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QUATERNIONS APPLIED TO PHYSICS IN NON-EUCLIDEAN SPACE.

I.—THE MATHEMATICAL METHODS.

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§1. The utility of quaternions in mathematical physics in non-Euclidean space is much the same as in Euclidean, that is to say they are suitable for establishing fundamental relations. Details must be worked out by some system of scalar coordinates. It is hoped that the applications, to Physics, for which the methods of this paper have been prepared, will appear in subsequent papers; but, quite naturally, it has been found that the methods alone demand for exposition more space than can be placed at the writer's disposal for the first paper of the series. As probably considerable intervals of time will elapse between the publication of successive papers, it is well to state at the outset that the foundations of the following have already been treated:—particle dynamics, rigid dynamics, hydrodynamics, elastic solids and electrodynamics in free ether; and also four types of wave-motion, (1) in elastic fluids, (2) and (3) the two types in elastic isotropic solids, (4) in free ether. As examples of the kind of results that emerge we may mention (1) that in fluids the famous vortex-motion theorems are true without any real modification (so that for instance a vortex theory of matter is just as applicable in non-Euclidean as in Euclidean space); (2) that except when wave-lengths are infinitesimal compared with the space constant, only one of the four types of wave-motion mentioned possesses velocity independent of wave-length (and therefore possesses equal velocities of propagation for waves and for groups of waves). The one of the four is wave-motion in ether.

When below we come to hyperbolic space we shall use complex quaternions [$p + p' \surd(-1)$ where p and p' are real quaternions]. If q is a complex quaternion, then, as Hamilton prescribed, the tensor $\mathrm{T}q$ and the unitat $\mathrm{U}q$ will mean $\mathrm{T}q = \surd(q\mathrm{K}q)$ and $\mathrm{U}q = q/\mathrm{T}q$, with the proviso that the scalar $\mathrm{T}q$ is that particular one of the two square roots whose real part is positive; or if the real part is zero, $\mathrm{T}q$ is I multiplied by a positive scalar, I standing for $\surd(-1)$. The arc $\mathrm{A}q$ of q will mean a complex scalar $a + \mathrm{I}a' = \cos^{-1}\mathrm{SU}q$ where a and a' are real, a ranging between 0 and π, and a' between $+\infty$ and $-\infty$; and when a is zero a' is positive. a will be called the angle of q and a' the advance. When the advance is zero the unitat is a versor; when the angle is zero the unitat is a translator; when neither angle nor advance is zero the unitat is the product of a versor and a co-axial translator.

Elliptic Space Preliminaries.

§2. The quaternions to be applied in elliptic space are never complex; real quaternions furnish a complete geometric method, and Clifford's bi-quaternions naturally come forward with a second allied method. For the present the quaternions are to be real ordinary quaternions.

Let, in the first instance, the axes of a complete system of quaternions be lines through a fixed point O in an elliptic space, and let any quaternion q of the system (in addition to its usual signification of a quotient of two vectors through O) signify a length measured along the axis of $\mathrm{V}q$ equal to $\mathrm{A}q$. If OP is in this direction and of this length, q may be called the position quaternion of P (origin O) and $\mathrm{U}q$ may be called the position unitat of P. Next assign yet another meaning to a quaternion according to the following description ; if u is the position unitat of P and v is the position unitat of Q, vu^{-1} is a unitat signifying a length PQ in the sense PQ measured along the straight line PQ ; and similarly if x is any scalar, xvu^{-1} is a quaternion signifying the same line-segment.

The following five fundamental statements are almost obvious :—

(1) $\mathrm{A}\,(vu^{-1})$ = length PQ, or what is the same thing $\mathrm{S}(vu^{-1}) = \cos \mathrm{PQ}$. For in elliptic space

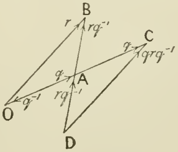
$$\cos \mathrm{PQ} = \cos \mathrm{OP} \,.\, \cos \mathrm{OQ} + \sin \mathrm{OP} \,.\, \sin \mathrm{OQ} \,.\, \cos \mathrm{POQ}$$

and we have $\mathrm{S}(v\mathrm{K}u) = \mathrm{S}u \,.\, \mathrm{S}v - \mathrm{S} \,.\, \mathrm{V}u\mathrm{V}v.$

[Important note on the establishment of our methods. Virtually all our proofs are based on the two cosine formulae for the two spaces. Certainly many tacit geometric assumptions are made below, (such as those referring to common perpendiculars between lines and planes), but these, did space permit, could easily be stated explicitly and proved by present methods.]

(2) If P is taken as origin in place of O, the new position unitat of Q is understood to be vu^{-1} If w is the position unitat (origin O) of a third point R ; then whether we take O for origin or P the line-segment QR is signified by the same unitat. For $wu^{-1}\,(vu^{-1})^{-1} = wv^{-1}$.

