Wilson Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory from Random Matrix Theory

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Abstract

Based on the AdS/CFT correspondence, string theory has given exact predictions for circular Wilson loops in $U(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to all orders in a $1/N$ expansion. These Wilson loops can also be derived from Random Matrix Theory. In this paper we show that the result is generically insensitive to details of the Random Matrix Theory potential. We also compute all higher k-point correlation functions, which are needed for the evaluation of Wilson loops in arbitrary irreducible representations of $U(N)$.

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1 Introduction

One of the surprising predictions of the Maldacena conjecture [1] relating type IIB string theory on an $AdS_5 \times S^5$ background to $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions concerns the static quark-antiquark potential. At strong 't Hooft gauge coupling $\lambda \equiv g^2 N$ this potential is predicted to be of strength $\sqrt{\lambda}$ [2, 3] rather than λ (as at weak coupling). Because the gauge coupling does not run in $\mathcal{N} = 4$ SYM this could in principle be tested by explicit gauge theory computations. Substantial progress in this direction was made last year by Erickson, Semenoff and Zarembo [4]. They computed the sum of all Feynman-gauge planar diagrams without internal vertices to the expectation value of a circular Wilson loop in $\mathcal{N} = 4$ SYM and found ($I_n(x)$ is the nth order modified Bessel function)

$$\langle W \rangle_{circle} \;=\; \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \;\sim\; \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \;, \tag{1}$$

in agreement with the leading-order (in λ) AdS/CFT prediction [5, 6]

$$\langle W \rangle_{AdS/CFT} \;=\; e^{\sqrt{\lambda}} \;. \tag{2}$$

By an explicit computation it was also shown in ref. [4] that the first corrections to (1) of order λ^2 coming from diagrams with internal vertices precisely cancel in four dimensions.

Recently, Drukker and Gross [7, 8] have made a quite remarkable extension of this result by (a) outlining a proof that the above sum of rainbow diagrams (1) actually gives the exact result to all orders in $\sqrt{\lambda}$ at $N = \infty$, and (b) showing that the calculation can be extended to all orders in a $1/N^2$ expansion. Comparing again with the AdS/CFT correspondence they find *exact* agreement to leading order in λ, at every order in $1/N^2$.

One of the key ideas of ref. [7] is to make efficient use of a Random Matrix Theory (RMT) interpretation [4] of the result (1). Consider the unitary ensemble of $N \times N$ hermitian matrices, and a corresponding partition function $\mathcal{Z}$ of Gaussian Boltzmann factor. Identifying N with that of the gauge group $U(N)$ of $\mathcal{N} = 4$ SYM, it was noticed in [4] that in the limit $N \to \infty$

$$\begin{aligned} \left\langle \frac{1}{N} \mathrm{Tr}\; e^M \right\rangle \;&\equiv\; \frac{1}{\mathcal{Z}} \int dM \; \frac{1}{N} \mathrm{Tr}\; e^M \; \exp\left[-\frac{2N}{\lambda} \mathrm{Tr}\; M^2 \right] \\ &=\; \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \;, \end{aligned} \tag{3}$$

precisely reproduces the gauge theory result (1). Intuitive reasons for why this is a correct representation of the circular Wilson loop in $\mathcal{N} = 4$ SYM have been given in refs. [4, 7]. Moreover, in ref. [7] it was argued that this particular Random Matrix Theory representation can be used to compute exactly the leading order (in $\sqrt{\lambda}$) circular Wilson loop to *all orders* in the $1/N^2$ expansion. It should thus be possible to completely by-pass the otherwise quite cumbersome computation in the $\mathcal{N} = 4$ supersymmetric gauge theory language.

The fact that the RMT representation was chosen with Gaussian Boltzmann weight is clearly linked directly to the fact that only rainbow graphs are being summed in the field theory context. An obvious question to ask is why this can yield the full answer. Although arguments have been given [4, 7], a full proof is still lacking. While the resulting effective theory of the circular Wilson loop can be argued to be zero-dimensional (basically on account of a "conformal anomaly" invalidating the conformal mapping from the line to the circle at just one space-time point [7]), there is *a priori* no guarantee that vertices cannot contribute at that particular point. Such interaction vertices would correspond to higher order terms in a more general RMT potential $V(M)$.

