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The Chiral Supereigenvalue Model

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ABSTRACT

A supereigenvalue model with purely positive bosonic eigenvalues is presented
and solved by considering its superloop equations. This model represents the
supersymmetric generalization of the complex one matrix model, in analogy to
the relation between the supereigenvalue and the hermitian one matrix model.
Closed expressions for all planar multi-superloop correlation functions are found.
Moreover an iterative scheme allows the calculation of higher genus contributions
to the free energy and to the correlators. Explicit results for genus one are given.

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1 Introduction

The term supersymmetric matrix model by now is used for a number of zero and one dimensional large N supersymmetric theories. $SU(N)$ Super-Yang-Mills theories in the large N limit reduced to one time dimension are relevant for the description of supermembranes [1], and recently this model has been proposed as a non-perturbative formulation of M-theory [2]. The zero dimensional version of this model has been proposed to be connected to IIB strings [3]. Scalar supersymmetric matrix models in zero dimensions have been applied to the investigation of branched polymers and the meander problem [4] and to $c = -2$ conformal field theories coupled to 2d gravity [5], for a review see [6].

However, following the successful application of random matrix models to 2d quantum gravity and lower dimensional bosonic strings [7], there still is no supersymmetric *matrix* model at hand which achieves a description of discretized super–Riemann surfaces. Nevertheless, on the level of eigenvalue models a disrete approach to 2d supergravity was established through the so-called supereigenvalue model [8], representing a supersymmetric generalization of the hermitian one matrix model. The investigation of its double scaling limit [9] revealed the relevance of the model for $N = 1$ super–Liouville theory [10]. Moreover in refs. [11] a complete iterative solution of the model was presented by one of the authors.

As the complex matrix model [12] is just as well suited for the description of 2d quantum gravity as the hermitian one, we wish to ask the question whether a supersymmetric generalization of the complex matrix model exists. The complex model is in some respect more simple than the hermitian model as it enjoys the following feature. In their pioneering work [13] Ambjørn, Jurkiewicz and Makeenko have been able to give a closed universal expression for all planar correlation functions. Such an expression has been found only for the complex model with a simple one-arc support of the spectral density, given by a single parameter. For the hermitian model [13, 14] or for a more complicated support [15, 16] the multi–loop correlators remain universal, but have to be calculated successively. Furthermore a complete iterative solution for higher genera of the complex matrix model has been given in [17], where explicit results have been presented up to and including genus three.

The present work generalizes the complex matrix model to a model we have called chiral supereigenvalue model, as it contains only the positive half of the bosonic real eigenvalues. In section 2 the model is defined and its superloop equations are given. These equations are solved in the planar limit in section 3. Moreover closed expressions for all planar multi–superloop correlation functions are derived here. The full iterative scheme for higher genera is presented in section 4 where explicit results are given for genus one.

2 Superloop Equations

The chiral supereigenvalue model is built out of a set of N Grassmann odd and even variables[2] θ_i and λ_i. Its partition function reads

$$\mathcal{Z} = (\prod_{i=1}^{N} \int_0^\infty d\lambda_i \int d\theta_i) \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) \exp\left(-N \sum_{i=1}^{N} [V(\lambda_i) - \theta_i \Psi(\lambda_i)]\right) \quad (2.1)$$

where

$$V(\lambda_i) = \sum_{k=0}^{\infty} g_k {\lambda_i}^k \quad \text{and} \quad \Psi(\lambda_i) = \sum_{k=0}^{\infty} \xi_{k+1/2} {\lambda_i}^k, \quad (2.2)$$

the g_k and $\xi_{k+1/2}$ being Grassmann even and odd coupling constants, respectively. Note that the bosonic integral over the λ_i runs over $\mathbb{R}_+$. One shows that this model obeys a set of super–Virasoro constraints

$$\mathcal{L}_n \, \mathcal{Z} = 0 \,, \quad \mathcal{G}_{n+1/2} \, \mathcal{Z} = 0 \,, \quad n \geq 0 \,, \quad (2.3)$$

where the operators $\mathcal{L}_n$ and $\mathcal{G}_{n+1/2}$ are given in [8]. The supereigenvalue model [8] obeys these constraints starting already at $n \geq -1$. Hence the partition function of the supereigenvalue model also solves the chiral constraints (2.3) but not vice versa. A similar relation holds between the loop equations of the hermitian and complex matrix model.

