New Critical Matrix Models and Generalized Universality

G. Akemann

CEA/Saclay, Service de Physique Théorique*
F-91191 Gif-sur-Yvette Cedex, France
e-mail: akemann@spht.saclay.cea.fr

and

G. Vernizzi

Department of Theoretical Physics, Oxford University
1 Keble Road, Oxford, OX1 3NP, United Kingdom
e-mail: vernizzi@thphys.ox.ac.uk

Abstract

We study a class of one-matrix models with an action containing nonpolynomial terms. By tuning the coupling constants in the action to criticality we obtain that the eigenvalue density vanishes as an arbitrary real power at the origin, thus defining a new class of multicritical matrix models. The corresponding microscopic scaling law is given and possible applications to the chiral phase transition in QCD are discussed. For generic coupling constants off-criticality we prove that all microscopic correlation functions at the origin of the spectrum remain in the known Bessel universality class. An arbitrary number of Dirac mass terms can be included and the corresponding massive universality is maintained as well. We also investigate the critical behavior at the edge of the spectrum: there, in contrast to the behavior at the origin, we find the same critical exponents as derived from matrix models with a polynomial action.

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*Unité associée CNRS/SPM/URA 2306

1 Introduction

The study of random matrix models as simplified versions of more complicated interacting Quantum field theories has often been very fruitful. For instance, there are zero-dimensional matrix models with applications ranging from Quantum Gravity to Quantum Chromodynamics (QCD). For a review on these topics we refer to [1] and [2], respectively, as well as to [3] for applications in other fields. In most of the cases it has been sufficient to consider the so-called Gaussian matrix model, where the random matrices (that describe for example an underlying Hamiltonian) have entries which are Gaussian random variables. The reason for such Gaussian matrix models being generic is their property of universality. Universality in random matrix models means that when reweighting the matrix elements differently within a certain class of perturbations around the Gaussian models the same correlation functions of the matrix eigenvalues are retained. Different proofs of universality [4, 5, 6, 7, 8, 9, 10, 11, 12] using various techniques have been obtained in distinct large-N regimes, where N is the size of the matrices.

In general one has to distinguish between the *macroscopic* large-N limit where the oscillations of the correlation functions are smoothed and the *microscopic* large-N limit, where the eigenvalues are rescaled such that a particular region of the spectral density is magnified. This region might be the origin, the bulk or the endpoint of the spectrum and microscopic universality classes emerge correspondingly. In fact the universal kernel of orthogonal polynomials that generates all the correlation functions is described by Bessel, Sine or Airy functions, respectively. The stability of universality under changes of the functional form of the measure has also been tested [13, 14] and again the microscopic and macroscopic regime behave differently. In this paper we will see another extension of universality by adding nonpolynomial terms to Gaussian (or more general) weight functions.

However, this is not the main motivation of the present work. In many applications one has to take into account extensions of the Gaussian model and to fine tune in order to reach critical points (for a review on phase transitions in matrix models we refer to [15]). There may be several different reasons for doing so. For instance, in Quantum Gravity the critical behavior is used to enhance the otherwise subleading contributions in the $1/N^2$ topological expansion and make them all contribute in the double scaling limit (for a review see [1]). In the study of chiral symmetry breaking in QCD [16, 2] the spectral density at the origin $\rho(0)$ is proportional to the order parameter, the chiral condensate. Terms of higher order than Gaussian may then be used to achieve a phase transition by requiring $\rho(0) = 0$ [17]. The corresponding scaling exponents and correlation functions will be those on top of the phase transition. Such matrix models have already been investigated in [17] where also universality was found within a given multicriticality class. But only very specific transition points could be reached in [17], where $\rho(\lambda = 0) \sim \lambda^{2m}$ with $m \in \mathbb{N}$. Our aim here is to complete this study and find new critical models where the density vanishes with an arbitrary real power. For that purpose we will add a nonpolynomial part to a Gaussian potential, or to a more general polynomial potential, and study its effect. It has been argued in [18] that at the temperature induced chiral phase transition in QCD the eigenvalue density precisely behaves as $\rho(\lambda = 0) \sim \lambda^{1/\delta}$, with mean field exponent $\delta = 3$.

Single nonpolynomial potentials have been already studied in [19, 20, 21, 22]. They have been shown to fall into the microscopic universality class of the sine kernel in the bulk and at the origin of the spectrum. It does therefore not come as a surprise that the extensions of the model we study here fall into the corresponding massless and massive Bessel universality classes [6, 9, 10] when Dirac mass terms are included. This holds as long as we keep away from criticality or in other words when we maintain $\rho(0) \neq 0$. The investigation of new multicritical points is however only possible with several interaction terms in the potential, polynomial and nonpolynomial. This is our starting point.

Nonpolynomial potentials have also been studied in the context of Quantum Gravity [23, 24]. We will analyze whether the terms we introduce may also change the critical exponents [25] in the

appropriate large-N limit. Finding real critical exponents would correspond to new representations of conformal field theories by simple one-matrix models. However, our findings in this respect are negative in the sense that the perturbations we consider here will not lead out of the known universality classes. Nevertheless it would be very interesting to find realizations of such simple one-matrix models instead of more complicated multi-matrix or $O(n)$ models [26, 27].

The matrix-model approach to the chiral phase transition we described so far is not the only possible one. The phase transition can also be studied by adding an external field [18, 28] or by allowing more substructure for the matrices such as additional spin or gauge group degrees of freedom [29]. While such extensions are more close to the phenomenology they are certainly more difficult to solve exactly without any further assumptions. However, let us emphasize that such a matrix model [30] was used to deduce the phase diagram of QCD realistic for two massless quarks in the temperature-density plane and that other approaches have confirmed this general picture (see e.g. [31]). It consists of a second-order line beginning at a critical temperature with zero density that joins a first-order line starting from a critical density at zero temperature in a tricritical point.

In the multicritical matrix models considered in [17] the phase transition is of third order. Therefore these models may become interesting precisely at the tricritical point. They may also be relevant for the chiral phase transition occurring when the number of flavors N_f is raised above the critical point, where the beta-function of QCD changes its sign. The relation of the large-N_f phase transition to the one occurring in the Gross-Witten model has been pointed out in [32]. The extension of [17] to nonpolynomial potentials we investigate here may also have a phase transition of different order. The fact that the critical exponents of [18] are included in our model suggests a possible second order transition. However, the continuity of our results at even integer powers suggests that the transition remains of third order. We have not been able to give a definite answer to that question.

