Errata.

Page 33, line 21—for "l" (where it first occurs), read "L."

Page 34. The third equation should read "$-2x + 14y = 600$."

THE THEORY OF THE QUOTA IN PROPORTIONAL REPRESENTATION—II.*

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(Read 19th May, 1913.)

* For Part I., see *Papers and Proceedings of the Royal Society of Tasmania*, 1912, pp. 49-77.

List Systems.

54. In §§ 22-25, I discussed, with the aid of the method of least squares, the problem of partitioning a given number of seats among three parties in proportion to their strengths. I had not then seen any mathematical discussions of this problem; but I have since obtained a copy of a paper published in 1910 by M. A. Sainte-Laguë, Professeur de Mathématiques spéciales au Lycée de Douai (and now of Besançon), in which he gives a discussion of the problem for any number of parties by the same method.([20]) The volume in which this paper is published is not accessible to many students in this part of the world, and I have therefore made a summary of M. Sainte-Laguë's results.

I use the notation of § 25 of my own paper, as corrected in the erratum slip.

55. Each elector, says M. Sainte-Laguë, has the right to be represented by a fraction of a deputy given by $m/v = 1/Q$. If he belongs to the party A, he is represented by the fraction x/p of a deputy; whence the error in representation for him is seen to be $x/p - m/v$. For the electors of the various parties, there are errors $e_1, e_2, e_3, \ldots e_s$. The various methods diverge from one another in the ways in which they endeavour to make these errors as small as possible.

56. To arrive at the best rule, M. Sainte-Laguë applies the method of least squares.

For each elector of party A the error in representation is—

$$\frac{x}{p} - \frac{m}{v}.$$

The sum of the squares of the errors for the p electors of this party is—

$$p\left(\frac{x}{p} - \frac{m}{v}\right)^2$$

and the sum of the squares of the errors for all v electors is—

$$\sigma = \Sigma p\left(\frac{x}{p} - \frac{m}{v}\right)^2$$

([20]) *La représentation proportionnelle et la méthode des moindres carrés*, Annales Scientifiques de l'Ecole Normale Supérieure, 3e. série, tome 27, December, 1910, pp. 530-542. M. Sainte-Laguë has given a more popular account of his results in the Revue Générale des Sciences pures et appliqueés of 30th October, 1910, pp. 846–852; and the "rule of least squares" is stated in a communication made to the Academy of Sciences of Paris on 8th August, 1910.

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whence it can be shown that—

$$\sigma = \Sigma \frac{x^2}{p} - \frac{m^2}{v}$$

The quantity to be made a minimum is, then, $\Sigma\ (x^2/p)$. M. Sainte-Laguë suggests a geometrical solution similar to the method used by me for three parties, but gives the following practical solution for any number of parties. The identity—

$$x^2 = 1 + 3 + 5 + \ldots + (2x - 3) + (2x - 1)$$

shows that the sum to be made a minimum is the sum of the x first numbers of (1), the y first numbers of (2), &c.; x, y, z being chosen so that the m smallest numbers are selected:—

$$\frac{1}{p}, \frac{3}{p}, \frac{5}{p} \ldots . \qquad (1)$$

$$\frac{1}{q}, \frac{3}{q}, \frac{5}{q} \ldots . \qquad (2)$$

$$\frac{1}{r}, \frac{3}{r}, \frac{5}{r} \ldots . \qquad (3)$$

As the same result would be obtained by inverting all these numbers and choosing the m largest we have the following rule:—

Rule of least squares: Divide p, q, r ... by the odd integers 1, 3, 5 ..., and in the various series of quotients so obtained select the largest, until m have been obtained. Party A receives as many members as the number of quotients taken from its series; and so with the other parties.

57. Next, consider only positive errors (*i.e.*, errors for electors who are over-represented). If the error for each elector of party A is positive, this party has at least $X + 1$ seats ($X + 1$ being the whole number next greater than X_0). According as the seats obtained are $X + 1$, $X + 2$, ..., the error for each elector of A is—

$$\frac{X+1}{p} - \frac{m}{v}, \quad \frac{X+2}{p} - \frac{m}{v} \ldots .$$

and so for the other parties; and the remaining seats have to be allotted so that these errors may be as small as possible. If the parties have obtained X, Y, Z ... seats, and

there are l seats left, these seats will be distributed according to the l smallest of these numbers, or according to the l largest of the numbers—

$$\frac{p}{X+1}, \frac{p}{X+2} \ldots, \frac{q}{Y+1}, \frac{q}{Y+2} \ldots, \frac{r}{Z+1} \ldots$$

If we notice that the numbers—

$$\frac{p}{1}, \frac{p}{2}, \frac{p}{3} \ldots \frac{p}{X}, \frac{q}{1}, \frac{q}{2} \ldots \frac{q}{Y}, \frac{r}{1} \ldots \frac{r}{Z}, \ldots,$$

each greater than those that follow in the same series, may be considered as corresponding to the seats already allotted, we are led to the *rule of D'Hondt*, of which the statement is the same as the rule of least squares, with the substitution of the consecutive integers 1, 2, 3, 4 .. as divisors in place of the odd integers 1, 2, 3, ...

