

Algorithms for Worst Case Identification in \mathcal{H}_∞ and in the ν -gap metric

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Abstract

This paper considers two robustly convergent algorithms for identification of a linear system from its (possibly) noisy frequency response data. Both algorithms are based on the same principle; obtaining a good worst case fit to the data under a smoothness constraint on the obtained model. However they differ in their notions of distance and smoothness. The first algorithm yields an FIR model of a stable system from and is optimal, in a certain sense for a finite model order. The second algorithm may be used for modelling unstable plants and yields a real rational approximation in the \mathcal{L}_2 -gap. Given a model and a controller stabilising the true plant, a procedure for winding number correction is also suggested.

Key words: Robust identification; identification for control, ν -gap metric

1 Introduction

Worst case identification has attracted a lot of attention since its definitive formulation in [7]. These identification algorithms use an assumption that the true, unknown plant belongs to a subset Ψ of the set of linear systems and that the additive measurement noise belongs to a bounded set Ξ . Suitable choices of Ψ and Ξ enable us to derive bounds on the worst case identification error. The identification algorithm is said to be untuned if it is independent of the definitions of Ψ and Ξ ; and is said to be tuned otherwise. Examples of untuned algorithms may be found in [7], [6] and [8]; while tuned algorithms are investigated in [1], [4], [5]. [10] and [9] provide a review of a variety of worst case \mathcal{H}_∞ identification techniques. [8] also gives some results related to identification in the gap metric and related metrics.

The approach presented here differs from most conventional algorithms in two important aspects:

- Provided the frequency response of the model transfer

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function is sufficiently smooth and under reasonable (qualitative) assumptions that the frequency response of the true plant is also smooth and the measurement noise does not have a periodic component, the worst case fit between the noisy frequency response samples and the frequency response of the model over a sufficiently dense grid of measurement frequencies is a good indication of the ‘true’ distance between the plant and the model. The algorithms presented here offer an explicit trade-off between a quantity related to smoothness of the frequency response of the model and the worst case fit achieved over the given frequency response samples. Two different algorithms presented differ in their notions of the worst case distance and smoothness.

- In most conventional untuned algorithms, the model order is a function of length of data. This restriction is removed in the work presented here, obtaining robustly convergent algorithms one of which is optimal, in a certain sense, for a finite model order.

Motivation for this work comes from the analysis of the optimal achievable worst case error by Zames et al in [14] and from a subsequent tuned FIR approximation scheme suggested by Glaum, Lin and Zames [4]. One of the two algorithms suggested here is a modification of a similar method discussed in [8]. This modification yields the property of optimality under finite model order as mentioned above.



The rest of the paper is organised as follows. Section 2 defines the notation used in the subsequent work. An untuned algorithm for identification in \mathcal{H}_∞ is proposed in section 3. In section 4, a related algorithm for in the ν -gap metric is described. Both the algorithms are illustrated using simulation examples in section 5.

2 Notation

Let \mathbb{R} and \mathbb{C} denote the sets of real and complex numbers respectively. \mathbb{C}^n denotes the space of $n \times 1$ complex vectors. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\partial\mathbb{D}$ denote the boundary of \mathbb{D} . \mathcal{R} represents the space of all real rational transfer functions. \mathcal{L}_∞ denotes the normed space of all functions essentially bounded on \mathbb{D} and having norm $\|f\|_{\mathcal{L}_\infty} := \text{ess sup}_\omega \bar{\sigma}(f(e^{j\omega}))$, where $\bar{\sigma}(\cdot)$ represents the maximum singular value. \mathcal{H}_∞ denotes the normed space of functions analytic in \mathbb{D} and having norm $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)| < \infty$. \mathcal{RH}_∞ represents the subspace of real rational transfer functions in \mathcal{H}_∞ . For $P \in \mathcal{RH}_\infty$, $P^*(z) := P^T(z^{-1})$. A subset of \mathcal{H}_∞ of special interest here is

$$\mathcal{P}_1(\alpha) = \{f : f \in \mathcal{H}_\infty, \|f'\|_\infty \leq \alpha\} \text{ where } f'(z) := \frac{df}{dz}.$$

l_∞ denotes the space of bounded sequences. A ball in l_∞ is defined by $\overline{Bl}_\infty(\epsilon) := \{v : v \in l_\infty, \|v\|_\infty \leq \epsilon\}$.

Kolmogorov n -width of a subset $\mathcal{P} \subset \mathcal{H}_\infty$ in \mathcal{H}_∞ is defined by [11]

$$d_n(\mathcal{P}) = \inf_{X_n} \sup_{x \in \mathcal{P}} \inf_{y \in X_n} \|x - y\|_\infty$$

where the left-most infimum is taken over all n -dimensional subspaces X_n of \mathcal{H}_∞ . Thus $d_n(\mathcal{P})$ represents the optimal worst case error achievable by approximating a system in \mathcal{P} by a linear combination of n basis functions in \mathcal{H}_∞ . A subspace which achieves this error for $\mathcal{P}_1(\alpha)$ is

$$S_n = \text{span}\{1, z, \dots, z^{n-1}\} \quad (1)$$

where *span* is over real field. Further, the exact value of n -width in this case is [11] $d_n(\mathcal{P}_1(\alpha)) = \frac{\alpha}{n}$. In representing LMIs, we use Q^T and Q^* respectively to denote transpose and complex conjugate transpose of a matrix Q . $0_{m \times n}$ and I_n respectively represent $m \times n$ zero matrix and $I_{n \times n}$ identity matrix. $\text{diag}(a_i)$ represents a complex block diagonal matrix with block matrices a_i along the diagonal.

3 Identification in \mathcal{H}_∞

3.1 Problem Formulation

A priori Information : The plant transfer function $P(z)$ to be identified is a single input single output

(SISO) system which belongs to $\mathcal{P}_1(\alpha)$. The noise v corrupting measurement belongs to $\overline{Bl}_\infty(\epsilon)$. Here, $\epsilon \in [0, \infty)$, $\alpha \in [0, \infty)$ are known constants.

