

# Multicritical Microscopic Spectral Correlators of Hermitian and Complex Matrices

G. AKEMANN

Centre de Physique Théorique CNRS  
Case 907 Campus de Luminy  
F-13288 Marseille Cedex 9  
France

P. H. DAMGAARD<sup>a</sup>, U. MAGNEA<sup>b</sup>

<sup>a</sup>The Niels Bohr Institute and <sup>b</sup>NORDITA  
Blegdamsvej 17  
DK-2100 Copenhagen Ø  
Denmark

S. M. NISHIGAKI

Institute for Theoretical Physics  
University of California  
Santa Barbara, CA 93106-4030  
USA

October 24, 2006

## Abstract

We find the microscopic spectral densities and the spectral correlators associated with multicritical behavior for both hermitian and complex matrix ensembles, and show their universality. We conjecture that microscopic spectral densities of Dirac operators in certain theories without spontaneous chiral symmetry breaking may belong to these new universality classes.

CPT-97/P.3526  
NBI-HE-97-45  
NORDITA-97/73  
NSF-ITP-97-137  
hep-th/9712006

arXiv:hep-th/9712006 v2 3 Dec 1997

# 1 Introduction

It has been conjectured that the spectra of massless Dirac operators near eigenvalues  $\lambda = 0$  display universal features that can be extracted from zero-dimensional theories of random matrices [1, 2]. The natural object to focus on is the so-called microscopic spectral density

$$\rho_S(\lambda) \equiv \lim_{V \rightarrow \infty} \frac{1}{V} \rho \left( \frac{\lambda}{V} \right), \quad (1.1)$$

which displays the behavior of the spectral density near the origin, as measured on the scale of the space-time volume  $V$ .

For QCD-like theories in  $(2n + 1)$ -dimensions (gauge groups  $SU(N_c)$  for  $N_c \geq 3$ ,  $N_f$  fermions in the fundamental representation of the gauge group) the relevant matrix ensemble is that of hermitian matrices, the unitary ensemble [2]. In even space-time dimensions, where chiral symmetry can be defined, its possibly spontaneous breakdown is known to be related to a non-vanishing spectral density at the origin [3]:  $\langle \bar{\psi}\psi \rangle \sim \rho(0)$ . In odd space-time dimensions the closest analogue to chiral symmetry is parity. This raises the obvious question of whether flavor or flavor-parity symmetries can be spontaneously broken in odd dimensions, similar to the breaking of chiral flavor-symmetries in even dimensions. While there is yet no direct evidence from, *e.g.*, Monte Carlo simulations that this actually occurs for QCD-like theories (see, however, ref. [4]), one can entertain this idea, and look at some of the consequences. It has been suggested that for an even number of flavors, and three space-time dimensions, the pattern of symmetry breaking may be of the flavor-breaking kind  $U(N_f) \rightarrow U(N_f/2) \times U(N_f/2)$  [5, 2]. An order parameter for such a transition is  $|\Sigma| \equiv |\langle \bar{\psi}\psi \rangle|$ , in complete analogy with the breaking of chiral symmetries in even dimensions. By the Banks-Casher relation [3], this order parameter is proportional to the spectral density at the origin,  $\rho(0)$ . In such a scenario one can evaluate the microscopic spectral density  $\rho_S(\lambda)$  by methods very analogous to the even-dimensional case [2]. The result has been found to be [6, 2]

$$\begin{aligned} \rho_S(\lambda) = & \frac{1}{4} \pi^2 \rho(0)^2 \lambda \left\{ J_{N_f + \frac{1}{2}}(\pi \rho(0) \lambda)^2 + J_{N_f - \frac{1}{2}}(\pi \rho(0) \lambda)^2 \right. \\ & \left. - J_{N_f + \frac{1}{2}}(\pi \rho(0) \lambda) J_{N_f - \frac{3}{2}}(\pi \rho(0) \lambda) - J_{N_f - \frac{1}{2}}(\pi \rho(0) \lambda) J_{N_f + \frac{3}{2}}(\pi \rho(0) \lambda) \right\} \end{aligned} \quad (1.2)$$

for the Gaussian distribution. As was recently proven in ref. [7], this microscopic spectral density is highly universal. It follows from hermitian random matrix models with arbitrary potentials  $V(M)$  that support a large- $N$  spectral density which is non-vanishing at the origin. It was confirmed in [2] that this microscopic spectral density is consistent with the appropriate generalization of the Leutwyler-Smilga spectral sum rules [8] to this odd-dimensional situation.

The question of whether the symmetry breaking  $U(N_f) \rightarrow U(N_f/2) \times U(N_f/2)$  actually occurs in, say,  $(2+1)$ -dimensional QCD, is of course a dynamical question that is not addressed in the above considerations. To achieve such a spontaneous symmetry breaking, the eigenvalues of the Dirac operator must accumulate, as the volume  $V$  is taken to infinity, sufficiently fast near  $\lambda = 0$ . A simple scaling argument shows that the accumulation must be such that the average level spacing  $\Delta\lambda$  among the eigenvalues must be roughly constant near  $\lambda = 0$ , and inversely proportional to the volume  $V$ . This is precisely what motivates the introduction of the microscopic density  $\rho_S(\lambda)$  as in eq. (1.1). An obvious question to ask, then, is to what extent universal microscopic spectral densities can be extracted if there is *no* spontaneous symmetry breaking. In other words, suppose the dynamics of the massless theory is such that the eigenvalues of the Dirac operator do *not* accumulate sufficiently fast for a condensate

to form. Then the whole connection to the Leutwyler-Smilga sum rules (and their odd-dimensional generalizations) is lost, and the argument in favor of universal microscopic spectral densities has to be reconsidered. This question is particularly interesting in the case of (2+1)-dimensions, because there the spontaneous breaking of flavor or flavor-parity symmetries is far from obvious.

But also the even-dimensional case (corresponding to integration over complex matrices, the so-called chiral unitary ensemble, [9] for QCD-like theories) merits closer attention in this connection. For example, the universal results of refs. [1, 7] are derived for an arbitrary number of flavors  $N_f$ . The dependence on  $N_f$  is simple, occurring only in the indices of the relevant Bessel functions. This rather mild variation as a function of  $N_f$  is totally consistent with the fact that the only essential flavor dependence enters in the precise chiral symmetry breaking pattern  $SU_L(N_f) \otimes SU_R(N_f) \rightarrow SU(N_f)$ , and not in detailed dynamical questions such as quark screening etc. Surely the assumption about this chiral symmetry breaking pattern must break down as the number of flavors is increased. For QCD an almost certain upper limit is (for quarks in the fundamental representation)  $N_f = 17$ , for which the leading-order coefficient of the beta function has changed sign. The renormalized theory in this case is presumable “trivial”, and almost certainly not in a phase of chiral symmetry breaking.<sup>1</sup> However, although the renormalized coupling for infinite cut-off may be vanishing, it is perhaps not *a priori* obvious that the spectrum of the Dirac operator is identical to that of a totally free theory. So although the eigenvalues may not accumulate fast enough near  $\lambda = 0$  to produce a chiral condensate, the accumulation rate may still be very different from that of a theory with no bare interactions at all. This raises the question whether also such spectral densities display universal features that are computable from random matrix theory. As mentioned, it is even quite likely that QCD possesses a large “conformal window” in a range of flavors, where  $N_f \leq 16$  but still greater than some critical value  $N_f^*$ . In such a phase, which is not expected to support the spontaneous breaking of chiral symmetry, the spectrum of the Dirac operator will definitely not be that of a totally free theory either. Finally, although the scenario is quite different in three dimensions (because the gauge theory is superrenormalizable there, with a trivial beta function), the outcome of adding more fermionic flavors to the theory is believed to be quite similar. More flavors imply more screening of charges, and hence reduced forces between the fermions. Eventually the forces should become too small to cause chiral symmetry breaking. So even if flavor symmetries are spontaneously broken in three dimensions for a low number of fermionic flavors, one does not expect this to persist as  $N_f$  is increased.

Yet another example may be general  $SU(N_c)$  gauge theories coupled to both fermions and Higgs fields. In the Higgs phases of such theories we again do not expect chiral symmetry breaking. In the absence of Yukawa couplings the Dirac operators of such theories are identical to those without Higgs fields, and the arguments of ref. [1] would naturally lead one to associate analogous matrix models *with vanishing spectral density at the origin* to such phases. In fact, if one mentally integrates out the Higgs fields, the resulting theories are formally of the same class as the theories without Higgs fields. Their only difference is that they belong to different phases. A totally different class of gauge theories in four dimensions which do not cause spontaneous chiral symmetry breaking are those of Abelian gauge groups, such as QED. Also for these much simpler theories one can ask whether their Dirac operator spectrum near the origin contain universal functions.

All of these physical considerations have led us to analyze the fate of the universal eigenvalue correlators (and in particular their microscopic spectral densities) when the the macroscopic spectral densities at the origin reach zero, *i.e.* when  $\rho(0) = 0$  but without a finite gap in eigenvalues around  $\lambda = 0$ .

---

<sup>1</sup> The number  $N_f = 17$  at which chiral symmetry breaking no longer occurs is probably too conservative. Lattice gauge theory simulations indicate that it may occur much before [10].

From a pure matrix model point of view, these questions are clearly posed, and may be considered in their own right. For example the proof for the macroscopic universality of density-correlators with such a multi-band support for the eigenvalues has been recently given for the hermitian and complex matrix model [11]. Their possible physical relevance in terms of applications to the Dirac operator spectrum near the origin can necessarily only be phrased in terms of conjectures. They are, however, highly non-trivial statements, and they are directly testable in, for example, lattice gauge theory simulations.<sup>2</sup>

To require that the (macroscopic) spectral density precisely just vanishes at the origin puts a constraint on the matrix model in question. This implies the fine tuning of at least one parameter (coupling) in the matrix model potential  $V(M)$ . From this point of view it is not surprising that if we associate the usual scaling (with no such constraint) with normal critical behavior, the just-vanishing of the spectral density at the origin corresponds to, in general, multicritical behavior. There are, for similar reasons, tight constraints on the precise manner in which the macroscopic spectral density can approach zero at the origin, in general labeled by an integer  $m$ , which counts the order of multicriticality. To achieve such multicriticality of  $m$ th order,  $m$  couplings must be tuned. This integer  $m$  in turn describes the rate at which eigenvalues accumulate near the origin, a rate which hence cannot be totally arbitrary.

With these preliminary remarks to serve as motivation, we now turn to the details of the actual matrix model calculations. Our analysis will be based on the orthogonal polynomial technique, which is known to greatly simplify many calculations in the unitary and chiral unitary matrix ensembles. As in ref. [7], we shall turn the three-step recursion relations for these orthogonal polynomials into a differential equation in the large- $N$  limit. From this we can derive the limiting forms of the orthogonal polynomials in the appropriate large- $N$  scaling regions. A recent paper by Kanzieper and Freilikher [13] shows very elegantly how the derivation of this differential equation can be simplified. Their approach is in fact more powerful, and leads more easily to generalizations. We shall essentially follow their approach here. As derived in ref. [13], the crucial property of the differential equation is that it does not explicitly depend on the chosen matrix model potential  $V(M)$  once the large- $N$  limit is taken, but only on the associated *macroscopic* spectral density  $\rho(\lambda)$ . As we shall see in section 2, this simplifying feature unfortunately does not survive in the particular microscopic large- $N$  limit that is required to probe the multicritical domain near  $\lambda=0$ . The resulting differential equation will therefore not manifestly depend only on the macroscopic spectral density  $\rho(\lambda)$ , but will also contain contributions from what are naively subdominant terms in the  $1/N$  expansion. Universality of the orthogonal polynomials, and hence universality of the microscopic spectral density and the spectral correlators, is hence not as obvious as in the usual case. Finding those naively subleading corrections that turn out to contribute in the microscopic multicritical limit entails solving an auxiliary differential equation, known in a different context as the string equation.<sup>3</sup> For the case at hand it turns out to be a generalized Painlevé II equation, which we show how to solve accurately by numerical methods. Both because the solution to the generalized Painlevé II equation cannot be found in closed analytical form, and because the underlying differential equation for the orthogonal polynomials themselves cannot be recast in a particularly simple form, it is not possible to give a closed analytical expression for the microscopic spectral correlators in this multicritical case. The solution can be found to any required accuracy by numerical means. We do however identify an intermediate “mesoscopic” range at which the spectral correlators can be computed analytically, expressed by some combination of Bessel

<sup>2</sup>For a beautiful comparison between recent lattice gauge theory data and the universal predictions from matrix models in the usual case of chiral symmetry breaking, see ref. [12].

<sup>3</sup>Only after properly solving this additional differential equation, and inserting its solution, does the differential equation for the orthogonal polynomials become manifestly universal.

functions. In section 3 we derive, for completeness, the analogous results for the chiral-unitary matrix ensembles. They may be relevant for Dirac operators of fermions in the fundamental representations of gauge groups  $SU(N_c \geq 3)$  in even space-time dimensions. These results follow as simple corollaries from the unitary ensemble. Section 4 contains our conclusions.

