ON THE GIERER-MEINHARDT SYSTEM WITH SATURATION

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ABSTRACT. We consider the following shadow system of the Gierer-Meinhardt system with saturation:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{\xi(1+kA^2)} & \text{in } \Omega \times (0,\infty), \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega A^2 \, dx & \text{in } (0,+\infty), \\ \frac{\partial A}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,\infty), \end{cases}$$

where $\epsilon > 0$ is a small parameter, $\tau \geq 0$, $k > 0$ and $\Omega \subset R^n$ is smooth bounded domain. The case $k = 0$ has been studied by many authors in recent years. Here we give some sufficient conditions on k for the existence and stability of stable spiky solutions. In the one-dimensional case we have a complete answer of the stability behavior. Central to our study are a parameterized ground-state equation and the associated nonlocal eigenvalue problem (NLEP) which is solved by functional analysis and the continuation method.

1. Introduction

Turing in his pioneering work in 1952 [36] proposed that a patterned distribution of two chemical substances, called the morphogens, could trigger the emergence of a complex cell structure leading to the development of a complete organism. He shows by linear stability analysis that the homogeneous state may be unstable which explains why a stable spatially complex pattern of the morphogens arises.

Since the work of Turing, a lot of models have been proposed and analyzed to explore this phenomenon, which is now called Turing instability. One of the most studied models is the Gierer-Meinhardt system which after suitable

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rescaling can be stated as follows: ([16], [25])

$$(GM) \qquad \begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{H(1+kA^2)}, \quad A > 0 \quad \text{in } \Omega \times (0,\infty), \\ \tau H_t = D\Delta H - H + A^2, \quad H > 0 \quad \text{in } \Omega \times (0,\infty), \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

The unknowns $A = A(x,t)$ and $H = H(x,t)$ represent the concentrations of the activator and inhibitor at a point $x \in \Omega \subset R^2$ and at a time $t > 0$; $\Delta := \sum_{j=1}^2 \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^2; Ω is a bounded and smooth domain in R^2; $\nu = \nu(x)$ is the outer normal at $x \in \partial\Omega$. The term $\frac{A^2}{1+kA^2}$ is the so-called Michaelis-Menton saturation term, where $k > 0$. This term describes saturation since for $A \to \infty$ the term $A^2/(1+kA^2)$ converges to $\frac{1}{k}$.

The Gierer-Meinhardt system without saturation (i.e. $k = 0$) has been the object of extensive studies in recent years which we now briefly summarize. We start with the shadow system [32] (which arises for $D = +\infty$):

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{\xi}, \quad A > 0 \quad \text{in } \Omega \times (0,\infty), \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|} \int_\Omega A^2 \, dx, \quad \text{in } (0,\infty), \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{1.1}$$

Since we a have purely power-like nonlinearity, the steady state of (1.1) can be conveniently rescaled to the following simple singularly perturbed equation:

$$\begin{cases} \epsilon^2 \Delta u - u + u^2 = 0, u > 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) has been studied by a lot of authors. It has been proved that problem (1.2) admits a rich set of multiple boundary and multiple interior spike solutions. See [1], [2], [3], [4], [7], [8], [11], [12], [13], [14], [18], [19], [20], [21], [24], [27], [28], [29], [40], [39], [41], [45], [46], and the references therein. (Recent surveys can be found in [26], [44].) At each spike, the solution resembles the following ground-state solution:

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in} \quad R^n, \\ w(0) = \max_{y \in R^n} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty \end{cases} \tag{1.3}$$

whose existence as well as uniqueness has been shown in [17] and [23], respectively.

The stability of multiple spike solutions with respect to the shadow system has been studied in [15], [30], [31], [42], [37], [38]. Central to understanding the stability is the following nonlocal eigenvalue problem (NLEP):

$$\begin{cases} \Delta\phi - \phi + 2w\phi - 2\frac{\int_{R^n} w\phi}{\int_{R^n} w^2} w^2 = \lambda\phi \quad \text{in} \quad R^n, \\ \phi \in H^1(R^n), \quad \lambda \in \mathcal{C}, \end{cases} \tag{1.4}$$

where $\mathcal{C}$ is the set of complex numbers. It was proved in [42] that problem (1.4) is stable if $n \leq 3$. Note that (1.4) is **not** self-adjoint and hence complex eigenvalues do occur (see [37]).

When $D < +\infty$, (GM) is quite difficult to solve in general. In recent years, for the case $k = 0$, the existence and stability of multiple spike solutions have been studied in one or two dimensions. See [22], [35], [47], [48], [49], and the references therein.

In this paper, we concentrate on the saturation case, i.e, $k > 0$. As far as the authors know, the only papers dealing with the saturation case to the Gierer-Meinhardt system are due to M. del Pino [9] and [10], where solutions with multiple layers are constructed. His assumption is that $\epsilon << 1$, but that k is **fixed**. Here we will allow k to depend on ϵ and we would like to understand the role of k on the existence and stability of **spiky** solutions. For simplicity, we consider the shadow system only. (The full system with $D > 0, k > 0$ is more difficult to analyze.) Namely, we study the following problem:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^2}{\xi(1+kA^2)}, \quad A > 0 \quad \text{in } \Omega \times (0, \infty), \\ \tau\xi_t = -\xi + \frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx, \; \xi > 0 \quad \text{in } (0, \infty), \\ \frac{\partial A}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{1.5}$$

Our first problem is the existence of steady states which, contrary to the case $k = 0$, can no longer be rescaled to (1.2). In fact, one has to consider a system of two equations – one of these is a PDE and the other is an algebraic equation:

$$\begin{cases} \epsilon^2 \Delta A - A + \frac{A^2}{\xi(1+kA^2)} = 0, \quad A > 0 \quad \text{in } \Omega, \\ \xi = \frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx, \quad \xi > 0, \\ \frac{\partial A}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

To obtain a steady-state solution for (1.6), we first have to solve the following parameterized ground-state equation:

$$\begin{cases} \Delta w_\delta - w_\delta + \frac{w_\delta^2}{1+\delta w_\delta^2} = 0, \quad w_\delta > 0 \quad \text{in} \quad R^n, \\ w_\delta(0) = \max_{y \in R^n} w_\delta(y), \quad w_\delta(y) \to 0 \text{ as } |y| \to \infty \end{cases} \tag{1.7}$$

and then solve the algebraic equation

$$\delta \left(\int_{R^n} w_\delta^2(y)\, dy \right)^2 = k_0, \tag{1.8}$$

where

$$k_0 = \lim_{\epsilon \to 0} 4k\epsilon^{-2n} |\Omega|^2. \tag{1.9}$$

We note that there is an immediate change of type of nonlinearities: a **convex** nonlinearity in (1.3) becomes a **bistable** nonlinearity in (1.7).

