ON A TWO DIMENSIONAL REACTION-DIFFUSION SYSTEM WITH HYPERCYCLICAL STRUCTURE

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ABSTRACT. We study a hypercyclical reaction-diffusion system which arises in the modeling of catalytic networks and describes the emerging of cluster states. We construct single cluster solutions in full two-dimensional space and then establish their stability or instability in terms of the number N of components. We provide a rigorous analysis around the single cluster solutions, which is new for systems of this kind. Our results show that as N increases, the system becomes unstable.

1. INTRODUCTION

Recently there has been a great interest in the study of self-replicating patterns observed in the many different types of models. We consider a hypercyclical reaction-diffusion system which arises as a spatial model concerning the origin of life similar to the one introduced by Eigen and Schuster [17]. A number of RNA-like polymers ("components") catalyse the replication of each other in a cyclic way. Examples in nature include Krebs and Bethe-Weizsäcker cycles. Eigen and Schuster argue that the hypercycle satisfies important criteria of natural selection: 1. Selective stability of each component due to favorable competition with error copies, 2. Cooperative behavior of the components integrated into the hypercycle, and 3. Favorable competition of the hypercycle unit with other less efficient systems.

We show rigorously that this may lead to compartmentation (i.e., the build-up of spatially small and essentially closed subsystems) due to spontaneous formation of clusters (also called "spots" or "spikes").

We first study a general system of $N+1$ equations, where N may be any positive integer representing the number of components. For this general

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system we provide the existence of solutions with clusters which for the different components have the same location but possibly different values.

Then we study the stability question for some particularly important examples.

At this point we should like to emphasize that we provide a rigorous analysis around cluster solutions, not around constant states. This approach is new for the kind of $(N+1)$-systems under investigation.

As suggested in [8], [9] we study the following reaction-diffusion system:

$$\begin{cases} \frac{\partial X_i}{\partial t} = D_X \Delta X_i - g_X X_i + M \sum_{j=1}^N k_{ij} X_i X_j, \quad i = 1, 2, \ldots, N, \quad x \in R^2, \\ \frac{\partial M}{\partial t} = D_M \Delta M + k_M - g_M M - LM \sum_{i,j=1}^N k_{ij} X_i X_j, \quad x \in R^2, \end{cases} \tag{1.1}$$

where X_i denotes the concentration of the polymers, and M is the concentration of activated monomers. N is the number of different polymer species. The replication of each polymer X_i is catalysed by each X_j at a rate constant k_{ij}. Linear (non-catalytic) growth terms are neglected. The activated monomers are produced at constant rate, k_M; g_X and g_M are decay rate constants. L is the number of monomers in each polymer, and D_X and D_M are constant diffusion coefficients.

If the coefficients k_{ij} are represented by a cyclical $N \times N$ matrix, namely (e.g., for $N = 5$)

$$\mathcal{K}_{hyper} = (k_{ij}^{hyper}) = \begin{pmatrix} 0 & 0 & 0 & 0 & k_0 \\ k_0 & 0 & 0 & 0 & 0 \\ 0 & k_0 & 0 & 0 & 0 \\ 0 & 0 & k_0 & 0 & 0 \\ 0 & 0 & 0 & k_0 & 0 \end{pmatrix}_{N \times N}, \quad k_0 > 0,$$

the system (1.1) is called “elementary hypercycle” by Eigen and Schuster [17] as the polymers interact in pairs only. There are more complex hypercycles if the polymers interact in triples, quadruples, etc. However, more complex hypercycles are likely to be of less importance for an efficient start of evolution than elementary hypercycles since they are more difficult to form in the first place.

While Eigen and Schuster [17] use an assumption of constant organisation, meaning that the total sum of all polymer concentrations is kept constant, in system (1.1) another mechanism for bounding the polymer concentrations is present: Since each polymer consists of L monomers the polymer concentrations are bounded by the limited supply of activated monomers. This is a nonlocal coupling in contrast to the local coupling in the model of Eigen and Schuster.

We pose the problem in two-dimensional space which on the one hand allows a rigorous analysis and on the other hand is relevant if the early biochemical reactions take place in very thin layers like for example on the surfaces of rocks.

A cluster may loosely be defined as a region of high concentration $\sum_{i=1}^{N} X_i$ of the polymers and low concentration of the monomer, as monomers are consumed by the replication of polymers (if the region shrinks to a point, then it is called point-condensation).

Let us mention some related results.

In [8] the parameter dependence of stability of clusters and spirals against parasites (i.e., rival polymers which receive catalytic support from the hypercycle but do not contribute to the catalysis of any other polymer) is studied numerically. A parasite may or may not destroy the hypercycle depending on the rate constants. In [9] clusters (for $N = 5$) are established numerically for the elementary N-hypercycle system,

In [7] for a closely related reaction-diffusion model the dependence of cluster states on diffusivities is shown numerically including the cluster size, their shape, and the distance between different clusters.

The effect of faulty replication on the hypercycle has been studied by an analysis of the geometry of bifurcations around steady states and numerical computations in the framework of an ODE reaction model [1].

For a cellular automata model it was shown numerically that a spiral wave structure may be stable against parasites [5]. The chaotic dynamics for this type of model has been investigated numerically in [30], [39].

There are a number of recent results on the special case $N = 1$ of our model, which is then also called Gray-Scott system [19], [20]. We would like to recall them here. In [13], by using Mel'nikov method, Doelman, Kaper and Zegeling constructed single and multiple pulse solutions for (1.1) in the one-dimensional case with $D_M = 1, D_X = \delta^2 << 1$, where $X_i = X$. In their paper [13], it is assumed that $k_M = g_M \sim \delta^2, g_X \sim \delta^{2\alpha/3}$, $k_{11} = 1$, $L = 1$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $M = O(\delta^\alpha), X = O(\delta^{-\frac{\alpha}{3}})$. Later the stability of single and multiple pulse solutions in 1-D are obtained in [11], [12]. (The techniques are extended to other reaction-diffusion equations in [14].) Some related results on the existence and stability of solutions to the Gray-Scott model in 1-D can be found in [15], [25], [26], [36] and [40].

In R^2 and R^3, Muratov and Osipov [31] have given some formal asymptotic analysis on the construction and stability of spiky solution. In [49], the system (1.1) for $N = 1$ is studied on the real axis in the shadow system case, namely, $D_M >> 1, D_X << 1$ and $k_M = g_M = O(1), g_X = O(1)$, $k_{11} = 1$, $L = 1$. The shadow system can be reduced to a single equation. For spike solutions for single equations, please see [3], [4], [10], [18], [21], [23], [24], [28], [29], [38], [32], [33], [34], [35], [37], [42], [44], [43], [45], [46], [47], [48], [51], [52], and the references therein.

In the general higher dimensional case, as far as we know, the only rigorous existence and stability results on the Gray-Scott system have been established in [50]. The existence of one-spike solutions is proved. Their stability is established and rests upon the derivation and analysis of a related NLEP (nonlocal eigenvalue problem).

In this paper, we study the existence and stability of a single-cluster solution in 2-D. Let us first reduce the system (1.1) to standard form. Dividing by g_X and g_M, respectively, gives

$$\frac{1}{g_X}\partial_t X_i = \frac{D_X}{g_X}\Delta X_i - X_i + \frac{M}{g_X}\sum_{j=1}^N k_{ij}X_iX_j,$$

$$\frac{1}{g_M}\partial_t M = \frac{D_M}{g_M}\Delta M + \frac{k_M}{g_M} - M - \frac{LM}{g_M}\sum_{ji,=1}^N k_{ij}X_iX_j.$$

Rescaling $M = (k_M/g_M)\hat{M}$, $X_i = \sqrt{L/g_M}\hat{X}_i$, we get

$$\frac{1}{g_X}\partial_t \hat{X}_i = \frac{D_X}{g_X}\Delta \hat{X}_i - \hat{X}_i + \frac{1}{g_X}\frac{k_M}{g_M}\hat{M}\sqrt{\frac{g_M}{L}}\sum_{j=1}^N k_{ij}\hat{X}_i\hat{X}_j,$$

$$\frac{1}{g_M}\partial_t \hat{M} = \frac{D_M}{g_M}\Delta \hat{M} + 1 - \hat{M} - \hat{M}\sum_{i,j=1}^{N} k_{ij}\hat{X}_i\hat{X}_j.$$

Rescaling space variables x and time variable t:

$$x = \sqrt{\frac{D_M}{g_M}}\hat{x}, \quad t = \frac{1}{g_M}\hat{t},$$

renaming constants:

$$A = \frac{k_M}{g_X g_M}\sqrt{\frac{g_M}{L}}, \quad \epsilon^2 = \frac{D_X}{D_M}\frac{g_X}{g_M}, \quad \tau = \frac{g_X}{g_M}$$

and dropping the hats, we finally arrive at the following standard form

$$\begin{cases} \partial_t X_i = \epsilon^2 \Delta X_i - X_i + AM\sum_{i=1}^{N} k_{ij}X_iX_j, \\ \tau\partial_t M = \Delta M + 1 - M - M\sum_{i=1,j}^{N} k_{ij}X_iX_j. \end{cases} \tag{1.2}$$

We shall study (1.2) in the whole R^2 for $\epsilon > 0$ small. Different choices of A might distinguish between stability and instability. Therefore we will treat it as a parameter. We shall construct solutions of (1.2) which are radially symmetric:

$$X_i = X_i(|x|) \in H^1(R^2), \quad i = 1, \ldots, N,$$
$$M = M(|x|) \in H^1(R^2).$$

The stationary equation of (1.2) becomes

$$\begin{cases} \epsilon^2 \Delta X_i - X_i + AM\sum_{j=1}^{N} k_{ij}X_iX_j = 0, \quad i = 1, ..., N, \\ \Delta M + 1 - M - M\sum_{i=1,j}^{N} k_{ij}X_iX_j = 0. \end{cases} \tag{1.3}$$

We first construct cluster solutions to (1.3). To this end, we need to introduce some assumptions and notations.

We assume that

$$\text{the matrix } (k_{ij}) \text{ is invertible.} \tag{1.4}$$

So the following equation has a unique solution $(\hat{\zeta}_1, ..., \hat{\zeta}_N)$:

$$\sum_{j=1}^{N} k_{ij}\hat{\zeta}_j = 1, \quad i = 1, \ldots, N. \tag{1.5}$$

We assume that

$$\hat{\zeta}_j > 0, \quad j = 1, \ldots, N. \tag{1.6}$$

(The $\hat{\zeta}_j$ will be the scale of the height of each X_j.) We shall also use the notation $\hat{\zeta} = \sum_{i=1}^{N} \hat{\zeta}_i$.

Let w be the unique solution of the following problem

$$\begin{cases} \Delta w - w + w^2 = 0, w > 0 \ \text{ in } R^2, \\ w(0) = \max_{y \in R^2} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{1.7}$$

(The solution of (1.7) is radial and unique. See [22] and [27].)