## 2 Multicritical Unitary Ensembles

We begin with the unitary matrix ensemble described by the partition function

$$\begin{aligned} \mathcal{Z} &= \int dM \det^{2\alpha} M e^{-N \text{Tr} V(M)} \\ &\sim \int_{-\infty}^{\infty} \prod_{i=1}^N \left( d\lambda_i |\lambda_i|^{2\alpha} e^{-NV(\lambda_i)} \right) \left| \det_{ij} \lambda_j^{i-1} \right|^2, \end{aligned} \quad (2.1)$$

with some even potential

$$V(M) = \sum \frac{g_{2k}}{2k} M^{2k}. \quad (2.2)$$

The matrices  $M$  are  $N \times N$  and hermitian, with real eigenvalues  $\lambda_i$ . The integration is over their associated Haar measure  $dM$ . This matrix model is, for generic potentials  $V(M)$  giving rise to a single cut containing the origin of eigenvalues  $\lambda=0$ , conjectured to be related to the Dirac operator spectrum of  $SU(N_c \geq 3)$  gauge theories coupled to an even number of flavors  $N_f = 2\alpha$ . For that application  $\alpha$  is thus restricted to be an integer. Using the formalism of Kanzieper and Freilikher [14] (see also refs. [15, 16]) this restriction can be lifted without difficulty, and we shall therefore in greater generality consider any real  $\alpha > -1/2$ . We shall largely employ the notation of ref. [14] in what follows below.

Using the method of orthogonal polynomials to analyze the above partition function, one has to find polynomials  $P_n(\lambda)$  orthogonal with respect to the weight function  $w(\lambda) \equiv |\lambda|^{2\alpha} \exp[-NV(\lambda)]$ . We choose them orthonormalized:

$$\int_{-\infty}^{\infty} d\lambda w(\lambda) P_m(\lambda) P_n(\lambda) = \delta_{mn}. \quad (2.3)$$

It is actually more convenient to work directly with the ‘‘wave functions’’

$$\psi_m(\lambda) \equiv |\lambda|^\alpha e^{-\left(\frac{N}{2}\right)V(\lambda)} P_m(\lambda), \quad (2.4)$$

because these are the objects that enter directly into the determination of the spectral kernel (two-point function):

$$\begin{aligned} K_N(\lambda, \mu) &= \sum_{n=0}^{N-1} \psi_n(\lambda) \psi_n(\mu) \\ &= c_N \frac{\psi_N(\mu) \psi_{N-1}(\lambda) - \psi_N(\lambda) \psi_{N-1}(\mu)}{\mu - \lambda}. \end{aligned} \quad (2.5)$$

Here the coefficients  $c_N$  are those of the three-step recursion relation for the orthogonal polynomials:

$$\lambda P_{n-1}(\lambda) = c_n P_n(\lambda) + c_{n-1} P_{n-2}(\lambda). \quad (2.6)$$

They can be determined from the following identity<sup>4</sup> (which in the context of 2-d quantum gravity is called the string equation) by partial integration, using eq. (2.6),

$$n = c_n \int_{-\infty}^{\infty} d\lambda w(\lambda) P_n'(\lambda) P_{n-1}(\lambda). \quad (2.7)$$

---

<sup>4</sup>By switching to the monic normalization  $\tilde{P}_n(\lambda) = \lambda^n + \dots$  one can easily derive that  $P_n'(\lambda) = \frac{n}{c_n} P_{n-1}(\lambda) + \mathcal{O}(\lambda^{n-2})$ .

The wave functions  $\psi_m(\lambda)$  are clearly orthonormalized on the usual (flat) Lebesgue measure.

In order to derive a differential equation for the wave functions it is convenient to express the differential operator acting on the orthogonal polynomials in the following way [14]:

$$P_n'(\lambda) \equiv A_n(\lambda)P_{n-1}(\lambda) - B_n(\lambda)P_n(\lambda) , \quad (2.8)$$

where the functions  $A_n$  and  $B_n$  have to be determined. Following Kanzieper and Freilikher [14], they are given by

$$A_n(\lambda) = N c_n \int_{-\infty}^{\infty} d\mu \frac{w(\mu)}{\mu - \lambda} [V'_{(\alpha)}(\mu) - V'_{(\alpha)}(\lambda)] P_n(\mu)^2 , \quad (2.9)$$

$$\begin{aligned} B_n(\lambda) &= N c_n \int_{-\infty}^{\infty} d\mu \frac{w(\mu)}{\mu - \lambda} [V'_{(\alpha)}(\mu) - V'_{(\alpha)}(\lambda)] P_n(\mu) P_{n-1}(\mu) \\ &\equiv B_n^{\text{reg}}(\lambda) + (1 - (-1)^n) \frac{\alpha}{\lambda} , \end{aligned} \quad (2.10)$$

where the  $\alpha$ -dependent part of the measure has been absorbed into the potential<sup>5</sup>:

$$V_{(\alpha)}(\lambda) \equiv V(\lambda) - \frac{2\alpha}{N} \ln |\lambda| . \quad (2.11)$$

In contrast to reference [14] we have chosen the usual matrix model convention eq. (2.1) which makes the dependence on  $N$  explicit. The evaluation of different powers in  $N$  will become very important in the multicritical case. Using the results eqs. (2.9) and (2.10) as well as the recursion relation eq. (2.6) the following identity is exactly satisfied [14]:

$$B_n(\lambda) + B_{n-1}(\lambda) + N V'_{(\alpha)}(\lambda) = \frac{\lambda}{c_{n-1}} A_{n-1}(\lambda) . \quad (2.12)$$

Differentiating eq. (2.8) for  $\psi_n(\lambda)$  and using relations (2.6) and (2.12) finally leads to the following exact differential equation for the wave functions  $\psi_n(\lambda)$ , valid for any  $n$  [14]:

$$\psi_n''(\lambda) - F_n(\lambda)\psi_n'(\lambda) + G_n(\lambda)\psi_n(\lambda) = 0 , \quad (2.13)$$

where

$$F_n(\lambda) \equiv \frac{A_n'(\lambda)}{A_n(\lambda)} \quad (2.14)$$

$$\begin{aligned} G_n(\lambda) &\equiv \frac{c_n}{c_{n-1}} A_n(\lambda) A_{n-1}(\lambda) - \left( B_n(\lambda) + \frac{N}{2} V'_{(\alpha)}(\lambda) \right)^2 \\ &+ \left( B_n(\lambda) + \frac{N}{2} V'_{(\alpha)}(\lambda) \right)' - \frac{A_n'(\lambda)}{A_n(\lambda)} \left( B_n(\lambda) + \frac{N}{2} V'_{(\alpha)}(\lambda) \right) . \end{aligned} \quad (2.15)$$

The rest of our paper will consist in analyzing the way in which suitable microscopic large- $N$  limits affect the above differential equation. Performing the usual large- $N$  limit the recursion coefficients  $c_n$  and the functions  $A_n(\lambda)$  and  $B_n^{\text{reg}}(\lambda)$  are assumed to approach smooth functions. Under these

---

<sup>5</sup>In  $A_n(\lambda)$  the term proportional to  $\alpha$  vanishes due to the potential  $V_{(\alpha)}(\lambda)$  being even, whereas for  $B_n(\lambda)$  it is given explicitly, following again [14].

conditions, with a generic potential  $V(\lambda)$ , the function  $A_n(\lambda)$  is directly related to the macroscopic spectral density  $\rho(\lambda)$  [13, 17]:

$$\rho(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N\pi} A_N(\lambda) \sqrt{1 - (\lambda/a)^2}, \quad (2.16)$$

where  $a = \lim_{N \rightarrow \infty} c_N$  is the endpoint of the spectrum. Consequently, because of eq. (2.12), the asymptotic differential equation (2.13) for  $\psi_N(\lambda)$  should only depend on the macroscopic spectral density. But precisely at multicriticality, and in the appropriate microscopical scaling limit, the simple relationship (2.16) unfortunately breaks down. This is due to the contribution of terms that are naively subleading in  $1/N$ . Because of this, the appropriate differential equation for the wave functions  $\psi_N(\lambda)$  in the multicritical microscopic limit will *not* just depend on the spectral density  $\rho(\lambda)$ . To see this, we first make the definition of multicriticality precise.

## 2.1 MULTICRITICAL POTENTIALS

Multicriticality can be reached at both the endpoint of the spectrum, and at the origin. Because of the physical motivations mentioned in the introduction, we here focus exclusively on multicriticality at the origin of the spectrum. The case of  $\rho(0) \neq 0$  will thus play, in this context, the rôle of being uncritical ( $m = 0$ ). As  $\rho(0) \rightarrow 0$ , the one-cut solution around the origin will turn into a two-cut solution, thereby defining the first criticality ( $m = 1$ ). As is well known, this happens in the simplest case [18, 19, 20] when the potential  $V(\lambda)$  has a double-well shape, as shown in fig. 1.

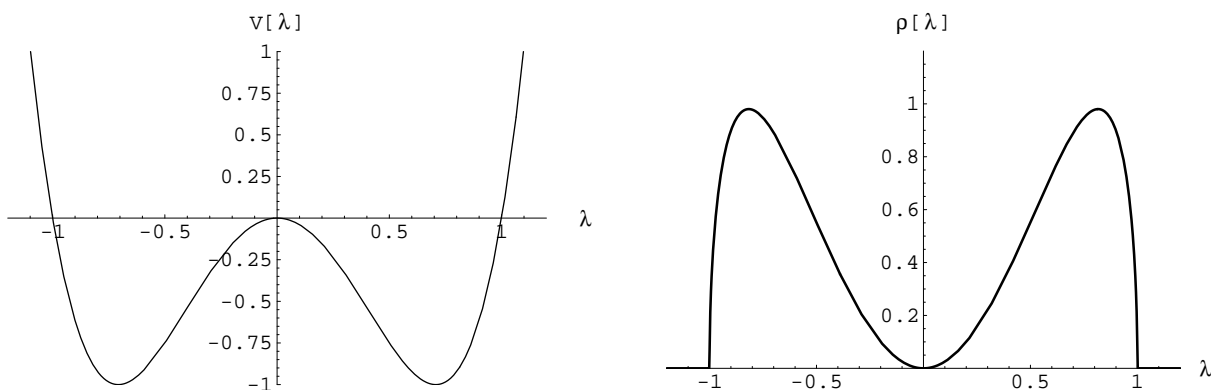


Figure 1: The critical  $m = 1$  potential (left) and the associated spectral density (right).

A characterization of potentials leading to multicriticality at the origin has been given by Crnković and Moore [21]. In general, for polynomial potentials of degree  $2k$ , the macroscopic spectral density is, for a symmetric one-cut solution, of the form

$$\rho(\lambda) = Q_{2k-2}(\lambda) \sqrt{1 - (\lambda/a)^2}, \quad (2.17)$$

where  $Q_k(\lambda)$  is a polynomial of degree  $k$ . Generically,  $Q_k(0) \neq 0$ , in which case  $\rho(\lambda) \sim \text{const.}$  near  $\lambda=0$ . To force  $\rho(0) \rightarrow 0$  thus entails tuning the potential  $V(\lambda)$  so that the polynomial  $Q(\lambda)$  vanishes at the origin. Because  $Q(\lambda)$  is a polynomial of even degree, this can be achieved by powers of an even integer,  $2m$ , where  $m$  labels the order of multicriticality:

$$\rho(\lambda) = \text{const.} \lambda^{2m} + \dots, \quad \text{as } \lambda \rightarrow 0. \quad (2.18)$$

Polynomial potentials  $V_m(\lambda)$  that lead to such behavior are thus at least of degree  $2m+2$ . A particular class of such potentials for which the behavior of the macroscopic spectral density is as above can be compactly described by [21]<sup>6</sup>

$$V'_m(\lambda) = k(m)\lambda^{2m+1} \left(1 - \frac{1}{\lambda^2}\right)^{1/2} \Big|_+, \quad (2.19)$$

where endpoint of the support  $a$  has been set to unity and the “+” subscript indicates that only the polynomial part in an expansion around  $\lambda = \infty$  is kept. The constant in front, chosen so as to normalize the macroscopic spectral density by

$$\int_{-1}^1 d\lambda \rho(\lambda) = 1 \quad (2.20)$$

is given by

$$k(m) = 2^{2m+1} \frac{(m+1)!(m-1)!}{(2m-1)!}. \quad (2.21)$$

The spectral density at the multicritical point is then given by

$$\rho_m(\lambda) = \frac{1}{2\pi} k(m) \lambda^{2m} \sqrt{1 - \lambda^2}. \quad (2.22)$$

Conforming with common usage, we shall denote such multicritical potentials as “minimal”. An infinity of other potentials can be constructed by adding higher powers of  $\lambda$  the polynomial  $Q_k(\lambda)$ , which lead to precisely the same multicritical behavior (2.18). For this reason one can speak about an  $m$ th multicritical universality class. One sees immediately that the  $\alpha$ -dependent term in our integration measure has no influence on the classification of these multicritical potentials, since this term is suppressed by one power of  $1/N$  in the planar limit (see eq. (2.11)).

## 2.2 THE BREAKDOWN OF THE USUAL PLANAR LIMIT

As it has been mentioned earlier in this section an asymptotic differential equation for  $\psi_N(\lambda)$  at multicriticality is no longer easy to obtain due to a breakdown of the relation (2.16). In order to show this let us briefly review how the result is obtained in the noncritical case, where the usual planar limit produces all relevant terms, and point out what breaks down at multicriticality.