A posteriori Information: A vector of (not necessarily uniformly spaced) noisy frequency response samples

$$P_\omega = [P_{\omega_1} P_{\omega_2} \dots P_{\omega_m}]^T, P_{\omega_i} = P(e^{j\omega_i}) + v_i \quad (2)$$

where $\omega_i \in (0, \pi)$, $i = 1, 2, \dots, m$. Also given is a corresponding vector of (angular) frequencies

$$W = [\omega_1 \ \omega_2 \ \dots \ \omega_m]^T. \quad (3)$$

$\bar{\delta} = \max_i |\omega_{i+1} - \omega_i|$ is the maximum separation between adjacent angular frequencies. The frequencies $\omega_1 < \omega_2 < \dots < \omega_m$ are distributed such that

$$\bar{\delta} \geq \max(\omega_1, \pi - \omega_m) \quad (4)$$

holds. Define

$$[0, \pi]^m := \{x : x \in \mathbb{R}^m, x_i \in [0, \pi] \forall i \in [1, m]\} \quad (5)$$

and let S_n be as defined in (1).

Find : An algorithm $A_{\bar{\delta}} : \mathbb{C}^m \times [0, \pi]^m \mapsto S_n$ such that the worst case error defined by

$$e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) = \sup_{P \in \mathcal{P}_1(\alpha), v \in \overline{Bl}_\infty(\epsilon)} \|A_{\bar{\delta}}(P_\omega, W) - P\|_\infty \quad (6)$$

converges as follows

$$\begin{aligned} \lim_{\bar{\delta} \rightarrow 0} e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) &\leq d_n(\mathcal{P}_1) + \gamma\epsilon, \\ \lim_{n \rightarrow \infty} e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) &\leq g(\alpha, \bar{\delta}, \epsilon). \end{aligned} \quad (7)$$

For a given α , g is a function monotonically increasing in both $\bar{\delta}$ and ϵ such that $\lim_{\epsilon \rightarrow 0} \lim_{\bar{\delta} \rightarrow 0} g(\alpha, \bar{\delta}, \epsilon) = 0$. γ is a constant independent of both data and *a priori* information. In addition, derive explicit bounds on the above error.

From (7), as $\epsilon \rightarrow 0$ and $\bar{\delta} \rightarrow 0$, $e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) \rightarrow d_n(\mathcal{P}_1)$. This is the optimality property alluded to earlier.

3.2 Identification Algorithm

$A_{\bar{\delta}}(P_\omega, W) \in S_n$ is a solution of the following optimisation problem:

minimise λ subject to $f \in S_n$ and

$$\max \left\{ \max_{i \in [1, m]} |f(e^{j\omega_i}) - P_{\omega_i}|, \tilde{k}_1 \|f'\|_\infty \right\} < \lambda \quad (8)$$



where $\tilde{k}_1 = k_1 (\bar{\delta})^r$, $0 < k_1 < \infty$ and $0 < r < 1$ are constants. For a finite data, user only needs to specify a single constant \tilde{k}_1 ; the decomposition of \tilde{k}_1 into a constant and $\bar{\delta}$ -dependent part will be used later in section 3.3 to prove robust convergence.

Recalling the definition of S_n in (1), $A_{\bar{\delta}}(P_\omega, W)$ must be of the form $A_{\bar{\delta}}(P_\omega, W) = \sum_{k=0}^{n-1} a_k z^k$, $a_k \in \mathbb{R}$. The steps to implement (8) are outlined below.

- Define

$$\begin{aligned} A &= \begin{bmatrix} 0_{1 \times n-3} & 0 \\ I_{n-3} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0_{n-3 \times 1} \end{bmatrix}, \\ C &= \begin{bmatrix} 2a_2 & 3a_3 & \dots & (n-1)a_{n-1} \end{bmatrix}, \quad D = a_1, \\ \phi_i &= \begin{bmatrix} 1 & e^{j\omega_i} & \dots & e^{(n-1)j\omega_i} \end{bmatrix}, \quad i = 1, 2, \dots, m \\ \theta &= \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} \end{bmatrix}^T, \\ L_1 &= \begin{bmatrix} \lambda(A^T X A - X) & A^T X B & C^T \\ B^T X A & \lambda(B^T X B - 1) & D \\ C & D & -\lambda \end{bmatrix}, \\ L_2 &= \text{diag} \left(\begin{bmatrix} -\tilde{k}_1 \lambda & \theta^T \phi_i^* - P_{\omega_i}^* \\ \phi_i \theta - P_{\omega_i} & -\tilde{k}_1 \lambda \end{bmatrix} \right). \end{aligned}$$

- Solve

$$\inf_{a_0, a_1, \dots, a_{n-1}} \lambda \text{ subject to } X > 0, L_i < 0, i = 1, 2.$$

- Let \hat{a}_k , $k = 0, 1, \dots, n-1$ be the values of decision variables which minimise λ . Then $A_{\bar{\delta}}(P_\omega, W) = \sum_{k=0}^{n-1} \hat{a}_k z^k$.

The next result shows that the procedure outlined above actually amounts to solving (8).

Theorem 1 Let $\hat{P}(z) = \sum_{k=0}^{n-1} \hat{a}_k z^k$, with \hat{a}_k obtained as above. Then

$$\begin{aligned} & \max \left\{ \max_i | \hat{P}(e^{j\omega_i}) - P_{\omega_i} |, \tilde{k}_1 \| \hat{P}' \|_\infty \right\} \\ &= \inf_{f \in S_n} \max \left\{ \max_i | f(e^{j\omega_i}) - P_{\omega_i} |, \tilde{k}_1 \| f' \|_\infty \right\} \quad (9) \end{aligned}$$

Proof : Let $f(z) = \sum_{k=0}^{n-1} a_k z^k$. From [3] and from definition of L_1 , the conditions $L_1 < 0$ and $X > 0$ correspond to $\|f'\|_\infty < \lambda$. Also, $\max_i | \tilde{k}_1^{-1} | f(e^{j\omega_i}) - P_{\omega_i} | < \lambda$ is enforced by $L_2 < 0$. ■

Remark 2 In [8], the authors pose the same problem as (8) with $r = 1$. Here, it is shown that choosing $r < 1$ yields a much stronger convergence property. Further, an analytic, LMI based solution for this problem is provided which may be easily generalised to multivariable case.