58. If we consider only negative errors (*i.e.*, errors for electors who are under-represented), and limit ourselves to cases in which the parties have at least X, Y, Z ... members, negative errors will occur for such of the lists as do not get any more seats, and we have therefore to choose for the allotment of the remaining seats the smallest of the numbers

$$\frac{X}{p} - \frac{m}{v}, \quad \frac{Y}{q} - \frac{m}{v}, \ldots$$

Now if—

$$p \equiv X.Q + r_p, \qquad q \equiv Y.Q + r_q, \ldots,$$

we have—

$$\frac{X}{p} - \frac{m}{v} = \frac{1}{Q} - \frac{r_p}{pQ} - \frac{m}{v} = \frac{-r_p}{pQ}, \text{ \&c.}$$

We must choose then the smallest of the numbers $-r_p/pQ$ &c., or the largest of the numbers r_p/p, &c., for the remaining seats. This is the *rule of the largest fractions*. This method, M. Sainte-Laguë points out, is not to be confused with the *rule of the largest remainders*, in which the remaining seats are allotted according to the largest of the remainders r_p, r_q . . .

59. Finally, the *rule of Hquer* results from making as small as possible the difference between the largest positive error and the largest negative error.

60. M. Sainte-Laguë then passes to methods which are based on the consideration, not of the error for each elector, but of the error for each member; a less valid basis, as he considers, for the apportionment of seats. A member should represent $v/m = Q$ electors; if he has been elected by party A he represents p/x, whence the error for him is $p/x - v/m$.

Applying the method of least squares to these errors, M. Sainte-Laguë finds that if each party secures at least one member, the seats are to be allotted by using divisors whose approximate values are 3, 5, 7, 9, 11, 13, 15 ..., one seat having been allotted to the largest party before applying the rule.

If, in place of considering the error for each member, we consider the error for each party $(x - pm/v)$, the method of least squares leads us to the *rule of the largest remainders*, also known as the *Swiss rule*.

Finally, M. Sainte-Laguë applies several tests to compare the *rule of least squares* with the *rule of D'Hondt*, and concludes that the former rule leads to fairer results than the rule of D'Hondt.

61. For the assistance of those who may wish to become acquainted with the views of French and Belgian writers on the principal rules proposed in recent years for partitioning seats among parties district by district, I have compiled the following list of references to the various rules. These are the principal rules; but there are many others, and these may be found in the reports of MM. Benoist and Groussier.

Système des moyennes, or *rule of Dietz*, adopted in the *Projet de loi portant modification aux lois organiques sur l'élection des Députés*, passed by the French Chamber of Deputies in July, 1912. (Each party gets, in the first place, as many seats as it has whole quotas of votes, the quota being the number of votes in the district divided by the number of seats. If there is a seat unallotted, the number of votes for each list is divided by one more than the number of seats already allotted, and the seat is given to the list which gives the largest quotient, and so for any other unallotted seats.) Groussier, p. 27 et sqq. Lachapelle, pp. 114-6.

D'Hondt rule (système du diviseur électoral). Benoist, pp. 19, 43 et sqq. Flandin, pp. 11 et sqq. Goblet d'Alviella, pp. 5-8. Lachapelle, pp. 94-98, 107-117, 208-217. La Chesnais Ch. VII. and App. I. Macquart, pp. 548-551. Moch, *passim*. Sainte Laguë, p. 534 et sqq. (see § 57 above). Van den Heuvel.

Hagenbach-Bischoff rule or *système Genevois* (used in the Canton of Geneva and in the town of Basel) (the Droop quota $v/(m + 1) + 1$ is used as divisor; if any seats remain unallotted, $v/(m + 2) + 1$ is used as divisor; if any still remain $v/(m + 3) + 1$ is the divisor; and so on). Macquart, pp. 551-4. Lachapelle, pp. 117-9.

System of the electoral quota (quotient électoral), unallotted seats going to the largest parties. (The quota is the number of votes in the district divided by the number of seats). Groussier, p. 28 et sqq. This system was used in the Neuchâtel electoral law of 1894, article 64 (Benoist, p. 126.)

System of the electoral quota (quotient électoral), unallotted seats going to the parties with largest remainders. Groussier, p. 28 et sqq. Macquart, 546-9. Sainte-Laguë, pp. 537-8 (see § 60 above). Lachapelle, pp. 103-107. Moch, §§ 5-13.

System of the electoral quota (quotient électoral), unallotted seats going to the parties for which the ratio of the remainder to the strength of the party is largest (méthode des plus fortes fractions). Sainte-Laguë, p. 535 (see § 58 above).

System of M. Maurice Equer (in which seats are partitioned so that the difference between the greatest and least of the quantities x/p—see § 55—may be as small as possible). Equer. Groussier, p. 33 et sqq. Sainte-Laguë, p. 535 (see § 59 above).

Methode des moindres carrés (see § 56 above). Sainte-Laguë, p. 531 et sqq.

Van de Walle's system (system of the electoral quota; remainders added together in a group of districts and remaining seats allotted by applying D'Hondt rule to the totals of the remainders). Lachapelle, pp. 221-230. Van de Walle, pp. 1-30. Goblet d'Alviella, pp. 13-15. Van den Heuvel.

General discussions of the problem will be found in—

Groussier, pp. 22-51.
Sainte-Laguë, pp. 529-542.
Macquart, pp. 545-554.

