SYMMETRY OF NODAL SOLUTIONS FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS ON A BALL

JUNCHENG WEI AND MATTHIAS WINTER

ABSTRACT. In [40], it was shown that the following singularly perturbed Dirichlet problem

$$\epsilon^2 \Delta u - u + |u|^{p-1}u = 0, \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega$$

has a nodal solution u_ϵ which has the least energy among all nodal solutions. Moreover, it is shown that u_ϵ has exactly one local maximum point P_1^ϵ with a positive value and one local minimum point P_2^ϵ with a negative value and, as $\epsilon \to 0$,

$$\varphi(P_1^\epsilon, P_2^\epsilon) \to \max_{(P_1,P_2)\in\Omega\times\Omega} \varphi(P_1, P_2),$$

where $\varphi(P_1, P_2) = \min(\frac{|P_1-P_2|}{2}, d(P_1, \partial\Omega), d(P_2, \partial\Omega))$. The following question naturally arises: where is the **nodal surface** $\{u_\epsilon(x) = 0\}$? In this paper, we give an answer in the case of the unit ball $\Omega = B_1(0)$. In particular, we show that for ϵ sufficiently small, P_1^ϵ, P_2^ϵ and the origin must lie on a line. Without loss of generality, we may assume that this line is the x_1-axis. Then u_ϵ must be even in $x_j, j = 2, ..., N$, and odd in x_1. As a consequence, we show that $\{u_\epsilon(x) = 0\} = \{x \in B_1(0)|x_1 = 0\}$. Our proof is divided into two steps: first, by using the method of moving planes, we show that P_1^ϵ, P_2^ϵ and the origin must lie on the x_1-axis and u_ϵ must be even in $x_j, j = 2, ..., N$. Then, using the Liapunov-Schmidt reduction method, we prove the uniqueness of u_ϵ (which implies the odd symmetry of u_ϵ in x_1). Similar results are also proved for the problem with Neumann boundary conditions.

1. INTRODUCTION

We consider **nodal** solutions to the following singularly perturbed semilinear elliptic problem

$$\begin{cases} \epsilon^2 \Delta u - u + |u|^{p-1}u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\epsilon > 0$ is a small constant, $\Delta := \sum_{j=1}^N \frac{\partial^2}{\partial x_j \partial x_j}$ denotes the Laplace operator in R^N, and

$$1 < p < \left(\frac{N+2}{N-2}\right)_+ \left(= \frac{N+2}{N-2} \text{ when } N \geq 3; = +\infty \text{ when } N = 1, 2\right).$$

1991 *Mathematics Subject Classification.* Primary 35J20, 35J65; Secondary 35J40.

Key words and phrases. symmetry of nodal solutions, singular perturbation problems, method of moving planes, Liapunov-Schmidt reduction.

Problem (1.1) arises in various applications, such as chemotaxis, population genetics, chemical reactor theory, etc. In the past few years the effect of the geometry or the topology of Ω on the solvability and /or the multiplicity of positive solutions of problem like (1.1) has been extensively studied, see [6], [7], [8], [11], [12], [14], [15], [17], [30], [37], and the references therein. In particular, in [37], Ni and Wei established that for ϵ sufficiently small problem (1.1) has a positive least-energy solution with one local (hence global) maximum point P_ϵ and $d(P_\epsilon, \partial\Omega)$ tends to $\max_{P\in\Omega} d(P, \partial\Omega)$, where $d(P, \partial\Omega)$ is the usual distance function of P to the boundary $\partial\Omega$. In [49] the second author showed a kind of converse of the result in [37], namely for every strict local maximum point of the distance function, say P, there exists a family of positive solutions u_ϵ of (1.1) with a single peak P_ϵ in Ω such that $P_\epsilon toP$ as $\epsilon \to 0$. The effect of the geometry on the existence of multi-peaked solutions of (1.1) has been studied in [8], [12], [14], [15], [17], [30], [38] and the references therein. Recent surveys can be found in [42] and [56].

In [40] Noussair and the first author established the existence of a "least energy" **nodal** solution and showed that, for small ϵ, it has exactly one local maximum point P_1^ϵ with a positive value and one local minimum point P_2^ϵ with a negative value. Moreover, as $\epsilon \to 0$, $\varphi(P_1^\epsilon, P_2^\epsilon) \to \max_{(P_1,P_2)\in\Omega\times\Omega} \varphi(P_1, P_2)$, where the function $\varphi(P_1, P_2)$ is defined by

$$\varphi(P_1, P_2) = \min(\frac{|P_1 - P_2|}{2}, d(P_1, \partial\Omega), d(P_2, \partial\Omega)). \tag{1.3}$$

A natural question is: Where is the **nodal surface (or nodal line)** $\{x \in \Omega | u_\epsilon(x) = 0\}$?

In this paper, we give an answer in the case of the domain Ω being the unit ball $B = \{x \in R^N | |x| < 1\}$. Naturally, one may ask: is the solution u_ϵ **odd** in one-direction (say x_1)? Our answer is yes.

In fact, we can give a complete characterization of all possible two-peaked nodal solutions. More precisely, a solution u_ϵ is called a two-peaked nodal solutions to (1.1) if the following holds:

(a) for ϵ sufficiently small, u_ϵ has only one local maximum point P_1^ϵ and one local minimum point P_2^ϵ, and $u_\epsilon(P_1^\epsilon) > 0, u_\epsilon(P_2^\epsilon) < 0$,

(b) the energy of u_ϵ is bounded, namely

$$\limsup_{\epsilon\to 0}(\epsilon^{-N} J_\epsilon[u_\epsilon]) < +\infty, \tag{1.4}$$

where $J_\epsilon[u]$ is the energy functional associated with (1.1):

$$J_\epsilon[u] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}, u \in H_0^1(\Omega). \tag{1.5}$$

The following is our first result:

Theorem 1.1. *Let ϵ be small enough and let u_ϵ be a two-peaked nodal solution of (1.1) with exactly one local maximum point P_1^ϵ which has positive value and exactly one local minimum point P_2^ϵ which has a negative value. Then the points P_1^ϵ, P_2^ϵ and the origin lie on a line. Without loss of generality, we may assume that this line is the x_1-axis. Then u_ϵ is even in $x_j, j = 2, ..., N$ and odd in x_1. As a consequence, $P_1^\epsilon = -P_2^\epsilon$, the nodal surface is given by $\{u_\epsilon(x) = 0\} = \{x \in B_1(0)|x_1 = 0\}$ and the two-peaked nodal solution to (1.1) is unique.*

Our method can also be applied to the corresponding Neumann problem:

$$\begin{cases} \epsilon^2\Delta u - u + |u|^{p-1}u = 0 \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.6}$$

In [41], Noussair and the first author proved the existence of a nodal solution to (1.6) which has the least energy among all nodal solutions. Moreover, it has exactly one local maximum point $P_1^\epsilon \in \partial\Omega$ which has a positive value and one local minimum point $P_2^\epsilon \in \partial\Omega$ which has a negative value. It is shown that, as $\epsilon \to 0$, $H(P_1^\epsilon) \to \max_{P\in\partial\Omega} H(P), H(P_2^\epsilon) \to \max_{P\in\partial\Omega} H(P), |P_1^\epsilon - P_2^\epsilon|/\epsilon \to +\infty$, where $H(P)$ is the mean curvature of the boundary $\partial\Omega$ at P. Since $H(P) = 1$ when $\Omega = B_1(0)$, we can only conclude that $|P_1^\epsilon - P_2^\epsilon|/\epsilon \to +\infty$. Now we have

Theorem 1.2. *Suppose that $\Omega = B_1(0)$. Let ϵ be small enough and let u_ϵ be a nodal solution of (1.6) with exactly one local maximum point $P_1^\epsilon \in \partial\Omega$ having positive value and one local minimum point $P_2^\epsilon \in \partial\Omega$ having negative value. Then we must have $P_1^\epsilon = -P_2^\epsilon$. Without loss of generality, we may assume that $P_1^\epsilon = (1, 0, ..., 0)$ Then u_ϵ is even in $x_j, j = 2, ..., N$ and is odd in x_1. As a consequence, the nodal surface satisfies $\{u_\epsilon(x) = 0\} = \{x \in B_1(0)|x_1 = 0\}$.*

Our proofs of Theorems 1.1 and 1.2 involve the use of the method moving planes (MMP) to **nodal solutions** and the method of Liapunov-Schmidt reduction.

MMP is a powerful method in showing symmetry for **positive solutions** to Dirichlet problems [20]. For positive solutions to Neumann problems, it has been used recently to show partial symmetry for blow-up and concentration problems [9], [31], [32]. In particular, we mention the results of Lin and Takagi [32] who showed that for the Neumann problem (1.6), (positive) single-boundary spike solutions must be axially symmetric, whereas single interior spike solutions must be radially symmetric. Further, for the two-boundary spike solution the two local maximum points $P_1^\epsilon \in \partial\Omega$, $P_2^\epsilon \in \partial\Omega$ must satisfy $P_1^\epsilon = -P_2^\epsilon$. By using this information, they showed the uniqueness of the single-boundary spike solution and of the two boundary spike solution, respectively. (We remark that the uniqueness of the single-boundary solutions and the single-interior spike solutions in general domains is studied in [5], [39], [53], [51].) As far as we know, there have been no previous results on the application of MMP to nodal solutions.

We adopt the method of [32] to **nodal** solutions. However, MMP alone can not establish the oddness of u_ϵ in x_1. To this end, we follow [34], where a combination of MMP and the Liapunov-Schmidt reduction method is used to show the uniqueness of two- and three-peaked positive solutions to singularly perturbed Neumann problems. The method of Liapunov-Schmidt reduction has been used in singularly perturbed problems to obtain existence and multiplicity of solutions ([2], [3], [4], [5], [10], [13], [14], [18], [24], [25], [27], [29], [43], [44], [54], [55]). As far as we know, the results of this paper are the first in using a combination of **both** methods to prove the partial symmetry for **nodal solutions**.

More precisely, our proof of Theorem 1.1 proceeds in two steps:

Step 1. We use MMP to show that $P_1^\epsilon, P_2^\epsilon$ and the origin must lie on a line (say the x_1-axis). Furthermore, u_ϵ is even in x_j, $j = 2, ..., N$. So, without loss of generality, we may assume that $P_1^\epsilon = (l_1^\epsilon, 0, ..., 0), P_2^\epsilon = (l_2^\epsilon, 0, ..., 0)$. This reduces our problem to one on R^2 with the two scalar variables l_1^ϵ and l_2^ϵ.

Step 2. We now show that u_ϵ is odd in x_1, namely $u_\epsilon(x_1, ..., x_N) = -u_\epsilon(-x_1, ..., x_N)$. To achieve this, we show the uniqueness of u_ϵ if ϵ is small enough. We have to compute the degree of u_ϵ restricted to the symmetry class obtained in Step 1. We use the Liapunov-Schmidt reduction method and asymptotic analysis to show that u_ϵ is nondegenerate and that the degree at u_ϵ is exactly $(-1)^0$. This proves the uniqueness.

Finally, we remark that our results are also true if we replace $|u|^{p-1}u$ by some more general nonlinearity $f(u)$ which satisfies some nondegeneracy conditions. We omit the details.

The structure of the paper is as follows:

In Section 2, we shall study some properties of nodal solutions with two peaks.

In Section 3, we use the well-known method of moving planes (MMP) to show that $P_1^\epsilon, P_2^\epsilon$ and the origin must lie on a line and that u_ϵ is axially symmetric about that line.

In Section 4, Section 5 and Section 6, we prove the uniqueness of nodal solutions in the partial symmetry class introduced in Section 3. As a consequence, we show that u_ϵ is odd in x_1.

In Section 4, we present some preliminaries on the reduction from the infinite dimensional space $H_0^1(\Omega)$ to a finite dimensional problem on the space of the locations of the maximum and minimum points. In Section 5, we compute the first and second order derivatives of reduced the problem.

In Section 6, we show the uniqueness of two-peaked nodal solutions by computing its Morse index (restricted to a certain symmetry class).

