# EXISTENCE AND STABILITY OF MULTIPLE SPOT SOLUTIONS FOR THE GRAY-SCOTT MODEL IN $R^2$

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ABSTRACT. In this paper, we rigorously prove the existence and stability of multiple spot patterns for the Gray-Scott system in a two dimensional domain which are far from spatial homogeneity. The Green's function and its derivatives together with two nonlocal eigenvalue problems both play a major role in the analysis. We establish a threshold behavior for stability: If a certain inequality for the parameters holds then we get stability, otherwise we get instability of multiple spot solutions. The exact asymptotics of the critical thresholds are obtained.

## 1. INTRODUCTION

The irreversible Gray-Scott model [9], [10] describes the kinetics of a simple autocatalytic reaction in an unstirred flow reactor. Substance $V$ whose concentration is kept fixed outside the reactor is supplied through the walls into the reactor with rate $F$ and the products of the reaction are removed from the reactor with the same rate. Inside the reactor $V$ undergoes a reaction involving an intermediate substance $U$. Furthermore, $V$ reacts at the rate $k$ to change into $P$. Both reaction are irreversible, so $P$ is an inert product. The reactions are summarized as follows:

$$U + 2V \to 3V$$

$$V \to P$$

The equations of chemical kinetics which describe the spatiotemporal changes of the concentrations of $U$ and $V$ in the reactor in dimensionless units are

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.
*Key words and phrases.* Pattern formation, Self-replication, Spotty solutions, Reaction-diffusion systems.

given by

$$\begin{cases} V_t = D_V \Delta V - (F+k)V + UV^2 & \text{in } \Omega, \\ U_t = D_U \Delta U + F(1-U) - UV^2 & \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The unknowns $U = U(x,t)$ and $V = V(x,t)$ represent the concentrations of the two biochemicals at a point $x \in \Omega \subset R^2$ and at a time $t > 0$, respectively; $\Delta := \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $R^2$; $\Omega$ is a bounded and smooth domain in $R^2$; $\nu(x)$ is the outer normal at $x \in \partial\Omega$; $D_U, D_V$ are the diffusion coefficients of $U$ and $V$ respectively.

The most interesting phenomenon related to Gray-Scott is the the so-called "self-replicating" pattern. To begin with, in 1993, Pearson [19] presented some numerical simulations on the Gray-Scott model in a square of size 2.5 in $R^2$ with periodic boundary conditions. By choosing $D_U = 2 \times 10^{-5}, D_V = 10^{-5}$ and varying the parameters $F$ and $k$, several interesting patterns were discovered. One of them is that the spot may self replicate in a self-sustaining fashion and develop into a variety of time-dependent and time-independent asymptotic states. Lin, McCormick, Pearson and Swinney [14] reported their experiments in a ferro-cyanide-iodate-sulfite reaction which showed strong qualitative agreement with the self-replication regimes of the simulations in [19]. Moreover, those same experiments led to the discovery of other new patterns, such as annular patterns emerging from circular spots. See [15] for more details on the set-up. In 1-D, numerical simulations were done by Reynolds, Pearson and Ponce-Dawson [21], [22], independently by Petrov, Scott and Showalter [20]. And again self-replication phenomena were observed. However, in 1-D, self-replication patterns were observed when $D_U = 1, D_V = \delta^2 = 0.01$. Some formal asymptotics and dynamics in 1-D were contained in [21] and [20]. Recent numerical simulations of [5] in 1-D and [18], [16] in 2-D show that the single spot may be stable in some very narrow parameter regimes.

The first rigorous result in constructing a single peak (or pulse or spike) solution is due to Doelman, Kaper and Zegeling in 1997. In [5], by using the Mel'nikov method, Doelman, Kaper and Zegeling constructed single and multiple pulse solutions for (1.1) in the case $N = 1, D_U = 1, D_V = \delta^2 << 1$. In their paper [5], it is assumed that $F \sim \delta^2, F + k \sim \delta^{2\alpha/3}$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $U = O(\delta^\alpha), V = O(\delta^{-\frac{\alpha}{3}})$. Later the stability of single and multiple pulse solutions in 1-D were obtained in [3], [4]. Hale, Peletier and Troy studied the case $D_U = D_V$ in 1-D and the existence of single and multiple pulse solutions are established in [11], [12]. In [18], Nishiura and Ueyama proposed a skeleton structure of self-replicating dynamics. Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can be found in [6] and [7].

In higher dimension, as far as the authors know, there are very few rigorous results on the existence or stability of spotty solutions for (1.1). In $R^2$ and $R^3$, Muratov and Osipov [16] have given some formal asymptotic analysis on the construction and stability of spotty solution. In [26], the first author studied (1.1) in a bounded domain for the shadow system case, namely, it is assumed that $D_U >> 1, D_V << 1$, $F = O(1)$, and $F + k = O(1)$. The shadow system can be reduced to a single equation. In [27], (1.1) is studied for $N = 2$ in $R^2$ and rigorous results on existence and stability of single spot ground states are established.

In the present paper, we study the Gray-Scott model in a bounded domain $\Omega \subset R^2$ (as first studied numerically by Pearson [19]). First we shall rigorously construct multiple interior spot solutions and then we shall prove results on the stability of such solutions.

We first write the equations (1.1) in standard form. We assume that the domain size is $l$, i.e., $\Omega = l\hat{\Omega}, |\hat{\Omega}| = 1$. Dividing the first equation in (1.1) by $F + k$ and dividing the second equation equation in (1.1) by $F$ we obtain

$$\begin{cases} \frac{1}{F+k} V_t = \frac{D_V}{F+k} \Delta V - V + \frac{1}{F+k} U V^2 \quad \text{in } \Omega, \\ \frac{1}{F} U_t = \frac{D_U}{F} \Delta U + 1 - U - \frac{1}{F} U V^2 \quad \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Setting $V = \sqrt{F}\hat{V}$ gives

$$\begin{cases} \frac{1}{F+k} \hat{V}_t = \frac{D_V}{F+k} \Delta \hat{V} - \hat{V} + \frac{\sqrt{F}}{F+k} U \hat{V}^2 \quad \text{in } \Omega, \\ \frac{1}{F} U_t = \frac{D_U}{F} \Delta U + 1 - U - U \hat{V}^2 \quad \text{in } \Omega, \\ \frac{\partial U}{\partial \nu} = \frac{\partial \hat{V}}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Rescaling time $t = \frac{\hat{t}}{F+k}$, space $x = l\hat{x}$, $\Omega = l\hat{\Omega}$ and introducing the variables $A = \frac{\sqrt{F}}{F+k}$, $\tau = \frac{F+k}{F} > 1$ we can rewrite

$$\begin{cases} \hat{V}_{\hat{t}} = \frac{D_V}{(F+k)l^2}\Delta_{\hat{x}}\hat{V} - \hat{V} + AU\hat{V}^2 \quad \text{in } \hat{\Omega}, \\ \tau U_{\hat{t}} = \frac{D_U}{Fl^2}\Delta_{\hat{x}}U + 1 - U - U\hat{V}^2 \quad \text{in } \hat{\Omega}, \\ \frac{\partial U}{\partial \nu} = \frac{\partial \hat{V}}{\partial \nu} = 0 \quad \text{on } \partial\hat{\Omega}. \end{cases} \tag{1.4}$$

Letting $\epsilon^2 = \frac{D_V}{(F+k)l^2}$, $D = \frac{D_U}{Fl^2}$ and dropping the hats we get

$$\begin{cases} v_t = \epsilon^2\Delta v - v + Auv^2 \quad \text{in } \Omega, \\ \tau u_t = D\Delta u + 1 - u - uv^2 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Throughout this paper, we assume that

$$\epsilon << 1 \ \text{ does not depend on } \ x,$$

$$\tau > 0 \ \text{ does not depend on } \ x \text{ or } \epsilon,$$

$$D, A > 0 \ \text{ do not depend on } \ x \text{ (but may depend on } \epsilon),$$

$$D << e^{C/\epsilon} \ \text{ for some } C < 1.$$

Let $w$ be the unique solution of the problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \quad \text{in } R^2, \\ w(0) = \max_{y \in R^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.6}$$

For existence and uniqueness of the solutions of (1.6) we refer to [8] and [13]. We also recall that

$$w(y) \sim |y|^{-1/2}e^{-|y|} \quad \text{as } |y| \to \infty. \tag{1.7}$$

We define two important parameters

$$\eta_\epsilon = \frac{|\Omega|}{2\pi D}\log\frac{1}{\epsilon}, \quad \alpha_\epsilon = \frac{\epsilon^2 \int_{R^2} w^2}{A^2|\Omega|}. \tag{1.8}$$

We first show for $K = 1, 2, \ldots$ that multiple interior $K-$spot solutions exist if and only if

$$\lim_{\epsilon \to 0} 4(\eta_\epsilon + K)\alpha_\epsilon < 1. \tag{1.9}$$

The locations of the spots are determined by using a certain Green's function and its derivatives.

Furthermore, concerning stability one has to study the eigenvalues of the order $O(1)$ which are called “large eigenvalues” and the eigenvalues of the order $o(1)$ which are called “small eigenvalues” separately. We show that the small eigenvalues are related to the Green’s function and its derivatives. Suppose these small eigenvalues, which are real, are all negative. Then for $K-$spot solutions the following result holds true: If

$$\lim_{\epsilon\to 0}\frac{(2\eta_\epsilon+K)^2}{\eta_\epsilon}\alpha_\epsilon<1 \tag{1.10}$$

then $K$-spot solutions are stable for a wide range of $\tau\geq 0$. On the other hand, if

$$\lim_{\epsilon\to 0}\frac{(2\eta_\epsilon+K)^2}{\eta_\epsilon}\alpha_\epsilon>1 \tag{1.11}$$

then $K-$spot solutions are unstable for a wide range of $\tau_0$. Precise statements may be found in Theorem 2.3.

The structure of the paper is as follows.

In Section 2 we rigorously state our main results.

In Section 3 we discuss the relevance and novelty of our results.

In Section 4 we provide some preliminary calculations on the heights of the spikes. In Section 5, we give a formal derivation of two nonlocal eigenvalue problems (NLEP). Sections 4 and 5 both provide some preliminary analysis which uses only the leading-order behavior of the steady state. Therefore this is done first.

In Section 6, we give a rigorous study on the two NLEPs derived in Section 5 in a sequence of lemmas. This section is the key in the proof of our main stability theorem, Theorem 2.3.

In Section 7 to Section 9, we give a rigorous account of the existence issue and prove Theorems 2.1 and 2.2. In Section 7 contains suitable approximate solutions are constructed. In Section 8 the inifinite-dimensional existence problem is reduced to a finite-dimensional one. (Liapunov-Schmidt reduction procedure). We then solve the reduced problem in Section 9.

Finally, in Section 10 we finish the stability proof by calculating the large eigenvalues with the help of Section 6 and the small eigenvalues with the help of Sections 7–9. This finishes the proof of Theorem 2.3.

To simplify our notation, we use $e.s.t.$ to denote exponentially small terms in the corresponding norms, more precisely, $e.s.t. = O(e^{-(1-d)/\epsilon})$ as $\epsilon \to 0$ for some $0 < d < 1$ (independent of $\epsilon$).

**Acknowledgments.** The research of JW is supported by an Earmarked Grant from RGC of Hong Kong. MW thanks the Department of Mathematics at CUHK for their kind hospitality. We thank two anonymous referees for carefully reading the manuscript and many critical suggestions which helped us to improve the presentation of the paper.

## 2. Main Results: Existence and Stability of $K-$spot Solutions

We now describe the results of the paper in detail.

Let $\beta^2 = \frac{1}{D}$ and assume that $\lim_{\epsilon\to 0}\beta = \beta_0 \in [0, +\infty)$. If $\beta_0 = 0$, we call it the **weak** coupling case. If $\beta_0 > 0$, we call it the **strong** coupling case. We also define

$$\eta_0 = \lim_{\epsilon\to 0} \eta_\epsilon \in [0, +\infty],\ \alpha_0 = \lim_{\epsilon\to 0} \alpha_\epsilon \in [0, +\infty], \tag{2.1}$$

where $\eta_\epsilon, \alpha_\epsilon$ are defined in (1.8).

Let $\mathbf{P} = (P_1, \dots, P_K) \in \Omega^K$, where $\mathbf{P}$ is arranged such that

$$\mathbf{P} = (P_1, P_2, \dots, P_K) \ \text{ with } P_i = (P_{i,1}, P_{i,2}) \quad \text{for } i = 1, 2, \dots, K.$$

For the rest of the paper we assume that $\mathbf{P} \in \bar{\Lambda}$, where for $\delta > 0$ fixed we define

$$\Lambda = \{(P_1, P_2, \dots, P_K) \in \Omega^K : |P_i - P_j| > 2\delta \text{ for } i \neq j$$

$$\text{and } d(P_i, \partial\Omega) > \delta \text{ for } i = 1, 2 \dots, K\}. \tag{2.2}$$

For $\beta > 0$ fixed let $G_\beta(x,\xi)$ be the Green's function

$$\begin{cases} \Delta G_\beta(x,\xi) - \beta^2 G_\beta(x,\xi) + \delta(x-\xi) = 0 & x, \xi \in \Omega, \\ \dfrac{\partial G_\beta(x,\xi)}{\partial \nu_x} = 0 & x \in \partial\Omega, \xi \in \Omega. \end{cases} \tag{2.3}$$

and let

$$H_\beta(x,\xi) = \frac{1}{2\pi} \log \frac{1}{|x-\xi|} - G_\beta(x,\xi)$$

be the regular part of $G_\beta(x,\xi)$. If $\beta = 0$, we define $G_0(x,\xi)$ to be the Green's function

$$\begin{cases} \Delta G_0(x,\xi) - \frac{1}{|\Omega|} + \delta(x-\xi) = 0 & x,\xi \in \Omega, \\ \int_\Omega G_0(x,\xi)\,dx = 0, \\ \dfrac{\partial G_0(x,\xi)}{\partial \nu_x} = 0 & x \in \partial\Omega,\ \xi \in \Omega. \end{cases} \tag{2.4}$$

and let

$$H_0(x,\xi) = \frac{1}{2\pi}\log\frac{1}{|x-\xi|} - G_0(x,\xi)$$

be the regular part of $G_0(x,\xi)$.

For $\mathbf{P} \in \overline{\Lambda}$ and $\beta \geq 0$ we define

$$F_\beta(\mathbf{P}) = \sum_{k=1}^{K} H_\beta(P_k,P_k) - \sum_{i,j=1,\ldots,K,i\neq j} G_\beta(P_i,P_j) \tag{2.5}$$

and

$$M_\beta(\mathbf{P}) = \nabla^2_{\mathbf{P}} F_\beta(\mathbf{P}). \tag{2.6}$$

Note that $F_\beta(\mathbf{P}) \in C^\infty(\overline{\Lambda})$.

Throughout the paper, we assume that

$$\lim_{\epsilon\to 0} 4(\eta_\epsilon + K)\alpha_\epsilon < 1, \tag{2.7}$$

and

$$(T1) \qquad \lim_{\epsilon\to 0} \frac{(2\eta_\epsilon + K)^2}{\eta_\epsilon}\alpha_\epsilon \neq 1. \tag{2.8}$$

We first consider the existence of $K-$spot solutions in the strong coupling case:

**Theorem 2.1.** *(Existence of $K$-spot solutions in the* **strong** *coupling case).*

*Suppose that $\lim_{\epsilon\to 0}\beta = \beta_0 \neq 0$. Assume that (2.7) and (2.8) hold. Let $\mathbf{P}_0 = (P_1^0, P_2^0, \ldots, P_K^0) \in \overline{\Lambda}$ be a nondegenerate critical point of $F_{\beta_0}(\mathbf{P})$ (defined by (2.5)). Then for $\epsilon$ sufficiently small problem (1.5) has two stationary solutions $(v_\epsilon^\pm, u_\epsilon^\pm)$ with the following properties:*

*(1) $v_\epsilon^\pm(x) = \sum_{j=1}^K \frac{1}{A\xi_\epsilon^\pm}(w(\frac{x-P_j^\epsilon}{\epsilon}) + O(\frac{1}{\log\frac{1}{\epsilon}}))$ uniformly for $x \in \bar\Omega$. Here*

$$\xi_\epsilon^\pm = \frac{1 \pm \sqrt{1-4\eta_\epsilon\alpha_\epsilon}}{2} + O(\alpha_\epsilon) \tag{2.9}$$

*(2)* $u_\epsilon^\pm(x) = \xi_\epsilon^\pm(1 + \frac{1}{\log\frac{1}{\epsilon}}))$ *uniformly for* $x \in \bar{\Omega}$.

*(3)* $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, ..., K$.

Next we consider the existence of $K-$spot solutions in the weak coupling case.

