon^2}{8}\rho^{(4)}(0)y_1^4\right]+\frac{\epsilon^2}{16}\left(\rho''(0)\right)^3\left(\frac{w'(|y|)}{|y|}\right)'\frac{y_1^6}{|y|}+O(\epsilon^3e^{-a|y|})\right.\\
&+\chi\left[-\rho'(\epsilon y_1)\frac{\partial v_1}{\partial y_1}-\frac{w'(|y|)}{|y|}\frac{1}{2}\rho''(0)y_1^2-(\rho'(\epsilon y_1))^2\frac{w'(|y|)}{|y|}\frac{1}{2}\rho''(0)y_1^2\right.\\
&-\epsilon\left[\rho'(\epsilon y_1)\frac{\partial v_2}{\partial y_1}-\rho''(0)y_1\frac{\partial v_1}{\partial y_1}-(\rho'(\epsilon y_1))^2\rho''(0)y_1\frac{\partial v_1}{\partial y_1}\right]\\
&-\epsilon\left[\rho'(\epsilon y_1)\frac{\partial v_3}{\partial y_1}+\frac{1}{3}\frac{w'(|y|)}{|y|}\rho'''(0)y_1^3+\frac{1}{3}(\rho'(\epsilon y_1))^2\frac{w'(|y|)}{|y|}\rho'''(0)y_1^3\right]\\
&-\epsilon^2\left[-\frac{w'(|y|)}{|y|}y_1^4[\frac{1}{2}(\rho''(0))^3-\frac{1}{8}\rho^{(4)}(0)]-\rho''(0)y_1\frac{\partial v_2}{\partial y_1}+\frac{1}{16}\left(\rho''(0)\right)^3\left(\frac{w'(|y|)}{|y|}\right)'\frac{y_1^6}{|y|}+O(\epsilon e^{-a|y|})\right]\\
&\left.\left.-\epsilon^2\left[-\rho''(0)y_1\frac{\partial v_3}{\partial y_1}-\frac{1}{2}\rho'''(0)y_1^2\frac{\partial v_1}{\partial y_1}+O(\epsilon e^{-a|y|})\right]\right]+E_\epsilon(\chi)\right]\\
= &\ g_\epsilon(x),
\end{aligned}$$

where again $E_\epsilon(\chi)$ denotes all the terms involving derivatives of χ. This implies that $g_\epsilon \le Ce^{-a|y|}$. Therefore,

$$\left|\epsilon\frac{\partial \Psi_{\epsilon,P}}{\partial v}\right| \le Ce^{-a|y|}.$$

Let $\tilde{\Psi}_{\epsilon,P}(z)=\Psi_{\epsilon,P}(x)$, where $x=P+\epsilon z$. Then,$\tilde{\Psi}_{\epsilon,P}(z)$ satisfies the following equation:

$$\begin{cases} \Delta\tilde{\Psi}_{\epsilon,P}-\tilde{\Psi}_{\epsilon,P}+f_\epsilon=0 \text{ in } \Omega_{\epsilon,P}\\ \frac{\partial\tilde{\Psi}_{\epsilon,P}}{\partial\nu}=g_\epsilon \text{ on } \partial\Omega_{\epsilon,P}, \end{cases}$$

where $f_\epsilon \in L^2(\Omega_{\epsilon,P})$ and $g_\epsilon \in L^2(\partial\Omega_{\epsilon,P})$ and both the corresponding norms are bounded, independent of ϵ. Proposition then follows from Lemma A. □

Appendix B: Proof of Proposition 3.1

We prove Proposition 3.1 in this appendix.

We first compute the equation for $\tilde{\phi}_{\epsilon,P}$:

$$\begin{aligned}
&-\epsilon^2\Delta_x\tilde{\phi}_{\epsilon,P}+\tilde{\phi}_{\epsilon,P}\\
= &-\frac{1}{\epsilon}\left[\epsilon^2\Delta_x(\Phi_1\chi)-\Phi_1\chi\right]\\
= &-\frac{1}{\epsilon}\left[\left[\Delta_y\Phi_1-\Phi_1-\epsilon\rho''(\epsilon y_1)\frac{\partial\Phi_1}{\partial y_2}-2\rho'(\epsilon y_1)\frac{\partial^2\Phi_1}{\partial y_1\partial y_2}+|\rho'(\epsilon y_1)|^2\frac{\partial^2\Phi_1}{\partial y_2^2}\right]+E_\epsilon(\chi)\right]\\
= &-\frac{1}{\epsilon}\left[\left[-\epsilon\rho''(\epsilon y_1)\frac{\partial\Phi_1}{\partial y_2}-2\rho'(\epsilon y_1)\frac{\partial^2\Phi_1}{\partial y_1\partial y_2}+|\rho'(\epsilon y_1)|^2\frac{\partial^2\Phi_1}{\partial y_2^2}\right]+E_\epsilon(\chi)\right]\\
= &\ f_\epsilon,
\end{aligned}$$