In the usual planar limit the following behavior is assumed for the recursion coefficients:

$$c_{n\pm 1} \rightarrow c_n + \mathcal{O}(1/N), \quad (2.23)$$

which is directly related to the fact that we are here considering a one-cut phase. Moreover, an implicit assumption in ref. [14] is that the following smooth limits are satisfied:

$$A_{n\pm 1}(\lambda) \rightarrow A_n(\lambda) + \mathcal{O}(1/N), \quad (2.24)$$

$$B_{n\pm 1}^{\text{reg}}(\lambda) \rightarrow B_n^{\text{reg}}(\lambda) + \mathcal{O}(1/N). \quad (2.25)$$

The requirement eq. (2.23) together with the leading term of the relation

$$2c_N = a + \dots \quad (2.26)$$

---

<sup>6</sup>Crnković and Moore have considered a spectral density with two symmetric cuts merging. This leads to the the same multicritical potentials  $V_m(\lambda)$  as a symmetric one-cut density developing zeros at the origin.



between the coefficient  $c_N$  and the endpoint of the support leads to the asymptotic relation (2.16) between  $A_N(\lambda)$  and the macroscopic spectral density  $\rho(\lambda)$ . Inserting the limit (2.25) into eq. (2.12) leads asymptotically to

$$B_N(\lambda) + \frac{N}{2}V'_{(\alpha)}(\lambda) \rightarrow \frac{\lambda}{2c_N}A_N(\lambda) - (-1)^N\frac{\alpha}{\lambda}. \quad (2.27)$$

Consequently  $G_N(\lambda)$  can be expressed asymptotically by  $A_N(\lambda)$  (and via eq. (2.16), if true, by  $\rho(\lambda)$ ) only<sup>7</sup>.

$$G_N(\lambda) \rightarrow A_N^2\left(1 - \frac{\lambda^2}{a^2}\right) + \frac{(-1)^N\alpha - \alpha^2}{\lambda^2} + (-1)^N\frac{\alpha}{\lambda}\frac{A'_N(\lambda)}{A_N(\lambda)}. \quad (2.28)$$

Looking at the noncritical case at the origin the terms proportional to  $\frac{A'_N(\lambda)}{A_N(\lambda)}$  are subleading in the microscopic limit at the origin so the last term in  $G_N(\lambda)$  and the  $F_N(\lambda)$ -term can be disregarded. This leads to the asymptotic differential equation

$$\psi''_N(x) + \left(\frac{A_N(0)^2}{N^2} + \frac{(-1)^N\alpha - \alpha^2}{x^2}\right)\psi_N(x) = 0, \quad (2.29)$$

where the appropriate scaling  $x = N\lambda$  has been used and  $\psi_N(\lambda = \frac{x}{N})$  is regarded a function of  $x$ . It has been derived and solved for nonnegative integers in [7], and generalized to real  $\alpha > -1/2$  in [14]. Apart from the independence of the two derivations, we have also checked the resulting solutions (certain Bessel functions, see ref. [7]) by numerically determining  $\psi_N(\lambda)$  directly from the recursion relations themselves, for very large values of  $N$ .

Moreover there is an analytic consistency check on the differential equation (2.29). It is straightforward to verify (see Appendix A) that the following exact symmetry holds for the wave functions:

$$\psi_{2n}^{(\alpha+1)}(\lambda) = \text{sign}(\lambda)\psi_{2n+1}^{(\alpha)}(\lambda). \quad (2.30)$$

The superscript indicates the corresponding value of  $\alpha$  in the measure. We will refer to this symmetry as  $\alpha$ -symmetry. Note that the macroscopic spectral density is independent of  $\alpha$ , since the  $\alpha$ -dependence can be considered as a subleading term in the potential. Due to relation (2.16) and the limit (2.24) the differential equation (2.29) is therefore manifestly invariant under the  $\alpha$ -symmetry. Indeed, the solutions of ref. [7] trivially satisfy the relation (2.30).

Let us now turn to the multicritical case. The appropriate rescaling of eigenvalues that is needed to obtain a non-trivial microscopic limit will now be of the form

$$x = N^{\frac{1}{2m+1}}\lambda \quad (2.31)$$

at the  $m$ th multicritical point ( $m \in \mathbf{N}$ ). We first suppose that the usual planar limit eqs. (2.23) - (2.25) still holds. One might object that the recursion coefficients now obey a period-two ansatz [19] instead of eq. (2.23). But in the planar limit, and just at the critical point  $n = N$ , the two branches merge to the one-cut value  $c_N = a/2$ , as we are right on top of the one-cut–two-cut transition. Therefore the argument in [14] relating  $A_N(\lambda)$  and  $\rho(\lambda)$  (eq. (2.16)) should still be expected to hold. Next, we have seen from the previous subsection that at the  $m$ th multicritical point  $\rho(\lambda)$  behaves like  $\lambda^{2m}$  at  $\lambda = 0$ . Consequently the last term in eq. (2.28) for  $G_N(\lambda)$  becomes  $(-)^N 2m\alpha/\lambda^2$  which is of the same (leading) order as the second term.

---

<sup>7</sup>The last term has been omitted in eq. (34) of ref. [14] because the authors did not consider the microscopic limit at the origin.

So from the usual planar limit we are left with the full  $G_N(\lambda)$  eq. (2.28) in the asymptotic differential equation for  $\psi_N(\lambda)$ . However, this differential equation is no longer invariant under the  $\alpha$ -symmetry. It therefore cannot be the correct equation. In order to confirm this, we have compared the prediction from this naive differential equation to the polynomials constructed by Gram-Schmidt orthonormalization. The mismatch is shown in fig. 2.

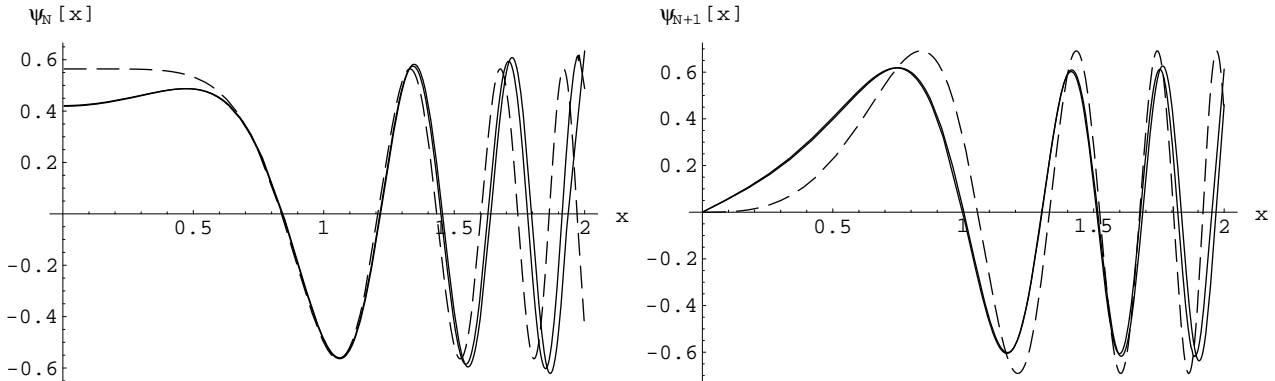


Figure 2: The microscopic wave functions for the  $m = 1$  critical potential (2.19) are plotted for  $N = 32$  and  $48$  ( $\alpha = 0$ ) (real lines). They converge to limiting functions which differ from the solution of the naive differential equation (2.28) (dotted lines) for small  $x$ .

The only explanation for this failure is that the simple relationship (2.16) breaks down or, in other words, that the  $\alpha$ -dependent corrections to the recursion coefficients that were subleading before have to be taken into account. Remarkably, this can also be established directly by analyzing the “string equation” for the coefficients  $c_n$ , as we will discuss in section 2.4. One finds that corrections to the  $c_n$ ’s are precisely of the order  $N^{-\nu}$  for  $\nu = 1/(2m + 1)$  at  $m$ th multicriticality, instead of  $N^{-1}$  (as for the case  $m = 0$ ). It is due to this rather devious compensation that the naive differential equation for the wave functions is incorrect.

Despite many efforts we have not been able to derive a general expression for the asymptotic differential equation containing all relevant contributions from the  $c_n$ ’s. The analytical formulas increase dramatically in complexity with increasing  $m$ . We will therefore restrict ourselves to a detailed analytical treatment of the  $m=1$  and  $m=2$  multicriticalities. Many aspects of the formalism do however readily generalize to arbitrary  $m$ , and we shall keep the discussion in as general form as possible. In Appendix B we write down a detailed conjecture for the differential equation for the wave functions at any multicriticality.

### 2.3 AN EXAMPLE: THE CASE $m = 1$

In order to treat all contributions to the differential equation at multicriticality we will restrict ourselves first to the simplest example, the case  $m = 1$ . This corresponds to a minimal potential of the simple Mexican-hat form

$$V_1(\lambda) = \frac{1}{2}g_2\lambda^2 + \frac{1}{4}g_4\lambda^4 \quad \text{with} \quad g_2 = -8, \quad g_4 = 16. \quad (2.32)$$

The corresponding eigenvalue density reads<sup>8</sup>

$$\rho_1(\lambda) = \frac{1}{2\pi} g_4 \lambda^2 \sqrt{1 - \lambda^2}. \quad (2.33)$$

We are now able to give exact expressions for the functions  $A_n(\lambda)$  and  $B_n(\lambda)$  only in terms of the recursion coefficients  $c_n$ . This will allow us to keep track where the correction to the  $c_n$ 's will enter. We will then be able to derive an asymptotic differential equation at  $m = 1$  multicriticality.

Making use of the completeness of the orthogonal polynomials we can explicitly express  $P_n'(\lambda)$  by lower order polynomials, as we have chosen a particular potential:

$$\begin{aligned} P_n'(\lambda) &= \sum_{k=0}^{n-1} P_k(\lambda) \int_{-\infty}^{\infty} d\mu w(\mu) N V'_{(\alpha)}(\mu) P_n(\mu) P_k(\mu) \\ &= N c_n \left( g_2 + g_4 (c_{n+1}^2 + c_n^2 + c_{n-1}^2) \right) P_{n-1}(\lambda) + N c_n c_{n-1} c_{n-2} g_4 P_{n-3}(\lambda) - (1 - (-1)^n) \frac{\alpha}{\lambda} P_n(\lambda) \end{aligned} \quad (2.34)$$

where the recursion relation (2.6) has been used for the potential eq. (2.32). Looking back at the definition (2.34)  $A_n(\lambda)$  must be an even function whereas  $B_n^{\text{reg}}(\lambda)$  is odd. A simple way of finding them is to assume that they are polynomial<sup>9</sup>, use the recursion relation and then comparing coefficients. In this way one finds

$$A_n(\lambda) = N c_n \left( g_2 + g_4 (c_{n+1}^2 + c_n^2) + g_4 \lambda^2 \right) \quad (2.35)$$

$$B_n(\lambda) = N c_n^2 g_4 \lambda + (1 - (-1)^n) \frac{\alpha}{\lambda}. \quad (2.36)$$

One can also derive these expressions directly, using the the definitions (2.9) and (2.10). If we would now set  $c_{n+1}^2 = c_n^2 = 1/4$  and use the critical couplings eq. (2.32) the combination  $g_2 + g_4 (c_{n+1}^2 + c_n^2)$  would vanish and we would be back at the relation (2.16), which no longer holds at multicriticality. From our example we are now in a position to determine the order and the explicit value of the corrections. Inserting the appropriate microscopic scaling limit at multicriticality, eq. (2.31), the whole differential equation (2.13) has to be rescaled by  $N^{-\frac{2}{2m+1}}$ , which is  $N^{-2/3}$  in our case. In order to get contributing corrections in the differential equation, the following terms have to be of order one:

$$\begin{aligned} u_{\pm} &\equiv N^{\frac{2}{3}} \left( g_2 + g_4 (c_{N\pm 1}^2 + c_N^2) \right) \\ v &\equiv N^{\frac{1}{3}} \left( g_2 + 2g_4 c_N^2 \right). \end{aligned} \quad (2.37)$$

In other words, for  $m=1$  the terms in the parenthesis must vanish like  $N^{-2/3}$  and  $N^{-1/3}$ , respectively. This is precisely what we get from eqs. (2.44) and (2.51) below. Expressed in terms of the above quantities, we get the following exact differential equation for the potential eq. (2.32) (from eqs. (2.35), (2.36) and (2.15)):

$$\psi_N''(x) - N^{-1/3} F_N(x) \psi_N'(x) + N^{-2/3} G_N(x) \psi_N(x) = 0. \quad (2.38)$$

<sup>8</sup>Here  $a = 1$  due to our conventions from subsect. 2.1.

<sup>9</sup>Remember that in the usual planar limit  $A_N(\lambda)$  becomes proportional to the polynomial  $Q_k(\lambda)$  in front of the square root of  $\rho(\lambda)$ .