**Theorem 2.2.** *(Existence of $K$-spot solutions in the* **weak** *coupling case).*

*Suppose that* $\lim_{\epsilon\to 0}\beta = 0$. *Assume that (2.7) and (2.8) hold. Let* $\mathbf{P}_0 = (P_1^0, P_2^0, \dots, P_K^0) \in \overline{\Lambda}$ *be a nondegenerate critical point of* $F_0(\mathbf{P})$ *(defined by (2.5)). Then for* $\epsilon$ *sufficiently small problem (1.5) has two stationary solutions* $(v_\epsilon^\pm, u_\epsilon^\pm)$ *with the following properties:*

*(1)* $v_\epsilon^\pm(x) = \sum_{j=1}^K \frac{1}{A\xi_\epsilon^\pm}(w(\frac{x-P_j^\epsilon}{\epsilon}) + O(h(\epsilon,\beta)))$ *uniformly for* $x \in \bar{\Omega}$*, where*

$$\xi_\epsilon^\pm = \begin{cases} \dfrac{1 \pm \sqrt{1-4K\alpha_0}}{2} + O(k(\epsilon,\beta)) & \text{if } \eta_\epsilon \to 0, \\ \dfrac{1 \pm \sqrt{1-4\lim_{\epsilon\to 0}\eta_\epsilon\alpha_\epsilon}}{2} + O(k(\epsilon,\beta)) & \text{if } \eta_\epsilon \to \infty, \\ \dfrac{1 \pm \sqrt{1-4(K+\eta_0)\alpha_0}}{2} + O(k(\epsilon,\beta)) & \text{if } \eta_\epsilon \to \eta_0 > 0, \end{cases} \tag{2.10}$$

$$k(\epsilon,\beta) = \begin{cases} \eta_\epsilon\alpha_\epsilon & \text{if } \eta_\epsilon \to 0, \\ \alpha_\epsilon & \text{if } \eta_\epsilon \to \infty, \\ \beta^2\alpha_\epsilon & \text{if } \eta_\epsilon \to \eta_0 \in (0,+\infty), \end{cases} \tag{2.11}$$

*and*

$$h(\epsilon,\beta) = \min\left\{\frac{1}{\log\frac{1}{\epsilon}}, \beta^2\right\}. \tag{2.12}$$

*(2)* $u_\epsilon^\pm(x) = \xi_\epsilon^\pm(1 + O(h(\epsilon,\beta))$ *uniformly for* $x \in \bar{\Omega}$.

*(3)* $P_j^\epsilon \to P_j^0$ *as* $\epsilon \to 0$ *for* $j = 1, ..., K$.

In each of Theorems 2.1 and 2.2, we have obtained two solutions. We call $(v_\epsilon^-, u_\epsilon^-)$ the **small** solution and the other one the **large** solution. When there is no confusion, we drop $\pm$ for simplicity.

Finally we study the stability and instability of the $K$-spot solutions constructed in Theorems 2.1 and 2.2. We say an eigenvalue problem is stable if there exists a constant $c_0 > 0$ such that for all eigenvalues $\lambda$, we have $\mathrm{Re}(\lambda) \le -c_0$. We say it is unstable if there exists an eigenvalue $\lambda$ with $\mathrm{Re}(\lambda) > 0$. We consider all $\tau \ge 0$.

**Theorem 2.3.** *(Stability of $K-$spot solutions). Assume that (2.7) and (2.8) hold. Let $0 < \epsilon << 1$ and let $\mathbf{P}_0$ be a nondegenerate critical point of $F_{\beta_0}(\mathbf{P})$ and let $(v_\epsilon, u_\epsilon)$ be the $K-$spot solutions constructed in Theorem 2.1 or Theorem 2.2 for $\epsilon$ sufficiently small, whose peaks approach $\mathbf{P}_0 \in \overline{\Lambda}$. Further assume that*

$$(*) \qquad \mathbf{P}^0 \text{ is a nondegenerate local maximum point of } F_{\beta_0}(\mathbf{P}).$$

*Then the large solutions are all linearly unstable for all $\tau \geq 0$. For the small solutions the following holds:*

**Case 1.** *$\eta_\epsilon \to 0$. (Then $\beta \to 0$.)*

*If $K = 1$, then there exists a unique $\tau_1 > 0$ such that for $\tau < \tau_1$, $(u_\epsilon, v_\epsilon)$ is linearly stable, while for $\tau > \tau_1$, $(u_\epsilon, v_\epsilon)$ is linearly unstable.*

*If $K > 1$, $(u_\epsilon, v_\epsilon)$ is linearly unstable for any $\tau \geq 0$.*

**Case 2.** *$\eta_\epsilon \to +\infty$. $(u_\epsilon, v_\epsilon)$ is linearly stable for any $\tau \geq 0$.*

**Case 3.** *$\eta_\epsilon \to \eta_0 \in (0, +\infty)$. (Then $\beta \to 0$.)*

*If $\alpha_0 < \frac{\eta_0}{(2\eta_0+K)^2}$, then $(u_\epsilon, v_\epsilon)$ is linearly stable for $\tau$ small enough or $\tau$ large enough.*

*If $K = 1$, $\alpha_0 > \frac{\eta_0}{(2\eta_0+1)^2}$, then there exist $\tau_2 > 0$, $\tau_3 > 0$ such that $(u_\epsilon, v_\epsilon)$ is linearly stable for $\tau < \tau_2$ and linearly unstable for $\tau > \tau_3$.*

*If $K > 1$ and $\alpha_0 > \frac{\eta_0}{(2\eta_0+K)^2}$, then $(u_\epsilon, v_\epsilon)$ is linearly unstable for any $\tau \geq 0$.*

The statements of Theorems 2.1, 2.2 and 2.3 are rather long. Let us therefore discuss our results in the following section.

## 3. Remarks and Discussions

Let us discuss what has been achieved in this paper and which important questions are still left open.

We have investigated the Gray-Scott system which is a very important reaction-diffusion system in the study of self-replicating phenomena. We study both the strong coupling case (i.e., $D$ is finite) and the weak coupling case (i.e., $D \to +\infty$), for small diffusion coefficients $\epsilon^2$ of the activator $V$. In a bounded domain we rigorously prove the existence of $K-$spot patterns and are able to locate the peaks in terms of the Green's function and its derivatives. Furthermore, we derive rigorous results on linear stability. There are small eigenvalues which are given to leading order in terms of the Green's function and its derivatives. We also have $O(1)$ eigenvalues which are given as eigenvalues of related nonlocal eigenvalue problems in $R^2$.

Roughly speaking, the following condition

$$\alpha_0 < \frac{1}{4(\eta_0 + K)} \tag{3.1}$$

guarantees the existence of two interior $K-$spot solutions – one is small and the other is large.

On the other hand, the inequality

$$\alpha_0 < \frac{\eta_0}{(2\eta_0 + K)^2} \tag{3.2}$$

gives the critical threshold for determining the stability of $K$-spot small solutions. (The large ones are always unstable.) So we have the following picture of $K-$spot solutions.

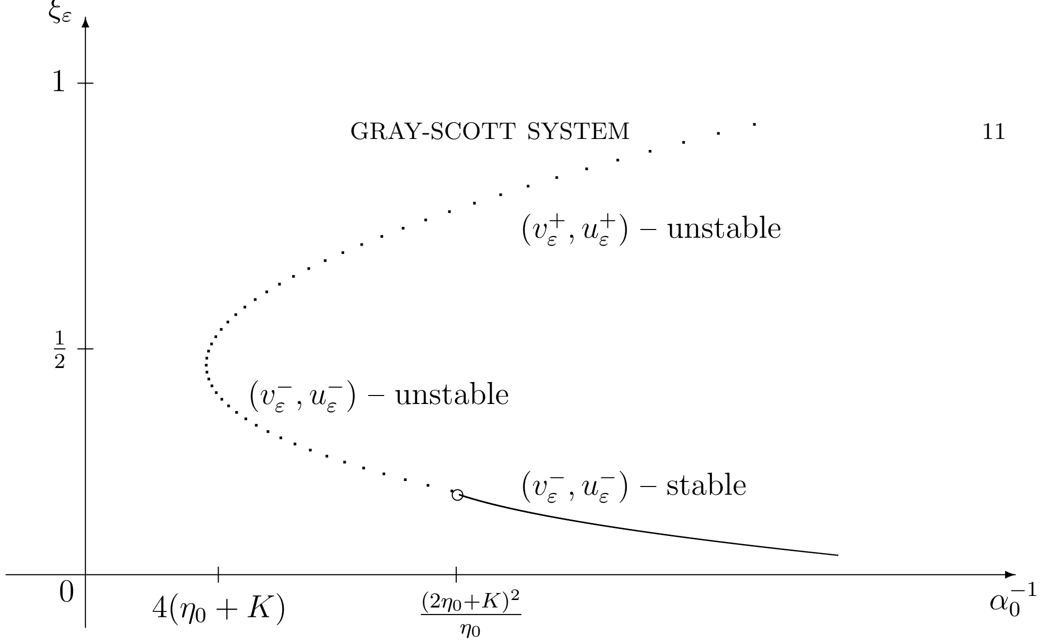


Fig. 1: Bifurcation diagram when $\tau$ is large.

A stability threshold also occurs for the Gierer-Meinhardt system, [30]. However, for the Gierer-Meinhardt system there are no large-amplitude solutions. Furthermore, the values for the thresholds are markedly different from here.

We first comment about the conditions on the locations of interior $K-$spots which are imposed for existence and stability. The condition on the locations of $\mathbf{P}_0$ is not so severe. For any bounded smooth domain $\Omega$, the functional $F_{\beta_0}(\mathbf{P})$ always admits a **global maximum** at some $\mathbf{P}_0 \in \overline{\Lambda}$. In fact, this can be seen very easily: if $|P_i - P_j| \to 0$ or $d(P_i, \partial\Omega) \to 0$, then $F_{\beta_0}(\mathbf{P}) \to -\infty$. (Note that as $d(P_i, \partial\Omega) \to 0$, $H_\beta(P_i, P_i) \to -\infty$.) This point $\mathbf{P}_0$ is a critical point of $F_{\beta_0}(\mathbf{P})$. If $\mathbf{P}_0$ is also a nondegenerate critical point of $F_{\beta_0}(\mathbf{P})$, then the matrix $M_{\beta_0}(\mathbf{P}_0)$ has only negative eigenvalues. (It is an interesting question to numerically compute the critical points of $F_{\beta_0}(\mathbf{P})$.)

Next we discuss our stability result.

Let us recall what has been proved in $R^1$. In [3], the stability of a single-pulse in $R^1$ is studied, though the scalings are quite different here. In a bounded interval in $R^1$, stability of multiple-peaked solutions for the Gray-Scott system is studied in [23] by a matched asymptotic analysis approach.

There it is shown that the critical thresholds are independent of $\epsilon$. Moreover, the critical thresholds arise in the computation of the small eigenvalues.

In $R^2$ the analysis is very different since it has to reflect the geometry of the domain, which is trivial for an interval on the real line. Here in $R^2$, the critical thresholds go to infinity as $\epsilon \to 0$. Furthermore, they are obtained in the study of the large eigenvalues. Since these are independent of the peak locations results about stability can be achieved without studying the higher-order terms of the equilibrium in detail. However, for the small eigenvalues this fine analysis is required.

Assuming that the eigenvalues of $M_{\beta_0}(\mathbf{P}_0)$ are all negative, the stability behavior for Case 1 ($\eta_\epsilon \to 0$) and Case 2 ($\eta_\epsilon \to \infty$) of the small-amplitude solutions for $\tau$ small or large is summarized in the following table:

| | Case 1. | Case 2. |
|---|---|---|
| $K = 1$, $\tau$ small | stable | stable |
| $K = 1$, $\tau$ large | unstable | stable |
| $K > 1$, $\tau$ small | unstable | stable |
| $K > 1$, $\tau$ large | unstable | stable |

In Case 3 for $\eta_0 < \alpha_0(2\eta_0 + K)^2$ the results are the same as in Case 1. In Case 3 for $\eta_0 > \alpha_0(2\eta_0 + K)^2$ the results are the same as in Case 2.

Case 1 resembles the shadow system case and Case 2 is similar to the strong coupling case.

Let us now discuss the role of $\tau$ for the stability.

In the Gray-Scott model, $\tau = \frac{F+k}{F} > 1$ is a very important control parameter and thus the effect of $\tau$ on the stability plays an important role in self-replicating phenomena.

In the strong coupling case (Case 2), $\tau$ has no effect on the stability or instability of $K-$spot solutions.

In the very weak coupling case (Case 1), $\tau$ has no effect on the instability of multiple-spot solutions. Only in the single-spot case, $\tau$ gives a critical threshold on the stability. Theorem 2.3 contains a new result on the **uniqueness** of the Hopf bifurcation point.

In the mild weak coupling case (Case 3), from Theorem 2.3, we see that large $\tau$ may increase stability: if $\alpha_0 < \frac{\eta_0}{(2\eta_0+K)^2}$ and $\tau$ large, $K-$spot solutions are stable. In Lemma 6.5, an explicit lower bound for $\tau$ in terms of $\eta_0$ and $\alpha_0$ is given.

In fact in Case 3, if $\alpha_0 < \frac{\eta_0}{(2\eta_0+K)^2}$, then for small $\tau$, $K-$spot solutions are also stable (Lemma 6.5). We conjecture that the solution is stable for all $\tau \geq 0$. If this is true, it will imply that $\tau$ has no effect on the stability (as in Case 1).

There are still many problems remaining open.

For many cases we can show that the $O(1)$ eigenvalues lie on the left- or right half of the complex plane. Some of the cases, in particular for finite $\tau > 0$, are still missing.

It would also be desirable to characterize the small eigenvalues not in terms of the Green's function and its derivatives but directly in terms of the domain $\Omega$ instead.

There are no results in either the weak or the strong coupling case on the dynamics of the full Gray-Scott system in a two-dimensional domain. Furthermore, there are no results at all about existence or stability of $K$-spot solutions in a three-dimensional domain. These important questions are still open.

## 4. Formal Analysis I: Calculating the heights of the peaks

In this section we are calculating the heights of the peaks as needed in the sections below. It is found that the heights depend on the number of peaks but not on their locations. This is a leading order asymptotic statement that is valid for $\epsilon \to 0$. A rigorous derivation for the heights $\xi_{\epsilon,j}$ will be given in Lemma 7.1 below.

For the rest of the paper, we always assume that $\mathbf{P}$, $\mathbf{P}_0 \in \overline{\Lambda}$, where $\Lambda$ was defined in (2.2), and that $|\mathbf{P}-\mathbf{P}_0| < r$ for some fixed and small enough number $r > 0$.

For $\beta > 0$ let $G_\beta(x,\xi)$ and $G_0(x,\xi)$ be the Green's functions defined in (2.3) and (2.4), respectively. Then we can derive a relation between $G_0$ and $G_\beta$ in the limit $\beta \to 0$ which is as follows. From (2.3) we get

$$\int_\Omega G_\beta(x,\xi)\,dx = \beta^{-2}.$$

Set

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + \overline{G}_\beta(x,\xi).$$

Then

$$\begin{cases} \Delta \bar{G}_\beta - \beta^2 \bar{G}_\beta - \frac{1}{|\Omega|} + \delta_\xi = 0 & \text{in } \Omega, \\ \displaystyle\int_\Omega \bar{G}_\beta(x,\xi)\,dx = 0, & \\ \dfrac{\partial \bar{G}_\beta}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

(2.4) and (4.1) imply that

$$\bar{G}_\beta(x,\xi) = G_0(x,\xi) + O(\beta^2) \quad \text{as } \beta \to 0$$

in the operator norm of $L^2(\Omega) \to H^2(\Omega)$. (Observe that the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ is compact.)

Hence

$$G_\beta(x,\xi) = \frac{\beta^{-2}}{|\Omega|} + G_0(x,\xi) + O(\beta^2) \tag{4.2}$$

in the operator norm of $L^2(\Omega) \to H^2(\Omega)$.

We define cut-off functions as follows: Let $r_0 = \frac{\delta}{4} > 0$, where $\delta$ was defined in (2.2), and let $\chi$ be a smooth cut-off function which is equal to 1 in $B_1(0)$ and equal to 0 in $R^2 \setminus \overline{B_2(0)}$.