Put

$$L := \frac{\sum_{i=1}^K \hat{\zeta}_i}{2\pi A^2} \epsilon^2 \log\left(\frac{1}{\epsilon}\right) \int_{R^2} (w(y))^2 dy, \tag{1.8}$$

where $\hat{\zeta}_j$ are given by (1.5).

If $0 < L < \frac{1}{4}$, then the following equation has two solutions:

$$\eta(1-\eta) = L. \tag{1.9}$$

We denote the smaller one by η^s, where $0 < \eta^s < \frac{1}{2}$ and the larger one by η^l, where $1 > \eta^l > \frac{1}{2}$.

We now state the existence result. In fact, this is quite easy. We search for solutions of the following type

$$X_i = \hat{\zeta}_i X_0, \quad i = 1, \ldots, N. \tag{1.10}$$

Substituting (1.10) into (1.3), we see that (X_0, M) satisfies

$$\begin{cases} \epsilon^2 \Delta X_0 - X_0 + AMX_0^2 = 0, \\ \Delta M + 1 - M - M \sum_{j=1}^N \hat{\zeta}_j X_0^2 = 0. \end{cases} \tag{1.11}$$

Applying Theorem 1.1 of [50], we obtain the following existence theorem:

Theorem 1.1. *Assume that*

$$\epsilon << 1 \tag{1.12}$$

and

$$\frac{1}{\log \frac{1}{\epsilon}} << L < \frac{1}{4} - \delta_0, \tag{1.13}$$

where $\delta_0 > 0$ is any small positive constant (independent of $\epsilon << 1$).

Then problem (1.3) admits two solutions $(X_\epsilon^s, M_\epsilon^s) = (X_{\epsilon,1}^s, \ldots, X_{\epsilon,N}^s, M_\epsilon^s)$ and $(X_\epsilon^l, M_\epsilon^l) = (X_{\epsilon,1}^l \ldots, X_{\epsilon,N}^l, M_\epsilon^l)$ with the following properties:

(1) all components are radially symmetric functions.

(2) $X^s_{\epsilon,i} = \frac{\hat{\zeta}_i}{AM^s_\epsilon(0)}(1+o(1))w(\frac{|x|}{\epsilon})$, $i = 1,\dots,N$,
$X^l_{\epsilon,i} = \frac{\hat{\zeta}_i}{AM^l_\epsilon(0)}(1+o(1))w(\frac{|x|}{\epsilon})$, $i = 1,\dots,N$,
where w is the unique solution of (1.7).

(3) $M^s_\epsilon(x) \to 1$ $M^l_\epsilon(x) \to 1$ *for all* $x \neq 0$ *and* $M^s_\epsilon(0)$, $M^l_\epsilon(0)$ *satisfy*

$$\begin{aligned} M^s_\epsilon(0) \sim \eta^s, \quad M^l_\epsilon(0) \sim \eta^l, \\ 0 < M^s_\epsilon(0) < M^l_\epsilon(0) < 1. \end{aligned} \tag{1.14}$$

(4) There exist $a > 0, b > 0$ *such that*

$$\begin{aligned} 1 - M^s_\epsilon(x) \le Ce^{-a|x|}, \quad 1 - M^l_\epsilon(x) \le Ce^{-a|x|}, \\ X^s_{\epsilon,i}(x) \le C\frac{1}{AM^s_\epsilon(0)}e^{-b\frac{|x|}{\epsilon}}, \quad X^l_{\epsilon,i}(x) \le C\frac{1}{AM^l_\epsilon(0)}e^{-b\frac{|x|}{\epsilon}} \end{aligned} \tag{1.15}$$

Finally, if $L > \frac{1}{4} + \delta_0$ *, then there are no single-cluster solutions.*

The main goal of this paper is to study the stability and instability of the cluster solution constructed in Theorem 1.1. To this end, we first linearize the equations (1.3) around $(X^s_\epsilon, M^s_\epsilon)$ or $(X^l_\epsilon, M^l_\epsilon)$, respectively. From now on we omit the superscripts s or l where this is possible without confusing the reader. The linearized operator is as follows:

$$\mathcal{L}\begin{pmatrix} \phi_{\epsilon,i} \\ \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon^2\Delta\phi_{\epsilon,i} - \phi_{\epsilon,i} + AM_\epsilon \sum_{j=1}^N k_{ij}(\phi_{\epsilon,j}X_{\epsilon,i} + X_{\epsilon,j}\phi_{\epsilon,i}) \\ +A\psi_\epsilon \sum_{j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} \\ \Delta\psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij}X_{\epsilon,i}X_{\epsilon,j} \\ -M_\epsilon \sum_{i,j=1}^N k_{ij}(\phi_{\epsilon,j}X_{\epsilon,i} + \phi_{\epsilon,i}X_{\epsilon,j}) \end{pmatrix}, \tag{1.16}$$

where $i = 1,\dots,N$. The eigenvalue problem becomes

$$\mathcal{L}\begin{pmatrix} \phi_{\epsilon,i} \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} \lambda_\epsilon\phi_{\epsilon,i} \\ \tau\lambda_\epsilon\psi_\epsilon \end{pmatrix}, \ i = 1,\dots,N. \tag{1.17}$$

We assume that the domain of $\mathcal{L}$ is $(H^2(R^2))^N$.

Certainly 0 is an eigenvalue of $\mathcal{L}$. The criterion for linearized stability of a cluster solution is that the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ (except for 0) lies in a left half plane $\{\lambda \in \mathcal{C} : \text{ Re } (\lambda) < -a_0\}$ where $a_0 > 0$, and that 0 is a semi-simple eigenvalue (with multiplicity 2), where $\mathcal{C}$ denotes the set of complex numbers.

It turns out that the stability and instability of cluster solutions depend highly on the matrix (k_{ij}). We now state various assumptions:

The first assumption is the most restrictive one:

$$\text{(H1)} \qquad \sum_{i=1}^{N} k_{ij}\hat{\zeta}_i = \gamma > 0. \tag{1.18}$$

To introduce the second assumption, we need to consider the following eigenvalue problem (EVP)

$$\begin{cases} \Delta\phi - \phi + \mu w\phi = 0, \\ \phi \in H^1(R^2) \end{cases}$$

By Lemma 4.1 of [43], (EVP) admits the following set of eigenvalues

$$\mu_1 = 1, \mu_2 = \ldots = \mu_{N+1} = 2, \mu_{N+1} > 2. \tag{1.19}$$

Put

$$\mathcal{B} = (b_{ij}), b_{ij} = k_{ij}\hat{\zeta}_j. \tag{1.20}$$

Observe that by (1.5) the matrix $\mathcal{B}$ has an eigenvalue 1 and the associated eigenvector is $\vec{e}_0 := (1, ..., 1)^\tau$.

The second assumption is the following:

$$\text{(H2)} \qquad 1 + \text{spec}(\mathcal{B}) \cap \text{spec}(\text{EVP}) = \{2\}. \tag{1.21}$$

Next, recall that $\eta^s < \eta^l$ are defined by (1.9). The third and fourth assumptions are:

$$\text{(H3)} \qquad \gamma \le 1, (1-\eta)(1+\gamma) > 1 + \sqrt{1-\gamma}. \tag{1.22}$$

and

$$\text{(H4)} \qquad (1-\eta)(1+\gamma) < \gamma. \tag{1.23}$$

The following is our main result on the stability.

Theorem 1.2. *Assume that*

$$\epsilon << 1, \qquad \frac{1}{\log\frac{1}{\epsilon}} << L < \frac{1}{4}, \tag{1.24}$$

and assumptions (H1) and (H2) hold.

Let $(X_\epsilon^s, M_\epsilon^s)$ *and* $(X_\epsilon^l, X_\epsilon^l)$ *be the solutions constructed in Theorem 1.1. Then for* $\epsilon << 1$*, we have the following.*

(1) (stability) Suppose that $(H3)$ *holds for* $\eta = \eta^s$. *Assume that* $\sigma = 1$ *is a simple eigenvalue of* $\mathcal{B}$ *and that all other eigenvalues* σ *of* $\mathcal{B}$ *satisfy* $\sigma = Re^{i\theta}$ *with some* $R > 0$ *and*

$$\theta \in \left(\frac{\pi}{2} - \theta_s^R, \frac{3\pi}{2} + \theta_s^R\right)$$

for some suitably chosen $\theta_s^R > 0$. *Then* $(X_\epsilon^s, M_\epsilon^s)$ *is linearly stable.*

(2) (Instability) Suppose that $(H3)$ *holds for* $\eta = \eta^s$. *If the eigenvalue* $\sigma = 1$ *of* $\mathcal{B}$ *is not simple or there exists* $\sigma = Re^{i\theta}$ *with* $\theta \in (-\theta_{us}^R, \theta_{us}^R)$ *for some* $\theta_{us}^R > 0$. *Then* $(X_\epsilon^s, M_\epsilon^s)$ *is linearly unstable.*

(3) (Instability) Suppose that $(H4)$ *holds for* $\eta = \eta^l$. *Then* $(X_\epsilon^l, M_\epsilon^l)$ *is linearly unstable.*

Remarks:

1. In many examples, $\gamma = 1$, so (1.22) holds automatically for $\eta = \eta^s$. When L is small (1.22) holds for $\eta = \eta^s$ and (1.23) holds for $\eta = \eta^l$.

2. The assumption (H1) allows that $\hat{\zeta}_i \neq \hat{\zeta}_j$ for some $i \neq j$. If all $\hat{\zeta}_i$ are equal, then necessarily $\gamma = 1$.

3. We do not know the optimal values for θ_s^R and θ_{us}^R. They are related to an eigenvalue problem with complex coefficients. See Lemma 3.3. We believe that in general, $\theta_s^R = \theta_{us}^R$.

4. By the same proof as in Theorem 1.2 of [50], we may relax the condition that $\tau = O(1)$ to $\tau \sim \epsilon^{-l}$. We will not pursue this generality since our main objective is to study the effect of the matrix (k_{ij}) on the stability of cluster solutions.

A direct application of Theorem 1.2 is the following stability result for the N-hypercycle case:

$$(k_{ij}^{hyper}) = \begin{pmatrix} 0 & 0 & 0 & ... & k_0 \\ k_0 & 0 & 0 & ... & 0 \\ 0 & k_0 & 0 & ... & 0 \\ ... & ... & ... & ... & 0 \\ 0 & 0 & ... & k_0 & 0 \end{pmatrix}_{N\times N}, \quad k_0 > 0.$$

Theorem 1.3. *For the* N*-hypercycle system, the small cluster solution is stable for* $N \leq 4$ *and is unstable for* $N > N_0$ *for some* $N_0 \geq 5$. *The large cluster solution is always unstable.*

Remark: The numerical computations in [9] suggest that, at least for $N = 5$, the cluster solution is numerically stable. This implies that at least numerically, $\theta_s^1 > \cos(\frac{2\pi}{5})$.

The structure of the paper is as follows:

In Section 2 we give some examples and make a few remarks about our results. In particular, Theorem 1.3 will be proved.

