MULTI-INTERIOR-SPIKE SOLUTIONS FOR THE CAHN-HILLIARD EQUATION WITH ARBITRARILY MANY PEAKS

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Abstract. We study the Cahn-Hilliard equation in a bounded smooth domain without any symmetry assumptions. We prove that for any fixed positive integer K there exist interior $K-$spike solutions whose peaks have maximal possible distance from the boundary and from one another. This implies that for any bounded and smooth domain there exist interior $K-$ peak solutions.

The central ingredient of our analysis is the novel derivation and exploitation of a reduction of the energy to finite dimensions (Lemma 5.5) with variables which are closely related to the location of the peaks. We do not assume nondegeneracy of the points of maximal distance to the boundary but can do with a global condition instead which in many cases is weaker.

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1. Introduction

The Cahn-Hilliard equation [7] was originally derived from the Helmholtz free energy of an isotropic two-component solid and can be written as follows:

$$E(u) = \int_\Omega [F(u(x)) + \frac{1}{2}\epsilon^2 |\nabla u(x)|^2] dx.$$

Here Ω is the region occupied by the body, $u(x)$ is a conserved order parameter typically representing the concentration of one of the components; $F(u)$ is the free energy density of a corresponding homogeneous solid which has a double well structure at low temperatures (the most common example is $F(u) = (1-u^2)^2$). The constant ϵ is proportional to the range of intermolecular forces and the gradient term is a contribution to the free energy describing spatial fluctuations.

We assume conservation of mass, i.e. there exists m with $0 < m < 1$ such that $m = \frac{1}{|\Omega|}\int_\Omega u\, dx$. Therefore, a stationary solution of $E(u)$ under $m = \frac{1}{|\Omega|}\int_\Omega u\, dx$ satisfies

$$\begin{cases} \epsilon^2 \Delta u - f(u) = \lambda_\epsilon & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \\ \int_\Omega u = m|\Omega| \end{cases} \tag{1.1}$$

where $f(u) = F'(u)$ and λ_ϵ is a constant.

In this paper we are concerned with solutions of (1.1) with spike layers. The one dimensional case was studied by Novick-Cohen and Segal [31], Bates and Fife [5], Grinfeld and Novick-Cohen [14],[15].

In [38] we constructed a boundary–spike–layer solution to (1.1) for $\epsilon << 1$ in the higher dimensional case when m is in the metastable region, i.e. $f'(m) > 0$. The spike is located near a nondegenerate critical point of the mean curvature of the boundary.

In [39] we constructed a multi–spike–layer solution to (1.1) where the spikes are each located near (different) nondegenerate critical points of the mean curvature of the boundary.

In [40] we constructed an interior–spike–layer solution to (1.1). The spike concentrates, as $\epsilon \to 0$ at a "nondegenerate peak point" (see [40] for the definition).

In this paper we continue our work along this line by constructing multi–interior–spike–layer solutions.

The existence of spike layer solutions as well as the location and the profile of the peaks for other problems arising in various models such as chemotaxis, pattern formation, chemical reactor theory, etc. have been studied by Lin, Ni, Pan, and Takagi [20, 26, 27, 28] for the Neumann problem and by Ni and Wei [30] for the Dirichlet problem. However, they do not have the volume constraint and the nonlinearity is simpler than here.

Naturally these stationary solutions are essential for the understanding of the global dynamics of the corresponding evolution process. While Bates and Fife [5] prove some results in this direction for the one dimensional case these questions are open for higher dimensions. After this work was completed we became aware of the preprint [6] which contains results similar to ours but using a dynamical systems approach.

Other important features of the Cahn-Hilliard equation with physical relevance are spinodal decomposition and pattern formation. In this respect see the recent work of Kielhöfer [18] and Maier-Paape and Wanner [23], [24].

From now on, we always assume that m is in the metastable region, i.e. $f'(m) > 0$.

Before stating our main result we first make the following transformations. For σ small enough let τ_σ be the unique solution of

$$f(m-\tau_\sigma) - f(m) - \sigma = 0 \tag{1.2}$$

which lies near zero. Obviously

$$\tau_\sigma = -\frac{\sigma}{f'(m)} + O(\sigma^2) \qquad \text{as } \sigma \to 0.$$

With this notation we further define

$$\begin{aligned} g_\sigma(v) &= f(m-\tau_\sigma - v) - f(m) - \sigma \\ &= -p_\sigma v + h_\sigma(v) \end{aligned}$$

where

$$v = m - \tau_\sigma - u,$$

$$p_\sigma = f'(m-\tau_\sigma),$$

$$h_\sigma(v) = f(m-\tau_\sigma - v) - f(m) - \sigma + f'(m-\tau_\sigma)v.$$

By the choice of h_σ

$$h_\sigma(v) = O(v^2)$$

as $v \to 0$. Note that in particular

$$\begin{aligned} g_0(v) &= f(m-v) - f(m) \\ &= -p_0 v + h_0(v) \end{aligned}$$

where

$$v = m - u,$$

$$p_0 = f'(m),$$

$$h_0(v) = f(m-v) - f(m) + f'(m)v.$$

Then equation (1.1) becomes

$$\begin{cases} \epsilon^2 \triangle v - p_0 v + h_0(v) - \frac{1}{|\Omega|}\int_\Omega h_0(v) = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

To accommodate more general nonlinearities we assume that for all $\sigma > 0$ which are sufficiently small

(g1) $h_0 \in C^2(R^+)$ and h_0 satisfies

$$h_0(v) = O(|v|^{p_1}), h_0^{'}(v) = O(|v|^{p_2 - 1}) \text{ as } |v| \to \infty$$

for some $1 < p_1, p_2 < \left(\frac{N+4}{N-4}\right)_+$ where $\left(\frac{N+4}{N-4}\right)_+ := \infty$ if $N \leq 4$ and $\left(\frac{N+4}{N-4}\right)_+ := \frac{N+4}{N-4}$ if $N > 4$. Furthermore, there exists $1 < p_3 < \left(\frac{N+4}{N-4}\right)_+$ such that

$$|h_0'(v+\phi) - h_0'(v)| \leq \begin{cases} C|\phi|^{p_3 - 1} \text{ if } p_3 > 2 \\ C(|\phi| + |\phi|^{p_3-1}) \quad \text{if } p_3 \leq 2. \end{cases}$$

(g2) For σ small enough the equation

$$\begin{cases} \triangle V_\sigma + g_\sigma(V_\sigma) = 0 \quad \text{in } R^N, \\ V_\sigma > 0, V_\sigma(0) = \max_{z \in R^n} \; V_\sigma(z), \\ V_\sigma \to 0 \quad \text{at } \infty \end{cases} \tag{1.4}$$

has a unique solution $V_\sigma(y)$ (by the results of [12], V_σ is radially symmetric, i.e., $V_\sigma = V_\sigma(r)$ and $V_\sigma^{'} < 0$ for $r = |y| \neq 0$). Further, V_σ is nondegenerate, namely the operator

$$L := \triangle + g_\sigma^{'}(V_\sigma) \tag{1.5}$$

is invertible in the space $H^2_r(R^N) := \Big\{ u = u(|y|) \in H^2(R^N) \Big\}$.

The assumptions (g1) and (g2) allow h_0 to be an unbounded real function. Since the solutions v_ϵ which are given by Theorem 1.1 are bounded uniformly with respect to ϵ, satisfying $D_1 \le v_\epsilon \le v_2$ with $D_1 < 0 < D_2$ and D_1, D_2 independent of ϵ, we can assume without loss of generality that in addition h_0 and its first two derivatives are bounded. (By changing h_0 on $R \setminus [D_1, D_2]$ this can be achieved and the bounded solution of the new equation (1.3) still exists.) For the rest of the paper we assume that h_0 is bounded.

In what follows, we state precisely our assumptions on the domain.

For any $\mathbf{P} = (P_1, ..., P_K) \in \Omega^K = \Omega \times \Omega \times ... \times \Omega$, we introduce the following function

$$\varphi(P_1, P_2, ..., P_K) = \min_{i,k,l=1,...,K; k \neq l} (d(P_i, \partial\Omega), \frac{1}{2}|P_k - P_l|).$$

We assume that there is an open subset Λ of Ω^K which satisfies

$$\max_{(P_1,...,P_K) \in \overline{\Lambda}} \varphi(P_1, ..., P_K) > \max_{(P_1,...,P_K) \in \partial\Lambda} \varphi(P_1, ..., P_K). \tag{1.6}$$

We emphasize that such a set Λ always exists . For example, we can take $\Lambda = \Omega^K$. We also observe that any such Λ can be modified so that for all $\mathbf{P} = (P_1, ..., P_K) \in \Lambda$ we have

$$\min_{i=1,...,K} d(P_i, \partial\Omega) > \delta > 0, \quad \min_{k,l=1,...,K; k \neq l} |P_k - P_l| > 2\delta > 0 \tag{1.7}$$

for some sufficiently small $\delta > 0$.

Next we discuss some other examples of Λ for some special domains. If $d(P, \partial\Omega)$ has K strict local maximum points $P_1, ..., P_K$ in Ω such that $\min_{i \neq j} |P_i - P_j| > 2 \max_{i=1,...,K} d(P_i, \partial\Omega)$, we can choose Λ such that (1.6) holds with $\max_{(P_1,...,P_K) \in \overline{\Lambda}} \varphi(P_1, ..., P_K)$ achieved at $\mathbf{P} = (P_1, ..., P_K)$. When $\Omega = B_R(0)$ and $K = 2$, one can take $P_1 = (R/2, 0, ..., 0), P_2 = (-R/2, ..., 0)$ and $\Lambda = \{(X_1, X_2) : R/2 - \delta < |X_i| < R/2 + \delta, i = 1, 2, |X_1 - X_2| > \delta\}$ with δ small. Then (1.6) holds and $\max_{(P_1,P_2) \in \overline{\Lambda}} \varphi(P_1, P_2) = R/2$ is achieved at $\mathbf{P} = (P_1, P_2)$.

Our main result can be stated as follows.

Theorem 1.1. *Assume that condition (1.6) holds. Let g satisfy assumptions (g1)-(g2). Then for ϵ sufficiently small problem (1.3) has a solution v_ϵ which possesses exactly K local maximum points $Q_1^\epsilon, ..., Q_K^\epsilon$ and $\mathbf{Q}^\epsilon = (Q_1^\epsilon, ..., Q_K^\epsilon) \in \Lambda$. Moreover, $\varphi(\mathbf{Q}^\epsilon) \to \max_{\mathbf{P}\in\overline{\Lambda}} \varphi(\mathbf{P})$ as $\epsilon \to 0$.*

More details about the asymptotic behavior of v_ϵ can be found in the proof of Theorem 1.1.

By taking $\Lambda = \Omega^K$, we have the following interesting corollary.

Corollary 1.2. *For any smooth and bounded domain and any fixed positive integer $K \in Z$, there always exists an interior K-peaked solution of (1.3) if ϵ is small enough.*

Remark 1.3. *It can be shown that the maximum of $\varphi(P_1, ..., P_K)$ in Ω^K is attained at some point $(Q_1, ..., Q_K)$ with $d(Q_i, \partial\Omega) = \max \varphi(P_1, ..., P_K)$ for some i. In other words, the distance between each pair of different Q_i's is always larger than or equal to twice the smallest $d(Q_i, \partial\Omega)$. (Otherwise the points Q_i can be moved in such a way that φ is increased.)*

If we connect the maximum point of $\varphi(P_1, ..., P_K)$ with the ball packing problem and call the set of the centers of K balls packed in Ω with the largest minimal radius a K packing center, then the K interior peaks of the above solution converge to a K packing center.

Remark 1.4. *The question of existence of spike layer solutions such that the peaks converge to a given K packing center is open if the K packing center is (locally) non-unique. For example if $\overline{\Omega}$ is constructed by connecting $B_1(0,0)$ by a thin tube to $B_{1+2/\sqrt{3}-\delta}(4,0) \setminus B_\delta(3 - 2/\sqrt{3} + \delta, 0)$ and smoothening the corners. Then, with $K = 3$, φ is maximized by having $P_1 = (0,0)$ and P_2, P_3 suitably in the second disk and the choice of P_2 and P_3 is non-unique. We conjecture that the only set of points which can be the limit of interior 3 peaks solutions are $P_1 = (0,0)$, $P_2 = (4, 1/2 + 1/\sqrt{3} - \delta/2)$, $P_3 = (4, -1/2 - 1/\sqrt{3} + \delta/2)$. We believe that our method can be refined to cover also such highly degenerate situations. The conditions in [6] also do not include this case.*

To introduce the most important ideas of the proof of Theorem 1.1, we need to give some necessary notations and definitions first.

For our approach it is essential to note that v is a solution of (1.3) if and only if v is a critical point of the constrained functional

$$J_\epsilon(v) = \frac{\epsilon^2}{2}\int_\Omega |\nabla v|^2 + \frac{p_0}{2}\int_\Omega v^2 - \int_\Omega H(v)$$

where

$$H(v) = \int_0^v h_0(s)ds, v \in X = \{v \in H^1(\Omega) | \int_\Omega v = 0\}.$$

It is important to note that in the defintion of X we require that

$$\int_\Omega v = 0$$

Recall on the other hand that for solutions of (1.3) this constraint does not have to be assumed a priori but follows automatically if the solutions are in $\{v \in H^2(\Omega) : \frac{\partial v}{\partial \nu} = 0 \text{ at } \partial\Omega\}$.