Let us assume the following ansatz for $(v_\epsilon, u_\epsilon)$:

$$\begin{cases} v_\epsilon \sim \sum_{j=1}^K \frac{1}{A\xi_{\epsilon,j}} w(\frac{x-P_j}{\epsilon}) \chi_{\epsilon,j}(x), \\ u_\epsilon(P_j) \sim \xi_{\epsilon,j}, \end{cases} \tag{4.3}$$

where $w$ is the unique solution of (1.6), $(P_1, ..., P_K) \in \Lambda$, $\xi_{\epsilon,j}$ is the height at $P_j$ , and

$$\chi_{\epsilon,j}(x) = \chi\left(\frac{x - P_j}{r_0}\right), \qquad x \in \Omega, \quad j = 1, \dots, K. \tag{4.4}$$

Then we can make the following calculations. Later we will rigorously prove Theorems 2.1 and 2.2 which includes the asymptotic relations given in (4.3) with error terms of the order $O(h(\epsilon, \beta))$, in suitable norms. Therefore the following calculations can be rigorously justified.

Let us first consider the case $\beta \to 0$ (weak coupling case). From the equation for $u_\epsilon$ in (1.5),

$$\Delta(1 - u_\epsilon) - \beta^2(1 - u_\epsilon) + \beta^2 u_\epsilon v_\epsilon^2 = 0,$$

we get by (4.2)

$$1 - u_\epsilon(P_i) = 1 - \xi_{\epsilon,i} = \int_\Omega G_\beta(P_i, \xi)\beta^2 u_\epsilon(\xi) v_\epsilon^2(\xi)\, d\xi$$

$$= \int_\Omega (\frac{1}{|\Omega|} + \beta^2 G_0(P_i, \xi) + O(\beta^4)) \left(\sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} w^2(\frac{\xi - P_j}{\epsilon}) \chi_{\epsilon,j}^2(\xi)\right) d\xi.$$

Thus

$$1 - \xi_{\epsilon,i} = \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy + \frac{1}{A^2 \xi_{\epsilon,i}} \beta^2 \int_\Omega G_0(P_i, \xi) w^2(\frac{\xi - P_i}{\epsilon}) \chi_{\epsilon,i}^2(\xi)\, d\xi$$

$$+ \beta^2 \sum_{j \neq i} G_0(P_i, P_j) \frac{1}{A^2 \xi_{\epsilon,j}} \epsilon^2 \int_{R^2} w^2(y)\, dy + \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} O(\beta^4 \epsilon^2 + \beta^2 \epsilon^4). \tag{4.5}$$

Using the decomposition for $G_0$ in (4.5) gives

$$1 - \xi_{\epsilon,i} = \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy$$

$$+ \frac{1}{A^2 \xi_{\epsilon,i}} \beta^2 \int_\Omega \left(\frac{1}{2\pi} \log \frac{1}{|P_i - \xi|} - H_0(P_i, \xi)\right) w^2(\frac{\xi - P_i}{\epsilon}) \chi_{\epsilon,i}^2(\xi)\, d\xi$$

$$+ \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} O(\beta^4 \epsilon^2 + \beta^2 \epsilon^4)$$

$$= \sum_{j=1}^K \frac{1}{A^2 \xi_{\epsilon,j}} \frac{\epsilon^2}{|\Omega|} \int_{R^2} w^2(y)\, dy$$

$$+\frac{1}{A^2\xi_{\epsilon,i}}\frac{\beta^2}{2\pi}\epsilon^2\log\frac{1}{\epsilon}\int_{R^2}w^2(y)\,dy+\sum_{j=1}^{K}\frac{1}{A^2\xi_{\epsilon,j}}O(\beta^2\epsilon^2). \tag{4.6}$$

Recall the definition of $\eta_\epsilon$ and $\alpha_\epsilon$ in (1.8). Then from (4.6) we obtain the following system of algebraic equations

$$1-\xi_{\epsilon,i}-\frac{\eta_\epsilon\alpha_\epsilon}{\xi_{\epsilon,i}}=\sum_{j=1}^{K}\frac{\alpha_\epsilon}{\xi_{\epsilon,j}}+O(\sum_{j=1}^{K}\frac{\beta^2\alpha_\epsilon}{\xi_{\epsilon,j}}). \tag{4.7}$$

A similar (and in fact simpler) calculation shows that in the case $\beta\to\beta_0$, $\beta_0\in(0,\infty)$, (strong coupling case) which is part of Case 2 below ($\eta_\epsilon\to\infty$) (4.7) holds with the error term replaced by

$$O(\sum_{j=1}^{K}\frac{\alpha_\epsilon}{\xi_{\epsilon,j}}).$$

Assuming asymptotically that

$$\lim_{\epsilon\to0}\frac{\xi_{\epsilon,i}}{\xi_{\epsilon,1}}=1,$$

$$\text{i.e., there exists } \xi_0>0 \text{ such that} \qquad \lim_{\epsilon\to0}\xi_{\epsilon,j}=\xi_j=\xi_0, \tag{4.8}$$

from (4.7) we get the basic equation for the height

$$1-\xi_0-\frac{(\eta_0+K)\alpha_0}{\xi_0}=O(\frac{\beta^2\alpha_\epsilon}{\xi_0}). \tag{4.9}$$

**Case 1:** $\eta_\epsilon\to0$.

Then (4.9) becomes

$$1-\xi_0=\frac{K\alpha_0}{\xi_0}+O(\frac{\eta_\epsilon\alpha_\epsilon}{\xi_0}).$$

This quadratic equation has a solution if and only if

$$4K\alpha_0<1 \tag{4.10}$$

and the solution is given by

$$\xi_0^{\pm}=\frac{1\pm\sqrt{1-4K\alpha_0}}{2}+O(k(\epsilon,\beta)),$$

where $k(\epsilon,\beta)$ is defined in Theorem 2.2.

**Case 2:** $\eta_\epsilon\to\infty$.

Then from (4.9) we get

$$1-\xi_0=\frac{\eta_\epsilon\alpha_\epsilon}{\xi_0}+O(\frac{\alpha_\epsilon}{\xi_0})$$

and so, in the same way as in Case 1, there exist solutions if and only if

$$4 \lim_{\epsilon \to 0} \eta_\epsilon \alpha_\epsilon < 1, \tag{4.11}$$

and solutions are given by

$$\xi_0^{\pm} = \frac{1 \pm \sqrt{1 - 4 \lim_{\epsilon \to 0} \eta_\epsilon \alpha_\epsilon}}{2} + O(k(\epsilon, \beta)).$$

**Case 3:** $\eta_\epsilon \to \eta_0, \quad (0 < \eta_0 < \infty)$.

We derive for $\xi_0$

$$1 - \xi_0 = \frac{(K + \eta_0)\alpha_0}{\xi_0} + O(\frac{\beta^2 \alpha_\epsilon}{\xi_0})$$

which has a root if and only if

$$4(\eta_0 + K)\alpha_0 < 1 \tag{4.12}$$

The solution is given by

$$\xi_0^{\pm} = \frac{1 \pm \sqrt{1 - 4(K + \eta_0)\alpha_0}}{2} + O(k(\epsilon, \beta)).$$

In conclusion, the results in this section show that the heights satisfy (2.9) anhd (2.10) in Theorem 2.1 and Theorem 2.2, respectively.

## 5. Formal Analysis II: Derivations of Two Nonlocal Eigenvalue Problems

Linearizing the system (1.5) around the equilibrium states $(v_\epsilon, u_\epsilon)$ given in Theroem 1.1, we obtain the following eigenvalue problem. Here we the leading-order approximation of the solution, i.e, that

$$\begin{cases} v_\epsilon \sim \sum_{i=1}^{K} \frac{1}{A\xi_{\epsilon,i}} w(\frac{x - P_i^\epsilon}{\epsilon}) \chi_{\epsilon,i}(x), \\ u_\epsilon(P_i^\epsilon) \sim \xi_{\epsilon,i}, \end{cases} \tag{5.1}$$

where the leading order of $\xi_{\epsilon,i} \sim \xi_\epsilon^{\pm} \to \xi_0^{\pm}$ is given in Section 4.

In this section, we derive two important nonlocal eigenvalue problems (NLEP). In Section 6 they will be studied which will give the critical thresholds for stability. In particular, we will show that the study of large eigenvalues is independent of the locations $P_j, j = 1, ..., K$.

Linearizing around the equilibrium states $(v_\epsilon, u_\epsilon)$

$$\begin{cases} v = v_\epsilon + \phi_\epsilon(y)e^{\lambda_\epsilon t}, \\ u = u_\epsilon + \psi_\epsilon(x)e^{\lambda_\epsilon t}, \end{cases}$$

and substituting the result into (1.5) we deduce the following eigenvalue problem

$$\begin{cases} \Delta_y\phi_\epsilon - \phi_\epsilon + 2Au_\epsilon v_\epsilon\phi_\epsilon + A\psi_\epsilon v_\epsilon^2 = \lambda_\epsilon\phi_\epsilon, \\ \frac{1}{\beta^2}\Delta\psi_\epsilon - \psi_\epsilon - 2u_\epsilon v_\epsilon\phi_\epsilon - \psi_\epsilon v_\epsilon^2 = \tau\lambda_\epsilon\psi_\epsilon. \end{cases} \tag{5.2}$$

Here $D = \frac{1}{\beta^2}$, $\lambda_\epsilon$ is some complex number and

$$\phi_\epsilon \in H_N^2(\Omega_\epsilon),\ \psi_\epsilon \in H_N^2(\Omega), \tag{5.3}$$

where the index $N$ indicates that $\phi_\epsilon$ and $\psi_\epsilon$ satisfy the no flux boundary condition and

$$\Omega_\epsilon = \{y \in R^2 | \epsilon y \in \Omega\}.$$

Let us study the large eigenvalues first, i.e., let us assume that $\liminf_{\epsilon\to 0}|\lambda_\epsilon| > 0$. We observe that if Re $(\lambda_\epsilon) \le -c_0$ for some $c_0 > 0$, then these eigenvalues only contribute to stability. (As $\epsilon \to 0$, $\lambda_\epsilon$ may approach the essential spectrum of the limiting operator on the entire space, which is contained in the interval $(-\infty, -c_0)$ with $c_0 > 0$.) Therefore, we have only to consider the behavior of eigenvalues satisfying the condition Re $(\lambda_\epsilon) \ge -c_0$. Furthermore, we may assume that $0 < c_0 < 1$. Let $\lambda_\epsilon \to \lambda_0 \ne 0$ as $\epsilon \to 0$ (possibly after taking a subsequence).

The second equation in (5.2) is equivalent to

$$\Delta_x\psi_\epsilon - \beta^2(1+\tau\lambda_\epsilon)\psi_\epsilon - 2\beta^2 u_\epsilon v_\epsilon\phi_\epsilon - \beta^2 v_\epsilon^2\psi_\epsilon = 0. \tag{5.4}$$

We introduce the following

$$\beta_{\lambda_\epsilon} = \beta\sqrt{1+\tau\lambda_\epsilon} \tag{5.5}$$

where in $\sqrt{1+\tau\lambda_\epsilon}$ we take the principal value. (This means that the real part of $\sqrt{1+\tau\lambda_\epsilon}$ is positive, which is possible because Re $(1+\tau\lambda_\epsilon) \ge \frac{1}{2}$).

Let us assume that

$$\|\phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1.$$

We cut off $\phi_\epsilon$ as follows: Define

$$\phi_{\epsilon,j}(y-\frac{P_j^\epsilon}{\epsilon})=\phi_\epsilon(y)\chi_{\epsilon,j}(x),$$

where $\chi_{\epsilon,j}(x)$ was introduced in (4.4).

From (5.2) using the fact that $\mathrm{Re}\sqrt{1+\lambda_\epsilon}>0$ and the exponential decay of $w$ it follows that

$$\phi_\epsilon=\sum_{j=1}^{K}\phi_{\epsilon,j}+e.s.t.\quad \text{in } H^2(\Omega_\epsilon).$$

Then by a standard procedure we extend $\phi_{\epsilon,j}$ to a function defined on $R^2$ such that

$$\|\phi_{\epsilon,j}\|_{H^2(R^2)}\le C\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)},\qquad j=1,\dots,K.$$

Then $\|\phi_{\epsilon,j}\|_{H^2(R^2)}\le C$. By taking a subsequence of $\epsilon$, we may also assume that $\phi_{\epsilon,j}\to\phi_j$ as $\epsilon\to 0$ in $H^2_{loc}(R^2)$ for $j=1,\dots,K$.

We have by (5.4)

$$\psi_\epsilon(x)=-\int_\Omega \beta^2 G_{\beta_{\lambda_\epsilon}}(x,\xi)(2u_\epsilon(\xi)v_\epsilon(\xi)\phi_\epsilon(\frac{\xi}{\epsilon})+\psi_\epsilon(\xi)v_\epsilon^2(\xi))\,d\xi. \tag{5.6}$$

In the case $\beta\to 0$ we calculate at $x=P_i^\epsilon$, $i=1,\dots,K$:

$$\psi_\epsilon(P_i^\epsilon)=-\beta^2\int_\Omega(\frac{(\beta_{\lambda_\epsilon})^{-2}}{|\Omega|}+G_0(P_i^\epsilon,\xi)+O(\beta^2))$$

$$\left(2\sum_{j=1}^{K}\frac{1}{A}w(\frac{\xi-P_j^\epsilon}{\epsilon})\phi_{\epsilon,j}(\frac{\xi-P_j^\epsilon}{\epsilon})+\sum_{j=1}^{K}\psi_\epsilon(P_j^\epsilon)\frac{1}{A^2\xi_{\epsilon,j}^2}w^2(\frac{\xi-P_j^\epsilon}{\epsilon})\right)\,d\xi\,(1+o(1))$$

$$=\left[\frac{1}{1+\tau\lambda_\epsilon}\left(-\frac{2\epsilon^2}{A|\Omega|}\sum_{j=1}^{K}\int_{R^2}w\phi_j-\frac{\epsilon^2\int_{R^2}w^2}{A^2|\Omega|}\sum_{j=1}^{K}\psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_j^2}\right)\right.$$

$$\left.+\frac{\beta^2\log\frac{1}{\epsilon}}{2\pi}\left(-\frac{2\epsilon^2}{A}\int_{R^2}w\phi_i-\frac{\epsilon^2\int_{R^2}w^2}{A^2}\psi_\epsilon(P_i^\epsilon)\frac{1}{\xi_i^2}\right)\right](1+o(1))$$

$$=\left[\frac{1}{1+\tau\lambda_\epsilon}\left(-2A\alpha_\epsilon\frac{\sum_{j=1}^{K}\int_{R^2}w\phi_j}{\int_{R^2}w^2}-\alpha_\epsilon\sum_{j=1}^{K}\psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_j^2}\right)\right.$$

$$\left.+\left(-2A\eta_\epsilon\alpha_\epsilon\frac{\int_{R^2}w\phi_i}{\int_{R^2}w^2}-\alpha_\epsilon\eta_\epsilon\psi_\epsilon(P_i^\epsilon)\frac{1}{\xi_i^2}\right)\right](1+o(1))$$

Let

$$\psi_\epsilon(P_j^\epsilon)\frac{1}{\xi_j^2} = \hat{\psi}_{\epsilon,j}, \quad \hat{\Psi}_\epsilon = (\hat{\psi}_{\epsilon,1}, ..., \hat{\psi}_{\epsilon,K}). \tag{5.7}$$

Then we have

$$\xi_i^2\hat{\psi}_{\epsilon,i} = \left[\frac{1}{1+\tau\lambda_0}\left(-2A\alpha_\epsilon\frac{\sum_{j=1}^K\int_{R^2} w\phi_{\epsilon,j}}{\int_{R^2} w^2} - \alpha_\epsilon\sum_{j=1}^K\hat{\psi}_{\epsilon,j}\right)\right.$$

$$\left. + \left(-2A\eta_\epsilon\alpha_\epsilon\frac{\int_{R^2} w\phi_i}{\int_{R^2} w^2} - \alpha_\epsilon\eta_0\hat{\psi}_{\epsilon,i}\right)\right](1+o(1)).$$

Writing in matrix form, we obtain

$$\left[(\xi_0^2+\alpha_0\eta_0)\mathcal{I} + \frac{\alpha_0}{1+\tau\lambda_0}\mathcal{E}\right]\lim_{\epsilon\to 0}\hat{\Psi}_\epsilon$$

$$= (-2A\eta_0\alpha_0\mathcal{I} - \frac{2A\alpha_0}{1+\tau\lambda_0}\mathcal{E})\frac{\int_{R^2} w\Phi}{\int_{R^2} w^2},$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in (H^2(R^2))^K,$$

$\mathcal{I}$ is the identity matrix, and

$$\mathcal{E} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}. \tag{5.8}$$

Thus for $\lambda_0 \neq 0$ in the limit $\epsilon \to 0$ from (5.2) we obtain the following nonlocal eigenvalue problem (NLEP):

$$\Delta\Phi - \Phi + 2w\Phi - 2\mathcal{B}\frac{\int_{R^2} w\Phi}{\int_{R^2} w^2}w^2 = \lambda_0\Phi, \tag{5.9}$$

where

$$\mathcal{B} = ((\xi_0^2+\alpha_0\eta_0)\mathcal{I} + \frac{\alpha_0}{1+\tau\lambda_0}\mathcal{E})^{-1}(\eta_0\alpha_0\mathcal{I} + \frac{\alpha_0}{1+\tau\lambda_0}\mathcal{E}). \tag{5.10}$$

More precisely, we have the following statement:

**Theorem 5.1.** *Assume that $(v_\epsilon, u_\epsilon)$ satisfies (5.1).*

*Let $\lambda_\epsilon$ be an eigenvalue of (5.2) such that $Re(\lambda_\epsilon) > -c_0$, where $0 < c_0 < 1$.*

*(1) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then $\lambda_0$ is an eigenvalue of the problem (NLEP) given in (5.9).*

*(2) Let $\lambda_0 \neq 0$ be an eigenvalue of the problem (NLEP) given in (5.9). Then for $\epsilon$ sufficiently small, there is an eigenvalue $\lambda_\epsilon$ of (5.2) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.*

**Proof:**

(1) of Theorem 5.1 follows the asymptotic analysis at the beginning of this section.