3.3 Convergence and a priori Error Bounds

Let $\omega^* \in [0, \pi)$. For convenience of notation, denote $\hat{P}(z) = (A_{\bar{\delta}}(P_\omega, W))(z)$. Using triangle inequality,

$$| \hat{P}(e^{j\omega^*}) - P(e^{j\omega^*}) | \leq \| \hat{P}' \|_\infty \frac{\bar{\delta}}{2} + \lambda + \epsilon + \| P' \|_\infty \frac{\bar{\delta}}{2} \quad (10)$$

where $\lambda = |P_{\omega_i} - \hat{P}(e^{j\omega_i})|$. Alongwith (10), the following result of Glaum et al ([4, lemma 3.1]) is central to the derivation of error bound:

Lemma 3 ([4]) For a priori information $v \in \overline{Bl}_\infty(\epsilon)$, $P \in \mathcal{P}_1$ and a posteriori information as in section 3.1, there exists a $\hat{P}(z) \in S_n$ which satisfies the following constraints:

$$\begin{aligned} | \hat{P}(e^{j\omega_i}) - P_{\omega_i} | &\leq \epsilon + \frac{\alpha}{n} + \frac{\alpha\delta}{4} \\ \| \hat{P}' \|_\infty &< \alpha + \frac{2\alpha}{n\delta} \end{aligned}$$

where $\delta > 0$ is arbitrary.

Theorem 4

$$\begin{aligned} \lim_{n \rightarrow \infty} e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) &\leq g(\alpha, \bar{\delta}, \epsilon) \\ \lim_{\bar{\delta} \rightarrow 0} e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) &\leq 2\epsilon + \frac{\alpha}{n} \quad (11) \end{aligned}$$

where $e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta})$ is as defined in (6) and $g(\cdot)$ is such that $\lim_{\bar{\delta} \rightarrow 0} \lim_{\epsilon \rightarrow 0} g(\alpha, \bar{\delta}, \epsilon) = 0$.

Proof: Let $\tilde{k}_1 = k_1 (\bar{\delta})^r$ where $0 < k_1 < \infty$ and $0 < r < 1$ are constants. From lemma 3,

$$\begin{aligned} & \min_{f \in S_n} \max \left\{ \max_{i \in [1, m]} | f(e^{j\omega_i}) - P_{\omega_i} |, k_1 \| f' \|_\infty (\bar{\delta})^r \right\} \\ &< \max \left(\epsilon + \frac{\alpha}{n} + \frac{\alpha\delta}{4}, k_1 \left\{ \alpha + \frac{2\alpha}{n\delta} \right\} (\bar{\delta})^r \right) \quad (12) \end{aligned}$$

for an arbitrary $\delta > 0$. Choose $\delta = \left(\frac{\bar{\delta}}{n}\right)^\zeta$ for arbitrary ζ , $0 < \zeta < r$ and let

$$\lambda_{opt} := \max \left(\epsilon + \frac{\alpha}{n} + \frac{\alpha}{4} \left(\frac{\bar{\delta}}{n}\right)^\zeta, k_1 \left\{ \alpha + \frac{2\alpha(\bar{\delta})^{-\zeta}}{n^{1-\zeta}} \right\} (\bar{\delta})^r \right).$$

Then it is easy to show that

$$e(A_{\bar{\delta}} : \alpha, \epsilon, n, \bar{\delta}) \leq \left(\frac{1}{k_1} \frac{(\bar{\delta})^{1-r}}{2} + 1 \right) \lambda_{opt} + \alpha \frac{\bar{\delta}}{2} + \epsilon$$

from which the result follows. ■



3.4 Choice of \tilde{k}_1

To choose constant \tilde{k}_1 in (8), the following procedure may be adopted:

- (1) Solve
- $$\begin{aligned} & \text{minimise } \lambda \text{ subject to } f \in S_n \text{ and} \\ & \max_{i \in [1, m]} |f(e^{j\omega_i}) - P_{\omega_i}| < \lambda. \end{aligned} \quad (13)$$

Let $\hat{\lambda}_1$ be the minimum cost and let $\hat{f}_1 \in S_n$ be the solution which achieves this cost.

- (2) If the slope of Nyquist plot of \hat{f}_1 is deemed too high by visual inspection, set $\tilde{k}_1 = \frac{\zeta \hat{\lambda}_1}{\|\hat{f}_1'\|_\infty}$ for some $\zeta > 1$ and solve (8). Let \hat{f}_2 be the solution and $\hat{\lambda}_2$ be the achieved cost. Then it is easy to see that $\max_{i \in [1, m]} |\hat{f}_2(e^{j\omega_i}) - P_{\omega_i}| < \zeta \hat{\lambda}_1$ and $\|\hat{f}_2'\|_\infty < \|\hat{f}_1'\|_\infty \cdot \zeta$ thus specifies the trade-off between the deterioration in the worst case fit and the reduction in worst case slope of the model.