The key to our construction is finding good approximating functions for the solutions. Our approach is by using a projection technique to obtain appropriate functions in the space X.

We have to study solutions in all of R^N first. Suppose that the function g_σ which was defined after (1.2) satisfies the conditions in (g2). As in (g2) let V_σ be the unique solution of the problem

$$\begin{cases} \triangle V_\sigma + g_\sigma(V_\sigma) = 0 \quad \text{in } R^N, \\ V_\sigma > 0, V_\sigma(0) = \max\limits_{z \in R^n} \ V_\sigma(z), \\ V_\sigma \to 0 \quad \text{at } \infty \end{cases} \tag{1.8}$$

where g_σ is defined after (1.2). It is known (see [12]) that V_σ is radially symmetric, decreasing and

$$\lim_{|y|\to\infty} V_\sigma(y) e^{\sqrt{p_\sigma}|y|}|y|^{\frac{N-1}{2}} = c_\sigma > 0.$$

Furthermore, we know from [40] that for σ sufficiently small $\dfrac{\partial V_\sigma}{\partial \sigma}$ exists and is continuous with respect to σ. It satisfies

$$\triangle(\frac{\partial V_\sigma}{\partial \sigma}) + f'(m - \tau_\sigma - V_\sigma)\left(-\frac{\partial V_\sigma}{\partial \sigma} - \frac{1}{f'(m)}\right) - 1 = 0. \tag{1.9}$$

For $P \in \overline{\Omega}$ let $\Omega_{\epsilon,P} := \{y|\epsilon y + P \in \Omega\}$ and $\Omega_\epsilon := \{y|\epsilon y \in \Omega\}$. Let U be any bounded smooth domain. We define a function $u = P_U V_\sigma$ as the unique solution of

$$\begin{cases} \Delta u - p_\sigma u + h_\sigma(V_\sigma) = 0 \text{ in } U, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U. \end{cases} \tag{1.10}$$

Fix $K \in N$ and choose $\mathbf{P} = (P_1, \dots, P_K) \in \overline{\Lambda}$. We take σ_0 such that

$$\int_\Omega \left(\tau_{\sigma_0} + \sum_{i=1}^K P_{\Omega_{\epsilon,P_i}} V_{\sigma_0} \left(\frac{x - P_i}{\epsilon} \right) \right) dx = 0.$$

We will shall show in Section 2 that σ_0 exists and is unique provided ϵ is small enough. We shall see that this choice of σ_0 is essential in dealing with the nonlocal integral term in (1.3).

We set

$$V_{\sigma,i}(y) = V_\sigma(y - \frac{P_i}{\epsilon}), \qquad PV_{\sigma,i}(y) = P_{\Omega_{\epsilon,P_i}} V_\sigma(y - \frac{P_i}{\epsilon}), \qquad y \in \Omega_\epsilon,$$

$$P^\epsilon V_{\sigma,i}(x) = P_{\Omega_{\epsilon,P_i}} V_\sigma \left(\frac{x - P_i}{\epsilon} \right), \qquad x \in \Omega,$$

$$w_{\epsilon,\mathbf{P}} = \tau_{\sigma_0} + \sum_{i=1}^K PV_{\sigma_0,i}.$$

We shall use $w_{\epsilon,\mathbf{P}}$ as our approximate solution. Further, denote

$$\mathcal{K}_{\epsilon,\mathbf{P}} = \text{span} \left\{ \frac{\partial(\tau_{\sigma_0} + \sum_{i=1}^K PV_{\sigma_0,i})}{\partial P_{i,j}}, \quad i = 1, ..., K, \; j = 1, ..., N \right\}.$$

(**Note:** Our definition of $P_{\Omega_{\epsilon,P_i}}$ is equivalent to the following: Let v be the unique solution of the boundary value problem

$$\begin{cases} \epsilon^2 \Delta v - p_\sigma v + h_\sigma(V_\sigma(\frac{x-P_i}{\epsilon})) = 0 \text{ in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.11}$$

(this is a problem on the domain Ω which is independent of of P_i). Then it is easy to see that

$$P_{\Omega_{\epsilon,P_i}} V_\sigma(y) = v(\epsilon y + P_i) \text{ for } y \in \Omega_{\epsilon,P_i}.$$

Hence $\frac{\partial(\tau_{\sigma_0} + \sum_{i=1}^K PV_{\sigma_0,i})}{\partial P_{i,j}}$ is well-defined.)

We will show that $\mathcal{K}_{\epsilon,\mathbf{P}}$ is an appropriate approximation to the kernel and cokernel, respectively, of the operator obtained from linearizing (1.3) at $w_{\epsilon,\mathbf{P}}$. Precise statements will be given in Propositions 5.1 and 5.2.

Then we solve for $\Phi_{\epsilon,\mathbf{P}}$ such that

$$\int_{\Omega_\epsilon} (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})\Psi = 0 \quad \text{for all} \quad \Psi \in \mathcal{K}_{\epsilon,\mathbf{P}},$$

$$\Delta(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - p_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + h_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})$$

$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} h_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})\, dy \in \mathcal{K}_{\epsilon,\mathbf{P}},$$

$$\frac{\partial}{\partial \nu}(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0 \quad \text{on} \quad \partial\Omega_\epsilon$$

using the Liapunov-Schmidt reduction method. Note that we obtain a family of "solutions" $\Phi_{\epsilon,\mathbf{P}}$ depending on $\mathbf{P} \in \overline{\Lambda}$. We will also write

$$v_\epsilon = w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}.$$

The method evolves from that of [11], [32] and [33] on the semi-classical (i.e. for small parameter $\overline{h}$) solution of the nonlinear Schrödinger equation

$$\frac{\overline{h}^2}{2}\Delta U - (V - E)U + U^p = 0 \tag{1.12}$$

in R^N where V is a potential function and E is a real constant. The method of Liapunov-Schmidt reduction was used in [11], [32] and [33] to construct solutions of (1.12) close to nondegenerate critical points of V for $\overline{h}$ sufficiently small. Note that in the present paper we do not assume nondegeneracy of the points of maximal distance to the boundary but can do with the global condition (1.6) instead which in many cases is weaker. For example if we take $\overline{\Omega} = B_1(0,0) \cup B_{K+2}(0,0) \cup [0, K+2] \times [-1, 1]$ then our method gives existence of K spike solutions whose peaks all approach the line $[0, K+2] \times \{0\}$. This case is not covered by the conditions in [6].

Then we show that $\Phi_{\epsilon,\mathbf{P}}$ is C^1 in the variable $\mathbf{P}$. After that, we define a novel functional

$$M_\epsilon(\mathbf{P}) = J_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}). \tag{1.13}$$

This says that we have also reduced the "energy" to finite dimensions. A large part of the paper is devoted to deriving an explicit expansion including error estimates for $M_\epsilon(\mathbf{P})$. This is a new result and it should be fundamental to a better understanding of qualitative and quantitative properties of the Cahn-Hilliard equation. It is a conceptual progress if not also a technial simplification compared with [6] where similar results are obtained by dynamical system/invariant manifold methods. We believe that it is more appropriate to derive static solutions by energy methods which are more static in nature than by dynamical system methods. We would like to mention that for all locations of K spike points considered in [6] our method also works by solving the finite-dimensional optimization problem on the union of suitable small balls around each of these spike points. On the other hand, the method in [6] can give more precise information about the location of the spikes.

We are convinced that our approach will help to shed more light on the problem of location the peaks of K spike solutions in particular in situations where the non-degeneracy is very weak. There are interesting open problems in this direction. See Remark 1.4.

We maximize $M_\epsilon(\mathbf{P})$ over $\overline{\Lambda}$. Condition (1.6) ensures that $M_\epsilon(\mathbf{P})$ attains its maximum in Λ. We show that the resulting solution has the properties of Theorem 1.1.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ϵ, for ϵ sufficiently small; $\delta > 0$ is a very small number; $o(1)$ means $|o(1)| \to 0$ as $\epsilon \to 0$.

For the construction of boundary spike solutions, we just need an algebraic order estimate. Here for the interior peak case, the nonlocal term $\int_\Omega h(v_\epsilon)$ is of algebraic order ϵ^N, but the term that really determines the location of interior spikes is exponentially small. We use the method of viscosity solutions as introduced in [22] to estimate exponentially small terms.

The paper is organized as follows. In Section 2 we show how to choose σ_0. In Section 3 we show some properties of the function $P_{\Omega_{\epsilon,\mathbf{P}}} V_\sigma$. In Section 4 we derive some key energy estimates which will be important to derive an explicit expansion including error estimates for $M_\epsilon(\mathbf{P})$. In Section 5 we first determine the function v_ϵ by the Liapunov-Schmidt reduction method. Then we Then we derive an expansion for $M_\epsilon(\mathbf{P})$, i.e., we reduce the energy to finite dimensions. After that show that $\Phi_{\epsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}$. Finally, in Section 6, we prove that the maximizing problem has a solution $\mathbf{P}^\epsilon \in \Lambda$ and that $w_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}$ is indeed a solution of (1.3) which satisfies all the properties of Theorem 1.1.

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2. Choosing σ

In this section we choose σ appropriately. Let $P^\epsilon V_{\sigma,i}$ be defined as after (1.10). We now choose $\sigma_0(\epsilon, \mathbf{P})$ such that

$$\int_\Omega (\tau_{\sigma_0} + \sum_{i=1}^K P^\epsilon V_{\sigma_0,i})\, dx = 0. \tag{2.1}$$

We will see in Section 4 that this choice of σ_0 is essential to get good estimates for the nonlocal terms in (1.3). We calculate (for $\sigma = \sigma_0(\epsilon, \mathbf{P})$)

$$\sum_{i=1}^K \int_{\Omega_{\epsilon,P_i}} P_{\Omega_{\epsilon,P_i}} V_\sigma \, dy = \sum_{i=1}^K \frac{1}{p_\sigma} \int_{\Omega_{\epsilon,P_i}} h_\sigma(V_\sigma)$$

$$= \frac{1}{p_\sigma} \left[K \int_{R^N} h_\sigma(V_\sigma) - \sum_{i=1}^K \int_{\Omega^C_{\epsilon,P_i}} h_\sigma(V_\sigma) \right].$$

This implies

$$\tau_\sigma = -\sum_{i=1}^K \frac{1}{|\Omega_{\epsilon,P_i}|} \int_{\Omega_{\epsilon,P_i}} P_{\Omega_{\epsilon,P_i}} V_\sigma$$

$$= -\frac{\epsilon^N}{p_\sigma |\Omega|} \left[K \int_{R^N} h_\sigma(V_\sigma) - \sum_{i=1}^K \int_{\Omega^C_{\epsilon,P_i}} h_\sigma(V_\sigma) \right]. \tag{2.2}$$

Setting

$$g_1(\sigma) = -p_\sigma \tau_\sigma,$$

$$g_2(\sigma) = K \int_{R^N} [h_\sigma(V_\sigma) - h_0(V_0)],$$

and

$$g_3(\sigma, \mathbf{P}) = \frac{\epsilon^N}{|\Omega|} \sum_{i=1}^K \int_{\Omega^C_{\epsilon,P_i}} h_\sigma(V_\sigma)$$

we can rewrite (2.2) as

$$g_1(\sigma) = \frac{\epsilon^N}{|\Omega|}\left(K\int_{R^N} h_0(V_0) + g_2(\sigma)\right) - g_3(\sigma,\mathbf{P}). \tag{2.3}$$

From now on we will frequently write g instead of g_0, h instead of h_0 and V instead V_0 thus dropping the index 0 if this can be done without causing confusion.

It is easy to show that

$$g_1(\sigma) = \sigma + O(\sigma^2), \qquad g_1'(\sigma) = 1 + O(\sigma) \qquad \text{as } \sigma \to 0,$$

$$g_1 \in C^1([0,\tilde\sigma]) \qquad \text{for some } \tilde\sigma > 0 \text{ small},$$

$$g_2(\sigma) = O(\sigma),\ g_2'(\sigma) = O(1) \qquad \text{as } \sigma \to 0,$$

$$g_2 \in C^1([0,\tilde\sigma]) \qquad \text{for some } \tilde\sigma > 0 \text{ small},$$

$$|g_3(\sigma,\mathbf{P})| = \left|\frac{\epsilon^N}{|\Omega|}\sum_{i=1}^K \int_{\Omega^C_{\epsilon,P_i}} h_\sigma(V_\sigma)\right|$$

$$\leq CK\frac{\epsilon^N}{|\Omega|}\int_{|y|\geq D_\epsilon/\epsilon}\left(|y|^{-(N-1)/2}\exp\left(-\sqrt{p_\sigma}|y|\right)\right)^2$$

$$= CK\frac{\epsilon^N}{|\Omega|}\int_{r=D_\epsilon/\epsilon}^\infty r^{-N+1}\exp(-2\sqrt{p_\sigma}r)r^{N-1}\,dr$$

$$= CK\frac{\epsilon^N}{|\Omega|}\frac{1}{\sqrt{p_\sigma}}\exp\left(-\frac{2\sqrt{p_\sigma}D_\epsilon}{\epsilon}\right)$$

$$\leq CK\frac{\epsilon^N}{|\Omega|}\exp\left(-\frac{2\sqrt{p_\sigma}D_\epsilon}{\epsilon}\right) \tag{2.4}$$

where $D_\epsilon = \min_{i=1,\dots,K} d(P_i,\partial\Omega)$ since

$$|V_{\sigma,i}(x)| \leq C\left|\frac{x-P_i}{\epsilon}\right|^{-(N-1)/2}\exp\left(-\frac{\sqrt{p_\sigma}|x-P_i|}{\epsilon}\right).$$

For ϵ small let $\sigma_1(\epsilon)$ be a solution of

$$g_1(\sigma) = \frac{\epsilon^N}{|\Omega|}\left(K\int_{R^N} h_0(V_0) + g_2(\sigma)\right).$$

Note that this equation is the same as (2.3) with the term $g_3(\sigma,\mathbf{P})$ dropped.