The paper is organized as follows. We will mainly concentrate on Hermitian matrix models also called the Unitary Ensemble. This model can be used to study flavor symmetry breaking in three dimensional QCD [33]. In the original approach to chiral symmetry breaking in four-dimensional QCD [16] a slightly different matrix model is needed, the chiral Unitary Ensemble. In order to translate our results to the chiral ensemble we will simply use the relations derived in [17] and [34] between the corresponding orthogonal polynomials and kernels. In Section 2 we first state our model with arbitrary polynomial and nonpolynomial power like terms in the potential. It includes Dirac mass terms motivated by the application to QCD. These terms are also studied under the name of characteristic polynomials. In the same section we calculate the macroscopic large-N spectral density away from criticality for potentials containing nonpolynomial parts using different techniques. We also prepare the ground for Section 3 by recalling the orthogonal polynomial method in the version as being reviewed in [35]. In Section 3 we present our results about generalized universality. We prove that away from criticality the correlation functions in the microscopic scaling limit at the origin fall into the same massless and massive universality classes as in [6, 9, 10], also in the presence of nonpolynomial terms. The tool [35] we use is to derive a universal differential equation for the orthogonal polynomials first without massive flavors. Then, in order to include an arbitrary number of Dirac masses we proceed along the lines of [12] without any restriction on the number of flavors. Our proof is valid for ensembles with chiral and nonchiral unitary invariance and is an alternative to the original proofs [9, 10] for the massive ensembles. In Section 4 we come to our main result by tuning the coupling constants of our model to criticality. After describing in detail the two simplest examples we introduce a general class of new multicritical densities. They behave as $\rho(\lambda = 0) \sim \lambda^{\kappa-1}$ with real noninteger $\kappa > 1$. For $2m-1 < \kappa < 2m+1$, $m \in \mathbb{N}$, we also give the corresponding minimal critical potentials containing m polynomial terms and one nonpolynomial term. In Section 5 we investigate possible critical points in the leading order of the free energy at large-N for the simplest of such multicritical models. We find a nonanalytic behavior at large-N when the spectral density is tuned to develop additional zeros at

the edge of the spectrum. The critical exponent $\gamma_{str} = -\frac{1}{2}$ we find for the free energy is the same as known from matrix models with polynomial potentials [25]. Section 6 contains our conclusions and in the Appendix we collect some results on the so-called two-cut spectral density, where the eigenvalues lie on two disjoint intervals that merge at the transition.

2 The model and its large-N solution

We start by describing the most general matrix model we wish to work with. In order to make sense of nonpolynomial terms we give the model right away in terms of the N eigenvalues of a Hermitian matrix:

$$\mathcal{Z}^{(2N_f,\alpha)}(\{m_f\}) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(d\lambda_i \, |\lambda_i|^{2\alpha} \prod_{f=1}^{N_f} \left(\lambda_i^2 + m_f^2\right) e^{-NV(\lambda_i)} \right) |\Delta(\lambda)|^2 \, , \tag{2.1}$$

$$V(\lambda) \equiv V_{pol}(\lambda) + V_{nonpol}(\lambda) \equiv \sum_{j=1}^{d_p} \frac{g_{2j}}{2j} \lambda^{2j} + \sum_{j=1}^{d_{np}} h_j |\lambda|^{\kappa_j} \quad , \tag{2.2}$$

where $1 < \kappa_1 < \ldots < \kappa_{d_{np}} \in \mathbb{R}\backslash\mathbb{N}$. Here we have restricted ourselves to $1 < \kappa_j$ as we do not only need a normalizable spectral density $\rho(\lambda)$ enforcing $0 < \kappa_j$, but also $\rho(0) < \infty$ which requires $1 < \kappa_j$, as we will see. The Vandermonde determinant $\Delta(\lambda) = \prod_{k>l}^{N}(\lambda_k - \lambda_l)$ originates from the diagonalization of the Hermitian matrix. The potential $V(\lambda)$ is chosen to be symmetric for simplicity and in view of applications to QCD (as we will show, odd powers are not necessary for obtaining multicritical potentials). It has been split into a polynomial part, $V_{pol}(\lambda)$, and a nonpolynomial part, $V_{nonpol}(\lambda)$. In the latter we have allowed for an arbitrary number d_{np} of real positive powers κ_j. In order to achieve a multicritical behavior at the origin we will actually need only one of such terms. However, for the large-N solution presented in this section as well as for the issue of universality in the next section any number of such terms may be present.

The Dirac mass terms in front of the exponential are added in order to study flavor symmetry breaking in three-dimensional QCD (see [33] for details). The $2N_f$ Dirac masses occur in N_f pairs $\pm i m_f$. Furthermore, we have included a massless Dirac determinant raised to some arbitrary real power $2\alpha > -1$. We will need this term later in Section 3 when we switch from the Unitary Ensemble of Hermitian matrices in eq. (2.1) to the chiral Unitary Ensemble which is needed in applications to four-dimensional QCD.

In the following we will determine the spectral density $\rho(\lambda)$ in the macroscopic large-N limit. A first argument how to calculate it in the presence of nonpolynomial terms comes from a saddle-point analysis. To begin with we observe that the preexponential factors containing the mass terms in eq. (2.1) can be formally included into the potential by shifting $V(\lambda) \to V(\lambda) - 1/N \ln[|\lambda|^{2\alpha} \prod_{f=1}^{N_f}(\lambda^2 + m_f^2)]$. Hence they are subleading in the large-N limit and we can set $\alpha = N_f = 0$ in the rest of this section. Only in the microscopic limit they will become important while the macroscopic spectral density does not depend on $\alpha, N_f \neq 0$.