The purpose of the present letter is two-fold. First, we shall turn the question around, and ask for the consequences of including higher-order terms in the RMT potential $V(M)$. Remarkably, there is a huge degree of universality at work here. To leading order in $1/N^2$ the precise result (1) changes, *but the leading-order contribution in $\sqrt{\lambda}$ remains generically unaffected*[1]. Moreover, all higher-order terms in $1/N^2$ are generically insensitive to the precise form of the potential $V(M)$, but depend only on a finite number of "moments" M_k. These statements of universality follow from a series of results derived in connection with the RMT approach to 2D quantum gravity [9, 10]. Our second purpose is to advocate the use of the so-called loop insertion method [11] to compute these circular Wilson loop expectation values. We believe this technique is superior to the orthogonal polynomial method in the present context, independently of whether one wants to restrict oneself to the Gaussian potential or not. To illustrate this, we show how to compute the most general k-loop correlation function for any potential $V(M)$, to all orders in the $1/N^2$ expansion. We first briefly, in the section below, recall some of the main results from Random Matrix Theory.

2 The k-point function to all orders in $1/N^2$

Our starting point is the observation that the loop expectation value $\langle (1/N)\mathrm{Tr}\exp[\sqrt{\lambda}M]\rangle$ is the Laplace transform $\mathcal{L}$ of the 1-point function or resolvent $G(p)$. Defining

$$G(p) \;\equiv\; \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{p-M}\right\rangle \tag{4}$$

we immediately have

$$\mathcal{L}^{-1}[G(p)](x) \;=\; \left\langle \frac{1}{N}\mathrm{Tr}\; e^{xM}\right\rangle \;\equiv\; W(x)\ . \tag{5}$$

Here, the expectation value with respect to the Random Matrix Theory partition function of the unitary ensemble

$$\mathcal{Z} \;\equiv\; \int dM \exp\left[-N\mathrm{Tr}V(M)\right] \quad , \qquad V(M) \;=\; \sum_{k=1}^{\infty}\frac{1}{k}g_k M^k\ . \tag{6}$$

is defined with a general potential $V(M)$ in the standard way.

To compute a Wilson loop in an arbitrary representation of $U(N)$ we need also more general expectation values

$$\left\langle \frac{1}{N}\mathrm{Tr}\; e^{x_1 M}\cdots\frac{1}{N}\mathrm{Tr}\; e^{x_k M}\right\rangle\ . \tag{7}$$

These can be computed from the corresponding connected loop expectation values

$$W(x_1,\ldots,x_k) \;\equiv\; N^{k-2}\left\langle \mathrm{Tr}\; e^{x_1 M}\cdots \mathrm{Tr}\; e^{x_k M}\right\rangle_{conn.} \;=\; \sum_{j=0}^{\infty}\frac{1}{N^{2j}}\; W_j(x_1,\ldots,x_k)\ . \tag{8}$$

The latter will follow similarly from the connected k-point correlation function (k-point resolvent)

$$G(p_1,\ldots,p_k) \;\equiv\; N^{k-2}\left\langle \mathrm{Tr}\frac{1}{p_1-M}\cdots\mathrm{Tr}\frac{1}{p_k-M}\right\rangle_{conn.} \;=\; \sum_{j=0}^{\infty}\frac{1}{N^{2j}}\; G_j(p_1,\ldots,p_k) \tag{9}$$

by k inverse Laplace transforms. For that purpose we will rely on the very powerful results for the connected k-point function which have been given explicitly in ref. [11] to all orders in the $1/N^2$ (genus) expansion. The reader familiar with this material can skip it, and go straight to section 3.

[1] We shall comment on the precise condition below.