where E_ϵ denotes all the terms involving derivatives of χ. Since $|\Phi_1| \le Ce^{-a|y|}$ for some constants C, $a > 0$, we have $f_\epsilon \in L^2(\Omega_{\epsilon,P})$ and $\int_{\Omega_{\epsilon,P}} f_\epsilon^2\, dx \le C$. On the other hand, for $x \in \omega_1$, it holds that

$$\frac{\partial\tilde{\phi}_{\epsilon,P}}{\partial\nu} = \frac{1}{\epsilon^2}\left[\frac{\partial\phi_{\epsilon,P}}{\partial\nu} - \epsilon\frac{\partial(\Phi_1\chi)}{\partial\nu}\right]. \tag{7.11}$$

Note that

$$\begin{aligned}\sqrt{1+(\rho'(\epsilon y_1))^2}\frac{\partial\phi_{\epsilon,P}}{\partial\nu} &= \frac{1}{\epsilon}\left[\rho'(\epsilon y_1)\frac{\partial\Phi_0}{\partial y_1}\chi + E_\epsilon(\chi)\right]\\ &= \left[\rho''(0)y_1 + \frac{1}{2}\rho'''(0)\epsilon y_1^2\right]\frac{\partial\Phi_0}{\partial y_1}\chi + E_\epsilon(\chi) + O(\epsilon^2),\end{aligned}$$

$$\sqrt{1+(\rho'(\epsilon y_1))^2}\frac{\partial(\Phi_1\chi)}{\partial\nu} = \frac{1}{\epsilon}\left[[\rho'(\epsilon y_1)\frac{\partial\Phi_1}{\partial y_1} - (1+(\rho'(\epsilon y_1)^2))\frac{\partial\Phi_1}{\partial y_2}]\chi + E_\epsilon(\chi)\right].$$

Therefore, we have

$$\begin{aligned}\epsilon\frac{\partial\tilde{\phi}_{\epsilon,P}}{\partial\nu} &= \frac{1}{\sqrt{1+(\rho'(\epsilon y_1))^2}}\frac{1}{\epsilon}\left[[\rho''(0)y_1 + \frac{1}{2}\rho'''(0)\epsilon y_1^2]\frac{\partial\Phi_0}{\partial y_1}\chi + O(\epsilon^2)\right.\\ &\quad\left. - [\rho'(\epsilon y_1)\frac{\partial\Phi_1}{\partial y_1} - (1+(\rho'(\epsilon y_1)))^2\frac{\partial\Phi_1}{\partial y_2}]\chi + E_\epsilon(\chi)\right]\\ &= \frac{1}{\sqrt{1+(\rho'(\epsilon y_1))^2}}\frac{1}{\epsilon}\left[[\frac{1}{2}\rho'''(0)\epsilon y_1^2\frac{\partial\Phi_0}{\partial y_1} - \rho'(\epsilon y_1)\frac{\partial\Phi_1}{\partial y_1} + O(\epsilon^2)]\chi + E_\epsilon(\chi)\right]\\ &= g_\epsilon,\end{aligned}$$

where again $E_\epsilon(\chi)$ denotes all the terms involving derivatives of χ. This implies that $g_\epsilon \le Ce^{-a|y|}$. Therefore,

$$\left|\epsilon\frac{\partial\tilde{\phi}_{\epsilon,P}}{\partial v}\right| \le Ce^{-a|y|}.$$

The rest is exactly the same as in the proof of Proposition 2.1. □

Appendix C: Proof of Lemma 4.1

In this appendix, we prove Lemma 4.1.