The spectral density for a single nonpolynomial potential $V(\lambda) = |\lambda|^\kappa$ has been calculated already in [19, 20, 21, 22]. In order to determine the full density $\rho(\lambda)$ for the potential eq. (2.2) we make use of the fact that the saddle-point equation for the spectral density is linear in the potential. We can therefore obtain solutions of the general problem by simply adding up solutions for the pure polynomial case with solutions for the nonpolynomial case. In order to see that in more details we repeat some well known facts about the saddle-point solution of matrix models (for a review see e.g. [1]). Defining the free energy as $\mathcal{F} \equiv -\ln[\mathcal{Z}]/N^2$ and switching from discrete eigenvalues to a continuous density

one has

$$\mathcal{F} \;=\; \int_\sigma d\lambda \rho(\lambda) V(\lambda) - \int_\sigma d\lambda\, d\mu \rho(\lambda)\rho(\mu) \ln|\lambda-\mu| + K\left(\int_\sigma d\lambda \rho(\lambda) - a\right) \;. \tag{2.3}$$

We have introduced a Lagrange multiplier K to enforce the normalization of the spectral density $\rho(\lambda)$ to a, which may differ from unity. Furthermore, we consider here the case where the support σ of $\rho(\lambda)$ is a compact interval, the so-called one-cut solution. For the two-cut solution we refer to the Appendix A. The saddle-point equation then reads

$$0 \;=\; V(\lambda) - 2\int_\sigma d\mu \rho(\mu) \ln|\lambda-\mu| + K \;, \qquad \lambda \;\in \sigma \equiv [-c,c] \;, \tag{2.4}$$

which is often written also after taking the derivative w.r.t. λ as

$$⨍_\sigma d\mu \frac{\rho(\mu)}{\lambda-\mu} = \;\frac{V'(\lambda)}{2}, \quad \lambda \in \sigma \,, \tag{2.5}$$

and $⨍$ is the Principal Value of the integral. The Lagrange multiplier K can be determined by choosing a particular value of λ in eq. (2.4), e.g. $\lambda = 0 \in \sigma$. Because of the linearity of eq. (2.4) two positive solutions ρ_1 and ρ_2 for two given potentials V_1 and V_2, respectively, can be simply added under the assumption that they have the same support σ. This gives a new solution $\rho = \rho_1 + \rho_2$ for the potential $V = V_1 + V_2$ which is nonnegative on the same support σ. Thus we can take known solutions of the saddle-point equation for polynomial and nonpolynomial potentials, fix their support to be equal and add them. The normalization to unity of the new spectral density, $\rho(\lambda) = \rho_{pol}(\lambda) + \rho_{nonpol}(\lambda)$, has to be imposed eventually. This condition uniquely determines the dependence of the end-point c on the coupling constants in the potential or, which is the same, it gives a relation among the coupling constants for any given fixed positive number c.

Let us give an example. Choosing the simplest polynomial potential, i.e. a Gaussian potential $V_{pol}(\lambda) = \frac{1}{2} g_2 \lambda^2$, the corresponding spectral density is the "Wigner semi-circle"

$$\rho_{Gauss}(\lambda) \;=\; \frac{g_2}{2\pi}\sqrt{c^2-\lambda^2} \;, \qquad \lambda \;\in [-c,c] \;. \tag{2.6}$$

If one fixes the normalization condition then g_2 and c must be related to each other. In particular, the normalization condition of the density to unity reads $\frac{1}{4} g_2 c^2 = 1$. Here we consider c as an arbitrary given positive parameter. For the nonpolynomial part we choose $V_{nonpol}(\lambda) = g|\lambda|^\kappa$, which is called Freud weight in the mathematical literature. The corresponding spectral density normalized to unity has been calculated in [19, 20, 22] (see also references therein). The un-normalized density sharing the same one-cut support $[-c,c]$ will be re-obtained in this section and can be written as (see also [21]):

$$\rho_{Freud}(\lambda) \;=\; \frac{g\kappa H c^{\kappa-1}}{\pi} \int_{\frac{|\lambda|}{c}}^1 ds \frac{s^{\kappa-1}}{\sqrt{s^2 - \frac{\lambda^2}{c^2}}} \tag{2.7}$$

$$\;=\; \frac{g\kappa H c^{\kappa-1}}{\pi} F\left(1-\frac{\kappa}{2}, 1; \frac{3}{2}; 1-\frac{\lambda^2}{c^2}\right)\sqrt{1-\frac{\lambda^2}{c^2}} \,, \qquad \lambda \;\in [-c,c] \;, \tag{2.8}$$

where

$$H \equiv \frac{\Gamma(\frac{\kappa}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{\kappa}{2})} \;=\; \frac{1}{B\left(\frac{\kappa}{2}, \frac{1}{2}\right)} \;, \tag{2.9}$$

B is the Beta function and F is the Hypergeometric function. It can be easily seen from the integral that for $0 < \kappa$ the density is normalizable and that for $1 < \kappa$ it holds $\rho(0) < \infty$. For a discussion

of $0 < \kappa \leq 1$ we refer to [20, 21]. Using some identities among Hypergeometric functions for $\kappa \neq 1, 3, 5, \ldots$, eq. (2.8) can also be written in the useful form:

$$\rho_{Freud}(\lambda) \;=\; \frac{g\kappa H c^{\kappa-1}}{\pi}\left(\frac{1}{2H}\tan\left[\frac{\kappa\pi}{2}\right]\left(\frac{\lambda}{c}\right)^{\kappa-1} + \frac{1}{\kappa-1}F\left(\frac{1}{2},\,\frac{1}{2}-\frac{\kappa}{2};\,\frac{3}{2}-\frac{\kappa}{2};\,\frac{\lambda^2}{c^2}\right)\right). \tag{2.10}$$

Imposing the condition $1 = Hgc^{\kappa}$ we would obtain a normalized density, reproducing [19, 20, 22]. By adding the two solutions eqs. (2.6) and (2.7) which now have the same support, we obtain for the full spectral density

$$\begin{aligned}\rho(\lambda) \;&=\; \rho_{Gauss}(\lambda) \;+\; \rho_{Freud}(\lambda)\\ &=\; \frac{g_2}{2\pi}\sqrt{c^2-\lambda^2} \;+\; \frac{g\kappa H c^{\kappa-1}}{\pi}\int_{\frac{|\lambda|}{c}}^{1} ds\frac{s^{\kappa-1}}{\sqrt{s^2-\frac{\lambda^2}{c^2}}}\,, \qquad \lambda \;\in [-c,c]\;.\end{aligned} \tag{2.11}$$