(3) Two interpretations of qrq^{-1}. If q and r are taken to represent quaternions (or line-segments) with axes through O there is first the usual interpretation of qrq^{-1} as a third quaternion with axis through O. It is what r becomes by conical rotation round the axis of q through an angle $2\mathrm{A}q$. But the diagram (in which the lines mean straight lines and OAC and DAB are both bisected at A so that the lines are all in one plane) shows that qrq^{-1} may also be interpreted as the line-segment (DC in the diagram) obtained by translating the line-segment r through O (OB in the diagram) along the axis of q (in the diagram OA is the line-segment q through O) through a distance equal to $2\mathrm{A}q$. The



line-segments, one through O, the other through C, each of which is represented by qrq^{-1}, are left parallels in Clifford's sense. This shows the connection between the (doubly infinite) several line-segment meanings we have assigned to a quaternion p. They have equal tensors and arcs and their axes are any left parallels. rq^{-1} represents AB in the diagram ; therefore it represents the left parallel of AB through O ; therefore the *right* parallel of AB through O (obtained from the left parallel by conical rotation round OA) must be $q^{-1}(rq^{-1})q = q^{-1}r$. Thus the two parallels through O, left and right, of the line-segment from the point q (origin O) to the point r, are rq^{-1} and $q^{-1}r$ respectively.

(4) If Q (position unitat, v), a given point, and P (position unitat, u) a variable point, are such that $S(vu^{-1}) = 0$, P is quadrantally distant from Q, that is to say the locus of P is the polar plane of Q. Hence $S.pKq = 0$ (p given, q current) is the equation of the plane polar to p.

(5) If u, v, w are the position unitats of three points P, Q, R ; if x, y, z are scalars ; and if

$$xu + yv + zw = 0 ;$$

then P, Q, R are collinear, and their mutual distances satisfy the sine formula

$$x^{-1} \sin QR = y^{-1} \sin RP = z^{-1} \sin PQ.$$

For $x + yvu^{-1} + zwu^{-1} = 0$, so that $yVvu^{-1} + zVwu^{-1} = 0$.

Clearly (4) and (5) show that one application of the present method is very similar to Joly's use of q as a point or plane symbol in Euclidean space (Joly's *Manual of Quaternions*, chap. xvii). As will be briefly explained below, Joly's method is much better adapted to interpretations in hyperbolic space than in Euclidean or elliptic. His (real) unit sphere has to be interpreted as the (real) absolute of hyperbolic space.

Hyperbolic Space Preliminaries.

§3. In hyperbolic space, our initial principles, as appearing in §2, of mathematical necessity, carry us on to a line calculus, a calculus precisely of the type of Clifford's bi-quaternion calculus for elliptic space.

Indeed these two line theories in hyperbolic and elliptic spaces are in a sense identical. Our formulae for hyperbolic space become formulae for elliptic space by a simple device. Let $I^2 = -1$ as in § 1 above; let E be a unit for which $E^2 = 1$ and such that E is commutative with quaternions. E is in fact Clifford's ω. Let J be put for either I or E; it being understood that when J = I our formulae have real interpretations in hyperbolic space, and when J = E in elliptic space. The reader is advised to ignore the second possibility (J = E) for the present, confining himself to J = I and hyperbolic space. After viewing the hyperbolic development let him return and observe how we might have put J = E from the beginning. The reason that the notation sinh c, cosh c, etc., is avoided below, and the equivalent notation $J^{-1} \sin Jc$, $\cos Jc$, etc., is used instead, is that thereby the formulae are left ready for interpretation with the meaning of J, J = E.

[Note to assist the reader in making the interpretation J = E. If $q = p + Ep'$ where p and p' are real quaternions so that q is a Clifford bi-quaternion, we must first, in fairly obvious ways, define, Vq, Sq, Kq, Tq, Uq, $Aq = \cos^{-1} SUq$. Putting $Aq = a + Ea'$ where a and a' are ordinary real scalars, called angle and advance; there is more difficulty than with complex quaternions in a precise and unambiguous use of these terms. SUq being $= b + Eb'$ we have $\cos a \cos a' = b$, $\sin a \sin a' = -b'$; and both b and b' may be positive or negative. Remembering that in elliptic space a twist with a and a' for advance and angle, about a given axis, is the same as a twist with a' and a for advance and angle, about the polar axis; we see that any rule for determining angle a and advance a' from b and b' ought to be ambiguous to the extent that the two are interchangeable. In view of all this I prefer the following rule:—Let one of the two a and a' be between 0 and π, and the other between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$; in the case of a pure translator or versor let the range of values be, $0, \pi$].

§4. We shall pass lightly over such parts of the treatment of hyperbolic space as are suggested by §2 above.

Let a translator

$$u = \cos Jc + \epsilon \sin Jc = \exp(\epsilon Jc)$$

where c is a real scalar and ϵ a real unit vector, represent a line-segment through a given point O of hyperbolic space ; and if OP is in the direction ϵ and of length c let u be taken as the position unitat of P (origin O). If with origin O, u, v are the position unitats of points P, Q vu^{-1} signifies the line-segment PQ. Expand vu^{-1} in full thus ; if

$$u = \cos Jc + \epsilon \sin Jc, \text{ and } v = \cos Jc' + \epsilon' \sin Jc'$$

then

$$vu^{-1} = v\mathrm{K}u$$

$$= (\cos Jc \cos Jc' - \sin Jc \sin Jc' \, \mathrm{S}\epsilon\epsilon')$$
$$+ (-\epsilon \sin Jc \cos Jc' + \epsilon' \sin Jc' \cos Jc$$
$$+ \mathrm{V}\epsilon\epsilon' \sin Jc \sin Jc')$$
$$= \cos Jc'' + \epsilon'' \sin Jc''$$

where c'' is a real scalar and ϵ'' is a unit vector (the statement meaning that $\epsilon''^2 = -1$), which in general is complex. It is important to note that the necessary and sufficient condition for ϵ'' to be real is

$$\mathrm{V}\epsilon\epsilon' . \sin Jc . \sin Jc' = 0$$

or P, Q and the origin are collinear. Since ϵ'' is in general complex vu^{-1} does not belong to the triply infinite system of u, v. On the other hand vu^{-1} is not a general unitat since $\mathrm{S}.vu^{-1}$ is real. It is in fact the general form of a translator ; satisfying the single scalar relation that its angle is zero.