To study the stability, we have to study the following new NLEP:

$$\begin{cases} \Delta \phi - \phi + \left(\frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2} \right) \phi - 2 \frac{\int_{R^n} w_\delta \phi}{\int_{R^n} w^2} \frac{w_\delta^2}{1+\delta w_\delta^2} = \lambda \phi \quad \text{in} \quad R^n, \\ \phi \in H^2(R^n), \quad \lambda \in \mathcal{C}. \end{cases} \tag{1.10}$$

Both problems (1.7)-(1.8) and (1.10) are not easy to solve because of non-power like nonlinearity. In this paper, we give a **complete** answer in one-dimensional case. In higher dimensions, we give **sufficient** conditions on k to ensure the existence and stability of solutions.

We state our result in one-dimensional case first. Without loss of generality, we may assume that $\Omega = [0, 1]$. We then have

Theorem 1.1. *Assume that*

$$\lim_{\epsilon \to 0} 4k\epsilon^{-2n} |\Omega|^2 = k_0 \in [0, +\infty). \tag{1.11}$$

Then for each k_0, and for ϵ sufficiently small, problem (1.6) admits a steady-state solution $(u_\epsilon, \xi_\epsilon)$ such that

(a) $A_\epsilon(x) = (1 + o(1))\xi_\epsilon w_{\delta_\epsilon}(\frac{x}{\epsilon})$, where $\delta_\epsilon \to \delta$, δ is the unique solution to (1.8) and w_{δ_ϵ} is the unique solution of (1.7), and

(b) $\xi_\epsilon = (2 + o(1))(\epsilon \int_{R^1} w_{\delta_\epsilon}^2)^{-1}$.

Moreover, $(A_\epsilon, \xi_\epsilon)$ is linearly stable for (1.5), provided τ is small.

In higher dimensions, the statement is more complicated. Let $Q \in \partial\Omega$. We use $H(Q)$ to denote the mean curvature function at Q. We say that Q is a nondegenerate critical point of $H(Q)$, if the following holds:

$$\partial_i H(Q) = 0, i = 1, ..., n-1, \quad \det(\partial_i \partial_j H(Q)) \neq 0,$$

where ∂_i denotes the $i-$th tangential derivative. We then have

Theorem 1.2. *Assume that*

$$\lim_{\epsilon \to 0} 4k\epsilon^{-2n}|\Omega|^2 = k_0 \in [0, +\infty) \tag{1.12}$$

and $Q_0 \in \partial\Omega$ is a nondegenerate critical point of $H(Q)$.

Then for each k_0, and for ϵ sufficiently small, problem (1.6) admits a steady-state solution $(A_\epsilon, \xi_\epsilon)$ such that

(a) $A_\epsilon(x) = (1 + o(1))\xi_\epsilon w_{\delta_\epsilon}(\frac{x - Q_\epsilon}{\epsilon})$, where $\delta_\epsilon \to \delta$ with δ being a solution to (1.8), and w_{δ_ϵ} is the unique solution of (1.7), and

(b) $Q_\epsilon \to Q_0$,

(c) $\xi_\epsilon = (2 + o(1))(\epsilon^n \int_{R^n} w_{\delta_\epsilon}^2)^{-1}$.

If Q_0 is a nondegenerate local maximum point of $H(Q)$, then there exists a $\hat{k}_0$ such that for all $k_0 \in (0, \hat{k}_0)$ the steady state $(A_\epsilon, \xi_\epsilon)$ is linearly stable for (1.5), provided τ is small and $n \leq 3$.

The organization of the paper is as follows: In Section 2, we study the parameterized ground-state problem (1.7) and the algebraic equation (1.8) and prove some preliminary results. In Section 3, we study the NLEP (1.10) for dimensions $n \leq 3$. In Section 4, we prove Theorems 1.1 and 1.2.

Finally, the proofs of some technical lemmas are given in Appendices A and B.

Some important questions are left open.

First, we have assumed that $k \leq C\epsilon^{2n}$ for some constant C. What happens if $\lim_{\epsilon \to 0} k\epsilon^{-2n} \to +\infty$? We believe that spikes do not exist. Does this mean that del Pino's result [10] holds in that case?

Secondly, our stability result in higher dimensions (Theorem 1.2) is incomplete. We conjecture that $\hat{k}_0$ should be infinity. It is also of interest to understand the stability behavior for dimensions $n \geq 4$ which leads to

NLEPs for a new parameter range. Another topic concerns the Hopf bifurcations occurring for τ large. For recent progress in this direction for the Gierer-Meinhardt system without saturation please see [37], [38].

Finally, the issue of existence and stability results for the case of **finite** D in one or two dimensions for the Gierer-Meinhardt system with saturation remains completely open.

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2. The Parameterized Ground-state

In this section, we consider (1.7) and (1.8). We first study (1.7). Note that when $\delta = 0$, (1.7) becomes (1.3).