By (2.6), equation (4.5) follows by using Taylor expansion:

$$\begin{aligned}&A(|\frac{x-P}{\epsilon}|)\\ =\;& A(|y|) + \sum_{i=1}^{2}\frac{\partial A(|y|)}{\partial y_i}(\frac{x_i-P_i}{\epsilon} - y_i) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 A(|y|)}{\partial y_i\partial y_j}(\frac{x_i-P_i}{\epsilon} - y_i)(\frac{x_j-P_j}{\epsilon} - y_j)\\ &+\frac{1}{6}\sum_{i,j,k}\frac{\partial^3 A(|y|)}{\partial y_i\partial y_j\partial y_k}(\frac{x_i-P_i}{\epsilon} - y_i)(\frac{x_j-P_j}{\epsilon} - y_j)(\frac{x_k-P_k}{\epsilon} - y_k) + O(\epsilon^4 e^{-a|y|}).\end{aligned}$$

Observe that

$$\begin{aligned}\frac{dA(|y|)}{d|y|^2} &= \frac{A'(|y|)}{2|y|},\\ \frac{d^2A(|y|)}{d|y|^2} &= \frac{1}{4}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3}),\\ \frac{d^3A(|y|)}{d|y|^3} &= \frac{1}{8}(\frac{A'''(|y|)}{|y|^3}-3\frac{A''(|y|)}{|y|^4}+3\frac{A'(|y|)}{|y|^5}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&A(\frac{x-P}{\epsilon})\\ =\ & A(|y|)+\frac{1}{2}\frac{A'(|y|)}{|y|}(|\frac{x-P}{\epsilon}|^2-|y|^2)+\frac{1}{2}(\frac{A''(y)}{|y|^2}-\frac{A'(|y|)}{|y|^3})\frac{1}{4}(|\frac{x-P}{\epsilon}|^2-|y|^2)^2\\ &+\frac{1}{6}(\frac{A'''(|y|)}{|y|^3}-3\frac{A''(|y|)}{|y|^4}+3\frac{A'(|y|)}{|y|^5})\frac{1}{8}(|\frac{x-P}{\epsilon}|^2-|y|^2)^3+O(\epsilon^4e^{-a|y|})\\ =\ & A(|y|)+\frac{1}{2}\frac{A'(|y|)}{|y|}(\epsilon\rho''(0)y_1^2y_2+\epsilon^2[\frac{1}{3}\rho'''(0)y_1^3y_2+\frac{1}{4}(\rho''(0))^2y_1^4]\\ &+\epsilon^3[\frac{1}{12}\rho^{(4)}(0)y_1^4y_2+\frac{1}{6}\rho''(0)\rho'''(0)y_1^5])\\ &+\frac{1}{8}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})(\epsilon^2(\rho''(0))^2y_1^4y_2^2+\epsilon^2[\frac{2}{3}\rho''(0)\rho'''(0)y_1^5y_2^2+\frac{1}{2}(\rho''(0))^3y_1^6y_2])\\ &+\frac{1}{48}(\frac{A'''(|y|)}{|y|^3}-3\frac{A''(|y|)}{|y|^4}+3\frac{A'(|y|)}{|y|^5})\epsilon^3(\rho''(0))^3y_1^6y_2^3+O(\epsilon^4e^{-a|y|}).\end{aligned}$$

Hence, we obtain (4.5).

Next we prove (4.6):

$$\begin{aligned}\int_\Omega A(\frac{x-P}{\epsilon})dx =\ & \epsilon^N\int_{\mathbf{R}^2_+}A(y)dy+\epsilon^{N+1}\int_{\mathbf{R}^2_+}[\frac{1}{2}\frac{A'(|y|)}{|y|}\rho''(0)y_1^2y_2]dy\\ &+\epsilon^{N+2}\int_{\mathbf{R}^2_+}[\frac{1}{2}\frac{A'(|y|)}{|y|}(\frac{1}{3}\rho'''(0)y_1^3y_2+\frac{1}{4}(\rho''(0))^2y_1^4)]dy\\ &+\epsilon^{N+2}\int_{\mathbf{R}^2_+}[\frac{1}{8}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})(\rho''(0))^2y_1^4y_2^2]dy\\ &+\epsilon^{N+3}\int_{\mathbf{R}^2_+}[\frac{1}{2}\frac{A'(|y|)}{|y|}(\frac{1}{12}\rho^{(4)}(0)y_1^4y_2+\frac{1}{6}\rho''(0)\rho'''(0)y_1^5)]dy\\ &+\epsilon^{N+3}\int_{\mathbf{R}^2_+}[\frac{1}{8}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})(\frac{2}{3}\rho''(0)\rho'''(0)y_1^5y_2^2+\frac{1}{2}(\rho''(0))^3y_1^6y_2)]dy\end{aligned}$$