§5. This property that vu^{-1} belongs to a quintuply infinite system and not to the triply infinite system of u, v is of course an important distinction from the case for elliptic space of §2 above. Nevertheless we have five fundamental statements similar to those numbered above for elliptic space.

(1) $J^{-1}\mathrm{A}(vu^{-1})$ equals distance PQ ; or, what is the same, $\mathrm{S}(vu^{-1}) = \cos(J.\mathrm{PQ})$.

(2) If w is the position unitat of a third point R ; wv^{-1} signifies the same line-segment QR, whether O or any other point P be taken as origin.

(3) uvu^{-1} is the line-segment v translated along the axis of u to a distance $2J^{-1}\mathrm{A}u$.

(4) In hyperbolic space, contrary to the case in elliptic space, we cannot have $S(vu^{-1}) = 0$ for two real points. To obtain the equation of a plane in the form $SpKq = 0$ we are led to define the position unitat of the plane through P (position unitat u) perpendicular to OP, as $uUVu$. P being given; and Q, with position unitat v, being variable; Q is on the plane provided

$$SvK(uUVu) = 0.$$

This is easily proved from (6) below by taking the origin at P, so that the position unitats of Q and O become vu^{-1} and u^{-1}. Expressing that cos QPO = 0 we get the the equation just written. The characteristic of such a position unitat of a plane is that the angle has the definite value $\frac{1}{2}\pi$; for a point the definite value was zero. Taking $u = \exp(\varepsilon Jc)$ we have in full

$$uUVu = \varepsilon \exp(\varepsilon Jc) = -\sin Jc + \varepsilon \cos Jc$$
$$= \exp(\varepsilon[\tfrac{1}{2}\pi + Jc])$$

from which we see that the scalar is a negative pure imaginary and the vector is real. The plane passes through the origin when c is zero. In this case its position unitat is the real vector ε.

(5) If u, v, w are the position unitats of three points P, Q, R; if x, y, z are scalars; and if

$$xu + yv + zw = 0$$

then P, Q, R are collinear and their mutual distances satisfy the sine formula

$$x^{-1} \sin (J.\,QR) = y^{-1} \sin (J.\,RP) = z^{-1} \sin (J.\,PQ).$$

§6. We have now to make some similar fundamental statements which are confined in application to the present method, being inapplicable or unnecessary with the method of §2.

(6) There is no special mathematical property distinguishing the origin O from any other origin P. At first sight this statement seems inconsistent with the prescription that vector parts of line-segments through O are pure imaginaries, whereas with other origins this is not so. But this is a mere question of terms, not one of contained mathematical meaning. If we translate i, j, k, three real rectangular unit vectors

through O, to the point whose position quaternion is q^2 they become

$$i' = qiq^{-1}, \quad j' = qjq^{-1}, \quad k' = qkq^{-1}.$$

If i, j, k are called real we must call i', j', k' complex because q is complex. But the calling one set real and the other complex is a mere naming of the two sets and does not imply any difference of contained meanings. All properties of the set i, j, k in the first place and of i', j', k' in the second are based on

$$i^2 = j^2 = k^2 = ijk = -1$$

and $$i'^2 = j'^2 = k'^2 = i'j'k' = -1$$

and no alteration would occur in any application if we called i', j', k' real and therefore i, j, k complex.

(7) When the origin is shifted to the point u_0, we have seen that the position unitat v of a point changes to vu_0^{-1}. The same rule holds for the position unitat v' of a plane; it likewise changes to $v'u_0^{-1}$; for the equation of the plane, v' given, is $Suv'^{-1} = 0$, or

$$S.(uu_0^{-1})\,(v'u_0^{-1})^{-1} = 0.$$

Although a change occurs in the position unitats themselves, no change occurs in the ratio of any two of them. Hence in interpreting the meaning of any such ratio we may take the origin wherever we please. Thus if $v'u'^{-1} = vu^{-1}$ where u, v, u', v' are the position unitats of four points, how are the points related? Take the point u for origin so that u becomes unity and v is of the form $\cos Jc'' + \epsilon'' \sin Jc''$ where ϵ'' passes through the origin and may be considered real. Thus

$$v'u'^{-1} = \cos Jc'' + \epsilon'' \sin Jc''.$$

Now when using ϵ'' above we saw that this implied that the points v', u' and the origin were collinear. Thus the original three v', u', u are collinear, and similarly v', u' and v are collinear. Hence regarding u and v as given, when $v'u'^{-1} = vu^{-1}$ the points u', v' are both on the given straight line joining the points u, v. And their distance apart is the given distance of u, v apart; because $Sv'u'^{-1} = Svu^{-1}$. This discussion shows that our present method is primarily a calculus of lines and not one of points and planes. To adapt it to points and planes an origin has to be selected.