Finally in Section 7, we show how the ideas can be adopted to prove the uniqueness of the two-boundary-peaked nodal solution and thus prove Theorem (1.2).

Several technical estimates are proved in Appendices A and B .

It is always assumed that $\epsilon > 0$ is small and $\delta > 0$ is a fixed but small constant. Throughout the paper, we use C to denote various constants independent of ϵ small. We use $P_{j,i}$ to denote the $i-$th component of P_j.

Acknowledgments. The research of JW is supported by an Earmarked Grant from RGC of Hong Kong. MW thanks the Department of Mathematics at CUHK for their kind hospitality.

2. Some Properties of u_ϵ

Let u_ϵ be a two-peaked nodal solution of (1.1) for $\Omega = B_1(0) =: B$ with one local maximum point P_1^ϵ having positive value and one local minimum point P_2^ϵ having negative value. In this section, we study some properties of u_ϵ, which will be useful in the next section.

The asymptotic behavior of u_ϵ can be characterized by the unique solution of the following ground-state equation

$$\begin{cases} \Delta w - w + w^p = 0, w > 0 \ \text{ in } R^N, \\ w(0) = \max_{y \in R^N} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{2.1}$$

It is well-known that problem (2.1) has a unique solution, called w, which is radially symmetric and nondegenerate, namely

$$\text{Kernel}(\Delta - 1 + pw^{p-1}) = \text{span}\left\{ \frac{\partial w}{\partial y_1}, ..., \frac{\partial w}{\partial y_N} \right\}. \tag{2.2}$$

The uniqueness of w is proved in [28] and the radial symmetry of w follows from the well-known result of Gidas, Ni and Nirenberg [21]. Moreover, we have the following asymptotic behavior of w:

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r} \left(1 + O\left(\frac{1}{r}\right)\right), \quad w^{'}(r) = -A_N r^{-\frac{N-1}{2}} e^{-r} \left(1 + O\left(\frac{1}{r}\right)\right), \tag{2.3}$$

for r large, where $A_N > 0$ is a generic constant.

We summarize the asymptotic behavior of u_ϵ as follows.

Lemma 2.1. *Let u_ϵ be a two-peaked nodal solution of (1.1) for $\Omega = B_1(0) =: B$ (with unique local maximum point P_1^ϵ and unique local minimum point P_2^ϵ. Then we have $\|u_\epsilon - w(\frac{x-P_1^\epsilon}{\epsilon}) + w(\frac{x-P_2^\epsilon}{\epsilon})\|_{L^\infty(\Omega)} \to 0$. Moreover, it holds that (a) there exists a $\delta > 0$ such that*

$$d(P_1^\epsilon, \partial\Omega) \geq \delta, \quad d(P_2^\epsilon, \partial\Omega) \geq \delta, \quad |P_1^\epsilon - P_2^\epsilon| \geq \delta, \ \textit{as } \epsilon \to 0, \tag{2.4}$$

(b) as a consequence,

$$P_1^\epsilon \to (\frac{1}{2}, 0, ..., 0), P_2^\epsilon \to (-\frac{1}{2}, 0, ..., 0) \quad \textit{as } \epsilon \to 0. \tag{2.5}$$

Proof: The proof of the first statement is standard. See [35], [36], and [37]. The proof of (2.4) is similar to that of Lemma 2.1 of [12].

To prove (2.5), we may assume without loss of generality that $P_1^\epsilon \to P_1^0 = (l_1, 0, ..., 0)$ for some $l_1 > 0$. Let $P_2^\epsilon \to P_2^0$. Then, similar to the proof of Theorem 1.1 of [48], we have

$$\int_{\partial\Omega} \frac{z - P_1^0}{|z - P_1^0|} d\mu_{P_1^0}(z) - c_1 \frac{P_1^0 - P_2^0}{|P_1^0 - P_2^0|} = 0, \tag{2.6}$$

$$\int_{\partial\Omega} \frac{z - P_2^0}{|z - P_2^0|} d\mu_{P_2^0}(z) - c_2 \frac{P_2^0 - P_1^0}{|P_2^0 - P_1^0|} = 0, \tag{2.7}$$

where $c_1, c_2 \geq 0$ and $d\mu_P(z) \in \Lambda_P$ which is defined by

$$\Lambda_P = \left\{ d\mu_P(z) \in M(\partial\Omega) \;\middle|\; \begin{array}{l} \text{there exist } \epsilon_k \to 0, P_{\epsilon_k} \to P, \text{ such that} \\ \lim_{\epsilon_k \to 0} \frac{e^{-\frac{2|z-P_{\epsilon_k}|}{\epsilon_k}} dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_{\epsilon_k}|}{\epsilon_k}} dz} \to d\mu_P(z) \end{array} \right\}$$

where $M(\partial\Omega)$ is the set of all bounded Borel measures on $\partial\Omega$ and the convergence is the weak-$*$ convergence of measures. In particular, if $L = |P_1^0 - P_2^0| > 2d(P_i^0, \partial\Omega)$, then $c_i = 0$.

Thus, we have that $L = |P_1^0 - P_2^0| = 2d(P_1^0, \partial\Omega) = 2d(P_2^0, \partial\Omega)$. Since P_1^0 is on the x_1-axis, we see that $d\mu_{P_1^0} = \delta_{(1,0,...,0)}$. From (2.6), we conclude that P_2^0 must also lie on the x_1-axis and hence $P_1^0 = -P_2^0 = (\frac{1}{2}, 0, ..., 0)$.

□

Our next result shows the existence of solutions having the properties of Theorem 1.1.

Lemma 2.2. *There exists a two-peaked nodal solution $\hat{u}_\epsilon$ of (1.1) for $\Omega = B_1(0) =: B$ such that $\hat{u}_\epsilon$ is even in $x_j, j = 2, ..., N$ and is odd in x_1. Moreover, the local maximum point P_1^ϵ and the local minimum point P_2^ϵ of $\hat{u}_\epsilon$ satisfy: $P_1^\epsilon = -P_2^\epsilon$, $P_1^\epsilon = (l_\epsilon, 0, ..., 0)$, $l_\epsilon \to \frac{1}{2}$.*

Proof: Let $\Omega^+ = B \cap \{x_1 > 0\}$ and v_ϵ be the least energy positive solution constructed in [37]. By the symmetry of Ω^+, we may assume that v_ϵ is even in $x_j, j = 2, ..., N$ and that the only maximum point of v_ϵ lies on the x_1-axis and approaches the point $(\frac{1}{2}, 0, ..., 0)$. Now let

$$\hat{u}_\epsilon = \begin{cases} v_\epsilon(x_1, x_2, ..., x_N), \text{ if } x_1 \geq 0, \\ -v_\epsilon(-x_1, x_2, ..., x_N), \text{ if } x_1 < 0. \end{cases} \tag{2.8}$$

It is easy to see that $\hat{u}_\epsilon$ is a two-peaked nodal solution of (1.1) and $\hat{u}_\epsilon$ satisfies the properties of Lemma 2.2.

□

In the rest of the paper, we shall prove the uniqueness of the nodal solution, namely that $u_\epsilon = \hat{u}_\epsilon$, provided that ϵ is sufficiently small.

3. MMP Applied to Nodal Solutions of (1.1)

In this section, we apply the well-known method of moving planes to a two-peaked nodal solution u_ϵ of (1.1) for $\Omega = B_1(0) =: B$. We follow the proofs given in Section 3 of [32], where it is shown that for two boundary spikes $P_1^\epsilon, P_2^\epsilon$ it holds that $P_1^\epsilon = -P_2^\epsilon$, provided that ϵ is sufficiently small.

Let $P_1^\epsilon, P_2^\epsilon$ be the local maximum and the local minimum point of u_ϵ, respectively. Our main result in this section says that $P_1^\epsilon, P_2^\epsilon$ and the origin must lie on a line and, moreover, u_ϵ is axially symmetric with respect to the line.

We prove this by contradiction. Suppose P_1^ϵ, P_2^ϵ and the origin are not on a line. (So they form a triangle.) Then $P_1^\epsilon, P_2^\epsilon$ and the origin lie in a two-dimensional hyperplane which without loss of generality is given by $\{(x_1, ..., x_N) | x_2 = ... = x_{N-1} = 0\}$. We may further assume that

$$t_\epsilon = P_{1,N}^\epsilon = -P_{2,N}^\epsilon > 0, P_{2,1}^\epsilon > 0. \tag{3.1}$$

Note that (3.1) is possible since $P_1^\epsilon, 0, P_2^\epsilon$ do not lie on a line.

Let $\theta_\epsilon = \arccos\left(\frac{P_{1,1}^\epsilon}{\sqrt{(P_{1,1}^\epsilon)^2 + (P_{1,N}^\epsilon)^2}}\right) \in (0, \pi - \arctan(\frac{-P_{2,N}^\epsilon}{P_{2,1}^\epsilon}))$.

Set $\mathbf{e}_\theta = (\sin\theta, 0, ..., 0, -\cos\theta)$, let Π_{N-1}^θ be the $(N-1)-$dimensional hyperplane perpendicular to the vector $\mathbf{e}_\theta$, and denote by x^θ the reflection of x with respect to Π_{N-1}^θ. Set

$$w_\epsilon^\theta(x) = u_\epsilon(x) - u_\epsilon(x^\theta) \text{ for } x \in \Sigma_\theta,$$

where Σ_θ is the connected component of $\Omega \backslash \Pi_{N-1}^\theta$ containing P_1^ϵ. Obviously, w_ϵ^θ satisfies

$$\begin{cases} \epsilon^2 \Delta w_\epsilon^\theta + c_\epsilon^\theta(x) w_\epsilon^\theta = 0 \text{ in } \Sigma_\theta, \\ w_\epsilon^\theta(x) = 0 \text{ on } \partial\Sigma_\theta, \end{cases} \tag{3.2}$$

where

$$c_\epsilon^\theta(x) = -1 + \frac{|u_\epsilon(x)|^{p-1}u_\epsilon(x) - |u_\epsilon(x^\theta)|^{p-1}u_\epsilon(x^\theta)}{u_\epsilon(x) - u_\epsilon(x^\theta)}. \tag{3.3}$$

We prove our claim in a series of three steps.

Step 1: We first prove that

$$w_\epsilon^0(x) > 0 \text{ for } x \in \Sigma_0 = \{x \in \Omega | x_N > 0\}. \tag{3.4}$$

Note that since P_1^ϵ is the only local maximum point of u_ϵ , P_1^ϵ is actually the global maximum point of u_ϵ. Similarly, P_2^ϵ is the global minimum point of u_ϵ. Let $\bar{P}_2^\epsilon$ be the reflection point of P_2^ϵ with respect to Π_{N-1}^0. Note that $\bar{P}_{2,N}^\epsilon = t_\epsilon = P_{1,N}^\epsilon$, by (3.1). (So $\bar{P}_2^\epsilon = (P_{2,1}^\epsilon, ..., -P_{2,N}^\epsilon)$.)

For a contradiction, we assume that the set

$$E_\epsilon := \{x \in \Sigma_0 | w_\epsilon^0(x) < 0\}$$

is non-empty. (The following argument is for a subsequence of $\epsilon_i \to 0$. For simplicity, we use the notation ϵ to denote ϵ_i.)

Case 1: $\frac{t_\epsilon}{\epsilon} \to +\infty$ as $\epsilon \to 0$.

In this case, it is easy to see that for arbitrarily large $R > 0$, we have $E_\epsilon \subset (B_{\epsilon R}(P_1^\epsilon) \cup B_{\epsilon R}(\bar{P}_2^\epsilon))^c$ for ϵ small enough, since P_1^ϵ is a global maximum point with a positive value and P_2^ϵ is a global minimum point with a negative value. Hence $|u_\epsilon| \leq \delta$ for $x \in E_\epsilon$ and ϵ small. Moreover, $w_\epsilon^0(P_1^\epsilon) > 0, w_\epsilon^0(\bar{P}_2^\epsilon) > 0$. This implies that

$$c_\epsilon^\theta(x) \leq -\frac{1}{2} \quad \text{for } x \in E_\epsilon. \tag{3.5}$$

Now by (3.2), the minimum value of w_ϵ^0, if it is negative, must be obtained on the boundary of Σ_0, which is impossible since $w_\epsilon^\theta = 0$ on $\partial\Sigma_0$. So E_ϵ is empty. By the Maximum Principle, $w_\epsilon^0 > 0$.