By the scaling

$$w_\delta(y) = \frac{1}{\sqrt{\delta}} v\left(\frac{y}{\delta^{\frac{1}{4}}}\right) \tag{2.1}$$

we see that (1.7) is equivalent to the following rescaled form:

$$\begin{cases} \Delta v + g(v) = 0, \quad v > 0 \quad \text{in} \quad R^n, \\ v(0) = \max_{y \in R^n} v(y), \quad v(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.2}$$

where

$$g(v) = -\sqrt{\delta} v + \frac{v^2}{1 + v^2}. \tag{2.3}$$

It is easy to see that, for each $\delta \in (0, \frac{1}{4})$, the equation $g(v) = 0$ has exactly two roots

$$t_1(\delta) = \frac{1 - \sqrt{1 - 4\delta}}{2\sqrt{\delta}}, \quad t_2(\delta) = \frac{1 + \sqrt{1 - 4\delta}}{2\sqrt{\delta}}. \tag{2.4}$$

Now we consider

$$c(\delta) = \int_0^{t_2(\delta)} g(s)\, ds. \tag{2.5}$$

To study $c(\delta)$, we introduce the function

$$\rho(t) = \frac{t - \arctan\,(t)}{t^2}.$$

Note that $\rho(t)$ is well-defined for $t \in [0, +\infty)$. The critical point of $\rho(t)$ is unique and is given by the solution of the equation

$$\arctan t = \frac{2t + t^3}{2(1 + t^2)}, \quad t > 0. \tag{2.6}$$

We denote the unique critical point of $\rho(t)$ by t_*. One computes numerically $t_* = 1.514... < \frac{\pi}{2}$. Let

$$\delta_* = (2\rho(t_*))^2. \tag{2.7}$$

Then it is easy to see that

$$c(\delta) \begin{cases} > 0 & \text{for } \delta < \delta_*, \\ = 0 & \text{for } \delta = \delta_*, \\ < 0 & \text{for } \delta > \delta_*. \end{cases} \tag{2.8}$$

Some important properties of the the function $g(v)$ are stated in the following lemma.

Lemma 2.1. *For each $\delta \in (0, \delta_*)$, the function $g(v)$ satisfies the follow conditions:*
(g1) $g \in C^3(R, R)$, $\quad g(0) = 0, \quad g'(0) = 0$.
(g2) There exist $b, c > 0$ such that $b < c$, $g(b) = g(c) = 0$, $g(v) > 0$ in $(-\infty, 0) \cup (b, c)$, and $g(v) < 0$ in $(0, b) \cup (c, +\infty)$.
(g3) $\int_0^c g(v)dv > 0$.
(g4) Let $\theta > b$ be the smallest positive number such that $G(u) = 0$, where

$$G(u) = \int_0^u g(s)ds,$$

and let $\rho > b$ be the smallest number such that $\frac{g(u)}{u-\rho}$ is nonincreasing for $u \in (\rho, c)$. Then either

(i) $\theta \geq \rho$, or

(ii) $\theta < \rho$ with $K_g(u)$ nonincreasing in $(\theta, \rho), K_g(u) \geq K_g(\theta)$ for $u \in (b, \theta)$ and $K_g(u) \leq K_g(\rho)$ for $u \in (0, b) \cup (\rho, c)$, where

$$K_g(u) = \frac{ug'(u)}{g(u)}.$$

The proof of Lemma 2.1 is elemantry and thus left to Appendix A.

In the following lemma we state some important properties of w_δ.

Lemma 2.2. *For each $\delta \in (0, \delta_*)$, problem (1.7) admits a unique solution, denoted by w_δ, which satisfies*

(i) $w_\delta \in C^\infty(R^n)$.

(ii) $w_\delta > 0$ is radially symmetric and $w_\delta'(r) < 0$ for $r \neq 0$.

(iii) w_δ and its derivatives decay exponentially at infinity, i.e., there exist $c_1, c_2 > 0$ such that

$$\left|\frac{\partial w_\delta}{\partial y_i}\right| \le c_1 e^{-c_2|y|}, \quad i = 1, \dots, N,$$

$$\left|\frac{\partial^2 w_\delta}{\partial y_i y_j}\right| \le c_1 e^{-c_2|y|}, \quad i, j, = 1, \dots, n.$$

(iv) The first eigenvalue of the following operator

$$L_\delta = \Delta - 1 + \frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2} \, : \, H^2(R^n) \to L^2(R^n), \tag{2.9}$$

denoted by $\lambda_1 = \lambda_1(L_\delta)$, is positive and simple; the corresponding eigenfunction ϕ can be made positive and radially symmetric.

(v) The second eigenvalue of L_δ is 0 and the dimension of its kernel is n. Namely, $\lambda_2(L_\delta) = 0$ and

$$\text{Kernel}\left(\Delta - 1 + \frac{2w_\delta}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2}\right) = \text{span}\left\{\frac{\partial w_\delta}{\partial y_1}, ..., \frac{\partial w_\delta}{\partial y_n}\right\}. \tag{2.10}$$

Proof: By Lemma 2.1, $g(v) = -\delta v + \frac{v^2}{1+v^2}$ satisfies conditions (g1)-(g4). By Proposition 1.3 of [2], Lemma 2.2 holds for equation (2.2). (See also [33], [34], [5]). Hence Lemma 2.2 also holds for (1.7).

□

The following lemma gives information about the dependence of w_δ on δ and provides some identities.

Lemma 2.3. *(1) w_δ is C^1 in δ,*

(2) As $\delta \to \delta_*, w_\delta(y) \to t_2(\delta_*)/\sqrt{\delta_*}$ in $C^2_{\text{loc}}(R^n)$.

(3) The following identities hold:

$$L_\delta w_\delta = \frac{w_\delta^2}{1+\delta w_\delta^2} - \frac{2\delta w_\delta^4}{(1+\delta w_\delta^2)^2}, \tag{2.11}$$

$$L_\delta \frac{dw_\delta}{d\delta} = \frac{w_\delta^4}{(1+\delta w_\delta^2)^2}, \tag{2.12}$$

$$L_\delta (y \cdot \nabla w_\delta) = 2\left(w_\delta - \frac{w_\delta^2}{1+\delta w_\delta^2}\right), \tag{2.13}$$

$$L_\delta (w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot w_\delta) = w_\delta, \tag{2.14}$$

$$L_\delta (w_\delta + 2\delta \frac{dw_\delta}{d\delta}) = \frac{w_\delta^2}{1+\delta w_\delta^2}. \tag{2.15}$$

Proof: (1) follows from the uniqueness of w_δ given in Lemma 2.2.