Here

$$\begin{aligned}
N^{-1/3}F_N(x) &= \frac{2g_4x}{u_+ + g_4x^2} \\
N^{-2/3}G_N(x) &= c_N^2 u_+ u_- + \left( (-1)^N \alpha + \frac{1}{2} \right) v + \left( c_N^2 (u_+ + u_-) g_4 - \frac{1}{4} v^2 + N^{-1/3} \left( (-1)^N \alpha + \frac{3}{2} \right) g_4 \right) x^2 \\
&\quad + \left( c_N^2 g_4^2 - \frac{1}{2} N^{-1/3} v g_4 \right) x^4 - \frac{1}{4} N^{-2/3} g_4^2 x^6 + \frac{(-1)^N 2g_4 \alpha - v g_4 x^2 - N^{-1/3} g_4^2 x^4}{u_+ + g_4 x^2} \\
&\quad + \frac{(-1)^N \alpha - \alpha^2}{x^2} .
\end{aligned} \tag{2.39}$$

Due to eq. (2.37) it is now clear that the terms being explicitly proportional to negative powers of  $N$  will vanish in the microscopic large- $N$  limit. Moreover, the dominant term  $\sim x^2$  in  $G_N(x)$  vanishes up to  $\mathcal{O}(N^{-1/3})$ :

$$v^2 = 4c_N^2 g_4 (u_+ + u_-) . \tag{2.40}$$

This follows directly from the string equation (2.7) at  $n = N$ , which in our example reads

$$\frac{n}{N} + (1 - (-1)^n) \frac{\alpha}{N} = c_n^2 \left( g_2 + g_4 (c_{n+1}^2 + c_n^2 + c_{n-1}^2) \right) , \tag{2.41}$$

and from inserting the critical values for the coupling constants. Finally the correct asymptotic differential equation for  $m = 1$  criticality is eq. (2.38) with

$$N^{-2/3}G_N(x) = c_N^2 u_+ u_- + \left( (-1)^N \alpha - \frac{1}{2} \right) v + \frac{u_+ v + (-1)^N 2\alpha g_4}{u_+ + g_4 x^2} + c_N^2 g_4^2 x^4 + \frac{(-1)^N \alpha - \alpha^2}{x^2} , \tag{2.42}$$

where one of the quantities  $u_+$ ,  $u_-$  or  $v$  can still be eliminated by means of eq. (2.40). After having determined the functional form of the differential equation, which is now no longer a Bessel equation, the only point left to be done is the analytical calculation of the corrections to, say,  $c_{n+1}^2$  and  $c_n^2$ . This will be the topic of the next subsection.

Let us add a final remark on the  $\alpha$ -symmetry of this new asymptotic differential equation. The function  $F_n(\lambda)$  can be seen to be invariant for finite  $n$  from the definition eq. (2.14) since one has

$$\frac{A_{2n}^{(\alpha+1)}(\lambda)}{c_{2n}^{(\alpha+1)}} = \frac{A_{2n+1}^{(\alpha)}(\lambda)}{c_{2n+1}^{(\alpha)}} . \tag{2.43}$$

This follows directly from eq. (2.30). An immediate consequence of this identity is the invariance of  $u_+$  under the  $\alpha$ -symmetry. Still, apart from the last two terms,  $G_N(x)$  is not obviously invariant. In order to prove its invariance under the  $\alpha$ -symmetry one would have to know precisely the behavior of  $u_-$  and  $v$  as functions of  $\alpha$ . As we shall see below, this is a highly non-trivial problem, related to the solution, for arbitrary  $\alpha$ , of a certain non-linear differential equation.

## 2.4 PAINLEVÉ II

The apparently subdominant terms that enter the differential equation for the wave functions  $\psi(\lambda)$  in the microscopic limit near multicriticality arise from corrections to the coefficients  $c_n$  as  $N \rightarrow \infty$ . It is well-known [19, 20] that in the case at hand the coefficients  $c_n$  do not approach one single smooth

function in the large- $N$  limit, but rather split into two functions depending on whether the index is odd or even. In all generality [21]<sup>10</sup>

$$\begin{aligned} c_{2n}^2 &= C + N^{-\nu} f_m(z) + N^{-2\nu} g_m(z) + N^{-3\nu} h_m(z) + \dots \\ c_{2n+1}^2 &= \bar{C} + N^{-\nu} \bar{f}_m(z - N^{-\nu}) + N^{-2\nu} \bar{g}_m(z - N^{-\nu}) + N^{-3\nu} \bar{h}_m(z - N^{-\nu}) + \dots \end{aligned} \quad (2.44)$$

for  $2n \approx N$ , where the scaling variable  $z$  is defined by

$$z \equiv N^{2m\nu} \left( 1 - \frac{2n}{N} \right) \quad (2.45)$$

with the exponent

$$\nu \equiv \frac{1}{2m+1} \quad (2.46)$$

at  $m$ th multicriticality. Here  $f_m(z), g_m(z), h_m(z)$  and the corresponding barred quantities are assumed to be smooth functions.

Consider first the case  $m=1$ . At the critical point, which is at  $\frac{2n}{N} = 1$ , the two branches for the even and odd recursion coefficients meet in the planar limit, which implies  $C = \bar{C}$ . Inserting eqs. (2.44) into the string equation (2.41) for even and odd  $n$ , and expanding consistently in  $N^{-\nu}$  up to third order, yields the following set of equations:

$$C = \bar{C} = \frac{1}{4}, \quad (2.47)$$

which is nothing else than the statement  $2c_N = a + \dots$ . Furthermore we get

$$\bar{f}_1(z) = -f_1(z) \quad (2.48)$$

$$g_1(z) + \bar{g}_1(z) = 2f_1(z)^2 - \frac{1}{8}z \quad (2.49)$$

$$0 = f_1(z) \left( 16f_1(z)^2 - z \right) - 2f_1''(z) + \frac{\alpha}{2}, \quad (2.50)$$

where the latter is a Painlevé II equation [20] for  $\alpha = 0$ . The dependence on  $h_1(z)$  and  $\bar{h}_1(z)$  has been eliminated using the equations from even and odd  $n$ . Due to the following observation [21]

$$c_{2n+q}^2 + c_{2n+q+1}^2 = 2C + N^{-2\nu} \left( (-1)^q f_m'(z) + g_m(z) + \bar{g}_m(z) \right) + \dots, \quad (2.51)$$

where  $2n+q \approx N$ , we just have to determine the combination  $g_1(z) + \bar{g}_1(z)$  and the function  $f_1(z)$  in order to get the quantities  $u_{\pm}$  and  $v$  from the last subsection. So we just have to solve the generalized Painlevé II equation (2.50).

Consider first the case  $\alpha=0$ . The solution to eq. (2.50) can then be shown to be unique [20, 22] once the following physical boundary conditions are imposed. First, in the usual planar limit  $z \rightarrow \infty$ , we take<sup>11</sup>  $f_1(z) = -\sqrt{z}/4$ . This follows directly from solving the string equation (2.41) in the large- $N$  limit. Next, we demand that there are no singularities on the real  $z$ -axis. (This is equivalent to enforcing  $f_1(z) = 0$  at  $z = -\infty$ ). Because the boundary conditions in this manner are given at the opposite ends of the interval (here the real line), and because the solution is unstable towards the development of singularities once slightly improper boundary conditions are imposed, it is not entirely trivial to compute the correct solution numerically. We have found that the discretized, iterative,

<sup>10</sup>Here we have used the most general ansatz for the coefficients  $c_n$ .

<sup>11</sup>With  $f_1(z)$  being a solution also  $-f_1(z)$  is a solution. We fix the sign by noting that for any finite  $N$  the branch of the coefficients  $c_n^2$  with odd  $n$  is always larger than the one with even  $n$  [23].

procedure proposed in ref. [24] for the odd-order generalization of Painlevé I equation works well here too. In fig. 3 we show a plot of the solution to eq. (2.50) found in this way. As a check, we display on the same figure a numerical determination of  $f_1(z)$  directly from the recursion coefficients  $c_n$ , evaluated to high order. For detailed definition of the parameters used to obtain this numerical solution, we refer to ref. [24].

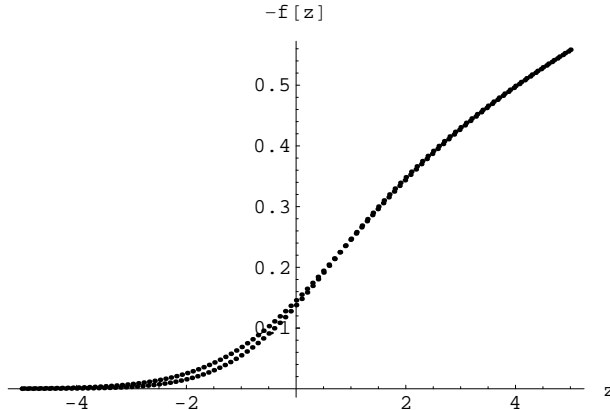


Figure 3: Numerical solution to Painlevé II equation  $f(z)(16f^2(z) - z) - 2f''(z) = 0$  (lower dots). We have used the parameters: cutoff= 5, mesh= 1/10, calibration parameter  $h = 1/300$ , number of iterations  $\nu = 400 \sim 500$ , and extrapolated the result to  $\nu = \infty$ . Explicit values of  $f_n = N^{\frac{1}{3}} (2c_n^2 - c_{n-1}^2 - c_{n+1}^2) / 4$  for the critical potential (2.32) with  $\alpha = 0$ ,  $N = 1024$  are also plotted (upper dots).

Inserting  $g_2 = -8$  and  $g_4 = 16$  for the minimal  $m = 1$  multicritical potential, we get for  $N$  even<sup>12</sup>

$$\begin{aligned} u_{\pm} &= 16[2f_1(0)^2 \pm f_1'(0)] \\ v &= 32f_1(0) . \end{aligned} \tag{2.52}$$

Note that the leading contributions (proportional to  $N^{\frac{2}{3}}$  and  $N^{\frac{1}{3}}$ , respectively) to the above quantities vanish at the critical point. The first corrections, from eq. (2.51), are precisely of the required power to render both  $u_{\pm}$  and  $v$  finite in the  $N \rightarrow \infty$  limit. Yet higher order corrections can indeed be ignored here. If we substitute the solution to the Painlevé II equation, the full differential equation (2.38) for the orthogonal polynomials in the microscopic limit is thus known in this case. Numerically we find from the above solution  $u_+ \simeq -1.105$ ,  $u_- \simeq 2.211$  and  $v \simeq -4.207$ . In fig. 4 we plot the solution to the equation (2.38) obtained in this way. On the same figure we also show the wave function for  $\psi_N(x)$  constructed directly by Gram-Schmidt orthonormalization. The two functions are seen to agree to very high precision. This gives us complete confidence in our procedure.

<sup>12</sup>For  $N$  odd one has to replace  $\pm \rightarrow \mp$  on the r.h.s. and  $v \rightarrow -v$ .

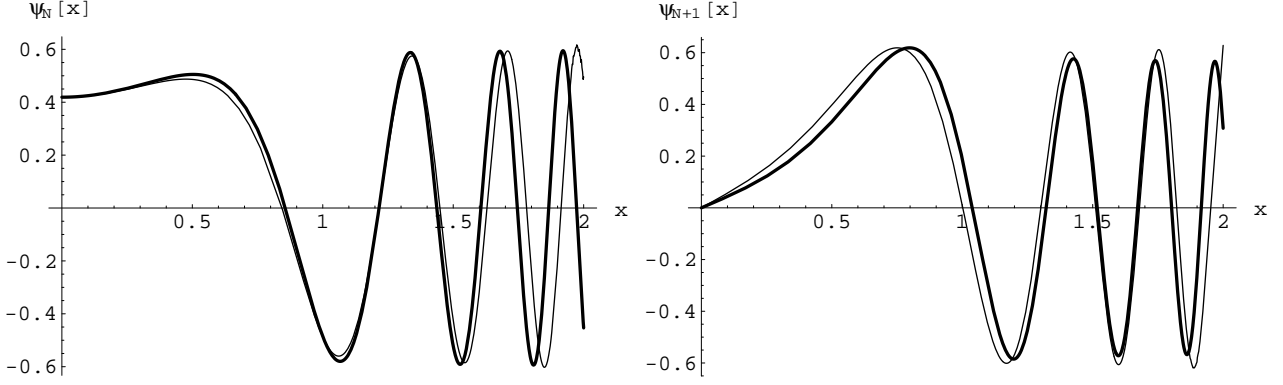


Figure 4: Numerical solutions to the linear differential equation (2.38) using data from Painlevé II (bold lines). Their vertical scales are chosen to optimize the matching with the microscopic wave functions ( $N = 48$ ) constructed from the critical potential with  $\alpha = 0$  by Gram-Schmidt orthonormalization (real lines).

The microscopic correlation functions are constructed out of these limiting functions for the wave functions. In fig. 5 we exhibit a microscopic spectral density  $\rho_S(x) \sim (\psi_N(x)\psi'_{N-1}(x) - \psi_{N-1}(x)\psi'_N(x))$ :

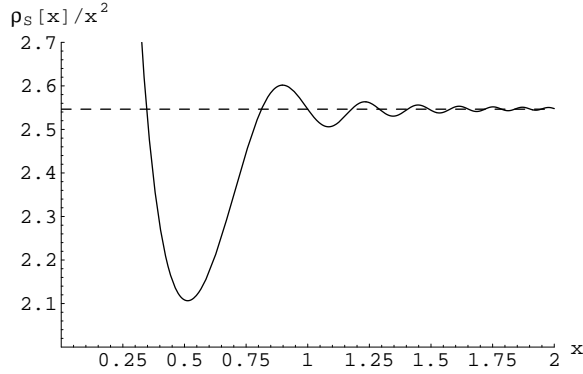


Figure 5: Microscopic spectral density for the  $m = 1$  criticality, divided by  $x^2$  in order to take out the trivial  $x^2$ -growth which is a consequence of the matching-on to the macroscopic spectral density. The normalization is determined by requiring  $\lim_{x \rightarrow \infty} \rho_S''(x) = \rho''(0) = 16/\pi$  (dotted line).