This finishes Case 1.

Case 2: $\frac{t_\epsilon}{\epsilon} \to s$ as $\epsilon \to 0$, where $s \in (0, +\infty)$.

In this case, since $\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon} \to +\infty$, we have $\frac{|P_1^\epsilon - \bar{P}_2^\epsilon|}{\epsilon} \to +\infty$. Let $x_\epsilon \in \bar{E}_\epsilon$ be such that

$$w_\epsilon^0(x_\epsilon) = \inf_{E_\epsilon} w_\epsilon^0(x) < 0. \tag{3.6}$$

Assume for the moment that

$$\text{limsup}_{\epsilon \to 0} \left\{ \frac{\min(|x_\epsilon - P_1^\epsilon|, |x_\epsilon - \bar{P}_2^\epsilon|)}{\epsilon} \right\} \to +\infty.$$

Then, by Lemma 2.1, $u_\epsilon(x_\epsilon) \to 0$, $c_\epsilon(x_\epsilon) < -\frac{1}{2} < 0$ and $0 \le \epsilon^2 \Delta w_\epsilon^0(x_\epsilon) = -c_\epsilon^0(x_\epsilon) w_\epsilon^0(x_\epsilon) < 0$, a contradiction. Therefore we conclude that

$$\min(|x_\epsilon - P_1^\epsilon|, |x_\epsilon - \bar{P}_2^\epsilon|) \le R\epsilon$$

for some $R > 0$.

Without loss of generality, we may assume that $|x_\epsilon - P_1^\epsilon| \le R\epsilon$. (The other case is exactly the same.) Let $\hat{P}_1^\epsilon$ be the projection point of P_1^ϵ on Π_{N-1}^0. That is, $\hat{P}_1^\epsilon = (P_{1,1}^\epsilon, 0, ..., 0)$.

Let

$$x = \hat{P}_1^\epsilon + \epsilon y, \quad v_\epsilon(y) = u_\epsilon(x), \quad x_\epsilon = \hat{P}_1^\epsilon + \epsilon y_\epsilon.$$

Set $y_\epsilon = (y_\epsilon^{'}, y_{\epsilon,N})$. Then $y_{\epsilon,N} \ge 0$ and let us assume that $y_{\epsilon,N} \to \eta_* \ge 0, y_\epsilon^{'} \to y_*^{'}$. We claim that $\eta_* > 0$. In fact, by our assumption, $\frac{P_{1,N}^\epsilon}{\epsilon} \to s > 0$. Then $v_\epsilon(y) \to w(y - se_N)$ in $C_{loc}^2(R^N)$, where $e_N = (0, ..., 0, 1)$. (Observe that $\frac{|P_1^\epsilon - \bar{P}_2^\epsilon|}{\epsilon} \to +\infty$.) If $\eta_* = 0$, then $\frac{\partial w_\epsilon^0(\epsilon y_\epsilon + P_1^\epsilon)}{\partial y_N} \to 2\frac{\partial w}{\partial y_N}(y_*^{'}, s) < 0$ which contradicts to the fact that $\nabla w_\epsilon^0(x_\epsilon) = 0$. So $\eta_* > 0$. In this case, $w_\epsilon^0(x_\epsilon) \to w(y_*^{'}, \eta_* - s) - w(y_*^{'}, -\eta_* - s) > 0$ if $\eta_* > 0$, for ϵ small. A contradiction again.

Case 3: $\frac{t_\epsilon}{\epsilon} = 0$.

This is the most complicated case.

Let $B_+ = B \cap \{x_N > 0\}$. Set $N_\epsilon := \max_{x \in B_+} |w_\epsilon^0(x)|$, and let $\tilde{x}_\epsilon \in \bar{B}_+$ be such that $|w_\epsilon^0(\tilde{x}_\epsilon)| = N_\epsilon$. Then it is easy to see that

$$\min(|\tilde{x}_\epsilon - P_1^\epsilon|, |\tilde{x}_\epsilon - \bar{P}_2^\epsilon|) \le R\epsilon \quad \text{for some } R > 0.$$

Without loss of generality we may assume that $|\tilde{x}_\epsilon - P_1^\epsilon| \le R\epsilon$. We rescale

$$x = \hat{P}_1^\epsilon + \epsilon y, \quad \tilde{w}_\epsilon^0(y) = \frac{1}{N_\epsilon} w_\epsilon^0(\hat{P}_1^\epsilon + \epsilon y). \tag{3.7}$$

Let $P_1^\epsilon = \hat{P}_1^\epsilon + \epsilon \zeta_\epsilon e_N$. Then similar to the proof of Case 3 of [32], we conclude that $\tilde{w}_\epsilon^0(y) \to c\frac{\partial w}{\partial y_N}$ in $C_{loc}^2(R^n)$, for some $c < 0$ and moreover,

$$C_0^{-1} \le \frac{\zeta_\epsilon}{N_\epsilon} \le C_0. \tag{3.8}$$

Next, we let $\hat{P}_2^\epsilon$ be the projection point of P_2^ϵ on Π_{N-1}^0 and rescale $\hat{w}_\epsilon^0(y) = \frac{1}{N_\epsilon} w_\epsilon^0(\hat{P}_2^\epsilon + \epsilon y)$. As in [32], we show that $\hat{w}_\epsilon^0(y) \to \gamma \frac{\partial w}{\partial y_N}$ in $C_{loc}^2(R^N)$ for some $\gamma < 0$.

Now let $x_\epsilon \in \bar{E}_\epsilon$ be such that (3.6) holds. Then as before,

$$\min(|x_\epsilon - P_1^\epsilon|, |x_\epsilon - \bar{P}_2^\epsilon|) \le R\epsilon$$

for some $R>0$. We may assume that $|x_\epsilon - P_1^\epsilon| \le R\epsilon$. Let

$$x = \hat{P}_1^\epsilon + \epsilon y, \quad x_\epsilon = \hat{P}_1^\epsilon + \epsilon y_\epsilon.$$

Then $y_\epsilon = (y_\epsilon^{'}, y_{\epsilon,N}) \to y_* = (y_*^{'}, \eta_*)$ with $\eta_* \ge 0$. Since $\tilde{w}_\epsilon^0(y_\epsilon) < 0, \tilde{w}_\epsilon^0(y_\epsilon^{'}, 0) = 0$, we conclude that $\eta_* = 0$ and by the mean value theorem

$$\frac{\partial \tilde{w}_\epsilon^0}{\partial y_N}(y_\epsilon^{'}, \xi_\epsilon) < 0$$

for some $\xi_\epsilon \in (0, y_{\epsilon,N})$. By letting $\epsilon \to 0$, we obtain that

$$0 \ge \lim_{\epsilon \to 0} \frac{\partial \tilde{w}_\epsilon^0}{\partial y_N}(y_\epsilon^{'}, \xi_\epsilon) = c \frac{\partial^2 w}{\partial y_N^2}(y_*^{'}, 0) > 0$$

since $c<0$. A contradiction.

The other case $|x_\epsilon - \bar{P}_2^\epsilon| \le R\epsilon$ can be ruled out in the same way.

This finishes Step 1.

Step 2. Let

$$\theta_0 = \sup\,\{\bar{\theta} | w_\epsilon^\theta > 0 \text{ for } x \ \in \Sigma_\theta \text{ and } 0 \le \theta \le \bar{\theta}\}.$$

By Step 1, $\theta_0 > 0$. By the definition of $\theta_0, w_\epsilon^{\theta_0} \ge 0$ in Σ_{θ_0} and if $w_\epsilon^{\theta_0}(x) > 0$ for some $x \in \Sigma_{\theta_0}$, then $w_\epsilon^{\theta_0} > 0$ in Σ_{θ_0} by the maximum principle. So $\theta_0 \ge \theta_\epsilon$ and $w_\epsilon^{\theta_0} \equiv 0$ on Σ_{θ_0}. Since P_1^ϵ is a local maximum point, we see that $\theta_0 \le \theta_\epsilon$. Hence $\theta_0 \equiv \theta_\epsilon$ and $w_\epsilon^{\theta_\epsilon}(x) \equiv 0$ for $x \in \Sigma_{\theta_\epsilon}$. Since u_ϵ has exactly one local maximum and one local minimum point, this implies that $P_1^\epsilon, P_2^\epsilon$ and the origin must lie on a line.

Step 3: By Step 2, $P_1^\epsilon, P_2^\epsilon$ and the origin must lie on a line. Without loss of generality, we may assume that this line is the x_1-axis. We now claim that u_ϵ is even in x_N. In fact, we prove that

$$w_\epsilon^0(x) \equiv 0 \text{ on } \Sigma_0.$$

Suppose that there exists $\epsilon \to 0$ such that

$$N_\epsilon = \sup_{x \in \Sigma_0} |w_\epsilon^0(x)| > 0.$$

Let $\tilde{x}_\epsilon \in \Sigma_0$ be such that $|w_\epsilon^0(\tilde{x}_\epsilon)| = N_\epsilon$. As before, we may assume that $\min(|\tilde{x}_\epsilon - P_1^\epsilon|, |\tilde{x}_\epsilon - P_2^\epsilon|) \le R\epsilon$ for some $R>0$. Without loss of generality, we may assume that $|\tilde{x}_\epsilon - P_1^\epsilon| \le R\epsilon$. As in Case 3 above, $\tilde{w}_\epsilon^0(y) = \frac{w_\epsilon^0(P_1^\epsilon + \epsilon y)}{N_\epsilon} \to c\frac{\partial w}{\partial y_N}(y)$ in $C_{loc}^2(R^N)$ for some constant $c \ne 0$. But $\nabla u_\epsilon(P_1^\epsilon) = 0$ and hence $\frac{\partial \tilde{w}_\epsilon^0}{\partial y_N}(0) = 0$, $c\frac{\partial^2 w}{\partial y_N^2}(0) = 0$ which forces $c=0$. A contradiction.

Similarly we can prove that u_ϵ is even in $x_j, j = 2, ..., N-1$.

4. Uniqueness Proof I: Reduction to Finite-Dimensional Problem

In this section, Section 5 and Section 6, we shall prove the uniqueness of two-peaked nodal solutions. Our main idea is to show that two-peaked nodal solutions are nondegenerate (in some symmetry class) and to compute the Morse index of such solutions. We remark that the uniqueness and Morse index of boundary spikes have been studied in [4] and [51].

We first introduce a general framework. This framework is a combination of the Liapunov-Schmidt reduction method and the variational principle. The Liapunov-Schmidt reduction method has been introduced and used in a lot of papers. See [1], [2], [3], [4], [5], [18], [24], [25], [27], [43], [44], [54], [55] and the references therein. A combination of the Liapunov-Schmidt reduction method and the variational principle was used in [3], [10], [13], [14], [24] and [25]. We shall follow the procedure in [24].

Step 1. Choose suitable approximate functions.

Recall that $\Omega = B$. Let w be the unique solution of (2.1). We fix a point $P \in \Omega$ and introduce the following functions as suitable approximate functions – the *"projection"* of w in $H_0^1(\Omega)$. This projection was first introduced in [37] and later studied in [49]. The idea of projecting a function has been used in other problems as well. See [3], [6], [37], [45], [54], [55] and the references therein.