To prove (2), we note that $w_\delta \le t_2(\delta)/\sqrt{\delta}$ and hence, as $\delta \to \delta_*, w_\delta$ approaches in $C^2_{\mathrm{loc}}(R^n)$ a solution of the equation

$$\Delta u - u + \frac{u^2}{1+\delta_* u^2} = 0, \quad y \in R^n, \quad u = u(|y|)$$

which admits only constant solutions. That constant must be $t_2(\delta_*)/\sqrt{\delta_*}$ since $w_\delta(0) \to t_2(\delta_*)/\sqrt{\delta_*}$. This proves (2).

The first two identities (2.11) and (2.12) follow from direct computations and the third one (2.13) follows from Pohozaev's identity. (2.14) – (2.15) follow from (2.11) – (2.14).

□

Now we can consider the following algebraic equation:

$$k_0 = \delta \left(\int_{R^n} w_\delta^2(y)\, dy \right)^2. \tag{2.16}$$

Lemma 2.4. *For each fixed $k_0 > 0$, there exists a $\delta \in (0, \delta_*)$ such that (2.16) holds.*

Proof: Let $\beta(\delta) = \delta \left(\int_{R^n} w_\delta^2(y)\, dy\right)^2$.

Certainly, $\beta(\delta)$ is a continuous function of δ and $\beta(0) = 0$. Now we consider the asymptotic behavior of w_δ as $\delta \to \delta_*$. By Lemma 2.3 (2), as $\delta \to \delta_*, w_\delta(|y|) \to t_2(\delta_*)/\sqrt{\delta_*}$ in $C^2_{\mathrm{loc}}(R^n)$. Thus we have

$$\beta(0) = 0, \quad \beta(\delta) \to \infty \text{ as } \delta \to \delta_*. \tag{2.17}$$

By the mean-value theorem, for each $k_0 \in (0, +\infty)$, there exists a $\delta \in (0, \delta_*)$ such that $\beta(\delta) = k_0$.

□

Remark 2.1: The uniqueness of δ is unclear. To show uniqueness, we have to compute

$$\frac{d\beta}{d\delta} = \left[\int_{R^n} w_\delta^2(y)\,dy + 4\delta\int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\,dy\right]\int_{R^n} w_\delta^2(y)\,dy. \tag{2.18}$$

We claim that

Lemma 2.5.

$$\int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\,dy\bigg|_{\delta=0} > 0. \tag{2.19}$$

Proof: By (2.12) and (2.14), we have

$$\int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\,dy\bigg|_{\delta=0} = \int_{R^n} w_0 L_0^{-1}(w_0^4)\,dy$$

$$= \int_{R^n} w_0^4(L_0^{-1}w_0)\,dy = \left(1-\frac{n}{10}\right)\int_{R^n} w_0^5\,dy > 0.$$

□

So, at least for k small, the solution to (2.16) is unique. We conjecture that Lemma 2.5 holds for any $\delta \in (0,\delta_*)$. This is true for the one-dimensional case:

Lemma 2.6. *Suppose that $n=1$. For any $\delta \in (0,\delta_*)$, we have*

$$\frac{d}{d\delta}\left(\int_{R^1} w_\delta^2\,dy\right) > 0. \tag{2.20}$$

The proof of Lemma 2.6 is technical and is left to Appendix B.

3. Stability

Let $(A_\epsilon,\xi_\epsilon)$ be the solution given in Theorems 1.1 and 1.2. We now consider the (linear) stability of $(A_\epsilon,\xi_\epsilon)$. We then have

$$\epsilon^2\Delta\phi - \phi + \frac{2A_\epsilon\phi}{\xi_\epsilon(1+kA_\epsilon^2)} - \frac{2kA_\epsilon^3\phi}{\xi_\epsilon(1+kA_\epsilon^2)^2} - \frac{A_\epsilon^2}{\xi_\epsilon^2(1+kA_\epsilon^2)}\cdot\eta = \lambda\phi, \tag{3.1}$$

$$-\eta + \frac{2}{|\Omega|}\int_\Omega A_\epsilon\phi\,dx = \tau\lambda\eta, \tag{3.2}$$

where $(\phi,\eta) \in H^2(\Omega)\times R$.

Assume first that $\tau = 0$. Then

$$\eta = \frac{2}{|\Omega|} \int_\Omega A_\epsilon \phi \, dx. \tag{3.3}$$

Substituting (3.3) into (3.1), we obtain the following nonlocal eigenvalue problem after the same re-scaling as above and after taking the limit for $\epsilon \to 0$:

$$\Delta\phi - \phi + \frac{2w_\delta \phi}{1 + \delta w_\delta^2} - \frac{2\delta w_\delta^3 \phi}{(1 + \delta w_\delta^2)^2} - \frac{2 \int_{R^N} w_\delta \phi \, dy}{\int_{R^2} w_\delta^2 \, dy} \cdot \frac{w_\delta^2}{1 + \delta w_\delta^2} = \lambda \phi. \tag{3.4}$$

The purpose of this section is to give a thorough study of (3.4). The following is the main theorem:

Theorem 3.1. *Assume that $\delta \in [0, \delta_{**})$, where $\delta_{**} > 0$ is such that*

$$\delta_{**} = \sup \left\{ \delta \in (0, \delta_*) \mid \int_{R^n} w_s \frac{dw_s}{ds} > 0, \text{ for } s \in (0, \delta) \right\}. \tag{3.5}$$

Assume also that $n \leq 3$. Then for all nonzero eigenvalues λ of (3.4), we must have $Re(\lambda) < -c_0 < 0$ for some $c_0 > 0$.

Remark 3.1: By Lemma 2.5, $\delta_{**} > 0$. By Lemma 2.6, $\delta_{**} = \delta_*$ when $n = 1$. So we arrive at the following corollary.