The works referred to above by authors' names are as follows:—

Benoist, Charles (Député). *Rapport fait au nom de la commission du suffrage universel chargée d'examiner diverses propositions de loi tendant à établir la représentation proportionnelle.* Chambre des Députés, Annexe au procès-verbal de la séance du 7 Avril, 1905. (Chamber of Deputies, No. 2376 of 1905.)

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Equer, Maurice. *Arithmétique et représentation proportionnelle.* Supplément à la Grande Revue de 25 Juin, 1910.

Flandin, Etienne (Député). *Rapport fait au nom de la commission du suffrage universel chargée d'examiner les propositions de loi: 1° de M. Dansette; 2° de M. Louis Martin et plusieurs de ses collègues; 3° de M. Massabuau; 4° de M. Etienne Flandin (Yonne), tendant à l'établissement du scrutin de liste avec représentation proportionnelle dans les élections à la Chambre des Députés.* Chambre des Députés, Annexe au procès-verbal de la séance du 22 Mars, 1907. (Chamber of Deputies, No. 883 of 1907.)

Goblet d'Alviella, M. le Comte Félix. *Quelques considérations sur la représentation proportionnelle.* Extrait de la Revue de Belgique. (Bruxelles: Société anonyme M. Weissenbruch, Imprimeur du Roi, 1910.)

Groussier, Arthur (Député). *Rapport fait du nom de la commission du suffrage universel chargée d'examiner le projet de loi et diverses propositions de loi portant modification aux lois organiques sur l'élection des députés et tendant à établir le scrutin de liste avec représentation proportionnelle.* Chambre des Députés, Annexe au procès-verbal de la 1re séance du 16 Mars, 1911. (Chamber of Deputies, No. 826 of 1911.)

Lachapelle, Georges (Secrétaire général du Comité républicain de la R.P.). *La représentation proportionnelle en France et en Belgique.* (Paris: Félix Alcan, 1911.)

La Chesnais, P.-G. *La représentation proportionnelle et les partis politiques.* (Paris: Georges Bellais, 1904.)

Macquart, Emile (Secrétaire général de la Ligue pour la Représentation proportionnelle). *Examen critique des divers procédés de répartition proportionnelle en matière électorale.* Revue Scientifique, 5e série, tome iv., 28 Octobre, 1905, pp. 545-554.

Moch, Gaston. *La représentation vraiment proportionnelle.* Collection de la Grande Revue. (Paris: Edouard Cornély et Cie, 1910.)

Sainte-Laguë, A. (Professeur au Lycée de Douai). *La représentation proportionnelle et la méthode des moindres carrés.* Annales scientifiques de l'Ecole Normale Supérieure, 3e. série, tome 27, December, 1910, pp. 530-542.

Van den Heuvel, Jules. *Le mécanisme de la représentation proportionnelle.* Extrait de la Revue Générale, février, 1911. (Bruxelles: Goemaere, 1911.)

Van de Walle, Victor. *La Représentation proportionnelle intégralement appliquée aux élections législatives: Proposition de loi (avant-projet).* (Bruxelles: Imprimerie du Progrès—V. Feron, 1910.)

List Systems—The Method of the Uniform Quota.

62. Finally, it remains to point out that the problem of apportioning seats among parties arises from fixing before the election the number of seats for each constituency. The problem can be avoided, and a partition of seats among parties as exact as the size of the legislature allows can be secured, if the number of seats in the legislature is fixed, but the number of seats for each constituency is determined after the polling by the number of votes polled in it. On this idea is based the system of *le nombre unique*, or the *uniform quota*—a system supported by the late Professor Henri Poincaré and other French mathematicians as the only exact method of proportional representation. ([20a])

63. Hare proposed to use for the quota the number obtained by dividing the total of the votes throughout the country by the number of members in the House of Commons. He also proposed that the whole country should be one constituency; a proposal which, with other notions contained in his works, is usually thought to have kept back for a generation the progress of proportional representation in England. The same quota is used in the system of *le nombre unique*; but the country is divided into districts, as in other systems of proportional representation, and these may be equal or unequal, as may be convenient. The system assumes that the same parties will contest the election in many districts or throughout the country; it would break down if there were many isolated candidatures, but these are not to be expected when the party system has become established.

The votes for all the candidates of each party throughout the country are totalled, and then the total number of votes for all parties is obtained. This total is divided by the number of members to be elected, and the result is *le nombre unique*, or the uniform quota.

The total number of votes for each party is then divided by the quota. The quotient so obtained is the share of representation of the party. If the sum of the quotients

([20a]) See note ([20a]), § 73.

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is not equal to the number of members to be elected, the remaining seats can go to the parties with the largest remainders, or otherwise as may be thought fit; it matters little how the remaining seats are dealt with if the House is at all large. The proportion of seats to voters in each party can thus be made as exact as the number of members will allow.

Next, the number of votes for each party in each district is divided by the quota, and members equal in number to the quotient so obtained are selected from the candidates of the party in that district. The total number of seats allotted to a party in the various districts will be less than the total number of seats to which it is entitled, for in some or all of the districts there will be remainders. The unallotted seats are then given to the districts with the largest remainders. Each party's share of representation in each district has now been ascertained. It only remains to choose the members for the party in each district from the candidates of the party in the district; the candidates to be chosen will be those of the party who are highest on the poll.([20b])

64. As an example, let us apply the system of the uniform quota to the Tasmanian General Election of 23rd January, 1913. The first choices (with the single transferable vote) obtained by the parties were as follow:—

General Election, Tasmania, 23rd January, 1913.—Votes for the Parties.