We introduce the one–superloop correlators

$$\widehat{W}(p \,|\,) = N \left\langle \sum_i \frac{p\,\theta_i}{p^2 - \lambda_i} \right\rangle \quad \text{and} \quad \widehat{W}(\,|\,p) = N \left\langle \sum_i \frac{p}{p^2 - \lambda_i} \right\rangle. \quad (2.4)$$

They may be obtained from the partition function $\mathcal{Z}$ by application of the superloop insertion operators $\delta/\delta V(p)$ and $\delta/\delta \Psi(p)$:

$$\widehat{W}(p_1, \ldots, p_n \,|\, q_1, \ldots, q_m) = \frac{1}{\mathcal{Z}} \frac{\delta}{\delta \Psi(p_1)} \cdots \frac{\delta}{\delta \Psi(p_n)} \frac{\delta}{\delta V(q_1)} \cdots \frac{\delta}{\delta V(q_m)} \, \mathcal{Z}, \quad (2.5)$$

where

$$\frac{\delta}{\delta V(p)} = -\sum_{k=0}^{\infty} \frac{1}{p^{2k+1}} \frac{\partial}{\partial g_k} \quad \text{and} \quad \frac{\delta}{\delta \Psi(p)} = -\sum_{k=0}^{\infty} \frac{1}{p^{2k+1}} \frac{\partial}{\partial \xi_{k+1/2}}. \quad (2.6)$$

However, it is convenient to work with the connected part of the $(n|m)$–superloop correlators, denoted by W. They may be obtained in the following way from the free energy $F = N^{-2} \ln \mathcal{Z}$:

$$W(p_1, \ldots, p_n \,|\, q_1, \ldots, q_m) = \frac{\delta}{\delta \Psi(p_1)} \cdots \frac{\delta}{\delta \Psi(p_n)} \frac{\delta}{\delta V(q_1)} \cdots \frac{\delta}{\delta V(q_m)} \, F \quad (2.7)$$

$$= N^{n+m-2} \left\langle \sum_{i_1} \frac{p_1 \theta_{i_1}}{p_1^2 - \lambda_{i_1}} \cdots \sum_{i_n} \frac{p_n \theta_{i_n}}{p_n^2 - \lambda_{i_n}} \sum_{j_1} \frac{q_1}{q_1^2 - \lambda_{j_1}} \cdots \sum_{j_m} \frac{q_m}{q_m^2 - \lambda_{j_m}} \right\rangle_C.$$

[2]N is even.

Note that correlation functions with $n \geq 2$ vanish due to the structure of F discussed below. With the normalizations chosen above, one assumes that these correlators enjoy the genus expansion

$$W(p_1, \ldots, p_n \mid q_1, \ldots, q_m) = \sum_{g=0}^{\infty} = \frac{1}{N^{2g}} W_g(p_1, \ldots, p_n \mid q_1, \ldots, q_m). \qquad (2.8)$$

Similarly one has the genus expansion

$$F = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} F_g \qquad (2.9)$$

for the free energy.

2.1 Superloop Equations

The superloop equations of our model are two Schwinger–Dyson equations. For the Grassmann–odd superloop equation we perform the shift

$$\lambda_i \to \lambda_i + \theta_i \frac{\epsilon \, \lambda_i}{p^2 - \lambda_i} \quad \text{and} \quad \theta_i \to \theta_i + \frac{\epsilon \, \lambda_i}{p^2 - \lambda_i} \qquad (2.10)$$

where ϵ is an odd constant. One then finds the odd equation

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{V}'(\omega) W(\omega \mid) + 2\,\omega^2 \, \bar{\Psi}(\omega) \, W(\mid \omega)}{p^2 - \omega^2} \;=\; W(p \mid) \, W(\mid p) + \frac{1}{N^2} W(p \mid p) \qquad (2.11)$$

and its counterpart, the Grassmann–even superloop equation, takes the form

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{V}'(\omega) \, W(\mid \omega) + \omega \, \bar{\Psi}'(\omega) \, W(\omega \mid)}{p^2 - \omega^2} - \frac{p}{2} \frac{d}{dp} \oint_C \frac{d\omega}{4\pi i} \frac{\bar{\Psi}(\omega) \, W(\omega \mid)}{p^2 - \omega^2} =$$

$$\frac{1}{2} \left[W(\mid p)^2 + \frac{1}{2p} W(p \mid) \, W'(p \mid) + \frac{1}{2\,N^2} \left(W(\mid p, p) - \frac{1}{2p} \frac{d}{dq} W(q, p \mid) \Big|_{p=q} \right) \right]. \qquad (2.12)$$

It is obtained through the shift

$$\lambda_i \to \lambda_i + \frac{\epsilon \, \lambda_i}{p^2 - \lambda_i} \quad \text{and} \quad \theta_i \to \theta_i + \frac{1}{2} \frac{\epsilon \, p^2 \, \theta_i}{(p^2 - \lambda_i)^2} \qquad (2.13)$$

with ϵ even and infinitesimal. Moreover we have introduced the potentials

$$\begin{aligned} \bar{V}'(\omega) &= 2\,\omega \, V'(\omega^2) \\ \bar{\Psi}'(\omega) &= 2\,\omega \, \Psi'(\omega^2) \\ \bar{\Psi}(\omega) &= \Psi(\omega^2). \end{aligned} \qquad (2.14)$$

In the derivation we have assumed that the loop correlators have one–cut structure, i.e. in the limit $N \to \infty$ we assume that the eigenvalues are contained in a finite interval $[-\sqrt{c}, \sqrt{c}]$. Moreover C is a curve around the cut.