(8) If u and v are intersecting planes take the origin on their line of intersection. If they are non-intersecting planes take the the origin at the point where their common perpendicular meets one of them. [If they are parallel planes $V.vu^{-1}$ as a matter of fact is a nullitat and $S.vu^{-1} = 1$; proof of which is left to reader.] If either u or v is a point take that point for origin. Then the following interpretations are rendered evident. (i) When u and v are intersecting planes vu^{-1} is an ordinary versor whose axis is definitely fixed in space (the axis is the line of intersection) and whose arc is the angle between the planes. (ii) When u and v are non-intersecting planes or when u and v are two points vu^{-1} is a translator whose axis is the common perpendicular of the planes or the line joining the points, and whose advance is the distance between the planes or the points. (iii) When u and v are, the one a point and the other a plane, vu^{-1} is a unitat whose angle is $\frac{1}{2}\pi$, whose advance is the distance between the point and plane, and whose axis is the perpendicular from point to plane. It will be seen that a unit vector, with definite axis fixed in space, occurs twice among these interpretations. Under (i) it occurs as the ratio of two intersecting perpendicular planes; under (iii) it occurs as the ratio of a plane and an incident point. With our present geometric interpretations this geometric concept of a directed unit line with axis fixed in space ought not to be called a vector; for the future we shall call it a unit rotor.

(9) Turn the unit rotor i through an angle a towards the rotor j; then translate the turned rotor along the rotor k to a distance a'. The first change of position is effected by the operator $e^{\frac{1}{2}ak}(\ \ )\,e^{-\frac{1}{2}ak}$ and the second by the operator $e^{\frac{1}{2}Ja'k}(\ \ )\,e^{-\frac{1}{2}Ja'k}$; that is to say i is first changed to $e^{ak}i$ and then to $e^{(a+Ja')k}i$. Since i and the final $e^{(a+Ja')k}i$ are any two unit rotors, we have here proved that the ratio $\epsilon'\epsilon^{-1}$ of any unit rotor ϵ' to any other ϵ is the unitat whose axis is the common perpendicular between ϵ and ϵ', and whose angle and advance are the angle and distance of the twist about this common perpendicular, which converts the one into the other. We have geometrically interpreted the general complex unitat and incidentally justified our terms angle, advance, versor, translator.

x and y being real scalars, xi is a rotor; $\mathrm{J}yi$ is a rotor-couple; $(x + \mathrm{J}y)\,i$ is a motor. [See §7 below.] Clearly multiplication by $x + \mathrm{J}y$ completes our interpretations. Without entering into details, it will be evident to the reader, that we have a unique geometric interpretation in hyperbolic space of the general complex quaternion as the ratio of two motors.

(10) $q(\ )q^{-1}$ where q is a complex quaternion shifts any motor (or complex quaternion) by a finite twist whose axis is that of q and whose angle and advance are twice those of q. [For proof take the motor as $\omega + \mathrm{J}\sigma$ where ω and σ are rotors through the point of intersection of the rotors i, j, k; and take q as $(x + \mathrm{J}y)e^{(a + \mathrm{J}a')k}$.]

(11) The rate of increase of a motor α, fixed in a rigid body can be expressed as $\mathrm{V}\gamma\alpha$ where γ is a motor expressing the rate of displacement of the rigid body. [$\gamma = 2\mathrm{V}\dot{q}q^{-1}$ where q is as in (10)].

(12) If α, β, γ are three motors such that $\alpha + \beta + \gamma = 0$ one straight line intersects all three axes perpendicularly (for $\mathrm{V}\beta\alpha^{-1} + \mathrm{V}\gamma\alpha^{-1} = 0$). If k is a unit rotor along this line, and i any unit rotor intersecting k perpendicularly, then

$$\mathrm{U}\alpha = e^{ak}i, \quad \mathrm{U}\beta = e^{bk}i, \quad \mathrm{U}\gamma = e^{ck}i$$

where a, b, c are complex scalars satisfying the sine formula,

$$\mathrm{T}\alpha \sin(c - b) = \mathrm{T}\beta \sin(a - c) = \mathrm{T}\gamma \sin(b - a).$$

[Interpret the condition $\mathrm{S}\alpha\beta\gamma = 0$.]

If α and β are two motors; then if $\mathrm{S}\alpha\beta = 0$ they intersect perpendicularly, and conversely; and if $\mathrm{V}\alpha\beta = 0$ they are co-axial, and conversely.

(13) If ω, σ are rotors through the origin, required the axis, pitch, etc., of the motor $\omega + \mathrm{J}\sigma$. First obtain the co-axial unit rotor by dividing by $\mathrm{T}(\omega + \mathrm{J}\sigma)$, that is, by $\sqrt{(-\omega^2 - \mathrm{J}^2\sigma^2 - 2\mathrm{J}\mathrm{S}\omega\sigma)}$. Let this unit rotor be $\omega_0 + \mathrm{J}\sigma_0$ where ω_0, σ_0 are rotors through the origin; (for which

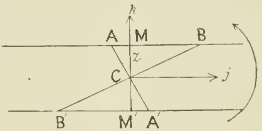
$$\omega_0^2 + \mathrm{J}^2\sigma_0^2 = -1, \quad \mathrm{S}\omega_0\sigma_0 = 0).$$

Then put $\omega_0 + \mathrm{J}\sigma_0 = e^{(a + \mathrm{J}b)k}i$ and deduce

$$\mathrm{J}\sigma_0\,\omega_0^{-1} = k \tan \mathrm{J}b.$$

*Straight Lines Symmetric About a Point; Components and Moments; Rotors and Rotor-couples.**