Corollary 3.2. *Let $n = 1$. Then for all nonzero eigenvalues λ of (3.4), we must have $Re(\lambda) < -c_0 < 0$ for some $c_0 > 0$.*

We now prove Theorem 3.1. This will be proved by a continuation method. We begin with $\delta = 0$. When $\delta = 0$, Theorem 3.1 has been proved in [42] and it follows from the following key inequality:

Lemma 3.3. *(Lemma 5.1 of [42]). Assume that $n \leq 3$. Then we have*

$$\int_{R^n} (|\nabla \phi|^2 + |\phi|^2 - 2w_0^2 |\phi|^2) \, dy + \frac{2 \int_{R^n} w_0 \phi_0 \, dy \int_{R^n} w_0^2 \phi \, dy}{\int_{R^n} w_0^2 \, dy}$$
$$- \frac{\left(\int_{R^n} w_0 \phi \, dy\right)^2}{\left(\int_{R^n} w_0^2 \, dy\right)^2} \int_{R^n} w_0^3 \, dy \geq c_1 d_{L^2}(\phi, X_1), \tag{3.6}$$

where

$$X_1 = \left\{ w_0, \frac{\partial w_0}{\partial y_j}, \, j = 1, ..., n \right\}$$

and d_{L^2} is the L^2-distance.

Proof of Theorem 3.1:

Suppose that $\delta \in [0, \delta_{**})$.

We use the continuation method to prove Theorem 3.1. We will find a suitable quadratic functional and show its positivity by varying δ.

We first note that we may restrict ϕ to a space of radially symmetric functions. (This follows by the same argument as in [6] and [51].) So we may assume that

$$\phi \in H_r^2(R^n) = H^2(R^n) \cap \{\phi(y) = \phi(|y|)\}.$$

To begin with, we multiply (3.4) by $\overline{\phi}$ – the conjugate function of ϕ and obtain

$$Q_\delta[\phi_R, \phi_R] + Q_\delta[\phi_I, \phi_I] = -\lambda \int_{R^n} |\phi|^2 \, dy, \tag{3.7}$$

where

$$Q_\delta[u,u] = \int_{R^n} \left(|\nabla u|^2 + u^2 - \frac{2w_\delta^2 u^2}{1+\delta w_\delta^2} + \frac{2\delta w_\delta^3 u^2}{(1+\delta w_\delta^2)^2} \right) dy \tag{3.8}$$
$$+2 \frac{\int_{R^n} w_\delta u \, dy}{\int_{R^n} w_\delta^2(y)\, dy} \cdot \int_{R^n} \frac{w_\delta^2 u}{1+\delta w_\delta^2} \, dy$$

and $\phi_R = \mathrm{Re}(\phi)$, $\phi_I = \mathrm{Im}(\phi)$ are the real and the imaginary parts of ϕ, respectively.

Therefore, to prove Theorem 3.1, it is enough to show that Q_δ is positive definite. We re-write Q_δ as follows:

$$Q_\delta[u,u] = -(\mathcal{L}_\delta u, u),$$

where

$$\mathcal{L}_\delta u = \Delta u - u + \frac{2w_\delta}{1+\delta w_\delta^2} u - \frac{2\delta w_\delta^3}{(1+\delta w_\delta^2)^2} u - \frac{\int_{R^n} w_\delta u \, dy}{\int_{R^n} w_\delta^2(y)\, dy} \cdot \frac{w_\delta^2}{1+\delta w_\delta^2} \, dy$$
$$- \frac{w_\delta}{\int_{R^n} w_\delta^2(y)\, dy} \cdot \int_{R^n} \frac{w_\delta^2 u}{1+\delta w_\delta^2} \, dy. \tag{3.9}$$

Clearly,

$$Q_\delta \quad \text{is positive definite} \quad \Longleftrightarrow \mathcal{L}_\delta \quad \text{has negative spectrum only.} \tag{3.10}$$

(3.6) implies that the principal eigenvalue of $\mathcal{L}_\delta$ is negative for $\delta = 0$. We now continue in δ. Assume that at some point $\delta \in (0, \delta_*)$, the principal eigenvalue of $\mathcal{L}_\delta$ becomes zero. That is, there exists a function $\phi \in H_r^2(R^n)$ such that

$$\mathcal{L}_\delta \phi = 0. \tag{3.11}$$

We re-write (3.11) as

$$L_\delta \phi = \frac{\int_{R^n} w_\delta \phi\, dy}{\int_{R^n} w_\delta^2\, dy} \cdot \frac{w_\delta^2}{1+\delta w_\delta^2} + \int_{R^n} \frac{w_\delta^2 \phi}{1+\delta w_\delta^2}\, dy \frac{w_\delta}{\int_{R^n} w_\delta^2\, dy},$$

and, applying L_δ^{-1} (which exists by Lemma 2.2) on both sides of the equation,

$$\phi = \frac{\int_{R^n} w_\delta \phi\, dy}{\int_{R^n} w_\delta^2\, dy} \cdot \left(L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} \right) + \int_{R^n} \frac{w_\delta^2 \phi}{1+\delta w_\delta^2}\, dy \frac{L_\delta^{-1} w_\delta}{\int_{R^n} w_\delta^2\, dy}. \tag{3.12}$$

Let $A = \int_{R^n} w_\delta \phi\, dy$ and Then (3.12) implies

$$\begin{cases} A = \dfrac{\int_{R^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy}{\int_{R^n} w_\delta^2\, dy} A + \dfrac{\int_{R^n} w_\delta L_\delta^{-1} w_\delta\, dy}{\int_{R^n} w_\delta^2\, dy} B \\[2ex] B = \dfrac{\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy}{\int_{R^n} w_\delta^2\, dy} A + \dfrac{\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta\, dy}{\int_{R^n} w_\delta^2\, dy} B. \end{cases} \tag{3.13}$$

Observe that $A^2 + B^2 \neq 0$ as otherwise $L_\delta \phi = 0$ and $\phi \in \text{Kernel}(L_\delta)$ which is impossible by Lemma 2.2 since $\phi \in H_r^2(R^n)$.