Let

$$f(u) = |u|^{p-1}u. \tag{4.1}$$

We define $w_{\epsilon,P}$ to be the unique solution of

$$\begin{cases} \epsilon^2 \Delta w_{\epsilon,P} - w_{\epsilon,P} + f(w(\frac{x-P}{\epsilon})) = 0 \text{ in } \Omega, \\ w_{\epsilon,P} > 0 \text{ in } \Omega, \ w_{\epsilon,P} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.2}$$

Set

$$\bar{w}_{\epsilon,P} = w(\frac{x-P}{\epsilon}), \quad w_{\epsilon,P} = \bar{w}_{\epsilon,P}(x) + \varphi_{\epsilon,P}(x). \tag{4.3}$$

Then $\varphi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2 \Delta \varphi_{\epsilon,P} - \varphi_{\epsilon,P} = 0 \text{ in } \Omega, \\ \varphi_{\epsilon,P} = \bar{w}_{\epsilon,P} \text{ on } \partial\Omega. \end{cases} \tag{4.4}$$

The asymptotic behavior of $\varphi_{\epsilon,P}$ has been studied in [37] and is related to the distance function: For $P \in \Omega$ we define

$$d_P := d(P, \partial\Omega) = 1 - |P|. \tag{4.5}$$

For $P \neq 0$, it is easy to compute that

$$\nabla_P d_P = -\frac{P}{|P|}, \tag{4.6}$$

$$\frac{\partial^2}{\partial P_i \partial P_j} d_P = -\frac{1}{|P|}\left(\delta_{ij} - \frac{P_i P_j}{|P|^2}\right), \tag{4.7}$$

where $P = (P_1, ..., P_N)$.

We state the following useful lemma about the properties of $\varphi_{\epsilon,P}$ and the computations of some integrals. Its proof is technical and thus delayed to Appendix A.

Lemma 4.1. *Let $\Omega = B$ and $P \in \Omega, P \neq 0$.*

(1) For ϵ sufficiently small, we have

$$\varphi_{\epsilon,P}(P + \epsilon y) = \varphi_{\epsilon,P}(P)(1 + o(1))e^{-<\nabla d_P, y>}, \text{ for } P + \epsilon y \in \bar{\Omega}, \tag{4.8}$$

$$\varphi_{\epsilon,P}(P) = (c_N + o(1))(d_P(1 - d_P))^{-\frac{N-1}{2}} \epsilon^{N-1} e^{-2d_P/\epsilon}, \tag{4.9}$$

where $c_N > 0$ is a generic constant (depending on N only), and

$$\int_\Omega f'(\bar{w}_{\epsilon,P}) \frac{\partial \bar{w}_{\epsilon,P}}{\partial P_i} \varphi_{\epsilon,P}(x) dx$$

$$= (-\gamma_1 + o(1))\epsilon^{N-1} \varphi_{\epsilon,P}(P)(\nabla d_P)_i + O(e^{-(2+\sigma)d_P/\epsilon}) \tag{4.10}$$

where $(\nabla d_P)_i$ denotes the $i-th$ component of ∇d_P (which is $-P_i/|P|$ in our case) and

$$\gamma_1 = \int_{R^N} f(w) e^{-y_1} dy > 0, \quad \sigma = \min(p - 1, 1). \tag{4.11}$$

(2) For ϵ sufficiently small and $P_1, P_2 \in \Omega$, $\frac{|P_1 - P_2|}{\epsilon} \to +\infty$, we have

$$\int_\Omega f'(\bar{w}_{\epsilon,P_1}) \bar{w}_{\epsilon,P_2} \frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}}$$

$$= \epsilon^{N-1}(-\gamma_1 + o(1)) w(\frac{|P_1 - P_2|}{\epsilon})(\nabla_{P_1}(|P_1 - P_2|))_i + O(e^{-(1+\sigma)|P_1 - P_2|/\epsilon}), \tag{4.12}$$

where γ_1 is given by (4.11).

Step 2. Finite-dimensional reduction.

We now describe the so-called Liapunov-Schmidt finite dimension reduction procedure. Most of the material is from Sections 3, 4 and 5 in [24]. See also Sections 4, 5 and 6 in [25].

We first introduce some notations.

We observe that solving (1.1) is equivalent to finding a zero of the following nonlinear equation:

$$S_\epsilon[u] := \Delta u - u + f(u) = 0, u \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon), \tag{4.13}$$

where

$$f(u) = |u|^{p-1}u, \quad \Omega_\epsilon = \{y|\epsilon y \in \Omega\}. \tag{4.14}$$

For any $u, v \in H_0^1(\Omega)$, we define the inner product and the norm as follows:

$$< u, v >_\epsilon = \epsilon^{-N} \int_\Omega (\epsilon^2 \nabla u \cdot \nabla v + u \cdot v)\, dx, \quad \|u\|_\epsilon = < u, v >_\epsilon^{1/2} .$$

Fix $\mathbf{P} = (P_1, P_2) \in \Omega \times \Omega$. Let $\varphi(\mathbf{P}) = \varphi(P_1, P_2)$ be defined in (1.3). We assume that

$$\mathbf{P} = (P_1, P_2) \in \Lambda_\delta = \{\mathbf{P} \in \Omega \times \Omega | \varphi(\mathbf{P}) \geq 2\delta\}, \tag{4.15}$$

where δ is a small but fixed positive constant.

Let

$$w_{\epsilon,\mathbf{P}} = w_{\epsilon,P_1} - w_{\epsilon,P_2}. \tag{4.16}$$

To simplify notations, we use the following simplified symbols:

$$\partial_{j,i} := \frac{\partial}{\partial P_{j,i}}, \quad j = 1, 2,\ i = 1, ..., N.$$

We remark that the variable of $w_{\epsilon,\mathbf{P}}$ is in Ω. Sometimes, we also consider $w_{\epsilon,\mathbf{P}}(\epsilon y)$ for $y \in \Omega_\epsilon$ and we denote $w_{\epsilon,\mathbf{P}}(\epsilon y)$ as $w_{\epsilon,\mathbf{P}}$ as well.

Now we define the approximate kernel and cokernel, respectively, as follows:

$$\mathcal{K}_{\epsilon,\mathbf{P}} := \text{span } \{\partial_{j,i} w_{\epsilon,\mathbf{P}} | j = 1, 2, i = 1, ..., N\} \subset H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon), \tag{4.17}$$

$$\mathcal{C}_{\epsilon,\mathbf{P}} := \text{span } \{\partial_{j,i} w_{\epsilon,\mathbf{P}} | j = 1, 2, i = 1, ..., N\} \subset L^2(\Omega_\epsilon). \tag{4.18}$$

We also need the following spaces

$$\mathcal{K}_{\epsilon,\mathbf{P}}^\perp = \{u \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon) | \int_{\Omega_\epsilon} u \partial_{j,i} w_{\epsilon,\mathbf{P}} = 0, j = 1, 2, i = 1, ..., N\}, \tag{4.19}$$

$$\mathcal{C}_{\epsilon,\mathbf{P}}^\perp = \{u \in L^2(\Omega_\epsilon) | \int_{\Omega_\epsilon} u \partial_{j,i} w_{\epsilon,\mathbf{P}} = 0, j = 1, 2, i = 1, ..., N\}. \tag{4.20}$$

Set for the linear operators

$$\tilde{L}_{\epsilon,\mathbf{P}}(\phi) = \Delta\phi - \phi + f'(w_{\epsilon,\mathbf{P}})\phi, \quad \mathcal{L}_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}}^\perp \circ \tilde{L}_{\epsilon,\mathbf{P}}, \tag{4.21}$$

for $\phi \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$, where $\pi_{\epsilon,\mathbf{P}}^\perp$ is the projection from $L^2(\Omega_\epsilon)$ into $\mathcal{C}_{\epsilon,\mathbf{P}}^\perp$.

The following lemma can be proved along the line of Propositions 3.1 and 3.2 in [49].

Lemma 4.2. *For $\epsilon << 1$ and $\mathbf{P} \in \Lambda_\delta$ (see (4.15)) the linear operator $\mathcal{L}_{\epsilon,\mathbf{P}} : \mathcal{K}_{\epsilon,\mathbf{P}}^\perp \to \mathcal{C}_{\epsilon,\mathbf{P}}^\perp$ is one-to-one and onto. Moreover, the inverse of $\mathcal{L}_{\epsilon,\mathbf{P}}$ exists and is bounded uniformly in ϵ and $\mathbf{P}$.*

Next, we have

Lemma 4.3. *For ϵ sufficiently small and $\mathbf{P} \in \Lambda_\delta$, there exists a unique $v_{\epsilon,\mathbf{P}} \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ such that*

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) \in \mathcal{C}_{\epsilon,\mathbf{P}}. \tag{4.22}$$

Moreover, $v_{\epsilon,\mathbf{P}}$ is C^2 in $\mathbf{P}$ and

$$\|v_{\epsilon,\mathbf{P}}\|_\epsilon \le Ce^{-(1+\sigma)\varphi(\mathbf{P})/\epsilon} \tag{4.23}$$

$$\|\partial_{j,i} v_{\epsilon,\mathbf{P}}\|_\epsilon \le C\epsilon^{-2} e^{-(1+\sigma)\varphi(\mathbf{P})/\epsilon}, \tag{4.24}$$

where $\sigma = \min(1, p-1)$.

Proof: The proof of this Lemma is similar to that of Lemma 2.4 of [53]. □

Step 3. Solve the finite dimensional problem.

Fix any $\mathbf{P} \in \Lambda_{2\delta}$. Let $v_{\epsilon,\mathbf{P}}$ be the unique solution of (4.22) given by Lemma 4.3. Now we define

$$M_\epsilon(\mathbf{P}) = M_\epsilon(P_1, P_2) := \epsilon^{-N} J_\epsilon[w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}] \tag{4.25}$$

$$M_\epsilon(\mathbf{P}) : \Lambda_{2\delta} \to R,$$

where J_ϵ is the energy functional introduced in (1.5) of Section 1.

By Lemma 4.3, $M_\epsilon(\mathbf{P}) \in C^2(\Lambda_{2\delta})$. Then we have the following reduction theorem, whose proof is similar to that of Proposition 4.1 of [24].

Lemma 4.4. *The function $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}, \mathbf{P}^\epsilon \in \Lambda_{2\delta}$ is a critical point of J_ϵ if and only if $\mathbf{P}^\epsilon$ is a critical point of $M_\epsilon(\mathbf{P})$.*

Therefore, to prove the existence and uniqueness of solutions of (1.1), we just need to concentrate on the study of critical points of $M_\epsilon(\mathbf{P})$, which is a finite-dimensional problem. We shall compute $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$ in the next two sections.

5. Uniqueness Proof II: computations of $\nabla M_\epsilon(\mathbf{P})$ and $\nabla^2 M_\epsilon(\mathbf{P})$

In this section, we compute the (first and second order) derivatives of $M_\epsilon(\mathbf{P})$.

By Lemma 2.1, if $P_1^\epsilon, j = 1, 2$ are the two local extrema of u_ϵ, then $\varphi(P_1^\epsilon, P_2^\epsilon) \geq \delta_0$ for some $\delta_0 > 0$. Now we choose $\delta = \frac{\delta_0}{4}$. By Lemma 4.4, $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a nodal solution with two spikes if and only if $\mathbf{P}^\epsilon$ is a critical point of M_ϵ, since $\mathbf{P}^\epsilon \in \Lambda_{2\delta}$.

The asymptotic expansion of $M_\epsilon(\mathbf{P})$ in Λ_δ is given in Lemma 4.4 of [40].

Lemma 5.1. *(Lemma 4.4. of* [40]*.) For ϵ sufficiently small and $\mathbf{P} \in \Lambda_\delta$, we have*

$$M_\epsilon(\mathbf{P}) = 2I(w) + \frac{1}{2}(\gamma_1 + o(1))(\sum_{i=1}^{2} \varphi_{\epsilon,P_i}(P_i)) + (\gamma_1 + o(1))w(|P_1 - P_2|/\epsilon) \tag{5.1}$$

where

$$I(w) = \frac{1}{2}\int_{R^N} |\nabla w|^2 + \frac{1}{2}\int_{R^N} w^2 - \frac{1}{p+1}\int_{R^N} |w|^{p+1} \tag{5.2}$$

and γ_1 is given by (4.11).