Then by the Implicit Function Theorem

$$\sigma_1(\epsilon) = \frac{\epsilon^N}{|\Omega|}K\int_{R^N} h_0(V_0) + O(\epsilon^{2N}) \quad \text{as} \quad \epsilon \to 0$$

and $\sigma_1(\epsilon)$ is unique if ϵ is small enough and it is independent of $\mathbf{P}$. For the solution $\sigma_0(\epsilon, \mathbf{P})$ of (2.3) we make the ansatz $\sigma_0(\epsilon, \mathbf{P}) = \sigma_1(\epsilon) + \eta(\epsilon, \mathbf{P})$. Then because of (2.4) the Implicit Function Theorem implies

$$\eta(\epsilon, \mathbf{P}) = O(g_3(\sigma, \mathbf{P})) = O\left(\epsilon^N \exp\left(-\frac{2\sqrt{p_\sigma} D_\epsilon}{\epsilon}\right)\right).$$

Since $|p_\sigma - p_0| = O(\sigma)$ we have proved

$$\sigma_0(\epsilon, \mathbf{P}) = \sigma_1(\epsilon) + O\left(\epsilon^N \exp\left(-\frac{2\sqrt{p_0} D_\epsilon}{\epsilon}\right)\right)$$

and $\sigma_0(\epsilon, \mathbf{P})$ is unique if ϵ is small enough for all $\mathbf{P} \in \overline{\Lambda}$.

3. Projection of V_σ

In this section, we study properties of the function V_σ introduced in Section 2. In particular, we consider the "projection" $P_{\Omega_{\epsilon,P}}$ of V_σ in $H^1_N(\Omega)$ onto the linear subspace of $H^1(\Omega)$ of functions satisfying the Neumann boundary condition and prove some estimates.

Recall that for $P \in \Omega$ we defined $P_{\Omega_{\epsilon,P}} V_\sigma$ as the unique solution of

$$\begin{cases} \triangle v - p_\sigma v + h_\sigma(V_\sigma) = 0 \text{ in } \Omega_{\epsilon,P}, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_{\epsilon,P} \end{cases} \tag{3.1}$$

where p_σ, h_σ are as defined in the introduction. Recall that

$$\begin{aligned} \Omega_{\epsilon,P} &: = \{y | \epsilon y + P \in \Omega\}, \\ \Omega_\epsilon &: = \{y | \epsilon y \in \Omega\}, \\ \varphi_{\epsilon,P}(x) &= V_\sigma\left(\frac{|x-P|}{\epsilon}\right) - P_{\Omega_{\epsilon,P}} V_\sigma(y), \ \epsilon y + P = x. \end{aligned}$$

Then $\varphi_{\epsilon,P}(x)$ satisfies

$$\begin{cases} \epsilon^2 \triangle v - p_\sigma v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} V_\sigma\left(\frac{|x-P|}{\epsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

It is immediately seen that on $\partial\Omega$

$$\frac{\partial}{\partial \nu} V_\sigma(\frac{|x-P|}{\epsilon}) = \frac{1}{\epsilon} V_\sigma'(\frac{|x-P|}{\epsilon}) \frac{< x-P, \nu >}{|x-P|}$$

$$= -\frac{1}{\epsilon}\left(|x-P|^{-(N-1)/2} \cdot \epsilon^{+\frac{N-1}{2}} e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}} \sqrt{p_\sigma}\Big(c_\sigma + O(\epsilon)\Big)\right) \frac{< x-P, \nu >}{|x-P|}$$

$$= -\epsilon^{\frac{N-3}{2}} e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}} \sqrt{p_\sigma}\Big(c_\sigma + O(\epsilon)\Big) \frac{< x-P, \nu >}{|x-P|^{\frac{N+1}{2}}}$$

for some $c_\sigma > 0$.

To analyze $P_{\Omega_{\epsilon,P}} V_\sigma$, we introduce another linear problem. Let $P^D_{\Omega_{\epsilon,P}} V_\sigma$ be the unique solution of

$$\begin{cases} \epsilon^2 \triangle v - p_\sigma v + h_\sigma(V_\sigma) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Set

$$\varphi^D_{\epsilon,P} = V_\sigma - P^D_{\Omega_{\epsilon,P}} V_\sigma, \psi^D_{\epsilon,P}(x) = -\epsilon \log \varphi^D_{\epsilon,P}(x).$$

Note that $\varphi_{\epsilon,P}$, $\varphi^D_{\epsilon,P}$ and $\psi^D_{\epsilon,P}$ depend on σ. Then $v = \psi^D_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon \triangle v - |\nabla v|^2 + p_\sigma = 0 & \text{in } \Omega, \\ v = -\epsilon \log(V_\sigma(\frac{|x-P|}{\epsilon})) & \text{on } \partial\Omega. \end{cases}$$

Note that for $x \in \partial\Omega$

$$\psi^D_{\epsilon,P}(x) = -\epsilon \log\left((\frac{|x-P|}{\epsilon})^{-\frac{N-1}{2}} e^{-\frac{\sqrt{p_\sigma}|x-P|}{\epsilon}} (c_\sigma + O(\epsilon)) \right)$$

$$= \sqrt{p_\sigma}|x-P| + \frac{N-1}{2}\epsilon \log(\frac{|x-P|}{\epsilon}) + O(\epsilon)$$

$$= \sqrt{p_0}|x-P| + \frac{N-1}{2}\epsilon \log(\frac{|x-P|}{\epsilon}) + O(\sigma) + O(\epsilon)$$

since $p_\sigma = p_0 + O(\sigma)$. The proof of Lemma 3.1 is based on this estimate.

For the rest of this section we assume that $\sigma = \sigma_0$.

Lemma 3.1. (1) $\frac{\partial \psi^D_{\epsilon,P}}{\partial \nu} = (\sqrt{p}_0 + o(1)) \frac{< x-P, \nu >}{|x-P|}$ *for all* $P \in \Omega$ *uniformly on* $\partial\Omega$,

(2) $\psi^D_{\epsilon,P}(x) \longrightarrow \psi^D_0(x) = \inf_{z \in \partial\Omega} \sqrt{p_0}(|z-x| + |z-P|)$ *as* $\epsilon \to 0$ *for all* $P \in \Omega$ *uniformly in* $\bar{\Omega}$. *In particular,* $\psi^D_0(P) = 2\sqrt{p}_0 d(P, \partial\Omega)$.

Note that ψ^D_0 is a viscosity solution of the Hamilton-Jacobi equation $|\nabla u| = \sqrt{p_0}$ in Ω (see [22]).

Proof. (1) Corollary 3.4 and Section 5 in [10] prove that

$$\frac{\partial \psi^D_{\epsilon,P}}{\partial \nu} = (1+O(\epsilon))\frac{\partial \psi^D_{0,P}}{\partial \nu} \quad \text{uniformly on } \partial\Omega.$$

For $x \in \partial\Omega$ and $\nu(x)$ its exterior unit normal vector consider the points $x+\lambda\nu(x)$ with λ small. The condition for $z \in \partial\Omega$ to be a critical point of

$$(|x-\lambda\nu(x)-z|+|z-P|)$$

is

$$\frac{< x \pm \lambda\nu(x) - P, \tau_j(z) >}{|x+\lambda\nu(x)-P|} = \frac{< z-P, \tau_j(z) >}{|z-P|}$$

where $\tau_1(x), \dots, \tau_{N-1}(x), \nu(x)$ is an orthonormal system of $N-1$ tangent vectors and the exterior normal vector at $x \in \partial\Omega$. The sign in the last equation depends on the location of P. It is easy to see that for λ small enough z in the unique point on $\partial\Omega$ for which $\psi^D_{0,P}(x+\lambda\nu) = \inf_{z\in\partial\Omega} \sqrt{p_0}(|z-(x+\lambda\nu)|+|z-P|)$ is attained. This implies that for a critical point z

$$(|x-\lambda\nu(x)-z|+|z-P|) = (|x+\lambda\nu(x)-z|+|z-P|)$$

$$= |x \pm \lambda\nu(x) - P|$$

$$= \left(\sum_{j=1}^{N-1} < x \pm \lambda\nu(x) - P, \tau_j(z) >^2 + < x+\lambda\nu(x)-P, \nu(z) >^2\right)^{1/2}$$

$$= \left(\sum_{j=1}^{N-1} < x-P, \tau_j(z) >^2 + < x-P, \nu(x) >^2 + 2\lambda < x-P, \nu(x) > + O(\lambda^2)\right)^{1/2}$$

$$= |x-P| + \lambda\frac{< x-P, \nu(x) >}{|x-P|} + O(\lambda^2) \quad \text{as } \lambda \to 0.$$

(Note that

$$|x \pm \lambda\nu(x) - z| = O(\lambda),$$

$$< \tau_j(x), \tau_j(z) > = 1 + O(\lambda), \quad j = 1, \dots, N-1,$$

$$< \nu(x), \nu(z) > = 1 + O(\lambda),$$

$$< \tau_i(x), \tau_j(z) > = O(\lambda), \quad i,j = 1, \dots, N-1,\ i \neq j,$$

$$< \tau_j(x), \nu(z) > = O(\lambda), \quad j = 1, \dots, N-1\,.)$$

This implies

$$\psi^D_0(x+\lambda\nu(x)) = \sqrt{p_0}(|x-P| + \frac{< x-P, \nu(x) >}{|x-P|}\lambda) + O(\lambda^2)$$

and

$$\frac{\partial \psi_{0,P}^D}{\partial \nu}(x) = \sqrt{p_0}\frac{< x - P, \nu >}{|x - P|}.$$

(2) see Lemma 4.4 in [30]. □

Let us now compare $\varphi_{\epsilon,P}(x)$ and $\varphi_{\epsilon,P}^D(x)$. To this end, we introduce another function. Let U_ϵ be the solution of the problem

$$\begin{cases} \epsilon^2 \Delta U_\epsilon - p_\sigma U_\epsilon = 0 \text{ in } \Omega, \\ U_\epsilon = 1 \text{ on } \partial\Omega. \end{cases}$$

Set

$$\Psi_\epsilon = -\epsilon \log(U_\epsilon).$$

Then by Lemma 4.1 of [10], we have

$$\Psi_\epsilon(x) = \sqrt{p_\sigma} d(x, \partial\Omega) + O(\epsilon) \quad \text{in } \Omega,$$

$$\frac{\partial \Psi_\epsilon}{\partial \nu}(x) = -\sqrt{p_\sigma} + O(\epsilon) \quad \text{on } \partial\Omega.$$

This implies

$$|U_\epsilon(x)| \le C e^{-\sqrt{p_\sigma}\frac{d(x,\partial\Omega)}{\epsilon}} \quad \text{in } \Omega \tag{3.3}$$

and

$$\frac{\partial U_\epsilon}{\partial \nu}(x) = \frac{\sqrt{p_\sigma}}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0 \quad \text{ at } \partial\Omega. \tag{3.4}$$

Moreover, for any $\delta_0 > 0$ we have

$$\frac{U_\epsilon(\epsilon y + P)}{U_\epsilon(P)} \le C e^{\sqrt{p_\sigma}(1+\delta_0)|y|} \tag{3.5}$$

for ϵ sufficiently small.

This leads to the following

Lemma 3.2. *There exist $\eta_0, \delta_0 > 0, \epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, we have*

$$-(1+\eta_0\epsilon)\varphi_{\epsilon,P}^D - C e^{-\frac{\sqrt{p_\sigma}}{\epsilon}(1+\delta_0)d(P,\partial\Omega)} U_\epsilon < \varphi_{\epsilon,P}$$

$$< -(1-\eta_0\epsilon)\varphi_{\epsilon,P}^D + C e^{-\frac{\sqrt{p_\sigma}}{\epsilon}(1+\delta_0)d(P,\partial\Omega)} U_\epsilon.$$

Proof. We first assume that Ω is strictly starlike with respect to P. Namely, there is a constant $c_0 > 0$ such that

$$\langle x - P, \nu(x) \rangle \ge c_0 > 0$$

for all $x \in \partial\Omega$. Then on $\partial\Omega$, we have

$$\begin{aligned}\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} &= e^{-\frac{\psi^D_{\epsilon,P}(x)}{\epsilon}}(-\frac{1}{\epsilon})\frac{\partial \psi^D_{\epsilon,P}(x)}{\partial \nu}\\ &= -\frac{1}{\epsilon}V_\sigma \frac{\partial \psi^D_{\epsilon,P}(x)}{\partial \nu}\\ &= \frac{1}{\epsilon}V_\sigma \sqrt{p_\sigma}(1+O(\epsilon))\frac{< x-P,\nu >}{|x-P|}\\ &= -(1+O(\epsilon))\frac{\partial \varphi_{\epsilon,P}}{\partial \nu}.\end{aligned}$$

Since Ω is strictly starlike with respect to P, we have $\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu} < 0$. The following are standard facts from elliptic partial differential equations: Assume that for any $v \in H^2(\Omega) \cap H^1(\partial\Omega)$ and

$$\Delta v - p_\sigma v = 0,$$

$$\frac{\partial v}{\partial \nu} < 0.$$

Then $v \geq 0$ in Ω ("positivity"). Furthermore, the solution depends linearly on its Neumann boundary condition ("linearity"). An analogous result holds for the corresponding Dirichlet problem. Using positivity and linearity we get

$$-(1+\eta_0\epsilon)\varphi^D_{\epsilon,P} \leq \varphi_{\epsilon,P} \leq -(1-\eta_0\epsilon)\varphi^D_{\epsilon,P}.$$

Now we consider any bounded smooth domain Ω thus dropping the strictly starlike condition.