To prove (2) of Theorem 5.1, we follow the argument given in Section 2 of [1], where the following eigenvalue problem was studied:

$$\begin{cases} \epsilon^2 \Delta h - h + p u_\epsilon^{p-1} h - \frac{qr}{s+1+\tau\lambda_\epsilon} \frac{\int_\Omega u_\epsilon^{r-1} h}{\int_\Omega u_\epsilon^r} u_\epsilon^p = \lambda_\epsilon h \text{ in } \Omega, \\ h = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.11}$$

where $u_\epsilon$ is a solution of the single equation

$$\begin{cases} \epsilon^2 \Delta u_\epsilon - u_\epsilon + u_\epsilon^p = 0 \text{ in } \Omega, \\ u_\epsilon > 0 \text{ in } \Omega, \ u_\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Here $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$ and $1 < p < +\infty$ if $n = 1, 2$, $\frac{qr}{(s+1)(p-1)} > 1$ and $\Omega \subset R^n$ is a smooth bounded domain. If $u_\epsilon$ is a single interior peak solution, then it can be shown ([25]) that the limiting eigenvalue problem is an NLEP

$$\Delta\phi - \phi + p w^{p-1}\phi - \frac{qr}{s+1+\tau\lambda_0} \frac{\int_{R^n} w^{r-1}\phi}{\int_{R^n} w^r} w^p = \lambda_0 \phi \tag{5.12}$$

where $w$ is the corresponding ground state solution in $R^n$:

$$\Delta w - w + w^p = 0, w > 0 \text{ in } R^n, w = w(|y|) \in H^1(R^n).$$

Dancer in [1] showed that if $\lambda_0 \neq 0$, $\mathrm{Re}(\lambda_0) > 0$ is an unstable eigenvalue of (5.12), then there exists an eigenvalue $\lambda_\epsilon$ of (5.11) such that $\lambda_\epsilon \to \lambda_0$.

We now follow his idea. Let $\lambda_0 \neq 0$ be an eigenvalue of problem (5.9) with $\mathrm{Re}(\lambda_0) > 0$. We first note that from the equation for $\psi_\epsilon$, we can express $\psi_\epsilon$

in terms of $\phi_\epsilon$. Now we write the first equation for $\phi_\epsilon$ as follows:

$$\phi_\epsilon = -R_\epsilon(\lambda_\epsilon)\left[2Au_\epsilon v_\epsilon \phi_\epsilon + A\psi_\epsilon v_\epsilon^2\right], \tag{5.13}$$

where $R_\epsilon(\lambda)$ is the inverse of $-\Delta + (1+\lambda_\epsilon)$ in $H^2(\Omega_\epsilon)$ (which exists if $\mathrm{Re}(\lambda_\epsilon) > -1$ or $\mathrm{Im}(\lambda_\epsilon) \neq 0$) and $\psi_\epsilon = T[\phi_\epsilon]$ is given by (5.6), where $T$ is a compact operator acting on $\phi_\epsilon$. (Note that we have assumed that Re $(\lambda_\epsilon) > -c_0 > -1$.) The important thing is that $R_\epsilon(\lambda_\epsilon)$ is a compact operator if $\epsilon$ is sufficiently small. The rest of the argument follows exactly that in [1]. For the sake of limited space, we omit the details here.

□

Therefore, the study of large eigenvalues can be reduced to the study of the system of nonlocal eigenvalue problems (5.9). We can further reduce the problem by computing the eigenvalues of $\mathcal{B}$.

The eigenvalues of $\mathcal{B}$ can be computed as follows:

$$b_1 = \frac{\eta_0\alpha_0(1+\tau\lambda_0) + K\alpha_0}{(\xi_0^2 + \eta_0\alpha_0)(1+\tau\lambda_0) + K\alpha_0}, \tag{5.14}$$

$$b_2 = \cdots = b_K = \frac{\eta_0\alpha_0}{\xi_0^2 + \eta_0\alpha_0}. \tag{5.15}$$

Thus the study of the large eigenvalue problem is reduced to the study of the following two NLEPs:

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2(\eta_0\alpha_0(1+\tau\lambda_0) + K\alpha_0)}{(\xi_0^2 + \eta_0\alpha_0)(1+\tau\lambda_0) + K\alpha_0}\frac{\int_{R^2} w\Phi}{\int_{R^2} w^2} w^2 = \lambda_0\Phi, \tag{5.16}$$

and

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2\eta_0\alpha_0}{\eta_0\alpha_0 + \xi_0^2}\frac{\int_{R^2} w\Phi}{\int_{R^2} w^2} w^2 = \lambda_0\Phi. \tag{5.17}$$

Note that in Case 1 ($\eta_0 = 0$), we have

$$b_1 = \frac{K\alpha_0}{\xi_0^2(1+\tau\lambda_0) + K\alpha_0}, \quad b_2 = ... = b_K = 0.$$

In Case 2 ($\eta_0 = +\infty$), we get

$$b_1 = b_2 = ... = b_K = 1.$$

In Case 3 ($\eta_0 \in (0,\infty)$), we study (5.16) and (5.17) directly. In the strong coupling case ($\beta \to \beta_0$) similar and in fact simpler calculations than in this section give the same result as in Case 2.

Problems (5.16) and (5.17) will be studied in the next section.

## 6. Study of Two nonlocal eigenvalue problems

In this section, we give a rigorous study of problems (5.16) and (5.17). To this end, we write them in a unified form:

$$L\phi := \Delta\phi - \phi + 2w\phi - f(\tau\lambda_0)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}w^2 = \lambda_0\phi, \quad \phi \in H^2(R^2), \tag{6.1}$$

where $w$ is the unique solution of (1.6) and $f(\tau\lambda_0)$ is a continuous function.

We first have

**Lemma 6.1.** *If $f(0) < 1$ and $0 < c \le f(\alpha)$ for $\alpha > 0$, then there exists a positive eigenvalue of (6.1) for any $\tau > 0$.*

**Proof:** First, we may assume that $\phi$ is a radially symmetric function, namely, $\phi \in H_r^2(R^2) = \{u \in H^2(R^2) | u = u(|y|)\}$. Let $L_0 = \Delta - 1 + 2w$. Then $L_0$ is invertible in $H_r^2(R^2)$. Let us denote the inverse as $L_0^{-1}$. On the other hand, $L_0$ has a unique positive eigenvalue (see Lemma 1.2 of [25]). We denote this eigenvalue by $\mu_1$. Let us assume that $\lambda_0 \ne \mu_1$. Otherwise the proof is already complete.

Then $\lambda_0 > 0$ is an eigenvalue of (6.1) if and only if it satisfies the following algebraic equation:

$$\int_{R^2} w^2 = f(\tau\lambda_0)\int_{R^2}[((L_0-\lambda_0)^{-1}w^2)w]. \tag{6.2}$$

Equation (6.2) can be simplified further to the following

$$\rho(\lambda_0) := (1 - f(\tau\lambda_0))\int_{R^2} w^2 - \lambda_0 f(\tau\lambda_0)\int_{R^2}[((L_0-\lambda_0)^{-1}w)w] = 0. \tag{6.3}$$

Note that $\rho(0) = (1 - f(0))\int_{R^2} w^2 > 0$. On the other hand, as $\lambda_0 \to \mu_1$, $0 < \lambda_0 < \mu_1$, we have $\int_{R^2}((L_0-\lambda_0)^{-1}w)w \to +\infty$ and hence $\rho(\lambda_0) \to -\infty$. By continuity, there exists $\lambda_0 \in (0,\mu_1)$ such that $\rho(\lambda_0) = 0$. Such a positive $\lambda_0$ will be an eigenvalue of (6.1).

□

Similarly, we have

**Lemma 6.2.** *If* $\lim_{\tau\to+\infty} f(\tau\lambda) = f_{+\infty} < 1$ *and* $0 < c \le f(\alpha)$ *for* $\alpha > 0$, *then there exists a positive eigenvalue of (6.1) for* $\tau > 0$ *large.*

**Proof:** Using the same notation as in the proof of Lemma 6.1, we fix a $\lambda_1 \in (0, \mu_1)$ so that $\lambda_1 \int_{R^2}[((L_0-\lambda_1)^{-1}w)w] < (1-f_{+\infty})\int_{R^2} w^2$. For $\tau$ large, it is easy to see that $\rho(\lambda_1) > 0$. Now the rest follows from the same proof as in Lemma 6.1.

□

Next we consider the case when $f(0) > 1$. To this end, we need the following lemma:

**Lemma 6.3.** *Consider the eigenvalue problem*

$$\Delta\phi - \phi + 2w\phi - \gamma\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}w^2 = \lambda_0\phi, \phi \in H^2(R^2), \tag{6.4}$$

*where* $w$ *is the unique solution of (1.6) and* $\gamma$ *is real.*

*(1) If* $\gamma > 1$, *then there exists a positive constant* $c_0$ *such that* $Re(\lambda_0) \le -c_0$ *for any nonzero eigenvalue* $\lambda_0$ *of (6.4).*

*(2) If* $\gamma < 1$, *then there exists a positive eigenvalue* $\lambda_0$ *of (6.4).*

*(3) If* $\gamma \neq 1$ *and* $\lambda_0 = 0$, *then* $\phi \in$ *span* $\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

*(4) If* $\gamma = 1$ *and* $\lambda_0 = 0$, *then* $\phi \in$ *span* $\{w, \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

**Proof:** (1), (3) and (4) have been proved in Theorem 5.1 of [26]. (2) follows from Lemma 6.1. □

We now consider the function $f(\tau\lambda) = \frac{\mu}{1+\tau\lambda}$. We then have

**Lemma 6.4.** *Let* $\gamma = \frac{\mu}{1+\tau\lambda_0}$ *where* $\mu > 0, \tau \ge 0$.

*(1) Suppose that* $\mu > 1$. *Then there exists a unique* $\tau = \tau_1 > 0$ *such that for* $\tau > \tau_1$, *(6.1) admits a positive eigenvalue, and for* $\tau < \tau_1$, *all eigenvalues of problem (6.1) satisfy* $Re(\lambda) < 0$. *At* $\tau = \tau_1$, $L$ *has a Hopf bifurcation.*

*(2) Suppose that* $\mu < 1$. *Then* $L$ *admits a real eigenvalue* $\lambda_0$ *with* $\lambda_0 \ge c_2 > 0$.

**Proof of Lemma 6.4:**

(2) follows from Lemma 6.1. We only need to prove (1).

Set

$$L\phi := \Delta\phi - \phi + 2w\phi - \frac{\mu}{1+\tau\lambda_0}\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}w^2 = \lambda_0\phi,\ \phi \in H^2(R^2). \tag{6.5}$$

Let $\mu > 1$. Observe that the eigenvalues depend on $\tau$ continuously and that there is no eigenvalue $\lambda_0$ such that Re $(\lambda_0) > 0$ and $|\lambda_0| \to +\infty$ as $\tau \to 0$. (This fact follows from the inequality (6.23) below.) By Lemma 6.3 (1), for $\tau = 0$ (and by perturbation, for $\tau$ small), all eigenvalues lie on the left half plane. By Lemma 6.2, for $\tau$ large, there exist unstable eigenvalues. Thus there must be an point $\tau_1$ at which $L$ has a Hopf bifurcation, i.e., $L$ has a purely imaginary eigenvalue $\lambda_0 = \sqrt{-1}\lambda_I$. To prove Lemma 6.4 (1), all we need to show is that $\tau_1$ is unique.

Let $\lambda_0 = \sqrt{-1}\lambda_I$ be an eigenvalue of $L$. Without loss of generality, we may assume that $\lambda_I > 0$. (Note that $-\sqrt{-1}\lambda_I$ is also an eigenvalue of $L$.) Let $\phi_0 = (L_0 - \sqrt{-1}\lambda_I)^{-1}w^2$. Then (6.5) becomes

$$\frac{\int_{R^2} w\phi_0}{\int_{R^2} w^2} = \frac{1+\tau\sqrt{-1}\lambda_I}{\mu}. \tag{6.6}$$

Let $\phi_0 = \phi_0^R + \sqrt{-1}\phi_0^I$. Then from (6.6), we obtain the two equations

$$\frac{\int_{R^2} w\phi_0^R}{\int_{R^2} w^2} = \frac{1}{\mu}, \tag{6.7}$$

$$\frac{\int_{R^2} w\phi_0^I}{\int_{R^2} w^2} = \frac{\tau\lambda_I}{\mu}. \tag{6.8}$$

Note that (6.7) is independent of $\tau$.

Let us now compute $\int_{R^2} w\phi_0^R$. Observe that $(\phi_0^R, \phi_0^I)$ satisfies

$$L_0\phi_0^R = w^2 - \lambda_I\phi_0^I, \quad L_0\phi_0^I = \lambda_I\phi_0^R.$$

So $\phi_0^R = \lambda_I^{-1}L_0\phi_0^I$ and

$$\phi_0^I = \lambda_I(L_0^2+\lambda_I^2)^{-1}w^2, \quad \phi_0^R = L_0(L_0^2+\lambda_I^2)^{-1}w^2. \tag{6.9}$$

Substituting (6.9) into (6.7) and (6.8), we obtain

$$\frac{\int_{R^2}[wL_0(L_0^2+\lambda_I^2)^{-1}w^2]}{\int_{R^2} w^2} = \frac{1}{\mu}, \tag{6.10}$$

$$\frac{\int_{R^2}[w(L_0^2+\lambda_I^2)^{-1}w^2]}{\int_{R^2}w^2}=\frac{\tau}{\mu}. \tag{6.11}$$

Let $h(\lambda_I)=\frac{\int_{R^2}wL_0(L_0^2+\lambda_I^2)^{-1}w^2}{\int_{R^2}w^2}$. Then integration by parts gives $h(\lambda_I)=\frac{\int_{R^2}w^2(L_0^2+\lambda_I^2)^{-1}w^2}{\int_{R^2}w^2}$. Note that $h'(\lambda_I)=-2\lambda_I\frac{\int_{R^2}w^2(L_0^2+\lambda_I^2)^{-2}w^2}{\int_{R^2}w^2}<0$. So since

$$h(0)=\frac{\int_{R^2}w(L_0^{-1}w^2)}{\int_{R^2}w^2}=1,$$

$h(\lambda_I)\to 0$ as $\lambda_I\to\infty$, and $\mu>1$, there exists a unique $\lambda_I>0$ such that (6.10) holds. Substituting this unique $\lambda_I$ into (6.11), we obtain a unique $\tau=\tau_1>0$.

□

Finally, we consider another NLEP:

$$\Delta\phi-\phi+2w\phi-f(\tau\lambda_0)\frac{\int_{R^2}w\phi}{\int_{R^2}w^2}w^2=\lambda_0\phi,\ \phi\in H^2(R^2), \tag{6.12}$$

where

$$f(\tau\lambda)=\frac{2(\eta_0\alpha_0(1+\tau\lambda)+K\alpha_0)}{(\eta_0\alpha_0+\xi_0^2)(1+\tau\lambda)+K\alpha_0} \tag{6.13}$$

and $0<\tau<+\infty$.