We now show that the asymptotic expansion in (5.1) holds true in the C^2 sense. Set

$$\tilde{M}_\epsilon(\mathbf{P}) := \frac{\gamma_1}{2}\sum_{j=1}^{2} \varphi_{\epsilon,P_j}(P_j) + \gamma_1 w(|P_1 - P_2|/\epsilon). \tag{5.3}$$

By (4.9) of Lemma 4.1 and (2.3), we see that if $|P_j^\epsilon| \geq \frac{1}{10}, j = 1, 2$, then we have

$$\tilde{M}_\epsilon(\mathbf{P}) := \frac{c_N(\gamma_1 + o(1))}{2}\epsilon^{\frac{N-1}{2}}\sum_{j=1}^{2} c(P_j^\epsilon)e^{-2d_{P_j^\epsilon}/\epsilon} \tag{5.4}$$

$$+A_N(\gamma_1 + o(1))\epsilon^{\frac{N-1}{2}}(|P_1 - P_2|)^{-\frac{N-1}{2}}e^{-|P_1-P_2|/\epsilon},$$

where the distance function d_P is given in (4.5), c_N is given in (4.9) of Lemma 4.1, $A_N > 0$ is given by (2.3), and

$$c(P) = (d_P(1 - d_P))^{-\frac{N-1}{2}}. \tag{5.5}$$

The following lemma is our key estimate.

Lemma 5.2. *Suppose that $\mathbf{P}^\epsilon \in \Lambda_\delta$ and ϵ is sufficiently small.*

(1) If $|P_j^\epsilon| \geq d_0$ for some j and $d_0 > 0$, then we have

$$\partial_{j,i}M_\epsilon(\mathbf{P}) = \partial_{j,i}\tilde{M}_\epsilon(\mathbf{P}) + O(\tilde{M}_\epsilon(\mathbf{P})), j = 1, 2, i = 1, ..., N. \tag{5.6}$$

(2) Suppose that $\mathbf{P}^\epsilon$ *is a critical point of* $M_\epsilon(\mathbf{P})$ *such that* $|P_j^\epsilon| \geq d_0, j = 1, 2$ *for some* $d_0 > 0$. *Then we have*

$$\partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\Big|_{\mathbf{P}=\mathbf{P}^\epsilon} = \partial_{l,m}\partial_{j,i}\tilde{M}_\epsilon(\mathbf{P})\Big|_{\mathbf{P}=\mathbf{P}^\epsilon} + O(\epsilon^{-1}\tilde{M}_\epsilon(\mathbf{P}^\epsilon)), j, l = 1, 2, i, m = 1, ..., N. \tag{5.7}$$

More precisely, we have

$$\partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\Big|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$= \epsilon^{N-2}(\gamma_1 + o(1))w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)e_{jl,m}^\epsilon e_{jl,i}^\epsilon + \epsilon^{N-2}(\gamma_1 + o(1))\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)e_{j,i}^\epsilon e_{l,m}^\epsilon \delta_{jl} \tag{5.8}$$

where

$$e_j^\epsilon = \frac{P_j^\epsilon}{|P_j^\epsilon|}, e_{jk}^\epsilon = \frac{P_j^\epsilon - P_k^\epsilon}{|P_j^\epsilon - P_k^\epsilon|}, j \neq k, \tag{5.9}$$

and $e_{j,i}^\epsilon$ *and* $e_{jk,i}^\epsilon$ *denote the* $i-th$ *component of the vectors* e_j^ϵ *and* e_{jk}^ϵ, *respectively.*

The proof of Lemma 5.2 is very technical and will be presented in Appendix B.

6. UNIQUENESS OF u_ϵ

In this section, we prove the uniqueness of the two-peaked nodal solution u_ϵ for ϵ sufficiently small. Let u_ϵ be a two-peaked nodal solution whose local maximum point and local minimum points are $\tilde{P}_j^\epsilon, j = 1, 2$, respectively.

By MMP (Section 3), the solution u_ϵ is even in $x_j, j = 2, ..., N$. Let

$$H_s^2(\Omega_\epsilon) = \{u \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon) | u \text{ is even with respect to } x_j, j = 2, ..., N\}. \tag{6.1}$$

Consider the following minimization problem

$$\min_{(P_1,P_2)\in\hat{\Lambda}_{2\delta}} \|u_\epsilon - w_{\epsilon,P_1} + w_{\epsilon,P_2}\|_{L^2(\Omega_\epsilon)} \tag{6.2}$$

where

$$\hat{\Lambda}_{2\delta} = \{(P_1, P_2) \ |\mathbf{P} \in \Lambda_{2\delta}, P_{j,i} = 0, i = 2, ..., N, j = 1, 2\}. \tag{6.3}$$

It is easy to see that the minimum in (6.2) is attained (say by $\mathbf{P}^\epsilon$) and thus we have

$$u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon \tag{6.4}$$

where $\mathbf{P}^\epsilon \in \hat{\Lambda}_\delta, \phi_\epsilon \in H_s^2(\Omega_\epsilon)$. Moreover, $\phi_\epsilon \in \mathcal{K}_{\epsilon,\mathbf{P}^\epsilon}^\perp$. Since

$$S[w_{\epsilon,\mathbf{P}^\epsilon} + \phi_\epsilon] = 0 \in \mathcal{C}_{\epsilon,\mathbf{P}^\epsilon}, \phi_\epsilon \in \mathcal{K}_{\epsilon,\mathbf{P}^\epsilon}^\perp,$$

by Lemma 4.3, we see that

$$\phi_\epsilon = v_{\epsilon,\mathbf{P}^\epsilon}, \tag{6.5}$$

where $v_{\epsilon,\mathbf{P}^\epsilon}$ is defined by Lemma 4.3. (Note that P_j^ϵ may not be a local maximum or local minimum point of u_ϵ. But it is easy to show that up to a permutation, $P_j^\epsilon = \tilde{P}_j^\epsilon + o(1)$, $j = 1, 2$.)

For $\mathbf{P} \in \hat{\Lambda}_\delta$, we may define $\mathbf{L} = (l_1, l_2)$, where $l_1 = P_{1,1}, l_2 = P_{2,1}$ and

$$\hat{M}_\epsilon(\mathbf{L}) = M_\epsilon(\mathbf{P}). \tag{6.6}$$

Similar to Lemma 4.4, we have that $\mathbf{L}^\epsilon$ is a critical point of $\hat{M}_\epsilon(\mathbf{L})$ if and only if $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}$ is a critical point of J_ϵ.

To avoid clumsy notation, we drop the hat from now on. Thus our problem is reduced to a two-dimensional problem. By Lemma 2.1, we only need to prove the uniqueness of the critical point of $M_\epsilon(\mathbf{L})$ for $\mathbf{L}$ in the set

$$\omega = \left\{ (l_1, l_2) | |l_1 - \frac{1}{2}| \leq \delta, |l_2 + \frac{1}{2}| \leq \delta \right\},$$

which is a two-dimensional problem.

We begin with the following lemma which computes how much $\mathbf{L}^\epsilon$ differs from $\mathbf{L}^0 = (\frac{1}{2}, -\frac{1}{2})$. This is a refinement of (2.7) of Lemma 2.1. This kind of estimate is needed for the uniqueness proof. See [5] and [51].

Lemma 6.1. *Let $\mathbf{L}^\epsilon = (l_1^\epsilon, l_2^\epsilon)$ be as above. Then there exists a unique constant a such that*

$$l_1^\epsilon = \frac{1}{2} + \epsilon a + o(\epsilon), \quad l_2^\epsilon = -\frac{1}{2} - \epsilon a + o(\epsilon). \tag{6.7}$$

Proof: Our main tool is (1) of Lemma 5.2. Note that $\mathbf{P}^\epsilon = (P_1^\epsilon, P_2^\epsilon), P_j^\epsilon = (l_j^\epsilon, 0, ..., 0)$ is a critical point of $\mathbf{M}_\epsilon$. By Lemma 2.1, $l_1^\epsilon \to \frac{1}{2}, l_2^\epsilon \to -\frac{1}{2}$. Now adding the two equations in (5.6) (and using (5.4)), we obtain that

$$\sum_{j=1}^{2} e^{-2|P_j^\epsilon|/\epsilon} \frac{P_j^\epsilon}{|P_j^\epsilon|} + o(\sum_{j=1}^{2} e^{-2|P_j^\epsilon|/\epsilon}) = 0. \tag{6.8}$$

which implies that

$$|P_2^\epsilon| = |P_1^\epsilon| + o(\epsilon). \tag{6.9}$$

Hence we deduce that

$$l_1^\epsilon = -l_2^\epsilon + o(\epsilon). \tag{6.10}$$

Next we examine equation (5.6) at $j = 1$. We have

$$\frac{P_1^\epsilon}{|P_1^\epsilon|} + a^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} \frac{P_2^\epsilon - P_1^\epsilon}{|P_2^\epsilon - P_1^\epsilon|} = o(1),$$

where $a^0 > 0$ is a generic constant. So we obtain

$$a^0 e^{(2d_{P_1^\epsilon} - |P_2^\epsilon - P_1^\epsilon|)/\epsilon} = 1 + o(1),$$

and hence

$$|P_2^\epsilon - P_1^\epsilon| = 2d_{P_1^\epsilon} + \epsilon a_0 + o(\epsilon), \tag{6.11}$$

where $a_0 = \log a^0$ is a generic constant. From (6.10) and (6.11), we see that Lemma 6.1 holds.

□

By Lemma 6.1, any critical point $\mathbf{L}^\epsilon$ of $M_\epsilon(\mathbf{L})$ in $B_\delta(\mathbf{L}^0)$ must satisfy $\mathbf{L}^\epsilon = \mathbf{L}^0 + \epsilon\mathbf{a} + o(\epsilon)$ for some fixed $\mathbf{a} = (a, -a)$. Let $\mathbf{Q}^\epsilon = \mathbf{L}^0 + \epsilon\mathbf{a}$.

Our next lemma shows that every critical point $\mathbf{L}^\epsilon$ must be nondegenerate.

Lemma 6.2. *Let $\mathbf{L}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ be a critical point of $M_\epsilon(\mathbf{L})$. Then for ϵ sufficiently small, we have*

$$\sum_{j,l=1}^{2} \partial_l \partial_j M_\epsilon(\mathbf{L})\Big|_{\mathbf{L}=\mathbf{L}^\epsilon} \eta_l \eta_j \geq C\epsilon^{N-2} e^{-2\varphi(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2 \tag{6.12}$$

where C is independent of $\epsilon, \eta = (\eta_1, \eta_2)$, and $|\eta|^2 = \eta_1^2 + \eta_2^2$.

Proof: We have by Lemma 5.2 (2) (for $i = m = 1$)

$$\sum_{j,l=1}^{2} \partial_l \partial_j M_\epsilon(\mathbf{L})\Big|_{\mathbf{L}=\mathbf{L}^\epsilon} \eta_l \eta_j$$

$$= (\gamma_1 + o(1))\epsilon^{N-2}[\varphi_{\epsilon, P_1^\epsilon}(P_1^\epsilon)|\eta_1|^2 + \varphi_{\epsilon, P_2^\epsilon}(P_2^\epsilon)|\eta_2|^2] \tag{6.13}$$
$$+(\gamma_1 + o(1))\epsilon^{N-2} w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)(1 + o(1))|\eta_1 - \eta_2|^2.$$

Since $\mathbf{L}^\epsilon \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, $|\mathbf{L}^\epsilon - \mathbf{Q}^\epsilon| \leq \epsilon\delta$, we have

$$\varphi_{\epsilon, P_j^\epsilon}(P_j^\epsilon) \sim w(|P_1^\epsilon - P_2^\epsilon|/\epsilon), \quad j = 1, 2.$$

(6.13) shows that

$$\sum_{j,l=1}^{2} \partial_l \partial_j M_\epsilon(\mathbf{L})\Big|_{\mathbf{L}=\mathbf{P}^\epsilon} \eta_l \eta_j$$

$$\geq C\epsilon^{N-2} e^{-2\varphi(\mathbf{P}^\epsilon)/\epsilon} |\eta|^2 \tag{6.14}$$

for some $C > 0$ independent of ϵ.

This proves the lemma.