In Section 3, we study some local and nonlocal eigenvalue problems associated with w.

In Section 4, we separate the eigenvalue problem into two cases: small eigenvalues and large eigenvalues. The case of large eigenvalues is then reduced to a nonlocal eigenvalue problem (NLEP).

In Section 5, we analyze the NLEP for the case of large eigenvalues.

Throughout this paper, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon\to 0} \frac{A}{B} = 1$ and $A = O(B)$ implies that $|A| \leq C|B|$.

2. Applications of Theorem 1.2: Examples and Remarks

In this section, we apply our stability results of Theorem 1.2 to some specific examples. We would like to point out that there are many matrices which satisfy the assumptions in Theorem 1.2.

Example 1. (**Proof of Theorem 1.3:**)

For the hypercyclical network we have

$$b_{ij} = \delta_{i,j+1} \quad \text{modulo } N.$$

The eigenvalues are $e^{2\pi j\sqrt{-1}/N}, \quad j = 1, \dots, N$ and are all simple. In this case, $\gamma = 1$ and (H1) is satisfied. It is easy to check that assumption (H2) holds. Assumption (H3) is satisfied since

$$(1 - \eta^s)(1 + \gamma) > 1.$$

By Theorem 1.2 (2), we obtain the stability of the small cluster solution for $N = 1, 2, 3, 4$. We do not know if stability still holds for $N = 5$. However, for N very large $e^{2\pi\sqrt{-1}/N}$ is close to 1. By Theorem 1.2 (3), the small cluster solution is unstable.

For the large cluster solution, it is easy to check that $(H4)$ holds and by Theorem 1.2 (4), the large cluster solution is unstable.

This proves Theorem 1.3.

□

Example 2. For the (cyclical) bidiagonal matrix

$$b_{ij} = ((1-\alpha)\delta_{ij} + \alpha\delta_{i,j+1}) \quad \text{modulo } N, \quad 0 \le \alpha \le 1$$

we obviously have (H1) with $\gamma = 1$. It is easy to calculate that the eigenvalues are $1-\alpha(1-e^{2\pi j\sqrt{-1}/N})$, $j = 1, \dots, N$ and are all simple. (H2) and (H3) hold so that the small cluster solution is stable if $(1-\cos(2\pi k/N))^{-1} \le \alpha \le 1$. The last condition if equivalent to $1/2 \le \alpha \le 1$ for $N = 2$, $2/3 \le \alpha \le 1$ for $N = 3$, and $\alpha = 1$ for $N = 4$. Since we do not know θ_s^R or θ_{us}^R explicitly, Theorem 1.2 does not give a stability or instability criterion for $N \ge 5$. For large N, however, $1-\alpha(1-e^{2\pi\sqrt{-1}/N})/2$ is close to 1 uniformly in α and by Theorem 1.2 (3) the small cluster solution is unstable for $0 \le \alpha \le 1$.

For the large cluster solution, it is easy to check that $(H4)$ holds and by Theorem 1.2 (4), the large cluster solution is unstable.

Example 3. For $b_{ij} = \delta_{ij}$ the conditions (H1) and (H3) hold with $\gamma = 1$. (H2) holds for $N = 1$ but not for $N \ge 2$. Arguing as in Example 2 we have stability of the small cluster solution for $N = 1$ but not for $N \ge 2$. Because of (H4) the large cluster solution is unstable. (For $N = 1$ this is the Gray-Scott system, for which stability and instability was established by [50]).

Example 4. For the (cyclical) tridiagonal matrix

$$b_{ij} = ((1-2\alpha)\delta_{ij} + \alpha\delta_{i,j+1} + \alpha\delta_{i,j-1}) \quad \text{modulo } N, \quad 0 \le \alpha \le 1$$

we obviously have (H1) with $\gamma = 1$. It is easy to calculate that the eigenvalues are $1-2\alpha(1-\cos(2\pi j/N))$, $j = 1, \dots, N$ and are all real and simple. (H2) and (H3) hold so that by Theorem 1.2 (3) the small cluster solution is stable if and only if $(2-2\cos(2\pi k/N))^{-1} \le \alpha \le 1$. The last condition is equivalent to $1/4 \le \alpha \le 1$ for $N = 2$, $1/3 \le \alpha \le 1$ for $N = 3$, $1/2 \le \alpha \le 1$ for $N = 4$,

$(2 - 2\cos(2\pi/5))$ for $N = 5$, and $\alpha = 1$ for $N = 6$. There are no possible values for α if $N \geq 7$.

For the large cluster solution, it is easy to check that (H4) holds and by Theorem 1.2 (4), the large cluster solution is unstable.

From all the previous examples, we see as a general trend that if the system is not too much dominated by diagonal terms we have stability of the small cluster solutions. Otherwise, an instability emerges. This means that cooperative behavior is needed to stabilise the cluster.

The results also indicate that for many configurations the small cluster solutions are stable if N is small but turn unstable as N increases. This is in correspondence with the result of Eigen and Schuster [17] that the constant nontrivial steady state for the hypercycle is stable if and only if $N \leq 4$.

3. Some Important Lemmas

In this section, we collect some important properties associated with the function w, which is defined by (1.7).

We first study some local eigenvalue problems.

Lemma 3.1. *(1) The linear operator*

$$\begin{cases} L_0\phi := \Delta\phi - \phi + 2w\phi \\ \qquad \phi \in H^1(R^2) \end{cases}$$

has the kernel

$$Ker\ (L_0) = span\ \left\{ \frac{\partial w}{\partial y_j} \middle| j = 1, \ldots, N \right\}.$$

(2) The eigenvalue problem (EVP)

$$\begin{cases} \Delta\phi - \phi + \mu w\phi = 0, \\ \qquad \phi \in H^1(R^2) \end{cases}$$

admits the following set of eigenvalues

$$\mu_1 = 1,\ v_1 = span\ \{w\},$$

$$\mu_2 = \ldots = \mu_{N+1} = 2,\ v_2 = Ker\ (L_0),$$

$$\mu_{N+1} > 2.$$

(3) If $\mu_R > 0$, then the following eigenvalue problem

$$\begin{cases} \Delta\phi - \phi + w\phi + \mu_R w\phi = \lambda\phi, \\ \mu_R > 0,\ \phi \in H^1(R^2) \end{cases}$$

admits a positive (principal) eigenvalue λ_1 such that

$$-\lambda_1 = \inf_{\phi\in H^1(R^2)\setminus\{0\}} \frac{\int_{R^2} |\nabla\phi|^2 + \phi^2 - (1+\mu_R)w\phi^2}{\int_{R^2}\phi^2} < 0.$$

(4) Let ϕ (complex-valued) satisfy the following eigenvalue problem

$$\begin{cases} \Delta\phi - \phi + w\phi + \sigma w\phi = \lambda\phi \\ Re\ (\sigma) \le 0, \quad \phi \in H^1(R^2), \quad \lambda \ne 0. \end{cases}$$

Then

$$Re\ (\lambda) \le -c_0 < 0.$$

Proof: For (1) and (2) please see Lemma 4.1 of [43].

(3) follows by the variational characterization of the eigenvalues:

$$-\lambda_1 = \inf_{\phi\in H^1(R^2),\phi\not\equiv 0} \frac{\int_{R^2} |\nabla\phi|^2 + \phi^2 - (1+\mu_R)w\phi^2}{\int_{R^2}\phi^2} < 0$$

since by the last inequality for $\phi = w$

$$-\mu_R \frac{\int_{R^2} w^3}{\int_{R^2} w^2} < 0.$$

To prove (4) note that

$$\sigma = \sigma_R + i\sigma_I,\ \phi = \phi_R + i\phi_I,\ \lambda = \lambda_R + i\lambda_I$$

and write the eigenvalue problem for real and imaginary parts separately:

$$\Delta\phi_R - \phi_R + (1+\sigma_R)w\phi_R - \sigma_I w\phi_I = \lambda_R\phi_R - \lambda_I\phi_I, \tag{3.1}$$

$$\Delta\phi_R - \phi_I + (1+\sigma_R)w\phi_I + \sigma_I w\phi_R = \lambda_R\phi_I + \lambda_I\phi_R. \tag{3.2}$$

Multiplying (3.1) by ϕ_R, (3.2) by ϕ_I, integrating over R^2, and adding up, we get

$$\int_{R^2}[-|\nabla\phi_R|^2 - \phi_R^2 + (1+\sigma_R)w\phi_R^2] + \int_{R^2}[-|\nabla\phi_I|^2 - \phi_I^2 + (1+\sigma_R)w\phi_I^2]$$
$$= \lambda_R \int_{R^2} \phi_R^2 + \phi_I^2.$$

Since in the last equation l.h.s.≤ 0 we also get r.h.s.≤ 0. Therefore $\lambda_R \le 0$. Now assume that $\lambda_R = 0$. Then by (2) we get $\phi_R = c_1 w$, $\phi_I = c_2 w$ (with

$c_1, c_2 \in R$) and $\sigma_R = 0$. But this implies $\lambda_I = 0$, $\sigma_I = 0$ and we get $\lambda = 0$, contrary to what we assumed. Therefore λ_R can not be zero and we conclude Re $\lambda \leq -c_0 < 0$.

□

By a perturbation to (3), (4) of Lemma 3.1 we immediately get the following.

Lemma 3.2. *Let ϕ (complex-valued) satisfy the following eigenvalue problem*

$$\begin{cases} \Delta\phi - \phi + w\phi + \sigma w\phi = \lambda\phi \\ \sigma = \sigma_R + i\sigma_I = Re^{i\theta}, \quad \phi \in H^1(R^2). \end{cases}$$

Then

(1) If

$$\theta \in \left(\frac{\pi}{2} - \theta_s^R, \frac{3\pi}{2} + \theta_s^R\right),$$

then

$$Re\ (\lambda) \leq -c_0 < 0.$$

(2) If

$$\theta \in (-\theta_{us}^R, \theta_{us}^R),$$

for some $\theta_{us}^R > 0$, then there exists an eigenvalue λ with Re $(\lambda) > 0$.

Proof: Since it is a straightforward perturbation we omit it. □

Remarks: (1) We do not know if $\theta_s^R = \theta_{us}^R$.

(2) It is an interesting and difficult problem to obtain the optimal values for θ_s^R and θ_{us}^R.

(3) By a continuity argument there is a $\theta = \theta_h^R$ such that we have a Hopf bifurcation at θ_h^R.

Next we study a nonlocal eigenvalue problem.

Lemma 3.3. *Consider the following eigenvalue problem*

$$\Delta_y\phi - \phi + (1+\gamma)w\phi - \mu\frac{\int_{R^2} w\phi\, dy}{\int_{R^2} w^2\, dy}w^2 = \lambda_0\phi,\ \phi \in H^2(R^2). \tag{3.3}$$

(1) Suppose that $0 < \gamma \leq 1, \mu > 1 + \sqrt{1-\gamma}$. Let $\lambda_0 \neq 0$ be an eigenvalue of (3.3). Then we have $Re(\lambda_0) \leq -c_1$ for some $c_1 > 0$.