District.	Liberal.	Labor.	Independent.	Grand Total.
Bass	6839	6932	—	13,771
Darwin	6174	6441	—	12,615
Denison	7717	7132	—	14,849
Franklin	8566	6677	—	15,243
Wilmot	6861	4451	977	12,289
All	36,157	31,633	977	68,767

([20b]) On the system of *le nombre unique*, see—

Le Système du Nombre Unique, a pamphlet of 8 pages published by the Comité Républicain de la R.P., 23 Rue Pasquier, Paris.
Lachapelle (see § 61), pp. 89-102, 230-5, 258-9.
Groussier (see § 61), pp. 103-4, 191-3.

I assume that these numbers represent the relative strengths of the parties. With a party-list system, as each elector would have several votes, the numbers would be multiples of these (subject to a slight disarrangement of the votes in Wilmot); but these numbers will serve for the illustration.

We first divide the total number of votes polled, 68,767, by the number of members to be elected, 30; the result is the quota, 2292.

The total of the votes for each party is then divided by 2292; the results are, Liberal, 15·78; Labour, 13·79; Independent, 0·43. The members to be allotted to the parties are therefore Liberal, 16; Labour, 14; Independent, 0 (which, it may be noticed, was the result given by the single transferable vote).

Next divide the totals of the votes for the parties in the various districts by the quota. The results are:—

General Election, Tasmania, 23rd January, 1913.—Share of Representation in each District according to the Method of the Uniform Quota.

District.	Liberal.	Labor.	Independent.	Grand Total.
Bass	2·98	3·02	—	6·00
Darwin.....	2·70	2·81	—	5·51
Denison	3·37	3·11	—	6·48
Franklin	3·74	2·91	—	6·65
Wilmot	2·99	1·94	0·43	5·36
All.....	15·78	13·79	0·43	30

Allotting seats first to whole quotas, the Liberals would get two seats in Bass, Darwin and Wilmot, and three in Denison and Franklin, total 12; and the remaining four seats would go to the districts in which there are the greatest remainders, namely Bass, Darwin, Franklin, and Wilmot. Similarly the Labour Party would get, from whole quotas, three seats in Bass and Denison, two in Darwin and Franklin, and one in Wilmot, total 11; and the remaining three seats would go to Darwin, Franklin, and Wilmot. The result of the election would therefore be:—

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General Election, Tasmania, 23rd January, 1913.—Result according to the Method of the Uniform Quota.

District.	Liberal	Labor.	Independent.	Grand Total.
Bass	3	3	—	6
Darwin	3	3	—	6
Denison	3	3	—	6
Franklin	4	3	—	7
Wilmot.....	3	2	0	5
All.....	16	14	0	30

As between the parties the result would be the same as with the single transferable vote; but in Franklin (in which 15,243 votes were polled) there would have been seven members instead of six, of whom the Liberals would have had four, and in Wilmot (in which only 12,289 votes were polled) there would have been only five members instead of six, and there would thus have been a greater approach to electoral equality between electors in these two districts than with the same number of members for each district.

It would remain only to choose the members from the candidates of each party in each district. For each party the candidates highest on the poll would be chosen; these, of course, would not necessarily be the same if each elector had several votes as when he had only one vote.

65. From this illustration two of the principal advantages of the method of the uniform quota can be seen; first, districts in which political interest is more active may get more members than districts in which, although the number of electors enrolled is the same, fewer voters go to the poll; second, it is no longer necessary to alter boundaries as the distribution of population changes, for the method (so far as the size of the House allows) will give proportional representation to the districts in spite of differences in their electoral populations. The method, then, gives proportional representation as between parties throughout the country; proportional representation as between parties in each district; and proportional representation as between districts of varying sizes.

Multiple Transferable Vote Systems.

66. In these systems a voter gives equal votes to a number of candidates less than the number of members to be elected, and marks other candidates in an order of preference. The name "multiple transferable vote" describes such systems; but they might also be called "limited vote systems with preferential voting."

67. To appreciate the relation of the multiple transferable vote to other systems used in many-membered constituencies, the following arrangement of these systems in order of development will be useful:—

(*a*) The single non-transferable vote. This is used in Japan.([21]).

(*b*) The single transferable vote.

(*c*) The limited vote, in which an elector gives equal votes to a number of candidates less than the number of members to be returned. The limited vote was used in England from 1867 to 1885 in thirteen three-member constituencies and one four-member constituency.([22])

(*d*) The limited vote with preferential voting, or the multiple transferable vote.

(*e*) The block vote, or *scrutin de liste*, in which an elector votes for as many candidates as are to be elected. This is the system used for the Federal Senate. The block vote (with the modification that the elector might vote for fewer than the number to be elected) was used in Tasmania for the House of Assembly from 1856 to 1870 in one five-member constituency and one three-member constituency, and from 1885 to 1896 in eight two-member constituencies.