Then we have

**Lemma 6.5.** *Let*

$$a=\frac{6\eta_0^2\alpha_0^2(\eta_0\alpha_0-\xi_0^2)^2\int_{R^2}w^4}{(\xi_0^2+\eta_0\alpha_0)^2\int_{R^2}w^2},\qquad b=\frac{8K\eta_0^2\alpha_0^3\xi_0^2\int_{R^2}w^4}{(\xi_0^2+\eta_0\alpha_0)^2\int_{R^2}w^2},$$

$$c=\frac{3}{2}((K+\eta_0)\alpha_0-\xi_0^2)^2 \tag{6.14}$$

*and* $0<\tau_2\le\tau_3$ *be the two solutions (if they exist) of the following quadratic equation*

$$a\tau^2-b\tau+c=0 \tag{6.15}$$

*(1) If* $\eta_0\alpha_0>\xi_0^2$*, then for* $\tau<\tau_2$ *or* $\tau>\tau_3$ *problem (6.12) is stable.*

*(2) If* $\eta_0\alpha_0<\xi_0^2$*, for* $\tau$ *small problem (6.12) is stable while for* $\tau$ *large it is unstable.*

**Remark:** Problem (6.15) may not have a solution if $\eta_0\alpha_0$ is large. It is also easy to see that

$$\tau_2 \le \tau_3 := \frac{4\xi_0^2 K\alpha_0}{3(\eta_0\alpha_0 - \xi_0^2)} \tag{6.16}$$

**Proof:** We prove (1) first. To this end, we apply the following inequality (Lemma B.1 in [26]): For any $\phi \in H_r^2(R^2)$, we have

$$\int_{R^2}(|\nabla\phi|^2 + \phi^2 - 2w\phi^2) + 2\frac{\int_{R^2} w\phi \int_{R^2} w^2\phi}{\int_{R^2} w^2} - \frac{\int_{R^2} w^3}{(\int_{R^2} w^2)^2}(\int_{R^2} w\phi)^2 \ge 0, \tag{6.17}$$

where equality holds if and only if $\phi$ is a multiple of $w$.

Now let $\phi = \phi_R + \sqrt{-1}\phi_I$ satisfy (6.12), i.e.

$$L_0\phi - f(\tau\lambda)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda\phi. \tag{6.18}$$

Multiplying (6.18) by $\bar{\phi}$ – the conjugate function of $\phi$ – and integrating over $R^2$, we obtain that

$$\int_{R^2}(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2) = -\lambda\int_{R^2}|\phi|^2 - f(\tau\lambda)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}\int_{R^2} w^2\bar{\phi}. \tag{6.19}$$

Multiplying (6.18) by $w$ and integrating over $R^2$, we obtain that

$$\int_{R^2} w^2\phi = \left(\lambda + f(\tau\lambda)\frac{\int_{R^2} w^3}{\int_{R^2} w^2}\right)\int_{R^2} w\phi. \tag{6.20}$$

Hence

$$\int_{R^2} w^2\bar{\phi} = \left(\bar{\lambda} + f(\tau\bar{\lambda}\right)\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\int_{R^2} w\bar{\phi}. \tag{6.21}$$

Substituting (6.21) into (6.19), we have that

$$\int_{R^2}(|\nabla\phi|^2 + |\phi|^2 - 2w|\phi|^2) \tag{6.22}$$

$$= -\lambda\int_{R^2}|\phi|^2 - f(\tau\lambda)\left(\bar{\lambda} + f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2}\right)\frac{|\int_{R^2} w\phi|^2}{\int_{R^2} w^2}.$$

We just need to consider the real part of (6.22). Now applying the inequality (6.17) and using (6.21) we arrive at

$$-\lambda_R \ge \left[\mathrm{Re}\left(f(\tau\lambda)\left(\bar{\lambda} + f(\tau\bar{\lambda})\frac{\int_{R^2} w^3}{\int_{R^2} w^2}\right)\right)\right.$$

$$-2\mathrm{Re}\left(\bar\lambda + f(\tau\bar\lambda)\frac{\int_{R^2} w^3}{\int_{R^2} w^2}\right) + \frac{\int_{R^2} w^3}{\int_{R^2} w^2}\Bigg]\frac{|\int_{R^2} w\phi|^2}{\int_{R^2}|\phi|^2\int_{R^2} w^2}.$$

Assuming that $\lambda_R \geq 0$, then we have

$$\frac{\int_{R^2} w^3}{\int_{R^2} w^2}|f(\tau\lambda)-1|^2 + \mathrm{Re}(\bar\lambda(f(\tau\lambda)-1)) \leq 0. \tag{6.23}$$

Using the fact that $\lambda_R \geq 0$, we arrive at the following inequality

$$\frac{3}{2}(K\alpha_0 + (\eta_0\alpha_0 - \xi_0^2))^2 + \left(\frac{3}{2}(\eta_0\alpha_0 - \xi_0^2)^2\tau^2 - 2\tau K\alpha_0\xi_0^2\right)\lambda_I^2 \leq 0 \tag{6.24}$$

since multiplying (1.6) by $rw'$ and integrating over $R^2$ implies $\int_{R^2} w^3 = \frac{3}{2}\int_{R^2} w^2$.

If $\tau \geq \tau_3$, then (6.24) does not hold. To study the case $\tau < \tau_3$, we need to have an upper bound for $\lambda_I$. From (6.19), we have

$$\lambda_I \int_{R^2}|\phi|^2 = \mathrm{Im}\left(f(\tau\lambda)\frac{\int_{R^2} w\phi}{\int_{R^2} w^2}\int_{R^2} w^2\bar\phi\right)$$

Hence

$$|\lambda_I| \leq |f(\tau\lambda)|\sqrt{\frac{\int_{R^2} w^4}{\int_{R^2} w^2}} \leq \frac{2\eta_0\alpha_0}{\xi_0^2+\eta_0\alpha_0}\sqrt{\frac{\int_{R^2} w^4}{\int_{R^2} w^2}}. \tag{6.25}$$

Substituting (6.25) into (6.24), we see that if

$$a\tau^2 - b\tau + c > 0$$

where $a, b, c$ are defined at (6.14), then (6.24) is impossible.

We next prove (2). For $\tau$ large, we see that $f(\tau\lambda) \to f_{+\infty} := \frac{2\eta_0\alpha_0}{\xi_0^2+\eta_0\alpha_0} < 1$, then the perturbation argument of Dancer [1] shows that there exists a real and positive eigenvalue of (6.12). For $\tau$ small, we follow the same argument as in (1). We omit the details.

Lemma 6.5 is thus proved.

□

## 7. Existence Proof I: Approximate Solutions

Let us start to prove Theorem 2.1 and Theorem 2.2. The first step is to choose a good approximate solution. The second step is to use the Liapunov-Schmidt process to reduce the problem to a finite dimensional problem. The last step is to solve the reduced problem. Such a procedure has been used in the study of the Gierer-Meinhardt system (both in the strong coupling case [28], [29] and in the weak coupling case [30]). We shall sketch it and leave the details to the reader.

Since the proof in the strong coupling case (i.e., $D = O(1)$) is exactly the same as in the Gierer-Meinhardt case, we consider the weak coupling case only. So we assume that $\beta \to 0$.

Motivated by the results in Section 4, we rescale

$$\hat{v}(y) = Av(\epsilon y), \quad y \in \Omega_\epsilon = \{y | \epsilon y \in \Omega\}. \tag{7.1}$$

Then an equilibrium solution $(\hat{v}, u)$ has to solve the following rescaled Gray-Scott system:

$$\begin{cases} \Delta_y \hat{v} - \hat{v} + \hat{v}^2 u = 0, \quad y \in \Omega_\epsilon, \\ \Delta_x u + \beta^2(1-u) - \frac{\beta^2}{A^2}\hat{v}^2 u = 0, \quad x \in \Omega. \end{cases} \tag{7.2}$$

For a function $\hat{v} \in H^1(\Omega)$, let $T[\hat{v}]$ be the unique solution of the following problem

$$\Delta T[\hat{v}] + \beta^2(1 - T[\hat{v}]) - \frac{\beta^2}{A^2}\hat{v}^2 T[\hat{v}] = 0 \text{ in } \Omega, \ \frac{\partial T[\hat{v}]}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{7.3}$$

In other words, we have

$$1 - T[\hat{v}](x) = \int_\Omega G_\beta(x,\xi) \frac{\beta^2}{A^2} \hat{v}(\frac{\xi}{\epsilon})^2 T[\hat{v}](\xi) d\xi. \tag{7.4}$$

System (7.2) is equivalent to the following equation in operator form:

$$S_\epsilon(\hat{v}, u) = \begin{pmatrix} S_1(\hat{v}, u) \\ S_2(\hat{v}, u) \end{pmatrix} = 0, \tag{7.5}$$

where

$$S_1(\hat{v}, u) = \Delta_y \hat{v} - \hat{v} + \hat{v}^2 u, \quad H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega_\epsilon),$$

$$S_2(\hat{v}, u) = \Delta_x u + \beta^2(1-u) - \frac{\beta^2}{A^2}\hat{v}^2 u, \quad H_N^2(\Omega_\epsilon) \times H_N^2(\Omega) \to L^2(\Omega).$$

Here the index $N$ indicates that the functions satisfy the no flux boundary conditions

$$\frac{\partial \hat{v}}{\partial \nu} = 0, \ y \text{ on } \partial\Omega_\epsilon, \qquad \frac{\partial u}{\partial \nu} = 0, \ x \text{ on } \partial\Omega.$$

Let $\mathbf{P} \in \Lambda$ and $(\xi_1, ..., \xi_K) = (\xi_0, ..., \xi_0)$ where $\xi_0$ is given by (2.10) for $\epsilon = 0$. We now choose a good approximate function. Let $(\xi_{\epsilon,1}, ..., \xi_{\epsilon,K})$ be such that $|\xi_{\epsilon,j} - \xi_j| \le \delta_0$ for $\delta_0$ small. Set

$$\hat{v}_{\epsilon,j}(y) := \frac{1}{\xi_{\epsilon,j}} w(\frac{\epsilon y - P_j}{\epsilon})\chi_{\epsilon,j}(\epsilon y), \quad y \in \Omega_\epsilon, \tag{7.6}$$

where $\chi_{\epsilon,j}$ was defined in (4.4). Note that $\xi_{\epsilon,j}$ is still undetermined.

We choose our approximate solutions:

$$v_{\epsilon,\mathbf{P}}(y) := \sum_{j=1}^{K} \hat{v}_{\epsilon,j}(y), \quad u_{\epsilon,\mathbf{P}}(x) := T[v_{\epsilon,\mathbf{P}}](x) \tag{7.7}$$

for

$$x \in \Omega, \quad y \in \Omega_\epsilon = \{y \in R^2 | \epsilon y \in \Omega\}.$$

Note that $u_{\epsilon,\mathbf{P}}$ satisfies

$$0 = \Delta u_{\epsilon,\mathbf{P}} + \beta^2(1 - u_{\epsilon,\mathbf{P}}) - \frac{\beta^2}{A^2} v_{\epsilon,\mathbf{P}}^2 u_{\epsilon,\mathbf{P}}$$

$$= \Delta u_{\epsilon,\mathbf{P}} + \beta^2(1 - u_{\epsilon,\mathbf{P}}) - \frac{\beta^2}{A^2} \sum_{j=1}^{K} \hat{v}_{\epsilon,j}^2 u_{\epsilon,\mathbf{P}} + e.s.t.$$

Let $\hat{\xi}_{\epsilon,j} = u_{\epsilon,\mathbf{P}}(P_j)$. Then we have

$$1 - \hat{\xi}_{\epsilon,j} = \frac{\beta^2}{A^2} \int_\Omega G_\beta(P_j, \xi) \sum_{j=1}^{K} \hat{v}_{\epsilon,j}^2(\frac{\xi}{\epsilon}) u_{\epsilon,\mathbf{P}} d\xi + e.s.t.$$

By way of computations similar to those in Section 4, we obtain

$$1 - \hat{\xi}_{\epsilon,i} = \sum_{j=1}^{K} \frac{\alpha_\epsilon \hat{\xi}_{\epsilon,j}}{\xi_{\epsilon,j}^2} + \frac{\eta_\epsilon \alpha_\epsilon \hat{\xi}_{\epsilon,i}}{\xi_{\epsilon,i}^2} + O\left(\sum_{j=1}^{K} \frac{\beta^2 \alpha_\epsilon \hat{\xi}_{\epsilon,j}}{\xi_{\epsilon,j}^2}\right), \quad i = 1, ..., K. \tag{7.8}$$

Now we have

**Lemma 7.1.** *Let $(\xi_1, ..., \xi_K) = (\xi_0, ..., \xi_0)$. Then, for $\epsilon$ sufficiently small, there exists a unique solution $(\xi_{\epsilon,1}, ..., \xi_{\epsilon,K})$ of (7.8) such that*

$$\hat{\xi}_{\epsilon,j} = \xi_{\epsilon,j} \quad j = 1, ..., K, \tag{7.9}$$

*and $\xi_{\epsilon,j} = \xi_0 + O(k(\epsilon, \beta))$, where $k(\epsilon, \beta)$ was defined in Theorem 2.2.*

**Proof:** Let $\xi = (\xi_0, ..., \xi_0), \xi_\epsilon = (\xi_{\epsilon,1}, ..., \xi_{\epsilon,K})$ and $\hat{\xi}_\epsilon = (\hat{\xi}_{\epsilon,1}, ..., \hat{\xi}_{\epsilon,K})$. Note that $\hat{\xi}_\epsilon$ is a function of $\xi_\epsilon$. We write (7.8) as a functional equation

$$\mathbf{G}(\epsilon, \xi_\epsilon, \hat{\xi}_\epsilon) = 0, \quad \|\xi - \xi_\epsilon\| < \delta_0, \tag{7.10}$$

where

$$\mathbf{G}(\epsilon, \xi_\epsilon, \hat{\xi}_\epsilon) = r.h.s. \text{ of } (7.8) - l.h.s. \text{ of } (7.8)$$

and the norm is the vector norm. Note that $\mathbf{G}(0, \xi, \hat{\xi})|_{\hat{\xi}=\xi=(\xi_0,\cdots,\xi_0)} = 0$. Now we claim that $\nabla_{\hat{\xi}}\mathbf{G}(0, \xi, \hat{\xi})|_{\xi=\hat{\xi}=(\xi_0,\cdots,\xi_0)}$ is nonsingular. Once this is proved, then the implicit function theorem gives the result.

Now it follows that

$$-\nabla_{\hat{\xi}}\mathbf{G}(0, \xi, \hat{\xi})|_{\xi=\hat{\xi}=(\xi_0,\cdots,\xi_0)} = \begin{pmatrix} 1 + \frac{\alpha_0\eta_0}{\xi_0^2} & & \\ & \ddots & \\ & & 1 + \frac{\alpha_0\eta_0}{\xi_0^2} \end{pmatrix} + \begin{pmatrix} \frac{\alpha_0}{\xi_0^2} & \cdots & \frac{\alpha_0}{\xi_0^2} \\ \vdots & \vdots & \vdots \\ \frac{\alpha_0}{\xi_0^2} & \cdots & \frac{\alpha_0}{\xi_0^2} \end{pmatrix}.$$

It is easy to see that $\nabla_{\hat{\xi}}\mathbf{G}(0, \xi, \hat{\xi})$ is strictly negative definite and hence nonsingular.

□

The reason for choosing the functions in (7.7) as approximations to stationary states lies in the following calculations: We insert our ansatz (7.7) into (7.5) and calculate

$$S_2(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = 0, \tag{7.11}$$

$$\begin{aligned} S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) &= \Delta_y v_{\epsilon,\mathbf{P}} - v_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}^2 u_{\epsilon,\mathbf{P}} \\ &= \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}} \Big[\Delta_y w(y - \frac{P_j}{\epsilon}) - w(y - \frac{P_j}{\epsilon})\Big] \\ &\quad + \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}^2} w^2(y - \frac{P_j}{\epsilon}) u_{\epsilon,\mathbf{P}} + e.s.t. \\ &= \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}^2} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}} - \xi_{\epsilon,j}) + e.s.t. \\ &= \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}^2} w^2(y - \frac{P_j}{\epsilon}) \underbrace{(\hat{\xi}_{\epsilon,j} - \xi_{\epsilon,j})}_{=0} \\ &\quad + \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}^2} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}}(x) - \hat{\xi}_{\epsilon,j}) + e.s.t. \\ &= \sum_{j=1}^K \frac{1}{\xi_{\epsilon,j}^2} w^2(y - \frac{P_j}{\epsilon})(u_{\epsilon,\mathbf{P}}(x) - u_{\epsilon,\mathbf{P}}(P_j)) + e.s.t. \end{aligned} \tag{7.12}$$

by Lemma 7.1.