§7. In elliptic space there is a special method of resolution (by left and right parallels) of forces and the like which is inapplicable to hyperbolic space; but resolutions in the latter easily translate to corresponding resolutions in the former. Let us, then, pay special attention to hyperbolic resolutions. The terms "equal and opposite" and "equal and similarly directed," as applied to rotors, may be based on the conception of the straight line symmetric, about a given point, to a given straight line. Let AB be a given straight line and C a given point. Join the points A, B, ... to C and produce to A′, B′, ..., making AC equal to CA′ etc. The locus of A′, B′, ... is the straight line symmetric to AB, about C. C is the centre of symmetry, and the straight line through C perpendicular to the plane CAB is the axis of symmetry; AB and B′A′ are similarly directed; AB and A′B′ are opposite. [In hyperbolic space, AB and B′A′ are any two *non-intersecting* coplanar straight lines; in elliptic space they are any two coplanar straight lines whatever. In hyperbolic space there is only one centre and one axis of symmetry. In elliptic space there are two axes, and for polar space two centres, for antipodal space two pairs of antipodal centres. The terms "similarly directed" and "opposite" give different meanings for the two axes; and except when the centre or axis of symmetry is given the terms ought not to be used in elliptic space; unless we elect to imply that the "near" centre and axis is to be understood, to the exclusion of the "far" centre and axis.] The centre of symmetry is the mid-point of the common perpendicular MM′ of the two straight lines. Below the length CM is taken to be z; the unit rotor along CM is taken to be k; and the unit rotor through C, perpendicular to k, in the plane CAB, is taken to be j. For application in §8 below, it is important to



* Probably the contents of §7 are well known. If that is so, no harm can come from again enunciating what happens to be important for our purposes.

remark, that the one straight line B′A′ may be obtained from the other AB by a translation 2AC through any point A of the second mentioned.

Equal and similarly directed unit rotors along AB and B′A′ are

$$e^{Jzk}j = j \cos Jz - i \sin Jz,$$

$$e^{-Jzk}j = j \cos Jz + i \sin Jz.$$

Equal and opposite unit rotors are obtained by reversing the first of these. Thus two equal and similarly directed unit rotors combine to a similarly directed rotor through the point of symmetry, whose magnitude is $2 \cos Jz$; and two equal and opposite unit rotors combine to a rotor-couple ($2i \sin Jz$) along the axis of symmetry whose magnitude is $2J^{-1} \sin Jz$ and whose sense is that which we should expect from Euclidean analogies. Call this sense the usual sense, and call the opposite sense the unusual sense. If we combine two equal and opposite rotor-couples by changing the above j to Jj we get $2Ji \sin Jz$. In hyperbolic space this gives us an unexpected result; namely, that a couple of rotor-couples has the unusual sense. The anomaly does not occur in elliptic space.

The anomaly is sufficiently important to deserve a kinematical comment. If we combine a right-handed velocity of rotation a about AB, in the diagram, with an equal one about A′B′ we get as we should expect a velocity of translation along the axis of symmetry, in the usual sense, equal to $2a \sinh z$. On the other hand, if we combine a velocity of translation a' along AB with an equal one along A′B′ we get in hyperbolic space, as we should scarcely expect, a velocity of rotation ($2a' \sinh z$) in the direction of the curved arrow in the diagram. [This is neither oversight nor nonsense. In Euclidean space, in a similar case the result would be zero. Let the reader ask himself what he means by the combination of two velocities of translation of a rigid body.]

A rotor or rotor-couple is given at a given point A, the magnitude being a. It is required to replace it by an equivalent rotor and rotor-couple at a second given point C. Join AC and produce AC to A′ making AC = CA′; and at A′ introduce two equal and opposite rotors or rotor-couples, each of magnitude $\frac{1}{2}a$. Then combine one of these with $\frac{1}{2}a$ at A and the other with the remaining $\frac{1}{2}a$ at A. The effect is easily deduced from the above and may be stated as follows. The rotor at A is equivalent to a *component* rotor at C, the magnitude of the component being a multiplied by cos [J. distance CA]; together with a *moment*, that is a rotor-couple through C, the magnitude of the moment being a multiplied by J^{-1} sin [J. distance CA].

§8. Let $\omega + J\sigma = \omega' + J\sigma'$ where ω, σ are rotors through O, and ω', σ' rotors through P, and let the position unitat of P (origin O) be u. By §7 the component and moment of ω, at P, are $\frac{1}{2}(\omega + u\omega u^{-1})$ and $\frac{1}{2}(\omega - u\omega u^{-1})$ respectively; and the component and moment of $J\sigma$, at P are $\frac{1}{2}J(\sigma + u\sigma u^{-1})$ and $\frac{1}{2}J(\sigma - u\sigma u^{-1})$ respectively. Hence

$$\left.\begin{aligned} \omega' &= \tfrac{1}{2}(\omega + J\sigma) + \tfrac{1}{2}u(\omega - J\sigma)u^{-1} \\ J\sigma' &= \tfrac{1}{2}(\omega + J\sigma) + \tfrac{1}{2}u(-\omega + J\sigma)u^{-1} \end{aligned}\right\} \text{ or}$$

$$\left.\begin{aligned} \omega' + J\sigma' &= \omega + J\sigma \\ \omega' - J\sigma' &= u(\omega - J\sigma)u^{-1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1)$$