2.1 *The 1-point function*

Starting with the 1-point function the general solution is of the following form [11]

$$G_k(p) \;=\; \sum_{n=1}^{3k-1}\left(A_k^{(n)}\chi^{(n)}(p)+D_k^{(n)}\psi^{(n)}(p)\right)\;,\quad k\geq 1. \tag{10}$$

The constants $A_k^{(n)}$ and $D_k^{(n)}$ are rational functions of the "moments"

$$\begin{aligned} M_k &\equiv \oint_C \frac{d\omega}{2\pi i}\frac{V'(\omega)}{(\omega-a)^{k+1/2}(\omega-b)^{1/2}} \;=\; g_{k+1}+g_{k+2}\left(\frac{1}{2}b+(k+\frac{1}{2})a\right)+\ldots \\ J_k &\equiv \oint_C \frac{d\omega}{2\pi i}\frac{V'(\omega)}{(\omega-a)^{1/2}(\omega-b)^{k+1/2}} \;=\; g_{k+1}+g_{k+2}\left(\frac{1}{2}a+(k+\frac{1}{2})b\right)+\ldots \end{aligned} \tag{11}$$

and depend on at most $2(3k-1)$ of them. These moments encode in a universal way the dependence on the infinite set of coupling constants $\{g_k\}$, and we will give a few examples below. The functions $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ are the basis functions needed for the solution of the loop equation. They are given explicitly in ref. [11]. All we need to know here is that $\chi^{(n)}(p)$ and $\psi^{(n)}(p)$ are linear combinations of the functions $\phi_a^{(k)}(p)=(p-a)^{-k-1/2}(p-b)^{-1/2}$ and $\phi_b^{(k)}(p)=(p-b)^{-k-1/2}(p-a)^{-1/2}$, respectively, up to order $k=n$. They also depend on the moments M_k and J_k up to $k=n$. Since the inverse Laplace transforms of $\phi_a^{(k)}(p)$ and $\phi_b^{(k)}(p)$ are known in closed form, we immediately read off the corresponding expansion of the Wilson loop $W(x)$. For a Gaussian potential $V(M)=\frac{1}{2}g_2M^2$ we have $b=-a$ and $M_1=J_1=4/a^2=g_2$. All higher moments $M_{k\geq 2}$ and $J_{k\geq 2}$ vanish. In particular the correlators then contain less terms and thus differ from the general expression.

The result given in eq. (10) gives only the $1/N^{2k}$ corrections to the 1-point function $G(p)$ for $k\geq 1$. The leading order result of $k=0$ is non-universal and depends explicitly on all the coupling constants in the potential $V(M)$. It can be written [12]

$$G_0(p) \;=\; \frac{1}{2}\oint_C \frac{d\omega}{2\pi i}\frac{V'(\omega)}{p-\omega}\sqrt{\frac{(\omega-b)(\omega-a)}{(p-b)(p-a)}} \;+\; \frac{1}{\sqrt{(p-b)(p-a)}}\;. \tag{12}$$

The boundary conditions that fix the end-points of the cut $[b,a]$ are

$$\delta_{1,k} \;=\; \frac{1}{2}\oint_C \frac{d\omega}{2\pi i}\omega^k V'(\omega)\frac{1}{\sqrt{(\omega-b)(\omega-a)}}\;,\quad k=0,1 \tag{13}$$

which follow from the requirement that $G_0(p)\sim\frac{1}{p}$ for $p\to\infty$. In order to illustrate the non-universality let us give an explicit example for a symmetric sixth-order potential:

$$G_0(p) \;=\; \frac{1}{2}\left[g_2p+g_4p^3+g_6p^5-\left(g_2+\frac{1}{2}g_4a^2+\frac{3}{8}g_6a^4+(g_4+\frac{1}{2}g_6a^2)p^2+g_6p^4\right)\sqrt{p^2-a^2}\right]\;. \tag{14}$$

The endpoint a of the cut (which is now symmetric) $[-a,a]$ is given by eq. (13):

$$1 \;=\; \frac{1}{4}g_2a^2+\frac{3}{16}g_4a^4+\frac{5}{32}g_6a^6\;. \tag{15}$$

2.2 *Higher k-point functions*

Higher k-point functions can be obtained by applying successively the loop insertion operator [11]

$$\frac{d}{dV(p)} \;\equiv\; -\sum_{k=1}^{\infty}\frac{k}{p^{k+1}}\frac{d}{dg_k}\;, \tag{16}$$

starting from the resolvent itself:

$$G(p_1,\ldots,p_k) \;=\; \frac{d}{dV(p_k)}\cdots\frac{d}{dV(p_2)}G(p_1)\ . \tag{17}$$