On the other hand, we see from (7.7) that for $i = 1, ..., K$ and $x = P_i + \epsilon y$, $|\epsilon y| < \delta$:

$$u_{\epsilon,\mathbf{P}}(x) - u_{\epsilon,\mathbf{P}}(P_i) = u_{\epsilon,\mathbf{P}}(P_i + \epsilon y) - u_{\epsilon,\mathbf{P}}(P_i)$$

$$= \frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_i,\xi) - G_\beta(P_i+\epsilon y,\xi))\sum_{j=1}^K \hat{v}^2_{\epsilon,j}(\frac{\xi}{\epsilon})u_{\epsilon,\mathbf{P}}d\xi + e.s.t.$$

$$= \frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_i,\xi) - G_\beta(P_i+\epsilon y,\xi))\hat{v}^2_{\epsilon,i}(\frac{\xi}{\epsilon})u_{\epsilon,\mathbf{P}}d\xi$$

$$+\frac{\beta^2}{A^2}\int_\Omega (G_\beta(P_i,\xi) - G_\beta(P_i+\epsilon y,\xi))\sum_{j\neq i} \hat{v}^2_{\epsilon,j}(\frac{\xi}{\epsilon})u_{\epsilon,\mathbf{P}}d\xi + e.s.t.$$

$$= \frac{|\Omega|\beta^2\alpha_\epsilon}{\xi_{\epsilon,i}}(\epsilon\frac{1}{2}\nabla_{P_i}F_0(\mathbf{P})\cdot y + O(\epsilon\beta^2|y| + \epsilon^2|y|^2))$$

$$+\frac{|\Omega|\beta^2\alpha_\epsilon}{\xi_{\epsilon,i}\int_{R^2}w^2}\int_{R^2}\log\frac{|y-z|}{|z|}w^2(z)dz. \tag{7.13}$$

(Recall the definition of $F_0$ in (2.5).)

Substituting (7.13) into (7.12) and noting that $S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = e.s.t.$ for $|x - P_j| \geq \delta$, $j = 1, 2, ..., K$, we have the following estimate

$$\|S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)} = O(\beta^2\alpha_\epsilon) = O(h(\epsilon,\beta)).$$

The last equality follows by considering the three cases separately.

In Case 1, we have $O(\beta^2\alpha_\epsilon) = O(\beta^2) = O(h(\epsilon,\beta))$ since $\alpha_\epsilon \to \alpha_0$ and $\beta^2 << \left(\log\frac{1}{\epsilon}\right)^{-1}$ due to $\eta_\epsilon \to 0$.

In Case 2, we have $O(\beta^2\alpha_\epsilon) = O(\beta^2\eta_\epsilon^{-1}) = O((\log\frac{1}{\epsilon})^{-1}) = O(h(\epsilon,\beta))$ since $\lim_{\epsilon\to 0}\eta_\epsilon\alpha_\epsilon$ exists, $O(\eta_\epsilon) = O(\beta^2\log\frac{1}{\epsilon})$, and $\beta^2 >> \left(\log\frac{1}{\epsilon}\right)^{-1}$ due to $\eta_\epsilon \to \infty$.

In Case 3, we have $O(\beta^2\alpha_\epsilon) = O(\beta^2) = O(h(\epsilon,\beta))$ since $\alpha_\epsilon \to \alpha_0$ and $O(\beta^2) = O((\log\frac{1}{\epsilon})^{-1})$ due to $\eta_\epsilon \to \eta_0 > 0$.

Summarizing the results, we have the following key lemma:

**Lemma 7.2.** *For $x = P_i + \epsilon y$, $|\epsilon y| < \delta$, we have*

$$S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = S_{1,1} + S_{1,2} \tag{7.14}$$

*where*

$$S_{1,1}(y) = |\Omega|\beta^2\alpha_\epsilon\frac{1}{\xi^3_{\epsilon,i}}w^2(y)(\epsilon\nabla_{P_i}F_0(\mathbf{P})\cdot y + O(\epsilon\beta^2|y| + \epsilon^2|y|^2)) \tag{7.15}$$

*and*

$$S_{1,2}(y) = \frac{|\Omega|\beta^2\alpha_\epsilon}{\xi_{\epsilon,i}^3 \int_{R^2} w^2} w^2(y) \int_{R^2} \log\frac{|y-z|}{|z|} w^2(z)dz. \tag{7.16}$$

*Furthermore,* $S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}}) = e.s.t.$ *for* $|x-P_j| \geq \delta,\ j = 1,2,...,K$ *and we have the estimate*

$$\|S_1(v_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon)} = O(h(\epsilon,\beta)). \tag{7.17}$$

## 8. Existence II: Reduction to finite dimensions

In this section, we use the Liapunov-Schmidt reduction method to reduce the problem of finding an equilibrium to a finite-dimensional problem.

We first study the linearized operator defined by

$$\tilde{L}_{\epsilon,\mathbf{P}} := S'_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} \end{pmatrix},$$

$$\tilde{L}_{\epsilon,\mathbf{P}} : H^2_N(\Omega_\epsilon) \times W^{2,2}_N(\Omega) \to L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $\epsilon > 0$ is small and $\mathbf{P} \in \bar{\Lambda}$.

To obtain the asymptotic form of $\tilde{L}_{\epsilon,\mathbf{P}}$ we cut off $\phi_\epsilon$ as follows: Introduce

$$\phi_{\epsilon,j}(y - \frac{P_i}{\epsilon}) := \phi_\epsilon(y)\chi_{\epsilon,j}(x),$$

where $\chi_{\epsilon,j}(x)$ was introduced in (4.4) and $y \in \Omega_\epsilon$. By taking a subsequence of $\epsilon$, we may also assume that $\phi_{\epsilon,j} \to \phi_j$ as $\epsilon \to 0$ in $H^2_{loc}(R^2)$ for $j = 1, \ldots, K$. Similar to Section 5, the asymptotic limit of $\tilde{L}_{\epsilon,\mathbf{P}}$ is the following system of linear operators

$$\mathcal{L}\Phi := \Delta\Phi - \Phi + 2w\Phi - 2\mathcal{B}_0 \frac{\int_{R^2} w\Phi}{\int_{R^2} w^2} w^2, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_K \end{pmatrix} \in (H^2(R^2))^K, \tag{8.1}$$

where

$$\mathcal{B}_0 = ((\xi_0^2 + \alpha_0\eta_0)\mathcal{I} + \alpha_0\mathcal{E})^{-1}(\eta_0\alpha_0\mathcal{I} + \alpha_0\mathcal{E}) \tag{8.2}$$

where $\mathcal{E}$ is in (5.8). The eigenvalues of $\mathcal{B}_0$ are given by

$$b_1 = \frac{\eta_0\alpha_0 + K\alpha_0}{\xi_0^2 + \eta_0\alpha_0 + K\alpha_0}, \quad b_2 = ... = b_K = \frac{\eta_0\alpha_0}{\xi_0^2 + \eta_0\alpha_0}.$$

It is easy to see that $2b_1 \neq 1$ and $2b_2 = 1$ if and only if $\alpha_0 = \frac{\eta_0}{(2\eta_0+K)^2}$.

Now we have the following key lemma which reduces the infinite dimensional problem to a finite dimensional one.

**Lemma 8.1.** *Assume that assumption (2.8) holds. Then*

$$Ker(\mathcal{L}) = Ker(\mathcal{L}^*) = X_0 \oplus X_0 \oplus \cdots \oplus X_0, \tag{8.3}$$

*where*

$$X_0 = span\left\{ \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2} \right\}$$

*and $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ under the $(L^2(R^2))^K$ inner product.*

*As a consequence, the operator*

$$\mathcal{L} : (H^2(R^2))^K \to (L^2(R^2))^K$$

*is an invertible operator if it is restricted as follows*

$$\mathcal{L} : (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(R^2))^K \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(R^2))^K.$$

*Moreover, $\mathcal{L}^{-1}$ is bounded.*

**Proof:** By (2.8) and the argument above, we see that $2b_i \neq 1$. If $\mathcal{L}\Phi = 0$, then by diagonalization, it can be reduced to (5.16) with $\tau = 0$, or to (5.17), respectively. By Lemma 6.3, $\Phi \in X_0 \oplus X_0 \oplus \cdots \oplus X_0$.

Next, let $\Psi \in Ker(\mathcal{L}^*)$. Then we have

$$\Delta\Psi - \Psi + 2w\Psi - 2\mathcal{B}_0^t \frac{\int_{R^2} w^2\Psi}{\int_{R^2} w^2} w = 0, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_K \end{pmatrix} \in (H^2(R^2))^K. \tag{8.4}$$

Multiplying the above equation by $w$ (componentwise) and integrating, we obtain

$$(\mathcal{I} - 2\mathcal{B}_0^t) \int_{R^2} w^2\Psi = 0. \tag{8.5}$$

Since $\mathcal{B}_0^t = \mathcal{B}_0$ we know that $\mathcal{I} - 2\mathcal{B}_0^t$ is nonsingular. This implies that $\int_{R^2} w^2\Psi = 0$. Thus all the nonlocal terms vanish and $\Psi \in X_0 \oplus X_0 \oplus \cdots \oplus X_0$.

The rest follows from the Fredholm Alternatives Theorem. $\square$

From the above lemma, we now define the approximate kernel and co-kernel as follows:

$$K_{\epsilon,\mathbf{P}} := \text{span}\ \{\frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\ l = 1, 2\} \subset H^2_N(\Omega_\epsilon)$$

and

$$C_{\epsilon,\mathbf{P}} := \text{span}\ \{\frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,l}} | j = 1, \ldots, K,\ l = 1, 2\} \subset L^2(\Omega_\epsilon),$$

$$\mathcal{K}_{\epsilon,\mathbf{P}} := K_{\epsilon,\mathbf{P}} \oplus \{0\} \subset H^2_N(\Omega_\epsilon) \times W^{2,2}_N(\Omega),$$

$$\mathcal{C}_{\epsilon,\mathbf{P}} := C_{\epsilon,\mathbf{P}} \oplus \{0\} \subset L^2(\Omega_\epsilon) \times L^2(\Omega).$$

We then define

$$\mathcal{K}^\perp_{\epsilon,\mathbf{P}} := K^\perp_{\epsilon,\mathbf{P}} \oplus W^{2,t}_N(\Omega) \subset H^2_N(\Omega_\epsilon) \times W^{2,2}_N(\Omega),$$

$$\mathcal{C}^\perp_{\epsilon,\mathbf{P}} := C^\perp_{\epsilon,\mathbf{P}} \oplus L^2(\Omega) \subset L^2(\Omega_\epsilon) \times L^2(\Omega),$$

where $C^\perp_{\epsilon,\mathbf{P}}$ and $K^\perp_{\epsilon,\mathbf{P}}$ denote the orthogonal complement for the scalar product of $L^2(\Omega_\epsilon)$ in $H^2_N(\Omega_\epsilon)$ and $L^2(\Omega_\epsilon)$, respectively.

Let $\pi_{\epsilon,\mathbf{P}}$ denote the projection in $L^2(\Omega_\epsilon) \times L^2(\Omega)$ onto $\mathcal{C}^\perp_{\epsilon,\mathbf{P}}$. (Here the second component of the projection is the identity map.)

We are going to show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} = 0$$

has the unique solution $\Sigma_{\epsilon,\mathbf{P}} = \begin{pmatrix} \Phi_{\epsilon,\mathbf{P}}(y) \\ \Psi_{\epsilon,\mathbf{P}}(x) \end{pmatrix} \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ if $\epsilon$ is small enough. That is equivalent to the following equation

$$S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, T[v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}]) \in C_{\epsilon,\mathbf{P}}, \Phi_{\epsilon,\mathbf{P}} \in K^\perp_{\epsilon,\mathbf{P}}. \tag{8.6}$$

The following two propositions show the invertibility of the corresponding linearized operator.

**Proposition 8.2.** *Let* $\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}}$. *There exist positive constants* $\overline{\epsilon}, \overline{\beta}, C$ *such that for all* $\epsilon \in (0, \overline{\epsilon}), \beta \in (0, \overline{\beta})$,

$$\|\mathcal{L}_{\epsilon,\mathbf{P}}\Sigma\|_{L^2(\Omega_\epsilon)\times L^2(\Omega)} \geq C\|\Sigma\|_{H^2(\Omega_\epsilon)\times H^2(\Omega)} \tag{8.7}$$

*for arbitrary* $\mathbf{P} \in \bar{\Lambda}$, $\Sigma \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$.

**Proposition 8.3.** *There exist positive constants* $\bar{\bar{\epsilon}}, \overline{\overline{\beta}}$ *such that for all* $\epsilon \in (0, \bar{\bar{\epsilon}}), \beta \in (0, \overline{\overline{\beta}})$ *the map*

$$\mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ \tilde{L}_{\epsilon,\mathbf{P}} : \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}} \to \mathcal{C}^{\perp}_{\epsilon,\mathbf{P}}$$

*is surjective for arbitrary* $\mathbf{P} \in \bar{\Lambda}$.

The proofs of Propositions 8.2 and 8.3 follow from Lemma 8.1 and are similar to those in [30]. We omit the details.

By using the Contraction Mapping Principle, we get from Lemma 7.2

**Lemma 8.4.** *There exist* $\bar{\epsilon} > 0, \overline{\beta}, C > 0$ *such that for every triple* $(\epsilon, \beta, \mathbf{P})$ *with* $0 < \epsilon < \bar{\epsilon}, 0 < \beta < \overline{\beta}$, $\mathbf{P} \in \bar{\Lambda}$ *there exists a unique* $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$ *satisfying* $S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \in \mathcal{C}_{\epsilon,\mathbf{P}}$ *and*

$$\|(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}})\|_{H^2(\Omega_\epsilon) \times H^2(\Omega)} \le C h(\epsilon, \beta). \tag{8.8}$$

More refined estimates for $\Phi_{\epsilon,\mathbf{P}}$ are needed. We recall that $S_1$ can be decomposed into two parts, $S_{1,1}$ and $S_{1,2}$. $S_{1,1}$ is an odd function and $S_{1,2}$ is an even function . Similarly, we can decompose $\Phi_{\epsilon,\mathbf{P}}$:

**Lemma 8.5.** *Let* $\Phi_{\epsilon,\mathbf{P}}$ *be defined in Lemma 8.4. Then for* $x = P_i + \epsilon y$, *we have*

$$\Phi_{\epsilon,\mathbf{P}} = \Phi^1_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}, \tag{8.9}$$

*where* $\Phi^2_{\epsilon,\mathbf{P}}$ *is an even function in* $y$ *and*

$$\Phi^1_{\epsilon,\mathbf{P}} = O(\epsilon h(\epsilon, \beta)). \tag{8.10}$$

**Proof:** Let $S[v] := S_1(v, T[v])$. We first solve

$$S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}] - S[v_{\epsilon,\mathbf{P}}] + \sum_{j=1}^{K} S_{1,2}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}}, \Phi^2_{\epsilon,\mathbf{P}} \in K^{\perp}_{\epsilon,\mathbf{P}}. \tag{8.11}$$

Then we solve

$$S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}} + \Phi^1_{\epsilon,\mathbf{P}}] - S[v_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}] + \sum_{j=1}^{K} S_{1,1}(y - \frac{P_j}{\epsilon}) \in C_{\epsilon,\mathbf{P}}, \Phi^1_{\epsilon,\mathbf{P}} \in K^{\perp}_{\epsilon,\mathbf{P}}. \tag{8.12}$$

Using the same proof as in Lemma 8.4, both equations (8.11) and (8.12) have unique solutions for $\epsilon << 1$. By uniqueness, $\Phi_{\epsilon,\mathbf{P}} = \Phi^1_{\epsilon,\mathbf{P}} + \Phi^2_{\epsilon,\mathbf{P}}$. Since $S_{11} = S^0_{11} + S^\perp_{11}$, where $\|S^0_{11}\|_{H^2(\Omega_\epsilon)} = O(\epsilon h(\epsilon,\beta))$ and $S^\perp_{11} \in C^\perp_{\epsilon,\mathbf{P}}$, it is easy to see that $\Phi^1_{\epsilon,\mathbf{P}}$ and $\Phi^2_{\epsilon,\mathbf{P}}$ have the required properties.

□

## 9. Existence III: The reduced problem

In this section, we solve the reduced problem and prove our main theorem on existence.

By Lemma 8.4 there exists a unique solution $(\Phi_{\epsilon,\mathbf{P}}, \Psi_{\epsilon,\mathbf{P}}) \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ such that

$$S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \in \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Our idea is to find $\mathbf{P} \in \overline{\Lambda}$ such that

$$S_\epsilon \begin{pmatrix} v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \\ u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} \end{pmatrix} \perp \mathcal{C}_{\epsilon,\mathbf{P}}.$$

Let

$$W_{\epsilon,j,i}(\mathbf{P}) := \frac{2\xi^4_{\epsilon,j}}{\alpha_\epsilon|\Omega|\beta^2\epsilon^2} \int_{\Omega_\epsilon} (S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}}) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}),$$
$$W_\epsilon(\mathbf{P}) := (W_{\epsilon,1,1}(\mathbf{P}), ..., W_{\epsilon,K,2}(\mathbf{P})),$$

where $\xi_{\epsilon,j}$ is given by Lemma 7.1.