The key to the solution of these complicated equations order by order in N^{-2} is the observation that the free energy F depends at most quadratically on fermionic coupling constants. This was proven in ref. [18] for the supereigenvalue model, but an inspection of the proof reveals that the same holds true for the chiral supereigenvalue model as well. Via eq. (2.7) this directly translates to the one-loop correlators, which we from now on write as

$$W(p \,|\,) = v(p) \tag{2.15}$$
$$W(\,|\, p) = u(p) + \widehat{u}(p). \tag{2.16}$$

Here $v(p)$ is of order one in fermionic couplings, whereas $u(p)$ is taken to be of order zero and $\widehat{u}(p)$ of order two in the fermionic coupling constants $\xi_{k+1/2}$. This observation allows us to split up the two superloop equations (2.11) and (2.12) into a set of four equations, sorted by their order in the $\xi_{k+1/2}$'s. Doing this we obtain

Order 0:

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{V}'(\omega)}{p^2 - \omega^2} u(\omega) = \frac{1}{2} u(p)^2 + \frac{1}{2 N^2} \frac{\delta}{\delta V(p)} u(p) + \frac{1}{4 \, p} \frac{1}{N^2} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v(q) \Big|_{p=q} \tag{2.17}$$

Order 1:

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{V}'(\omega)}{p^2 - \omega^2} v(\omega) + \oint_C \frac{d\omega}{4\pi i} \frac{2 \, \omega^2 \, \bar{\Psi}(\omega)}{p^2 - \omega^2} u(\omega) = v(p) \, u(p) + \frac{1}{N^2} \frac{\delta}{\delta V(p)} v(p) \tag{2.18}$$

Order 2:

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{V}'(\omega)}{p^2 - \omega^2} = \widehat{u}(\omega) + \oint_C \frac{d\omega}{4\pi i} \frac{\omega \, \bar{\Psi}'(\omega)}{p^2 - \omega^2} v(\omega) - \frac{p}{2} \frac{d}{dp} \oint_C \frac{d\omega}{4\pi i} \frac{\bar{\Psi}(\omega)}{p^2 - \omega^2} v(\omega)$$

$$+ u(p) \, \widehat{u}(p) + \frac{1}{4p} v(p) \frac{d}{dp} v(p) + \frac{1}{2 \, N^2} \frac{\delta}{\delta V(p)} \widehat{u}(p) \tag{2.19}$$

Order 3:

$$\oint_C \frac{d\omega}{4\pi i} \frac{2 \, \omega^2 \bar{\Psi}(\omega)}{p^2 - \omega^2} \widehat{u}(\omega) = v(p) \, \widehat{u}(p). \tag{2.20}$$

Plugging the genus expansions into these equations lets them decouple partially, in the sense that the equation of order 0 at genus g only involves u_g and lower genera contributions. The order 1 equation then only contains v_g, u_g and lower genera results and so on. The first thing to do, however, is to find the solution for $g = 0$.

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3 The Planar Solution

3.1 Solution for $u_0(p)$ and $v_0(p)$

In the limit $N \to \infty$ the order 0 equation may be solved:

$$u_0(p) = \oint_C \frac{d\omega}{4\pi i} \frac{\omega \bar{V}'(\omega)}{p^2 - \omega^2} \left[\frac{p^2 - c}{\omega^2 - c} \right]^{1/2}, \tag{3.1}$$

where the endpoint c of the cut on the positive real axis is determined by the requirement:

$$1 = \oint_C \frac{d\omega}{4\pi i} \frac{\omega \bar{V}'(\omega)}{\sqrt{\omega^2 - c}}, \tag{3.2}$$

deduced from our knowledge that $W(\,|\,p) = 1/p + \mathcal{O}(p^{-2})$.

The order 1 equation (2.18) in the $N \to \infty$ limit determining the odd loop correlator $v_0(p)$ may now also be solved to obtain

$$v_0(p) = 2 \oint_C \frac{d\omega}{4\pi i} \frac{\omega^2 \bar{\Psi}(\omega)}{p^2 - \omega^2} \left[\frac{\omega^2 - c}{p^2 - c} \right]^{1/2} + \frac{\chi}{\sqrt{p^2 - c}}. \tag{3.3}$$

Here χ is a constant not determined by eq. (2.18), in fact $\chi = N^{-1} \langle \sum_i \theta_i \rangle$ in the planar limit.

3.2 Moments and Basis Functions

Instead of the couplings g_k we introduce the bosonic moments M_k and I_k defined by [17][3]

$$M_k = \oint_C \frac{d\omega}{4\pi i} \frac{\omega \bar{V}'(\omega)}{\omega^{2k+2}} \frac{1}{[\omega^2 - c]^{1/2}}, \quad k \geq 0 \tag{3.4}$$

$$I_k = \oint_C \frac{d\omega}{4\pi i} \frac{\omega \bar{V}'(\omega)}{(\omega^2 - c)^k} \frac{1}{[\omega^2 - c]^{1/2}}, \quad k \geq 0, \tag{3.5}$$

and the couplings $\xi_{k+1/2}$ are replaced by the fermionic moments

$$\Lambda_k = \oint_C \frac{d\omega}{4\pi i} \frac{\omega^2 \bar{\Psi}(\omega)}{w^{2k+2}} [\omega^2 - c]^{1/2}, \quad k \geq 0$$

$$\Xi_k = \oint_C \frac{d\omega}{4\pi i} \frac{\omega^2 \bar{\Psi}(\omega)}{(\omega^2 - c)^k} [\omega^2 - c]^{1/2}, \quad k \geq 0. \tag{3.6}$$

The main motivation for introducing these new variables is that, for each term in the genus expansion of the free energy and the correlators, the dependence on

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[3]Note the different sign in c chosen here.

an infinite number of coupling constants arranges itself nicely into a function of a *finite* number of moments.

