

MICROSCOPIC AND BULK SPECTRA OF DIRAC OPERATORS FROM FINITE-VOLUME PARTITION FUNCTIONS

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The microscopic spectrum of the QCD Dirac operator is shown to obey random matrix model statistics in the bulk region of the spectrum close to the origin using finite-volume partition functions.

1 Introduction

The spectral density $\rho(\lambda)$ of the Dirac operator in QCD contains interesting informations as it is for example directly proportional to the chiral condensate Σ at the origin through the Banks-Casher relation ¹. While the full spectrum is only accessible numerically in QCD on the lattice many analytic results for parts of the spectrum have been obtained during the past years using random matrix theory (RMT), chiral perturbation theory and finite-volume partition functions (for a recent review see ²). Within these results two different regimes have been investigated: (i) *unscaled* macroscopic correlations and (ii) microscopic correlations between eigenvalues *rescaled* by the mean spectral density. In the region (ii) one furthermore has to distinguish between scaling at the *origin* and in the *bulk* of the spectrum. In the macroscopic regime (i) the slope of the spectral density at the origin

$$\rho'(\lambda = 0) = \frac{\Sigma^2}{16\pi^2 F_\pi^4} \frac{(N_f - 2)(N_f + \beta)}{\beta N_f}, \quad (1)$$

has been calculated by Smilga and Stern ³ for fundamental $SU(N_c \geq 3)$ fermions ($\beta = 2$) and very recently by Toublan and Verbaarschot ⁴ for fundamental $SU(2)$ fermions ($\beta = 1$) and adjoint fermions ($\beta = 4$) with any gauge group. Here F_π denotes the pion decay constant.

Coming to the rescaled correlations (ii) at the origin all higher order correlation functions of scaled eigenvalues $x = \lambda/\Sigma V$ are known. They are given analytically as functions of the chiral condensate Σ and scaled sea-quark masses $\mu_f = m_f/\Sigma V$ (for details see ²). This description holds in the limit of Leutwyler and Smilga ⁵ $1/\Lambda_{QCD} \ll V^{1/4} \ll 1/m_\pi$, where m_π is the pion mass, up to a certain scale $\lambda_{Th} \sim V^{1/2}$ called Thouless energy ⁶ in analogy to disordered systems. Within this region the analytic description provides an

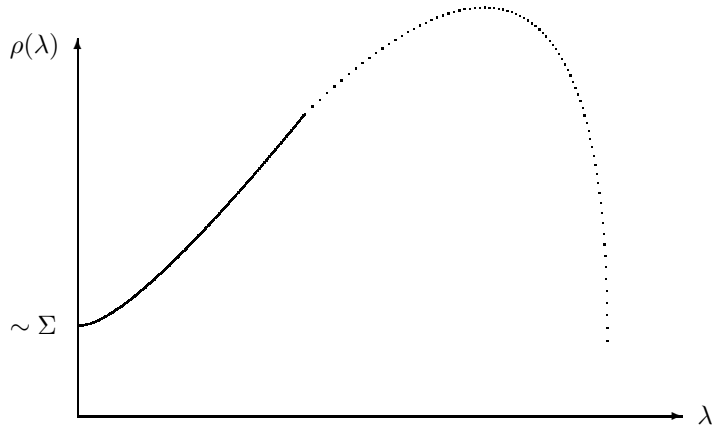


Figure 1. The unscaled spectral density plotted schematically (compare Ref.⁸).

exact nonperturbative limit of QCD, dealing, however, with an unphysically heavy pion that does not fit into the finite-volume of the system.

Away from the origin much less is known about correlations on the microscopic scale. Up to now it has been found only empirically that also there correlations follow a (non-chiral) RMT description^{7,8}. The aim of the results presented here is to understand analytically why this is possible. Therefore, the correlation functions are expressed in terms of finite-volume partition functions⁹ and explored in the bulk region but still in the vicinity of the origin. The region of validity is therefore given by the one of finite-volume partition functions. It does not hold all the way up to the unphysical turn-over in the spectrum (dotted line in Fig. 1) which is due to cut-off and finite-size effects on the lattice. This intermediate region remains unexplained so far.