This is applicable to elliptic space by putting $J = E$. In the notation of §2 where E is not used, the same reasoning applies; rotor ω at O gives rotor $\frac{1}{2}(\omega + u\omega u^{-1})$ and rotor-couple $\frac{1}{2}(\omega - u\omega u^{-1})$ at P, and rotor-couple σ at O gives rotor-couple $\frac{1}{2}(\sigma - u\sigma u^{-1})$ and couple of rotor-couples (i.e. rotor) $\frac{1}{2}(\sigma - u\sigma u^{-1})$ at P; or

$$\left.\begin{aligned} \omega' &= \tfrac{1}{2}(\omega + \sigma) + \tfrac{1}{2}u(\omega - \sigma)u^{-1} \\ \sigma' &= \tfrac{1}{2}(\omega + \sigma) + \tfrac{1}{2}u(-\omega + \sigma)u^{-1} \end{aligned}\right\} \text{ or}$$

$$\left.\begin{aligned} \omega' + \sigma' &= \omega + \sigma \\ \omega' - \sigma' &= u(\omega - \sigma)u^{-1} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

In addition to (2) we have the statements from §2 above that left and right parallels at P, of ω at O, are denoted by ω and $u\omega u^{-1}$; and similarly for σ. Hence component and moment, [at P], of ω at O, are $\frac{1}{2}$(left + right parallels) of ω at O, and $\frac{1}{2}$(left — right parallels) of ω at O, respectively; and similarly for σ.

Real Linear Transformations of Points and Planes.

§9. The explanation of the meaning of (1) of §8 is more concise than that of (2); and when motors naturally present themselves, as in most physical applications, then equations after the model of (1), which treat the motors as wholes, are to be preferred to the others. But in pure geometry, where three scalar coordinates, in place of six, are more natural, the method of §2, and the corresponding method for hyperbolic space, namely Joly's, seem to the writer superior.

If ϕ is a quaternion linity, the conjugate ϕ' of ϕ is defined by $Sq\phi r = Sr\phi' q$ for any two quaternions q, r. The K-conjugate $\phi^{\backprime}$ of ϕ is defined by $S.qK\phi r = S.rK\phi^{\backprime} q$; that is, the bilinear scalar $S.pKq$ is used in the definition of $\phi^{\backprime}$ in the same manner as the bilinear scalar $S.pq$ is used in the definition of ϕ'. It follows from the definitions that $\phi^{\backprime} = K\phi' K$; K itself being a quaternion linity which is both self-conjugate and self-K-conjugate. In elliptic space $\phi^{\backprime}$ is of importance, and not ϕ'. This is due to the fact that the equation of the absolute is $S.qKq = 0$, not $S.q^2 = 0$. The equation of any quadric is $S.qK\phi q = 0$ where ϕ is self-K-conjugate. The quadric being real, and therefore ϕ real, it can, by proper choice of origin invariably be expressed in the (wholly real) form

$$\phi q = g\,Sq - aiSiq - bjSjq - ckSkq$$

i, j, k intersecting in the origin. [For an application below it should be added that when we do not permit choice of origin the real form is

$$\phi q = (gS.qp^{-1} - aiS.iqp^{-1} - bjS.jqp^{-1} - ckS.kqp^{-1}).\,p,$$

Putting $p = ip'$ the last changes to

$$\phi q = (aS.qp'^{-1} - giS.iqp'^{-1} - cjS.jqp'^{-1} - bkS.kqp'^{-1}).p',$$

there being no real distinction between the four mutually quadrantally distant points whose position unitats are $1, i, j, k$].

The general real K-skew linity ($\phi^{\backprime} = -\phi$) has the very simple form $\phi q = aq + q\beta$ where α, β are real given vectors. The general ϕ for which $\phi^{\backprime}\phi = 1 = \phi\phi^{\backprime}$ has the equally simple form

$$\phi q = pqp'^{-1}, \text{ with } Tp = Tp'.$$

In elliptic space the point (or plane) transformation $r = \phi q$, $\phi^{\backprime}\phi = 1$, means kinematically the general finite twist; and the transformation $r = \phi q$, $\phi^{\backprime} = -\phi$, converts a position quaternion q into the rate of change of position quaternion due to a changing twist. Similarly below, for hyperbolic space, we have the interpretations of $\phi'\phi = 1$ and $\phi' = -\phi$.

The real linear transformation in hyperbolic space requires that we translate to Joly's notation and then if we please back to our own. Let C be the complex self-conjugate quaternion linity $C = S + IV$. [Elementary properties of C. C is a square root of K, that is $C^2 = K$; $C^4 = K^2 = 1$; $C^3 = C^{-1} = KC$; $C' = C = C^{\backprime}$; $C^{-1} = S - IV$. Since $C - 1$ annuls scalars, and $C - I$ annuls vectors, C satisfies the quadratic $(C - 1)(C - I) = 0$. If

$$q = Cp, \qquad q' = Cp'$$

then $$S.qKq' = S.pp', \qquad qKq = S.p^2.$$

Thus K-conjugacy in the system q, q' (our system) corresponds to conjugacy in the system p, p' (Joly's system)]. If p is one of Joly's point or plane symbols (according to Joly, interpreted in Euclidean space) then $q = Cp$ is our corresponding hyperbolic position quaternion. Joly's plane $S.pp_0 = 0$ becomes our plane $S.qKq_0 = 0$; Joly's ϕ becomes our $C\phi C^{-1} = \psi$; so that Joly's ϕ' becomes our $\psi^{\backprime}$. For real linear transformations, it is Joly's ϕ, not our ψ, which is a real linity.