We can choose a constant $R = (1+2\delta_0)d(P,\partial\Omega)$ for some $\delta_0 > 0$ such that $\Omega_1 := B_R(P) \cap \Omega$ is strictly starlike with respect to P, i.e.

$$\langle x - P, \nu(x)\rangle \geq c_0 > 0, \quad x \in \partial\Omega_1.$$

Then on $\partial\Omega_1 \cap \partial\Omega = \Gamma_1$, we have

$$\frac{\partial \varphi_{\epsilon,P}}{\partial \nu} = -(1+O(\epsilon))\frac{\partial \varphi^D_{\epsilon,P}}{\partial \nu}$$

as above.

Now we construct functions $\tilde{\varphi}_{\epsilon,P}$ and $\tilde{\varphi}^D_{\epsilon,P}$ which are close to $\varphi_{\epsilon,P}$ and $\varphi^D_{\epsilon,P}$, respectively. We define $\tilde{V_\sigma}\left(\frac{x-P}{\epsilon}\right) = V_\sigma\left(\frac{x-P}{\epsilon}\right)\chi_{\delta_0}(x)$ where χ_{δ_0} is a smooth (cutoff) function such that $\chi_{\delta_0}(x)=1$ for $x\in\overline{B_{d(P,\partial\Omega)+\delta_0}(P)}$ and $\chi_{\delta_0}(x)=0$ for $x\in\overline{R^N\setminus B_{d(P,\partial\Omega)+2\delta_0}(P)}$. As above we define

$$\tilde{\varphi}_{\epsilon,P}(x)=\tilde{V_\sigma}\left(\frac{x-P}{\epsilon}\right)-P_{\Omega_{\epsilon,P}}\tilde{V_\sigma}\left(\frac{x-P}{\epsilon}\right),$$
$$\tilde{\varphi}^D_{\epsilon,P}(x)=\tilde{V_\sigma}\left(\frac{x-P}{\epsilon}\right)-P^D_{\Omega_{\epsilon,P}}\tilde{V_\sigma}\left(\frac{x-P}{\epsilon}\right).$$

It is immediately seen that

$$\|V_\sigma-\tilde{V_\sigma}\|_{L^\infty(\partial\Omega)}\le C\exp\left(-\sqrt{p_\sigma}/\epsilon(1+\delta_0)d(P,\partial\Omega)\right),$$
$$\|\frac{\partial V_\sigma}{\partial\nu}-\frac{\partial\tilde{V_\sigma}}{\partial\nu}\|_{L^\infty(\partial\Omega)}\le C\frac{1}{\epsilon}\exp\left(-\sqrt{p_\sigma}/\epsilon(1+\delta_0)d(P,\partial\Omega)\right).$$

Using positivity and linearity for the Dirichlet and Neumann problems, respectively, and (3.4) we get

$$|\tilde{\varphi}^D_{\epsilon,P}-\varphi^D_{\epsilon,P}|\le C\exp\left(-\sqrt{p_\sigma}/\epsilon(1+\delta_0)d(P,\partial\Omega)\right)U_\epsilon,$$
$$|\tilde{\varphi}_{\epsilon,P}-\varphi_{\epsilon,P}|\le C\exp\left(-\sqrt{p_\sigma}/\epsilon(1+\delta_0)d(P,\partial\Omega)\right)U_\epsilon$$

a.e. in Ω.

Combining this with the result for Ω stricty starlike with respect to P we conclude the proof of Lemma 3.2.

□

By Lemma 3.2 we have that

$$\Psi_\epsilon(P):=-\epsilon\log\left(-\varphi_{\epsilon,P}(P)\right)\to 2\sqrt{p_0}d(P,\partial\Omega)\qquad\text{as }\epsilon\to 0$$

since

$$\varphi_{\epsilon,P}(P)=(-1+O(\epsilon))\varphi^D_{\epsilon,P}(P)+O(e^{-\sqrt{p_\sigma}/\epsilon(1+\delta_0)d(P,\partial\Omega)}).$$

Let

$$\bar{V}_{\epsilon,P}(y)=\frac{1}{\varphi_{\epsilon,P}(P)}\cdot\varphi_{\epsilon,P}(x)$$

where $x=\epsilon y+P$.

Then $\bar{V}_{\epsilon,P}(0)=1$ (hence $\bar{V}_{\epsilon,P}(y)>0$). Furthermore, we have

Lemma 3.3. *For every sequence $\epsilon_k\to 0$, there is a subsequence $\epsilon_{k\ell}\to 0$ such that under the assumption $\sigma=\sigma_0(\epsilon)$, $\bar{V}_{\epsilon_{k\ell},P}\to\bar{V}$ uniformly on every compact set of R^N where $\bar{V}$ is a positive solution of*

$$\begin{cases}\triangle u-p_0u=0 & \text{in } R^N,\\ u>0 & \text{in } R^N \text{ and } u(0)=1.\end{cases}\tag{3.6}$$

Moreover for any $c_1 > 0$, $\sup_{z\in\Omega_{\epsilon_{k\ell},P}} e^{-(\sqrt{p_0}+c_1)|z|}\left|\bar V_{\epsilon_{k\ell},P}(z)-\bar V\right| \to 0$ *as* $\epsilon_{k\ell}\to 0$.

Proof. Assume that $\sigma=\sigma_0(\epsilon)$. By Lemma 3.2, we have

$$\begin{aligned}|\bar V_{\epsilon,P}(y)| &= |(V_\sigma - P_{\Omega_{\epsilon,P}}V_\sigma)\frac{1}{\varphi_{\epsilon,P}(P)}| \\ &\le C\frac{\varphi^D_{\epsilon,P}(P)}{\varphi_{\epsilon,P}(P)} + C\frac{1}{\varphi_{\epsilon,P}(P)}e^{-\frac{\sqrt{p_\sigma}}{\epsilon}(1+\delta_0)d(P,\partial\Omega)}U_\epsilon \\ &\le Ce^{\sqrt{p_\sigma}(1+\delta_0)|y|} + Ce^{-\frac{\sqrt{p_\sigma}}{\epsilon}(1+\delta_0)d(P,\partial\Omega)}U_\epsilon \quad (\text{ by Lemma 4.6 in [30] }) \\ &\le Ce^{\sqrt{p_\sigma}(1+\delta_0)|y|} + CU_\epsilon(x)/U_\epsilon(P) \quad (\text{since } U_\epsilon(P)\le Ce^{-\frac{\sqrt{p_\sigma}}{\epsilon}d(P,\partial\Omega)}) \\ &\le Ce^{\sqrt{p_\sigma}(1+\delta_0)|y|}.\end{aligned}$$

By a local compactness argument, we have that $\lim_{\epsilon\to 0} V_{\epsilon_{k\ell},P} = \bar V$ and $\bar V$ satisfies (3.6). Furthermore, the exponential decay estimate at the end of Lemma 3.3 follows immediately from this argument.

□

4. Key Energy Estimates

In this section, we derive some key energy estimates. We first state some useful lemmas about the interactions of two V's.

Lemma 4.1. *Let* $f\in C(R^N)\cap L^\infty(R^N), g\in C(R^N)$ *be radially symmetric and satisfy for some* $\alpha\ge 0,\beta\ge 0,\gamma_0\in R$

$$f(x)\exp(\alpha|x|)|x|^\beta \to \gamma_0 \ \text{as}\ |x|\to\infty$$
$$\int_{R^N}|g(x)|\exp(\alpha|x|)(1+|x|^\beta)\,dx<\infty.$$

Then

$$\exp(\alpha|y|)|y|^\beta\int_{R^N} g(x+y)f(x)\,dx \to \gamma_0\int_{R^N} g(x)\exp(-\alpha x_1)\,dx \ \text{as}\ |y|\to\infty.$$

For the proof, see [4].

We then have the following estimates. Recall that $V_{\sigma,i}$ was defined in the introduction.

Lemma 4.2. $\frac{1}{V_0(\frac{|P_1-P_2|}{\epsilon})}\int_{R^N} h(V_{0,1})V_{0,2}\to\gamma>0$ *as* $\epsilon\to 0$ *where*

$$\gamma=\int_{R^N} h(V_0(y))e^{-\sqrt{p_0}y_1}dy. \tag{4.1}$$

The next lemma is the key result in this section.

Lemma 4.3. *For any* $\mathbf{P} = (P_1, \ldots, P_K) \in \overline{\Lambda}$ *and* ϵ *sufficiently small*

$$J_\epsilon(\tau_\sigma + \sum_{i=1}^{K} PV_{\sigma,i}) = \epsilon^N [K I(V_0) - \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^{K} (e^{-\frac{1}{\epsilon}\Psi_\epsilon(P_i)})$$
$$-(\gamma + o(1)) \sum_{i,l=1, i\neq l}^{K} V_0(\frac{|P_i - P_l|}{\epsilon}) + O(\sigma)] \tag{4.2}$$

where $\sigma = \sigma_0(\epsilon)$*,* γ *is defined by (4.1),*

$$I(V_0) = \int_{R^N} |\nabla V_0|^2 + \frac{p_0}{2} \int_{R^N} |V_0|^2 - \int_{R^N} H(V_0),$$

with $H(t) = \int_0^t h_0(s)\, ds$ *and* $O(\sigma)$ *is to be understood as a term which is independent of* $\mathbf{P} \in \overline{\Lambda}$*.*

Proof.

We shall prove the cases when $K = 1$ and $K = 2$. The other cases are similar. Throughout the proof we assume that $\sigma = \sigma_0(\epsilon)$.

We begin with the case $K = 1$. Recall that $P_{\Omega_{\epsilon,P}} V_\sigma$ satisfies

$$\Delta P_{\Omega_{\epsilon,P}} V_\sigma - p_\sigma P_{\Omega_{\epsilon,P}} V_\sigma + h_\sigma(V_\sigma) = 0 \qquad \text{in } \Omega_{\epsilon,P}.$$

Hence

$$\epsilon^2 \int_\Omega |\nabla P_{\Omega_{\epsilon,P}} V_\sigma|^2 + p_0 \int_\Omega |\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma|^2$$
$$= \epsilon^2 \int_\Omega |\nabla P_{\Omega_{\epsilon,P}} V_\sigma|^2 + p_0 \int_\Omega |P_{\Omega_{\epsilon,P}} V_\sigma|^2 - p_0 \tau_\sigma^2 |\Omega|$$
$$\text{(by the definition of } \sigma)$$
$$= \epsilon^N \int_{\Omega_{\epsilon,P}} h_\sigma(V_\sigma) P_{\Omega_{\epsilon,P}} V_\sigma + \epsilon^N (p_0 - p_\sigma) \int_{\Omega_{\epsilon,P}} |P_{\Omega_{\epsilon,P}} V_\sigma|^2 - p_0 \tau_\sigma^2 |\Omega|.$$

On the other hand, we note that

$$H(\tau_\sigma + u) = \int_0^{\tau_\sigma + u} h_0(t)\, dt = H(\tau_\sigma) + \int_0^u h_0(t + \tau_\sigma)\, dt$$
$$= H(\tau_\sigma) + \int_0^u [p_0(t + \tau_\sigma) + \sigma + h_\sigma(t) - p_\sigma t]$$
$$= H(\tau_\sigma) + H_\sigma(u) + (p_0 - p_\sigma)\frac{1}{2} u^2$$
$$+ (p_0 \tau_\sigma + \sigma) u$$

where $H_\sigma(u) = \int_0^u h_\sigma(t) dt$.