Of these, the single transferable vote gives approximately proportional representation (§§ 6-21); the single untransferable vote and the limited vote give representation to minorities, but not in proportion to their strengths; while the multiple transferable vote, with suitable rules, gives, as is shown below, the same approximation to proportional representation as the single transferable vote.

([21]) See J. H. Humphreys, *Proportional Representation* (London, 1911) pp. 283-9.

([22]) Benoist (see § 61) collects other instances of the use of the limited vote (p. 13).

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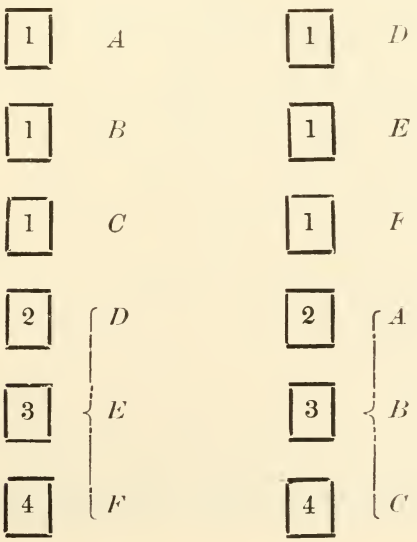
68. As an example of the multiple transferable vote, let us take a contest in a six-member district between two parties each nominating six candidates, in which each voter has three first choices, and second, third and fourth choices; and let us suppose that there is neither cross-voting between the parties nor short-voting within a party.

The first stage of the scrutiny will be to count the first choices obtained by each candidate. The total of these for all candidates will be three times the number of voters. The quota will be one-sixth (if the Hare quota is used) or one-seventh (if the Droop quota is used) of the total number of first choices; *i.e.*, three-sixths (Hare) or three-sevenths (Droop) of the number of voters. Some candidates will have surpluses above the quota, and there must be rules for transferring these; when all surpluses have been transferred, there may be one or more seats unfilled, and there must be rules for excluding the candidates lowest on the poll and distributing their votes.

Various sets of rules have been proposed. To illustrate the importance of the differences between the rules, let us take an extreme case based upon the following ballot-papers (*A*, *B*, *C*, *D*, *E*, *F* being supposed to be the candidates of one party)—



Let us suppose that *A*, *B*, *C* are the candidates of one section, and *D*, *E*, *F* the candidates of another section, of the same party; the supporters of the first section give a first choice to each of *A*, *B*, *C*, and their subsequent choices to the candidates of the second section (as in the first ballot-paper); the supporters of the second section give their first choices to *D*, *E*, *F*, and their subsequent choices to *A*, *B*, *C* (as in the second ballot-paper).

Let us further suppose that this party has two-thirds of all the voters, and is therefore entitled to four out of the six members; and that on making up the totals of the first choices, *A*, *B*, *C* are found to have each just a quota (either Hare or Droop); that all but three of the candidates of the other party have been excluded; and that *D*, *E*, *F* each have fewer votes than the three remaining candidates of the other party. There will, then, be just a quota of papers marked like the first ballot-paper; and the other ballot-papers for the party will be marked like the second ballot-paper.

A, *B*, *C* having just a quota each, there are no votes to be transferred from them, and *D*, *E*, *F* get no benefit from the second, third, and fourth choices given to them on the quota of papers on which *A*, *B*, *C* have first choices. *D*, *E*, *F* being lowest on the poll, one of them has to be excluded; let it be *D*. Suppose that the rules provide ([23]) that the votes of an excluded candidate are to be divided among the candidates having second or next available choices on the papers on which the excluded candidate has a first choice. On *D*'s papers, the second, third, and fourth choices have been given only to candidates who are already returned. Consequently there is no candidate available to receive *D*'s votes; and all of his votes are lost. One of *E*, *F*, say *E*, is now lowest on the poll; his votes cannot be transferred and are also lost; finally the votes of the third candidate *F* are lost. We are left, then, with three elected candidates of the party we have been considering and three candidates of the other party. The other party, numbering only one-third of the voters, and so entitled only to two members, thus gets three.

The failure to obtain proportional representation has arisen in this case because there were no candidates to

([23]) The rules of the "Launceston Voting System" for pre-elections contained in *The Tasmanian Workers' Political League Election Manual* (Tasmanian News Printing Works, Hobart, 1912) have this provision.

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whom (with the rules supposed) the votes of the excluded candidate could be transferred. With the single transferable vote (and no cross-voting or short-voting), an excluded candidate's votes can always be transferred if there is another candidate of the same party unelected, but in the cases just considered, although *E*, *F* are still unelected, the rules do not permit of *D*'s votes going to them.

69. The following table shows the representation which the Launceston Voting System may give to a party having 60 per cent. of the voters in a two-party contest in a six-member district. In the cases in which disproportional representation is shown to be possible, it should be remembered that the assumptions necessary for these cases are not likely to be realised very frequently.

Multiple Transferable Vote according to the Launceston Voting System in a Six-Member District—Possible Representation of a Party having 60% of the Voters, and so entitled to Four Members.

Number of First Choices allowed to each Elector.	Members Returned.
4	The party must get four members.
	If three members are returned each with just a quota, the party may fail to return a fourth (see §68); total 3.
3	If two members are returned each with just a quota, the party may return only one more; total 3.
	If one member is returned with just a quota, the party will get three others; total 4.
2	The party must get four members.