Next, consider *e.g.*, the case  $\alpha = 1$ . Appropriate boundary conditions for the generalized Painlevé II equation are easily established. In the planar limit, here reached at  $z \rightarrow \pm\infty$ , the  $\alpha$ -term does not contribute, as it is suppressed by  $1/N$  compared with the rest of the potential  $V(\lambda)$ . The boundary conditions are therefore precisely the same as for  $\alpha = 0$ :

$$f_1(z) \rightarrow 0 \quad \text{as } z \rightarrow -\infty, \quad (2.53)$$

$$f_1(z) \rightarrow -\sqrt{z}/4 \quad \text{as } z \rightarrow \infty. \quad (2.54)$$

For numerical purposes more refined boundary conditions are convenient:

$$f_1(z) = \alpha/(4z) \quad \text{as } z \rightarrow -\infty, \quad (2.55)$$

$$f_1(z) = -\frac{1}{8} \left( \frac{z}{3y(z)} + y(z) \right) + \frac{i\sqrt{3}}{8} \left( \frac{z}{3y(z)} - y(z) \right) \quad \text{as } z \rightarrow \infty,$$

$$\text{with } y(z) \equiv \left( -\alpha + \sqrt{\alpha^2 - z^3/27} \right)^{\frac{1}{3}} \quad (2.56)$$

This last condition is found from solving the algebraic 3rd order equation obtained from eq. (2.50), self-consistently setting  $f_1''(z) \sim 0$  as  $z \rightarrow \infty$ , and choosing the root which reduces to  $f_1(z) = -\sqrt{z}/4$  in the limit  $z \rightarrow \infty$ . In fact, because the solution (2.54), extended to the negative  $z$ -axis by

$$f_1(z) = \frac{z}{12y(z)} + \frac{1}{4}y(z) \quad \text{for } z < 0, \quad (2.57)$$

has  $f_1''(z) \approx 0$  everywhere, and automatically satisfies boundary condition (2.53), it is almost exact. Numerically it is therefore an extremely efficient starting point for finding the full solution to eq. (2.50). It becomes exact in the limit  $\alpha \rightarrow \infty$  (see also below).

In fig. 6 we plot the solution of the differential equation (2.38) for  $\alpha = 1$ , and compare it with the solution explicitly solving the recursions relations, using very for large values of  $N$ . We again find perfect agreement.

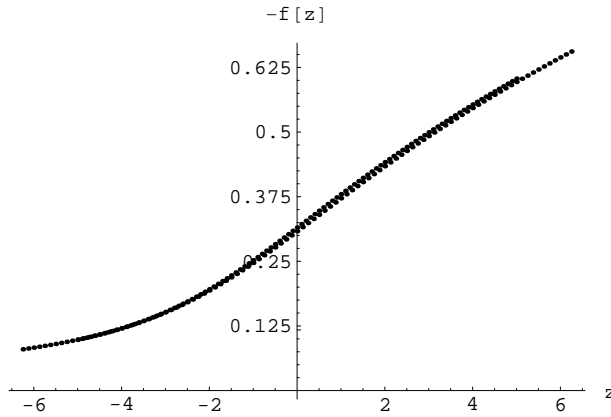


Figure 6: Numerical solution to Painlevé II equation  $f(z)(16f^2(z) - z) - 2f''(z) + 1/2 = 0$  (lower dots). We have used the same parameters as in fig. 3. Explicit values of  $f_n$  for the critical potential with  $\alpha = 1$ ,  $N = 512$  are also plotted (upper dots).

This is also a good point to check that the  $\alpha$ -symmetry (2.30) is satisfied once we take the solution to the generalized Painlevé II equation into account. It follows from eqs. (2.30) and (2.50) that the required relation is (indicating the  $\alpha$ -dependence explicitly by a subscript):

$$f'_{1(\alpha)}(z) + 2f_{1(\alpha)}(z)^2 = -f'_{1(\alpha-1)}(z) + 2f_{1(\alpha-1)}(z)^2. \quad (2.58)$$

This is obviously a highly non-trivial identity which the appropriate solution to the generalized Painlevé II equation (2.50) must satisfy. In the limit  $\alpha \rightarrow \infty$  the simple algebraic solution (2.56) approaches, for fixed  $z$ , the full solution to the generalized Painlevé II equation (2.50). It is straightforward to verify that the algebraic solution (2.56) indeed satisfies the identity (2.58), order by order in a large- $\alpha$  expansion. Moreover, by performing two Miura transformations we can actually prove that the relation (2.58) is exact. The two transformations are

$$M(z) = f'_1(z) + 2f_1(z)^2 \quad (2.59)$$

$$M(z) = -f'_1(z) + 2f_1(z)^2 \quad \text{and shift } \alpha \rightarrow \alpha - 1. \quad (2.60)$$

Under these two different transformations the generalized Painlevé II equation (2.50) turns into the *same* equation:

$$\alpha(1 - \alpha) + 2z^2M(z) - 32zM(z)^2 + 128M(z)^3 - 4M'(z) + 16[M'(z)]^2 + 4zM''(z) - 32M(z)M''(z) = 0. \quad (2.61)$$



The two functions (2.59) and (2.60) thus satisfy the same differential equation. The two boundary conditions are of course inherited from those of  $f_1(z)$ , and one can readily check from eqs. (2.53)-(2.54) that the two functions (2.59) and (2.60) also satisfy the same boundary conditions at  $z \rightarrow \pm\infty$ . The two functions must therefore be identical, and the relation (2.58) has been proven.

Note that, to the order at which we are working, the identity (2.58) implies the following relation among the recursion coefficients (indicating the  $\alpha$ -dependence explicitly by a superscript):

$$c_{2n}^{(\alpha)2} + c_{2n+1}^{(\alpha)2} = c_{2n+1}^{(\alpha-1)2} + c_{2n+2}^{(\alpha-1)2} . \quad (2.62)$$

In this form one easily proves that the differential equation for the wave functions (2.38) respects the  $\alpha$ -symmetry (2.30).

## 2.5 UNIVERSALITY

Up to now we have only been investigating the most simple example,  $m = 1$  multicriticality with the minimal potential. In order to prove universality, one has to show that all perturbations to the  $m = 1$  minimal potential that remain within the  $m = 1$  class lead to the same universal differential equation (2.38). It should be parameterized by only one universal constant,  $\rho^{(2m)}(0)$ , in direct analogy with the noncritical case  $m = 0$  in [7]. Since there is no longer a simple relationship as eq. (2.16) at hand, this is a rather involved task. Still, we have strong evidence that universality holds precisely in the sense just described. In order to illustrate this, we will give as an example a nonminimal  $m = 1$  multicritical potential. We will also briefly outline the situation for  $m \geq 2$  below.

When adding a sextic term to the minimal potential eq. (2.32),

$$V_1(\lambda) = \frac{1}{2}g_2\lambda^2 + \frac{1}{4}g_4\lambda^4 + \frac{1}{6}g_6\lambda^6 , \quad (2.63)$$

the conditions for  $m = 1$  multicriticality and the corresponding eigenvalue density read

$$0 = g_2 + \frac{1}{2}g_4 + \frac{3}{8}g_6 , \quad (2.64)$$

$$1 = \frac{1}{16}(g_4 + g_6) , \quad (2.65)$$

$$\rho(\lambda) = \frac{1}{2\pi} \left( (g_4 + \frac{1}{2}g_6)\lambda^2 + g_6\lambda^4 \right) \sqrt{1 - \lambda^2} .$$

The outcome of this modification is that the calculations of the previous 2 subsections exactly go through when replacing

$$g_4 \rightarrow g_* \equiv g_4 + \frac{1}{2}g_6 , \quad (2.66)$$

or, in other words, it depends only on one single parameter  $\sim \rho''(0)$ .

Without going through the details, where one can use the results for  $m = 2$  from appendix B, we define  $u_{\pm}$  and  $v$  as in eq. (B.5), but with a scaling in powers of  $N$  as in eq. (2.37). The final result reads

$$\begin{aligned} u_{\pm} &= g_*(2f^2(0) \pm f'(0)) , \\ v &= 2g_*f(0) , \end{aligned} \quad (2.67)$$

where the generalized Painlevé II equation is now given by

$$0 = g_* f(z)^3 - z f(z) - g_* \frac{1}{8} f''(z) + \frac{\alpha}{2}. \quad (2.68)$$

Using eq. (2.67) and  $g_*$  instead of  $g_4$  in eq. (2.42) for  $G_N$  and eq. (2.39) for  $F_N$  provides the differential equation (2.38) for the  $m = 1$  nonminimal potential eq. (2.63), where  $g_* = \rho''(0)/\pi$ . Eqs. (2.67) and (2.68) and thus the differential equation (2.38) with (2.42) will hold also for all higher order perturbations of the minimal potential.

The essential ingredient in obtaining universality is thus the fact that we still have the freedom of modifying the potential while remaining in the same universality class. The only quantity that enters in the differential equation is, in the  $m = 1$  case, the second derivative of the macroscopic spectral density at the origin,  $\rho''(0)$ . All other details of the specific  $m = 1$  potential have disappeared. By a simple rescaling of  $\rho''(0)$ , all  $m = 1$  critical potentials should thus lead to the same universal functions.

The way this should generalize to the higher- $m$  case is as follows. An alternative way of characterizing multicritical potentials is by the scaling behavior of the recursion coefficients  $c_n^2$  near the “critical point”, where the two branches of the period-two solution merge. Here, in the usual microscopic limit[21],

$$f_m(z) = - \frac{z^{1/2m}}{(2^{2m+1}(m+1))^{1/2m}}, \quad \text{for } z \rightarrow \infty \quad (2.69)$$

which is the generalization to higher  $m$  of the behavior  $\lim_{z \rightarrow \infty} f_1(z) \sim -\sqrt{z}/4$  we found explicitly in the  $m = 1$  case. This scaling of the recursion coefficients near the point where they merge, cf. eqs. (2.44) and (2.69),

$$c_n^2 - C \sim \pm \text{const.} \left(1 - \frac{n}{N}\right)^{\frac{1}{2m}} \quad (2.70)$$

is *universal* in the sense that its power-law scaling depends only on the class  $m$  of multicriticality (the positive solution corresponds to odd  $n$ , while the negative one corresponds to even  $n$ ). The coefficients  $C = \bar{C}$  of eq. (2.44) will change with the chosen  $m$ th multicritical potential, and so will the particular normalization of (2.69), but the scaling (2.70) will remain. The non-universal coefficients  $C = \bar{C}$  drop out of the differential equation for the orthogonal polynomials simply by the requirement of  $m$ th order multicriticality. The next term of the expansion (2.51) is what will enter in the differential equation. Our statements about universality above translate into saying that for all  $m$  this particular combination of solutions to the generalized Painlevé II equations for  $f_m(z)$  (and higher functions of the expansion (2.44)) will be proportional to  $\rho^{(2m)}(0)$ . We have not proven this.

## 2.6 A DOUBLE-SCALING LIMIT

As we have shown above, apparently non-leading terms from the coefficients  $c_n$  turn out to contribute in the multicritical microscopic limit. At the core of this phenomenon stands the breakdown in this limit of the simple relation (2.16) between the function  $A_n$  (which enters directly into the differential equation for the polynomials), and the macroscopic spectral density  $\rho(\lambda)$ . The way this relation breaks down at multicriticality makes it clear that the phenomenon is more general, and directly related to the very definition of multicriticality. So far we have used the most naive definition, where we first tune the couplings of the potential  $V(\lambda)$  to their values at the  $N = \infty$  multicriticality, and *then* take the  $N \rightarrow \infty$  limit in the differential equation for the polynomials. What we implicitly have shown above is that the final result will depend on this precise procedure. This should come as no surprise,

since we in contrast to the conventional case of a finite  $\rho(0)$  really have to approach a critical point at some particular critical coupling(s). The whole analysis is then quite analogous to finite-size scaling in statistical mechanics [25]. Finite  $N$  here plays the rôle of finite volume  $V$ , and as in statistical mechanics the notion of a (multi)critical point at finite  $N$  is ambiguous. One can define it to be precisely at the position of the infinite- $N$  (multi)critical point, or one can choose other definitions (at the peak of the specific heat, or otherwise) that all agree in the  $N = \infty$  limit.

In order to remedy this situation, we clearly have to consider more general ways of reaching multicriticality. Instead of starting out with the couplings fixed precisely at the values corresponding to  $N = \infty$  multicriticality, we can consider a continuum of different schemes in which the couplings are tuned towards the multicritical values at just such a rate as to contribute non-trivial terms to the differential equation in the microscopic  $N \rightarrow \infty$  limit. Because  $m$  couplings must be tuned to reach  $m$ th-order multicriticality, this program is however not very practical from an analytical point of view. A much more economical procedure is to tune only one parameter, which in a generic manner takes us away from and toward the multicritical points. Following the conventions of matrix models applied to 2-d quantum gravity, we shall call this a double-scaling limit. It is, in a very precise sense, equivalent to finite-size scaling theory in statistical mechanics [26].