We further introduce the basis functions $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$ recursively

$$\chi^{(n)}(p) \;=\; \frac{1}{I_1} \Big(\phi^{(n)}(p) - \sum_{k=1}^{n-1} \chi^{(k)}(p)\, I_{n-k+1} \Big), \tag{3.7}$$

$$\Psi^{(n)}(p) \;=\; \frac{1}{M_0} \Big(\Omega^{(n)}(p) - \sum_{k=1}^{n-1} \Psi^{(k)}(p)\, M_{n-k} \Big), \tag{3.8}$$

where

$$\phi^{(n)}(p) \;=\; \frac{1}{(p^2 - c)^{n+1/2}} \tag{3.9}$$

$$\Omega^{(n)}(p) \;=\; \frac{1}{p^{2n}\,(p^2 - c)^{1/2}}, \tag{3.10}$$

following again [17]. It is easy to show that for the linear operator $\widehat{\overline{V}}'$ defined by

$$\widehat{\overline{V}}' \circ f(p) = \oint_C \frac{d\omega}{4\pi i}\, \frac{\omega\,\bar{V}'(\omega)}{p^2 - \omega^2}\, f(\omega) \;-\; u_0(p)\, f(p) \tag{3.11}$$

we have

$$\widehat{\overline{V}}' \circ \chi^{(n)}(p) \;=\; \frac{1}{(p^2 - c)^n}, \tag{3.12}$$

$$\widehat{\overline{V}}' \circ \Psi^{(n)}(p) \;=\; \frac{1}{p^{2n}}. \tag{3.13}$$

3.3 Solution for $\widehat{u}_0$ and χ

Next consider the order 2 equation (2.19) at genus 0. Plugging eq. (3.3) into the right hand side of this equation yields after a somewhat lengthy calculation

$$\widehat{\overline{V}'} \circ \widehat{u}_0 = \frac{\Xi_2\,(2\,\Xi_1 - \chi\,)}{p^2 - c} \tag{3.14}$$

With eqs. (3.12) and (3.13) this immediately tells us that

$$\widehat{u}_0(p) = \frac{\Xi_2\,(2\,\Xi_1 - \chi\,)}{I_1}\, \frac{1}{(p^2 - c)^{3/2}}. \tag{3.15}$$

Finally we determine the odd constant χ. One may think that this is done by employing the order 3 equation for $g = 0$, however it turns out that (2.20) is

identically fulfilled with the above results for $v_0(p)$ and $\widehat{u}_0(p)$. We instead use the consistency requirement

$$W_0(p, p \mid) = \frac{\delta}{\delta\Psi(p)} \, v_0(p) \; = \; 0 \tag{3.16}$$

due to anti–commutativity. From this one has

$$\frac{\delta}{\delta\Psi(p)} \, \chi = p - \frac{\sqrt{p^2 - c}}{2} - \frac{\sqrt{p^2 - c}\, p^2}{2 \, (p^2 - c)}, \tag{3.17}$$

from which one deduces

$$\chi = \Lambda_0 + \Xi_1, \tag{3.18}$$

by using some of the formulas in the appendix. Putting it all together, we may now write down the complete genus 0 solution for the one–superloop correlators $W(\mid p)$ and $W(p \mid)$:

$$W_0(\mid p) \;=\; \oint_C \frac{d\omega}{4\pi i} \, \frac{\omega \, \bar{V}'(\omega)}{p^2 - \omega^2} \left[\frac{p^2 - c}{\omega^2 - c} \right]^{1/2} + \frac{\Xi_2 \, (\Xi_1 - \Lambda_0)}{I_1 \, (p^2 - c)^{3/2}}$$

$$W_0(p \mid) \;=\; 2 \oint_C \frac{d\omega}{4\pi i} \, \frac{\omega^2 \, \bar{\Psi}(\omega)}{p^2 - \omega^2} \left[\frac{\omega^2 - c}{p^2 - c} \right]^{1/2} + \frac{\Xi_1 + \Lambda_0}{[p^2 - c]^{1/2}}. \tag{3.19}$$

3.4   All planar correlation functions

According to the definition (2.7) all planar correlation functions can be obtained from eq. (3.19) by taking functional derivatives w.r.t. $V(p)$ or $\Psi(p)$. In this subsection we prove that this successive application can be reduced to a closed algebraic expression for all multi-superloop correlators.