By imposing $\int_\sigma d\lambda\rho(\lambda) = 1$ we obtain the following relation between the coupling constants and the endpoint c of the support:

$$1 \;=\; \frac{1}{4}g_2c^2 + Hgc^{\kappa}\;. \tag{2.12}$$

This relation can be easily checked by comparing to the known polynomial potential with quadratic and quartic coupling [36] by setting $\kappa = 4$. We can immediately draw two consequences. First, both parts of the density, $\rho_{Gauss}(\lambda)$ and $\rho_{Freud}(\lambda)$ have an expansion in powers of λ^2 around the origin with the latter having an extra term $\sim \lambda^{\kappa-1}$ as one can see from eq. (2.10). Consequently a critical behavior at the origin can be achieved by tuning the coupling constants g_2 and g such that the leading constant term, $\rho(0)$, vanishes. This will be analyzed in great detail in Section 4. Second, it is easy to see how to generalize the above adding of spectral densities. Since the density for a polynomial potential is already know to most generality [4] we can use the above procedure to add successively single nonpolynomial potentials of higher and higher real power to obtain the density for the potential eq. (2.2).

A standard way to obtain eqs. (2.7) and (2.8), is by introducing the resolvent $G(z)$:

$$G(z) \equiv \int_\sigma d\mu\frac{\rho(\mu)}{z-\mu}\,, \quad z \in \mathbb{C}\backslash\sigma \tag{2.13}$$

in terms of which the saddle point equation (2.5) reads $G(\lambda+i0)+G(\lambda-i0) = V'(\lambda)$, $\lambda \in \sigma$. Clearly, $G(z)$ has a cut at $z \in \sigma$. The (unique) solution which is bounded at the end points and which behaves like $G(z) \sim 1/z$ at large $|z|$ is (see e.g. [37])

$$\begin{aligned}G(z) \;&=\; \sqrt{z^2-c^2}\int_{-c}^{c}\frac{dt}{2\pi}\frac{V'(t)}{(z-t)\sqrt{c^2-t^2}}\\ &=\; gHc^{\kappa-1}\frac{c^2}{z^2}F\left(\frac{1+\kappa}{2},1;\frac{\kappa}{2}+1;\frac{c^2}{z^2}\right)\sqrt{\frac{z^2}{c^2}-1}\,,\end{aligned} \tag{2.14}$$

in the one-cut case, for $V(\lambda) = g|\lambda|^{\kappa}$. For any real positive κ, the spectral density eq. (2.8) is then recovered by means of $2\pi i\rho(\lambda) = G(\lambda - i0) - G(\lambda + i0)$, $\lambda \in [-c, c]$ (after applying an identity for the Hypergeometric function corresponding to the transformation $\frac{c^2}{z^2} \to 1 - \frac{z^2}{c^2}$). The normalization condition is readily obtained from the large-z asymptotic behavior of eq. (2.14). In fact by using that $F(\alpha, \beta, \gamma, 0) = 1$ one has $G(z) \sim gHc^{\kappa}/z = 1/z$ which is the previously stated formula $gHc^{\kappa} = 1$.

In the remaining part of this section we obtain the spectral density by using another technique, the method of orthogonal polynomials. It will turn out to be very useful also in the next section where we prove the universality of all microscopic correlation functions away from criticality. Let us begin by repeating a few well known formulas [38] in order to fix our notation. We choose a set of orthonormal polynomials,

$$\int_{-\infty}^{\infty} d\lambda\, w(\lambda) P_n^{(2N_f,\alpha)}(\lambda)\, P_m^{(2N_f,\alpha)}(\lambda) \,=\, \delta_{nm} \ , \tag{2.15}$$

where the weight function is taken from eq. (2.1):

$$w(\lambda) \;\equiv\; |\lambda|^{2\alpha} \prod_{f=1}^{N_f} \left(\lambda^2 \;+\; m_f^2\right)\, e^{-NV(\lambda)} \ . \tag{2.16}$$

The k-point correlation function of the eigenvalues,

$$R_N^{(2N_f,\alpha)}(\lambda_1,\ldots,\lambda_k) \;\equiv\; \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} \prod_{i=k+1}^{N} d\lambda_i \;\prod_{j=1}^{N} w(\lambda_j)\; |\Delta(\lambda)|^2 \ , \tag{2.17}$$

defined as the $(N-k)$-fold integral over the integrand of eq. (2.1), can be expressed through the kernel of the orthogonal polynomials [38]

$$R_N^{(2N_f,\alpha)}(\lambda_1,\ldots,\lambda_k) \;=\; \frac{N!}{(N-k)!} \det_{1\le i,j\le k} \left[K_N^{(2N_f,\alpha)}(\lambda_i,\lambda_j)\right] \ . \tag{2.18}$$

The kernel is given by

$$\begin{aligned} K_N^{(2N_f,\alpha)}(\lambda,\eta) \;&=\; w(\lambda)^{\frac{1}{2}} w(\eta)^{\frac{1}{2}} \sum_{i=0}^{N-1} P_i^{(2N_f,\alpha)}(\lambda) P_i^{(2N_f,\alpha)}(\eta) \\ &=\; w(\lambda)^{\frac{1}{2}} w(\eta)^{\frac{1}{2}}\; r_N \frac{P_N^{(2N_f,\alpha)}(\lambda) P_{N-1}^{(2N_f,\alpha)}(\eta) - P_N^{(2N_f,\alpha)}(\eta) P_{N-1}^{(2N_f,\alpha)}(\lambda)}{\lambda-\eta} \ , \end{aligned} \tag{2.19}$$

where in the last step we have used the Christoffel-Darboux identity. It follows from a three step recursion relation, which any set of orthogonal polynomials obeys. For any symmetric potential $V(\lambda)$, polynomial or not, it reads

$$\lambda P_n^{(2N_f,\alpha)}(\lambda) \;=\; r_{n+1} P_{n+1}^{(2N_f,\alpha)}(\lambda) \;+\; r_n P_{n-1}^{(2N_f,\alpha)}(\lambda) \ , \tag{2.20}$$

under the assumption that the integrals over the $P_n^{(2N_f,\alpha)}(\lambda)$ exists. The recursion coefficients r_n are determined from the so-called string equation

$$n \;=\; -r_n \int_{-\infty}^{\infty} d\lambda\, w'(\lambda) P_n^{(2N_f,\alpha)}(\lambda)\, P_{n-1}^{(2N_f,\alpha)}(\lambda) \ . \tag{2.21}$$

In the case of $\alpha = N_f = 0$, which is all we need in order to determine the macroscopic spectral density, one can show that by setting $r_N = 2c$ eq. (2.21) is equivalent to the normalization condition of the spectral density arising from the saddle-point analysis (e.g. see eq. (2.12)).