70. Disproportional representation through a division of a party into two sections (as in the cases just considered) can be avoided if the rules provide that on the distribution of a surplus or of the votes of an excluded candidate the votes shall go to the other candidates having first

choices on the same papers if there are no candidates available with second or subsequent choices.([24])

In the case supposed in § 68, neither *A*, *B*, or *C* are available, and *D*'s votes would be divided between *E* and *F*.

If *E* and *F* were still lowest on the poll, one of them would be excluded, and his votes (including those he had obtained from *D*) would go to *F*, who would have all the first choices given to *D*, *E*, and himself. Thus no votes would be lost, and the party, if the total of the first choices polled for *D*, *E*, *F* entitled it to another seat, would get the seat.

Let us suppose, then, that we have rules which provide for the transfer of a vote so long as there is an unexcluded candidate, whether with a first choice or a subsequent choice, marked on the same paper.([25]) With such rules no votes are lost by a party, and it will be found that the argument in regard to the single transferable vote contained in §§ 6-21 is applicable. Either the Hare quota or the Droop quota will give representation approximately proportional, and the Droop quota will be preferable to the Hare quota.

([24]) There is an objection to such a rule, however, at all events in pre-elections. One of the reasons for preferring the multiple transferable vote to the single transferable vote in an election such as the pre-election of the candidates of a party is that with the single vote a section as small as the quota (one-seventh), and possibly out of sympathy with the rest of the party, may return a candidate who will stand for the party as a whole; whereas with the multiple vote, the quota (if there are three first choices) is three-sevenths, and the rules are intended to prevent the return of any candidate with less than a quota of supporters. If *D*, *E*, *F* are the candidates of a small section, and if *D*'s votes go to *E*, and *E*'s to *F*, *F* has as many votes as if each voter of the section had given him three first choices, and so a section as small as one-seventh is enabled to return a candidate.

([25]) A set of rules providing for all possible cases would be rather complicated. The scrutiny, too, would be difficult. Mr. J. H. Humphreys (*Minutes of Evidence taken before the Royal Commission on Systems of Election*, Stationery Office, London, 1910, Cd. 5352, at p. 40) has pointed out that counting of votes is more laborious when there are several votes on a paper than when there is only one. "Whenever the ballot-paper (as in the Belgian system and with the single transferable vote) represents but one vote only, the process of counting consists of sorting papers according to the votes given, and then in counting the heaps of papers so formed. Whenever there is more than one vote recorded upon a ballot-paper it becomes necessary to extract the particulars of each vote upon recording sheets." With the multiple transferable vote and fractional transfers, fractional values add a further complication. These difficulties are avoided in the Launceston Voting System of the Labour Party in Tasmania, in which no choice can have a fractional value.

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Single Transferable Vote Systems.

71. The argument in §§ 6-21 has been restated more fully and clearly by Mr. F. W. Barford, of Melbourne, in a paper *A Study in Proportional Representation* read at the meeting of the Australasian Association for the Advancement of Science held in Melbourne in January, 1913.

Close Contests.

72. In a note to § 4, I collected some information as to the frequency of close contests. Students interested in this aspect of representation will find further information in the following papers in the Journal of the Royal Statistical Society:—

John Biddulph Martin: *Electoral Statistics: A Review of the Working of our Representative System from* 1832 *to* 1881, *in view of Prospective Changes therein.* Journal of the Royal Statistical Society, March, 1884 (XLVII., 75-115).

J. A. Baines: *Parliamentary Representation in England, illustrated by the Elections of* 1892 *and* 1895. Journal of the Royal Statistical Society, March, 1896 (LIX., 38-118). Table D shows the distribution of seats according to the majority per cent. in 1892 and 1895; Table G the percentage of the majority in each constituency at these elections.

F. Y. Edgeworth: *Miscellaneous Applications of the Calculus of Probabilities.* Journal of the Royal Statistical Society, September, 1898 (LXI., 534-544).

Sir Richard Biddulph Martin: *The Electoral "Swing of the Pendulum."* Journal of the Royal Statistical Society, December, 1906 (LXIX., 655-707).

Single-Member Constituencies.

73. It is well known that single-member constituencies usually fail to give proportional representation to parties, and sometimes put a minority in power.([26]). But it is commonly said that this defect will be remedied if the

([26]) The case against the single-member system is stated by J. H. Humphreys (*Proportional Representation* (London, 1911), Ch. v.); and by Professor J. R. Commons, in his *Proportional Representation* (2nd edition, New York, 1907) at pp. 36-85, his illustrations being taken mainly from elections in the United States.

constituencies are approximately equal in electoral population and if members are elected by a system of preferential voting ensuring that the candidate returned has received votes from a majority of the electors (as in the elections for the Legislative Council in Tasmania). The following paragraphs will show that equality of constituencies and an absolute majority system are insufficient to secure even that the majority in the country shall have a majority in the House, to say nothing of a majority proportional to its strength.([26a])

([26a]) Mr. L. F. Giblin, M.H.A., has pointed out to me that it is possible even with proportional representation for the minority in the country to win a majority of seats. With proportional representation the total representation of each party depends to some extent (though to a much less extent than with single-member districts) on the distribution of the strengths of the parties among the districts. If members are allotted to parties district by district, the representation, although proportional as nearly as possible in each district, may become disproportional for the country as a whole. The only sure way to secure exactly proportional representation is to allot seats in proportion to votes throughout the country, as in the method of the uniform quota. Mr. Giblin writes :—