(2) Suppose that $\mu < \gamma$, then problem (3.3) admits a real eigenvalue λ_0 with $\lambda_0 \geq c_2 > 0$ for some $c_2 > 0$.

Proof:

(1). When $\gamma = 1$, this has been proved in Theorem 2.1 of [49]. For $0 < \gamma < 1$, we proceed by the same proof. The key is to use the following inequality (Lemma 2.3 of [49]): there exists a positive constant $a_1 > 0$ such that

$$\begin{aligned} &L_1(\phi,\phi) \\ &:= \int_{R^2}(|\nabla\phi|^2+\phi^2-2w\phi^2)+\frac{2\int_{R^2}w\phi\int_{R^2}w^2\phi}{\int_{R^2}w^2}-\frac{\int_{R^2}w^3}{(\int_{R^2}w^2)^2}(\int_{R^2}w\phi)^2 \\ &\geq a_1 d^2_{L^2(R^2)}(\phi,X_1) \end{aligned} \tag{3.4}$$

for all $\phi \in H^1(R^2)$, where $X_1 :=$ span $\{w, \frac{\partial w}{\partial y_j} | j = 1, ..., N\}$ and $d_{L^2(R^2)}$ means the distance in L^2-norm.

Suppose that (α_0, ϕ) satisfies (3.3) and $\alpha_0 \neq 0$. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp$ Ker (L_0). Then we obtain two equations

$$L_0\phi_R + (\gamma-1)w\phi_R - \mu\frac{\int_{R^2}w\phi_R}{\int_{R^2}w^2}w^2 = \alpha_R\phi_R - \alpha_I\phi_I, \tag{3.5}$$

$$L_0\phi_I + (\gamma-1)w\phi_I - \mu\frac{\int_{R^2}w\phi_I}{\int_{R^2}w^2}w^2 = \alpha_R\phi_I + \alpha_I\phi_R. \tag{3.6}$$

Multiplying (3.5) by ϕ_R, (3.6) by ϕ_I, integrating over R^2, and adding together, we obtain

$$\begin{aligned} &-\alpha_R\int_{R^2}(\phi_R^2+\phi_I^2) \\ &= L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)+(1-\gamma)\int_{R^2}(w\phi_R^2+w\phi_I^2) \\ &+(\mu-2)\frac{\int_{R^2}w\phi_R\int_{R^2}w^2\phi_R+\int_{R^2}w\phi_I\int_{R^2}w^2\phi_I}{\int_{R^2}w^2} \\ &+\frac{\int_{R^2}w^3}{(\int_{R^2}w^2)^2}[(\int_{R^2}w\phi_R)^2+(\int_{R^2}w\phi_I)^2] \end{aligned}$$

Multiplying (3.5) by w, (3.6) by w, and integrating over R^2 we obtain

$$\gamma\int_{R^2}w^2\phi_R - \mu\frac{\int_{R^2}w\phi_R}{\int_{R^2}w^2}\int_{R^2}w^3 = \alpha_R\int_{R^2}w\phi_R - \alpha_I\int_{R^2}w\phi_I,$$

$$\gamma\int_{R^2} w^2\phi_I-\mu\frac{\int_{R^2} w\phi_I}{\int_{R^2} w^2}\int_{R^2} w^3=\alpha_R\int_{R^2} w\phi_I+\alpha_I\int_{R^2} w\phi_R.$$

Hence we have

$$\gamma\int_{R^2} w\phi_R\int_{R^2} w^2\phi_R+\gamma\int_{R^2} w\phi_I\int_{R^2} w^2\phi_I$$
$$=(\alpha_R+\mu\frac{\int_{R^2} w^3}{\int_{R^2} w^2})((\int_{R^2} w\phi_R)^2+(\int_2 w\phi_I)^2).$$

Therefore we get

$$-\alpha_R\int_{R^2}(\phi_R^2+\phi_I^2)$$
$$=L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)+(1-\gamma)\int_{R^2}(w\phi_R^2+w\phi_I^2)$$
$$+(\mu-2)(\frac{1}{\gamma}\alpha_R+\frac{\mu}{\gamma}\frac{\int_{R^2} w^3}{\int_{R^2} w^2})\frac{(\int_{R^2} w\phi_R)^2+(\int_{R^2} w\phi_I)^2}{\int_{R^2} w^2}$$
$$+\frac{\int_{R^2} w^3}{(\int_{R^2} w^2)^2}[(\int_{R^2} w\phi_R)^2+(\int_{R^2} w\phi_I)^2].$$

Set

$$\phi_R=c_Rw+\phi_R^\perp,\quad \phi_R^\perp\perp X_1,$$
$$\phi_I=c_Iw+\phi_I^\perp,\quad \phi_I^\perp\perp X_1.$$

Then

$$\int_{R^2} w\phi_R=c_R\int_{R^2} w^2,\int_{R^2} w\phi_I=c_I\int_{R^2} w^2,$$
$$d^2_{L^2(R^2)}(\phi_R,X_1)=\|\phi_R^\perp\|^2_{L^2},d^2_{L^2(R^2)}(\phi_I,X_1)=\|\phi_I^\perp\|^2_{L^2}.$$

By some simple computations we have

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)+(1-\gamma)\int_{R^2}(w\phi_R^2+w\phi_I^2)$$
$$+(\frac{\mu-2}{\gamma}+1)\alpha_R(c_R^2+c_I^2)\int_{R^2} w^2+(\frac{\mu^2-2\mu+\gamma}{\gamma})(c_R^2+c_I^2)\int_{R^2} w^3$$
$$+\alpha_R(\|\phi_R^\perp\|^2_{L^2}+\|\phi_I^\perp\|^2_{L^2})=0.$$

Note that since $\mu>1+\sqrt{1-\gamma}$, we have

$$\frac{\mu-2}{\gamma}+1>0,\mu^2-2\mu+\gamma>0.$$

Hence by (3.4), we must get

$$\alpha_R\le -c_1<0$$

for some $c_1>0$.

This proves (1) of Lemma 3.3.

(2). Assume that $\mu < \gamma$. Let

$$L_{\gamma-1} = L_0 + (\gamma - 1)w = \Delta - 1 + (1+\gamma)w.$$

By Lemma 3.1 (3), $L_{\gamma-1}$ has a positive eigenvalue $a_\gamma > 0$. Consider the following function

$$h(\alpha) = \int_{R^2} ((L_{\gamma-1} - \alpha)^{-1} w) w.$$

It is easy to see that

$$h^{'}(\alpha) > 0, \lim_{\alpha \to a_\gamma} h(\alpha) = +\infty.$$

Hence there must exist an $\alpha_0 > 0$ such that

$$(\frac{\gamma}{\mu} - 1) \int_{R^2} w^2 - \alpha_0 \int_{R^2} ((L_{\gamma-1} - \alpha_0)^{-1} w) w = 0.$$

It is easy to see that this $\alpha_0 > 0$ is an eigenvalue of (3.3).

□

4. Reduction to NLEP

Let (X_ϵ, M_ϵ) be one of the two solutions constructed in Section 1. We now study the eigenvalue problem associated with (X_ϵ, M_ϵ). We assume that

$$\frac{1}{\log \frac{1}{\epsilon}} << L < \frac{1}{4}.$$

We need to analyze the following eigenvalue problem (letting $x = \epsilon y$)

$$\begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + AM \sum_{j=1}^N k_{ij} (X_j \phi_{\epsilon,i} + \phi_{\epsilon,j} X_i) \\ +A\psi_\epsilon \sum_{j=1}^N k_{ij} X_i X_j = \lambda_\epsilon \phi_{\epsilon,i}, \ y \in R^2, \\ \Delta \psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{i,j=1}^N k_{ij} X_i X_j \\ -M \sum_{i,j=1}^N k_{ij} (X_j \phi_{\epsilon,i} + X_i \phi_{\epsilon,j}) = \tau \lambda_\epsilon \psi_\epsilon, \ x \in R^2, \\ \lambda_\epsilon \in \mathcal{C}. \end{cases} \tag{4.1}$$

We assume that $(\phi_{\epsilon,1}, ..., \phi_{\epsilon,N}, \psi_\epsilon) \in (H^2(R^2))^N \oplus H^2(R^2)$. Here we equip $(H^2(R^2))^N \oplus H^2(R^2)$ with the following norm

$$\|(X,u)\|^2_{(H^2(R^2))^N \oplus H^2(R^2)} = \|X(y)\|^2_{(H^2(R^2))^N} + \|u(x)\|^2_{H^2(R^2)}.$$

Since $X_i = \hat{\zeta}_i X_0$, problem (4.1) becomes

$$\begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + AMX_0 \sum_{j=1}^N k_{ij}(\hat{\zeta}_i \phi_{\epsilon,j} + \hat{\zeta}_j \phi_{\epsilon,i}) \\ +A\psi_\epsilon \hat{\zeta}_i X_0^2 = \lambda_\epsilon \phi_{\epsilon,i}, \\ \Delta \psi_\epsilon - \psi_\epsilon - \psi_\epsilon \sum_{j=1}^N \hat{\zeta}_j X_0^2 \\ -M \sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i \phi_{\epsilon,j} + \hat{\zeta}_j \phi_{\epsilon,i}) X_0 = \tau \lambda_\epsilon \psi_\epsilon. \end{cases} \tag{4.2}$$

Let us first formally derive the limiting eigenvalue problems.