Hence we have

$$\int_\Omega H(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma)$$
$$= H(\tau_\sigma)|\Omega| + \epsilon^N \int_{\Omega_{\epsilon,P}} H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma) + \frac{1}{2}(p_0 - p_\sigma)\epsilon^N \int_{\Omega_{\epsilon,P}} [P_{\Omega_{\epsilon,P}} V_\sigma]^2$$
$$+(p_0\tau_\sigma + \sigma) \int_{\Omega_{\epsilon,P}} P_{\Omega_{\epsilon,P}} V_\sigma$$
$$= H(\tau_\sigma)|\Omega| + \epsilon^N \int_{\Omega_{\epsilon,P}} H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma) + \frac{1}{2}(p_0 - p_\sigma)\epsilon^N \int_{\Omega_{\epsilon,P}} [P_{\Omega_{\epsilon,P}} V_\sigma]^2$$
$$-(p_0\tau_\sigma + \sigma)\tau_\sigma|\Omega|.$$

Now we combine and calculate

$$J_\epsilon(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma)$$
$$= \epsilon^N \int_{\Omega_{\epsilon,P}} \frac{1}{2} h_\sigma(V_\sigma) P_{\Omega_{\epsilon,P}} V_\sigma - H(\tau_\sigma)|\Omega| - \epsilon^N \int_{\Omega_{\epsilon,P}} H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma)$$
$$+\sigma\tau_\sigma|\Omega| + \frac{1}{2} p_0 \tau_\sigma^2 |\Omega|$$
$$= \epsilon^N \int_{\Omega_{\epsilon,P}} [\frac{1}{2} h_\sigma(V_\sigma) P_{\Omega_{\epsilon,P}} V_\sigma - H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma)]$$
$$+(\sigma\tau_\sigma - H(\tau_\sigma))|\Omega| + \frac{1}{2} p_0 \tau_\sigma^2 |\Omega|$$
$$= \epsilon^N \int_{\Omega_{\epsilon,P}} [\frac{1}{2} h_\sigma(V_\sigma) P_{\Omega_{\epsilon,P}} V_\sigma - H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma)] + O(\sigma^2)$$

where the "$O(\sigma)$"–terms do not depend on P explicitly. Note that by Lemma 3.1 and similar arguments as in the proof of Lemma 5.1 of [30] we have

$$\epsilon^N \int_{\Omega_{\epsilon,P}} h_\sigma(V_\sigma) P_{\Omega_{\epsilon,P}} V_\sigma$$
$$= \epsilon^N \int_{\Omega_{\epsilon,P}} h_\sigma(V_\sigma) V_\sigma + \epsilon^N \int_{\Omega_{\epsilon,P}} h_\sigma(V_\sigma)[P_{\Omega_{\epsilon,P}} V_\sigma - V_\sigma]$$
$$= \epsilon^N \left[\int_{R^N} h_0(V_0)V_0 + O(\sigma) + o(\varphi_{\epsilon,P}(P))\right] - \epsilon^N \varphi_{\epsilon,P}(P) \left[\int_{\Omega_{\epsilon,P}} h_0(V_0)\overline{V}_{\epsilon,P} + O(\sigma)\right]$$
$$= \epsilon^N \left[\int_{R^N} h_0(V_0)V_0 - \varphi_{\epsilon,P}(P)\gamma + O(\sigma) + o(\varphi_{\epsilon,P}(P))\right] \qquad (4.3)$$

where $O(\sigma)$ is independent of $P \in \overline{\Lambda}$ and

$$\gamma = \int_{R^N} h_0(V_0)\overline{V} = \int_{R^N} h_0(V_0) e^{-\sqrt{p_0} y_1}$$

for any solution $\overline{V}$ of (3.6) (see Lemma 4.7 in [30]) where it is also shown that γ is independent of the choice of the solution $\overline{V}$ of (3.6). Recall that

$$\bar{V}_{\epsilon,P}(y) = \frac{1}{\varphi_{\epsilon,P}(P)} \cdot \varphi_{\epsilon,P}(x)$$

where $x = \epsilon y + P$ as defined before Lemma 3.3.

For the last estimate note that because of the exponential decay of V_σ (see the equation before (1.9)) we have

$$\frac{\partial}{\partial \sigma} \int_{R^N} h_\sigma(V_\sigma) V_\sigma$$
$$= \int_{R^N} [h'_\sigma(V_\sigma)V_\sigma + h_\sigma(V_\sigma)] \frac{\partial V_\sigma}{\partial \sigma}$$
$$+ \int_{R^N} \{[f'(m - \tau_\sigma - V_\sigma)\left(-\frac{1}{p_\sigma}\right) - 1$$
$$+ f''(m - \tau_\sigma)\frac{1}{p_\sigma}]V_\sigma\}$$

Because of

$$\left|\frac{\partial V_\sigma}{\partial \sigma}\right| \le C$$

this implies

$$\left|\frac{\partial}{\partial \sigma} \int_{R^N} h_\sigma(V_\sigma) V_\sigma\right|_{\sigma=0} \le C \int_{R^N} \{h'_0(V_0)V_0 + h_0(V_0)$$
$$+ [f'(m - V_0)\left(-\frac{1}{p_0}\right) - 1 + f''(m)\frac{1}{p_0}]V_0\} \le C.$$

Similarly, we have

$$\int_{\Omega_{\epsilon,P}} H_\sigma(P_{\Omega_{\epsilon,P}} V_\sigma) = \int_{R^N} H(V_0) + (\gamma + o(1))\varphi_{\epsilon,P}(P) + O(\sigma)$$

where $O(\sigma)$ is independent of $P \in \Omega$.

Combining all together we obtain

$$J_\epsilon(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma) = \epsilon^N (\frac{1}{2} \int_{\Omega_{\epsilon,P}} |\nabla P_{\Omega_{\epsilon,P}} V_\sigma|^2 + \frac{p_0}{2} \int_{\Omega_{\epsilon,P}} |\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma|^2$$
$$- \int_{\Omega_{\epsilon,P}} H(\tau_\sigma + P_{\Omega_{\epsilon,P}} V_\sigma))$$
$$= \epsilon^N I(V) + \frac{1}{2} \epsilon^N \varphi_{\epsilon,P}(P)\,[\gamma + o(1)] + \epsilon^N (O(\sigma) + o(\varphi_{\epsilon,P}(P))).$$

This proves Lemma 4.3 for $K = 1$. Now we consider $K = 2$.

Similarly, as before we have

$$\epsilon^{-N} J_\epsilon(\tau_\sigma + PV_{\sigma,1} + PV_{\sigma,2})$$

$$= \int_{\Omega_{\epsilon,P}} \frac{1}{2}[|\nabla(PV_{\sigma,1}+PV_{\sigma,2})|^2 + p_0(PV_{\sigma,1}+PV_{\sigma,2})^2] - \int_{\Omega_{\epsilon,P}} H_\sigma(PV_{\sigma,1}+PV_{\sigma,2})$$

$$+O(\sigma) + o(\varphi_{\epsilon,P_1}(P_1) + \varphi_{\epsilon,P_2}(P_2) + V_0(\frac{|P_1-P_2|}{\epsilon})).$$

By Lemmas 4.1 and 4.2 we have

$$\int_\Omega h_\sigma(V_{\sigma,1}) PV_{\sigma,2} = \epsilon^N(\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon})$$

$$+\epsilon^N o(\varphi_{\epsilon,P_2}(P_2)) + O(\sigma).$$

$$\int_\Omega h_\sigma(PV_{\sigma,1}) PV_{\sigma,2} = \epsilon^N(\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon})$$

$$+\epsilon^N o(\sum_{i=1}^{2} \varphi_{\epsilon,P_i}(P_i)) + O(\sigma)$$

where $O(\sigma)$ is independent of P_1, $P_2 \in \Omega$. Let $\delta > 0$ be a sufficiently small number. We then have

$$\int_\Omega H_\sigma(PV_{\sigma,1} + PV_{\sigma,2}) = \int_{\Omega_1} H_\sigma(PV_{\sigma,1} + PV_{\sigma,2})$$

$$+\int_{\Omega_2} H_\sigma(PV_{\sigma,1} + PV_{\sigma,2}) + \int_{\Omega_3} H_\sigma(PV_{\sigma,1} + PV_{\sigma,2})$$

$$= I_1 + I_2 + I_3$$

where $I_i, i = 1,2,3$ are defined by the last equation and

$$\Omega_1 = \{|x-P_1| \le \frac{1-\delta}{2}|P_1-P_2|\}, \Omega_2 = \{|x-P_2| \le \frac{1-\delta}{2}|P_1-P_2|\},$$

$$\Omega_3 = \Omega \backslash (\Omega_1 \cup \Omega_2).$$

We also set

$$(\Omega_i)_\epsilon = \epsilon^{-1}\Omega_i, \quad i = 1,2,3.$$

For I_3, we have

$$|I_3| \le \int_{\Omega_3} (V_{0,1} + V_{0,2})^2 = O(e^{-\sqrt{p_0}2\frac{1}{\epsilon}|P_1-P_2|}).$$

For I_1, we have (reasoning as for $K=1$ above)

$$\begin{aligned} I_1 =& \int_{\Omega_1} H_\sigma(PV_{\sigma,1}+PV_{\sigma,2}) \\ =& \epsilon^N [\int_{R^N} H(V_0) + \gamma \varphi_{\epsilon,P_1}(P_1) + \int_{(\Omega_1)_\epsilon} h_0(V_{0,1})V_{0,2} \\ & + O(e^{-\sqrt{p_0}2\frac{1}{\epsilon}|P_1-P_2|}) + o(\sum_{i=1}^2 \varphi_{\epsilon,P_i}(P_i)) + O(\sigma)]. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 =& \epsilon^N [\int_{R^N} H(V_0) + \gamma \varphi_{\epsilon,P_2}(P_2) + \int_{(\Omega_2)_\epsilon} h_0(V_{0,2})V_{0,1} \\ & + O(e^{-\sqrt{p_0}2\frac{1}{\epsilon}|P_1-P_2|}) + o(\sum_{i=1}^2 \varphi_{\epsilon,P_i}(P_i)) + O(\sigma)]. \end{aligned}$$

Hence

$$\begin{aligned} \epsilon^{-N} J_\epsilon(\tau_\sigma + \sum_{i=1}^2 PV_{\sigma,i}) &= 2I(V_0) + \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^2 \varphi_{\epsilon,P_i}(P_i) + \int_{\Omega_\epsilon} h_0(V_{0,1})PV_{0,2} \\ & - \int_{(\Omega_1)_\epsilon} h_0(V_{0,1})V_{0,2} - \int_{(\Omega_3)_\epsilon} h_0(V_{0,2})V_{0,1} \\ & + \epsilon^N \gamma V_0 \left(\frac{|P_1-P_2|}{\epsilon} \right) + o(\sum_{i=1}^2 \varphi_{\epsilon,P_i}(P_i)) + o(V_0(\frac{|P_1-P_2|}{\epsilon})) \\ &= 2I(V_0) + \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^2 \varphi_{\epsilon,P_i}(P_i) \\ & - (\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon}) + O(\sigma). \end{aligned}$$

Here we have used

$$\int_{\Omega_\epsilon} h_0(V_{0,1})V_{0,2} = (\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon}),$$

$$\int_{(\Omega_1)_\epsilon} h_0(V_{0,1})V_{0,2} = (\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon}),$$

and

$$\int_{(\Omega_3)_\epsilon} h_0(V_{0,2})V_{0,1} = (\gamma + o(1)) V_0(\frac{|P_1-P_2|}{\epsilon}).$$

□

5. Liapunov-Schmidt Reduction

In this section, we reduce problem (1.3) to finite dimensions by the Liapunov-Schmidt method. We first introduce some notation.

$$X = \{v \in H^2(\Omega_\epsilon) | \int_{\Omega_\epsilon} v = 0, \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_\epsilon\},$$

$$Y = \{v \in L^2(\Omega_\epsilon) | \int_{\Omega_\epsilon} v = 0\}.$$

For $v \in X$ define

$$S_\epsilon(v) = \Delta v - p_0 v + h_0(v) - \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} h_0(v)$$

where

$$S_\epsilon : X \to Y.$$

Then solving equation (1.3) is equivalent to

$$S_\epsilon(v) = 0, v \in X.$$

Fix $\mathbf{P} = (P_1, ..., P_K) \in \overline{\Lambda}$.

Recall that $w_{\epsilon,\mathbf{P}} = \tau_\sigma + \sum_{i=1}^K PV_{\sigma,i}$ where τ_σ is defined after (1.2) and $\sigma = \sigma_0$ (see Section 2). Hence $\tau_\sigma = O(\epsilon^N)$.

Consider the linearized operator

$$S'_\epsilon(w_{\epsilon,\mathbf{P}}) = L_\epsilon : u \mapsto \Delta u - p_\sigma u + h'_\sigma(\sum_{i=1}^K PV_{\sigma,i})u$$

$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} h'_\sigma(\sum_{i=1}^K PV_{\sigma,i})u$$

where

$$L_\epsilon : X \to Y.$$

We denote $\mathbf{P} = (P_1, \ldots, P_K) = ((P_{1,1}, \ldots, P_{1,N}), \ldots, (P_{K,1}, \ldots, P_{K,N}))$ and choose the approximate kernel as

$$\mathcal{K}_{\epsilon,\mathbf{P}} = \text{span}\left\{ \frac{\partial(\tau_\sigma + \sum_{i=1}^K PV_{\sigma,i})}{\partial P_{i,j}} \middle| i = 1, \ldots, K,\ j = 1, \ldots, N \right\}$$

Let $\pi_{\epsilon,\mathbf{P}}$ denote the orthogonal projection in Y onto $\mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ with respect to the norm of $L^2(\Omega_\epsilon)$. Our goal in this section is to show that the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0$$

has a unique solution $\Phi_{\epsilon,\mathbf{P}}$ such that $\Phi_{\epsilon,\mathbf{P}} \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \cap X$ (here we mean the orthogonal complement in X to the finite–dimensional linear subspace $\mathcal{K}_{\epsilon,\mathbf{P}}$ with respect to the norm of $L^2(\Omega_\epsilon)$) if ϵ is small enough and $\mathbf{P} = (P_1, ..., P_K) \in \overline{\Lambda}$.