4 Identification in the ν -gap metric

4.1 The ν -gap metric

Consider a plant $P_i = N_i M_i^{-1}$ with $\{N_i, M_i\}$ right co-prime and $G_i = \begin{bmatrix} N_i^T & M_i^T \end{bmatrix}^T$ inner. G_i is called the normalised right graph symbol of P_i . The ν -gap between two plants P_0 and P_1 is defined as [12]

$$\begin{aligned} \delta_\nu(P_0, P_1) &= \inf_{Q, Q^{-1} \in \mathcal{L}_\infty} \|G_0 - G_1 Q\|_\infty \text{ if } I(P_0, P_1) = 0 \\ &= 1 \text{ otherwise} \end{aligned} \quad (14)$$

where $I(P_0, P_1) := \text{wno det}(G_1^* G_0)$ and $\text{wno}(g)$ denotes the winding number of $g(z)$ evaluated on the standard Nyquist contour indented around any poles on $\partial\mathbb{D}$. When the winding number condition is satisfied, $\delta_\nu(P_0, P_1)$ equals the \mathcal{L}_2 -gap, defined by

$$\delta_{\mathcal{L}_2}(P_0, P_1) := \sup_\omega \kappa(P_0, P_1)(e^{j\omega}) \quad (15)$$

where $\kappa(P_0, P_1)(e^{j\omega})$ is the pointwise *chordal* distance,

$$\begin{aligned} \kappa(P_0, P_1)(e^{j\omega}) &:= \inf_{Q \in \mathbb{C}} \bar{\sigma}(G_0 - G_1 Q)(e^{j\omega}) \\ &= \bar{\sigma}\left((I + P_1 P_1^*)^{-\frac{1}{2}}(P_0 - P_1)(I + P_0^* P_0)^{-\frac{1}{2}}\right)(e^{j\omega}). \end{aligned}$$

Given a nominal controller C that stabilises a (possibly frequency weighted) plant P_0 , a useful closed loop performance measure is $b(P_0, C) = \|H(P_0, C)\|_\infty^{-1}$ where the closed loop transfer function $H(P_0, C)$ is defined by

$$H(P_0, C) = \begin{bmatrix} P_0 \\ I \end{bmatrix} (I - C P_0)^{-1} \begin{bmatrix} -C & I \end{bmatrix}.$$

$\delta_\nu(P_0, P_1)$ is a measure of difference in closed loop performance (as expressed by $b(P_i, C)$ of two feedback loops $H(P_0, C)$ and $H(P_1, C)$). It is known that [12] any controller C stabilising P_0 and achieving $b(P_0, C) > \alpha$ stabilises the plant set $\{P_1 : \delta_\nu(P_0, P_1) \leq \alpha\}$. Further, given P_0, P_1 and C such that $b(P_0, C) > \delta_\nu(P_0, P_1)$, the following properties hold [12]:

$$\begin{aligned} \kappa(P_0, P_1)(e^{j\omega}) &\leq \bar{\sigma}(H(P_0, C) - H(P_1, C))(e^{j\omega}) \\ &\leq \kappa(P_0, P_1)(e^{j\omega}) \bar{\sigma}(H(P_0, C))(e^{j\omega}) \bar{\sigma}(H(P_1, C))(e^{j\omega}), \\ b(P_1, C) &\geq b(P_0, C) - \delta_\nu(P_0, P_1). \end{aligned} \quad (16)$$

From (16), it follows that any controller C that stabilises a model P_1 with a good $b(P_1, C)$ also stabilises the true plant P_0 , *without any significant deterioration in performance*, provided $\delta_\nu(P_0, P_1)$ is small. In the subsequent sections, an untuned algorithm is presented which attempts to achieve a small ν -gap between the plant and the model.

4.2 Problem Formulation

A priori Information: The true plant to be identified belongs to the plant set defined by

$$\mathcal{P}_2(\beta) = \{P : G_P \in \mathcal{C}^1, \|G'_P\|_\infty \leq \beta\} \quad (17)$$

where $G'_P = \frac{dG_P}{dz}$ and \mathcal{C}^1 represents the space of functions with a continuous derivative on $\partial\mathbb{D}$. The noise corrupting measurement belongs to $\overline{Bl}_\infty(\epsilon)$. Here, $\epsilon \in [0, \infty)$, $\beta \in [0, \infty)$ are known constants.

A posteriori Information: A SISO plant is considered for notational simplicity. A vector of (not necessarily uniformly spaced) noisy frequency response samples of the normalised right graph symbol of the true plant $P_0(z)$ is given:

$$G_\omega := \begin{bmatrix} G_{\omega_1}^T & G_{\omega_2}^T & \dots & G_{\omega_m}^T \end{bmatrix}^T, \quad G_{\omega_i} = G_0(e^{j\omega_i}) + \begin{bmatrix} v_{1,i} \\ v_{2,i} \end{bmatrix} \quad (18)$$

where $\omega_i \in (0, \pi)$, $i = 1, 2, \dots, m$ are such that (4) holds and $v_1, v_2 \in \overline{Bl}_\infty(\epsilon)$. Also given is a frequency vector as in (3).

An affinely parameterised model structure is used for identification which in SISO case is defined by

$$S_{n,2} = \{f : f = [f_1 \ f_2]^T, f_1 \in S_n, f_2 \in S_n\}. \quad (19)$$

For a constant matrix (resp. transfer function matrix) $X = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$ with x_2 a square and invertible (resp.



invertible in \mathcal{R}), let $\text{Quot}(X) := x_1 x_2^{-1}$. For the information and model set above and for an algorithm $A_{\bar{s}} : \mathbb{C}^{2m} \times [0, \pi)^m \mapsto S_{n,2}$, define the worst case error by

$$e(A_{\bar{s}} : \beta, \epsilon, n, \bar{\delta}) = \sup_{\substack{v_1 \in \bar{B}l_{\infty}(\epsilon) \\ v_2 \in \bar{B}l_{\infty}(\epsilon)}} \sup_{P_0 \in \mathcal{P}_2(\beta)} \delta_{\mathcal{L}_2}(P_0, \text{Quot}(A_{\bar{s}}(G_{\omega}, W))) \quad (20)$$

where $\bar{\delta} = \max_i |\omega_{i+1} - \omega_i|$, as before. Finally, define an optimisation problem

$$\min_{\substack{f \in S_{n,2} \\ Q_i \in \mathbb{C}, |Q_i| \leq h}} \max \left(\max_i \bar{\sigma}((G_{\omega_i} - f)(e^{j\omega_i})Q_i), \lambda_f \right) \quad (21)$$

where $\lambda_f = \tilde{k}_2 \|f'\|_{\infty}$, $\tilde{k}_2 = k_2 (\bar{\delta})^r$, $0 < k_2 < \infty$, $0 < r < 1$ and $1 < h < \infty$ are constants.