Let us begin with the highest degree in fermions, the two–fermion loop correlator, which takes the universal form

$$W_0(p, q \mid) = \frac{1}{2(p^2 - q^2)} \Big[p^2 \sqrt{\frac{q^2 - c}{p^2 - c}} \; + \; q^2 \sqrt{\frac{p^2 - c}{q^2 - c}} \; - \; 2pq \Big]. \tag{3.20}$$

¿From comparing it to the well known two–loop correlator $W_0^{\mathrm{cmm}}(p, q)$ of the complex matrix model [13],

$$W_0(p, q \mid) \;=\; 2(p^2 - q^2) W_0^{\mathrm{cmm}}(p, q) \,, \tag{3.21}$$

a closed expression can be immediately written down following [13]:

$$W_0(p, q \mid r_1, \ldots, r_n) = \Big(\frac{1}{I_1} \frac{\partial}{\partial c} \Big)^{n-1} \frac{c(p^2 - q^2)}{4 I_1 (p^2 - c)^{3/2} (q^2 - c)^{3/2}} \prod_{k=1}^{n} \frac{c}{(r_k^2 - c)^{3/2}}, \;\; n \geq 1. \tag{3.22}$$

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For the one–fermion loop correlators we first calculate

$$W_0(p \mid q) = \frac{1}{2I_1(p^2-c)^{3/2}(q^2-c)^{3/2}}\Big[(p^2-c)\,c\,\Xi_2 - p^2(\Xi_1-\Lambda_0)\Big], \quad (3.23)$$

using some formulas of the appendix. Due to the fact that the fermionic moments Ξ_k and Λ_k depend on the bosonic potential $V(p)$ only via c, we can apply the following theorem proven in [13]

$$\frac{\delta}{\delta V(p)}\left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)^n \frac{1}{I_1}h(c) = \left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)^{n+1}\frac{1}{I_1}h(c)\frac{c}{(p^2-c)^{3/2}}, \quad n \geq 0, \quad (3.24)$$

where $h(c)$ is an arbitrary function of c. This immediately leads to the closed form

$$W_0(p \mid q_1,\ldots,q_n) = \left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)^{n-1}\frac{(p^2-c)\,c\,\Xi_2 - p^2(\Xi_1-\Lambda_0)}{2cI_1(p^2-c)^{3/2}}\prod_{k=1}^{n}\frac{c}{(q_k^2-c)^{3/2}}, \quad n \geq 1. \quad (3.25)$$

In order to calculate the remaining correlators we first calculate $W_0(\mid p,q)$. The bosonic part is twice the known universal two–loop correlator of the complex matrix model whereas for the fermionic part we apply once more theorem (3.24) to eq. (3.19) and obtain

$$\begin{aligned}
W_0(\mid p,q) &= \frac{1}{2(p^2-q^2)^2}\Big[p^2\sqrt{\frac{q^2-c}{p^2-c}} + q^2\sqrt{\frac{p^2-c}{q^2-c}} - 2pq\Big] \\
&+ \left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)\frac{c\,\Xi_2(\Xi_1-\Lambda_0)}{I_1(p^2-c)^{3/2}(q^2-c)^{3/2}}.
\end{aligned} \quad (3.26)$$

The general result can be derived using the results of [13] for the bosonic part and theorem (3.24) for the fermionic part:

$$W_0(\mid p_1,\ldots,p_n) = \left[\left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)^{n-3} + \left(\frac{1}{I_1}\frac{\partial}{\partial c}\right)^{n-1}4\,\Xi_2\,(\Xi_1-\Lambda_0)\right]\frac{1}{4cI_1}\prod_{k=1}^{n}\frac{c}{(p_k^2-c)^{3/2}} \quad (3.27)$$

for $n \geq 3$.

4 The Iterative Procedure

4.1 The Iteration for u_1 and v_1

As already mentioned the structure of the superloop eqs. (2.17) and (2.18) allow for an iterative solution in the genus, similar in spirit to the situation for

the supereigenvalue model [11]. Let us demonstrate how this works for $g = 1$. According to eq. (2.17) for $u_1(p)$ we have

$$\widehat{V}' \circ u_1(p) = \frac{1}{2} \left[\frac{\delta}{\delta V(p)} u_0(p) + \frac{1}{2p} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v_0(q) \mid_{p=q} \right]. \qquad (4.1)$$

With the help of some formulas in the appendix, one shows that

$$\frac{\delta}{\delta V(p)} u_0(p) = \frac{1}{2p} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v_0(q) \mid_{p=q} = \frac{c^2}{8 (p^2 - c)^2 p^2}. \qquad (4.2)$$

And therefore

$$u_1(p) = -\frac{1}{8} \Psi^{(1)}(p) + \frac{1}{8} \chi^{(1)}(p) + \frac{c}{8} \chi^{(2)}(p). \qquad (4.3)$$