Since (X_0, M) satisfies (1.11), we have

$$X_0(y) \sim (AM(0))^{-1}(1 + o(1))w(y). \tag{4.3}$$

and

$$M(0)(1 - M(0)) \sim L := \frac{\hat{\zeta}}{2\pi A^2} \epsilon^2 \log\left(\frac{1}{\epsilon}\right) \int_{R^2} w(y)^2 dy. \tag{4.4}$$

The eigenvalue problem is changed into

$$\begin{cases} \Delta_y \phi_{\epsilon,i} - \phi_{\epsilon,i} + \sum_{j=1}^N k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j} + \hat{\zeta}_j w \phi_{\epsilon,i}) \\ +\frac{1}{AM(0)^2} \psi_\epsilon \hat{\zeta}_i w^2 = \lambda_\epsilon \phi_{\epsilon,i}, \\ \Delta \psi_\epsilon - \psi_\epsilon - \frac{\psi_\epsilon}{A^2 M(0)^2} \sum_{i=1}^N \hat{\zeta}_i w^2 \\ -\frac{M}{AM(0)} \sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j} + \hat{\zeta}_j w \phi_{\epsilon,i}) = \tau \lambda_\epsilon \psi_\epsilon. \end{cases} \tag{4.5}$$

From the equation for ψ_ϵ, we formally have

$$\psi_\epsilon(0) \sim -\frac{\psi_\epsilon(0)}{A^2 M(0)^2} \frac{\epsilon^2 \log\frac{1}{\epsilon}}{2\pi} \int_{R^2} \hat{\zeta} w^2$$

$$-\frac{M(0)}{AM(0)} \frac{\epsilon^2 \log\frac{1}{\epsilon}}{2\pi} \sum_{i,j=1}^N k_{ij} \int_{R^2} (\hat{\zeta}_i w \phi_{\epsilon,j} + \hat{\zeta}_j w \phi_{\epsilon,i}).$$

This implies

$$\psi_\epsilon(0) \sim -\frac{A^{-1}}{1 + (AM(0))^{-2} \hat{\zeta} \frac{\epsilon^2 \log\frac{1}{\epsilon}}{2\pi} \int_{R^2} w^2} \frac{\epsilon^2 \log\frac{1}{\epsilon}}{2\pi} \sum_{i,j=1}^N k_{ij} \int_{R^2} (\hat{\zeta}_i w \phi_{\epsilon,j} + \hat{\zeta}_j w \phi_{\epsilon,i}).$$

By (4.4), we have

$$\psi_\epsilon(0) \sim -M(0) A^{-1} \frac{\epsilon^2 \log\frac{1}{\epsilon}}{2\pi} \sum_{i,j=1}^N k_{ij} \int_{R^2} (\hat{\zeta}_i w \phi_{\epsilon,j} + \hat{\zeta}_j w \phi_{\epsilon,i}).$$

Substituting this relation into the equation for ϕ_ϵ, we obtain the following nonlocal eigenvalue problem:

$$\Delta \phi_{\epsilon,i} - \phi_{\epsilon,i} + w\phi_{\epsilon,i} + \sum_{j=1}^N k_{ij} \hat{\zeta}_i \phi_{\epsilon,j} w \tag{4.6}$$

$$-M(0)(AM(0))^{-2}\hat{\zeta}_i\frac{\epsilon^2\log\frac{1}{\epsilon}}{2\pi}\frac{\int_{R^2}(\sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w\phi_{\epsilon,j}+\hat{\zeta}_j w\phi_{\epsilon,i})}{\int_{R^2}w^2}w^2=\lambda_\epsilon\phi_{\epsilon,i}.$$

By (4.4), we have

$$M(0)(AM(0))^{-2}\hat{\zeta}_i\frac{\epsilon^2\log\frac{1}{\epsilon}}{2\pi}=\frac{(1-M(0))\hat{\zeta}_i}{\hat{\zeta}\int_{R^2}w^2}$$

Set

$$\lim_{\epsilon\to 0}(1-M(0))\frac{\hat{\zeta}_i}{\hat{\zeta}}=(1-\eta)\frac{\hat{\zeta}_i}{\hat{\zeta}}:=\Lambda_i$$

and

$$\lim_{\epsilon\to 0}\phi_{\epsilon,i}:=\phi_i,\quad i=1,...,N.$$

Then we obtain the following nonlocal eigenvalue problem (NLEP)

$$\Delta\phi_i-\phi_i+w\phi_i+w\sum_{j=1}^N k_{ij}\hat{\zeta}_i\phi_j \tag{4.7}$$

$$-\Lambda_i\frac{\int_{R^2}(\sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w\phi_j+\hat{\zeta}_j w\phi_i)}{\int_{R^2}w^2}w^2=\lambda_0\phi_i, i=1,...,N,$$

where

$$\lambda_0=\lim_{\epsilon\to 0}\lambda_\epsilon.$$

In fact, we can rigorously prove the following separation of eigenvalues.

Theorem 4.1. *Let λ_ϵ be an eigenvalue of (4.2).*

(1) Suppose that $\lambda_\epsilon\to 0$ as $\epsilon\to 0$. Then we have $\lambda_\epsilon=0$ if ϵ is small enough and

$$(\phi_\epsilon,\psi_\epsilon)\in span\ \{(\partial_{y_1}X_\epsilon,\partial_{y_1}M_\epsilon),(\partial_{y_2}X_\epsilon,\partial_{y_2}M_\epsilon)\}.$$

(2) Suppose that $\lambda_\epsilon\to\lambda_0\neq 0$. Then λ_0 is an eigenvalue of NLEP (4.7).

Proof:

(1). If $\lambda_\epsilon\to 0$, we can proceed exactly as in Section 4 of [49]. Let us denote the linear operator on the left hand side of (4.7) as $\mathcal{L}$, where $\mathcal{L}:H^2(R^2)\to L^2(R^2)$. The key point is to prove the following lemma:

Lemma 4.2. *(1). Let ϕ be an eigenfunction of (4.7) with $\lambda_0=0$. Then we have*

$$\phi\in\mathcal{K}_0:=span\ \{\partial_{y_1}w\vec{e}_0,\partial_{y_2}w\vec{e}_0\},$$

where $\vec{e}_0 = (1, \dots, 1)^\tau$. *(This implies that Ker* $(\mathcal{L}) = \mathcal{K}_0$*.)*

(2). The operator $\mathcal{L}$ *is an invertible operator if restricted as follows*

$$\mathcal{L} : \mathcal{K}_0^{\perp,1} \to \mathcal{K}_0^{\perp,2},$$

where

$$\mathcal{K}_0^{\perp,1} = \{u \in (H^2(R^2))^N | \int_{R^2} u \partial_{y_i} w \vec{e}_0 = 0, i = 1, 2\},$$
$$\mathcal{K}_0^{\perp,2} = \{u \in (L^2(R^2))^N | \int_{R^2} u \partial_{y_i} w \vec{e}_0 = 0, i = 1, 2\}.$$

Proof: (1). Recall that $L_0 = \Delta - 1 + 2w$. It is easy to check that $\partial_{y_1} w \vec{e}_0, \partial_{y_2} w \vec{e}_0 \in$ Ker $(\mathcal{L})$. All we need to show is that the dimension of Ker $(\mathcal{L})$ is at most 2. To this end, let $\phi \in$ Ker $(\mathcal{L})$. We first show that the nonlocal term vanishes. In fact, summing all the equations together, we obtain

$$\Delta(\sum_{j=1}^N \phi_j) - (\sum_{j=1}^N \phi_j) + (1+\gamma) w (\sum_{j=1}^N \phi_j) - (1+\gamma)(1-\eta) \frac{\int_{R^2} w (\sum_{j=1}^N \phi_j)}{\int_{R^2} w^2} w^2 = 0.$$

That is

$$\Delta(\sum_{j=1}^N \phi_j - cw) - (\sum_{j=1}^N \phi_j - cw) + (1+\gamma) w (\sum_{j=1}^N \phi_j - cw) = 0, \tag{4.8}$$

where

$$c = \frac{1}{\gamma}(1+\gamma)(1-\eta) \frac{\int_{R^2} w (\sum_{j=1}^N \phi_j)}{\int_{R^2} w^2}.$$

By assumption (H2), either $\gamma = 1$ or $\gamma \neq \mu_3, \mu_4, \dots$. So we have either

$$\sum_{j=1}^N \phi_j - cw \in \text{Ker } (L_0)$$

or

$$\sum_{j=1}^N \phi_j - cw = 0.$$

In any case, we have

$$\int_{R^2} (\sum_{j=1}^N \phi_j - cw) = 0.$$

Putting this into (4.8) we get

$$\int_{R^2} w \sum_{j=1}^N \phi_j = 0,$$

since

$$\frac{1}{\gamma}(1+\gamma)(1-\eta) \neq 1.$$

Thus the nonlocal term vanishes and we obtain the following system of equations

$$\Delta\phi_i - \phi_i + w\phi_i + \sum_{j=1}^{N} b_{ij} w\phi_j = 0, \quad i = 1, ..., N.$$

Decompose

$$b_{ij} = \sum_{k,l=1}^{N} p_{ik} d_{kl} p^{-1}_{lj},$$

where d_{kl} has Jordan form (i.e., it is composed of Jordan blocks

$$\begin{pmatrix} \sigma & 1 & 0 & \cdots & 0 \\ 0 & \sigma & 1 & \cdots & 0 \\ 0 & 0 & \sigma & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & \sigma \end{pmatrix})$$

and $\sum_{k=1}^{N} p_{ik} p^{-1}_{kj} = \delta_{ij}$. Set

$$\Phi_i = \sum_{j=1}^{N} p^{-1}_{ij} \phi_j.$$

Then the operator L can be expressed in terms of Φ as follows:

$$\Delta\Phi_i - \Phi_i + w\Phi_i + \sum_{j=1}^{N} d_{ij} \Phi_j w = 0.$$

If $1 + \sigma \notin$ spec (EVP) (recall that (EVP) was defined in Lemma 3.1 (2)) then by the last line of the corresponding Jordan block we get $\Phi_i = 0$ using Lemma 3.1. Using this in the previous line we get $\Phi_{i-1} = 0$, etc. This implies all components of Φ corresponding to the Jordan block vanish.

If $\sigma = 1$ then by Lemma 3.1 we get $\Phi_i \in$ span $\left\{ \frac{\partial w}{\partial y_j} \middle| j = 1, \ldots, N \right\}$. However the $(i-1)$-th line gives

$$L_0 \Phi_{i-1} + \Phi_i = 0,$$

which is impossible since Ker $(L_0) =$ Coker (L_0).

Thus $\Phi_i = 0$. Going backwards with respect to the lines of the Jordan block, we see that

$$\Phi_i = \Phi_{i-1} = ... = \Phi_2 = 0, \quad L_0\Phi_1 = 0$$

Thus we have $\Phi_1 \in \text{Ker } (L_0)$.

In conclusion, we have proved that except for one $i \in \{1, \ldots, N\}$, where $\Phi_i \in \text{Ker } (L_0)$, for all other $i \in \{1, \ldots, N\}$, $\Phi_i = 0$. This implies that the dimension of $\mathcal{L}$ is at most 2.

This finishes the proof of (1).