Theorem 5 *Let $A_{\bar{s}}(G_{\omega}, W)$ and $\{\hat{Q}_i \in \mathbb{C}, i = 1, \dots, m\}$ be such that, $A_{\bar{s}}(G_{\omega}, W) = \text{Quot}(f_{\bar{s}})$, $f_{\bar{s}} \in S_{n,2}$ and*

$$\max \left(\max_i \bar{\sigma} \left(G_{\omega_i} - f_{\bar{s}}(e^{j\omega_i})\hat{Q}_i \right), \tilde{k}_2 \|f'_{\bar{s}}\|_{\infty} \right) = \min_{\substack{f \in S_{n,2} \\ Q_i \in \mathbb{C}, |Q_i| \leq h}} \max \left(\max_i \bar{\sigma} \left(G_{\omega_i} - f(e^{j\omega_i})Q_i \right), \tilde{k}_2 \|f'\|_{\infty} \right).$$

Here, $\tilde{k}_2 = k_2 (\bar{\delta})^r$, $0 < k_2 < \infty$ and $0 < r < 1$ and $h > 1$ are real constants.

$$\begin{aligned} \text{Then, } \lim_{\epsilon \rightarrow 0} \lim_{\bar{\delta} \rightarrow 0} e(A_{\bar{s}} : \beta, \epsilon, n, \bar{\delta}) &\leq h_1(\beta, n, \epsilon) \\ \text{and } \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} e(A_{\bar{s}} : \beta, \epsilon, n, \bar{\delta}) &\leq h_2(\beta, \bar{\delta}, \epsilon). \end{aligned} \quad (22)$$

Here $h_1(\cdot), h_2(\cdot)$ satisfy $\lim_{\bar{\delta} \rightarrow 0} \lim_{\epsilon \rightarrow 0} h_2(\beta, \bar{\delta}, \epsilon) = 0$, $\lim_{n \rightarrow \infty} \lim_{\epsilon \rightarrow 0} h_1(\beta, n, \epsilon) = 0$.

Proof: Let $P_{\bar{s}} = A_{\bar{s}}(G_{\omega}, W)$. At any point $\omega^* \in [0, \pi)$, it may be shown that the following bound holds:

$$\begin{aligned} \kappa(P_0, P_{\bar{s}})(e^{j\omega^*}) &\leq \frac{\beta \bar{\delta}}{2} + \sqrt{2}\epsilon + \bar{\sigma}(G_{\omega_i} - f_{\bar{s}}(e^{j\omega_i})\hat{Q}_i) \\ &\quad + \|f'_{\bar{s}}\|_{\infty} \frac{h\bar{\delta}}{2}. \end{aligned} \quad (23)$$

The proof from here onwards in along the same lines as the proof of theorem 4 and is omitted. ■

Some remarks on this result are in order.

- First, note the similarity of results in theorems 5 and 4. Both the algorithms presented offer a trade-off between worst case fit and worst case complexity and yield a worst case error which converges independently with respect to model order and the number of data points. Both algorithms also use a similar, LMI-based numerical implementation.

- In a realistic situation, P_{ω} as defined in (2) will be known. A procedure to re-cast (21) in terms of P_{ω} is described in the next section.
- A good approximation in \mathcal{L}_2 -gap is not necessarily a good approximation in ν -gap, since the winding number condition may not be satisfied. Given a controller stabilising the true plant, a winding number correction can be carried out, as shown in section 4.6.

Here, the problem of identification in ν -gap metric is solved in three steps:

- (1) Re-write the problem (21) in terms of P_{ω} .
- (2) Solve

$$\min_{\substack{f \in S_{n,2} \\ |Q_i| \leq h}} \max \left\{ \max_i \bar{\sigma}(G_{\omega_i} - f)(e^{j\omega_i})Q_i, \tilde{k}_2 \|f'\|_{\infty} \right\} \quad (24)$$

where \tilde{k}_2 and $h > 1$ are user chosen constants. Let \hat{f} solve this problem and let $P_1 = \text{Quot}(\hat{f})$.

- (3) Given P_1 and a controller stabilising P_0 , solve

$$\min_{P \in \mathcal{R}, I(P, P_0)=0} \delta_{\mathcal{L}_2}(P_1, P)$$

Let P_2 be the solution. Then by triangle inequality,

$$\begin{aligned} \max_i \kappa(P_{\omega_i}, P_2(e^{j\omega_i})) &\leq \max_i \kappa(P_{\omega_i}, P_1(e^{j\omega_i})) \\ &\quad + \delta_{\mathcal{L}_2}(P_1, P_2). \end{aligned} \quad (25)$$

These three steps are discussed in detail in sections 4.3, 4.4 and 4.6 respectively.

4.3 Reformulating the Problem in Terms of Plant Frequency Response

Let $\mathcal{R}^{m \times n}$ denote real rational transfer functions with m outputs and n inputs. Let $P_{\omega_i} = \text{Quot}(G_{\omega_i})$. Then P_{ω_i} represents the available experimental frequency response data for the true plant P_0 at the measurement frequency ω_i . The following result allows us to re-phrase the problem (21) in terms of P_{ω_i} :

Lemma 6 *Given $P_1 \in \mathcal{R}^{m \times n}$, $P_{\omega_i} \in \mathbb{C}^{m \times n}$, $\exists \hat{Q} \in \mathbb{C}^{n \times n}$, $F_i \in \mathbb{C}^{(m+n) \times n}$ such that*

$$\kappa(P_{\omega_i}, P_1(e^{j\omega_i})) = \bar{\sigma} \left(F_i - G_1(e^{j\omega_i})\hat{Q} \right)$$

and at any ω_i , the matrix F_i can be written as a function of the point frequency response matrix P_{ω_i} .