Now we solve the order 1 eq. (2.18) at $g = 1$ for $v_1(p)$

$$\widehat{V}' \circ v_1(p) = \frac{\delta}{\delta V(p)} v_0(p) - \widehat{\widehat{\Psi}} \circ u_1(p), \qquad (4.4)$$

where we have introduced the operator $\widehat{\widehat{\Psi}}$ defined by

$$\widehat{\widehat{\Psi}} \circ f(p) = \oint_C \frac{d\omega}{4\pi i} \frac{2 \omega^2 \bar{\Psi}(\omega)}{p^2 - \omega^2} f(\omega) - v_0(p) f(p). \qquad (4.5)$$

Note that generally eq. (4.4) and its higher g analogues fix $v_g(p)$ only up to a zero–mode contribution $\kappa_g \phi^{(0)}(p)$, which will be determined later on. In order to evaluate the right hand side of eq. (4.4) we need to know how the operator $\widehat{\widehat{\Psi}}$ acts on the basis functions $\phi^{(n)}(p)$ and $\Omega^{(n)}(p)$. A straightforward calculation yields

$$\widehat{\widehat{\Psi}} \circ \phi^{(n)}(p) = \sum_{r=1}^{n+1} \frac{2 \Xi_r}{(p^2 - c)^{n+2-r}} - \frac{\Xi_1 + \Lambda_0}{(p^2 - c)^{n+1}}$$

$$\widehat{\widehat{\Psi}} \circ \Omega^{(n)}(p) = \frac{\Xi_1 - \Lambda_0}{c^n} \frac{1}{p^2 - c} - \sum_{r=1}^{n} \sum_{l=1}^{r} \frac{2 \Lambda_{l-1}}{c^{n+1-r}} \frac{1}{p^{2(r+1-l)}}$$

$$\sum_{r=1}^{n} \frac{\Xi_1 + \Lambda_0}{c^{n+1-r}} \frac{1}{p^{2r}}. \qquad (4.6)$$

Moreover we need the quantity

$$\frac{\delta v_0(p)}{\delta V(p)} = W_0(p \mid p) = \frac{c (\Lambda_0 - \Xi_1)}{2 I_1 (p^2 - c)^3} + \frac{\Lambda_0 - \Xi_1 + c \Xi_2}{2 I_1 (p^2 - c)^2}, \qquad (4.7)$$

which may be obtained from eq. (3.23) by taking the limit $q \to p$.

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We now have collected all the ingredients needed to evaluate the right hand side of eq. (4.4). We find

$$
\begin{aligned}
v_1(p) \;=\; & -\frac{(\Xi_1 - \Lambda_0)}{8\,c\,M_0}\,\Psi^{(1)}(p) - \frac{5\,c(\Xi_1 - \Lambda_0)}{8\,I_1}\,\chi^{(3)}(p) \\
& -\Big[\frac{3\,(\Xi_1 - \Lambda_0)}{8\,I_1} - \frac{c\,I_2(\Xi_1 - \Lambda_0)}{8\,I_1{}^2} - \frac{c\,\Xi_2}{4\,I_1}\Big]\chi^{(2)}(p) \\
& -\Big[\frac{(\Xi_1 - \Lambda_0)}{8\,c\,M_0} - \frac{\Xi_2}{4\,I_1} - \frac{c\,I_2\,\Xi_2}{4\,I_1{}^2} + \frac{c\,\Xi_3}{4\,I_1}\Big]\chi^{(1)}(p) + \kappa_1\,\phi^{(0)}(p)
\end{aligned}
\tag{4.8}
$$

Yet the zero mode κ_1 is still undetermined.

4.2 The Computation of F_1 and κ_1

The bosonic part of the free energy of genus one F_1^{bos} can be obtained by integrating eq. (4.3):

$$
u_1(p) \;=\; \frac{\delta\,F_1^{\text{bos}}}{\delta V(p)}.
\tag{4.9}
$$

The result is twice the free energy of the complex matrix model, when comparing eq. (4.3) to [17], i.e.

$$
F_1^{\text{bos}} \;=\; -\frac{1}{12}\ln I_1 \;-\; \frac{1}{4}\ln M_0 \;-\; \frac{1}{3}\ln c \; .
\tag{4.10}
$$

In order to obtain the fermionic contribution to the free energy we rewrite the known part of $v_1(p)$ in the following form:

$$
v_1(p) - \kappa_1\,\phi^{(0)}(p) = \frac{2}{c}\,\kappa_1\,\frac{\delta\,(\Xi_1 - \Lambda_0)}{\delta\Psi(p)} \;+\; \frac{\delta\,F_1}{\delta\Psi(p)}.
\tag{4.11}
$$