(2). To show that $\mathcal{L}$ is invertible from $\mathcal{K}_0^{\perp,1} \to \mathcal{K}_0^{\perp,2}$, we just need to show that the conjugate operator of $\mathcal{L}$ – denoted by $\mathcal{L}^*$ – has the kernel $\mathcal{K}_0$. In fact, let $\phi \in ker(\mathcal{L}^*)$. Then we have

$$\Delta\phi_i - \phi_i + w\phi_i + w\sum_{j=1}^N k_{ji}\hat{\zeta}_j\phi_j$$

$$-\frac{\int_{R^2} w^2 \sum_{i=1}^N \Lambda_i\phi_i(1+\sum_{j=1}^N k_{ji}\hat{\zeta}_j)}{\int_{R^2} w^2} w = 0, i = 1, ..., N.$$

Recall that

$$\Lambda_i = \frac{(1-\eta)\hat{\zeta}_i}{\hat{\zeta}}.$$

Let

$$\hat{\zeta}_i\phi_i = \hat{\phi}_i.$$

By assumption (H1) we have

$$\Delta\hat{\phi}_i - \hat{\phi}_i + w\hat{\phi}_i + w\sum_{j=1}^N k_{ji}\hat{\zeta}_i\hat{\phi}_j$$

$$-\hat{\zeta}_i\frac{\int_{R^2}(1-\eta)w^2 \sum_{i=1}^N \hat{\phi}_i(1+\gamma)}{\hat{\zeta}\int_{R^2} w^2} w = 0, i = 1, ..., N.$$

Summing all the equation together, we have

$$\Delta\sum_{i=1}^N \hat{\phi}_i - \sum_{i=1}^N \hat{\phi}_i + 2w\sum_{i=1}^N \hat{\phi}_i - (1-\eta)(1+\gamma)\frac{\int_{R^2} w^2 \sum_{i=1}^N \hat{\phi}_i}{\int_{R^2} w^2} w = 0. \tag{4.9}$$

Multiplying (4.9) by w and then integrating over R^2, we obtain

$$(1-(1-\eta)(1+\gamma))\int_{R^2} w^2 \sum_{i=1}^N \hat{\phi}_i = 0$$

By the assumption (H3) or (H4), $(1-\eta)(1+\gamma) \neq 1$, and

$$\int_{R^2} w^2 \sum_{i=1}^{N} \hat{\phi}_i = 0.$$

That is the nonlocal term vanishes. The rest of the proof of Theorem 4.1 is similar to (1) since spec $\mathcal{B}) =$ spec $(\mathcal{B}^\tau)$.

□

The rest of the proof is exactly the same as in Section 4 of [49]. For the sake of limited space, we omit the details here. However, we shall sketch it in the appendix.

5. Analysis of NLEP

In this section we analyze the nonlinear eigenvalue problem (NLEP) which we have obtained in Section 4. We will discuss the case of $X_\epsilon^s, M_\epsilon^s$ in detail and prove stability in certain situations. Modifying the argument it can easily be seen that the solution $X_\epsilon^l, M_\epsilon^l$ is always unstable.

By Lemma 3.1, it is enough to exclude the eigenvalues of (4.7) with Re $(\lambda_0) \geq 0$ and $\lambda_0 \neq 0$.

We first take care of the nonlocal terms.

Adding these equation for $i = 1, \ldots, N$, we get

$$\Delta(\sum_{i=1}^{N} \phi_i) - (\sum_{i=1}^{N} \phi_i) + (1+\gamma) w (\sum_{i=1}^{N} \phi_i)$$

$$-(1+\gamma)(1-\eta) \frac{\int_{R^2} (\sum_{i=1}^{N} \phi_i w)}{\int_{R^2} w^2} w^2 = \lambda_0 \phi_i.$$

Since $(1+\gamma)(1-\eta) > 1 + \sqrt{1-\gamma}$ by Lemma 3.3 we have

$$\sum_{i=1}^{N} \phi_i = 0 \qquad \text{if Re } (\lambda_0) \geq 0. \tag{5.1}$$

Therefore the nonlocal terms in (NLEP) all vanish. We end up with the following:

$$\Delta \phi_i - \phi_i + w \phi_i + \sum_{j=1}^{N} b_{ij} \phi_j w = \lambda_0 \phi_i. \tag{5.2}$$

To finish the proof we have to transform this to Jordan form. We will see that the stability of (NLEP) can be expressed in terms of the eigenvalues of $\mathcal{B}$.

Decompose

$$b_{ij} = \sum_{k,l=1}^{N} p_{ik} d_{kl} p_{lj}^{-1},$$

where d_{kl} has Jordan form (i.e., it is composed of Jordan blocks

$$\begin{pmatrix} \sigma & 1 & 0 & \cdots & 0 \\ 0 & \sigma & 1 & \cdots & 0 \\ 0 & 0 & \sigma & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & \sigma \end{pmatrix})$$

and $\sum_{k=1}^{N} p_{ik} p_{kj}^{-1} = \delta_{ij}$. Set

$$\Phi_i = \sum_{j=1}^{N} p_{ij}^{-1} \phi_j.$$

Then (NLEP) can be expressed in terms of Φ as follows:

$$\Delta \Phi_i - \Phi_i + w \Phi_i + \sum_{j=1}^{N} d_{ij} \Phi_j \frac{\hat{\zeta}}{N} w = \lambda_0 \Phi_i.$$

We have the following theorem.

Theorem 5.1. *(1) Assume that $\sigma = 1$ is a simple eigenvalue of b_{ij} and that all other eigenvalues σ of b_{ij} satisfy Re $\sigma \le 0$. Then (NLEP) is stable.*

(2) Assume that the eigenvalue $\sigma = 1$ of b_{ij} is not simple or there exists $\sigma > 0$ with $\sigma \neq 1$. Then (NLEP) is unstable.

Proof. We have to study the eigenvalue problems for each Jordan block.

For stability our argument basically is as follows: Suppose that λ_0 is an eigenvalue with Re $(\lambda_0) \ge 0$. Then for the corresponding components of the eigenfunction Φ we conclude that they vanish. This is a contradiction. Therefore λ_0 can not be an eigenvalue.

Assume that σ with Re $\sigma \le 0$ is a simple eigenvalue of $\mathcal{B}$. Suppose that the corresponding i-th component Φ_i of the eigenfunction satisfies

$$\Delta \Phi_i - \Phi_i + (1+\sigma) w \Phi_i = \lambda_0 \Phi_i \tag{5.3}$$

with Re $(\lambda_0) \geq 0$. Then from Lemma 3.1 we know that $\Phi = 0$. This is a contradiction. Therefore Re $(\lambda_0) \leq -c_0 < 0$. We have stability. We argue in the same way if σ with Re $(\sigma) \leq 0$ has multiplicity bigger than 1 and is semi-simple. Suppose now that the multiplicity of the eigenvalue σ with Re$(\sigma) \leq 0$ of $\mathcal{B}$ is larger than 1 and it is not semi-simple. Then we end up with the Jordan block

$$\begin{pmatrix} \sigma & 1 & 0 & \cdots & 0 \\ 0 & \sigma & 1 & \cdots & 0 \\ 0 & 0 & \sigma & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 0 & 0 & 0 & \cdots & \sigma \end{pmatrix}.$$

The eigenvalue problem corresponding to the last line is (dropping the index of the eigenfunction)

$$\Delta\Phi_i - \Phi_i + (1+\sigma)w\Phi_i = \lambda_0\Phi_i.$$

But from Lemma 3.1 we know that $\Phi_i = 0$.

Putting this into the $(i-1)$-th equation we get (for the eigenfunction Φ_{i-1})

$$\Delta\Phi_i - \Phi_i + (1+\sigma)w\Phi_{i-1} = \lambda_0\Phi_{i-1} \tag{5.4}$$

and we conclude $\Phi_{i-1} = 0$. Continuing in the same way we see that those components of Φ corresponding to the Jordan block of σ all vanish. Finally we have shown for the corresponding components that they are all zero. Therefore Re $\lambda_0 \geq 0$ is not possible for Re $\sigma \leq 0$. We must have Re $\lambda_0 \leq -c_0 < 0$. We get stability of (NLEP).

By assumption we know that $\sigma = 1$ is an eigenvalue of $\mathcal{B}$ with eigenvector $\vec{e}_0$. After transformation (5.3) has an eigenvalue $\lambda_0 = 1$ with corresponding eigenfunction $\Phi_i = w$. However, condition (5.1) is equivalent to $\Phi_i = 0$. This excludes the eigenfunction w. If $\sigma = 1$ is a simple eigenvalue we get stability of (NLEP).

If $\sigma = 1$ is a multiple eigenvalue we get from (5.4) $\Phi_{i-1} = w$ with corresponding eigenvalue $\lambda_0 = 1$ and (NLEP) becomes unstable. To summarize, if $\sigma = 1$ is a simple eigenvalue of b_{ij} we have stability of (NLEP). However, if the multiplicity of $\sigma = 1$ is strictly greater than 1, then (NLEP) becomes unstable.

(2) If $\sigma > 0$ with $\sigma \neq 1$ is an eigenvalue of b_{ij} then by Lemma 3.1 (3) the eigenvalue problem

$$\Delta\Phi_i - \Phi_i + (1+\sigma)w\Phi_i = \lambda_0\Phi_i$$

admits a positive real eigenvalue. This results in instability of (NLEP). Theorem 5.1 is proved. □

6. The Appendix: Proof of Theorem 4.1

In this appendix, we shall give a proof of Theorem 4.1 (1) by using Lemma 4.2. This is similar to Section 4 of [50]. We shall give a sketch.

The purpose of this section is to study the small eigenvalues of (4.2):

$$\begin{cases} \Delta_y\phi_{\epsilon,i} - \phi_{\epsilon,i} + AMX_0\sum_{j=1}^N k_{ij}(\hat{\zeta}_i\phi_{\epsilon,j} + \hat{\zeta}_j\phi_{\epsilon,i}) \\ +A\psi_\epsilon\hat{\zeta}_i X_0^2 = \lambda_\epsilon\phi_{\epsilon,i}, \\ \Delta\psi_\epsilon - \psi_\epsilon - \psi_\epsilon\sum_{j=1}^N \hat{\zeta}_j X_0^2 \\ -M\sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i\phi_{\epsilon,j} + \hat{\zeta}_j\phi_{\epsilon,i})X_0 = \tau\lambda_\epsilon\psi_\epsilon. \end{cases}$$

Assume that $\lambda_\epsilon \to 0$. Moreover, we consider $(X, M) = (X^s, M^s)$ only. It is easy to see that $(\Phi_i^l, \Psi^l) := A\eta(\hat{\zeta}_i\frac{\partial X_0}{\partial x_l}, \frac{\partial M}{\partial x_l}), \quad i = 1,\dots,N,\ l = 1,2$ are solutions of (4.2) with $\lambda_\epsilon = 0$. We also denote this solution by (Φ^l, Ψ^l). Since X_0, M are radially symmetric functions, we have that $(\Phi^1, \Psi^1) \perp (\Phi^2, \Psi^2)$ in $(L^2(R^2))^N \oplus L^2(R^2)$. Here we equip $(L^2(R^2))^N \oplus L^2(R^2)$ with the following inner product

$$< (X_1, M_1), (X_2, M_2) >= \epsilon^{-2}\int_{R^2}\sum_{i=1}^N (X_1)_i(X_2)_i\, dx + \int_{R^2} M_1M_2\, dx.$$

We denote

$$\|(X, M)\|^2 =< (X, M), (X, M) > .$$

Again we let $x = \epsilon y$. The proof of Theorem 4.1 (1) consists of the following steps:

Step 1: We first decompose $(\phi_\epsilon, \psi_\epsilon)$ as follows

$$\phi_{\epsilon,i}^l = a_1\epsilon A\eta\hat{\zeta}_i\frac{\partial X_0}{\partial x_1} + a_2\epsilon A\eta\hat{\zeta}_i\frac{\partial X_0}{\partial x_2} + \phi_{\epsilon,i}^\perp, \tag{6.1}$$

$$\psi_\epsilon = a_1 \epsilon A \eta \frac{\partial M}{\partial x_1} + a_2 \epsilon A \eta \frac{\partial M}{\partial x_2} + \psi_\epsilon^\perp, \tag{6.2}$$

where $(\phi_\epsilon^\perp, \psi_\epsilon^\perp) = ((\phi_\epsilon^i)^\perp, \psi_\epsilon^\perp) \perp \text{span}\,\{(\Phi^1, \Psi^1), (\Phi^2, \Psi^2)\}$. We assume that

$$\|(\phi_\epsilon, \psi_\epsilon)\| = 1. \tag{6.3}$$

Since Ψ^l satisfies

$$\begin{array}{c} \Delta \Psi^l - \Psi^l - \frac{1}{A^2\eta^2} \Psi^l \sum_{j=1}^N \hat{\zeta}_j X_0^2 \\ -\frac{2}{A\eta} M \sum_{j=1}^N \hat{\zeta}_j X_0 \Phi_j^l = 0, \quad l = 1, 2, \end{array} \tag{6.4}$$

we have

$$|\Psi^l| = O(A\eta \frac{1}{\epsilon \log \frac{1}{\epsilon}}). \tag{6.5}$$

Estimate (6.5) implies

$$\int_{R^2} (A\eta\epsilon \frac{\partial M}{\partial x_l})^2 \, dx = O(A^2\eta^2 \frac{1}{(\log \frac{1}{\epsilon})^2}) = O(\epsilon^2), \tag{6.6}$$

since

$$A^2 = O(\frac{\epsilon^2 \log \frac{1}{\epsilon}}{\eta}).$$

By (6.6) and the fact that $\epsilon \frac{\partial X_0}{\partial x_l} = \frac{\partial w}{\partial y_l} + o(1)$, $l = 1, 2$, in $H^1(R^2)$, we obtain that

$$\|\epsilon \Phi_i^1\|^2 = (\hat{\zeta}_i)^2 \int_{R^2} \left(\frac{\partial w}{\partial y_1} \right)^2 dy + o(1),$$

$$\|\epsilon \Phi_i^2\|^2 = (\hat{\zeta}_i)^2 \int_{R^2} \left(\frac{\partial w}{\partial y_2} \right)^2 dy + o(1).$$

This implies

$$a_1 = O(1),\ a_2 = O(1);\ \|(\phi_\epsilon^\perp, \psi_\epsilon^\perp)\| = O(1).$$

Step 2: We now estimate ψ_ϵ. We calculate

$$\psi_\epsilon(0) = -\Big(\frac{\epsilon^2}{A^2\eta^2} (\log \frac{1}{\epsilon}) \psi_\epsilon(0) \sum_{j=1}^N \hat{\zeta}_j \int_{R^2} X_0(\epsilon y)^2 \, dy$$

$$- \frac{\epsilon^2}{A\eta} (\log \frac{1}{\epsilon}) \eta \sum_{i,j=1}^N k_{ij} \int_{R^2} (\hat{\zeta}_i \phi_{\epsilon,j} + \hat{\zeta}_j \phi_{\epsilon,i}) X_0(\epsilon y) \, dy \Big) (1 + O(\frac{1}{\log \frac{1}{\epsilon}})) + o(\|\phi_\epsilon\|^2).$$

This implies

$$\frac{1}{A\eta^2} \psi_\epsilon(0) = - \frac{(1 - M(0)) \sum_{i,j=1}^N k_{ij} \int_{R^2} (\hat{\zeta}_i \phi_{\epsilon,j} + \hat{\zeta}_j \phi_{\epsilon,i}) X_0(\epsilon y) \, dy}{A\eta \sum_{j=1}^N \hat{\zeta}_j \int_{R^2} X_0(\epsilon y)^2 \, dy}$$

$$\times(1+O(\frac{1}{\log\frac{1}{\epsilon}}))+o(\|\phi_\epsilon\|_{L^2_y})$$

$$=-\frac{(1-M(0))\sum_{i,j=1}^N k_{ij}\int_{R^2}(\hat{\zeta}_i\phi^\perp_{\epsilon,j}+\hat{\zeta}_j\phi^\perp_{\epsilon,i})X_0(\epsilon y)\,dy}{A\eta\sum_{j=1}^N\hat{\zeta}_j\int_{R^2}X_0(\epsilon y)^2\,dy}$$

$$\times(1+O(\frac{1}{\log\frac{1}{\epsilon}})+o(\|\phi_\epsilon\|_{L^2_y})$$

$$=O(\|\phi^\perp_\epsilon\|_{L^2_y})+o(\|\phi_\epsilon\|_{L^2_y})$$

and

$$\frac{1}{A\eta^2}(\psi_\epsilon(x)-\psi_\epsilon(0))=O(\frac{1}{\eta\log\frac{1}{\epsilon}}\|\Phi_\epsilon\|_{L^2_y}\log(1+\frac{|x|}{\epsilon})). \qquad (6.7)$$

Step 3: From (4.5) we see that the equation for $(\phi^\perp_\epsilon,\psi^\perp_\epsilon)$ is

$$\begin{cases}\Delta_y\phi^\perp_{\epsilon,i}-\phi^\perp_{\epsilon,i}+\sum_{j=1}^N k_{ij}(\hat{\zeta}_i w\phi^\perp_{\epsilon,j}+\hat{\zeta}_j w\phi^\perp_{\epsilon,i})\\ +\frac{1}{AM(0)^2}\psi^\perp_\epsilon\hat{\zeta}_i w^2=\lambda_\epsilon\phi^\perp_{\epsilon,i}+\lambda_\epsilon(a_1\epsilon A\eta\hat{\zeta}_i\frac{\partial X_0}{\partial x_1}+a_2\epsilon A\eta\hat{\zeta}_i\frac{\partial X_0}{\partial x_2}),\\ \Delta\psi^\perp_\epsilon-\psi^\perp_\epsilon-\frac{\psi^\perp_\epsilon}{A^2M(0)^2}\sum_{i=1}^N\hat{\zeta}_i w^2-\frac{M}{AM(0)}\sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w\phi^\perp_{\epsilon,j}+\hat{\zeta}_j w\phi^\perp_{\epsilon,i})\\ =\tau\lambda_\epsilon\psi^\perp_\epsilon+\tau\lambda_\epsilon(a_1\epsilon A\eta\frac{\partial M}{\partial x_1}+a_2\epsilon A\eta\frac{\partial M}{\partial x_2}).\end{cases}$$

Now we study the equation for $\psi^\perp_\epsilon$. By the representation formula,

$$\psi^\perp_\epsilon(x)=-\epsilon^2\frac{1}{A^2\eta^2}\int_{R^2}K_\beta(|x-\epsilon y|)\psi^\perp_\epsilon\sum_{i=1}^N\hat{\zeta}_i w^2$$

$$-\epsilon^2\frac{1}{A\eta}M(0)\int_{R^2}K_\beta(|x-\epsilon y|)\sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w\phi^\perp_{\epsilon,j}+\hat{\zeta}_j w\phi^\perp_{\epsilon,i})$$

$$-\tau\lambda_\epsilon\int_{R^2}K_\beta(|x-z|)(a_1\epsilon A\eta\frac{\partial M}{\partial x_1}+a_2\epsilon A\eta\frac{\partial M}{\partial x_2})\,dz$$

$$=-E_1(x)-E_2(x)-E_3(x),$$

where $K_\beta(|x-z|)=K(\beta|x-z|)$ is the fundamental solution of $-\Delta+\beta^2$ in R^2, $\beta^2=1+\tau\lambda_\epsilon=1+o(\epsilon)$, and E_i, $i=1,2,3$, are defined by the last equality.

We now estimate each of these terms. First,

$$E_1(0)=-\epsilon^2\log\frac{1}{\epsilon}\frac{1}{A^2\eta^2}\psi^\perp_\epsilon(0)\sum_{i=1}^N\hat{\zeta}_i\int_{R^2}w^2\,dy$$

$$=-\frac{1-\eta}{\eta}\psi^\perp_\epsilon(0).$$

Furthermore, we have

$$E_1(x) - E_1(0) = O(\frac{1}{\eta \log \frac{1}{\epsilon}} \psi_\epsilon^\perp(0) \log(1+|y|)).$$

Here we have used the lemma

Lemma 6.1. *Let $g(y)$ be a function in $L^2(R^2)$ such that*

$$|g(y)| \leq Ce^{-a|y|}.$$

Then we have

$$|\int_{R^2} \log \frac{|y-\bar{z}|}{|\bar{z}|} g(\bar{z}) | d\bar{z}| \leq C \log(1+|y|).$$

Proof: This follows from standard potential analysis. See e.g., [6]. □

For E_2, we have

$$E_2(0) = A\eta(1-\eta) \frac{\int_{R^2} \sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j}^\perp + \hat{\zeta}_j w \phi_{\epsilon,i}^\perp)}{\hat{\zeta} \int_{R^2} w^2 \, dy,}$$

$$E_2(x) - E_2(0) = O(A\eta \frac{1}{\log \frac{1}{\epsilon}} \|\phi_\epsilon^\perp\|_{L^2} \log(1+\frac{|x|}{\epsilon})).$$

E_3 can be estimated as follows: E_3 satisfies the equation

$$\Delta E_3 - \beta^2 E_3 = \tau \lambda_\epsilon (a_1 \epsilon A \eta \frac{\partial M}{\partial x_1} + a_2 \epsilon A \eta \frac{\partial M}{\partial x_2})$$

in R^2. Hence,

$$|E_3| \leq |\tau \lambda_\epsilon \beta^{-2}| |a_1 \epsilon A \eta \frac{\partial M}{\partial x_1} + a_2 \epsilon A \eta \frac{\partial M}{\partial x_2}|_{L^\infty}$$

$$= O(|\lambda_\epsilon| A\eta \frac{1}{\log \frac{1}{\epsilon}} (|a_1| + |a_2|)) \quad \text{(by (6.5))}.$$