Proof :

$$\text{Let } F_i = \left[(P_{\omega_i}(I + P_{\omega_i}^* P_{\omega_i})^{-\frac{1}{2}})^T ((I + P_{\omega_i}^* P_{\omega_i})^{-\frac{1}{2}})^T \right]^T. \quad (26)$$



Then it may be shown that $\kappa(P_{\omega_i}, P_1(e^{j\omega_i})) = \bar{\sigma}(F_i - G_1(G_1^* F_i))$. Details are omitted. ■

For a stable SISO plant P_0 , $F_i = \left[\frac{P_{\omega_i}}{\sqrt{1+|P_{\omega_i}|^2}} \frac{1}{\sqrt{1+|P_{\omega_i}|^2}} \right]^T$.

Given P_{ω_i} as defined in (2) and F_i as defined above, $\kappa(P_{\omega_i}, P_0(e^{j\omega_i})) \leq \bar{\sigma}(F_i - G_0(e^{j\omega_i})) \leq \sqrt{2}$. It is not clear how to relate this bound to assumptions on measurement noise. From a practical point of view, a proper measure of noise in the present context will be

$$\kappa(P_{\omega_i}, P_0(e^{j\omega_i})) \leq \frac{\bar{\sigma}(P_{\omega_i} - P_0(e^{j\omega_i}))}{\bar{\sigma}(I + P_{\omega_i}^* P_{\omega_i})^{\frac{1}{2}}} \leq \epsilon'. \text{ If solving (24)}$$

yields $\hat{\lambda}$ as a local optimum and if $\hat{f} \in S_{n,2}$ achieves this cost, then $\kappa(P_0(e^{j\omega_i}), \text{Quot}(\hat{f})(e^{j\omega_i})) \leq \hat{\lambda} + \epsilon'$.

4.4 Approximation in the \mathcal{L}_2 -gap

An iterative procedure for approximation in the \mathcal{L}_2 -gap is given below. SISO case is discussed for ease of notation.

Given : A vector of frequency response samples P_{ω} defined as in (2).

Initialisation : Set $k = 1$, $\hat{Q}_{0,i} = 1 \forall i \in [1, m]$.

Step A : Solve

$$\min_{f_k \in S_{n,2}} \left(\max_i \bar{\sigma}(F_i - f_k(e^{j\omega_i}) \hat{Q}_{k-1,i}), \tilde{k}_2 \|f_k'\|_{\infty} \right) \quad (27)$$

where \tilde{k}_2 is a user chosen constant, F_i is obtained from P_{ω_i} using (26) and $f_k' = \frac{df_k}{dz}$. Both the constraints $\bar{\sigma}(F_i - f_k(e^{j\omega_i}) \hat{Q}_{k-1,i}) \leq \lambda$ and $\tilde{k}_2 \|f_k'\|_{\infty} < \lambda$ may be written as affine constraints in the parameters of f_k , in a manner similar to the constraints in section 3.2. Thus, (27) may be written as an LMI optimisation problem. Let \hat{f}_k be the solution to (27).

Step B : Solve LMI optimisation

$$\min_{\substack{Q_{k,i} \in \mathbb{C} \\ |Q_{k,i}| \leq h}} \max_{i \in [1, m]} \bar{\sigma}(F_i - \hat{f}_k(e^{j\omega_i}) Q_{k,i}) \quad (28)$$

where $h > 1$ is a user chosen constant. Let $\hat{Q}_{k,i}, i = 1, \dots, m$ be the solution to (28). If $\max_i \bar{\sigma}(F_i - \hat{f}_k(e^{j\omega_i}) \hat{Q}_{k,i})$ is less than a specified tolerance, stop; otherwise set $k := k + 1$ and go to step A.

Define $v_{k,l} := \max_i \bar{\sigma}(F_i - \hat{f}_k(e^{j\omega_i}) \hat{Q}_{l,i})$. Thus $v_{k,k-1}$ and $v_{k,k}$ give the achieved cost at steps A and B respectively at k^{th} iteration. The achieved cost in the above procedure is non-increasing with each iteration.

Lemma 7 For $k \geq 1$, $\max(v_{k+1,k}, v_{k,k}) \leq v_{k,k-1}$.

Proof : Omitted.

4.5 Numerical Implementation

During initial stages of iteration, it is better to solve simpler versions of (27)-(28):

$$\min_{f_k \in S_{n,2}} \left(\max_i \bar{\sigma}(F_i - f_k(e^{j\omega_i}) \hat{Q}_{k-1,i}) \right) \quad (29)$$

$$\min_{Q_{k,i} \in \mathbb{C}} \max_i \bar{\sigma}(F_i - \hat{f}_k(e^{j\omega_i}) Q_{k,i}) \quad (30)$$

The first step here avoids the constraint on $\|f'\|_{\infty}$ and the second step has a closed-form solution

$$\hat{Q}_{k,i} = \frac{\hat{f}_k^*(e^{j\omega_i}) F_i}{\hat{f}_k^*(e^{j\omega_i}) \hat{f}_k(e^{j\omega_i})}. \quad (31)$$

This choice speeds up iterations considerably. As the iterations approach convergence, (29) may be replaced by (27), with \tilde{k}_2 chosen using $\|\hat{f}_k'\|_{\infty}$, using a procedure similar to one in section 3.4. The condition $|\hat{Q}_{k,i}| \leq h$ is necessary to prove worst case convergence (see (23)), but in practice the use of (28) in place of (30) is rarely required.