In order to determine the spectral density one has to consider also differentiation of orthogonal polynomials. Here we will use the Shohat method, which is reviewed in great detail in [35]. It consists in a particularly useful rewriting of $P_n'(\lambda)$, which is in general a linear combination of all lower polynomials, in terms of two auxiliary functions $A_n(\lambda)$ and $B_n(\lambda)$.

For the rest of this section we will restrict ourselves to the massless case, $N_f = 0$, (and suppress the index). The massive case with $N_f \neq 0$ will be treated later in Section 3. Furthermore we absorb the massless prefactor $|\lambda|^{2\alpha}$ from the weight function into the potential

$$V_\alpha(\lambda) \;=\; V(\lambda) - \frac{2\alpha}{N}\ln|\lambda|\,. \tag{2.22}$$

Defining the auxiliary functions[1]

$$A_n(\lambda) \;=\; N\; r_n \int_{-\infty}^{\infty} dt\; w(t)\frac{tV_\alpha'(t) - \lambda V_\alpha'(\lambda)}{t^2-\lambda^2} P_n^{(\alpha)}(t)^2\;, \tag{2.23}$$

$$B_n(\lambda) \;=\; N\; r_n \int_{-\infty}^{\infty} dt\; w(t)\frac{\lambda V_\alpha'(t) - tV_\alpha'(\lambda)}{t^2-\lambda^2} P_n^{(\alpha)}(t)P_{n-1}^{(\alpha)}(t)\;, \tag{2.24}$$

it is easy to show using the Christoffel-Darboux identity that the following relation for differentiation holds for any finite n and for any potential $V_\alpha(\lambda)$

$$\frac{d\; P_n^{(\alpha)}(\lambda)}{d\lambda} \;\equiv\; A_n(\lambda)P_{n-1}^{(\alpha)}(\lambda) \;-\; B_n(\lambda)P_n^{(\alpha)}(\lambda)\,. \tag{2.25}$$

Furthermore these functions obey the following identity at any finite n [35]:

$$B_n(\lambda) + B_{n-1}(\lambda) \;+\; N\; V_\alpha'(\lambda) \;=\; \frac{\lambda}{r_{n-1}} A_{n-1}(\lambda)\;. \tag{2.26}$$

The relation (2.25) will be the key to derive an exact differential equation for the wave functions

$$\psi_n^{(\alpha)}(\lambda) \;\equiv\; w(\lambda)^{\frac{1}{2}} P_n^{(\alpha)}(\lambda) \tag{2.27}$$

in the next section and to prove universality.

Now we are using eq. (2.25) to derive the spectral density for the general potential defined in eq. (2.2). From eqs. (2.18) and (2.19) the one-point function or spectral density is simply given by a single kernel at equal arguments:

$$R_N^{(\alpha)}(\lambda) \;=\; w(\lambda)\; r_N \left({P_N^{(\alpha)}}'(\lambda)P_{N-1}^{(\alpha)}(\lambda) - {P_N^{(\alpha)}}'(\lambda)P_{N-1}^{(\alpha)}(\lambda)\right)\;. \tag{2.28}$$

Using the recursion relation (2.20) and the identities (2.25) and (2.26) we arrive at [35]

$$\begin{aligned} R_N^{(\alpha)}(\lambda) \;&=\; r_N \left[A_N(\lambda)\psi_{N-1}^{(\alpha)}(\lambda)^2 + \frac{r_N}{r_{N-1}} A_{N-1}(\lambda)\psi_N^{(\alpha)}(\lambda)^2 \right. \\ &\quad \left. - \left(\frac{\lambda}{r_{N-1}} A_{N-1}(\lambda) + B_N(\lambda) - B_{N-1}(\lambda)\right)\psi_N^{(\alpha)}(\lambda)\psi_{N-1}^{(\alpha)}(\lambda)\right]\;, \end{aligned} \tag{2.29}$$

which is exact for any finite N. We will now take the macroscopic large-N limit where we smooth the oscillations of the correlators. By considering smoothed moments of the orthogonal polynomials at large-N the following result has been shown in [35]. For any set of orthogonal polynomials of a given measure $w(\lambda)$ which first leads to a large-N spectral density with single interval support σ and second has recursion coefficients obeying the condition $r_{N\pm1,\pm2,\ldots} \to r_N = 2c$ when $N \to \infty$, it holds

$$\begin{aligned} \overline{\psi_N^{(\alpha)}(\lambda)^2} \;&=\; \frac{1}{\pi}\frac{1}{\sqrt{c^2-\lambda^2}}\;\theta(c^2-\lambda^2)\;, \\ \overline{\psi_N^{(\alpha)}(\lambda)\psi_{N-1}^{(\alpha)}(\lambda)} \;&=\; \frac{1}{c\pi}\frac{\lambda}{\sqrt{c^2-\lambda^2}}\;\theta(c^2-\lambda^2)\;. \end{aligned} \tag{2.30}$$

[1] Compared to [35] we have already used the symmetry of the potential (see also appendix A there).