Suppose, then, that we have chosen our potential  $V(\lambda)$  to be multicritical according to eq. (2.19) for some given  $M$ . Consider now the partition function defined by

$$\mathcal{Z} = \int dM \det^{2\alpha} M e^{-(N/g) \text{Tr} V(M)} . \quad (2.71)$$

When  $g = 1$  this coincides with our previous definition, eq. (2.1). We can tune  $g$  towards  $g_c = 1$  in such a way that

$$y \equiv N^{2m\nu} (g - 1) \quad (2.72)$$

is kept fixed. This defines our double-scaling limit. The previous analysis was simply restricted to  $y = 0$ . Now all solutions will be parameterized by (functions of) this number  $y$ , which is kept fixed in the large- $N$  limit. The appropriate scaling variable is then

$$z \equiv N^{2m\nu} \left( 1 - \frac{2n}{N} g \right) \quad (2.73)$$

instead of eq. (2.45). This is completely analogous to the treatment of multicritical limits at the edge of the spectrum by Bowick and Brezin [27]. We can now probe a double-scaling region where  $g$  differs from unity only by an amount of the order  $N^{-2m\nu}$ , while the eigenvalues  $\lambda$  differ from zero by an amount of the order  $N^{-\nu}$ . Essentially all of the previous discussion goes through unchanged in this more general setting, with just some minor redefinitions. It follows from the above that the sole effect of including a non-vanishing  $y$  is to shift the scaling variable by  $z \rightarrow z + y$ . Accordingly, all of the previous discussion goes through unchanged in this more general setting, except that we should substitute the numerical values of the Painlevé II transcendent at  $z = y$  instead of  $z = 0$  into the linear differential equation (2.38).

## 2.7 A MULTICRITICAL MESOSCOPIC RANGE

It is of some interest to compare the correct solution for the wave functions  $\psi_N(x)$  with what we would have obtained if we had ignored the apparently subleading terms in the defining differential equation (2.38). Consider again for simplicity the case  $m = 1$  and  $\alpha = 0$ , for which we can compare the two solutions visually (see fig.2). It is clear that although the two solutions differ in detail, especially very

close to the origin, their qualitative features are nevertheless quite similar. This has a simple algebraic explanation. The coefficient of the leading power of  $A_N(\lambda)$  is proportional to  $k(m)$ , which can be seen from eq. (2.19) and the definition (2.9). Already for  $m = 1, 2$   $k(m)$  is large compared to one, a fact which is improving for growing  $m$ :

$$1 \ll k(m) \longrightarrow 4\sqrt{2\pi}m^{\frac{3}{2}} \quad \text{for} \quad m \rightarrow \infty . \quad (2.74)$$

If now the leading power of  $A_N(\lambda)$  is dominating all the lower terms, we will get back the relation (2.16) to the eigenvalue density as in the noncritical case and will therefore recover the “naive” differential equation from (2.28). The appropriate condition for  $x$  in this mesoscopic limit is

$$1 \ll k(m) x^{2m} , \quad (2.75)$$

with the simplified differential equation reading

$$\psi_N''(x) - \frac{2m}{x}\psi_N'(x) + \left( c_N^2 k(m) x^{4m} + \frac{(-1)^N \alpha(2m+1) - \alpha^2}{x^2} \right) \psi_N(x) = 0 . \quad (2.76)$$

This mesoscopic differential equation can be solved analytically in terms of Bessel functions. Note that since it is an *approximate* equation it is no longer  $\alpha$ -symmetric. At growing multicriticality  $m$  it will approximate the true differential equation (B.9) for smaller and smaller values of  $x$ , due to the behavior of  $k(m)$  eq. (2.74).

As it has been mentioned eq. (2.76) can be mapped to the following equation of Bessel type

$$\left( X^{-\beta} \frac{d}{dX} X^\beta \frac{d}{dX} + 1 \right) X^{-\frac{\beta}{2}} \psi_N(X) = 0 , \quad (2.77)$$

where

$$X = \frac{k(m)}{2(2m+1)} x^{2m+1} = \frac{\pi \rho^{(2m)}(0)}{(2m+1)!} x^{2m+1} , \quad \beta = \frac{2(-1)^N \alpha}{2m+1} . \quad (2.78)$$

Boundary conditions are however not easily established. The solution satisfying the usual boundary conditions, namely regularity at the origin and normalizability, reads

$$\psi_N(x) \sim \sqrt{X} J_{\frac{\alpha}{2m+1} - \frac{(-1)^N}{2}}(X) , \quad (2.79)$$

which still has to be normalized. (Since the differential equation cannot be trusted for small values of  $x$ , it is not obvious that the above boundary conditions are suitable; we shall for illustration stick to them here.) The approximate kernel and the approximate spectral density are then given by

$$\begin{aligned} K_{meso}(x, x') &= \lim_{N \rightarrow \infty} N^{-\frac{1}{2m+1}} K_N \left( N^{-\frac{1}{2m+1}} x, N^{-\frac{1}{2m+1}} x' \right) \\ &= \frac{\pi \rho^{(2m)}(0)}{2(2m+1)!} \frac{(xx')^{m+\frac{1}{2}}}{(x-x')} \left( J_{\frac{\alpha}{2m+1} + \frac{1}{2}}(X) J_{\frac{\alpha}{2m+1} - \frac{1}{2}}(X') - J_{\frac{\alpha}{2m+1} - \frac{1}{2}}(X) J_{\frac{\alpha}{2m+1} + \frac{1}{2}}(X') \right) , \end{aligned} \quad (2.80)$$

and

$$\rho_{meso}(x) = \left( \frac{\pi \rho^{(2m)}(0)}{2(2m)!} \right)^2 \frac{x^{4m+1}}{2m+1} \times \left( J_{\frac{\alpha}{2m+1} + \frac{1}{2}}(X)^2 + J_{\frac{\alpha}{2m+1} - \frac{1}{2}}(X)^2 - J_{\frac{\alpha}{2m+1} + \frac{1}{2}}(X) J_{\frac{\alpha}{2m+1} - \frac{3}{2}}(X) - J_{\frac{\alpha}{2m+1} - \frac{1}{2}}(X) J_{\frac{\alpha}{2m+1} + \frac{3}{2}}(X) \right). \quad (2.81)$$

Here the normalization has been fixed by requiring the matching condition

$$\lim_{x \rightarrow \infty} \rho_{meso}^{(2m)}(x) = \rho^{(2m)}(0) = (2m)! \frac{k(m)}{2\pi}. \quad (2.82)$$

The above ‘‘mesoscopic’’ spectral density of course has only approximate validity. For  $\alpha=0$  it simply coincides with the macroscopic spectral density near the origin (a feature it shares with the usual non-critical microscopic spectral density (1.2)), and is in this sense ‘‘exact’’. For  $\alpha/(2m+1)$  non-zero, but small, it does define an intermediate range where the set in of the universal mesoscopic oscillations is correctly encoded, and the macroscopic spectral density still not reached. To give an example of the approximation achieved, we show below, for  $\alpha=1$  and  $m=1$ , the full microscopic spectral density (fig. 7) and the approximate mesoscopic spectral density (fig. 8). The full microscopic spectral density has been computed using the method described above, where, from the solution to the Painlevé II equation, we find  $u_+ = 2.141$ ,  $u_- = 4.218$  and  $v = -10.087$ .

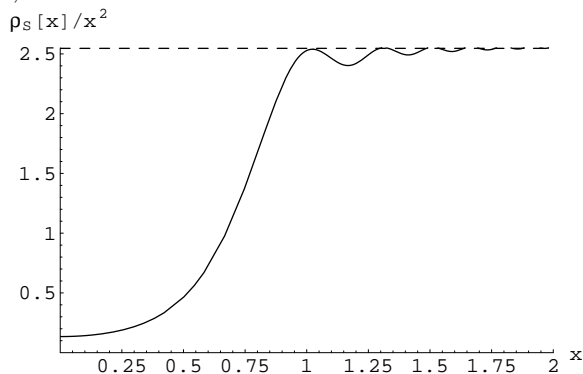


Figure 7: Microscopic spectral density for  $m=1$  criticality with  $\alpha=1$ .

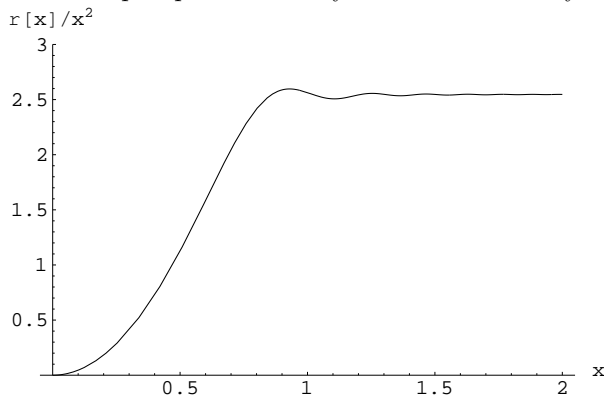


Figure 8: The approximate mesoscopic spectral density  $\rho_{meso}(x)/x^2$  for  $m=1$ ,  $\alpha=1$ .

For  $m=0$  ( $k(0) = 4$ ) the mesoscopic density reduces to the exact, noncritical *microscopic* density [7].

### 3 Multicritical Chiral Unitary Ensembles

Once the above results have been established, it is only a small step to extend them from the unitary ensemble to the chiral unitary ensemble. This is due to the fact that the partition functions, and their associated orthogonal polynomials, for the two different ensembles can be related to each other through appropriate shifts of the arguments. Let us recall [1] that the pertinent matrix integral for theories in even dimensions, gauge groups  $SU(N \geq 3)$ , and fermions in the fundamental representations of these gauge groups, is described by complex matrices [9]:

$$\mathcal{Z} = \int dM \det^{N_f} M \exp \left[ -\frac{N}{2} \text{Tr} \tilde{V}(M^2) \right], \quad (3.1)$$

with an even potential

$$\tilde{V}(M^2) = \sum \frac{\tilde{g}_k}{k} M^{2k}. \quad (3.2)$$

Here  $M$  is a  $(2N+|\nu|) \times (2N+|\nu|)$  block hermitian matrix of the form

$$M = \begin{pmatrix} 0 & W^\dagger \\ W & 0 \end{pmatrix} \quad (3.3)$$

where  $W$  itself is a rectangular complex matrix of size  $N \times (N+|\nu|)$ . The translation of parameters into gauge theory language is that  $\nu$  represents the topological charge, and the space-time volume equals  $2N+|\nu|$ , the size of the matrix  $M$ . The integration measure in eq. (3.1) is the Haar measure of  $W$ . This measure and the integrand of eq. (3.1) both are invariant under left and right multiplications:

$$W \rightarrow TWU, \quad U \in U(N), \quad T \in U(N+|\nu|). \quad (3.4)$$

It is convenient to rewrite the matrix integral (3.1) in terms of the eigenvalues of the hermitian matrix  $H \equiv W^\dagger W$  [9]:

$$\begin{aligned} \mathcal{Z} &= \int dW dW^\dagger \det^{N_f}(W^\dagger W) \exp \left[ -N \text{Tr} \tilde{V}(W^\dagger W) \right], \\ &\sim \int_0^\infty \prod_{i=1}^N \left( d\lambda_i \lambda_i^\alpha e^{-N\tilde{V}(\lambda_i)} \right) \left| \det_{ij} \lambda_j^{i-1} \right|^2, \end{aligned} \quad (3.5)$$

$$\sim \int_{-\infty}^\infty \prod_{i=1}^N \left( dz_i |z_i|^{2\alpha+1} e^{-N\tilde{V}(z_i^2)} \right) \left| \det_{ij} z_j^{2i-2} \right|^2, \quad (3.6)$$

where  $\alpha \equiv N_f + |\nu|$ . Here the fact has been used, that one can switch from real positive eigenvalues  $\lambda_i$  to real eigenvalues  $z_i$  [9]. In the latter form, the matrix integral can be analyzed directly in terms of the unitary ensemble considered in the previous section.

Let us introduce a set of even polynomials  $\tilde{P}_l^{(\alpha)}(z^2)$  orthonormal with respect to the measure  $d\omega(z) \equiv dz |z|^{2\alpha+1} \exp(-NV(z^2))$  on the real line. Then we can replace the Vandermonde determinant as usual, but this time by a determinant of only even polynomials  $\tilde{P}_l^{(\alpha)}(z^2)$ . In doing so we can directly use the even subset<sup>13</sup> of the orthonormal polynomials eq. (2.3), which we have already calculated, by shifting  $\alpha \rightarrow \alpha + \frac{1}{2}$ :

$$\tilde{P}_l^{(\alpha)}(z^2) \equiv P_{2l}^{(\alpha+\frac{1}{2})}(z). \quad (3.7)$$

---

<sup>13</sup>We will also encounter the odd polynomials when calculating the kernel via the Christoffel-Darboux formula.