"It should be noted that proportional voting with the single transferable vote may result in putting a minority in power, and, when the number of constituencies is small, the chance is not a remote one. In Tasmania under the present system, in which the Droop quota is used, assume that the quotas are the same in each division. Let party A return 16 members and party B 14 members. There being 35 quotas in the five districts, the 14 members for party B will represent a majority in the country if B's votes are more than $17\frac{1}{2}$ quotas, i.e., if the sum of its remainders is more than $3\frac{1}{2}$ quotas. If B's votes are more than $17\frac{1}{2}$ quotas, A's are less than $17\frac{1}{2}$, and the sum of its remainders is less than $1\frac{1}{2}$. That is party A (the minority in the country) will get a majority in Parliament if the votes not absorbed as quotas are divided between the parties A and B in less than the ratio 3 : 7. In practice, the parties are fairly equally divided and no party is likely to be represented in any division by less than two members out of six. Within the range thus indicated, it may be assumed approximately that any remainder from 0 to one quota is equally likely to occur. On these conditions, the chance of a majority of 16 members being returned by a minority of voters may be stated roughly as 1 in 14. That is to say, in every 14 elections in which the result was 16 to 14, there would be on an average one in which the minority of the voters returned the majority in Parliament. This, however, is but a small matter compared to the case in which 15 members are returned on each side, when the odds are more than 2 to 1 that one party is entitled to an additional member. If the number of members was altered to 7 in each district, making 35 members in all, this high probability of disproportional representation would be removed, but the chance of a majority of 18 to 17 being returned by a minority of voters would, under the same assumptions as above, be nearly 1 in 4. It should be noted that this possibility of disproportional representation, though not negligible, is small compared to the possibilities with single-member districts, and that it is equally present in the different List systems, excepting only the List system with the Uniform Quota, by which proportional representation is made certain under all circumstances so far as the number of members will allow."

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74. The following is an investigation of the conditions in which the larger of two nearly equal parties may get only a minority of seats in a country divided into single-member constituencies. It is assumed that in each district there are only two parties each with one candidate, or that if there are more than two candidates a system of preferential voting is used to secure that the candidate returned has received votes from an absolute majority of the voters; and that the constituencies are equal in the number of voters.

Let S be the party which has less than half of the votes polled throughout the country, and let L be the party which has more than half; and let the strengths of S, L throughout the country be $(50 - a)\%$ and $(50 + a)\%$. Let there be 100 constituencies and v voters in each.

Consider the constituencies in each of which S has a majority, and consequently wins the seat; let there be x of these, and let the average strength of S in them be $(50 + s)\%$. Consider also the constituencies in which L has a majority; let there be y of these, and let the average strength of l in them be $(50 + l)\%$.

Then, considering the total number of votes obtained by S, we have:

$$x.\frac{50+s}{100}.v + y.\frac{50-l}{100}.v = \frac{50-a}{100}.100v,$$

or

$$\frac{50+s}{100}.x + \frac{50-l}{100}.y = 50-a \qquad (1)$$

Similarly, from L's votes we get—

$$\frac{50-s}{100}.x + \frac{50+l}{100}.y = 50+a \qquad (2)$$

Also—

$$x + y = 100. \qquad (3)$$

Subtracting (1) from (2), we get—

$$ly - sx = 100a \qquad (4)$$

The condition for equal representation of the larger and smaller parties ($x = y = 50$) is

$$l - s = 2a. \qquad (5)$$

As an example, let the average strength of S in the districts in which it is in a majority be 51 %, and let the average strength of L in the districts in which it is in a majority be 57 %, and let the average strengths of the two parties throughout the country be 47 %, 53 %. The values of s, l, a are then $s = 1$, $l = 7$, $a = 3$, and $l - s = 2a$.

Then we have

$$\frac{51}{100} \cdot x + \frac{43}{100} y = 47$$

and

$$\frac{49}{100} \cdot x + \frac{57}{100} y = 53$$

whence $2x - 14y = 600.$

Also $x + y = 100.$

From these equations we find $x = y = 50$; *i.e.*, the representation of the parties is equal, and there is a deadlock in the House, in spite of the 6 % majority which L has over S in the country.

If $l - s > 2a$, we can see from (4) that $x > 50$, *i.e.*, the smaller party gets a majority of the seats.

Thus, if the average strength of S in the districts in which it has a majority is 51 %, and the average strength of L in the districts in which it has a majority is just over 57 %, S, the smaller party, will get more seats than L, the larger party, and so the smaller party will have a majority over the larger in the House, although the larger has a majority of over 6 % above the smaller in the country.

75. J. R. Commons ([27]) gives the following illustration of a distribution of votes which would give the smaller party the majority of seats. A country is divided into 40 districts, and in each of these 5500 electors vote. In 25 districts the smaller party obtains 2800 votes, and the larger 2700 votes; in 15 districts, the smaller party obtains 2000 votes and the larger 3000 votes. The votes polled are, then:

Smaller Party.

2800 in 25 districts	70,000
2000 in 15 districts	30,000
	100,000

Larger Party.