Here the bar denotes the smoothed large-N limit. Let us stress that the derivation of eq. (2.30) in the Appendix A of [35] is purely algebraic. No assumption about the analytic structure of the spectral density in the complex plane has been made, in particular no square-root behavior has been assumed. Using these results we can take the same smoothed large-N limit for the functions $A_N(\lambda)$ and $B_N(\lambda)$ defined in eqs. (2.23) and (2.24). Smoothing inside the integrand we obtain

$$A(\lambda) \equiv \frac{1}{N}\overline{A_N(\lambda)} = \frac{c}{\pi}\int_0^c dt\, \frac{tV'(t)-\lambda V'(\lambda)}{t^2-\lambda^2}\, \frac{1}{\sqrt{c^2-t^2}}\,, \tag{2.31}$$

$$B(\lambda) \equiv \frac{1}{N}\overline{B_N(\lambda)} = \frac{1}{\pi}\int_0^c dt\, \frac{\lambda V'(t)-tV'(\lambda)}{t^2-\lambda^2}\, \frac{t}{\sqrt{c^2-t^2}}\,. \tag{2.32}$$

Note that compared to eqs. (2.23) and (2.24) we have dropped the index α in the potential $V(\lambda)$ or in other words the term $2\alpha \ln|\lambda|/N$ from eq. (2.22). This can be seen as follows. Because of parity it drops out in $A_N(\lambda)$ already at finite N. In B_N one can split [35] $B_N(\lambda) = B_{N,\,reg}(\lambda)+(1-(-1)^N)\alpha/\lambda$ where the first term only contains V instead of V_α. The latter term obviously has no smooth limit but it is suppressed by $1/N$ in eq. (2.32).

We can now apply the smooth limit to the density eq. (2.29) itself and we obtain

$$\rho(\lambda) \equiv \frac{1}{N}\overline{R_N^{(\alpha)}(\lambda)} = \frac{1}{c\pi}A(\lambda)\sqrt{c^2-\lambda^2}\,, \quad \lambda \in [-c,c]\,. \tag{2.33}$$

In particular it holds that

$$\rho(0) = \frac{1}{\pi}\,A(0)\,. \tag{2.34}$$

Eq. (2.33) together with eq. (2.31) constitute our main result of this section. Let us stress that it does not necessarily imply that $\rho(\lambda)$ has a simple square-root cut as a function in the complex plane. The function $A(\lambda)$ having nonpolynomial parts inside the potential may precisely cancel the factor $\sqrt{c^2-\lambda^2}$, as we shall see in a simple example now. Let us choose again the nonpolynomial potential

$$V_{Freud}(\lambda) = g|\lambda|^\kappa\,, \quad 1<\kappa\in\mathbb{R}\,. \tag{2.35}$$

From eq. (2.31) we obtain for the auxiliary function

$$\begin{aligned} A(\lambda) &= \frac{c}{\pi}\, g\kappa \int_0^c dt\, \frac{t^\kappa-\lambda^\kappa}{t^2-\lambda^2}\frac{1}{\sqrt{c^2-t^2}} \\ &= cg\kappa\left(\frac{\lambda^{\kappa-1}}{2\sqrt{c^2-\lambda^2}} - \frac{c^\kappa H}{\kappa\lambda^2}F\left(1,\,\frac{1}{2}+\frac{\kappa}{2};\,1+\frac{\kappa}{2};\,\frac{c^2}{\lambda^2}\right)\right). \end{aligned} \tag{2.36}$$

In order to be able to expand the density around the origin and to see the square root cancelling in eq. (2.33) we still have to use some identities for Hypergeometric function (and to analytically continue to $\lambda^2<c^2$)

$$\begin{aligned} \frac{1}{\lambda^2}F\left(1,\,\frac{1}{2}+\frac{\kappa}{2};\,1+\frac{\kappa}{2};\,\frac{c^2}{\lambda^2}\right) &= \frac{1}{\sqrt{\lambda^2-c^2}}\,\frac{1}{\lambda}F\left(\frac{\kappa}{2},\,\frac{1}{2};\,1+\frac{\kappa}{2};\,\frac{c^2}{\lambda^2}\right) \\ &= \frac{1}{\sqrt{\lambda^2-c^2}}\left(\frac{\Gamma(1+\frac{\kappa}{2})\Gamma(\frac{1}{2}-\frac{\kappa}{2})}{\sqrt{\pi}}e^{i\frac{\pi}{2}\kappa}\frac{\lambda^{\kappa-1}}{c^\kappa}\right. \\ &\quad \left. + \frac{\Gamma(1+\frac{\kappa}{2})\Gamma(\frac{\kappa}{2}-\frac{1}{2})}{\Gamma(\frac{\kappa}{2})\Gamma(\frac{\kappa}{2}+\frac{1}{2})}\frac{i}{c}F\left(\frac{1}{2},\,\frac{1}{2}-\frac{\kappa}{2};\,\frac{3}{2}-\frac{\kappa}{2};\,\frac{\lambda^2}{c^2}\right)\right). \end{aligned} \tag{2.37}$$

Inserting this back into eq. (2.36) we obtain with eq. (2.33) the final expression for our density as given in eq. (2.10):

$$\rho_{Freud}(\lambda) \;=\; \frac{g\kappa H c^{\kappa-1}}{\pi}\left(\frac{1}{2H}\tan\left[\frac{\kappa\pi}{2}\right]\left(\frac{\lambda}{c}\right)^{\kappa-1}+\frac{1}{\kappa-1}F\left(\frac{1}{2},\,\frac{1}{2}-\frac{\kappa}{2};\,\frac{3}{2}-\frac{\kappa}{2};\,\frac{\lambda^2}{c^2}\right)\right). \tag{2.38}$$

This solution for the density we have derived can be seen to be equivalent to the first representation given in eq. (2.8), by using the identity relating Hypergeometric functions with argument z and $1-z$ respectively[2]. The advantage of eq. (2.10) is that we can immediately read off the expansion of the density at the origin. For noninteger $\kappa > 1$ it consists of a single term $\sim \lambda^{\kappa-1}$ plus a power series in λ^2. It is the first term which will be responsible for the new critical behavior to be discussed in Section 4.