Upon using the relations

$$
\begin{aligned}
\phi^{(0)}(p) \;&=\; \frac{2}{c}\,\frac{\delta\,(\Xi_1 - \Lambda_0)}{\delta\Psi(p)} \\
\phi^{(k)}(p) \;&=\; -\frac{2}{c}\,\frac{\delta}{\delta\Psi(p)}\Big[\sum_{l=2}^{k+1}\frac{\Xi_l}{(-c)^{k-l+1}} + \frac{(\Xi_1 - \Lambda_0)}{(-c)^k}\Big] \\
\Omega^{(n)}(p) \;&=\; 2\,\frac{\delta}{\delta\Psi(p)}\Big[\frac{\Lambda_r}{c^{n+1-r}} - \frac{(\Xi_1 - \Lambda_0)}{c^{n+1}}\Big],
\end{aligned}
\tag{4.12}
$$

one may integrate eq. (4.11) to obtain the fermionic piece of the free energy at genus one:

$$
\begin{aligned}
F_1^{\text{ferm}} \;=\; & \frac{\Xi_2\,\Xi_3}{2\,I_1{}^2} + (\Xi_1 - \Lambda_0)\Big[\frac{\Lambda_1}{4\,c^2\,M_0{}^2} - \Big(\frac{1}{2\,c^2\,I_1{}^2} + \frac{3\,I_2}{4\,c\,I_1{}^3}\Big) \\
& + \frac{3\,I_2{}^2}{2\,I_1{}^4} - \frac{5\,I_3}{4\,I_1{}^3} + \frac{1}{4\,c^2\,I_1\,M_0}\Big)\Xi_2 + \frac{\Xi_3}{2\,c\,I_1{}^2} + \frac{3\,I_2\,\Xi_3}{2\,I_1{}^3} - \frac{5\,\Xi_4}{4\,I_1{}^2}\Big].
\end{aligned}
\tag{4.13}
$$

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Similarly the zero–mode κ_1 follows immediately:

$$
\begin{aligned}
\kappa_1 \;=\;& (\Xi_1 - \Lambda_0)\left[-\frac{5\,I_3}{8\,I_1{}^3} - \frac{1}{8\,c^2\,M_0{}^2} + \frac{1}{8\,c^2\,I_1\,M_0} + \frac{3\,I_2{}^2}{4\,I_1{}^4}\right.\\
& \left. +\frac{3\,I_2}{8\,c\,I_1{}^3} + \frac{1}{4\,c^2\,I_1{}^2}\right] + \frac{5\,c\,\Xi_4}{8\,I_1{}^2} - \frac{\Xi_3}{2\,I_1{}^2} - \frac{3\,c\,I_2\,\Xi_3}{4\,I_1{}^3}\\
& +\Xi_2\left(\frac{3\,I_2}{8\,I_1{}^3} + \frac{3\,c\,I_2{}^2}{4\,I_1{}^4} - \frac{5\,c\,I_3}{8\,I_1{}^3} + \frac{1}{8\,c\,I_1\,M_0} + \frac{1}{4\,c\,I_1{}^2}\right)\\
& -\frac{\Lambda_1}{8\,c\,M_0{}^2}.
\end{aligned}
\tag{4.14}
$$

4.3 The Computation of $\widehat{u}_1(p)$

The remaining quantity $\widehat{u}_1(p)$ is now obtained from (4.13) by application of the $\delta/\delta V(p)$ operator. We find

$$
\widehat{u}_1(p) = \sum_{k=1}^{4} A_k^1\,\chi^{(k)}(p) \;+\; B_1^1\,\Psi^{(1)}(p),
\tag{4.15}
$$

where

$$
A_4^1 \;=\; \frac{-35\,c\,(\Xi_1 - \Lambda_0)\,\Xi_2}{8\,I_1{}^2}
$$

$$
A_3^1 \;=\; \frac{-15\,(\Xi_1 - \Lambda_0)\,\Xi_2}{8\,I_1{}^2} + \frac{25\,c\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_2}{8\,I_1{}^3} - \frac{15\,c\,(\Xi_1 - \Lambda_0)\,\Xi_3}{4\,I_1{}^2}
$$

$$
\begin{aligned}
A_2^1 \;=\;& \frac{3\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_2}{4\,I_1{}^3} - \frac{3\,c\,(\Xi_1 - \Lambda_0)\,I_2{}^2\,\Xi_2}{2\,I_1{}^4} + \frac{5\,c\,(\Xi_1 - \Lambda_0)\,I_3\,\Xi_2}{4\,I_1{}^3} - \frac{3\,(\Xi_1 - \Lambda_0)\,\Xi_2}{8\,c\,I_1\,M_0}\\
& -\frac{3\,(\Xi_1 - \Lambda_0)\,\Xi_3}{2\,I_1{}^2} + \frac{3\,c\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_3}{I_1{}^3} + \frac{3\,c\,\Xi_2\,\Xi_3}{2\,I_1{}^2} - \frac{15\,c\,(\Xi_1 - \Lambda_0)\,\Xi_4}{4\,I_1{}^2}
\end{aligned}
$$