$$\begin{aligned}&+\epsilon^{N+3}\int_{\mathbf{R}^2_+}[\frac{1}{48}(\frac{A'''(|y|)}{|y|^3}-3\frac{A''(|y|)}{|y|^4}+3\frac{A'(|y|)}{|y|^5})(\rho''(0))^3y_1^6y_2^3]dy+O(\epsilon^{N+4})\\ =\ & I_1+I_2+I_3+I_4+I_5+I_6+I_7+O(\epsilon^{N+4}),\end{aligned}$$

where I_1,I_2,I_3,I_4,I_5,I_6 and I_7 are defined by the last equality. Note that

$$
\begin{aligned}
I_3+I_4 &= \epsilon^{N+2}\{\int_{\mathbf{R}^2_+}\frac{1}{6}\frac{A'(|y|)}{|y|}\rho'''(0)y_1^3y_2+\frac{1}{8}[\frac{A'(|y|)}{|y|}(\rho''(0))^2y_1^4+\frac{1}{|y|}(\frac{A'(|y|)}{|y|})'(\rho''(0))^2y_1^4y_2^2]dy\}\\
&= \frac{\epsilon^{N+2}}{8}\{\int_{\mathbf{R}^2_+}\frac{A'(|y|)}{|y|}(\rho''(0))^2y_1^4dy+\int_{\mathbf{R}^2_+}\frac{1}{|y|}(\frac{A'(|y|)}{|y|})'(\rho''(0))^2y_1^4y_2^2]dy\}\\
&= \frac{\epsilon^{N+2}}{8}(\rho''(0))^2\{\int_{\mathbf{R}^2_+}\frac{A'(|y|)}{|y|}y_1^4dy+\int_{\mathbf{R}^2_+}\frac{\partial}{\partial y_2}(\frac{A'(|y|)}{|y|})y_1^4y_2dy\}\\
&= \frac{\epsilon^{N+2}}{8}(\rho''(0))^2\int_{\mathbf{R}^2_+}\frac{\partial}{\partial y_2}(\frac{A'(|y|)}{|y|}y_1^4y_2)dy=0,\\
I_5 &= \epsilon^{N+3}\int_{\mathbf{R}^2_+}\frac{A'(|y|)}{2|y|}\frac{1}{12}\rho^{(4)}(0)y_1^4y_2dy,\\
I_6 &= \frac{\epsilon^{N+3}}{16}\int_{\mathbf{R}^2_+}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})(\rho''(0))^3y_1^6y_2dy,\\
I_7 &= \frac{\epsilon^{N+3}}{48}\int_{\mathbf{R}^2_+}\frac{1}{|y|}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})'(\rho''(0))^3y_1^6y_2^3dy\\
&= \frac{\epsilon^{N+3}}{48}(\rho''(0))^3\int_{\mathbf{R}^2_+}\frac{\partial}{\partial y_2}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})y_1^6y_2^2dy\\
&= -\frac{\epsilon^{N+3}}{48}(\rho''(0))^3\int_{\mathbf{R}^2_+}y_1^62y_2(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})dy\\
&= -\frac{\epsilon^{N+3}}{24}(\rho''(0))^3\int_{\mathbf{R}^2_+}y_1^6y_2(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})dy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_6+I_7 &= \frac{\epsilon^{N+3}}{48}(\rho''(0))^3\int_{\mathbf{R}^2_+}(\frac{A''(|y|)}{|y|^2}-\frac{A'(|y|)}{|y|^3})y_1^6y_2dy\\
&= \frac{\epsilon^{N+3}}{48}(\rho''(0))^3\int_{\mathbf{R}^2_+}\frac{1}{|y|}(\frac{A'(|y|)}{|y|})'y_1^6y_2dy.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_\Omega A(\frac{x-P}{\epsilon})dx &= \epsilon^N\int_{\mathbf{R}^2_+}A(y)dy+\frac{\epsilon^{N+1}}{2}\int_{\mathbf{R}^2_+}\frac{A'(|y|)}{|y|}\rho''(0)y_1^2y_2dy\\
&\quad+\frac{\epsilon^{N+3}}{24}\int_{\mathbf{R}^2_+}\frac{A'(|y|)}{|y|}\rho^{(4)}(0)y_1^4y_2dy+\frac{\epsilon^{N+3}}{48}\int_{\mathbf{R}^2_+}(\rho''(0))^3\frac{1}{|y|}(\frac{A'(|y|)}{|y|})'y_1^6y_2dy+O(\epsilon^{N+4})
\end{aligned}
$$