As a preparation the following two propositions give the invertibility of the corresponding linearized operator.

Proposition 5.1. *Let* $L_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ L_\epsilon$. *There exist positive constants* $\overline{\epsilon}, \overline{C}$ *such that for all* $\epsilon \in (0, \overline{\epsilon})$ *and* $\mathbf{P} = (P_1, \ldots, P_K) \in \overline{\Lambda}$

$$\|L_{\epsilon,\mathbf{P}}\Phi\|_{L^2(\Omega_\epsilon)} \geq \overline{C}\|\Phi\|_{H^2(\Omega_\epsilon)} \tag{5.1}$$

for all $\Phi \in \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \cap X$.

Proposition 5.2. *For any* $\epsilon \in (0, \tilde{\epsilon})$ *and* $\mathbf{P} = (P_1, \ldots, P_K) \in \overline{\Lambda}$ *the map*

$$L_{\epsilon,\mathbf{P}} = \pi_{\epsilon,\mathbf{P}} \circ L_\epsilon : \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \cap X \to \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} \cap Y$$

is surjective.

Proof of Proposition 5.1. We will follow the method used in [11], [32], [33], and [38]. Suppose that (5.1) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\mathbf{P^k}\} = \{(P_1^k, \ldots, P_K^k)\} = \{(P_{1,1}^k, \ldots, P_{1,N}^k), \ldots, (P_{K,1}^k, \ldots, P_{K,N}^k)\}$, and $\{\Phi_k\}$ $(k = 1, 2, \ldots)$ with $\epsilon_k > 0$, $\mathbf{P^k} \in \overline{\Lambda}$, $\Phi_k \in \mathcal{K}_{\epsilon_k,\mathbf{P^k}}^{\perp} \cap X$ such that

$$\epsilon_k \to 0, \tag{5.2}$$

$$\mathbf{P^k} \to \mathbf{P} \in \overline{\Lambda}, \tag{5.3}$$

$$\|L_{\epsilon_k,\mathbf{P^k}}\Phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0, \tag{5.4}$$

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1, \qquad k = 1,\, 2,\, \ldots . \tag{5.5}$$

For $i = 1, 2, \ldots, K$, $j = 1, 2, \ldots, N$ and $k = 1, 2, \ldots$ denote

$$e_{ij,k} = \frac{\partial(\tau_\sigma + \sum_{i=1}^K PV_{\sigma,i,k})}{\partial P_{i,j}^k} \Big/ \left\| \frac{\partial(\tau_\sigma + \sum_{i=1}^K PV_{\sigma,i,k})}{\partial P_{i,j}^k} \right\|_{L^2(\Omega_{\epsilon_k})}$$

where

$$PV_{\sigma,i,k}(y) = P_{\Omega_{\epsilon_k,P_i^k}} V_{\sigma_k}\left(y - \frac{P_i^k}{\epsilon_k}\right), \quad y \in \Omega_{\epsilon_k}.$$

Note that

$$< e_{i_1 j_1,k}, e_{i_2 j_2,k} >= \delta_{i_1 i_2}\delta_{j_1 j_2} + O(\epsilon_k) \qquad \text{as } k \to \infty$$

by the symmetry of the function V and the fact that $\mathbf{P} \in \overline{\Lambda}$ (recall that $V(\frac{|P_k - P_l|}{\epsilon}) \leq \eta\epsilon$). Here $\delta_{i_1 i_2}$ is the Kronecker symbol. Furthermore, because of (5.4),

$$\|L_{\epsilon_k}\Phi_k\|^2_{L^2(\Omega_{\epsilon_k})} - \sum_{i=1}^{K}\sum_{j=1}^{N}\left(\int_{\Omega_{\epsilon_k}}(L_{\epsilon_k}\Phi_k)e_{ij,k}\right)^2 \to 0 \tag{5.6}$$

as $k \to \infty$. For $i = 1, 2, \dots, N$ we introduce new sequences $\{\varphi_{i,k}\}$ by

$$\varphi_{i,k}\,(y) = \chi(\epsilon_k y)\Phi_k\left(y + \frac{P_i^k}{\epsilon_k}\right), \quad y \in \Omega_{\epsilon_k, P_i^k} \tag{5.7}$$

where $\chi(z)$ is a smooth cut-off function such that $\chi(z) = 1$ for $|z| \leq \delta$ and $\chi(z) = 0$ for $|z| > 2\delta$ for some small δ (actually we choose δ as in (1.7)).

Extend $\varphi_{i,k}$ to a function on R^N by setting $\varphi_{i,k}(y) = 0$ for $y \in R^N \setminus B_{2\delta}(0)$. It follows from (5.5) and the smoothness of χ that

$$\left\|\varphi_{i,k}\left(\cdot - \frac{P_i^k}{\epsilon_k}\right)\right\|_{H^2(R^N)} \leq C$$

for all k sufficiently large. (Note that the functions $\chi(\cdot + P_i^k i/\epsilon_k)$, $i = 1, \dots, K$ and $1 - \sum_{i=1}^K \chi(\cdot + P_i^k/\epsilon_k)$ consitute a partition of unity in Ω_ϵ.) The constants in the extension theorem (see [13] Lemma 6.37 and Theorem 7.25) can be chosen independent of ϵ whenever $\epsilon < 1$. Therefore, there exists a subsequence, again denoted by $\{\varphi_{i,k}\}$ which converges weakly in $H^2(R^N)$ to a limit $\varphi_{i,\infty}$ as $k \to \infty$. We are now going to show that $\varphi_{i,\infty} \equiv 0$. As a first step we deduce

$$\int_{R^N}\varphi_{i,\infty}\frac{\partial V_0}{\partial y_j} = 0, \qquad j = 1, \dots, N. \tag{5.8}$$

This statement is shown as follows

$$\int_{R^N}\varphi_{i,\infty}(y - P_i^k/\epsilon_k)\frac{\partial V_0}{\partial y_j}(y)\,dy$$

$$= \lim_{k\to\infty}\int_{R^N}\varphi_{i,k}(y - P_i^k)\frac{\partial V_0}{\partial y_j}(y)\,dy$$

$$= \lim_{k\to\infty}\int_{\Omega_{\epsilon_k}, P_i^k}\chi(\epsilon_k y)\Phi_k\left(y + \frac{P_i^k}{\epsilon_k}\right)\left(\sum_{i=1}^{K}\frac{\partial PV_{\sigma,i,k}}{\partial P_{i,j}^k}\left(y + \frac{P_i^k}{\epsilon_k}\right) + \frac{\partial \tau_\sigma}{\partial P_{i,j}^k}\right)dy$$

$$= \lim_{k\to\infty} \int_{\Omega_{\epsilon_k}} (\chi(\epsilon_k y - P_i^k) - 1)\Phi_k(y) \left(\frac{\sum_{i=1}^K \partial PV_{\sigma,i,k}}{\partial P_{i,j}^k} + \frac{\partial \tau_\sigma}{\partial P_{i,j}^k} \right) dy$$
$$= o(1).$$

Here we have used the facts that $V_0(\frac{|x-P_i^k|}{\epsilon})$ and $\partial PV_{\sigma,i,k}/\partial P_{i,j}^k$ have exponential decay outside $B_\delta(P_i^k)$, $\partial\tau_\sigma/\partial P_{i,j}^k \to 0$ as $k\to\infty$, $\Phi_k \in \mathcal{K}_{\epsilon_k,\mathbf{P}^k}^\perp$, and $\frac{\sum_{i=1}^K \partial PV_{\sigma,i,k}}{\partial P_{i,j}^k} + \frac{\partial\tau_\sigma}{\partial P_{i,j}^k} \in \mathcal{K}_{\epsilon_k,\mathbf{P}^k}$. This implies (5.8).

Let $\mathcal{K}_0$ be the kernel and cokernel of the linear operator $S_0'(V)$ which is the Fréchet derivative at V of

$$S_0(v) = \Delta v - p_0 v + h(v),$$
$$S_0 : H^2(R^N) \to L^2(R^N).$$

Note that

$$S_0'(V)v = \Delta v - p_0 v + h'(V)v$$

and

$$\mathcal{K}_0 = \text{span}\left\{ \frac{\partial V}{\partial y_j} \,|j = 1, \ldots, N \right\}.$$

Equation (5.8) implies that $\varphi_{i,\infty} \in \mathcal{K}_0^\perp$. By the exponential decay of V and by (5.4) we have after possibly taking a further subsequence that

$$\Delta\varphi_{i,\infty} - p_0\varphi_{i,\infty} + h'(V)\varphi_{i,\infty} = 0,$$

i.e. $\varphi_{i,\infty} \in \mathcal{K}_0$. Therefore $\varphi_{i,\infty} = 0$.

Hence

$$\varphi_{i,k} \rightharpoonup 0 \qquad \text{weakly in } H^2(R^N) \qquad \text{as } k \to \infty \tag{5.9}$$

for $i = 1, 2, \ldots, K$.

Furthermore, consider

$$\varphi_{0,k}(y) = \Phi_k(y) - \sum_{i=1}^K \varphi_{i,k}(y), \quad , y \in \Omega_\epsilon.$$

Now extend Φ_k from Ω_ϵ to R^N such that

$$\|\Phi_k\|_{H^2(R^N)} \le C$$

for all k sufficiently large and define the extension of $\varphi_{0,k}$ by

$$\varphi_{0,k}(y) = \Phi_k(y) - \sum_{i=1}^K \varphi_{i,k}(y), \quad y \in R^N$$

where $\varphi_{i,k}$ are the extensions before (5.8).

Then obviously

$$\|\varphi_{0,k}\|_{H^2(R^N)} \leq C$$

and we have for a subsequence

$$\varphi_{0,k} \rightharpoonup \varphi_{0,\infty} \quad \text{weakly in } H^2(R^N) \quad \text{as } k \to \infty$$

where $\varphi_{0,\infty}$ satisfies

$$\Delta\varphi_{0,\infty} - p_0\varphi_{0,\infty} = 0,$$
$$\varphi_{0,\infty} \in H^2(R^N).$$

Therefore $\varphi_{0,k} \rightharpoonup 0$ weakly in $H^2(R^N)$ as $k \to \infty$.

Since

$$\varphi_{i,k} \rightharpoonup 0 \quad \text{weakly in } H^2(R^N) \text{ as } k \to \infty \quad \text{for } i = 0, 1, \ldots, K$$

we conclude that

$$\Phi_k \rightharpoonup 0 \quad \text{weakly in } H^2(R^N) \quad \text{as } k \to \infty$$

for the extended function Φ_k which was defined after (5.9). By Sobolev embedding,

$$\|\Phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0 \qquad \text{as } k \to \infty.$$

By Cauchy Schwarz inequality,

$$\|h'_\sigma(w_{\epsilon,\mathbf{P}} - \tau_\sigma)\Phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0.$$

By (5.4) and (5.6),

$$\|(\Delta - p_\sigma)\Phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0 \qquad \text{as } k \to \infty.$$

Since

$$\int_{\Omega_{\epsilon_k}} |\nabla\Phi_k|^2 + p_\sigma\Phi_k^2 = \int_{\Omega_{\epsilon_k}} [(p_\sigma - \Delta)\Phi_k]\Phi_k$$
$$\leq C\|(\Delta - p_\sigma)\Phi_k\|_{L^2(\Omega_{\epsilon_k})}$$

we have that

$$\|\Phi_k\|_{H^1(\Omega_{\epsilon_k})} \to 0 \qquad \text{as } k \to \infty.$$

In summary:

$$\|\Delta\Phi_k\|_{L^2(\Omega_{\epsilon_k})} \to 0 \text{ and } \|\Phi_k\|_{H^1(\Omega_{\epsilon_k})} \to 0. \tag{5.10}$$

From (5.10) and the following elliptic regularity estimate (for a proof see Appendix B in [38])

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} \le C(\|\Delta\Phi_k\|_{L^2(\Omega_{\epsilon_k})} + \|\Phi_k\|_{H^1(\Omega_{\epsilon_k})}) \tag{5.11}$$

for $\Phi_k \in H^2_N(\Omega_{\epsilon_k})$ we deduce that

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} \to 0 \qquad \text{as } k \to \infty.$$

This contradicts the assumption

$$\|\Phi_k\|_{H^2(\Omega_{\epsilon_k})} = 1$$

and the proof of Proposition 5.1 is completed. □

Proof of Proposition 5.2.

We define a linear operator T from $L^2(\Omega_\epsilon)$ to itself by

$$T = \pi_{\epsilon,\mathbf{P}} \circ L_\epsilon \circ \pi_{\epsilon,\mathbf{P}}$$

Its domain of definition is $H^2_N(\Omega_\epsilon) \cap X$. By the theory of elliptic equations and by integration by parts it is easy to see that T is an (unbounded) self-adjoint and hence also a closed operator. The L^2 estimates of elliptic equations imply that the range of T is closed in $L^2(\Omega_\epsilon)$. Then by the Closed Range Theorem ([41], page 205) we know that the range of T is the orthogonal complement of its kernel with respect to the L^2 norm. This implies Proposition 5.2.