3 Microscopic universality

In this section we prove that in the microscopic large-N scaling limit at the origin all correlation functions of our model (2.1) are universal, as long as the couplings in the potential $V(\lambda)$ eq. (2.2) are generic (=noncritical) and we stay in the phase of a single interval support. They belong to different universality classes depending on the number of massless and massive flavors, α and N_f respectively, as well as on the rescaled masses m_f. The universal parameter will be $\rho(0)$ which encodes the influence of all the coupling constants in the potential via eq. (2.34). Here we will make use of the powerful results on universality for polynomial potentials which have been obtained for massless [6, 8, 17] and massive [9, 10, 12] unitary and chiral unitary matrix models. Our extension to nonpolynomial potentials works as follows. We start with the unitary model without massive flavors, $N_f = 0$, resuming the method of orthogonal polynomials used in the last section. A differential equation for the asymptotic of the polynomials will be derived for arbitrary $\alpha > -1$ following [8]. Once we have established massless universality we can use the method of [12] to extend it to an arbitrary number of massive flavors N_f. In contrast to the chiral case no restriction on N_f to be even is needed here, as has been noted already in the extension to the non Hermitian case [40]. In a final step we use a direct relation between the polynomials of the unitary and the chiral unitary ensemble [17] to prove universality also in the chiral case, for arbitrary α and N_f.

Let us start with the unitary case and $N_f = 0$ as in the end of the last section. Following [35], eq. (2.25) for differentiation on the polynomials can be used together with the recursion relation (2.20) and the identity (2.26) to derive an exact second order differential equation for the wave functions $\psi_n^{(\alpha)}(\lambda)$ eq. (2.27) for any finite n (for the present form see [17]):

$$\psi_n^{(\alpha)}(\lambda)'' - F_n(\lambda)\psi_n^{(\alpha)}(\lambda)' + G_n(\lambda)\psi_n(\lambda) \;=\; 0\;, \tag{3.1}$$

where

$$\begin{aligned} F_n(\lambda) \;&\equiv\; \frac{A_n'(\lambda)}{A_n(\lambda)}\;, \\ G_n(\lambda) \;&\equiv\; \frac{r_n}{r_{n-1}}A_n(\lambda)A_{n-1}(\lambda) - \left(B_n(\lambda)+\frac{N}{2}V_\alpha'(\lambda)\right)^2 \\ &\quad + \left(B_n(\lambda)+\frac{N}{2}V_\alpha'(\lambda)\right)' - \frac{A_n'(\lambda)}{A_n(\lambda)}\left(B_n(\lambda)+\frac{N}{2}V_\alpha'(\lambda)\right)\;. \end{aligned} \tag{3.2}$$

[2]When κ is not an odd integer number. For κ an odd integer number the formula is slightly more complicated [39]. However, formula (2.10) is completely fine here because as we will see in Section 4, odd integers κ do not affect the issue of multicriticality we are considering.

Imposing a smooth limit for the recursion coefficients r_N as well as for $A_N(\lambda)$ and the regular part of $B_N(\lambda)$, the function $G_N(\lambda)$ can be entirely expressed in terms of the limiting function $A(\lambda)$, due to the identity (2.26). We obtain [17]

$$\psi_N^{(\alpha)}(\lambda)'' - \frac{A'(\lambda)}{A(\lambda)}\psi_N^{(\alpha)}(\lambda)' + \left(N^2A^2(\lambda)\left(1-\frac{\lambda^2}{c^2}\right) + \frac{(-1)^N\alpha-\alpha^2}{\lambda^2} + (-1)^N\frac{\alpha A(\lambda)'}{\lambda A(\lambda)}\right)\psi_N(\lambda) \;=\; 0\;. \tag{3.3}$$

The microscopic large-N limit at the origin is defined[3] as

$$\xi \;=\; N\lambda\;, \tag{3.4}$$

where ξ is kept fixed at large-N. This rescaling is the only consistent way to obtain a finite large-N differential equation from eq. (3.3) and it agrees with the rescaling in the pure Freud case obtained in [19, 20, 21, 22] when translating to their conventions. Eq. (3.4) holds despite the fact that $A'(\lambda)$ diverges $\sim \lambda^{\kappa-2}$ for $1<\kappa<2$. Even in that case the logarithmic derivative of $A(\lambda)$ (as well as $\lambda^2A(\lambda)^2$) is suppressed and we arrive at

$$\psi_N(\xi)'' + \left(A(0)^2 + \frac{(-1)^N\alpha-\alpha^2}{\xi^2}\right)\psi_N(\xi) \;=\; 0\;. \tag{3.5}$$

This equation of Bessel type entirely determines the asymptotic large-N behavior of the wave function, distinguishing between even and odd polynomials. Moreover it is universal since it only depends on the potential $V(\lambda)$ eq. (2.2) through the universal parameter $A(0)=\pi\rho(0)$. We refer to [6, 8] for the explicit solution of the differential equation as we are only interested in proving an extension of universality to nonpolynomial potentials here. Similarly we refer to [6] for the correlation functions, where detailed expressions can be found. We only wish to stress here that from eq. (3.5) the rescaled microscopic kernel

$$K_S^{(\alpha)}(\xi,\zeta) \;\equiv\; \lim_{N\to\infty}\frac{1}{N}K_N^{(\alpha)}\left(\frac{\xi}{N},\frac{\zeta}{N}\right) \tag{3.6}$$

follows as it can be entirely expressed in terms of the asymptotic wave functions (see eq. (2.19)). The microscopic correlation functions defined as

$$\rho_S^{(\alpha)}(\xi_1,\ldots,\xi_k) \;\equiv\; \lim_{N\to\infty}\frac{1}{N^k}R_N^{(\alpha)}\left(\frac{\xi_1}{N},\ldots,\frac{\xi_k}{N}\right) \tag{3.7}$$

can then be obtained from the rescaled version of eq. (2.18).

Up to now we have determined all microscopic correlation functions from eq. (3.5) for an arbitrary $\alpha>-1$ and proved their universality. In the next step we reintroduce the mass terms from eq. (2.1) and determine their universal correlations. Following [12] these masses can be put into the Vandermonde determinant leading to a relation between the massive k-point correlator and the massless $(k+N_f)$-point correlator analytically continuated in some of the arguments.