2 Correlation functions from finite-volume partition functions

Before taking the bulk limit we have to recall how the correlators can be expressed using the Leutwyler and Smilga partition functions in the microscopic limit at the origin. The generation function for the correlators is the kernel of orthogonal polynomials. For gauge group $SU(N_c \geq 3)$ with fermions in the fundamental representation it is given by⁹

$$K_S(x, y; \{\mu_f\}) = c\sqrt{xy} \prod_{f=1}^{N_f} \sqrt{(x^2 + \mu_f^2)(y^2 + \mu_f^2)} \frac{\mathcal{Z}_\nu^{(N_f+2)}(\{\mu_f\}, ix, iy)}{\mathcal{Z}_\nu^{(N_f)}(\{\mu_f\})} \quad (2)$$

$$\mathcal{Z}_\nu^{(N_f)}(\{\mu_f\}) = \frac{\det \mathcal{A}(\{\mu_f\})}{\Delta(\mu_f^2)}, \quad \mathcal{A}_{ij} = \mu_i^{j-1} I_{\nu+j-1}(\mu_i), \quad (3)$$

where $\mathcal{Z}_\nu^{(N_f)}$ is the partition function of N_f flavors with rescaled masses μ_f , ν zero-modes and Δ being the Vandermonde determinant. The rescaled correlation functions can then be obtained as

$$\rho_S^{(N_f, \nu)}(x_1, \dots, x_k; \{\mu_f\}) = \det_{1 \leq a, b \leq k} K_S(x_a, x_b; \{\mu_f\}). \quad (4)$$

This description is equivalent to a unitary chiral RMT description ($\beta = 2$). Since we want to go to the bulk of the spectrum we should expect that the chiral properties no longer play a role for the correlations. We will find that this is indeed the case even in the vicinity of the origin. It is therefore instructive to compare to 3-dimensional QCD with flavor symmetry breaking, which is described by non-chiral RMT¹⁰. The corresponding kernel can be written in the same way as in Eq. (2)⁹ after dropping the prefactor $c\sqrt{xy}$, where the product now runs over $N_f/2$ flavors occurring in pairs $\pm\mu_f$. The corresponding partition function reads

$$\mathcal{Z}^{(N_f)}(\{\mu\}) = \det \begin{pmatrix} A(\{\mu_f\}) & A(\{-\mu_f\}) \\ A(\{-\mu_f\}) & A(\{\mu_f\}) \end{pmatrix} \Delta(\{\mu\})^{-1}, \quad \mathcal{A}_{ij} = (\mu_i)^{j-1} e^{\mu_i}. \quad (5)$$

3 The bulk limit

In RMT it is known for massless flavors how to “invert” the origin scaling limit where $V \rightarrow \infty$, $\lambda \rightarrow 0$ and $x = \lambda/\Sigma V$ is kept fixed:

$$\lim_{\substack{x, y \rightarrow \infty \\ x - y = O(1)}} K_{\text{origin}}(x, y) = K_{\text{bulk}}(x, y) = c' \frac{\sin(x - y)}{x - y}. \quad (6)$$

It takes us from the Bessel-kernel at the origin (Eq. (2) for $m_f = 0$) to the sine-kernel in the bulk. We will see in the following that the same result holds including the masses. In RMT the translational invariance of the spectrum then implies that the sine-kernel holds everywhere in the bulk. Since we cannot make such a strong statement about the QCD Dirac spectrum we will be restricted to the domain of validity of finite-volume partition functions. In taking the bulk limit of the full kernel Eq. (2) we also have to scale back all masses $\mu_f \rightarrow \infty$ at the same time since they would otherwise drop out

trivially. Using the asymptotic expansion of Bessel functions we obtain

$$\lim_{\substack{x, y, \mu_f \rightarrow \infty \\ \text{differences } O(1)}} K_S(x, y; \{\mu_f\}) = c' \prod_{f=1}^{N_f} \frac{(\mu_f^2 + xy)}{\sqrt{(x^2 + \mu_f^2)(y^2 + \mu_f^2)}} \frac{\sin(x-y)}{x-y}. \quad (7)$$

Expanding the prefactor around one of the arguments it is unity up to higher orders and thus we find the same result as in the massless case Eq. (6). Taking the same limit of the non-chiral theory Eqs. (2),(5) we obtain precisely the same result and thus find, that the chiral properties are lost in the bulk even in the vicinity of the origin. Another consequence is that in the bulk any mass scale given by the sea-quarks drops out on the microscopic scale.

In Ref. ⁸ it has been stated that the Thouless energy scale at the origin is larger or equal to the one in the bulk. Looking at the bulk but close to the origin it is clear that if we take $\lambda_1 < \lambda_{Th}$ and $\lambda_2 > \lambda_{Th}$ we are outside the domain of validity although it may still be $|\lambda_1 - \lambda_2| < \lambda_{Th}$.

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