□

We are now in a position to solve the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0. \tag{5.12}$$

We first rewrite the equation

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0$$

and calculate

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) =$$

$$\Delta(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - p_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) + h_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) - \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} h_0(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})$$

$$= \Delta\Phi_{\epsilon,\mathbf{P}} - p_\sigma\Phi_{\epsilon,\mathbf{P}} + h_\sigma^{'}(w_{\epsilon,\mathbf{P}} - \tau_\sigma)\Phi_{\epsilon,\mathbf{P}} - \frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon} h_\sigma^{'}(w_{\epsilon,\mathbf{P}} - \tau_\sigma)\Phi_{\epsilon,\mathbf{P}}$$

$$+[h_\sigma(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} - \tau_\sigma) - h_\sigma(w_{\epsilon,\mathbf{P}} - \tau_\sigma) - h_\sigma^{'}(w_{\epsilon,\mathbf{P}} - \tau_\sigma)\Phi_{\epsilon,\mathbf{P}}]$$

$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}[h_\sigma(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma(w_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}]$$
$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}h_\sigma(\sum_{i=1}^K PV_{\sigma,i})$$
$$+\sum_{i=1}^K[\Delta PV_{\sigma,i}-p_\sigma(PV_{\sigma,i})]+h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-p_\sigma\tau_\sigma$$
$$=\Delta\Phi_{\epsilon,\mathbf{P}}-p_\sigma\Phi_{\epsilon,\mathbf{P}}+h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}$$
$$+[h_\sigma(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma(w_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}]$$
$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}[h_\sigma(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma(w_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}]$$
$$+h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-\sum_{i=1}^K h_\sigma(V_{\sigma,i})$$
$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}[h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-\sum_{i=1}^K h_\sigma(V_{\sigma,i})-p_\sigma\tau_\sigma]$$
$$=L_\epsilon\Phi_{\epsilon,\mathbf{P}}+N_\epsilon^1(\Phi_{\epsilon,\mathbf{P}})+N_\epsilon^2(\Phi_{\epsilon,\mathbf{P}})+E_\epsilon$$

where

$$L_\epsilon\Phi_{\epsilon,\mathbf{P}}=\Delta\Phi_{\epsilon,\mathbf{P}}-p_\sigma\Phi_{\epsilon,\mathbf{P}}+h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}$$
$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}},$$
$$N_\epsilon^1(\Phi_{\epsilon,\mathbf{P}})=[h_\sigma(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma(w_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}]$$
$$N_\epsilon^2(\Phi_{\epsilon,\mathbf{P}})=-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}[h_\sigma(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma(w_{\epsilon,\mathbf{P}}-\tau_\sigma)-h_\sigma^{'}(w_{\epsilon,\mathbf{P}}-\tau_\sigma)\Phi_{\epsilon,\mathbf{P}}],$$
$$E_\epsilon=h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-\sum_{i=1}^K h_\sigma(V_{\sigma,i})$$
$$-\frac{1}{|\Omega_\epsilon|}\int_{\Omega_\epsilon}[h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-\sum_{i=1}^K h_\sigma(V_{\sigma,i})]-p_\sigma\tau_\sigma.$$

Since $L_{\epsilon,\mathbf{P}}:\ X\cap\mathcal{K}_{\epsilon,\mathbf{P}}^\perp\to Y\cap\mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ is invertible (call the inverse $L_{\epsilon,\mathbf{P}}^{-1}$) we can rewrite (5.12) as

$$\Phi=-L_{\epsilon,\mathbf{P}}^{-1}\circ\pi_{\epsilon,\mathbf{P}}\circ N_\epsilon^1(\Phi)$$
$$-L_{\epsilon,\mathbf{P}}^{-1}\circ\pi_{\epsilon,\mathbf{P}}\circ N_\epsilon^2(\Phi)-L_{\epsilon,\mathbf{P}}^{-1}\circ\pi_{\epsilon,\mathbf{P}}\circ E_\epsilon(\Phi)$$
$$\equiv G_{\epsilon,\mathbf{P}}(\Phi) \tag{5.13}$$

where the operator $G_{\epsilon,\mathbf{P}}$ is defined by the last equation for $\Phi \in X$. We are going to show that the operator $G_{\epsilon,\mathbf{P}}$ is a contraction on

$$B_{\epsilon,\delta} \equiv \{\Phi \in X \cap \mathcal{K}_{\epsilon,\mathbf{P}}^{\perp} | \|\Phi\|_{H^2(\Omega_\epsilon)} < \delta\}$$

if δ is small enough.

The following error estimates are essential for the rest of the proof.

Lemma 5.3. *For ϵ small enough, we have*

$$\|N_\epsilon^1(\Phi)\|_{L^2(\Omega_\epsilon)} \le C_\delta \|\Phi\|_{L^2(\Omega_\epsilon)} \quad \textit{for all } \Phi \in B_{\epsilon,\delta}, \tag{5.14}$$

$$|N_\epsilon^2(\Phi)| \le C_\delta \epsilon^{N/2} \|\Phi\|_{L^2(\Omega_\epsilon)} \quad \textit{for all } \Phi \in B_{\epsilon,\delta}, \tag{5.15}$$

$$\|E_\epsilon\|_{L^2(\Omega_\epsilon)} \le C e^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(P_1,\dots,P_K)} \tag{5.16}$$

Proof. By the remark on page 5 we may assume that h_0 together with its first two derivatives is bounded. This implies

$$\|N_\epsilon^1(\Phi)\|_{L^2(\Omega)} \le C_\delta \|\Phi\|_{L^2(\Omega)}$$

for $\Phi \in B_{\epsilon,\delta}$. (5.14) is proved. Furthermore, by Cauchy Schwarz inequality,

$$|N_2^\epsilon(\Phi)| \le C\epsilon^N \int_{\Omega_\epsilon} |\Phi| \le C\epsilon^N \left(\int_{\Omega_\epsilon} \Phi^2\right)^{1/2} \left(\int_{\Omega_\epsilon} 1\right)^{1/2}$$
$$\le C\epsilon^{N/2} \|\Phi\|_{L^2(\Omega_\epsilon)}$$

for $\Phi \in B_{\epsilon,\delta}$. (5.15) is proved.

To prove (5.16), we divide the domain into $(K+1)$ parts: let $\Omega = \cup_{i=1}^{K+1} \Omega_i$ where

$$\Omega_i = \{|x - P_i| \le \frac{1-\delta}{2} \min_{k \ne l} |P_k - P_l|\}, i = 1, ..., K, \Omega_{K+1} = \Omega \setminus \cup_{i=1}^K \Omega_i.$$

We now estimate E_ϵ in each domain.

In Ω_{K+1}, we have

$$|E_\epsilon| \le C(V_{0,1} + ... + V_{0,K})^2 \le O(e^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\min_{k,l}|P_k - P_l|}).$$

Hence

$$\|E_\epsilon\|_{L^2(\Omega_{K+1})_\epsilon)} \le O(e^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(\mathbf{P})}).$$

In $\Omega_i, i = 1, ..., K$, we have

$$|E_\epsilon| \le \sum_{j \ne i} (|h_\sigma'(V_{\sigma,i}) V_{\sigma,j}| + |h_\sigma'(V_{\sigma,i})(PV_{\sigma,j} - V_{\sigma,j})|)$$

$$+O(\sum_{j\neq i}(|PV_{\sigma,j}|^2+|V_{\sigma,j}|^2))+O(|PV_{\sigma,i}-V_{\sigma,i}|^2).$$

Using Lemma 3.2 and the facts that $PV_{\sigma,j}$ and $V_{\sigma,j}$ decay exponentially, we obtain

$$\|E_\epsilon\|_{L^2((\Omega_i)_\epsilon)} \le Ce^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(P_1,\dots,P_K)}.$$

□

Thus

$$\|G_{\epsilon,\mathbf{P}}(\Phi)\|_{H^2(\Omega_\epsilon)} \le \overline{C}^{-1}(\|\pi_{\epsilon,\mathbf{P}}\circ N^1_\epsilon(\Phi)\|_{L^2(\Omega_\epsilon)}$$
$$+\|\pi_{\epsilon,\mathbf{P}}\circ N^2_\epsilon(\Phi)\|_{L^2(\Omega_\epsilon)}+\|\pi_{\epsilon,\mathbf{P}}\circ E_\epsilon\|_{L^2(\Omega_\epsilon)}$$
$$\le \overline{C}^{-1}C(c(\delta)\delta+\delta_\epsilon)$$

where $\overline{C}>0$ is independent of $\delta>0$, $\delta_\epsilon = e^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(\mathbf{P})}$ and $c(\delta)\to 0$ as $\delta\to 0$. Similarly we show

$$\|G_{\epsilon,\mathbf{P}}(\Phi)-G_{\epsilon,\mathbf{P}}(\Phi')\|_{H^2(\Omega_\epsilon)} \le \overline{C}^{-1}Cc(\delta)\|\Phi-\Phi'\|_{H^2(\Omega_\epsilon)}$$

where $c(\delta)\to 0$ as $\delta\to 0$. Therefore $M_{\epsilon,\mathbf{P}}$ is a contraction on B_δ. The existence of a fixed point $\Phi_{\epsilon,\mathbf{P}}$ now follows from the Contraction Mapping Principle and $\Phi_{\epsilon,\mathbf{P}}$ is a solution of (5.13).

Because of Lemma 5.3,

$$\|\Phi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \le (\|\pi_{\epsilon,\mathbf{P}}\circ N^1_\epsilon(\Phi_{\epsilon\mathbf{P}})\|_{L^2(\Omega_\epsilon)}$$
$$+\|\pi_{\epsilon,\mathbf{P}}\circ N^2_\epsilon(\Phi_{\epsilon,\mathbf{P}})\|_{L^2(\Omega_\epsilon)}+\|\pi_{\epsilon,\mathbf{P}}\circ E_\epsilon\|_{L^2(\Omega_\epsilon)}$$
$$\le \overline{C}^{-1}(C\delta_\epsilon + c(\delta)\|\Phi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)})$$

we have

$$\|\Phi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \le C(\delta_\epsilon).$$

We have proved

Lemma 5.4. *There exists $\overline{\epsilon}>0$ such that for every $(N+1)$-tuple $\epsilon, P_1,\dots,P_K$ with $0<\epsilon<\overline{\epsilon}$ and $\mathbf{P}=(P_1,\dots,P_K)\in\overline{\Lambda}$ there is a unique $\Phi_{\epsilon,\mathbf{P}}\in X\cap\mathcal{K}^\perp_{\epsilon,\mathbf{P}}$ satisfying $S_\epsilon(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})\in Y\cap\mathcal{K}_{\epsilon,\mathbf{P}}$ and*

$$\|\Phi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \le C(e^{-\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(\mathbf{P})}). \tag{5.17}$$

The next lemma is our main estimate.

Lemma 5.5. *Let $\Phi_{\epsilon,\mathbf{P}}$ be defined by Lemma 5.4. Then we have*

$$\begin{aligned} &J_\epsilon(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}}) \\ &=\epsilon^N\Bigg[KI(V)-\frac{1}{2}(\gamma+o(1))\sum_{i=1}^K e^{-\frac{1}{\epsilon}\Psi_\epsilon(P_i)} \\ &-\sum_{k,l=1,\dots,K,k\neq l}(\gamma+o(1))V(\frac{|P_k-P_l|}{\epsilon})+O(\sigma)\Bigg] \end{aligned} \tag{5.18}$$

where γ is defined by (4.1) and the terms of order $O(\sigma)$ do not explicitly depend on $\mathbf{P}\in\overline{\Lambda}$.

Proof.

In fact for any $\mathbf{P}\in\overline{\Lambda}$, we have

$$\epsilon^{-N}J_\epsilon(w_{\epsilon,\mathbf{P}}+\Phi_{\epsilon,\mathbf{P}})=\epsilon^{-N}J_\epsilon(w_{\epsilon,\mathbf{P}})+g_{\epsilon,\mathbf{P}}(\Phi_{\epsilon,\mathbf{P}})+O(\|\Phi_{\epsilon,\mathbf{P}}\|^2_{H^2(\Omega_\epsilon)})$$

where

$$\begin{aligned} &g_{\epsilon,\mathbf{P}}(\Phi_{\epsilon,\mathbf{P}}) \\ &=\int_{\Omega_\epsilon}(\sum_{i=1}^K\nabla PV_{\sigma,i}\nabla\Phi_{\epsilon,\mathbf{P}}+p_0(\tau_\sigma+\sum_{i=1}^K PV_{\sigma,i})\Phi_{\epsilon,\mathbf{P}} \\ &\quad-\int_{\Omega_\epsilon}h_0(\tau_\sigma+\sum_{i=1}^K PV_{\sigma,i})\Phi_{\epsilon,\mathbf{P}} \\ &=\int_{\Omega_\epsilon}[\sum_{i=1}^K h_\sigma(V_{\sigma,i})+p_0(\tau_\sigma+\sum_{i=1}^K PV_{\sigma,i})-p_\sigma\sum_{i=1}^K PV_{\sigma,i}-h_0(\tau_\sigma+\sum_{i=1}^K PV_{\sigma,i})]\Phi_{\epsilon,\mathbf{P}} \\ &=\int_{\Omega_\epsilon}[\sum_{i=1}^K h_\sigma(V_{\sigma,i})-h_\sigma(\sum_{i=1}^K PV_{\sigma,i})-\sigma]\Phi_{\epsilon,\mathbf{P}}+O(\sigma) \\ &\le\|\sum_{i=1}^K h_\sigma(V_{\sigma,i})-h_\sigma(\sum_{i=1}^K PV_{\sigma,i})\|_{L^2}\|\Phi_{\epsilon,\mathbf{P}}\|_{L^2(\Omega_\epsilon)}+O(\sigma) \\ &=O(e^{-2\sqrt{p_\sigma}\frac{1}{\epsilon}\varphi(\mathbf{P})})+O(\sigma) \end{aligned} \tag{5.19}$$

as in the proof of Lemma 5.3.