In going from eigenvalues  $\lambda$  to  $z$  in eq. (3.6) we have an even potential  $V(z^2)$  as we should, in contrast to the real positive picture (3.5). In order to justify the identification (3.7) also in the multicritical case we still need a relation between the respective critical potentials. It is given by [28]

$$\tilde{V}_m(\lambda^2) = 2V_m(\lambda) , \quad (3.8)$$

The critical potential for the chiral unitary ensemble has been calculated by imposing

$$\rho_m(z) = \frac{1}{4\pi} k(m) z^{2m} \sqrt{1-z^2} . \quad (3.9)$$

for the eigenvalue density on  $\mathbb{R}$ , exactly as in the unitary case eq. (2.19). The factor of two in eq. (3.8) can be traced back to the difference in the saddle point equations of the two ensembles [28]. Consequently the wavefunctions of the chiral unitary ensemble at  $m$ -th multicriticality can be taken from the unitary ones eq. (2.4) simply by shifting  $\alpha \rightarrow \alpha + \frac{1}{2}$ :

$$\begin{aligned} \tilde{\psi}_l^{(\alpha)}(z^2) &\equiv |z|^{\alpha+\frac{1}{2}} e^{-\frac{N}{2}\tilde{V}_m(z^2)} \tilde{P}_l^{(\alpha)}(z^2) \\ &= |z|^{\alpha+\frac{1}{2}} e^{-NV_m(z)} P_{2l}^{(\alpha+\frac{1}{2})}(z) \\ &= \psi_{2l}^{(\alpha+\frac{1}{2})}(z) , \end{aligned} \quad (3.10)$$

where we have identified the couplings  $\tilde{g}_k \equiv g_k$  according to eq. (3.8). The wavefunctions of the unitary ensemble  $\psi_{2l}^{(\alpha+\frac{1}{2})}(z)$  now have to be evaluated at  $2N$  instead of  $N$ . They are determined by the corresponding universal differential equation as being discussed in great detail in the previous section.

The eigenvalue density in the microscopic limit is obtained from the kernel expressed in terms of even and odd wave functions  $\psi_k^{(\alpha+\frac{1}{2})}(z)$ :

$$\begin{aligned} K_N(z^2, w^2) &\equiv |zw|^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}N(\tilde{V}_m(z^2)+\tilde{V}_m(w^2))} \sum_{l=0}^{N-1} \tilde{P}_l^{(\alpha)}(z^2) \tilde{P}_l^{(\alpha)}(w^2) \\ &= \sum_{l=0}^{N-1} \psi_{2l}^{(\alpha+\frac{1}{2})}(z) \psi_{2l}^{(\alpha+\frac{1}{2})}(w) \\ &= c_{2N} c_{2N-1} \frac{1}{z^2 - w^2} \left( \psi_{2N}^{(\alpha+\frac{1}{2})}(z) \psi_{2N-2}^{(\alpha+\frac{1}{2})}(w) - \psi_{2N-2}^{(\alpha+\frac{1}{2})}(z) \psi_{2N}^{(\alpha+\frac{1}{2})}(w) \right) \\ &= c_{2N} \frac{1}{z^2 - w^2} \left( \psi_{2N}^{(\alpha+\frac{1}{2})}(z) w \psi_{2N-1}^{(\alpha+\frac{1}{2})}(w) - z \psi_{2N-1}^{(\alpha+\frac{1}{2})}(z) \psi_{2N}^{(\alpha+\frac{1}{2})}(w) \right) . \end{aligned} \quad (3.11)$$

Here we have made use of the Christoffel-Darboux identity and the recursion relation for the wave functions. The advantage of the last expression is that we have avoided to expand  $\psi_{2N-2}^{(\alpha+\frac{1}{2})}(x)$  in  $1/N$ . Imposing the same scaling for the real eigenvalues  $z$  as in eq. (2.31),

$$x = N^{\frac{1}{2m+1}} z \quad (3.12)$$

we obtain

$$\begin{aligned} \rho_s(x) &= \lim_{y \rightarrow x} \lim_{N \rightarrow \infty} N^{-\frac{1}{2m+1}} K_N \left( x^2 N^{-\frac{2}{2m+1}}, y^2 N^{-\frac{2}{2m+1}} \right) \\ &= \lim_{N \rightarrow \infty} \frac{c_{2N}}{2} \left( \psi_{2N}^{(\alpha+\frac{1}{2})}(x)' \psi_{2N-1}^{(\alpha+\frac{1}{2})}(x) - \psi_{2N}^{(\alpha+\frac{1}{2})}(x) \psi_{2N-1}^{(\alpha+\frac{1}{2})}(x)' - \frac{1}{x} \psi_{2N}^{(\alpha+\frac{1}{2})}(x) \psi_{2N-1}^{(\alpha+\frac{1}{2})}(x) \right) . \end{aligned} \quad (3.13)$$

### 3.1 THE MULTICRITICAL MESOSCOPIC RANGE

Since in the unitary case we could obtain analytic expressions for the wave functions in an intermediate, mesoscopic range we can do the same here, using the results of subsection 2.7. The range of applicability is thus given again by eq. (2.75). The unnormalized, analytic expressions for the mesoscopic wave functions eq. (2.79), which are needed here read (see our discussion about boundary conditions in the previous section):

$$\begin{aligned}\psi_{2N}^{(\alpha+\frac{1}{2})}(x) &\sim \sqrt{X} J_{\beta-\frac{1}{2}}(X), \\ \psi_{2N-1}^{(\alpha+\frac{1}{2})}(x) &\sim \sqrt{X} J_{\beta+\frac{1}{2}}(X),\end{aligned}\quad (3.14)$$

where

$$X \equiv \frac{k(m)}{2(2m+1)} x^{2m+1} = 2\pi\rho^{(2m)}(0) \frac{x^{2m+1}}{(2m+1)!}, \quad \beta \equiv \frac{2\alpha+1}{2(2m+1)}. \quad (3.15)$$

Inserting these expressions into eqs. (3.11) and (3.14) we obtain the following expressions for the approximate mesoscopic kernel and spectral density

$$\begin{aligned}K_{meso}(x, x') &\equiv \lim_{N \rightarrow \infty} N^{-\frac{1}{2m+1}} K_N \left( x^2 N^{-\frac{2}{2m+1}}, x'^2 N^{-\frac{2}{2m+1}} \right) \\ &= \frac{2\pi\rho^{(2m)}(0)}{(2m+1)!} \frac{(xx')^{m+\frac{1}{2}}}{x^2 - x'^2} \left( x J_{\beta-\frac{1}{2}}(X') J_{\beta+\frac{1}{2}}(X) - x' J_{\beta-\frac{1}{2}}(X) J_{\beta+\frac{1}{2}}(X') \right)\end{aligned}\quad (3.16)$$

and

$$\rho_{meso}(x) = \frac{\pi\rho^{(2m)}(0)x^{2m}}{2(2m)!} \left( X \left( J_{\beta+\frac{1}{2}}(X)^2 + J_{\beta-\frac{1}{2}}(X)^2 \right) - \frac{2\alpha}{2m+1} J_{\beta+\frac{1}{2}}(X) J_{\beta-\frac{1}{2}}(X) \right). \quad (3.17)$$

The normalization has been fixed by requiring again

$$\lim_{x \rightarrow \infty} \rho_{meso}^{(2m)}(x) = \rho^{(2m)}(0) = (2m)! \frac{k(m)}{4\pi}. \quad (3.18)$$

For  $m = 0$  the above mesoscopic density (3.17) reduces to the exact noncritical *microscopic* density [7].

In fig. 9 we exhibit the exact microscopic density (3.14) for  $m = 1$  and  $\alpha = 0$ , constructed by solving the Painlevé II equation for  $\alpha = 1/2$ , and the mesoscopic approximation to it eq. (3.17).

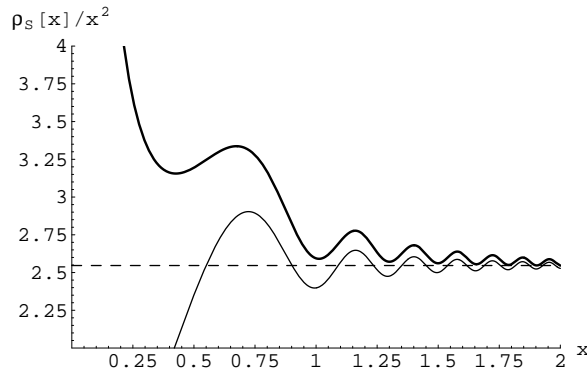


Figure 9: Microscopic spectral density for the  $m = 1$  critical chiral unitary ensemble with  $\alpha = 0$  (bold line) and its mesoscopic approximation (real line). The normalization is determined by requiring  $\lim_{x \rightarrow \infty} \rho_S''(x) = \rho''(0) = 8/\pi$  (dotted line).



## 4 Conclusions

In this paper we have investigated random matrix models which may be related to QCD-like gauge theories with unbroken chiral or flavor symmetries. From the Banks-Casher relation [3] the vanishing of the corresponding order parameter  $\langle \bar{\psi}\psi \rangle$  implies a vanishing spectral density of the Dirac operator at the origin. The case where this density vanishes as an even power can be mapped to the multicritical points in the appropriate matrix models if one follows the same arguments as in [1] for the broken phase.

The new universal results we find for multicritical matrix models in the microscopic large- $N$  limit may suggest a universal behavior for the microscopic density of the Dirac operator itself. In contrast to the case of broken symmetries, there are here no known spectral sum rules of these theories with which to compare the results from random matrix theory. The relevance of these results for the Dirac spectrum of theories with unbroken chiral/flavor symmetries is therefore still far from obvious, and we can at present only view it as an intriguing conjecture. It would be particularly interesting to investigate if an approach analogous to that of a chiral Lagrangian [8] exists in this case.

The detailed universality predictions, the listing of universality classes, and the “critical exponents”  $\nu = 1/(2m + 1)$  can all be tested in lattice gauge theory with an appropriate fermion prescription. In contrast to the noncritical case [7], the universal results for the microscopic spectral densities are no longer given in terms of elementary functions. Instead, we derive universal differential equations for the orthogonal polynomials of the multicritical models, which then yield the spectral density from its kernel. For the practical purpose of comparison with lattice gauge theory data, the generalization of the present formulation to that of massive fermions may be highly advantageous. Recently the massive case for non-critical potentials was worked out in detail for the chiral unitary ensemble and proven to be universal [29] in the appropriate scaling limit (see also ref. [30] for an explicit computation in the Gaussian case). The same universality of the massive case extends also to unitary ensemble [31]. The multicritical cases for both ensembles with non-zero masses can be worked out completely analogously using the wave functions derived in this paper. The universality arguments also generalize accordingly.

We have discussed here in great detail the solution for first order multicriticality with an arbitrary number of massless fermions, giving evidence for the universality of the determining differential equation and thus for the microscopic spectral density. For higher order criticality we have derived the general form of the differential equation and conjectured its universality. With growing multicriticality  $m$  the analysis becomes increasingly involved, but all the necessary ingredients for carrying the computations through for arbitrary  $m$  have been presented here. Moreover, in a certain mesoscopic large- $N$  limit we have been able to give approximate, analytic expressions for the universality classes of the orthogonal polynomials as well as the microscopic density for any order of multicriticality and any number of fermions.

In the noncritical case, the relevant scale that determines the microscopic limit is the spectral density at the origin,  $\rho(0)$ . All eigenvalues are in that case blown up near  $\lambda=0$  by the scale  $\rho(0)N$ . As could have been guessed from the outset, in the  $m$ th multicritical case the relevant scale is set by the first non-vanishing derivative of the macroscopic spectral density at the origin,  $\rho^{(2m)}(0)$ . All eigenvalues must be blown up by a scale determined by this number. In addition, and this is what requires a more detailed analysis, we have seen that the eigenvalues must be blown up with the volume  $N$  raised to the exponent  $\nu$ . The noncritical case is simply that of  $m=0$ . Viewed from this perspective, the universality classes derived in this paper are the completely natural generalizations of those derived

previously [1, 2]. Universality simply translates into the freedom of redefining only one scale,  $\rho^{(2m)}(0)$ , in order to leave all results invariant.

One might think that the most obvious application of the universal results described in this paper would be in connection with chiral symmetry restoration at finite temperature or density. Because one parameter needs to be tuned in each case (temperature  $T$ , or chemical potential  $\mu$ ), one could believe that these two cases, when occurring at continuous phase transitions, would be good candidates for the  $m=1$  universality class. We are mildly skeptical about this, and have instead, in the introduction, suggested what seem to us to be more promising candidates. There has already been a large amount of work on possible extensions of the by now essentially established  $T=0$  results to the  $T \neq 0$  and  $\mu \neq 0$  situations (see, *e.g.*, ref. [32] for just a limited selection of papers, and also others cited in ref. [33]).

## Acknowledgments

We thank T.R. Morris for discussions at an early stage of this work. The work of G.A. is supported by European Community grant no. ERBFMBICT960997. Furthermore he wishes to thank the Niels Bohr Institute for its warm hospitality during several stays. The work of S.M.N. is supported in part by JSPS Postdoctoral Fellowships for Research Abroad, by the Nishina Memorial Foundation, and by NSF Grant PHY94-07194.

## A A symmetry among the orthonormal polynomials

In this appendix we prove a lemma relating polynomials orthonormal with respect to measures  $d\lambda w(\lambda)$  and  $d\lambda \lambda^2 w(\lambda)$ . It follows from a more general statement in the book of Szegö ([34], theorem 3.1.4). We have referred to this relation as the  $\alpha$ -symmetry in the main text.