The planar 2-point function obtained in this way from eq. (12) is universal, the well-known result of Ambjørn, Jurkiewicz and Makeenko [9]:

$$G_0(p,q) \;=\; \frac{1}{4(p-q)^2}\left[\frac{(p-b)(q-a)+(p-a)(q-b)}{\sqrt{(p-b)(p-a)(q-b)(q-a)}} \;-\; 2\right]\ . \tag{18}$$

As a consequence all k-point functions for $k\geq 2$ are universal to all orders in $1/N^2$, because the derivatives $d/dV(p)$ acting on the universal variables b,a,M_k,J_k can be expressed again in terms of these variables and the functions $\phi_b^{(k)}(p)$ and $\phi_a^{(k)}(p)$. They are again given explicitly in [11].

Apart from $G_0(p,q)$ all universal k-point functions $G_j(p_1,\ldots,p_k)$ factorize with respect to their arguments into a finite number of terms to all orders in $1/N^2$. This follows from the fact that once a function $G_j(p_1,\ldots,p_{k-1})$ is factorized, the application of the loop insertion operator $d/dV(p_k)$ only adds factors which are functions of p_k. At genus zero the 3-point function factorizes

$$\begin{aligned} G_0(p,q,r) \;&=\; \frac{a-b}{8}\Bigg[\frac{1}{M_1}\frac{1}{(p-a)^{3/2}(p-b)^{1/2}}\,\frac{1}{(q-a)^{3/2}(q-b)^{1/2}}\,\frac{1}{(r-a)^{3/2}(r-b)^{1/2}} \\ &\quad -\frac{1}{J_1}\frac{1}{(p-a)^{1/2}(p-b)^{3/2}}\,\frac{1}{(q-a)^{1/2}(q-b)^{3/2}}\,\frac{1}{(r-a)^{1/2}(r-b)^{3/2}}\Bigg]\ . \end{aligned} \tag{19}$$

By applying successively the loop insertion operator one sees that all $G_0(p_1,\ldots,p_k)$ also factorize for $k>3$. At higher genus factorization follows immediately from the 1-point function eq. (10) (see the example for genus one below). This factorization property makes the successive application of k inverse Laplace transformations particularly simple. The only exception is $G_0(p,q)$, which requires a little more work.

Let us finally mention that the same procedure we have described here applies immediately to the complex matrix model [13] and to the so-called reduced hermitian matrix model. There, a closed expression has been derived for all planar k-point correlators with $k\geq 2$ [9, 14]. All higher genus corrections in $1/N^2$ are also available for both the complex [15] and reduced hermitian models [16]. Precisely as for the hermitian case treated below, we can simply read off the corresponding loop correlation functions from the inverse Laplace transformations.

3 Wilson loops from Laplace transforms

We are now ready to compute Wilson loop expectation values by means of inverse Laplace transforms. We begin with the Wilson loop in the fundamental representation of $U(N)$. As already mentioned, this expectation value as computed in Random Matrix Theory is not universal. Consider as an example the sixth-order (symmetric) potential from eq. (14). To leading order in $1/N^2$ we get

$$W_0(x) \;=\; \frac{2}{ax}I_1(ax) \;+\; \frac{3}{8}\left(g_4+\frac{5}{4}g_6a^2\right)\frac{a^3}{x}I_3(ax) \;+\; \frac{5}{32}g_6\,\frac{a^5}{x}I_5(ax)\ . \tag{20}$$

This shows that the natural variable is the combination ax, and we identify $a\equiv\sqrt{\lambda}$. The asymptotic behavior for large λ appears to depend in a complicated way on the coupling constants g_4 and g_6. But from the constraint (15) it follows that in order to achieve $a\to\infty$ we require at least[2] $g_{2n}\sim a^{-2n}$ (or

[2]For an arbitrary symmetric potential the constraint (13) can be written $1=\frac{1}{2}\sum_{k=0}^{\infty}g_{2k}\binom{2k}{k}\frac{a^{2k}}{4^k}$.

even higher suppression in 1/a). Because the leading term in the asymptotic Bessel-function expansion at large argument is $I_n(z) \sim \exp[z]/\sqrt{2\pi z}$ for all fixed n, the large-λ behavior is therefore

$$W_0(x) \sim const.\ \lambda^{-3/4} \exp[x\sqrt{\lambda}]\ , \tag{21}$$

in agreement with the Gaussian result [4, 7] when we set $x = 1$. So the *leading* large-λ behavior is unaffected by higher order terms in the potential $V(M)$.