Combining the estimates for E_i, $i = 1, 2, 3$, we have

$$\psi_\epsilon^\perp(0) = A\eta^2 2(1-\eta)(1+o(1)) \frac{\int_{R^2} \sum_{i,j=1}^N k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j}^\perp + \hat{\zeta}_j w \phi_{\epsilon,i}^\perp)}{\hat{\zeta} \int_{R^2} w^2 \, dy} \tag{6.8}$$

and

$$\frac{1}{A\eta^2}(\psi_\epsilon^\perp(x) - \psi_\epsilon^\perp(0)) = O(\eta \frac{1}{\log \frac{1}{\epsilon}} \|\phi_\epsilon^\perp\|_{L_y^2} \log(1+\frac{|x|}{\epsilon}))$$

$$+O(\eta \frac{1}{\log \frac{1}{\epsilon}} (|a_1| + |a_2|)).$$

Substituting this into the equation for $\phi_{\epsilon,i}^{\perp}$, we get

$$\Delta_y \phi_{\epsilon,i}^{\perp} - \phi_{\epsilon,i}^{\perp} + \sum_{j=1}^{N} k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j}^{\perp} + \hat{\zeta}_j w \phi_{\epsilon,i}^{\perp}) - \tilde{\mu} \frac{\int_{R^2} \sum_{i,j=1}^{N} k_{ij}(\hat{\zeta}_i w \phi_{\epsilon,j}^{\perp} + \hat{\zeta}_j w \phi_{\epsilon,i}^{\perp})}{\hat{\zeta} \int_{R^2} w^2 \, dy} \hat{\zeta}_i w^2$$

$$+ o(\|\phi_{\epsilon}^{\perp}\|^2)$$

$$= \lambda_\epsilon \phi_{\epsilon,i}^{\perp} + \lambda_\epsilon (a_1 \epsilon A \eta \hat{\zeta}_i \frac{\partial X_0}{\partial x_1} + a_2 \epsilon A \eta \hat{\zeta}_i \frac{\partial X_0}{\partial x_2})$$

$$+ O\left(\eta \frac{1}{\log \frac{1}{\epsilon}} \|\phi_{\epsilon}^{\perp}\|_{L_y^2} \log(1 + \frac{|x|}{\epsilon}) + \eta \frac{1}{\log \frac{1}{\epsilon}} (|a_1| + |a_2|) \right) w^2,$$

where

$$\tilde{\mu} = (1 - \eta + o(1)).$$

By our assumption

$$\epsilon^{-2} \sum_{i=1}^{N} \int_{R^2} \phi_{\epsilon,i}^{\perp} A \eta \frac{\partial X_0}{\partial x_l} \, dx + \int_{R^2} \psi_{\epsilon}^{\perp} A \eta \frac{\partial M}{\partial x_l} \, dx = 0, \quad l = 1, 2,$$

which implies that that

$$\int_{R^2} \phi_{\epsilon,i}^{\perp}(\epsilon y) \frac{\partial w}{\partial y_l} \to 0, \quad l = 1, 2.$$

Therefore we get the following equation for $\phi_{\epsilon,i}^{\perp}$:

$$\tilde{L}_{\tilde{\mu},i} \phi_{\epsilon,i}^{\perp}(\epsilon y) = \lambda_\epsilon \phi_{\epsilon,i}^{\perp} + \lambda_\epsilon (a_1 \epsilon A \eta \hat{\zeta}_i \frac{\partial X_0}{\partial x_1} + a_2 \epsilon A \eta \hat{\zeta}_i \frac{\partial X_0}{\partial x_2}) \tag{6.9}$$

$$+ O(\eta \frac{1}{\log \frac{1}{\epsilon}} \|\phi_{\epsilon}^{\perp}\|_{L_y^2} \log(1 + \frac{|x|}{\epsilon}) + \eta \frac{1}{\log \frac{1}{\epsilon}} (|a_1| + |a_2|)) w^2,$$

where

$$\tilde{L}_{\tilde{\mu},i} \phi = \Delta_y \phi - \phi + \sum_{j=1}^{N} k_{ij}(\hat{\zeta}_i w \phi_j + \hat{\zeta}_j w \phi_i) - \tilde{\mu} \frac{\int_{R^2} \sum_{i,j=1}^{N} k_{ij}(\hat{\zeta}_i w \phi_j + \hat{\zeta}_j w \phi_i)}{\hat{\zeta} \int_{R^2} w^2 \, dy} \hat{\zeta}_i w^2$$

and

$$\int_{R^2} \phi_{\epsilon,i}^{\perp}(\epsilon y) \frac{\partial w}{\partial y_l} = o(1), \quad l = 1, 2. \tag{6.10}$$

Note that the linear operator on the left hand side of (6.9) is asymptotically close to the limit linear operator $\mathcal{L}$ in (4.7). Furthermore, from (6.10) we know that $\phi_{\epsilon,i}^{\perp}$ is almost perpendicular to Ker ($\mathcal{L}$). By a perturbation argument similar to Lyapunov-Schmidt reduction (compare Lemma 4.2 of

[50]), which is based on Lemma 4.2 of the present paper, we can invert the equation (6.9) to get

$$\|\phi_\epsilon^\perp\|_{H^2(R^2)} \le C\|\pi_\epsilon \circ \tilde{L}_{\tilde{\mu}}\phi_\epsilon^\perp\|_{L^2_y}$$
$$= O(\eta\frac{1}{\log\frac{1}{\epsilon}}(\|\phi_\epsilon^\perp\|_{L^2_y} + |a_1| + |a_2|)),$$

where π_ϵ is the projection of L^2 onto $(\mathrm{span}\{\epsilon\frac{\partial X_0}{\partial x_1}, \epsilon\frac{\partial X_0}{\partial x_2}\})^\perp$ (componentwise). This implies

$$\|\phi_\epsilon^\perp\|_{H^2(R^2)} = O(\eta\frac{1}{\log\frac{1}{\epsilon}}(|a_1| + |a_2|)). \tag{6.11}$$

From (6.8) and (6.11), we get

$$\frac{1}{A\eta^2}|\psi_\epsilon^\perp(x)| = O(\eta\frac{1}{\log\frac{1}{\epsilon}}(|a_1| + |a_2|)). \tag{6.12}$$

Step 4: Multiplying the equation for $\phi_{\epsilon,i}^\perp$ by $\frac{\partial X_0}{\partial x_1}$ and integrating over R^2, we have

$$\lambda_\epsilon\hat{\zeta}_i(\int_{R^2} a_1\epsilon A\eta(\frac{\partial X_0}{\partial x_1})^2\,dx) + \lambda_\epsilon\int_{R^2}\phi_{\epsilon,i}^\perp\frac{\partial X_0}{\partial x_1}\,dx \tag{6.13}$$

$$= \int_{R^2}\frac{\partial X_0}{\partial x_1}[\epsilon^2\Delta_x\phi_{\epsilon,i}^\perp - \phi_{\epsilon,i}^\perp + \frac{1}{A\eta^2}\psi_\epsilon^\perp\hat{\zeta}_i w^2 + \sum_{j=1}^N k_{ij}(\hat{\zeta}_i w\phi_{\epsilon,j} + \hat{\zeta}_j w\phi_{\epsilon,i})]\,dx + o(1).$$

The left hand side of (6.13) is

$$\text{l.h.s.} = \epsilon\lambda_\epsilon\hat{\zeta}_i(\int_{R^2} a_1(\frac{\partial w}{\partial y_1})^2\,dy + o(a_1)) + O(\epsilon\lambda_\epsilon(|a_1| + |a_2|)\frac{1}{\eta\log\frac{1}{\epsilon}}).$$

The right hand side of (6.13) is

$$\text{r.h.s.} = \int_{R^2}[\frac{\partial X_0}{\partial x_1}\frac{1}{A\eta^2}\psi_\epsilon^\perp\hat{\zeta}_i w^2 - \phi_{\epsilon,i}^\perp\frac{1}{\eta}\frac{\partial M}{\partial x_1}\hat{\zeta}_i w^2]\,dy$$
$$= O(\epsilon(|a_1| + |a_2|)\frac{1}{\eta\log\frac{1}{\epsilon}})$$

(by (6.12), (6.11), (6.5)).

Hence, from (6.13) we obtain

$$\lambda_\epsilon|a_1| \le O((|a_1| + |a_2|)\frac{1}{\eta\log\frac{1}{\epsilon}}). \tag{6.14}$$

In the same way, we have

$$\lambda_\epsilon|a_2| \le O((|a_1| + |a_2|)\frac{1}{\eta\log\frac{1}{\epsilon}}). \tag{6.15}$$

From (6.14) and (6.15), we get

$$\lambda_\epsilon(|a_1| + |a_2|) \le O(\frac{1}{\eta \log \frac{1}{\epsilon}}(|a_1| + |a_2|)|). \tag{6.16}$$

Now (6.16) implies that

$$\lambda_\epsilon(|a_1| + |a_2|) = 0$$

if ϵ is small enough. Thus (6.12) and (6.11) give

$$\phi_\epsilon^\perp \equiv 0, \quad \psi_\epsilon^\perp \equiv 0.$$

In conclusion, we have

$$\lambda_\epsilon = 0, \quad (\phi_\epsilon, \psi_\epsilon) \in \mathrm{span}\{(\Phi^1, \Psi^1), (\Phi^2, \Psi^2)\}.$$

The proof of Theorem 4.1 (1) is now completed.

The proof of Theorem 4.1 (2) uses very similar estimates and is omitted.

□

7. Discussion

We have studied a general system of $N + 1$ equations describing the interaction of N polymer species which catalyse each other in a cyclic way and are all composed of the same type of monomer. In the special case $N = 1$ the system reduces to the well-known Gray-Scott system.

Although there have rigorous been results in 1-D and formal results in 2-D on existence and stability of concentrated solutions these are first rigorous results in 2-D. We study the case of single-cluster solutions in the whole 2-D space. These are in some sense the simplest concentrated solutions in 2-D. This case appears to be relevant if the early biochemical reactions take place in very thin layers for example on the surface of rocks.

At this point we would like to summarize the various conditions we put on the coupling matrix $\mathcal{K}$. We start with the elementary hypercycle which is given explicitly on page 2. The assumptions for the existence result (Theorem 1.1) are more general: We merely assume that $\mathcal{K}$ is invertible and positive in some sense given in equation (1.6). This condition determines the relative concentration of different polymers uniquely. Thus the system reduces to a system of just two equations and existence follows by existence results on the

Gray-Scott system. The existence result gives two types of solutions: Large ones and small ones.

Regarding stability the story is not so easy: The problem is truly $(N+1)$-dimensional. Stability of solutions is determined by the spectrum of certain nonlocal eigenvalue problems in N variables which essentially depends on the spectrum of the matrix $\mathcal{K}$. These nonlocal eigenvalue problems are derived in Section 4 (with some technicalities postponed to Section 6, The Appendix) and analyzed in Section 5 (with the help of a few lemmas which are proved in Section 3). To make any treatment possible the additional conditions (H1) – (H4) on the matrix $\mathcal{K}$ and the closely related matrix $\mathcal{B}$ are assumed. Interestingly enough for the hypercyclical system the conditions (H1) – (H4) are satisfied. The same is true for (cyclical) bidiagonal and tridiagonal matrices $\mathcal{B}$ (see Section 2).

Under these assumptions the stability result reveals that the small solution is stable if $N \leq 4$. On the other hand, we show that the small solution is unstable if N is big enough. We do know the exact threshold value of N for which stability turns into instability. We also show that the large solution is always unstable.

Finally, let us recall attention to the point made in the introduction numerically it is known that parasites may destroy stable cluster states. Our results complement the picture by the rigorously proved fact that even pure cluster states may turn unstable if they become two large. This implies that the hypercycle although it has some very preferable properties (see the beginning of the introduction) on the other hand it has an inherent instability behaviour which may be an obstruction to the evolution of large biological systems.

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