Estimate (5.18) now follows from Lemmas 4.3 and 5.4.

□

Finally, we show that $\Phi_{\epsilon,\mathbf{P}}$ is actually smooth in $\mathbf{P}$.

Lemma 5.6. *Let $\Phi_{\epsilon,\mathbf{P}}$ be defined by Lemma 5.4. Then $\Phi_{\epsilon,\mathbf{P}}\in C^1$ in $\mathbf{P}$.*

Proof. Recall that $\Phi_{\epsilon,\mathbf{P}}$ is a solution of the equation

$$\pi_{\epsilon,\mathbf{P}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0 \tag{5.20}$$

such that

$$\Phi_{\epsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}. \tag{5.21}$$

By definition we easily conclude that the functions $PV_{\sigma,i}$, τ_σ, $\frac{\partial^2 PV_{\sigma,i}}{\partial P_{i,j}\partial P_{i,k}}$ and $\partial\tau_\sigma/\partial P_{i,j}$ are C^1 in $\mathbf{P}$. This implies that the projection $\pi_{\epsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}$. Applying $\partial/\partial P_{i,j}$ to (5.20) gives

$$\begin{aligned}\pi_{\epsilon,\mathbf{P}} \circ DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \left(\sum_{i=1}^{K} \frac{\partial PV_{\sigma,i}}{\partial P_{i,j}} + \frac{\partial \tau_\sigma}{\partial P_{i,j}} + \frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right) \\ + \frac{\partial \pi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0.\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = \Delta - p_0 + h_0'(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \\ - \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} h_0'(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}).. \end{aligned}$$

We decompose $\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}$ into two parts:

$$\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}} = \left(\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1 + \left(\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_2$$

where $\left(\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1 \in \mathcal{K}_{\epsilon,\mathbf{P}}$ and $\left(\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_2 \in \mathcal{K}^{\perp}_{\epsilon,\mathbf{P}}$. We can easily show that $\left(\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1$ is continuous in $\mathbf{P}$ since

$$\int_{\Omega_\epsilon} \Phi_{\epsilon,\mathbf{P}} \left(\frac{\partial PV_{\sigma,k}}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}}\right) = 0, \quad k = 1, ..., K, \quad l = 1, ..., N$$

and

$$\begin{aligned}\int_{\Omega_\epsilon} \frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}} \left(\frac{\partial PV_k}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}}\right) + \int_{\Omega_\epsilon} \Phi_{\epsilon,\mathbf{P}} \left(\frac{\partial^2 PV_k}{\partial P_{i,j}\partial P_{k,l}} + \frac{\partial^2 \tau_\sigma}{\partial P_{i,j}\partial P_{k,l}}\right) = 0, \\ k, i = 1, ..., K, \quad l, j = 1, ..., N.\end{aligned}$$

We can write equation (5.22) as

$$\begin{aligned}\pi_{\epsilon,\mathbf{P}} \circ DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \left((\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}})_2\right) \\ + \pi_{\epsilon,\mathbf{P}} \circ DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) \left(\sum_{i=1}^{K} \frac{\partial PV_{\sigma,i}}{\partial P_{i,j}} + (\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}})_1 + \frac{\partial \tau_\sigma}{\partial P_{i,j}}\right)\end{aligned}$$

$$+\frac{\partial \pi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}} \circ S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = 0. \tag{5.23}$$

As in the proof of Propositions 5.1 and 5.2, we can show that the operator

$$\pi_{\epsilon,\mathbf{P}} \circ DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) : \, X \cap \mathcal{K}^\perp_{\epsilon,\mathbf{P}} \to Y \cap \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$$

is invertible. Then we can take the inverse of $\pi_{\epsilon,\mathbf{P}} \circ DS_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})$ in the above equation and the inverse is continuous in $\mathbf{P}$.

Since $\frac{\partial PV_i}{\partial P_{i,j}}, \frac{\partial \tau_{\sigma,i}}{\partial P_{i,j}}, (\frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}})_1 \in \mathcal{K}_{\epsilon,\mathbf{P}}$ are continuous in $\mathbf{P}$ and so is $\frac{\partial \pi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}}$, we conclude that $(\partial \Phi_{\epsilon,\mathbf{P}}/(\partial P_{i,j}))_2$ is also continuous in $\mathbf{P}$. This is the same as the C^1 dependence of $\Phi_{\epsilon,\mathbf{P}}$ in $\mathbf{P}$. The proof is finished. □

6. The reduced problem: A Maximizing Procedure

In this section, we study a maximizing problem.

Fix $\mathbf{P} \in \overline{\Lambda}$. Let $\Phi_{\epsilon,\mathbf{P}}$ be the solution given by Lemma 5.4. We define a new functional

$$M_\epsilon(\mathbf{P}) = J_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) : \overline{\Lambda} \to R. \tag{6.1}$$

We shall prove

Proposition 6.1. *For ϵ small, the following maximizing problem*

$$\max\{M_\epsilon(\mathbf{P}) : \mathbf{P} \in \overline{\Lambda}\} \tag{6.2}$$

has a solution $\mathbf{P}^\epsilon \in \Lambda$.

Proof. Since $J_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})$ is continuous in $\mathbf{P}$, the maximizing problem has a solution. Let $M_\epsilon(\mathbf{P}^\epsilon)$ be the maximum where $\mathbf{P}^\epsilon \in \overline{\Lambda}$.

We claim that $\mathbf{P}^\epsilon \in \Lambda$.

In fact for any $\mathbf{P} \in \overline{\Lambda}$, by Lemma 5.5 we have

$$M_\epsilon(\mathbf{P}) = \epsilon^N[KI(V_0) - \frac{1}{2}(\gamma + o(1))(\sum_{i=1}^K e^{-\frac{1}{\epsilon}\Psi_\epsilon(P_i)}) - (\gamma + o(1)) \sum_{k \neq l} V_0(\frac{|P_k - P_l|}{\epsilon}) + O(\sigma)]$$

where $O(\sigma)$ is a term which does not depend on $\mathbf{P}$.

Since $M_\epsilon(\mathbf{P}^\epsilon)$ is the maximum, we have

$$\frac{1}{2}\sum_{i=1}^K e^{-\frac{1}{\epsilon}\psi_\epsilon(P_i^\epsilon)} + \sum_{k \neq l} V_0(\frac{|P_k^\epsilon - P_l^\epsilon|}{\epsilon}) \le \frac{1}{2}\sum_{i=1}^K e^{-\frac{1}{\epsilon}\psi_\epsilon(P_i)} + \sum_{k \neq l} V_0(\frac{|P_k - P_l|}{\epsilon}) + o(1)$$

for any $\mathbf{P} = (P_1, ..., P_K) \in \overline{\Lambda}$. This implies that

$$\varphi(P_1^\epsilon, ..., P_K^\epsilon) \geq \max_{\mathbf{P}\in\overline{\Lambda}} \varphi(P_1, ..., P_K) - \delta$$

for any $\delta > 0$.

So $\varphi(P_1^\epsilon, ..., P_K^\epsilon) \to \max_{\mathbf{P}\in\overline{\Lambda}} \varphi(P_1, ..., P_K)$ as $\epsilon \to 0$. By condition (1.6), we conclude $\mathbf{P}^\epsilon \in \Lambda$. This completes the proof of Proposition 6.1.

7. Proofs of Theorem 1.1 and Corollary 1.2

In this section section, we apply results of Section 3 and Section 4 to prove Theorem 1.1 and Corollary 1.2.

Proofs of Theorem 1.1 and Corollary 1.2. By Lemma 5.4 and Lemma 5.6, there exists ϵ_0 such that for $\epsilon < \epsilon_0$ we have a C^1 map which, to any $\mathbf{P} \in \overline{\Lambda}$, associates $\Phi_{\epsilon,\mathbf{P}} \in \mathcal{K}_{\epsilon,\mathbf{P}}^\perp$ such that

$$S_\epsilon(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}) = \sum_{k=1,...,K;l=1,...,N} \alpha_{kl} \left(\frac{\partial PV_{\sigma,k}}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}} \right) \tag{7.1}$$

for some constants $\alpha_{kl} \in R^{K(N-1)}$.

By Proposition 6.1, we have $\mathbf{P}^\epsilon \in \Lambda$, achieving the maximum of the maximization problem in Proposition 6.1. Let $\Phi_\epsilon = \Phi_{\epsilon,\mathbf{P}^\epsilon}$ and $u_\epsilon = w_{\epsilon,\mathbf{P}^\epsilon} + \Phi_{\epsilon,\mathbf{P}^\epsilon}$. Then we have

$$\frac{\partial}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon} M_\epsilon(\mathbf{P}^\epsilon) = 0, i = 1, ..., K, j = 1, ..., N.$$

Hence we have

$$\int_{\Omega_\epsilon} [\nabla u_\epsilon \nabla \frac{\partial(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon} + p_0 u_\epsilon \frac{\partial(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$-h(u_\epsilon)\frac{\partial(w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon}] = 0.$$

Since

$$\frac{\partial PV_{\sigma,i_1}}{\partial P_{i_2,j}} = 0 \quad \text{for} \quad i_1 \neq i_2$$

we get

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \nabla \frac{\partial(PV_{\sigma,i} + \tau_\sigma + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon}$$
$$+p_0 u_\epsilon \frac{\partial(PV_{\sigma,i} + \tau_\sigma + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon} - h(u_\epsilon)\frac{\partial(PV_{\sigma,i} + \tau_\sigma + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}}|_{\mathbf{P}=\mathbf{P}^\epsilon} = 0$$

for $i = 1, ..., K$ and $j = 1, ..., N$. Because of

$$w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}} \in X$$

we have

$$\int_{\Omega_\epsilon} [w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}] = 0.$$

Differentiating both sides, we get

$$\int_{\Omega_\epsilon} \frac{\partial (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}} = 0.$$

This implies that

$$\int_{\Omega_\epsilon} S_\epsilon(u_\epsilon) \frac{\partial (w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}} = 0.$$

Therefore we have

$$\sum_{k=1,...,K; l=1,...,N} \alpha_{kl} \int_{\Omega_\epsilon} \left(\frac{\partial PV_{\sigma,k}}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}} \right) \frac{\partial (PV_{\sigma,i} + \tau_\sigma + \Phi_{\epsilon,\mathbf{P}})}{\partial P_{i,j}} = 0. \tag{7.2}$$

Since $\Phi_{\epsilon,\mathbf{P}} \in \mathcal{K}^\perp_{\epsilon,\mathbf{P}}$, we have that

$$\begin{aligned} \left| \int_{\Omega_\epsilon} \left(\frac{\partial PV_{\sigma,k}}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}} \right) \frac{\partial \Phi_{\epsilon,\mathbf{P}}}{\partial P_{i,j}} \right| &= \left| - \int_{\Omega_\epsilon} \left(\frac{\partial^2 PV_{\sigma,i}}{\partial P_{k,l} \partial P_{i,j}} + \frac{\partial^2 \tau_\sigma}{\partial P_{k,l} \partial P_{i,j}} \right) \Phi_{\epsilon,\mathbf{P}} \right| \\ &\le \| \left(\frac{\partial^2 PV_{\sigma,i}}{\partial P_{k,l} \partial P_{i,j}} + \frac{\partial^2 \tau_\sigma}{\partial P_{k,l} \partial P_{i,j}} \right) \|_{L^2} \| \Phi_{\epsilon,\mathbf{P}} \|_{L^2} \\ &= O(\sigma + e^{-\sqrt{p_0} \frac{1}{\epsilon} \phi(\mathbf{P})}). \end{aligned}$$

Note that

$$\int_{\Omega_\epsilon} \left(\frac{\partial PV_{\sigma,k}}{\partial P_{k,l}} + \frac{\partial \tau_\sigma}{\partial P_{k,l}} \right) \left(\frac{\partial PV_{\sigma,i}}{\partial P_{i,j}} + \frac{\partial \tau_\sigma}{\partial P_{i,j}} \right) = \frac{1}{\epsilon^2} \delta_{ik} \delta_{lj} (A + o(1))$$

where

$$A = \int_{R^N} (\frac{\partial V}{\partial y_1})^2 > 0.$$

Thus (7.2) becomes a system of homogeneous equations for α_{kl} and the matrix of the system is nonsingular since it is diagonally dominant. So $\alpha_{kl} \equiv 0, k = 1, ..., K, l = 1, ...N$.

Hence $u_\epsilon = w_{\epsilon,\mathbf{P}} + \Phi_{\epsilon,\mathbf{P}}$ is a solution of (1.2).

By our construction, it is easy to see that $\epsilon^{-N} J_\epsilon(u_\epsilon) \to KI(V)$ and u_ϵ has only K local maximum points $Q_1^\epsilon, ..., Q_K^\epsilon$ and $Q_i^\epsilon \in \Lambda$. By the structure of u_ϵ we see that (up to a permutation) $Q_i^\epsilon - P_i^\epsilon = o(1)$. This proves Theorem 1.1.

□

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