We now turn to the first $1/N^2$ correction to this result. For simplicity, let us here again restrict ourselves to symmetric potentials (but see the Appendix for the most general expressions). The universal function W_1 follows from the genus-one 1-point function $G_1(p)$ for a symmetric potential

$$G_1(p) \;=\; \frac{a^2}{4M_1}\,\frac{1}{(p^2-a^2)^{5/2}} \;-\; \frac{aM_2}{8(M_1)^2}\,\frac{1}{(p^2-a^2)^{3/2}}\ . \tag{22}$$

Taking the inverse Laplace transform, we get the general result

$$W_1(x) \;=\; \frac{1}{12M_1}\,x^2 I_2(ax) \;-\; \frac{M_2}{8(M_1)^2}\,xI_1(ax)\ . \tag{23}$$

Specifically, for the same symmetric sixth-order potential as above we have

$$M_1 \;=\; g_2 \;+\; \frac{3}{2}a^2 g_4 \;+\; \frac{15}{8}a^4 g_6\ , \qquad M_2 \;=\; 2ag_4 \;+\; 5a^3 g_6 \quad . \tag{24}$$

For a Gaussian potential this reproduces the result of ref. [7]: $W_1(\lambda) = \lambda I_2(\sqrt{\lambda})/48$. Again we find that the leading large-λ behavior is unaffected by higher order terms in the potential, after using $M_1 \sim a^{-2}, M_2 \sim a^{-3}$ as $a \to \infty$: $W_1(\lambda) \sim const.\lambda^{3/4}\exp[\sqrt{\lambda}]$. Note that the second term in (23) is subleading in this limit. We have also computed the general genus-two ($1/N^4$) contribution $G_2(p)$, and confirmed the result of ref. [7] for the special case of a Gaussian potential. Clearly, the leading λ-behavior is also here of the same form for any generic potential.

We next turn to higher k-point correlation functions of these fundamental Wilson loops, needed for the evaluation of Wilson loops in arbitrary representations of $U(N)$. The universal 2-loop result (18) is seen to not factorize in p and q. One may use the convolution theorem to evaluate the needed double inverse Laplace transform, but it is simpler to note that

$$\frac{\partial}{\partial a^2}G_0(p,q) \;=\; \frac{1}{4}\frac{pq+a^2}{(p^2-a^2)^{3/2}(q^2-a^2)^{3/2}} \tag{25}$$

does factorize. It is thus elementary to get the inverse Laplace transforms,

$$\frac{\partial}{\partial a^2}W_0(x,y) \;=\; \frac{xy}{4}\left[I_0(ax)I_0(ay) \;+\; I_1(ax)I_1(ay)\right] \tag{26}$$

which we integrate up to

$$W_0(x,y) \;=\; \frac{xy}{2}\int_0^a du\ u\left[I_0(ux)I_0(uy) \;+\; I_1(ux)I_1(uy)\right]\ , \tag{27}$$

after fixing the integration constant by comparing [9]

$$G_0(p,p) \;=\; \frac{a^2}{4(p^2-a^2)^2} \;=\; \frac{1}{p^4}\,\frac{a^2}{4} \;+\; \mathcal{O}\left(\frac{1}{p^6}\right) \;=\; \frac{1}{p^4}\langle \mathrm{Tr}M\mathrm{Tr}M\rangle_{conn.} \;+\; \mathcal{O}\left(\frac{1}{p^6}\right) \tag{28}$$

and

$$W_0(x,y) \;=\; xy\ \langle \mathrm{Tr}M\mathrm{Tr}M\rangle_{conn.} \;+\; \mathcal{O}\left(x^2y^2, xy^3, x^3y\right)\ . \tag{29}$$

Performing the integral in eq. (27) we get for an arbitrary symmetric potential

$$W_0(x,y) \;=\; \frac{axy}{2(x+y)}\left[I_0(ay)I_1(ax) \;+\; I_0(ax)I_1(ay)\right] \;. \tag{30}$$