Note that $P_{j,i}$ denotes the $i$-th component of the $j$-th point. Then $W_\epsilon(\mathbf{P})$ is a map which is continuous in $\mathbf{P}$ and our problem is reduced to finding a zero of the vector field $W_\epsilon(\mathbf{P})$.

To simplify our notation, we let $\tilde{u}_{\epsilon,\mathbf{P}} = u_{\epsilon,\mathbf{P}} + \Psi_{\epsilon,\mathbf{P}} = T[v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}]$ and

$$\Omega_{\epsilon,P_j} = \{y | \epsilon y + P_j \in \Omega\}. \tag{9.1}$$

We calculate

$$\int_{\Omega_\epsilon} S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, \tilde{u}_{\epsilon,\mathbf{P}}) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$
$$= \int_{\Omega_\epsilon} S_1(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}, \tilde{u}_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$
$$+ \int_{\Omega_\epsilon} (v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2 (\tilde{u}_{\epsilon,\mathbf{P}}(x) - \tilde{u}_{\epsilon,\mathbf{P}}(P_j)) \frac{\partial v_{\epsilon,\mathbf{P}}}{\partial P_{j,i}}$$
$$= I_1 + I_2$$

where $I_1$ and $I_2$ are defined by the last equality.

For $I_1$, we have using integration by parts,

$$I_1 = \epsilon \int_{\Omega_{\epsilon,P_j}} [\Delta(v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - (v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + (v_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j))](-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial y_i})dy + O(\epsilon h(\epsilon,\beta))$$

$$= \epsilon \int_{\Omega_{\epsilon,P_j}} [(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j)) - 2w(v_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,P})](-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial y_i})\, dy + O(\epsilon h(\epsilon,\beta))$$

$$= \epsilon \int_{\Omega_{\epsilon,P_j}} 2\Phi^1_{\epsilon,\mathbf{P}} w^2(\tilde{u}_{\epsilon,\mathbf{P}}(P_j))(-\frac{1}{\xi_{\epsilon,j}}\frac{\partial w}{\partial y_i})\, dy + o(\epsilon h(\epsilon,\beta))$$

$$= O(\epsilon h(\epsilon,\beta))$$

by Lemma 8.5.

For $I_2$, we have similar to the computation in (7.13):

$$\tilde{u}_{\epsilon,\mathbf{P}}(P_j + \epsilon y) - \tilde{u}_{\epsilon,\mathbf{P}}(P_j) = \frac{|\Omega|\beta^2\alpha_\epsilon}{2\xi_{\epsilon,j}}(\epsilon\nabla_{P_j}F_0(\mathbf{P})\cdot y + O(\epsilon\beta^2|y| + \epsilon^2|y|^2)) + \frac{|\Omega|\beta^2\alpha_\epsilon}{\xi_{\epsilon,j}\int_{R^2} w^2}\int_{R^2}\log\frac{|y-z|}{|z|}w^2(z)dz$$

Hence

$$I_2 = \frac{|\Omega|\beta^2\alpha_\epsilon\epsilon^2}{2\xi^2_{\epsilon,j}}\int_{\Omega_{\epsilon,P_j}}(\frac{1}{\xi_{\epsilon,j}}w + \Phi_{\epsilon,\mathbf{P}})^2(\nabla_{P_j}F_0(\mathbf{P})\cdot y + O(\epsilon\beta^2|y| + \epsilon^2|y|^2)) \times(-\frac{\partial w}{\partial y_i} + O(\epsilon+\beta^2)|y|)$$

$$= -\frac{|\Omega|\beta^2\alpha_\epsilon\epsilon^2}{2\xi^4_{\epsilon,j}}[\int_{R^2} w^2\frac{\partial w}{\partial y_i}y_i\nabla_{P_{j,i}}F_0(\mathbf{P}) + O(\epsilon+\beta^2))]. \tag{9.2}$$

Combining $I_1$ and $I_2$, we obtain

$$W_\epsilon(\mathbf{P}) = c_0\nabla_{\mathbf{P}}F_0(\mathbf{P})(1 + O(\epsilon+\beta^2)),$$

where

$$c_0 = -\int_{R^2} w^2\frac{\partial w}{\partial y_i}y_i = \frac{1}{3}\int_{R^2} w^3.$$

Suppose for $\mathbf{P}_0$ we have $\nabla_{\mathbf{P}}F_0(\mathbf{P}_0) = 0$, $\det(\nabla_{\mathbf{P}}\nabla_{\mathbf{P}}(F_0(\mathbf{P}_0)) \neq 0$, then, since $W_\epsilon$ is continuous and for $\epsilon$, $\beta$ small enough maps balls into (possibly larger) balls, standard Brouwer's fixed point theorem shows that for $\epsilon << 1$ there exists a $\mathbf{P}^\epsilon$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$ and $\mathbf{P}^\epsilon \to \mathbf{P}_0$.

Thus we have proved the following proposition.

**Proposition 9.1.** *For $\epsilon$ sufficiently small there exist points $\mathbf{P}^\epsilon$ with $\mathbf{P}^\epsilon \to \mathbf{P_0}$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$.*

Finally, we prove Theorem 2.2.

**Proof of Theorem 2.2:** By Proposition 9.1, there exists $\mathbf{P}^\epsilon \to \mathbf{P}_0$ such that $W_\epsilon(\mathbf{P}^\epsilon) = 0$. In other words, $S_1(v_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}, u_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}) = 0$. Let $v_\epsilon = \frac{1}{A}(v_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}), u_\epsilon = u_{\epsilon,\mathbf{P}^\epsilon} + \Psi_{\epsilon,\mathbf{P}^\epsilon}$. It is easy to see that $u_\epsilon = \xi_{\epsilon,j}(1 + O(h(\epsilon,\beta)))$ and hence $v_\epsilon \geq 0$. By the Maximum Principle, $v_\epsilon > 0$. Therefore $(v_\epsilon, u_\epsilon)$ satisfies Theorem 2.2.

□

## 10. Stability Analysis

We now study the eigenvalue problem (5.2) for the solutions $(v_\epsilon, u_\epsilon)$ which we have constructed in Section 9. Let $\hat{v}_\epsilon = \frac{1}{A} v_\epsilon$. Then (5.2) becomes

$$\begin{cases} \Delta_y \phi_\epsilon - \phi_\epsilon + 2\hat{v}_\epsilon u_\epsilon \phi_\epsilon + \hat{v}_\epsilon^2 \psi_\epsilon = \lambda_\epsilon \phi_\epsilon, \\ \frac{1}{\beta^2}\Delta \psi_\epsilon - \psi_\epsilon - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \phi_\epsilon - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_\epsilon = \tau \lambda_\epsilon \psi_\epsilon, \end{cases}$$

where

$$\phi_\epsilon \in H^2(\Omega_\epsilon),\ \psi_\epsilon \in H^2_N(\Omega),$$

and finish the proof of Theorem 2.3.

We divide it into two cases: $\lambda_\epsilon \to \lambda_0 \neq 0$ and $\lambda_\epsilon \to 0$. In the first case, by Theorem 5.1, the problem is reduced to the study of two (NLEP)s:

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2(\eta_0\alpha_0(1+\tau\lambda_0) + K\alpha_0)}{(\xi_0^2 + \eta_0\alpha_0)(1+\tau\lambda_0) + K\alpha_0} \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0 \Phi, \tag{10.1}$$

and

$$\Delta\Phi - \Phi + 2w\Phi - \frac{2\eta_0\alpha_0}{\eta_0\alpha_0 + \xi_0^2} \frac{\int_{R^2} w\phi}{\int_{R^2} w^2} w^2 = \lambda_0 \Phi. \tag{10.2}$$

If $\xi_0 = \xi_0^+$, then it is easy to see that when $\tau = 0$,

$$b_1 = \frac{(K+\eta_0)\alpha_0}{\xi_0^2 + (K+\eta_0)\alpha_0} < \frac{1}{2}$$

and hence by Lemma 6.1, the large solution is unstable for all $\tau > 0$. Therefore only the small solutions $\xi_0 = \xi_0^-$ can be stable.

For the small solutions, in Case 1 ($\eta_0 = 0$) we have therefore $b_2 = ... = b_K = 0$. So if $K > 1$, the small solution is unstable for all $\tau > 0$. If $\eta_0 = 0$ and $K = 1$, we have

$$2b_1 = \frac{2\alpha_0}{\xi_0^2(1+\tau\lambda_0)+\alpha_0} = \frac{\mu_0}{1+\tau_0\lambda_0},$$

where $\mu_0 = \frac{2\alpha_0}{\xi_0^2+\alpha_0} > 1, \tau_0 = \frac{\tau\xi_0^2}{\xi_0^2+\alpha_0} > 0$. By Lemma 6.4(1), there exists $\tau_1 > 0$ such that for $\tau < \tau_1$, we have stability of large eigenvalues and for $\tau > \tau_1$, we have instability of large eigenvalues.

In Case 2, we have $\eta_0 = +\infty$ and $b_1 = ... = b_K = \frac{\eta_0\alpha_0}{\xi_0^2+\eta_0\alpha_0} > \frac{1}{2}$. So by Lemma 6.3(1), we have the stability of large eigenvalues for all $\tau > 0$.

Finally, we consider Case 3. By Lemma 6.3, problem (10.2) admits only stable eigenvalues if and only if

$$\eta_0\alpha_0 > \xi_0^2 \tag{10.3}$$

which is equivalent to (3.2).

So if $\alpha_0 > \frac{\eta_0}{(2\eta_0+K)^2}$ and $K > 1$, problem (10.2) admits a positive eigenvalue $\lambda_0$. So we have instability. If $\alpha_0 > \frac{\eta_0}{(2\eta_0+K)^2}$ and $K = 1$, we need to consider problem (10.1) only. By Lemma 6.5, problem (10.1) has only stable eigenvalues if $\tau < \tau_2$ or $\tau > \tau_3$, where $\tau_2$ and $\tau_3$ are given in Lemma 6.5.

Suppose $\alpha_0 < \frac{\eta_0}{(2\eta_0+K)^2}$. Then since (10.2) has only stable eigenvalues we need to consider problem (10.1) only. By Lemma 6.5, problem (10.1) is stable for $\tau$ small enough and unstable for $\tau$ large enough.

This finishes the study of large eigenvalues.

It remains to study small eigenvalues. Namely, we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. This part of the analysis is similar as in [29]. Therefore to save space, we shall only give a sketch.

Again let $(v_\epsilon, u_\epsilon)$ be the equilibrium state of (1.5). Since $\lambda_\epsilon \to 0$ and $\tau$ is finite, $\tau\lambda_\epsilon << 1$. So in (5.2) we have $\tau\lambda_\epsilon\psi_\epsilon << \psi_\epsilon$. Therefore without loss of generality we may take $\tau = 0$.

Let us define

$$\tilde{v}_{\epsilon,j}(y - \frac{P_j^\epsilon}{\epsilon}) = \chi_{\epsilon,j}(x)\hat{v}_\epsilon(y), \quad j = 1, ..., K, \quad y \in \Omega_\epsilon,$$

where $\chi_{\epsilon,j}$ was defined in (4.4).

Then it is easy to see that

$$\hat{v}_\epsilon(y) = \sum_{j=1}^{K} \tilde{v}_{\epsilon,j}(y - \frac{P_j^\epsilon}{\epsilon}) + e.s.t. \quad \text{in } H^2(\Omega_\epsilon).$$

As in [29], we decompose $\phi_\epsilon$ as follows:

$$\phi_\epsilon = \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} + \phi_\epsilon^\perp \tag{10.4}$$

with real numbers $a_{j,k}^\epsilon$, where

$$\phi_\epsilon^\perp \perp K_{\epsilon,\mathbf{P}^\epsilon}^{new} = \text{span}\,\{\frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} | j = 1, \ldots, K,\ k = 1, 2\} \subset H_N^2(\Omega_\epsilon).$$

Accordingly, we put

$$\psi_\epsilon(x) = \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \psi_{\epsilon,j,k} + \psi_\epsilon^\perp,$$

where $\psi_{\epsilon,j,k}$ is the unique solution of the problem

$$\begin{cases} \frac{1}{\beta^2}\Delta\psi_{\epsilon,j,k} - \psi_{\epsilon,j,k} - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_{\epsilon,j,k} = 0 & \text{in } \Omega, \\ \frac{\partial \psi_{\epsilon,j,k}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\psi_\epsilon^\perp$ satisfies

$$\begin{cases} \frac{1}{\beta^2}\Delta\psi_\epsilon^\perp - \psi_\epsilon^\perp - \frac{2}{A^2}\hat{v}_\epsilon u_\epsilon \phi_\epsilon^\perp - \frac{1}{A^2}\hat{v}_\epsilon^2 \psi_\epsilon^\perp = 0 & \text{in } \Omega, \\ \frac{\partial \psi_\epsilon^\perp}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose that $\|\phi_{\epsilon,j}\|_{H^2(\Omega_\epsilon)} = 1$. Then $|a_{j,k}^\epsilon| \le C$.

The idea then is that first we show that $\phi_\epsilon^\perp$ is small and then we obtain the algebraic equations for $a_{j,k}^\epsilon$.

We divide our proof into two steps.

**Step 1:** Estimates for $\phi_\epsilon^\perp$.

Substituting the decompositions of $\phi_\epsilon$ and $\psi_\epsilon$ into (5.2) we have

$$\sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon (\tilde{v}_{\epsilon,j})^2 \left[\psi_{\epsilon,j,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k}\right]$$

$$+\Delta_y \phi_\epsilon^\perp - \phi_\epsilon^\perp + 2\hat{v}_\epsilon u_\epsilon \phi_\epsilon^\perp + (\hat{v}_\epsilon)^2 \psi_\epsilon^\perp - \lambda_\epsilon \phi_\epsilon^\perp$$

$$= \lambda_\epsilon \sum_{j=1}^{K} \sum_{k=1}^{2} a_{j,k}^\epsilon \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} \quad \text{in } H^2(\Omega_\epsilon). \tag{10.5}$$

Set

$$I_3=\sum_{j=1}^{K}\sum_{k=1}^{2}a^{\epsilon}_{j,k}(\tilde v_{\epsilon,j})^2\left[\psi_{\epsilon,j,k}-\epsilon\frac{\partial u_\epsilon}{\partial x_k}\right]$$

and

$$I_4=\Delta_y\phi_\epsilon^\perp-\phi_\epsilon^\perp+2\hat v_\epsilon u_\epsilon\phi_\epsilon^\perp+(\hat v_\epsilon)^2\psi_\epsilon^\perp-\lambda_\epsilon\phi_\epsilon^\perp$$

Since $\phi_\epsilon^\perp\perp K^{new}_{\epsilon,\mathbf{P}^\epsilon}$, then similar to the proof of Proposition 8.3 it follows that

$$\|\phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)}\le C\|I_1\|_{L^2(\Omega_\epsilon)}. \tag{10.6}$$

Let us now compute $I_3$. The key is to estimate $\psi_{\epsilon,l,k}-\epsilon\frac{\partial u_\epsilon}{\partial x_k}$ near $x\in B_{r_0}(P_l^\epsilon)$.