$$
\begin{aligned}
&= \epsilon^N \int_{\mathbf{R}^2_+} A(y)dy + \frac{\epsilon^{N+1}}{2}\rho''(0)\int_{\mathbf{R}^2_+} \frac{\partial A(|y|)}{\partial y_2} y_1^2 dy \\
&\quad + \frac{\epsilon^{N+3}}{24}\rho^{(4)}(0)\int_{\mathbf{R}^2_+} \frac{\partial A(|y|)}{\partial y_2} y_1^4 dy + \frac{\epsilon^{N+3}}{48}(\rho''(0))^3 \int_{\mathbf{R}^2_+} \frac{\partial}{\partial y_2}\left(\frac{A'(|y|)}{|y|}\right) y_1^6 dy + O(\epsilon^{N+4}) \\
&= \epsilon^N \int_{\mathbf{R}^2_+} A(y)dy - \frac{\epsilon^{N+1}}{2}\rho''(0)\int_{\partial\mathbf{R}^2_+} A(|y|) y_1^2 dy_1 \\
&\quad - \frac{\epsilon^{N+3}}{24}\rho^{(4)}(0)\int_{\partial\mathbf{R}^2_+} A(|y|) y_1^4 dy_1 - \frac{\epsilon^{N+3}}{48}(\rho''(0))^3 \int_{\partial\mathbf{R}^2_+} A'(|y|)|y_1|^5 dy_1 + O(\epsilon^{N+4}).
\end{aligned}
$$

This finishes the proof of Lemma 4.1. □

References

[1] Adimurthi, G. Mancinni, and S.L. Yadava, *The role of mean curvature in a semilinear Neumann problem involving the critical Sobolev exponent*, Comm. Partial Differential Equations **20** (1995), 591-631.

[2] Adimurthi, F. Pacella, and S.L. Yadava, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity*, J. Funct. Anal. **113** (1993), 318-350.

[3] Adimurthi, F. Pacella, and S. L. Yadava, *Characterization of concentration points and L^∞-estimates for solutions involving the critical Sobolev exponent*, Differential Integral Equations **8(1)** (1995), 41-68.

[4] N. Alikakos and M. Kowalczyk, *Critical points of a singular perturbation problem via reduced energy and local linking*, J. Differential Equations **159** (1999), 403-426.

[5] P. Bates, E.N. Dancer, and J. Shi, *Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability,* Adv. Differential Equations **4** (1999), 1-69.

[6] P. Bates and G. Fusco, *Equilibria with many nuclei for the Cahn-Hilliard equation,* J. Differential Equations **4** (1999), 1-69 .

[7] P. Bates and J. Shi, *Existence and instability of spike layer solutions to singular perturbation problems,* J. Funct. Anal., 196 (2002), 211-264.

[8] C.C. Chen and C.S. Lin, *Uniqueness of the ground state solution of $\Delta u + f(u) = 0$ in $R^N, N \geq 3$,* Comm. Partial Differential Equations **16** (1991), 1549-1572.

[9] G. Cerami and J. Wei, *Multiplicity of multiple interior spike solutions for some singularly perturbed Neumann problem,* International Math. Research Notes **12** (1998), 601-626.

[10] E.N. Dancer and J. Wei, *On the effect of domain topology in some singular perturbation problems,* Top. Meth. Nonlinear Analysis **11** (1998), 227-248.

[11] E.N. Dancer and S. Yan, *Multipeak solutions for a singular perturbed Neumann problem,* Pacific J. Math. **189** (1999), 241-262.

[12] E.N. Dancer and S. Yan, *Interior and boundary peak solutions for a mixed boundary value problem,* Indiana Univ. Math. J. **48** (1999), 1177-1212.

[13] M. del Pino and P. Felmer, *Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting,* Indiana Univ. Math. J. **48** (1999), 883-898.

[14] M. del Pino, P. Felmer, and J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems,* SIAM J. Math. Anal. **31** (1999), 63-79.

[15] M. del Pino, P. Felmer, and J. Wei, *On the role of distance function in some singularly perturbed problems,* Comm. Partial Differential Equations **25** (2000), 155-177.

[16] A. Gierer and H. Meinhardt, *A theory of biological pattern formation,* Kybernetik (Berlin) **12** (1972), 30-39.