□

The inequality (6.12) shows that the matrix $(\partial_l \partial_j M_\epsilon(\mathbf{L})\big|_{\mathbf{L}=\mathbf{L}^\epsilon})$ is positive definite. Thus the Morse index is 0.

Finally we have

Lemma 6.3. *For $\delta > 0$ small, there exists a unique critical point of $M_\epsilon(\mathbf{L})$ over $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.*

Proof:

By Lemma 2.2, there exists a critical point $\mathbf{L}^\epsilon$ of $M_\epsilon(\mathbf{L})$. By Lemma 6.1, $\mathbf{L}^\epsilon = \mathbf{L}^0 + \epsilon \mathbf{a} + o(\epsilon)$ and any other critical point of $M_\epsilon(\mathbf{L})$ is in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$.

We now show that $\mathbf{L}^\epsilon$ is unique.

By Lemma 6.2, there is only a finite number of critical points of $M_\epsilon(\mathbf{L})$ in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (since each critical point is nondegenerate). Let k_ϵ be the number of critical points. At each critical point, we have by Lemma 6.2,

$$\deg(\nabla M_\epsilon, B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon), 0) = (-1)^0 = 1,$$

where $\delta_i > 0$ are small constants so that $B_{\delta_i\epsilon}(\mathbf{Q}_i^\epsilon)$ contains only one critical point (i.e. $\mathbf{Q}_i^\epsilon$) of $M_\epsilon(\mathbf{L})$.

Hence by the additivity of the degree we have

$$\deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = k_\epsilon(-1)^0. \tag{6.15}$$

On the other hand, it is easy to see that $\tilde{M}_\epsilon(\mathbf{L})$ has only one critical point in $B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$ (because of the nondegeneracy of $(\nabla^2 \tilde{M}_\epsilon(\mathbf{P})))$. For $\mathbf{L} \in B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$, we have

$$e^{-2d_{P_i}/\epsilon} = (1 + O(\delta))e^{-2d_{Q_i^\epsilon}}, w(|P_1 - P_2|/\epsilon) = (1 + O(\delta))w(|Q_1^\epsilon - Q_2^\epsilon|/\epsilon),$$

$$M_\epsilon(\mathbf{L}) = (1 + O(\delta))M_\epsilon(\mathbf{Q}^\epsilon).$$

By (1) of Lemma 5.2, we have $\nabla M_\epsilon(\mathbf{L}) = \nabla \tilde{M}_\epsilon(\mathbf{L}) + O(\tilde{M}_\epsilon(\mathbf{L}))$. Note that $\nabla M_\epsilon(\mathbf{L}) \neq 0$ and $\nabla \tilde{M}_\epsilon(\mathbf{L}) \neq 0$ on $\partial B_{\delta\epsilon}(\mathbf{Q}^\epsilon)$. By a continuity argument, we obtain that

$$\deg(\nabla M_\epsilon, B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = \deg(\nabla \tilde{M}_\epsilon(\mathbf{L}), B_{\delta\epsilon}(\mathbf{Q}^\epsilon), 0) = 1. \tag{6.16}$$

Comparing (6.15) and (6.16), we deduce that $k_\epsilon = 1$.

□

Lemma 6.3 shows that the two-peaked nodal solution is unique, up to a rotation, provided that ϵ is sufficiently small.

7. The Neumann Case: Proof of Theorem (1.2)

In this section, we consider the Neumann case. Note that in [41], the existence of a nodal solution u_ϵ with unique local maximum point $P_1^\epsilon \in \partial\Omega$ (having positive value) and unique local minimum point $P_2^\epsilon \in \partial\Omega$ (having negative value) is proved. Moreover, $\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon} \to +\infty$.

We now use MMP to prove the following result.

Lemma 7.1. *Let $P_1^\epsilon, P_2^\epsilon$ be a local maximum and a local minimum point of u_ϵ, respectively. Then, for ϵ sufficiently small, $P_1^\epsilon = -P_2^\epsilon$. Moreover, suppose that $P_1^\epsilon, P_2^\epsilon$ lie on the x_1- axis, then u_ϵ is even in $x_j, j = 2, ..., N$.*

Proof: The proof is similar to that in Section 3. Suppose $P_1^\epsilon, P_2^\epsilon$, and the origin are not on a line. Note that since $P_1^\epsilon, P_2^\epsilon \in \partial\Omega$, we may assume that $P_1^\epsilon = (\sqrt{1-t_\epsilon^2}, 0, ..., 0, t_\epsilon), P_2^\epsilon = (\sqrt{1-t_\epsilon^2}, 0, ..., 0, -t_\epsilon)$ with $t_\epsilon > 0$, $\frac{t_\epsilon}{\epsilon} \to +\infty$. We may just follow the proof of Case 1 in Section 3. The rest is exactly the same.

□

¿From Lemma 7.1, we see that $P_1^\epsilon = -P_2^\epsilon$. Without loss of generality, we may assume that $P_1^\epsilon = (1, 0, ..., 0), P_2^\epsilon = (-1, 0, ..., 0)$.

Our next result shows the existence of solutions having the properties of Theorem 1.2.

Lemma 7.2. *There exists a two-peaked nodal solution $\hat{u}_\epsilon$ to (1.1) such that $\hat{u}_\epsilon$ is even in $x_j, j = 2, ..., N$ and is odd in x_1. Moreover, the local maximum point P_1^ϵ and the local minimum point P_2^ϵ of $\hat{u}_\epsilon$ satisfy: $P_1^\epsilon = -P_2^\epsilon, P_1^\epsilon = (1, 0, ..., 0)$.*

Proof: Let $\Omega^+ = B \cap \{x_1 > 0\}$ and let v_ϵ be the least energy positive solution of the following mixed Neumann-Dirichlet problem

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0, \ u > 0 \ \text{ in } \Omega_+, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial(\Omega^+) \cap \partial(B), \ u = 0 \text{ on } \partial(\Omega_+) \cap \{x_1 = 0\}. \end{cases} \tag{7.1}$$

The existence of v_ϵ is standard: We consider the following energy functional

$$E_\epsilon[u] = \frac{\epsilon^2}{2} \int_{\Omega^+} |\nabla u|^2 + \frac{1}{2} \int_{\Omega^+} |u|^2 - \frac{1}{p+1} \int_{\Omega^+} u_+^{p+1},$$

where $u_+ = \max(u, 0)$, $u \in H^1_{0,s,\Gamma}(\Omega^+) = H^1(\Omega^+) \cap \{u \text{ is even in } x_j, j = 2, ..., N, \text{ and } u = 0 \text{ on } \Gamma\}$, and $\Gamma = \partial(\Omega^+) \cap \{x_1 = 0\}$. By arguments similar to [35], there exists a mountain-pass solution v_ϵ which satisfies (7.1). Moreover, for ϵ sufficiently small, v_ϵ has only one local maximum which lies on the boundary. By the symmetry of v_ϵ, the only maximum point of v_ϵ must lie on the x_1-axis and hence equals $(1, 0, ..., 0)$. Now let

$$\hat{u}_\epsilon = \begin{cases} v_\epsilon(x_1, x_2, ..., x_N), \text{ if } x_1 \geq 0, \\ -v_\epsilon(-x_1, x_2, ..., x_N), \text{ if } x_1 < 0. \end{cases} \tag{7.2}$$

It is easy to see that $\hat{u}_\epsilon$ is a two-peaked nodal solution of (1.1) and $\hat{u}_\epsilon$ satisfies the properties of Lemma 7.2.

□

It remains to prove that $u_\epsilon = \hat{u}_\epsilon$. In this case, it is easier that for the Dirichlet problem. The proof is similar to that in [32], where the uniqueness of two-boundary (positive) solutions is proved.

8. Appendix A: Proof of Lemma 4.1

In this appendix, we prove Lemma 4.1 of Section 4 which follows from computations done in [52].

As in [37], set $\varphi_{\epsilon,P} = e^{-\Psi_{\epsilon,P}(x)/\epsilon}$, where $\Psi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \Delta \Psi_{\epsilon,P}(x) - |\nabla \Psi_{\epsilon,P}(x)|^2 + 1 = 0 \text{ in } \Omega, \\ \Psi_{\epsilon,P} = -\epsilon \log w(\frac{x-P}{\epsilon}) \text{ on } \partial\Omega. \end{cases} \tag{8.1}$$

By Lemma 3.6 of [37], we see that

$$\Psi_{\epsilon,P}(P) \to 2d_P \quad \text{as } \epsilon \to 0. \tag{8.2}$$

It is also proved in [52] that

$$\frac{\partial \Psi_{\epsilon,P}(x)}{\partial \nu} = (-1 + O(\epsilon))\frac{\partial}{\partial \nu}|x - P| = (-1 + O(\epsilon))\frac{< x - P, \nu >}{|x - P|} \text{ on } \partial\Omega. \tag{8.3}$$

To compute the exact asymptotic expansion of $\varphi_{\epsilon,P}(P)$, we follow [52]. Let $G_\epsilon(x, z)$ be the Green's function which is the unique solution of the problem

$$\begin{cases} \epsilon^2 \Delta G_\epsilon(x, z) - G_\epsilon(x, z) + \delta(z - x) = 0 \;\text{ in } \Omega, \\ G_\epsilon(x, z) = 0 \text{ on } \partial\Omega. \end{cases} \tag{8.4}$$

Then we have

$$\varphi_{\epsilon,P}(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu} G_\epsilon(x, z) \varphi_{\epsilon,P}(z) dz. \tag{8.5}$$

We decompose

$$G_\epsilon(x, z) = K_\epsilon(|x - z|) - H_\epsilon(x, z)$$

where $K_\epsilon(r)$ is the fundamental solution of $\epsilon^2\Delta - 1$ in $R^N \backslash \{0\}$.

Then H_ϵ satisfies

$$\begin{cases} \epsilon^2 \Delta H_\epsilon - H_\epsilon = 0 \ \text{ in } \Omega, \\ H_\epsilon(x,z) = -K_\epsilon(|x-z|) \text{ on } \partial\Omega. \end{cases} \tag{8.6}$$

By using (8.3), it has been shown in [52] that

$$\frac{\partial H_\epsilon}{\partial \nu} = (-1+o(1))\frac{\partial K_\epsilon}{\partial \nu}. \tag{8.7}$$

So we have

$$\varphi_{\epsilon,P}(P) = \int_{\partial\Omega} (2+o(1))\frac{\partial}{\partial\nu} K_\epsilon(|z-P|)\varphi_{\epsilon,P} dz$$
$$= (c_N + o(1))\epsilon^{-1} \int_{\partial\Omega} (\frac{\epsilon}{|z-P|})^{N-1} e^{-2|z-P|/\epsilon} \frac{< z-P, \nu >}{|z-P|} dz$$
$$= (c_N + o(1))\epsilon^{N-2} \int_{\partial\Omega} (\frac{1}{|z-P|})^{N-1} e^{-2|z-P|/\epsilon} \frac{< z-P, \nu >}{|z-P|} dz. \tag{8.8}$$

Let P be such that $|P| \geq d_0$ for some $d_0 > 0$. Then the integral in (8.8) is a typical Laplace integral and can computed by the classical Laplace method: namely, we let $z = \sqrt{\epsilon} y$ and then obtain

$$\varphi_{\epsilon,P}(P) = (c_N + o(1))(d_P(1-d_P))^{-\frac{N-1}{2}} \epsilon^{\frac{3N}{2}-\frac{5}{2}} e^{-2d_P/\epsilon}$$

for some positive constant $c_N > 0$. This proves (4.9) of Lemma 4.1.

Next we prove (4.8) of Lemma 4.1. To this end, we note that for $x = P + \epsilon y$

$$\varphi_{\epsilon,P}(x) = \int_{\partial\Omega} \frac{\partial}{\partial\nu} G_\epsilon(x,z)\varphi_{\epsilon,P}(z) dz$$
$$= \epsilon^{-1}(c_N + o(1)) \int_{\partial\Omega} (\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}} (\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} e^{-\frac{|z-x|+|z-P|}{\epsilon}} \frac{< z-P, \nu >}{|z-P|} dz$$
$$= \epsilon^{-1}(c_N + o(1)) \int_{\partial\Omega} (\frac{\epsilon}{|z-x|})^{-\frac{N-1}{2}} (\frac{\epsilon}{|z-P|})^{-\frac{N-1}{2}} e^{-\frac{2|z-P|}{\epsilon}} e^{-\frac{<z-P,y>}{|z-P|}} \frac{< z-P, \nu >}{|z-P|} dz$$
$$= (1+o(1))\varphi_{\epsilon,P}(P) e^{-<\nabla d_P, y>}$$

which proves (4.8) of Lemma 4.1.