4.6 Satisfying the Winding Number Constraint

Let $\eta(H(P, C))$ represent the number of closed loop unstable poles of $H(P, C)$ and let $b_{\mathcal{L}_2}(P, C) = \|H(P, C)\|_{\mathcal{L}_{\infty}}^{-1}$, regardless of stability of $H(P, C)$. $b_{\mathcal{L}_2}(P, C) = 0$ if $H(P, C)$ has a pole on $\partial\mathbb{D}$. Let $\hat{f}_k = [f_{1,k} \ f_{2,k}]^T$ be the solution to (27) at k^{th} iteration, and let $P_1 = f_{1,k} f_{2,k}^{-1}$. Then the above procedure ensures that $\max_i \kappa(P_1(e^{j\omega_i}), P_{\omega_i}) < v_{k,k}$, but it doesn't guarantee that $I(P_1, P_0) = 0$. Suppose, a controller C which stabilises the true plant P_0 results in $\eta(H(P_1, C)) = k$ and $b_{\mathcal{L}_2}(P_1, C) = \alpha$. From [13], if another model P_2 satisfies $I(P_2, P_1) \leq -k$ and $\delta_{\mathcal{L}_2}(P_1, P_2) < \alpha$, then $\eta(H(P_2, C)) = 0$. A method for finding P_2 which guarantees $I(P_2, P_1) \leq -k$ is outlined in the next theorem.

Theorem 8 Given P_1 and $\gamma < 1$, the following are equivalent.

$$(1) \exists P_2 : I(P_2, P_1) \leq -k \text{ and } \delta_{\mathcal{L}_2}(P_1, P_2) < \gamma$$

$$(2) \gamma > \sqrt{1 - \sigma_k^2(\mathbf{H}_{G_1})}$$

where $\sigma_k(\mathbf{H}_G)$ is the k^{th} largest singular value of Hankel operator \mathbf{H}_G .

When the condition 2 is satisfied, such a P_2 may be constructed as $P_2 = \{ \text{Quot}(X) \}^*$ where $X = [x_1 \ x_2]^T$ satisfies $x_2^{-1} \in \mathcal{R}$, $X \in \mathcal{H}_{\infty(-), n-k}$ and $\|G_3 - X\|_{\infty} < \gamma$. Here G_3 is the normalised right graph symbol for $P_3 := P_1^*$, $\deg(P_1) = n$ and $\mathcal{H}_{\infty(-), n-k}$ denotes the space of functions analytic outside the unit disk except for at most $n - k$ stable poles.



Proof : Omitted for brevity; see [2].

As illustrated in the proof of this result [2], one can construct a P_2 which solves

$$\inf_{P_2 \in \mathcal{R}} \delta_{\mathcal{L}_2}(P_1, P_2). \quad (32)$$

$$I(P_2, P_1) \leq -k$$

Then $\max_i \kappa(P_{\omega_i}, P_2(e^{j\omega_i}))$ may be bounded as in (25). From ([13], chapter 3), it may be shown that $I(P_2, P_0) = 0$ provided $b_{\mathcal{L}_2}(P_2, C) > \delta_{\mathcal{L}_2}(P_0, P_2)$ and $b_{\mathcal{L}_2}(P_1, C) > \delta_{\mathcal{L}_2}(P_1, P_2)$. If both P_0 and P_1 are stable (*i.e.* stabilised by $C = 0$), $I(P_1, P_0) = 0$ provided $b_{\mathcal{L}_2}(P_1, 0) > \delta_{\mathcal{L}_2}(P_1, P_0)$. This may be approximately verified and the above procedure is unnecessary in this case. If the plant is unstable, the stabilising controller used to collect any operational data may also be used to build model P_2 .

It is instructive at this stage to compare briefly the two algorithms presented so far. The main advantages of the procedure in section 3.2 is its numerical and conceptual simplicity and the fact that it can incorporate *a priori* information in terms of known poles. On the other hand, the algorithm presented in sections 4.3-4.6 may be used to model unstable plants, even those with poles on the imaginary axis.

5 Simulation Examples

To illustrate the algorithm for identification in \mathcal{H}_∞ , consider a plant

$$P_0(s) = \frac{5(s+0.1)}{s^2+3s+2}.$$

Frequency response samples at 30 frequencies, logarithmically spaced between 0.1 rad/s and 10 rad/s are used for estimation. These frequencies are mapped to $\partial\mathbb{D}$ by Tustin transformation, with sampling period 0.28s. Gaussian distributed complex noise is added to the frequency response, with variance approximately 2% of $|P_0(e^{j\omega_i})|$ at each ω_i . This data yields $\epsilon = 0.2634$ and $\bar{\delta} = 0.158$. An FIR model of order 16 is identified by minimising λ subject to (13). This yields a worst case error of 0.0866 and the worst case slope of the obtained model is 3143. The problem (8) is then solved with $\zeta = 4$. The worst case error in this case is 0.0923 and the worst case slope is 837.1. In fig. 1, the solid line shows the noisy frequency response of the true plant, the dotted line shows the response of the first approximate FIR model and the dash-dot line shows the response of model with smoothing constraint. The slope of Bode plot in the later case is somewhat smoother at frequencies beyond 4 rad/s.

As an example for the algorithm for identification in the ν -gap metric, consider an unstable plant