As a check on this result, we note that it vanishes when one or both of the arguments x and y become equal to zero. Finally, at equal arguments we have

$$W_0(x,x) \;=\; \frac{ax}{2}I_0(ax)I_1(ax) \;. \tag{31}$$

As an example, let us evaluate $N^{-2}\langle(\mathrm{Tr}\exp[M])^2\rangle$, which in ref. [7] is denoted by $W_{1,1}$. We get

$$W_{1,1} \;=\; (W_0(1))^2 + \frac{1}{N^2}[W_0(1,1) + 2W_0(1)W_1(1)] + \mathcal{O}\left(\frac{1}{N^4}\right) \;, \tag{32}$$

where all functions on the right hand side have been given above. In particular, we find

$$W_{1,1} \;=\; \frac{4}{\lambda}I_1(\sqrt{\lambda})^2 + \frac{\sqrt{\lambda}}{2N^2}\left[I_0(\sqrt{\lambda})I_1(\sqrt{\lambda}) + \frac{1}{6}I_1(\sqrt{\lambda})I_2(\sqrt{\lambda})\right] + \mathcal{O}\left(\frac{1}{N^4}\right) \tag{33}$$

in the Gaussian case. The leading large-λ behavior of the $1/N^2$ correction goes like $\exp[2\sqrt{\lambda}]$, which is now ready to be compared with string theory through the AdS/CFT correspondence.

It is straightforward to go on to arbitrarily high order in $1/N^2$ for any n-point correlation function. As a last example, consider the universal 3-point function $W_0(p,q,r)$ for an arbitrary symmetric potential. We find:

$$\begin{aligned} W_0(x,y,z) \;=\; xyz\frac{a}{2M_1}&\left[I_1(ax)I_0(ay)I_0(az) \;+\; I_0(ax)I_1(ay)I_0(az)\right.\\ &\left.+I_0(ax)I_0(ay)I_1(az) \;+\; I_1(ax)I_1(ay)I_1(az)\right] \end{aligned} \tag{34}$$

where for the Gaussian case $M_1 = g_2 = 4/a^2 = 4/\lambda$. The most general result for the genus expansion of the resolvent can be written

$$W_k(x) \;=\; \sum_{n=1}^{3k-1}\left(A_k^{(n)}\mathcal{L}^{-1}[\chi^{(n)}(p)](x) + D_k^{(n)}\mathcal{L}^{-1}[\psi^{(n)}(p)](x)\right) \;,\quad k \geq 1. \tag{35}$$

where the inverse Laplace transforms are known explicitly. Any n-point function for $n \geq 2$ can then be found to all orders in the $1/N^2$ expansion by the iteration procedure described above. We have thus succeeded in determining all n-point Wilson loop correlation functions to all orders in the $1/N^2$ expansion. They are all given by appropriate products of n modified Bessel functions.

4 Other universality classes

By fine-tuning the RMT potential one can reach new (multi-critical) universality classes that would invalidate the above conclusions. To see what happens in such cases, it suffices to focus on the 1-point function, the Wilson loop itself, in the planar limit. First note that, with $\rho(\zeta)$ indicating the eigenvalue density

$$\left\langle\frac{1}{N}\mathrm{Tr}\; e^{xM}\right\rangle \;=\; \frac{1}{\mathcal{Z}}\int dM\;\frac{1}{N}\mathrm{Tr}\;e^{xM}\;\exp\left[-N\mathrm{Tr}V(M)\right] \;=\; \int_{-1}^{1}d\zeta_1\;\exp(ax\zeta_1)\;\rho(\zeta_1) \tag{36}$$

where for simplicity we have restricted ourselves to even potentials $V(M)$, and the cut has been rescaled to lie on the interval [-1,1]. By inserting the Wigner semi-circle law $\rho(\zeta) = (2/\pi)(1-\zeta^2)^{1/2}$

corresponding to a Gaussian potential we of course just recover the result (1) with $a = \sqrt{\lambda}$. (The form (36) also immediately shows the non-universality of this result, away from the large-λ limit). Multicritical densities belonging to multicritical universality classes near the soft edge can be chosen

$$\rho_m(\zeta) \;\sim\; (1-\zeta^2)^{m+1/2} \;. \tag{37}$$

According to eq. (36) the corresponding mth multicritical Wilson loop is

$$W_0(x) \;=\; \frac{(k+1)!2^{k+1}}{(ax)^{k+1}} I_{k+1}(ax) \;\sim\; const.\frac{e^{x\sqrt{\lambda}}}{\lambda^{(2k+3)/4}} \;. \tag{38}$$