From the above standard forms when $\phi^{\backprime} = \pm\phi$ we may derive standard forms for $\phi' = \pm\phi$ by noting the following statements; if $\phi^{\backprime} = \pm\phi$, then $(K\phi)' = \pm K\phi$ and conversely; and also, if $\phi^{\backprime} = \pm\phi$, then $(\phi K)' = \pm\phi K$ and conversely. I do not see how, similarly, to obtain a standard form for ϕ when $\phi'\phi = 1$; but such a form may be obtained by translating from our notation for a finite twist $\psi q = pqp'^{-1}$, into Joly's notation. First effect a conical rotation about a line through the origin; then effect a translation along a line through the origin. The first converts, in Joly's notation, q to rqr^{-1}; the second converts q to $-Kq + 2pS.pKq/S.p^2$ so that when $\phi'\phi = 1$ we have

$$\phi q = -r.Kq.r^{-1} + 2pS.prKqr^{-1}/S.p^2.$$

This is very different from Joly's standard indeterminate form for this case.

Non-Euclidean Space Integrals, Curl, &c.

§10. If r and $r + dr$ are the position quaternions of two neighbouring points then $(r + dr)r^{-1}$ has been interpreted in §2 as a quaternion whose angle and axis indicate the elementary line-segment joining the points; in §§3-8 as a bi-quaternion (in Clifford's sense for elliptic space, in Hamilton's sense for hyperbolic space) indicating the same thing. Tracing the interpretation a step further we may say that in all three cases the rotor element joining the points is

$$\mathrm{J}^{-1}\mathrm{V}drr^{-1} = d\lambda = \mathrm{J}^{-1}duu^{-1},$$

if u, $u + du$ are the position unitats; provided J means unity in the real quaternion method, $\surd(-1)$ in the complex quaternion method, and E in the Clifford-bi-quaternion method. This extension of the meaning of J will be henceforth understood. A general function of r is a function of position and of an independent scalar Tr. [To fix the ideas take Tr as any given function of the time (such as e^t), the same for every point of space.]

Define the scalars w, x, y, z and the operators ϑ, Θ, by

$$\left.\begin{aligned} r &= w + \mathrm{J}(ix + jy + kz), \\ \vartheta &= \mathrm{D}_w - \mathrm{J}^{-1}(i\mathrm{D}_x + j\mathrm{D}_y + k\mathrm{D}_z), \\ \Theta &= -\mathrm{J}\mathrm{V}r\vartheta \end{aligned}\right\} \quad \ldots \quad (1)$$

Space differentiations are effected by Θ. ϑ has been introduced merely to suggest to the reader the definition of Θ and also the proofs of the fundamental properties (2), (3), (4) below of Θ.

For space differentiation we have

$$d. = -\mathrm{S}d\lambda\Theta. \quad \ldots \ldots \ldots \ldots \ldots \quad (2)$$

ϑ is a symbolic quaternion passing through the origin, whose coordinates D_w, etc. may be treated as constants. Θ is a symbolic rotor passing through the point r, whose coordinates may *not* be treated as constants, because of the r included in the definition of Θ.

u, as above, standing for $\mathrm{U}r$, and $[p, p']$ for any function linear in each of two quaternions we have

$$[\theta_1, u_1] = \mathrm{J}[\zeta, \zeta u] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

the suffixes and the (ζ, ζ) having the usual quaternion significations. The following are examples of the use of (3) :—

$$\theta \mathrm{S} u = \mathrm{J}\zeta\mathrm{S}\zeta u = -\mathrm{J}\mathrm{V}u,$$

$$\theta \,.\, u^{-1} = -\theta_1 \,.\, u^{-1}u_1u^{-1} = \mathrm{J}(3\mathrm{S}u + \mathrm{V}u).$$

Let q be a function of position only, not of $\mathrm{T}r$; let the tri-linear expression $[\theta_1, \theta_1, q_1]$ imply that the differentiations of each θ affect q, but that they do not affect the variable factors r of either θ; and let $[\theta_1, \theta_1', q_1]$ imply that, in addition, the variable factor of second θ, namely θ_1', is affected by the differentiations of the other θ, namely θ_1. Then we have

$$[\theta_1, \theta_1', q_1] = [\theta_1, \theta_1, q_1] + \mathrm{J}[\zeta, \mathrm{V}\zeta\theta_1, q_1] \quad \ldots \ (4)$$

$\theta^2 q$ of course means $\theta(\theta q)$, that is $\theta_1\,\theta_1' q_1$; whereas $\theta_1^{\,2} q_1$ of course means $\theta_1\,\theta_1\,q_1$. Thus $\theta_1^{\,2}$ (but not θ^2) is, like the ∇^2 of quaternions in Euclidean space, a scalar operator. An important special case of (4) is $\theta^2 q = \theta_1^{\,2} q_1 - 2\mathrm{J}\theta q$, or

$$\left.\begin{aligned} &\theta(\theta + 2\mathrm{J})q = \theta_1^{\,2} q_1 \\ \text{and} \quad &\theta(\theta + 2\mathrm{J}) . = \mathrm{S}.\theta^2 . \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

Hence $\theta(\theta + 2\mathrm{J})$ is a scalar operator, and therefore so also is $(\theta + \mathrm{J})^2$.