From (3.13), we have

$$\begin{vmatrix} 1 - \dfrac{\int_{R^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy}{\int_{R^n} w_\delta^2\, dy} & -\dfrac{\int_{R^n} w_\delta L_\delta^{-1} w_\delta\, dy}{\displaystyle\int_{R^n} w_\delta^2\, dy} \\[2ex] -\dfrac{\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy}{\int_{R^n} w_\delta^2\, dy} & 1 - \dfrac{\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta\, dy}{\int_{R^n} w_\delta^2\, dy} \end{vmatrix} = 0, \tag{3.14}$$

which is equivalent to

$$\left(1 - \frac{\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta\, dy}{\int_{R^n} w_\delta^2\, dy} \right)^2$$

$$- \frac{1}{\left(\int_{R^n} w_\delta^2\, dy\right)^2} \left(\int_{R^n} w_\delta L_\delta^{-1} w_\delta\, dy \right) \left(\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy \right) = 0. \tag{3.15}$$

Now we simplify (3.15). We make use of the identities (2.11)–(2.15) in Lemma 2.3 and obtain

$$\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_\delta^{-1} w_\delta dy = \int_{R^n} w_\delta L_\delta^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2} dy = \int_{R^n} w_\delta (w_\delta + 2\delta \frac{dw_\delta}{d\delta})\, dy$$

$$= \int_{R^n} w_\delta^2\, dy + 2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy, \tag{3.16}$$

$$\int_{R^n} w_\delta L_\delta^{-1} w_\delta\, dy = \int_{R^n} w_\delta \left(w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot \nabla w_\delta \right) dy$$

$$= \left(1 - \frac{n}{4}\right) \int_{R^n} w_\delta^2\, dy + 2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy, \tag{3.17}$$

$$\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} L_0^{-1} \frac{w_\delta^2}{1+\delta w_\delta^2}\, dy = \int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \left(w_\delta + 2\delta \frac{dw_\delta}{d\delta} \right) dy$$

$$= \int_{R^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + 2\delta \int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy. \tag{3.18}$$

Multiplying (2.13) by $\frac{dw_\delta}{d\delta}$, using (2.12), and integrating, we obtain

$$\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy - \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy = \int_{R^n} \frac{w_\delta^4}{(1+\delta w_\delta^2)^2} \left(-\frac{1}{2} y \cdot \nabla w_\delta \right) dy$$

or

$$\int_{R^n} \frac{w_\delta^2}{1+\delta w_\delta^2} \frac{dw_\delta}{d\delta}\, dy = \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy + \frac{n}{2} \int_{R^n} \gamma_\delta(w_\delta)\, dy, \tag{3.19}$$

where

$$\gamma_\delta(w_\delta) = \int_0^{w_\delta} \frac{s^5}{(1+\delta s^2)^2}\, ds.$$

Let

$$h(\delta) := \left(2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)^2 - \left(\left(1 - \frac{n}{4}\right) \int_{R^n} w_\delta^2\, dy + 2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)$$

$$\times \left(\int_{R^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{R^n} \gamma_\delta(w_\delta)\, dy + 2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \right)$$

$$= -2\delta \int_{R^n} w_\delta \frac{dw_\delta}{d\delta}\, dy \left(\left(1 - \frac{n}{4}\right) \int_{R^n} w_\delta^2\, dy + \int_{R^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{R^n} \gamma_\delta(w_\delta)\, dy \right)$$

$$- \left(1 - \frac{n}{4}\right) \int_{R^n} w_\delta^2\, dy \left(\int_{R^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\, dy + n\delta \int_{R^n} \gamma_\delta(w_\delta)\, dy \right). \tag{3.20}$$

Hence (3.15) becomes

$$h(\delta) = 0. \tag{3.21}$$

But observe that for $0 \le \delta \le \delta_{**}$ we have $h(\delta) < 0$. Hence, $\delta > \delta_{**}$, which is a contradiction to our assumption that $\delta \in [0, \delta_{**})$.

This finishes the proof of Theorem 3.1.

□

Remark 3.2:

1). From the proof of Theorem 3.1, we see that the number δ_{**} can be replaced by

$$\delta_{***} = \sup\{\delta \in (0, \delta_0) \,:\, h(s) < 0, \quad s \in (0, \delta)\}. \tag{3.22}$$

2). Let us give another sufficient condition for stability. Note that

$$\int_{R^n} \frac{w_\delta^3}{1+\delta w_\delta^2}\,dy = \int_{R^n} w_\delta^2\,dy + \int_{R^n} |\nabla w_\delta|^2\,dy > \int_{R^n} w_\delta^2\,dy. \tag{3.23}$$

So

$$\frac{\left(1-\frac{n}{4}\right)\int_{R^n} w_\delta^2\,dy\left(\int_{R^n}\frac{w_\delta^3}{1+\delta w_\delta^2}\,dy + \int_{R^N} n\delta\gamma_\delta(w_\delta)\,dy\right)}{\left(1-\frac{n}{4}\right)\int_{R^n} w_\delta^2\,dy + \int_{R^n}\frac{w_\delta^3}{1+\delta w_\delta^2}\,dy + \int_{R^N} n\delta\gamma_\delta(w_\delta)\,dy}$$

$$> \frac{\left(1-\frac{n}{4}\right)\int_{R^n} w_\delta^2\,dy}{\left(2-\frac{n}{4}\right)} = \frac{4-n}{8-n}\int_{R^n} w_\delta^2\,dy.$$

Therefore, in order that $h(\delta) < 0$, it suffices to have

$$\frac{4-n}{8-n}\int_{R^n} w_\delta^2\,dy + 2\delta\int_{R^n} w_\delta\frac{dw_\delta}{d\delta}\,dy > 0. \tag{3.24}$$

Thus, if we define

$$\delta_{****} = \sup\left\{\delta \in (0, \delta_*) \,:\, \frac{4-n}{8-n}\int_{R^n} w_s^2\,dy + 2s\int_{R^n} w_s\frac{dw_s}{ds}\,dy > 0, \quad s \in (0,\delta)\right\} \tag{3.25}$$

then Theorem 3.1 holds true for $\delta \in (0, \delta_{****})$.

4. Proof of Theorem 1.1 and Theorem 1.2

In this section, we prove both Theorems 1.1 and 1.2.