The leading large-λ behavior thus differs in the prefactor from that of generic non-critical potentials, although the exponential form $\sim \exp[x\sqrt{\lambda}]$ is retained. We have explicitly checked eq. (38) for $m = 1, 2$ by tuning the potential accordingly in eq. (14). For this special choice of coupling constants the Bessel functions in eq. (20) precisely combine to give eq. (38).

It is tempting to speculate that the universality classes of multicritical points could correspond to circular Wilson loops of different gauge theories at conformal points. But we have no evidence to support such a claim.

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A Appendix

In this appendix we give, for completeness, a few examples to illustrate that the method we have presented also works for an arbitrary potential that is not restricted to be even. We start with a closed expression for the non-universal $W_0(p)$ which can be derived from eq. (12)

$$\begin{aligned} W_0(x) \;&=\; \int_0^x du\; \exp\left(u\frac{a+b}{2}\right) I_0\left(u\frac{a-b}{2}\right) \frac{1}{2}\oint_C \frac{d\omega}{2\pi i}\; V'(\omega)\sqrt{\frac{(\omega-b)(\omega-a)}{(p-b)(p-a)}}\exp[\omega(x-u)] \\ &+\; \exp\left(x\frac{a+b}{2}\right) I_0\left(x\frac{a-b}{2}\right) \end{aligned} \tag{39}$$

Next, we find the $1/N^2$ contribution:

$$\begin{aligned} W_1(x) \;&=\; x^2\frac{1}{16}\exp\left(x\frac{a+b}{2}\right)\left\{\frac{1}{M_1}\left[\frac{1}{2}I_0(xc) + \frac{2}{3}I_1(xc) + \frac{1}{6}I_2(xc)\right]\right. \\ &\qquad\qquad\left. +\; \frac{1}{J_1}\left[\frac{1}{2}I_0(xc) - \frac{2}{3}I_1(xc) + \frac{1}{6}I_2(xc)\right]\right\} \\ &-\; x\frac{1}{16}\exp\left(x\frac{a+b}{2}\right)\left\{\left(\frac{M_2}{(M_1)^2} + \frac{1}{cM_1}\right)\left[I_0(xc) + I_1(xc)\right]\right. \\ &\qquad\qquad\left. +\; \left(\frac{J_2}{(J_1)^2} - \frac{1}{cJ_1}\right)\left[I_0(xc) - I_1(xc)\right]\right\} \;, \end{aligned} \tag{40}$$

where we have defined $c \equiv (a-b)/2$. Note the appearance of the exponential prefactor $\exp[x(a+b)/2]$ which disappears for the symmetric potential with $b = -a$. The result for a generic symmetric potential eq. (23) is easily recovered by setting $J_k = (-1)^{k+1}M_k$ and $c = a$. As a last example let us also give the planar connected 3-loop function

$$W_0(x,y,z) \;=\; xyz\frac{a-b}{8}\exp\left(xyz\frac{a+b}{2}\right)\left\{\frac{1}{M_1}\left[I_0(xc) + I_1(xc)\right]\left[I_0(yc) + I_1(yc)\right]\left[I_0(zc) + I_1(zc)\right]\right.$$

$$
-\frac{1}{J_1}\left[I_0\left(xc\right)-I_1\left(xc\right)\right]\left[I_0\left(yc\right)-I_1\left(yc\right)\right]\left[I_0\left(zc\right)-I_1\left(zc\right)\right]\Big\}. \tag{41}
$$

Also the general expression for the 2-loop function can be evaluated for general asymmetric potentials, using a variant of the method described above for the symmetric case. Proceeding iteratively, this again determines all n-point functions to all orders in the $1/N^2$-expansion, now for potentials that are not necessarily symmetric.

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