Finally, we prove (4.10) and (4.12) of Lemma 4.1.

For $P \in \Omega$, we define

$$\Omega_{\epsilon,P} := \{y | \epsilon y + P \in \Omega\}. \tag{8.9}$$

If $P = 0$, we denote $\Omega_{\epsilon,P}$ as Ω_ϵ.

For $P \in \Omega$, we have

$$\int_\Omega f'(\bar{w}_{\epsilon,P}) \frac{\partial \bar{w}_{\epsilon,P}}{\partial P_i} \varphi_{\epsilon,P}(x) dx$$

$$= (-1+o(1))\varphi_{\epsilon,P}(P)\epsilon^{N-1}\int_{R^N} f^{'}(w)\frac{\partial w}{\partial y_i}e^{-<\nabla d_P, y>}dy \quad \text{(by Lemma 4.1 (4.8))}$$

$$= (-\gamma_1+o(1))\epsilon^{N-1}\varphi_{\epsilon,P}(P)(\nabla d_P)_i + O(e^{-(2+\sigma)d_P/\epsilon}), \tag{8.10}$$

where γ_1 is given in (4.11). This proves (4.10).

For $P_1, P_2 \in \Omega$ with $|P_1 - P_2|/\epsilon \to +\infty$, we have

$$\int_\Omega f^{'}(\bar{w}_{\epsilon,P_1})\bar{w}_{\epsilon,P_2}\frac{\partial \bar{w}_{\epsilon,P_1}}{\partial P_{1,i}}$$

$$= (-1+o(1))\epsilon^{N-1}\int_{\Omega_{\epsilon,P_1}} f^{'}(w(y))\frac{\partial w}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

$$= \epsilon^{N-1}\int_{R^N} f(w)\frac{\partial}{\partial y_i}w(y+\frac{P_1-P_2}{\epsilon})dy + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon})$$

$$= \epsilon^{N-1}(-\gamma_1+o(1))w(\frac{|P_1-P_2|}{\epsilon})(\nabla_{P_1}(|P_1-P_2|))_i + O(e^{-(1+\sigma)|P_1-P_2|/\epsilon}). \tag{8.11}$$

This proves (4.12).

□

9. Appendix B: Proof of Lemma 5.2

In this appendix, we prove Lemma 5.2.

Proof of (1) of Lemma 5.2: Observe that

$$\nabla_{j,i}M_\epsilon(\mathbf{P}) =< w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}}) >_\epsilon -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}}+v_{\epsilon,\mathbf{P}})$$

$$=< w_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}})$$

$$+ < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(w_{\epsilon,\mathbf{P}}) >_\epsilon -\epsilon^{-N}\int_\Omega f^{'}(w_{\epsilon,\mathbf{P}})v_{\epsilon,\mathbf{P}}\partial_{j,i}(w_{\epsilon,\mathbf{P}})$$

$$+ < w_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}})$$

$$+ < v_{\epsilon,\mathbf{P}}, \partial_{j,i}(v_{\epsilon,\mathbf{P}}) >_\epsilon -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(v_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon})$$

$$=< w_{\epsilon,\mathbf{P}}, \partial_{j,i}w_{\epsilon,\mathbf{P}} >_\epsilon -\epsilon^{-N}\int_\Omega f(w_{\epsilon,\mathbf{P}})\partial_{j,i}(w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon})$$

$$= \epsilon^{-N}\int_\Omega[(f(\bar{w}_{\epsilon,P_1}) - f(\bar{w}_{\epsilon,P_2}) - f(w_{\epsilon,P_1}-w_{\epsilon,P_2})](\partial_{j,i}w_{\epsilon,\mathbf{P}}) + O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon})$$

$$= \epsilon^{-N} \int_\Omega [\sum_{l=1}^{2} (-1)^{l-1} (f(\bar{w}_{\epsilon,P_l}) - f(w_{\epsilon,P_l})) + (f'(w_{\epsilon,P_1}) w_{\epsilon,P_2} - f'(w_{\epsilon,P_2}) w_{\epsilon,P_1})] \partial_{j,i} w_{\epsilon,\mathbf{P}}$$
$$+ O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon})$$
$$= \epsilon^{-N} \int_\Omega [f(\bar{w}_{\epsilon,P_j}) - f(w_{\epsilon,P_j})] \partial_{j,i} w_{\epsilon,P_j} + \epsilon^{-N} \sum_{l\neq j} \int_\Omega f'(w_{\epsilon,P_j}) w_{\epsilon,P_l} \partial_{j,i} w_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon}).$$
$$= \int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j})(-\varphi_{\epsilon,P_j}) \partial_{j,i} \bar{w}_{\epsilon,P_j} + \sum_{l\neq j} \int_{\Omega_{\epsilon,P_j}} f'(\bar{w}_{\epsilon,P_j}) \bar{w}_{\epsilon,P_l} \partial_{j,i} \bar{w}_{\epsilon,P_j} + O(e^{-(2+\sigma)\varphi(\mathbf{P})/\epsilon}) \tag{9.1}$$

Since $|P_j^\epsilon| \geq d_0$ for some $d_0 > 0$, (9.1) equals

$$\epsilon^{N-1}(\gamma_1 + o(1)) \varphi_{\epsilon,P_j}(P_j)(\nabla d_{P_j})_i + \epsilon^{N-1}(\gamma_1 + o(1)) \sum_{l\neq j} w(|P_j - P_l|/\epsilon)(\nabla |P_j - P_l|)_i \tag{9.2}$$

by (4.10) and (4.12) of Lemma 4.1.

By using Lemma 4.1, we see that (5.6) holds.

□

Proof of (2) of Lemma 5.2: Let $\mathbf{P}^\epsilon$ be a critical point of $M_\epsilon(\mathbf{P})$ in Λ_δ such that $|P_j^\epsilon| \geq d_0, j = 1, 2$ for some $d_0 > 0$. We now expand,

$$\partial_{l,m} \partial_{j,i} M_\epsilon(\mathbf{P})\Big|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$= < \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$+ < w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}, \partial_{l,m} \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$-\epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$-\epsilon^{-N} \int_\Omega f(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$= < \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}), \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}}) >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$-\epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i}(w_{\epsilon,\mathbf{P}} + v_{\epsilon,\mathbf{P}})|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

(since $\mathbf{P}^\epsilon$ is a critical point of $M_\epsilon(\mathbf{P})$)

$$= < \partial_{l,m} w_{\epsilon,\mathbf{P}}, \partial_{j,i} w_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i} w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$+ < \partial_{l,m} w_{\epsilon,\mathbf{P}}, \partial_{j,i} v_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i} v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$+ < \partial_{l,m} v_{\epsilon,\mathbf{P}}, \partial_{j,i} w_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i} w_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$+ < \partial_{l,m} v_{\epsilon,\mathbf{P}}, \partial_{j,i} v_{\epsilon,\mathbf{P}} >_\epsilon |_{\mathbf{P}=\mathbf{P}^\epsilon} - \epsilon^{-N} \int_\Omega f'(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon} \partial_{j,i} v_{\epsilon,\mathbf{P}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$= I_1 + I_2 + I_3 + I_4$$

where $I_i, i = 1, ..., 4$ are defined at the last equality.

We now estimate each term. By Lemma 4.3,

$$I_4 = O(\|\partial_{l,m} v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon \|\partial_{j,i} v_{\epsilon,\mathbf{P}^\epsilon}\|_\epsilon) = O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}). \tag{9.3}$$

Certainly the estimate of I_2 is the same as that of I_3. We consider I_2:

$$I_2 = \epsilon^{-N} \int_\Omega [f(\bar{w}_{\epsilon,P_l^\epsilon}) \partial_{l,m} \bar{w}_{\epsilon,P_l^\epsilon} - f(w_{\epsilon,\mathbf{P}^\epsilon} + v_{\epsilon,\mathbf{P}^\epsilon}) \partial_{l,m} w_{\epsilon,P_l^\epsilon}] \partial_{j,i} w_{\epsilon,P_j^\epsilon} = O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}). \tag{9.4}$$

Similarly, we have

$$I_3 = O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}). \tag{9.5}$$

Hence it remains to compute I_1 only. Without loss of generality, we may assume that $j = 1$. We consider two cases separately: $l = 2$ and $l = 1$.

When $l = 2$, we have by Lemma 4.1

$$\begin{aligned} I_1 &= \epsilon^{-N} \int_\Omega [-f'(\bar{w}_{\epsilon,P_2^\epsilon}) \partial_{2.m} \bar{w}_{\epsilon,P_2^\epsilon} + f'(w_{\epsilon,\mathbf{P}^\epsilon}) \partial_{2,m} w_{\epsilon,P_2^\epsilon}] \partial_{1,i} w_{\epsilon,P_1^\epsilon} \\ &= -\epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_2^\epsilon}) \partial_{2,m} \bar{w}_{\epsilon,P_2^\epsilon} - (f'(w_{\epsilon,P_2^\epsilon}) + f'(w_{\epsilon,P_1^\epsilon})) \partial_{2,m} w_{\epsilon,P_2^\epsilon}] \partial_{1,i} \bar{w}_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \\ &= -\epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_2^\epsilon}) \partial_{2,m} \bar{w}_{\epsilon,P_2^\epsilon} - f'(w_{\epsilon,P_2^\epsilon}) \partial_{2,m} w_{\epsilon,P_2^\epsilon}] \partial_{1,i} \bar{w}_{\epsilon,P_1^\epsilon} \\ &\quad + \epsilon^{-N} \int_\Omega f'(w_{\epsilon,P_1^\epsilon}) \partial_{2,m} w_{\epsilon,P_2^\epsilon} \partial_{1,i} \bar{w}_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \\ &= \epsilon^{-N} \int_\Omega f'(w_{\epsilon,P_1^\epsilon}) \partial_{2,m} w_{\epsilon,P_2^\epsilon} \partial_{1,i} \bar{w}_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \\ &= \epsilon^{-2} w(\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon}) \int_{\Omega_{\epsilon,P_1^\epsilon}} f'(w) \frac{\partial w}{\partial y_i} e^{-<e_{12}^\epsilon, y>} (e_{12}^\epsilon)_m + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \\ &= \epsilon^{-2} w(|P_1^\epsilon - P_2^\epsilon|/\epsilon)(\gamma_1 + o(1))(e_{12}^\epsilon)_m (e_{12}^\epsilon)_i + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \end{aligned} \tag{9.6}$$

For $l = 1$, we have

$$\begin{aligned} I_1 &= \epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_1^\epsilon}) \partial_{1,m} \bar{w}_{\epsilon,P_1^\epsilon} - f'(w_{\epsilon,\mathbf{P}^\epsilon}) \partial_{1,m} w_{\epsilon,P_1^\epsilon}] \partial_{1,i} w_{\epsilon,P_1^\epsilon} \\ &= \epsilon^{-N} \int_\Omega [f'(\bar{w}_{\epsilon,P_1^\epsilon}) \partial_{1,m} \bar{w}_{\epsilon,P_j^\epsilon} - f'(w_{\epsilon,P_1^\epsilon}) \partial_{1,m} w_{\epsilon,P_1^\epsilon}] \partial_{1,i} w_{\epsilon,P_1^\epsilon} \\ &\quad - \int_\Omega f''(\bar{w}_{\epsilon,P_1^\epsilon}) w_{\epsilon,P_2^\epsilon} \partial_{1,m} w_{\epsilon,P_1^\epsilon} \partial_{1,i} w_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon}) \\ &= I_{1,1} - I_{1,2}. \end{aligned}$$