We first consider existence of solutions to (1.6). By the scaling

$$A = \xi u, \quad \xi^{-1} = \frac{1}{|\Omega|} \int_\Omega u^2 \, dx, \tag{4.1}$$

it is easy to see that (1.6) is equivalent to the following two equations:

$$\begin{cases} \epsilon^2 \Delta u - u + \frac{u^2}{1+\delta u^2} = 0, u > 0, \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

and

$$\delta (2\epsilon^{-n} \int_\Omega u^2)^2 = k_\epsilon := 4k\epsilon^{-2n} |\Omega|^{-2}. \tag{4.3}$$

By assumption (1.11), $\lim_{\epsilon\to 0} k_\epsilon = k_0 \in [0, +\infty)$. By Lemma 2.4, there exists a $\delta_1 \in (0, \delta_*)$ such that

$$\delta_1 \left(\int_{R^n} w_{\delta_1}^2 \, dy \right)^2 = k_0. \tag{4.4}$$

Observe that w_δ is uniformly bounded in $H^1(R^n)$ for $\delta \in (0, \delta_1)$ (the bound may depend on δ_1).

Fox each fixed $\delta \in (0, \delta_1)$, by Lemma 2.2, w_δ is nondegenerate. Then Theorem 1.1 of [45] and Theorem 1.1 of [43] (see also Theorem 4.5 of [4]) imply that for ϵ sufficiently small, problem (4.2) admits a single boundary spike solution, $u_{\epsilon,\delta}$ which is unique and nondegenerate and possesses a unique local maximum point $Q_{\epsilon,\delta}$ which converges to Q_0 as $\epsilon \to 0$. (In the one-dimensional case, this follows from the implicit function theorem. In higher dimension, we have to use Liapunov-Schmidt reduction.)

It remains to solve the following algebraic equation:

$$\beta_\epsilon(\delta) := \delta \left(2\epsilon^{-n} \int_\Omega u_{\epsilon,\delta}^2 \, dx \right)^2 = k_\epsilon. \tag{4.5}$$

Since, $\beta_\epsilon(0) = 0$ and, $\lim_{\epsilon\to 0} \beta_\epsilon(\delta) \to \beta(\delta) = \delta(\int_{R^n} w_\delta^2 \, dy)^2$. (The convergence is uniform in $\delta \in (0, \delta_1)$.) So $\lim_{\epsilon\to 0} \beta_\epsilon(\delta_1) \to \delta_1 (\int_{R^n} w_{\delta_1}^2)^2 = k_0$. Since $u_{\epsilon,\delta}$ is unique, β_ϵ is a continuous function of δ. By the mean-value theorem, for $k_\epsilon \in (0, k_0)$, there exists a $\delta_\epsilon \in (0, \delta_1)$ such that $\beta_\epsilon(\delta_\epsilon) = k_\epsilon$. ($\delta_\epsilon$ may not be unique.) Since $k_0 \in [0, \infty)$ may be chosen arbitrarily we get a solution for any $k_\epsilon \in [0, \infty)$.

Then the solution $A_\epsilon = \xi_\epsilon u_{\epsilon,\delta_\epsilon}$, $\xi_\epsilon = \left(\frac{1}{|\Omega|} \int_\Omega u_{\epsilon,\delta_\epsilon}^2 \, dx \right)^{-1}$ satisfies the properties in Theorems 1.1 and 1.2.

This finishes the existence part.

Concerning stability of $(A_\epsilon, \xi_\epsilon)$, we have to study the following eigenvalue problem:

$$\begin{cases} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + \left(\frac{2A_\epsilon}{\xi_\epsilon(1+kA_\epsilon^2)} - \frac{2kA_\epsilon^3}{\xi_\epsilon(1+kA_\epsilon^2)}\right)\phi_\epsilon - \frac{A_\epsilon^2}{\xi_\epsilon^2(1+kA_\epsilon^2)}\eta_\epsilon = \lambda_\epsilon\phi_\epsilon \text{ in } \Omega, \\ -\eta_\epsilon + \frac{1}{|\Omega|}\int_\Omega (2A_\epsilon\phi_\epsilon)\,dx = \tau\lambda_\epsilon\eta_\epsilon. \end{cases} \tag{4.6}$$

We follow the method in [42] and consider two cases. In Case 1, we assume that $\lambda_\epsilon \to \lambda_0 \in \mathcal{C}$ and $\lambda_0 \neq 0$. (These are the so-called large eigenvalues.) Then similar to [42], λ_0 satisfies

$$\Delta\phi_0 - \phi_0 + \left(\frac{2w_\delta}{1+\delta w_\delta^2} - \frac{\delta w_\delta^3}{(1+\delta w_\delta^2)^2}\right)\phi_0 - \frac{2}{1+\tau\lambda_0}\frac{w_\delta^2}{1+\delta w_\delta^2}\frac{\int_{R^n} w_\delta\phi_0}{\int_{R^n} w_\delta^2} = \lambda_0\phi_0. \tag{4.7}$$

By Theorem 3.1, for $n \leq 3$ and $\delta \in (0, \delta_{**})$, problem (4.7) is stable for τ small, i.e., for all eigenvalues of (4.7) with $\lambda_0 \neq 0$ we must have $\mathrm{Re}(\lambda_0) < -c_0 < 0$ for some $c_0 > 0$. In the one-dimensional case, by Corollary 3.2, we can take $\delta_{**} = \delta_*$. This shows that the large eigenvalues are all stable.

It remains to consider Case 2, $\lambda_\epsilon \to 0$. We call these eigenvalues small eigenvalues. Note that in the one-dimensional case, λ_ϵ is bounded away from zero. So we just need to consider the higher dimensional case. In this situation, the proof is exactly the same as in the proof of Theorem 1.3 of [42]. We omit the details.

This finishes the stability part.

□

Appendix A: Proof of Lemma 2.1

Proof: The conditions (g1)- (g3) are easy to verify. (Here $b = t_1(\delta), c = t_2(\delta)$.) We only consider (g4).

We first compute ρ. By definition, there exists an $u_0 > b$ such that

$$g(u_0) = g'(u_0)(u_0 - \rho) \tag{4.8}$$

and

$$(g(u) - g'(u)(u-\rho))'|_{u=u_0} = 0. \tag{4.9}$$

(4.9) implies that $g^{''}(u_0)=0$ and therefore $u_0=\frac{1}{\sqrt{3}}$. By (4.8), we calculate

$$\rho=\frac{1}{3\sqrt{3}-8\sqrt{\delta}}$$

If $\theta\geq\rho$, we are done. (This is the case when δ is close to δ_*.)