LEMMA: Let  $P_n^{(\alpha)}(\lambda)$  denote the polynomials orthonormal w.r.t. the measure

$$d\lambda w^{(\alpha)}(\lambda) \equiv d\lambda |\lambda|^{2\alpha} \exp(-NV(\lambda)) \quad , \quad \alpha > -\frac{1}{2} \quad , \quad \lambda \in \mathbf{R} \quad , \quad (\text{A.1})$$

where  $V(\lambda^2)$  is an even polynomial with positive highest coefficient. The following identity then holds:

$$\frac{P_{2n+1}^{(\alpha)}(\lambda)}{\lambda} = P_{2n}^{(\alpha+1)}(\lambda) \quad . \quad (\text{A.2})$$

PROOF: Since  $\tilde{P}_{2n}^{(\alpha+1)}(\lambda) \equiv \frac{P_{2n+1}^{(\alpha)}(\lambda)}{\lambda}$  is an even polynomial of degree  $2n$  it is orthonormal to all odd polynomials w.r.t.  $d\lambda w^{(\alpha+1)}(\lambda)$ . Because of

$$\begin{aligned} \int_{-\infty}^{\infty} d\lambda w^{(\alpha+1)}(\lambda) \tilde{P}_{2m}^{(\alpha+1)}(\lambda) \tilde{P}_{2n}^{(\alpha+1)}(\lambda) &= \int_{-\infty}^{\infty} d\lambda w^{(\alpha+1)}(\lambda) \frac{P_{2m+1}^{(\alpha)}(\lambda) P_{2n+1}^{(\alpha)}(\lambda)}{\lambda^2} \\ &= \int_{-\infty}^{\infty} d\lambda w^{(\alpha)}(\lambda) P_{2m+1}^{(\alpha)}(\lambda) P_{2n+1}^{(\alpha)}(\lambda) \\ &= \delta_{mn} \end{aligned} \quad (\text{A.3})$$

one has

$$\tilde{P}_{2n}^{(\alpha+1)}(\lambda) = P_{2n}^{(\alpha+1)}(\lambda) \quad . \quad (\text{A.4})$$

Inserting the above result into the definition (2.4) of the wave functions  $\psi_n(\lambda)$  immediately leads to the  $\alpha$ -symmetry as stated in eq. (2.30).

## B The $m \geq 2$ multicritical case

In this appendix we give the explicit result for the asymptotic form of the differential equation (2.13) for  $m = 2$ , and conjecture the form for general  $m$ . The  $m = 2$  minimal multicritical potential reads

$$V_2(\lambda) = \frac{1}{2}g_2\lambda^2 + \frac{1}{4}g_4\lambda^4 + \frac{1}{6}g_6\lambda^6, \quad g_2 = -4, \quad g_4 = -16, \quad g_6 = 32, \quad (\text{B.1})$$

with

$$\rho_2(\lambda) = \frac{1}{2\pi}g_6\lambda^4\sqrt{1-\lambda^2}. \quad (\text{B.2})$$

Performing the calculations exactly along the same lines as in subsection 2.3 we get the following exact expressions

$$\begin{aligned} A_n(\lambda) &= Nc_n \left( g_2 + g_4(c_{n+1}^2 + c_n^2) + g_6(c_{n+1}^2(c_{n+2}^2 + c_{n+1}^2 + c_n^2) + c_n^2(c_{n+1}^2 + c_n^2 + c_{n-1}^2)) \right) \\ &\quad + Nc_n \left( g_4 + g_6(c_{n+1}^2 + c_n^2) \right) \lambda^2 + Nc_n g_6 \lambda^4 \\ B_n(\lambda) &= Nc_n^2 \left( g_4 + g_6(c_{n+1}^2 + c_n^2 + c_{n-1}^2) \right) \lambda + Nc_n^2 g_6 \lambda^3 + (1 - (-1)^n) \frac{\alpha}{\lambda} \end{aligned} \quad (\text{B.3})$$

Inserting them into the exact form for  $G_n(\lambda)$  eq. (2.15) and rescaling variables according to

$$x = N^{\frac{1}{5}}\lambda, \quad (\text{B.4})$$

we expect that the following non-trivial contributions to be of order 1:

$$\begin{aligned} u_{\pm} &\equiv N^{\frac{4}{5}} \left( g_2 + g_4(c_{N\pm 1}^2 + c_N^2) + g_6(c_{N\pm 1}^2(c_{N\pm 2}^2 + c_{N\pm 1}^2 + c_N^2) + c_N^2(c_{N+1}^2 + c_N^2 + c_{N-1}^2)) \right) \\ v &\equiv N^{\frac{3}{5}} \left( g_2 + 2g_4c_N^2 + 2g_6c_N^2(c_{N+1}^2 + c_N^2 + c_{N-1}^2) \right) \\ q_{\pm} &\equiv N^{\frac{2}{5}} \left( g_4 + g_6(c_{N\pm 1}^2 + c_N^2) \right) \\ p &\equiv N^{\frac{1}{5}} \left( g_4 + 2g_6c_N^2 \right) \end{aligned} \quad (\text{B.5})$$

This can be checked explicitly from the string equation for  $V_2(\lambda)$  when inserting the ansatz (2.44) for  $m = 2$ . We do not display the differential equations for the scaling functions  $f_2(z)$ ,  $g_2(z)$ ,  $h_2(z)$  etc. here. Finally the asymptotic expression for the differential equation of the wave functions reads in the microscopic large- $N$  limit

$$\psi_N''(x) - N^{-1/5}F_N(x)\psi_N'(x) + N^{-2/5}G_N(x)\psi_N(x) = 0, \quad (\text{B.6})$$

with

$$\begin{aligned} N^{-1/5}F_N(x) &= \frac{2q_+x + 4g_6x^3}{u_+ + q_+x^2 + g_6x^4} \\ N^{-2/5}G_N(x) &= c_N^2u_+u_- + ((-1)^N\alpha - \frac{3}{2})v + \left( c_N^2(u_+q_- + u_-q_+) - \frac{v^2}{4} + ((-1)^N\alpha - \frac{1}{2})p \right) x^2 \\ &\quad + \frac{(2u_+ + q_+x^2)(v + px^2) + (-1)^N 2\alpha(q_+ + 2g_6x^2)}{u_+ + q_+x^2 + g_6x^4} + c_N^2g_6^2x^8 + \frac{(-1)^N\alpha - \alpha^2}{x^2} \end{aligned} \quad (\text{B.7})$$

where only the dominant contributions are displayed. Moreover we have used that the quartic and sextic terms in  $G_N(x)$  are subdominant due to the following identities, which are valid up to subleading terms:

$$\begin{aligned} 0 &= 2c_N^2 g_6(u_+ + u_-) + 2c_N^2 q_+ q_- - pv \\ 0 &= 4c_N^2 g_6(q_+ + q_-) - p^2 . \end{aligned} \quad (\text{B.8})$$

The former follows directly from the string equation (2.7) whereas the latter can be seen to be valid when expressing it in terms of  $v$ .

For the general case of  $m$ th multicriticality we obtain the following form:

$$\psi_N''(x) - N^{-\frac{1}{2m+1}} F_N(x) \psi_N'(x) + N^{-\frac{2}{2m+1}} G_N(x) \psi_N(x) = 0 . \quad (\text{B.9})$$

The  $F_N$ -term does not simplify:

$$N^{-\frac{1}{2m+1}} F_N(x) = \frac{N^{-\frac{2}{2m+1}} A_N'(x)}{N^{-\frac{1}{2m+1}} A_N(x)} , \quad (\text{B.10})$$

where  $N^{-\frac{1}{2m+1}} A_N(x)$  is a polynomial in  $x = N^{\frac{1}{2m+1}} \lambda$  of the order  $2m$ . The  $G_N$ -term which we conjecture to be of the following form will simplify considerably:

$$N^{-\frac{2}{2m+1}} G_N(x) = Q(x) + \frac{R(x)}{N^{-\frac{1}{2m+1}} A_N(x)} + c_N^2 g_{2m+2} x^{4m} + \frac{(-1)^N \alpha - \alpha^2}{x^2} . \quad (\text{B.11})$$

Here  $Q(x)$  and  $R(x)$  are both polynomials of order  $2m - 2$  in  $x$ . They depend on certain universal combinations of coupling constants and recursion coefficients, as it has been explicitly shown in the two examples for  $m = 1$  and  $m = 2$  in eq. (2.42) and (B.7). The latter can be brought into the conjectured form when dividing out the quartic term in the denominator.

## References

- [1] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306.  
J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852.  
J.J.M. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531; Phys. Lett. **B329** (1994) 351; Nucl. Phys. **B426** (1994) 559.
- [2] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73** (1994) 2288.
- [3] T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
- [4] S. Hands, A. Kocic and J. Kogut, Nucl. Phys. **B390** (1993) 355.
- [5] R.D. Pisarski, Phys. Rev. **D29** (1984) 2423.  
T. Appelquist, M.J. Bowick, E. Cohler and L.C.R. Wijewardhana, Phys. Rev. Lett. **55** (1985) 1715.  
M.C. Diamantini, P. Sodano and G.W. Semenoff, Phys. Rev. Lett. **70** (1993) 3848.
- [6] T. Nagao and K. Slevin, J. Math. Phys. **34** (1993) 2075.
- [7] G. Akemann, P.H. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B487** (1997) 721.

- [8] H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.
- [9] T.R. Morris, Nucl. Phys. **B356** (1991) 703.
- [10] J.B. Kogut, J. Polonyi, H.W. Wyld and D.K. Sinclair, Phys. Rev. Lett. **54** (1985) 1475.  
J.B. Kogut and D.K. Sinclair, Nucl. Phys. **B295** (1988) 465.  
F. Brown *et al.*, Phys. Rev. **D46** (1992) 6555.  
P.H. Damgaard, U.M. Heller, A. Krasnitz and P. Olesen, Phys. Lett. **B400** (1997) 169.
- [11] G. Akemann and J. Ambjørn, J. Phys **A29** (1996) L555.  
G. Akemann, Nucl. Phys. **B482** (1996) 403; hep-th/9702005 (Nucl. Phys. **B** in press).
- [12] M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, hep-lat/9704018.
- [13] E. Kanzieper and V. Freilikher, Phys. Rev. Lett. **78** (1997) 3806.
- [14] E. Kanzieper and V. Freilikher, cond-mat/9704149.
- [15] J. Shohat, C. Rend. Acad. Sci. Paris, Ser. I Math. **191** (1930) 989; Duke Math. J. **5** (1939) 401.  
S.S. Bonan and D.S. Clark, J. Appr. Theory **46** (1986) 408; *ibid.* **63** (1990) 210.
- [16] C.A. Tracy and H. Widom, Commun. Math. Phys. **163** (1994) 33.
- [17] S. M. Nishigaki, talk presented at the NATO Workshop “*New Developments in Quantum Field Theory*”, Zakopane June 1997 (to appear in the proceedings)
- [18] Y. Shimamune, Phys. Lett. **108B** (1982) 407,  
G.M. Cicuta, L. Molinari and E. Montaldi, Mod. Phys. Lett. **A1** (1986) 125; J.Phys. **A23** (1990) L421.
- [19] L. Molinari, J. Phys. **A21** (1988) 1.
- [20] M. Douglas, N. Seiberg and S. Shenker, Phys. Lett. **B244** (1990) 381.
- [21] Č. Crnković and G. Moore, Phys. Lett. **B257** (1991) 322.  
G. Moore, Prog. Theor. Phys. Suppl. **102** (1990) 255.
- [22] Č. Crnković, M. Douglas and G. Moore, Nucl. Phys. **B360** (1991) 507.
- [23] O. Lechtenfeld, Int. J. Mod. Phys. **A7** (1992) 2335.  
D. Sénéchal, Int. J. Mod. Phys. **A7** (1992) 1491.
- [24] E. Brézin, E. Marinari and G. Parisi, Phys. Lett. **B242** (1990) 35.
- [25] M.N. Barber, *in* Phase Transitions and Critical Phenomena, Vol. 8, ed. C. Domb and J.L. Lebowitz (Academic Press, London, 1983).
- [26] P.H. Damgaard and U.M. Heller, Nucl. Phys. **B410** (1993) 494.
- [27] M.J. Bowick and E. Brézin, Phys. Lett. **B268** (1991) 21.
- [28] S. Dalley, C. Johnson and T. Morris, Nucl. Phys. **B368** (1992) 625.
- [29] P.H. Damgaard and S.M. Nishigaki, hep-th/9711023.
- [30] T. Wilke, T. Guhr and T. Wettig, hep-th/9711057.

- [31] P.H. Damgaard and S.M. Nishigaki, hep-th/9711096.
- [32] A.D. Jackson and J.J.M. Verbaarschot, Phys. Rev. **D53** (1996) 7223.  
M. Stephanov, Phys. Lett. **275** (1996) 249; Phys. Rev. Lett. **76** (1996) 4472.  
M.A. Nowak, G. Papp and I. Zahed, Phys. Lett. **B389** (1996) 137.
- [33] J.J.M. Verbaarschot, hep-th/9710114.  
R.A. Janik, M.A. Nowak, G. Papp and I. Zahed, hep-th/9710103.
- [34] G. Szegő, *Orthogonal Polynomials* (Am. Math. Soc., Providence, 1939).