From the equation for $\psi_{\epsilon,j,k}$, we obtain that

$$\psi_{\epsilon,j,k}(x)=-\frac{\beta^2}{A^2}\int_\Omega G_\beta(x,\xi)[2\hat v_\epsilon u_\epsilon\frac{\partial\tilde v_{\epsilon,j}}{\partial y_k}+\hat v_\epsilon^2\psi_{\epsilon,j,k}] \tag{10.7}$$

Similar to Section 4, we have

$$\psi_{\epsilon,j,k}(P_l^\epsilon)=O(h(\epsilon,\beta)\alpha_\epsilon\epsilon)-\sum_{s=1}^{K}\frac{\alpha_\epsilon}{\xi^2_{\epsilon,s}}\psi_{\epsilon,j,k}(P_s^\epsilon)-\frac{\eta_\epsilon\alpha_\epsilon}{\xi^2_{\epsilon,l}}\psi_{\epsilon,j,k}(P_l^\epsilon),\quad l=1,...,K$$

which implies that

$$\psi_{\epsilon,j,k}(P_l^\epsilon)=O(h(\epsilon,\beta)\alpha_\epsilon\epsilon),\quad l=1,...,K. \tag{10.8}$$

For $x=P_l^\epsilon+\epsilon y\in B_{r_0}(P_l^\epsilon)$ we calculate

$$\psi_{\epsilon,j,k}(P_l^\epsilon+\epsilon y)-\psi_{\epsilon,j,k}(P_l^\epsilon)$$
$$=\frac{\beta^2}{A^2}\int_\Omega(G_\beta(P_l^\epsilon,\xi)-G_\beta(P_l^\epsilon+\epsilon y,\xi))[2\hat v_\epsilon u_\epsilon\frac{\partial\tilde v_{\epsilon,j}}{\partial y_k}+\hat v_\epsilon^2\psi_{\epsilon,j,k}]d\xi$$
$$=\frac{\beta^2}{A^2}\int_\Omega(G_\beta(P_l^\epsilon,\xi)-G_\beta(P_l^\epsilon+\epsilon y,\xi))[2\hat v_{\epsilon,j}u_\epsilon\frac{\partial\tilde v_{\epsilon,j}}{\partial y_k}+{\hat v_\epsilon}^2\psi_{\epsilon,j,k}]+e.s.t.$$

If $l\ne j$, then we have

$$\psi_{\epsilon,j,k}(P_l^\epsilon+\epsilon y)-\psi_{\epsilon,j,k}(P_l^\epsilon)$$
$$=-\frac{\beta^2}{A^2}\sum_m\nabla_{P_l^\epsilon}\nabla_{P_j^\epsilon}G_\beta(P^\epsilon_{l,m},P^\epsilon_{j,k})\epsilon^2y_m\frac{\epsilon^2}{\xi_{\epsilon,j}}\int_{R^2}2z_kw(z)\frac{\partial w(z)}{\partial z_k}dz$$
$$+O(\beta^4A^{-2}\epsilon^4|y|+\beta^2A^{-2}\epsilon^5|y|^2)+O(\beta^2\alpha_\epsilon\epsilon|y|\sum_{l=1}^{K}|\psi_{\epsilon,j,k}(P_l^\epsilon)|)$$

$$= \frac{\beta^2|\Omega|\alpha_\epsilon}{\xi_{\epsilon,j}}\epsilon^2 \sum_m \nabla_{P^\epsilon_{l,m}} \nabla_{P^\epsilon_{j,k}} G_{\beta_0}(P^\epsilon_l, P^\epsilon_j) y_m + O(\beta^4 A^{-2}\epsilon^4|y| + \beta^2 A^{-2}\epsilon^5 |y|^2), \tag{10.9}$$

by the definition of $\alpha_\epsilon$ and (10.8).

For $l = j$, similar arguments show that

$$\psi_{\epsilon,j,k}(P^\epsilon_j + \epsilon y) - \psi_{\epsilon,j,k}(P^\epsilon_j) = -\frac{\beta^2|\Omega|\alpha_\epsilon}{\xi_{\epsilon,j}}\epsilon^2 \sum_m \nabla_{P^\epsilon_{j,m}} \nabla_{P^\epsilon_{j,k}} H_{\beta_0}(P^\epsilon_j, P^\epsilon_j) y_m$$

$$+ \frac{\beta^2}{A^2 \xi_{\epsilon,j}}\epsilon^2 \int_{R^2} 2\log\frac{|y-z|}{|z|} w \frac{\partial w}{\partial z_k} + O(\beta^4 A^{-2}\epsilon^4|y| + \beta^2 A^{-2}\epsilon^5|y|^2). \tag{10.10}$$

Next we compute $\epsilon\frac{\partial u_\epsilon}{\partial x_k}(x)$ for $x = P^\epsilon_l + \epsilon y \in B_{r_0}(P^\epsilon_l)$:

$$\epsilon\frac{\partial u_\epsilon}{\partial x_k}(x) = -\frac{\beta^2}{A^2}\int_\Omega \frac{\partial}{\partial x_k} G_\beta(x,\xi)(\epsilon \hat{v}^2_\epsilon u_\epsilon)\, d\xi.$$

So

$$\epsilon(\frac{\partial u_\epsilon}{\partial x_k}(x) - \frac{\partial u_\epsilon}{\partial x_k}(P^\epsilon_l)) = -\frac{\beta^2}{A^2}\int_\Omega [\frac{\partial}{\partial x_k} G_\beta(x,\xi) - \frac{\partial}{\partial x_k} G_\beta(x,\xi)|_{x=P^\epsilon_l}](\epsilon \hat{v}^2_\epsilon u_\epsilon)\, d\xi$$

$$= \frac{\beta^2}{A^2\xi_{\epsilon,j}}\epsilon^2 \int_{R^2} 2\log\frac{|y-z|}{|z|} w\frac{\partial w}{\partial z_k} + o(\beta^2 A^{-2}\epsilon^4|y|) \tag{10.11}$$

since

$$\nabla_{P^\epsilon_j} F_{\beta_0}(\mathbf{P}^\epsilon) = o(1).$$

Combining (10.10) and (10.11), we obtain that

$$[\psi_{\epsilon,j,k}(P^\epsilon_l + \epsilon y) - \epsilon\frac{\partial u_\epsilon}{\partial x_k}(P^\epsilon_l + \epsilon y)] - [\psi_{\epsilon,j,k}(P^\epsilon_l) - \epsilon\frac{\partial u_\epsilon}{\partial x_k}(P^\epsilon_l)]$$

$$= -\frac{\beta^2|\Omega|\alpha_\epsilon}{\xi_{\epsilon,j}}\epsilon^2 \sum_m \nabla_{P^\epsilon_{l,m}}\nabla_{P^\epsilon_{j,k}} F_{\beta_0}(\mathbf{P}^\epsilon) y_m + o\left(\beta^2 A^{-2}\epsilon^4|y|\right) \tag{10.12}$$

Hence we have

$$\|I_3\|_{L^2(\Omega_\epsilon)} = o(\beta^2 A^{-2}\epsilon^4 \sum_{j=1}^K \sum_{k=1}^2 |a^\epsilon_{j,k}|)$$

and by (10.6)

$$\|\phi^\perp_\epsilon\|_{H^2(\Omega_\epsilon)} = o(\beta^2 A^{-2}\epsilon^4 \sum_{j=1}^K \sum_{k=1}^2 |a^\epsilon_{j,k}|). \tag{10.13}$$

As in (7.23) of [30] it is easy to show that

$$\int_{\Omega_{\epsilon,P_j^\epsilon}} (I_4 \frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}) d\xi = \int_\Omega \tilde{v}_{\epsilon,l}^2 (\epsilon \frac{\partial u_\epsilon}{\partial x_m} \phi_\epsilon^\perp - \frac{\partial \tilde{v}_{\epsilon,l}}{\partial x_m} \psi_\epsilon^\perp) d\xi$$

$$= o(\beta^2 A^{-2} \epsilon^4 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|). \tag{10.14}$$

**Step 2:** Algebraic equations for $a_{j,k}^\epsilon$.

Multiplying both sides of (10.5) by $-\frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}$ and integrating over $\Omega_{\epsilon,P_l^\epsilon}$, we obtain

$$r.h.s. = \lambda_\epsilon \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_{\Omega_{\epsilon,P_l^\epsilon}} \frac{\partial \tilde{v}_{\epsilon,j}}{\partial y_k} \frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}$$

$$= \frac{1}{\xi_{\epsilon,l}^2} \lambda_\epsilon \sum_{j,k} a_{j,k}^\epsilon \delta_{jl} \delta_{km} \int_{R^2} \left( \frac{\partial w}{\partial y_1} \right)^2 dy \, (1 + o(1))$$

$$= \frac{1}{\xi_{\epsilon,l}^2} \lambda_\epsilon a_{l,m}^\epsilon \int_{R^2} \left( \frac{\partial w}{\partial y_1} \right)^2 (1 + o(1))$$

and

$$l.h.s. = \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_{\Omega_{\epsilon,P_l^\epsilon}} (\tilde{v}_{\epsilon,j})^2 \left[ \psi_{\epsilon,j,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k} \right] \frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}$$

$$+ \int_{\Omega_{\epsilon,P_l^\epsilon}} (I_4 \frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}) d\xi$$

$$= \epsilon^2 \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \int_{\Omega_{\epsilon,P_l^\epsilon}} (\tilde{v}_{\epsilon,l})^2 \left[ \psi_{\epsilon,l,k} - \epsilon \frac{\partial u_\epsilon}{\partial x_k} \right] \frac{\partial \tilde{v}_{\epsilon,l}}{\partial y_m}$$

$$+ o(\beta^2 A^{-2} \epsilon^4 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

by (10.14). Using (10.12), we obtain

$$l.h.s. = \frac{\epsilon^2 |\Omega| \beta^2 \alpha_\epsilon}{\xi_{\epsilon,j}^3} \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon$$

$$\times \int_{\Omega_{\epsilon,P_l^\epsilon}} w^2 \left( -\frac{\partial^2}{\partial P_{l,m}^\epsilon \partial P_{j,k}^\epsilon} F_{\beta_0}(\mathbf{P}^\epsilon) \epsilon y_m \right) \frac{\partial w}{\partial y_m}$$

$$+ o(\beta^2 A^{-2} \epsilon^4 \sum_{j=1}^K \sum_{k=1}^2 |a_{j,k}^\epsilon|)$$

$$= \frac{\epsilon^2 |\Omega| \beta^2 \alpha_\epsilon}{\xi_{\epsilon,j}^3} \int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m \sum_{j=1}^K \sum_{k=1}^2 a_{j,k}^\epsilon \left( -\frac{\partial}{\partial P_{l,m}^\epsilon} \frac{\partial}{\partial P_{j,k}^\epsilon} F_{\beta_0}(\mathbf{P}^\epsilon) \right)$$

$$+o(\beta^2 A^{-2}\epsilon^4 \sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|).$$

Note that

$$\int_{R^2} w^2 \frac{\partial w}{\partial y_m} y_m = \int_{R^2} w^2 w' \frac{y_m^2}{|y|}$$

$$= \frac{1}{2}\int_{R^2} w^2 w' |y| < 0.$$

Thus we have

$$l.h.s. = \frac{\epsilon^2|\Omega|\beta^2\alpha_\epsilon}{2\xi_{\epsilon,l}^3}(-\int_{R^2} w^2 w' |y|)\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^\epsilon \left(\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon} F_{\beta_0}(\mathbf{P}^\epsilon)\right)$$

$$+o(\beta^2 A^{-2}\epsilon^4 \sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|).$$

Combining the $l.h.s.$ and $r.h.s$, we have

$$\frac{\epsilon^2|\Omega|\alpha_\epsilon\beta^2}{2\xi_{\epsilon,l}}(-\int_{R^2} w^2 w' |y|)\sum_{j=1}^{K}\sum_{k=1}^{2} a_{j,k}^\epsilon \left(\frac{\partial}{\partial P_{l,m}^\epsilon}\frac{\partial}{\partial P_{j,k}^\epsilon} F_{\beta_0}(\mathbf{P}^\epsilon)\right)$$

$$+o(\beta^2 A^{-2}\epsilon^4 \sum_{j=1}^{K}\sum_{k=1}^{2}|a_{j,k}^\epsilon|)$$

$$= \lambda_\epsilon a_{l,m}^\epsilon \int_{R^2}\left(\frac{\partial w}{\partial y_1}\right)^2. \tag{10.15}$$

We have shown that the small eigenvalues with $\lambda_\epsilon \to 0$ satisfy $\lambda_\epsilon \sim C\epsilon^2\alpha_\epsilon\beta^2$ with some $C \neq 0$. Furthermore, (asymptotically) they are eigenvalues of the matrix $M_{\beta_0}(\mathbf{P}_0)$ and the coefficients $a_{j,k}^\epsilon$ are the corresponding eigenvectors. If condition $(*)$ holds, then the matrix $M_{\beta_0}(\mathbf{P}_0)$ is strictly negative definite and it follows that Re $\lambda_\epsilon < 0$. Therefore the small eigenvalues $\lambda_\epsilon$ are stable if $\epsilon$ is small enough.

Combining the estimates of the large eigenvalues and of the small eigenvalues, we have completed the proof of Theorem 2.3.

□

## References


[1] E.N. Dancer, On stability and Hopf bifurcations for chemotaxis systems, *Methods Appl. Anal.*, 8 (2001), 245-256.

[2] A. Doelman, R.A. Gardner, and T.J. Kaper, Large stable pulse solutions in reaction-diffusion equations, *Indiana Univ. Math. J.* 50 (2001), 443-507.

[3] A. Doelman, A. Gardner and T.J. Kaper, Stability analysis of singular patterns in the 1-D Gray-Scott model: A matched asymptotic approach, *Phys. D* 122 (1998), 1-36.

[4] A. Doelman, A. Gardner and T.J. Kaper, A stability index analysis of 1-D patterns of the Gray-Scott model, *Mem. Amer. Math. Soc.* 155 (2002), no. 737, xii+64 pp.

[5] A. Doelman, T. Kaper, and P. A. Zegeling, Pattern formation in the one-dimensional Gray-Scott model, *Nonlinearity* 10 (1997), 523-563.

[6] S.-I.Ei and Y. Nishiura, private communication.

[7] S.-I. Ei, Y. Nishiura and B. Sandstede, in preparation.

[8] B. Gidas, W.M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $R^N$, *Adv. Math. Suppl. Stud.* 7A (1981), 369-402.

[9] P. Gray and S.K. Scott, Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability, *Chem. Eng. Sci.* 38 (1983), 29-43.

[10] P. Gray and S.K. Scott, Autocatalytic reactions in the isothermal, continuous stirred tank reactor: oscillations and instabilities in the system $A + 2B \to 3B, B \to C$, *Chem. Eng. Sci.* 39 (1984), 1087-1097.

[11] J.K. Hale, L.A. Peletier and W.C. Troy, Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocatalysis, *SIAM J. Appl. Math.* 61 (2000), 102-130.

[12] J.K. Hale, L.A. Peletier and W.C. Troy, Stability and instability of the Gray-Scott model: the case of equal diffusivities, *Appl. Math. Letters* 12 (1999), 59-65.

[13] M.K. Kwong and L. Zhang, Uniqueness of positive solutions of $\Delta u + f(u) = 0$ in an annulus, *Differential Integral Equations* 4 (1991), 583-599.

[14] K. J. Lee, W. D. McCormick, J. E. Pearson, and H. L. Swinney, Experimental observation of self-replicating spots in a reaction-diffusion system, *Nature* 369 (1994), 215-218.

[15] K. J. Lee, W. D. McCormick, Q. Ouyang, and H. L. Swinney, Pattern formation by interacting chemical fronts, *Science* 261 (1993), 192-194.

[16] C.B. Muratov, V.V. Osipov, Static spike autosolitons in the Gray-Scott model, *J. Phys. A* 33 (2000), 8893-8916.

[17] Y. Nishiura, Global structure of bifurcating solutions of some reaction-diffusion systems, *SIAM J. Math. Anal.* 13 (1982), 555-593.

[18] Y. Nishiura and D. Ueyama, A skeleton structure of self-replicating dynamics, *Physica D* 130 (1999), 73-104.

[19] J.E. Pearson, Complex patterns in a simple system, *Science* 261 (1993), 189-192.

[20] V. Petrov, S.K. Scott and K. Showalter, Excitability, wave reflection, and wave splitting in a cubic autocatalysis reaction-diffusion system, *Phil. Trans. Roy. Soc. Lond., Series A* 347 (1994), 631-642.

[21] J. Reynolds, J. Pearson and S. Ponce-Dawson, Dynamics of self-replicating spots in reaction-diffusion systems, *Phy. Rev. E* 56 (1997), 185-198.

[22] J. Reynolds, J. Pearson and S. Ponce-Dawson, Dynamics of self-replicating patterns in reaction diffusion systems, *Phy. Rev. Lett.* 72 (1994), 2797-2800.

[23] M. J. Ward and J. Wei, The stability of multiple spikes for the one-dimensional Gray-Scott model, preprint.
[24] J. Wei, On the construction of single-spot solutions to a singularly perturbed Neumann problem, *J. Differential Equations* 129 (1996), 315-333.
[25] J. Wei, On single interior spike solutions of the Gierer-Meinhardt system: uniqueness and spectrum estimates, *Europ. J. Appl. Math.* 10 (1999), 353-378.
[26] J. Wei, Existence, stability and metastability of point condensation patterns generated by Gray-Scott system, *Nonlinearity* 12 (1999), 593-616.
[27] J. Wei, Pattern formations in two-dimensional Gray-Scott model: existence of single-spot solutions and their stability, *Physica D* 148 (2001), 20-48.
[28] J. Wei and M. Winter, On the two-dimensional Gierer-Meinhardt system with strong coupling, *SIAM J. Math. Anal.* 30 (1999), 1241-1263.
[29] J. Wei and M. Winter, Spikes for the two-dimensional Gierer-Meinhardt system: The strong coupling case, *J. Differential Equations* 178 (2000), 478-518.
[30] J. Wei and M. Winter, Spikes for the two-dimensional Gierer-Meinhardt system: The weak coupling case, *J. Nonlinear Science* 11 (2001), 415-458.

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