Suppose that $\theta<\rho$. We need to calculate

$$K_g(u)=\frac{u(-\sqrt{\delta}+\frac{2u}{(1+u^2)^2})}{-\sqrt{\delta}u+\frac{u^2}{1+u^2}}.$$

It is instructive to introduce $t=\arctan u$. Then it follows by straightforward computations that

$$K_g(u)=1+\frac{\sin(4t)}{2(-2\sqrt{\delta}+\sin(2t))}:=\hat{K}_g(t).$$

We then compute

$$\frac{d}{dt}(\hat{K}_g(t))=\frac{-8\sqrt{\delta}-4\sin^3(2t)+16\sqrt{\delta}\sin^2(2t)}{2(-2\sqrt{\delta}+\sin(2t))^2}.$$

Since $\delta<\delta_*$, it is easy to see that $\frac{d}{dt}(\hat{K}_g(t))<0$ and hence $\frac{d}{du}(K_g(u))<0$. This implies that $K_g(u)$ is nonincreasing in (b,c). Moreover

$$K_g(u)\leq K_g(0)=1,\ \text{for } u\in(0,b)$$

but

$$K_g(\rho)=1+\frac{u(1-u^2)}{u(1+u^2)-\sqrt{\delta}(1+u^2)^2}>1.$$

Hence $K_g(u)\leq K_g(\rho)$ for $u\in(0,b)\cup(\rho,c)$.

This shows that (g4) holds.

□

Appendix B: Proof of Lemma 2.6

Proof:

We assume that $n=1$. Then (1.7) becomes an ODE and it is easy to see that

$$w_\delta^{'}=-\sqrt{w_\delta^2-2F(\delta,w_\delta)},$$

where

$$F(\delta,t)=\int_0^t\frac{s^2}{1+\delta s^2}ds=\frac{1}{\delta}(t-\frac{1}{\sqrt{\delta}}\ \arctan(\sqrt{\delta}t)).$$

Let $t_\delta > 0$ be the unique solution of

$$t_\delta^2 - 2F(\delta, t_\delta) = 0, \quad t_\delta > 0.$$

Thus

$$\int_{R^1} w_\delta^2 \, dy = 2\int_0^{+\infty} w_\delta^2 \, dy = 2\int_0^{t_\delta} \frac{t^2 dt}{\sqrt{t^2 - 2F(\delta, t)}}$$

$$= \frac{1}{2}\eta^{-\frac{3}{2}} \int_0^\gamma \frac{s ds}{\sqrt{\eta - \rho(s)}}, \tag{4.10}$$

where

$$\rho(t) = \frac{t - \arctan(t)}{t^2}, \quad \eta = \sqrt{\delta}/2, \quad \text{and } \gamma = \gamma(\eta)$$

is the unique solution of

$$\rho(\gamma) = \eta, \gamma < t_*$$

(given after (2.6)).

It is easy to compute that

$$\rho^{'}(t) = \frac{1}{1+t^2} - \frac{2\rho}{t}$$

$$\rho^{''}(t) = -\frac{2t}{(1+t^2)^2} - \frac{2}{t(1+t^2)} + \frac{6\rho}{t^2}. \tag{4.11}$$

We first claim that

$$\rho^{''}(t) < 0 \quad \text{for} \quad 0 < t < t_*. \tag{4.12}$$

In fact, from (4.11), we see that (4.12) is equivalent to

$$\beta(t) := \frac{t^5}{(1+t^2)^2} + \frac{t^3}{1+t^2} - 3(t - \arctan(t)) > 0. \tag{4.13}$$

It is easy to see that $\beta(0) = 0$ and

$$\beta^{'}(t) = \frac{t^4(3-t^2)}{(1+t^2)^3} > 0$$

for $t < t_* < \sqrt{3}$. Hence $\beta(t) > 0$ for $t < t_*$. (4.13) is thus proved.

From (4.12), it is easy to prove that

$$\rho^{'}(t) > 0, \quad \text{and} \quad t\rho^{'}(t) < \rho(t) \text{ for } 0 < t < t_*. \tag{4.14}$$

We now re-write the integral in (4.10):

$$\int_0^\gamma \frac{sds}{\sqrt{\eta-\rho(s)}} = \gamma^2 \int_0^{\frac{\pi}{2}} \frac{\cos(\theta)\sin(\theta)d\theta}{\sqrt{\rho(\gamma)-\rho(\gamma\cos(\theta))}}. \tag{4.15}$$

Since $\frac{d\gamma}{d\eta} = \frac{1}{\rho'(\gamma)}$, by differentiating (4.15) and after some simple computations, we obtain that

$$\frac{d}{d\delta}\left(\int_{R^1} w_\delta^2\right) = 2^{-1/2}\delta^{-7/4}\gamma\int_0^{\frac{\pi}{2}}\left[\frac{\cos(\theta)\sin(\theta)}{(\rho(\gamma)-\rho(\gamma\cos(\theta)))^{\frac{3}{2}}} I(\gamma\cos(\theta))d\theta\right], \tag{4.16}$$

where $I(t)$ is given by

$$I(t) = \left[\frac{2\rho(\gamma)}{\rho'(\gamma)} - \frac{3\gamma}{2}\right](\rho(\gamma)-\rho(t)) - \frac{\rho(\gamma)}{2\rho'(\gamma)}(\gamma\rho'(\gamma)-\rho'(t)t). \tag{4.17}$$

Certainly, $I(\gamma)=0$. We now compute

$$I'(t) = \frac{\rho(\gamma)}{\rho'(\gamma)}\left[\frac{3}{2}(\frac{\gamma\rho'(\gamma)-\rho(\gamma)}{\rho(\gamma)})\rho'(t) + \frac{t}{2}\rho''(t)\right].$$

By (4.12) and (4.14), we deduce that

$$I'(t) < 0, \text{ for } 0 < t < \gamma.$$

Thus $I(t) > I(\gamma) = 0$ for $t \in (0,\gamma)$, which implies that by (4.16),

$$\frac{d}{d\delta}\left(\int_{R^1} w_\delta^2\, dy\right) > 0.$$

□

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