ϕ being any quaternion linity which is a function of position, the line-surface integral is

$$\int \phi d\lambda = \iint \phi_1 \mathrm{V} d\nu(\theta_1 + 2\mathrm{J}) \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

which is proved by first proving that $\int d\lambda = 2\mathrm{J}\iint d\nu$ by §7 above, and then proceeding as in the Euclidean case. [*Utility of quaternions in Physics*, §6.] Here $d\nu$ is a rotor element of any surface and $d\lambda$ a rotor element of the complete boundary, the relative senses of the two being as usual. The surface-volume integral is

$$\iint \phi d\nu = \iiint \phi_1 \Theta_1 db \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where db is an element of any given volume and $d\nu$ a rotor element of the complete boundary, pointing away from the bounded volume. [That $\iint d\nu = 0$ for a closed surface follows from $\int d\lambda = 2\text{J}\iint d\nu$ for an open surface.]

σ being any vector function of position (rotor, rotor-couple or motor; but generally to be thought of as a rotor), and $\phi = \text{S}(\ )\sigma$, (6) and (7) become

$$\int \text{S}\sigma d\lambda = \iint \text{S}.d\nu\ \Theta + 2\text{J})\sigma \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\iint \text{S}\sigma d\nu = \iiint \text{S}\Theta\sigma db \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

(8), (9) and (2) give us the proper forms for curl convergence and gradient (denoted below by crl, cnv, and grd). In Euclidean space, if q is a quaternion function of position we conveniently define thus

$$\text{crl}q = \text{V}\nabla\text{V}q, \quad \text{cnv}q = \text{S}\nabla q, \quad \text{grd}q = \nabla\text{S}q$$

so that $$\nabla q = (\text{crl} + \text{cnv} + \text{grd})q.$$

Similarly in non-Euclidean space a symbolic quaternion *linity* Λ takes the place of the symbolic linity $\nabla(\ )$. Let

$$\Lambda q = (\Theta + 2\text{JV})q \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$\left.\begin{array}{ll} \text{Then} & \text{crl}q = \text{V}\Lambda\text{V}q = \text{V}.\ \Theta + 2\text{J})\text{V}q \\ & \text{cnv}q = \text{S}\Lambda q = \text{S}\Theta q \\ & \text{grd}q = \Lambda\text{S}q = \Theta\text{S}q \\ \text{whence} & \Lambda q = (\text{crl} + \text{cnv} + \text{grd})q \end{array}\right\} \quad \ldots \quad (11)$$

It is easy to prove that

$$\begin{aligned}(\text{crl} + \text{grd} + \text{cnv})^2 &= \Lambda^2 \\ &= \text{crl}^2 + \text{cnv}.\text{grd} + \text{grd}.\text{cnv} \quad \ldots \quad (12)\end{aligned}$$

the last equation implying the six zero relations

$$\begin{aligned}0 &= \text{crl}.\text{grd} = \text{crl}.\text{cnv} = \text{grd}.\text{crl} \\ &= \text{grd}.\text{grd} = \text{cnv}.\text{crl} = \text{cnv}.\text{cnv} \quad \ldots \quad (13)\end{aligned}$$

When q, the quaternion function of a point, consists of a rotor through the point and a real scalar we will call it local. From (2), (8) and (9) it is obvious that

Λq and each of its three named parts crl$q$, cnv$q$, grd$q$ are local when q is local. The operators Λ, etc., will therefore be called localizing operators.

The reader is recommended to prove the following. In (1) §8 above, ω, σ are constant rotors through the origin and $\omega + J\sigma = \omega' + J\sigma'$ is the velocity motor of a rigid body; ω' and σ' being local rotors expressing the angular velocity and the linear velocity at the point u. With these meanings

$$\omega' = \tfrac{1}{2}\,\mathrm{crl}\,\sigma', \qquad \sigma' = \tfrac{1}{2}\,J^2\,\mathrm{crl}\,\omega',$$

$$\omega' = \tfrac{1}{4}\,J^2\,\mathrm{crl}^2\,\omega', \quad \sigma' = \tfrac{1}{4}\,J^2\,\mathrm{crl}^2\,\sigma'.$$

Since $\mathrm{cnv}\,.\,\mathrm{crl} = 0$ we have $\mathrm{cnv}\sigma' = \mathrm{cnv}\omega' = 0$ as might have been expected in the case of σ'.

Some additional formulae referring to differentiation are collected here for reference, proofs being left to the reader. In addition to the forms of (5) we have

$$\left.\begin{aligned}\Theta(\Theta + 2J) &= \Lambda^2 - 2J\mathrm{crl}\\ &= \mathrm{crl}(\mathrm{crl} - 2J) + \mathrm{grd}\,.\,\mathrm{cnv} + \mathrm{cnv}\,.\,\mathrm{grd}\,.\end{aligned}\right\} \quad (14)$$

in which last again we may write crl $-$ 2JV in place of crl $-$ 2J; and also we may write

$$\begin{aligned}\mathrm{crl}^2 - 2J\mathrm{crl} &= \Theta\mathrm{crl} = \mathrm{crl}\Theta\\ &= (\Lambda - 2J)\mathrm{crl} = \mathrm{crl}(\Lambda - 2J) \quad \ldots \quad (15)\end{aligned}$$

The next formula is especially useful for wave propagation of curl; p being a quaternion function of the point whose position unitat is u

$$\begin{aligned}(\Theta + J)(pu)\,.\,u^{-1} &= (\Lambda - 2JS)p\\ &= (\Theta - 2JK)p \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)\end{aligned}$$

Though $\Theta(\Theta + 2J)$ is a scalar operator, it appears from the Jcrl in the middle expression of (14) that it is not a localizing operator It should perhaps be noted that though we pay careful attention to localization and often assume q to be local, all our general formulae are true independently of any such supposition.