[17] B. Gidas, W.M. Ni, and L. Nirenberg, "Symmetry of positive solutions of nonlinear elliptic equations in R^N" in *Mathematical Analysis and Applications, Part A*, ed. L. Nachbin, Adv. Math. Suppl. Stud. **7**, Academic Press, New York, 1981, 369-402.

[18] M. Grossi, A. Pistoia, and J. Wei, *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory,* Cal. Var. Partial Differential Equations **11** (2000), 143-175.

[19] C. Gui, *Multipeak solutions for a semilinear Neumann problem,* Duke Math. J. **84** (1996), 739-769.

[20] C. Gui and J. Wei, *Multiple interior peak solutions for some singular perturbation problems,* J. Differential Equations **158** (1999), 1-27.
[21] C. Gui and J. Wei, *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems,* Can. J. Math. **52** (2000), 522-538.
[22] C. Gui, J. Wei, and M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems,* Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), 47-82.
[23] M. Kowalczyk, *Multiple spike layers in the shadow Gierer-Meinhardt system: existence of equilibria and approximate invariant manifold,* Duke Math. J. **98** (1999), 59-111.
[24] M.K. Kwong, *Uniquness of positive solutions of $\Delta u - u + u^p = 0$ in R^N,* Arch. Rational Mech. Anal. **105** (1989), 243-266.
[25] Y.-Y. Li, *On a singularly perturbed equation with Neumann boundary condition,* Comm. Partial Differential Equations **23** (1998), 487-545.
[26] Y.-Y. Li and L. Nirenberg, *The Dirichlet problem for singularly perturbed elliptic equations,* Comm. Pure Appl. Math. **51** (1998), 1445-1490.
[27] C.-S. Lin, W.-M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis systems,* J. Differential Equations **72** (1988), 1-27.
[28] W.-M. Ni and I. Takagi, *On the shape of least energy solution to a semilinear Neumann problem,* Comm. Pure Appl. Math. **41** (1991), 819-851.
[29] W.-M. Ni and I. Takagi, *Locating the peaks of least energy solutions to a semilinear Neumann problem,*Duke Math. J. **70** (1993), 247-281.
[30] W.-M. Ni and I. Takagi, *Point-condensation generated by a reaction-diffusion system in axially symmetric domains,* Japan J. Industrial Appl. Math. **12** (1995), 327-365.
[31] W.-M. Ni, I. Takagi, and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems: intermediate solutions,* Duke Math. J. **94** (1998), 597-618.
[32] W.-M. Ni and J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems,* Comm. Pure Appl. Math. **48** (1995),731-768.
[33] W.-M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states,* Notices Amer. Math. Soc. **45** (1998), 9-18.
[34] L.A. Peletier and J. Serrin, *Uniqueness of positive solutions of semilinear equations in R^N,* Arch. Rational Mech. Anal. **81** (1983), 181-197.
[35] J. Wei, *On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem,* J. Differential Equations **129** (1996), 315-333. .
[36] J. Wei, *On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem,* J. Diffential Equations **134** (1997), 104-133.
[37] J. Wei, *On the interior spike layer solutions for some singular perturbation problems,* Proc. Royal Soc. Edinburgh Sect. A **128** (1998), 849-874.
[38] J. Wei, *Uniqueness and eigenvalue estimates of boundary spike solutions,* Proc. R. Soc. Edinburgh Sect. A **131** (2001), 1457-1480.
[39] J. Wei, "Point-condensation generated by the Gierer-Meinhardt system: a brief survey" in *New Trend In Partial Differential Equations 2000*, ed. Y. Morita, H. Ninomiya, E. Yanagida, and S. Yotsutani, 2000, 46-59.
[40] J. Wei and M. Winter, *Stationary solutions for the Cahn-Hilliard equation,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **15** (1998), 459-492.
[41] J. Wei and M. Winter, *Multiple boundary spike solutions for a wide class of singular perturbation problems,* J. London Math. Soc. **59** (1999), 585-606.
[42] J. Wei and M. Winter, *Higher order energy expansions for some singularly perturbed Neumann problems,* C. R. Acad. Sci. Paris, Ser. I. **337** (2003), 37-42.
[43] J. Wei and M. Winter, *Higher order energy expansions and spike locations,* Calc. Var. Partial Differential Equations, to appear.

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk

Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: winter@mathematik.uni-stuttgart.de

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wkyeung@math.cuhk.edu.hk