$$
\begin{aligned}
A_1^1 \;=\;& \frac{\Xi_2\,\Lambda_1}{8\,c\,M_0{}^2} - \frac{(\Xi_1 - \Lambda_0)\,\Xi_2}{4\,c^2\,I_1{}^2} - \frac{3\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_2}{8\,c\,I_1{}^3} - \frac{3\,(\Xi_1 - \Lambda_0)\,I_2{}^2\,\Xi_2}{4\,I_1{}^4}\\
& +\frac{5\,(\Xi_1 - \Lambda_0)\,I_3\,\Xi_2}{8\,I_1{}^3} + \frac{9\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_3}{8\,I_1{}^3} - \frac{9\,c\,(\Xi_1 - \Lambda_0)\,I_2{}^2\,\Xi_3}{4\,I_1{}^4}\\
& -\frac{3\,(\Xi_1 - \Lambda_0)\,\Xi_3}{8\,c\,I_1\,M_0} + \frac{5\,\Xi_2\,\Xi_3}{4\,I_1{}^2} + \frac{3\,c\,I_2\,\Xi_2\,\Xi_3}{4\,I_1{}^3} - \frac{15\,(\Xi_1 - \Lambda_0)\,\Xi_4}{8\,I_1{}^2}\\
& +\frac{15\,c\,(\Xi_1 - \Lambda_0)\,I_2\,\Xi_4}{4\,I_1{}^3} + \frac{5\,c\,\Xi_2\,\Xi_4}{8\,I_1{}^2} - \frac{35\,c\,(\Xi_1 - \Lambda_0)\,\Xi_5}{8\,I_1{}^2} + \frac{15\,c\,(\Xi_1 - \Lambda_0)\,I_3\,\Xi_3}{8\,I_1{}^3}
\end{aligned}
$$

$$
B_1^1 \;=\; -\frac{(\Xi_1 - \Lambda_0)\,\Lambda_1}{4\,c^2\,M_0{}^2} + \frac{(\Xi_1 - \Lambda_0)\,\Xi_2}{8\,c^2\,I_1\,M_0}.
\tag{4.16}
$$

This completes our computation of the genus one contributions $W_1(p|\,)$, $W_1(\,|p)$ and F_1.

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5 Conclusions

We have constructed and completely solved a chiral supereigenvalue model away from the double scaling limit. The set of superloop equations determining the correlation functions could be solved by an iterative procedure in genus, in a similar way as for the supereigenvalue model [11]. In addition all planar multi–superloop correlators were determined explicitly, generalizing the results for the complex matrix model [13].

It may be expected, that in the double scaling limit the model presented here becomes equivalent to the double scaled supereigenvalue model [9, 11], as the same relation holds for the complex and hermitian model [14]. This open question is left for further investigation.

Appendix

Here we collect a number of important functional derivatives w.r.t. $V(p)$

$$\frac{\delta\,c}{\delta V(p)} = \frac{c}{I_1}\,\phi^{(1)}(p)$$

$$\frac{\delta\,I_k}{\delta V(p)} = (k+\frac{1}{2})\,\frac{c\,I_{k+1}}{I_1}\,\phi^{(1)}(p) - (k+\frac{1}{2})\,c\,\phi^{(k+1)}(p) - k\,\phi^{(k)}$$

$$\frac{\delta\,M_k}{\delta V(p)} = (\frac{1}{2}-k)\,\Omega^{(k+1)}(p) + \sum_{r=1}^{k}\frac{1}{2\,c^{k+1-r}}\,\Omega^{(r)}(p)$$

$$-\sum_{r=0}^{k}\frac{M_r}{2\,c^{k-r}\,I_1}\,\phi^{(1)}(p)$$

$$\frac{\delta\,\Lambda_k}{\delta V(p)} = \Big[\sum_{r=0}^{k}\frac{\Lambda_r}{2\,c^{k-r}\,I_1} - \frac{\Xi_1}{2\,c^k\,I_1}\Big]\,\phi^{(1)}(p)$$

$$\frac{\delta\,\Xi_k}{\delta V(p)} = \frac{(k-\frac{1}{2})\,c}{I_1}\,\Xi_{k+1}\,\phi^{(1)}(p) \tag{A.1}$$

and w.r.t. $\Psi(p)$ using $\frac{\delta\,\bar\Psi(q)}{\delta\Psi(p)} = \frac{q}{p^2-q^2}$:

$$\frac{\delta\,\Lambda_k}{\delta\Psi(p)} = \delta_{k,0}\,\frac{p}{2} - \frac{1}{2}\,\frac{\sqrt{p^2-c}}{p^{2k}}$$

$$\frac{\delta\,\Xi_k}{\delta\Psi(p)} = \delta_{k,1}\,\frac{p}{2} - \frac{1}{2}\,\frac{p^2\,\sqrt{p^2-c}}{(p^2-c)^k}. \tag{A.2}$$

The derivatives of the moments w.r.t. c read

$$\frac{\partial I_k}{\partial c} = (k+\frac{1}{2})I_{k+1}$$

$$\frac{\partial M_k}{\partial c} = -\sum_{r=0}^{k}\frac{M_r}{2c^{k-r+1}} + \frac{M_1}{2c^{k+1}}$$

$$\frac{\partial \Xi_k}{\partial c} = (k-\frac{1}{2})\Xi_{k+1}$$

$$\frac{\partial \Lambda_k}{\partial c} = \sum_{r=0}^{k}\frac{\Lambda_r}{2c^{k-r+1}} - \frac{\Lambda_1}{2c^{k+1}}. \tag{A.3}$$

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