For $I_{1,1}$, we have

$$I_{1,1} = \epsilon^{-N} \int_\Omega [\partial_{1,m} f(\bar{w}_{\epsilon,P_1^\epsilon}) - \partial_{1,m} f(w_{\epsilon,P_1^\epsilon})] \partial_{1,i} \bar{w}_{\epsilon,P_1^\epsilon}$$

$$= \epsilon^{-N} \int_\Omega [(-\frac{\partial}{\partial x_m} f(\bar{w}_{\epsilon,P_1^\epsilon}) - f(w_{\epsilon,P_1^\epsilon}))](-\frac{\partial}{\partial x_i}\bar{w}_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-N} \int_\Omega (f(w_{\epsilon,P_1^\epsilon}) - f(\bar{w}_{\epsilon,P_1^\epsilon})(\frac{\partial^2}{\partial x_i \partial x_m}\bar{w}_{\epsilon,P_1^\epsilon}) + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} \int_{\Omega_{\epsilon,P_1^\epsilon}} f'(w)(y)\varphi_{\epsilon,P_1^\epsilon}(P_1^\epsilon + \epsilon y)\frac{\partial^2 w}{\partial y_i \partial y_m} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2}\varphi_{\epsilon,P_1^\epsilon}(P_1^\epsilon) \int_{R^N} f'(w)e^{-<\nabla d_{P_1^\epsilon},y>}\frac{\partial^2 w}{\partial y_i \partial y_m}dy + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

(by (4.8) of Lemma 4.1)

$$= \epsilon^{-2}\varphi_{\epsilon,P_1^\epsilon}(P_1^\epsilon) \int_{R^N} (-f''(w)\frac{\partial w}{\partial y_i}\frac{\partial w}{\partial y_m} e^{-<\nabla d_{P_1^\epsilon},y>}dy + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2}\varphi_{\epsilon,P_1^\epsilon}(P_1^\epsilon)e_{1,i}^\epsilon e_{1,m}^\epsilon(\gamma_1 + o(1)). \tag{9.7}$$

For $I_{1,2}$, we have

$$I_{1,2} = \epsilon^{-N}\int_\Omega f''(\bar{w}_{\epsilon,P_1^\epsilon})\bar{w}_{\epsilon,P_2^\epsilon}\partial_{1,m}\bar{w}_{\epsilon,P_1^\epsilon}\partial_{1,i}\bar{w}_{\epsilon,P_1^\epsilon} + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2}\int_{R^N} f''(w)\frac{\partial w}{\partial y_i}\frac{\partial w}{\partial y_m} w(y + \frac{P_1^\epsilon - P_2^\epsilon}{\epsilon})dy + O(e^{-(2+\sigma)\varphi(\mathbf{P}^\epsilon)/\epsilon})$$

$$= \epsilon^{-2} w(\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon})e_{12,i}^\epsilon e_{12,m}^\epsilon(\gamma_1 + o(1)). \tag{9.8}$$

Combining all together, we have

$$\partial_{l,m}\partial_{j,i}M_\epsilon(\mathbf{P})\Big|_{\mathbf{P}=\mathbf{P}^\epsilon}$$

$$= \epsilon^{-2}(\gamma_1 + o(1))w(\frac{|P_1^\epsilon - P_2^\epsilon|}{\epsilon})e_{jl,m}^\epsilon e_{jl,i}^\epsilon$$

$$+\epsilon^{-2}(\gamma_1 + o(1))\varphi_{\epsilon,P_j^\epsilon}(P_j^\epsilon)e_{j,i}^\epsilon e_{l,m}^\epsilon \delta_{jl}$$

which is exactly (5.7).

□

ACKNOWLEDGMENTS

This research of the first author is partially supported by an Earmarked Grant from RGC of Hong Kong. The second author thanks the Department of Mathematics at CUHK for their kind hospitality.

References

[1] Alikakos, N., Fusco, G., and Kowalczyk, M., Finite dimensional dynamics and interfaces intersecting the boundary I, *Indiana Univ. Math. J.* 45 (1996), 1119-1155.

[2] N. Alikakos and M. Kowalczyk, Critical points of a singular perturbation problem via reduced energy and local linking, *J. Differential Equations* 159 (1999), 403-426.

[3] P. Bates, E.N. Dancer and J. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, *Adv. Differential Equations* 4 (1999), 1-69.

[4] P. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, *J. Differential Equations* 4 (1999), 1-69.

[5] P. Bates and J. Shi, Existence and instability of spike layer solutions to singular perturbation problems, *J. Funct. Anal.*, 196 (2002), 211-264.

[6] V. Benci and G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, *Arch. Rat. Mech. Anal.* 99 (1987), 283-300.

[7] D. Cao and E.S. Noussair, Multiplicity of positive and nodal solutions for nonlinear elliptic problems in R^N, *Ann. Inst. H. Poincaré Analyse Anal. Non Linéaire* 13 (1996), 567-588.

[8] D. Cao, N.E. Dancer, E.S. Noussair, and S. Yan, On the existence of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems, *Discrete Contin. Dyn. Syst.* 2 (1996), 221-236.

[9] C.C. Chen and C.S. Lin, On the symmetry of blowup solutions to a mean field equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 18 (2001), 271-296.

[10] G. Cerami and J. Wei, Multiplicity of multiple interior spike solutions for some singularly perturbed Neumann problem, *Internat. Math. Res. Notices* 12 (1998), 601-626.

[11] E.N. Dancer, The effect of the domain shape on the number of positive solutions of certain nonlinear equations. II, *J. Differential Equations*, 87 (1990), 316-339.

[12] E.N. Dancer and J. Wei, On the effect of domain topology in some singular perturbation problems, *Topol. Methods Nonlinear Anal.* 11 (1998), 227-248.

[13] E.N. Dancer and S. Yan, Multipeak solutions for a singular perturbed Neumann problem, *Pacific J. Math.* 189 (1999), 241-262.

[14] E.N. Dancer and S. Yan, Interior and boundary peak solutions for a mixed boundary value problem, *Indiana Univ. Math. J.* 48 (1999), 1177-1212.

[15] M. del Pino, P. Felmer, and J. Wei, Multi-peak solutions for some singular perturbation problems, *Cal. Var. Partial Differential Equations* 10 (2000), 119-134.

[16] M. del Pino, P. Felmer and J. Wei, On the role of mean curvature in some singularly perturbed Neumann problems, *SIAM J. Math. Anal.* 31 (1999), 63-79.

[17] M. del Pino, P. Felmer and J. Wei, On the role of distance function in some singularly perturbed problems, *Comm. Partial Differential Equations* 25 (2000), 155-177.

[18] Floer, A. and Weinstein, A., Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69 (1986), 397-408.

[19] A. Gierer and H. Meinhardt, A theory of biological pattern formation. *Kybernetik (Berlin)* 12 (1972), 30-39.

[20] B. Gidas, W.M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* 68 (1979), 209-243.

[21] B. Gidas, W.M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^N, *Adv. Math. Suppl Stud.* 7A (1981), 369-402.

[22] M. Grossi, Uniqueness of least-energy solution for a semilinear Neumann problem, *Proc. Amer. Math. Soc.* 128 (2000), 1665-1672.

[23] M. Grossi, A. Pistoia and J. Wei, Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory, *Cal. Var. Partial Differential Equations* 11(2000) 143-175.

[24] C. Gui and J. Wei, Multiple interior peak solutions for some singular perturbation problems, *J. Differential Equations* 158 (1999), 1-27.

[25] C. Gui and J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, *Can. J. Math.* 52(2000), 522-538.

[26] C. Gui, J. Wei and M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (2000), 47-82.

[27] M. Kowalczyk, Multiple spike layers in the shadow Gierer-Meinhardt system: existence of equilibria and approximate invariant manifold, *Duke Math. J.* 98 (1999), 59-111.

[28] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^N, *Arch. Rat. Mech. Anal.* 105 (1991), 243-266.

[29] Y.-Y. Li, On a singularly perturbed equation with Neumann boundary condition, *Comm. Partial Differential Equations* 23 (1998), 487-545.

[30] Y.-Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, *Comm. Pure Appl. Math.* 51 (1998), 1445–1490.

[31] C.-S. Lin, Locating the peaks of solutions via the maximum principle, I. Neumann problem, *Comm. Pure Appl. Math.* 54 (2001), 1065–1095.

[32] C.-S. Lin and I. Takagi, Method of rotating planes applied to a singularly perturbed Neumann problem, *Cal. Var. Partial Differential Equations* 13 (2001), 519-536.

[33] C.-S. Lin and W.-M. Ni, On the diffusion coefficient of a semilinear Neumann problem, *Calculus of variations and partial differential equations (Trento, 1986)* 160–174, Lecture Notes in Math., 1340, Springer, Berlin-New York, 1988.

[34] C.-S. Lin and J. Wei, Uniqueness of multiple-spike solutions via the method of moving planes, submitted.

[35] W.-M. Ni and I. Takagi, On the shape of least energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.* 41 (1991), 819-851.

[36] W.-M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993), 247-281.

[37] W.-M. Ni and J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.* 48 (1995), 731-768.

[38] E. Noussair, On the number of nodal solutions in singular perturbation problems, *Differential Integral Equations* 14 (2001), 1093-1109.

[39] E. Noussair and S. Yan, The effect of the domain geometry in singular perturbation problems, *Proc. London Math. Soc.* 76 (1998), 427-452.

[40] E.Z. Noussair and J. Wei, On the effect of domain geometry on the existence and profile of nodal solutions of some singularly perturbed semilinear Dirichlet problem, *Indiana Univ. Math. J.* 46 (1997), 1255-1272.

[41] E.Z. Noussair and J. Wei, On the existence and profile of nodal solutions of some singularly perturbed semilinear Neumann problem, *Comm. Partial Differential Equations* 23 (1998), 793-816.

[42] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices Amer. Math. Soc.* 45 (1998), 9-18.

[43] Y.G. Oh, Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$, *Comm. Partial Differential Equations* 13 (1988), 1499-1519.

[44] Y.G. Oh, On positive multi-bump bound states of nonlinear Schrödinger equations under multiple-well potentials, *Comm. Math. Phys.* 131 (1990), 223-253.

[45] O. Rey, The role of Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.* 89 (1990), 1-52.

[46] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, *J. Differential Equations* 129 (1996), 315-333.

[47] J. Wei, On the boundary apikw layer solutions of singularly perturbed semilinear Neumann problem, *J. Diffential Equations* 134 (1997), 104-133.

[48] J. Wei, Conditions for two-peaked solutions of singularly perturbed elliptic equations, *Manuscripta Mathematica* , 96 (1998), 113-136.

[49] J. Wei, On the interior spike layer solutions for some singular perturbation problems, *Proc. Roy. Soc. Edinburgh Sect. A* 128 (1998), 849-874.
[50] J. Wei, On the interior spike layer solutions of singularly perturbed semilinear Neumann problem, *Tohoku Math. J.* 50 (1998), 159-178.
[51] J. Wei, Uniqueness and eigenvalue estimates of boundary spike solutions, *Proc. Roy. Soc. Edinburgh Sect. A* 131 (2001), 1457-1480.
[52] J. Wei, On the effect of domain geometry in singular pertrubation problems, *Differential Integral Equations* 13(2000), 15-45.
[53] J. Wei, On single interior spike solutions of Gierer-Meinhardt system: uniqueness, spectrum estimates, *Europ. J. Appl. Math.* 10 (1999), 353-378.
[54] J. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15 (1998), 459-492.
[55] J. Wei and M. Winter, Multiple boundary spike solutions for a wide class of singular perturbation problems, *J. London Math. Soc.* 59 (1999), 585-606.
[56] J. Wei, Point-condensation generated by the Gierer-meinhardt system: a brief survey, book chapter in *New Trends In Partial Differential Equations 2000*, (Y. Morita, H. Ninomiya, E. Yanagida, and S. Yotsutani editors), pp. 46-59.

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: `wei@math.cuhk.edu.hk`

Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: `winter@mathematik.uni-stuttgart.de`