2700 in 25 districts	67,500
3500 in 15 districts	52,500
	120,000

([27]) *Proportional Representation*, 2nd edition (New York, 1911), pp. 48-49.

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The smaller party thus obtains 25 seats with only 100,000 votes, while the larger party with 120,000 votes obtains only 15 seats.

76. Professor Edgeworth ([28]) has shown that in the general elections of 1886, 1892, and 1895 in Great Britain the ratios, (number of Unionist supporters) ÷ (number of Unionists and number of Gladstonians) in the various constituencies, were distributed about an average in accordance with the normal law of error. Such a distribution will give a majority to the larger party in the House which will be a greater percentage of the House than the strength of the larger party in the country is of the total number of electors, as appears from a comparison of the actual result of these elections with the percentages of representation calculated from the curve of error representing the distribution.

The Right Hon. J. Parker Smith, in his evidence before the British Royal Commission on Systems of Election ([29]), gives reasons why the majority is usually exaggerated if single-member constituencies are used. He mentions a calculation by Major Macmahon, who has shown that if in a two-party contest the voters are in the ratio of A to B, then the members elected may be expected to be at least in the ratio of A^3 to B^3. Thus if the strengths of the parties are 55 % to 45 % ($A : B = 11 : 9$), the members may be expected to be in the ratio of 11^3 to 9^3, or nearly 2 to 1; *i.e.*, a party with 5 % more than half the electors may be expected to get nearly two-thirds of the members.

77. Statistics of elections in single-member constituencies in which the conditions of § 74—approximate equality between the constituencies in number of votes, and either only two candidates or else preferential voting—are fulfilled, are scarce. It is one of the disadvantages of single-member constituencies, especially in new countries, that redistribution is required much more frequently than with grouped districts, and the sizes of the constituencies are usually by no means equal when a few years have elapsed since the last redistribution. Also, the proportion of electors who vote varies largely from one constituency to another. In Australia at the

([28]) *Miscellaneous Applications of the Calculus of Probabilities*, Journal of the Royal Statistical Society, September, 1898 (LXI., 534 544).

([29]) *Minutes of Evidence taken before the Royal Commission on Systems of Election* (Stationery Office. London, 1910, Cd. 5352), Question 1253, p. 81.

House of Representatives election in 1910 ([30]) the numbers of electors enrolled in the constituencies in each State varied up to about 15 % from the mean for the State. The number of voters varied even more: leaving out a few very large and a few very small constituencies, the numbers of voters varied up to about 20 % or 25 % from the mean for the State. These numbers, however, were much more nearly equal than is usually the case with single-member districts, and this election is consequently more suitable as an illustration of the arguments in §§ 73-76 than any other of which I have statistics.

In New South Wales, at this election, there were contests between one Liberal candidate and one Labour candidate (or between two such candidates and a third who obtained so few votes that he need not be taken into account) in 24 constituencies. In these the Labour candidates polled 245,000 votes and the Liberal candidates 203,000; and consequently the Labour party, in proportion to its strength, was entitled to 13 seats and the Liberal party to 11. The seats won by the parties were Labour 17, Liberal 7.

In Queensland, there were contests between one Labour candidate and one Liberal candidate in each of the nine constituencies. The Labour candidates polled 89,000 votes and the Liberal candidates 76,000; the members to which the parties were entitled were consequently 5 to 4. The seats won by the parties were Labour 7, Liberal 2.

In Victoria, there were contests between one Labour candidate and one Liberal candidate in 19 constituencies. The Labour candidates polled 216,000, and the Liberal candidates 192,000; the members to which the parties were entitled were consequently 10 and 9; and these were the numbers of seats actually won by the parties.

The results in New South Wales and Queensland, then, confirm the predictions referred to in § 76, that single-member constituencies will usually exaggerate the majority obtained by the larger party. In Victoria the election gave exactly proportional representation in the 19 constituencies considered. But it is to be noted on the one hand that a loss by Liberal candidates in Victoria of only 800 votes, spread over four constituencies in which their majorities were very small, would have given the Labour party 14 members and reduced the Liberal mem-

([30]) See *Elections, 1910. Statistics relating to the Senate Election; the General Election for the House of Representatives* (&c.). (Papers of the Parliament of the Commonwealth of Australia. No. 1 of 1910).

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bers to 5; and on the other hand that a gain of only 700 votes in a constituency in which the Labour majority was small would have given the Liberal party 10 seats and the Labour party only 9—that is, a party having only 47 per cent. of the voters would have had a majority of the seats.

Remark as to the Conclusions of this Paper.

78. In discussing the various methods of proportional representation noticed in this paper, with the exception of the method of the uniform quota, I have considered the result that may be expected to occur in a single district. But in estimating the probability that an election throughout the country will give proportional or disproportional representation, it must be remembered that under- or over-representation in some districts is likely to be balanced by over- or under-representation in others, unless the system used has been deliberately constructed (as was the D'Hondt) with the object of favouring one party (the larger party in that case); and consequently that the result of an election in many districts is more likely to be in proportion to the strengths of the parties than an election in one district.([31]) This qualification, however, does not apply to the argument against the single-member system, in which the country has been considered as a whole.

([31]) But see § 73, note ([26a]), by Mr. L. F. Giblin.