$$P_0(s) = \frac{2(s-1)}{s(s^2+0.4s+1)}.$$

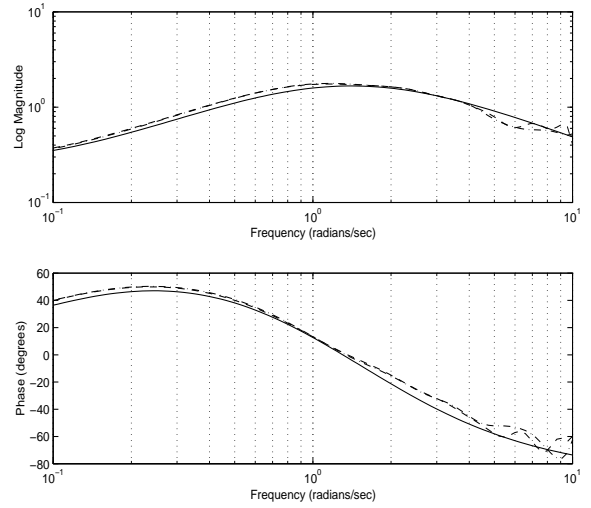


Fig. 1. Bode Plots: Frequency response samples and FIR approximation

Frequency response samples of this plant at 40 frequencies, logarithmically spaced between 0.1 rad/s and 30 rad/s are used for estimation. These frequencies are mapped to $\partial\mathbb{D}$ by Tustin transformation with sampling period 0.105s. Gaussian distributed complex noise is added to the frequency response, with variance approximately 2% of $|P_0(e^{j\omega_i})|$ at each ω_i . This data yields $\max_i \kappa(P_{\omega_i}, P_0(e^{j\omega_i})) = 0.0749$ and $\bar{\delta} = 0.2$. An initial model P_n is obtained using *fitsys* command in MATLAB which yields $\max_i \kappa(P_{\omega_i}, P_n(e^{j\omega_i})) = 0.7755$. Coprime factors of P_n are used to generate an initial $Q_{0,i}$. 50 iterations of (29)- (30) yield a model P_1 with $\max_i \kappa(P_{\omega_i}, P_1(j\omega_i)) = 0.045$. P_1 is de-stabilised by a controller C_0 which stabilises the true plant and has $b(P_0, C_0) = 0.1029$. Using the procedure described in section 4.6, another model P_2 stabilised by C_0 is obtained. Finally, $\max_i \kappa(P_{\omega_i}, P_2(j\omega_i)) = 0.066$. P_2 obtained is given by

$$P_2(s) = 0.0475 + \frac{-0.0863s^2 + 2.3101s - 1.6950}{s^3 + 0.6263s^2 + 0.9987s + 0.1009}$$

and $\delta_\nu(P_0, P_2) = 0.06$. In fig. (2), the solid line shows the noisy frequency response of the true plant. The dashed line indicates the response of the model.

6 Conclusion

Two robustly convergent algorithms for identification of linear systems from noisy frequency response data are suggested. In both the algorithms, the central idea is to provide an explicit trade-off between the complexity of the model (in terms of the worst case slope of the model or its coprime factors) and the quality of approximation (in terms of the worst case fit to frequency response data). The first algorithm yields an FIR model of a stable system and is optimal, in a certain sense for a fi-



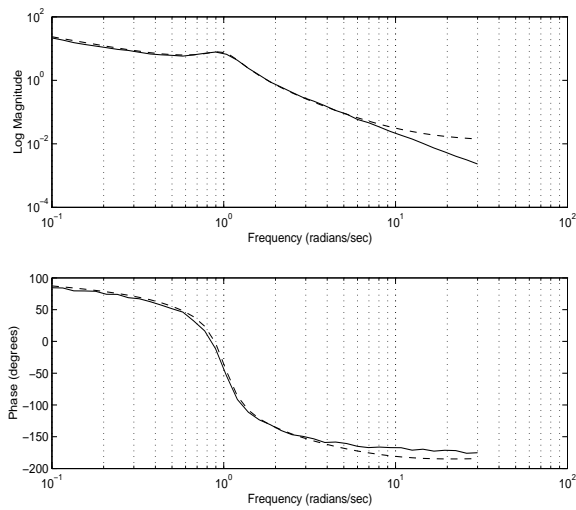


Fig. 2. Bode Plots: Frequency response samples and ν -gap approximation

nite model order. The second algorithm may be used for modelling unstable plants and yields a real rational approximation in the \mathcal{L}_2 -gap. Both algorithms are based on LMI optimisation. Given a model and a controller stabilising the true plant, a procedure for winding number correction is also suggested.

References

- [1] J. Chen, C.N. Nett, and M.K.H. Fan. Worst case system identification in \mathcal{H}_∞ : Validation of a priori information, essentially optimal algorithms, and error bounds. *IEEE Trans. Automat. Contr.*, 40:1260–1265, 1995.
- [2] P. Date and G. Vinnicombe. An algorithm for identification in the ν -gap metric. Proc. of 38th Conf. on Dec. and Contr., 1999.
- [3] P. Gahinet and P. Apkarian. A linear matrix inequality approach to \mathcal{H}_∞ control. *Int. Journal of Robust and Nonlin. Contr.*, 4:421–448, 1994.
- [4] M. Glaum, L. Lin, and G. Zames. Optimal \mathcal{H}_∞ approximation by systems of prescribed order using frequency response data. In *Proc. 35th IEEE Conf. Des. Contr.*, pages 2318–2321, 1996.
- [5] G. Gu. Modelling of normalised coprime factors with ν -metric uncertainty. *IEEE Trans. Automat. Contr.*, 44:1498–1511, 1999.
- [6] G. Gu and P.P. Khargonekar. A class of algorithms for identification in \mathcal{H}_∞ . *Automatica*, 28:299–312, 1992.
- [7] A.J. Helmicki, C.A. Jacobson, and C.N. Nett. Control oriented system identification: A worst case/deterministic approach in \mathcal{H}_∞ . *IEEE Trans. Automat. Contr.*, 36:1163–1176, 1991.
- [8] P. M. Mäkilä and J. R. Partington. Robust identification of strongly stabilizable systems. *IEEE Trans. Automat. Contr.*, 37:1709–1716, 1992.
- [9] P.M. Mäkilä, J.R. Partington, and T.K. Gustafsson. Worst-case control-relevant identification. *Automatica*, 31:1799–1819, 1995.
- [10] J.R. Partington. *Interpolation, Identification and Sampling*. Oxford Univ. Press, 1997.
- [11] A. Pinkus. *n-widths in Approximation Theory*. Springer-Verlag, 1985.
- [12] G. Vinnicombe. Frequency domain uncertainty and the graph topology. *IEEE Trans. Automat. Contr.*, 38:1371–1383, 1993.
- [13] G. Vinnicombe. *Uncertainty and Feedback: \mathcal{H}_∞ loop-shaping and the ν -gap metric*. Imperial College Press, 2000.
- [14] G. Zames, L. Lin, and L.Y. Wang. Fast identification n -widths and uncertainty principles for LTI and slowly varying systems. *IEEE Trans. Automat. Contr.*, 39:1827–1838, 1994.