To be more precise we can rewrite the mass terms times the Vandermonde in the integrand of eq. (2.1) as

$$\prod_{j=1}^{N}\prod_{f=1}^{N_f}(\lambda_j-im_f)(\lambda_j+im_f)\;|\Delta_N(\lambda_1,\ldots,\lambda_N)|^2 \;=\; \frac{|\Delta_{N+N_f}(\lambda_1,\ldots,\lambda_N,im_1,\ldots,im_{N_f})|^2}{|\Delta_{N_f}(im_1,\ldots,im_{N_f})|^2}\;. \tag{3.8}$$

[3]For $1<\kappa<2$ the slope of $\rho(\lambda)$ is infinite at the origin (see Figure 1) and thus the eigenvalues have to be unfolded before rescaling. We thank K. Splittorff for raising this point.

Here we have explicitly given all the arguments of the Vandermonde determinant. Because of the pairing of the $2N_f$ masses in pairs $\pm im_f$ coming from QCD3 we do not need to impose an extra degeneracy to the power of the Dyson index $\beta = 2$ as it was done in the chiral case [12]. This observation valid for the unitary case was made in [40]. Inserting eq. (3.8) into the definition (2.17) we see that the right hand side is proportional to a massless $k + N_f$-point function, with no mass terms. Inserting all normalization factors we arrive at [40][4]

$$R_N^{(2N_f,\alpha)}(\lambda_1,\ldots,\lambda_k) \;=\; \frac{R^{(0,\alpha)}_{N+N_f}(\lambda_1,\ldots,\lambda_k,im_1,\ldots,im_{N_f})}{R^{(0,\alpha)}_{N+N_f}(im_1,\ldots,im_{N_f})}\;. \tag{3.9}$$

Since for the right hand side we have already shown its universality in the microscopic limit the same statement holds for the left hand side. Here the masses have to be rescaled the same way as the eigenvalues, keeping $\mu_f \equiv Nm_f$ fixed. We have thus calculated the massive correlation functions and at the same time given a proof for their universality. Explicit expressions in different equivalent forms can be found in [10, 41]. We have thus not only extended the universality proof of [10] to potentials of the form in eq. (2.2) but also considerably shortcut proof and calculations.

After having completed the universality proof for the full model eq. (2.1) it remains to repeat the analysis for the corresponding chiral model. The partition function can be written in two ways:

$$\begin{aligned}
\mathcal{Z}^{(N_f,\alpha)}_{chiral}(\{m_f\}) &= \int_0^\infty \prod_{i=1}^N \left(d\lambda_i\; \lambda_i^\alpha \prod_{f=1}^{N_f} \left(\lambda_i \;+\; m_f^2\right)\; e^{-NV_{chiral}(\lambda_i)} \right) |\Delta(\lambda)|^2 \\
&= \int_{-\infty}^\infty \prod_{i=1}^N \left(d\lambda_i\; |\lambda_i|^{2\alpha+1} \prod_{f=1}^{N_f} \left(\lambda_i^2 \;+\; m_f^2\right)\; e^{-NV_{chiral}(\lambda_i^2)} \right) |\Delta(\lambda^2)|^2\;, \qquad (3.10)\\
V_{chiral}(\lambda^2) &\equiv \sum_{j=1}^{d_p} \frac{g_{2j}}{j}\lambda^{2j} \;+\; \sum_{j=1}^{d_{np}} 2h_j|\lambda|^{\kappa_j} \quad , \qquad (3.11)
\end{aligned}$$

where in the second line we have simply changed variables $\lambda \to \lambda^2$. In order to calculate the correlation functions we introduce again orthogonal polynomials in analogy to eq. (2.15):

$$\begin{aligned}
\delta_{nm} &= \int_0^\infty d\lambda\, \lambda^\alpha \prod_{f=1}^{N_f}(\lambda + m_f^2)\; e^{-NV_{chiral}(\lambda)} P^{(N_f,\alpha)}_{n,chiral}(\lambda)\, P^{(N_f,\alpha)}_{m,chiral}(\lambda) \qquad (3.12)\\
&= \int_{-\infty}^\infty dz\, |\lambda|^{2\alpha+1} \prod_{f=1}^{N_f}(\lambda^2 + m_f^2)\; \mathrm{e}^{-NV_{chiral}(\lambda^2)} P^{(N_f,\alpha)}_{n,chiral}(\lambda^2)\, P^{(N_f,\alpha)}_{m,chiral}(\lambda^2)\;.
\end{aligned}$$

By comparing to the unitary orthogonal polynomials eq. (2.15) we can immediately read off that the two sets are related. This relation has been already found in [17] and trivially extends to the massive case [34]:

$$P_{2N}^{(2N_f,\alpha+1/2)}(\lambda) \;=\; P^{(N_f,\alpha)}_{N,chiral}(\lambda^2)\;, \tag{3.13}$$

where we also have to identify the corresponding potentials

$$2V(\lambda) \;\equiv\; V_{chiral}(\lambda^2)\;. \tag{3.14}$$

[4]To be precise the relations holds for finite-N only in the normalization of [12]. At large-N this becomes irrelevant.

In other words the even subset of the polynomials of the Unitary Ensemble at shifted $\alpha+\frac{1}{2}$ is sufficient to construct the full set polynomials of the chiral ensemble. The same is true for the wave functions. Since we have started with a real α this shift is possible. We can thus borrow the full machinery for the asymptotic of the polynomials in the unitary case, including universality. We have shown already that through eq. (3.5) the asymptotic of the wave functions in the massless unitary case are the same for polynomial and nonpolynomial potentials and that they are universal. Through eq. (3.13) the same statements immediately translate to the wave functions of the massless chiral ensemble and thus through eqs. (2.19) and (2.18) to all massless correlation functions. For the massive correlations we have to invoke another relation since for the Unitary Ensemble we did not determine the massive orthogonal polynomials. We directly determined the massive correlation functions through eq. (3.9) and thus we cannot use eq. (3.13) to read off the massive chiral orthogonal polynomials. Fortunately the corresponding massive kernels of the two ensembles, chiral and non chiral, can be directly related as well by using eq. (3.13) [34]:

$$K_{N,chiral}^{(N_f,\alpha)}(\lambda^2,\eta^2) \;=\; \frac{1}{2}\left(K_{2N}^{(2N_f,\alpha+1/2)}(\lambda,\eta) \;+\; K_{2N}^{(2N_f,\alpha+1/2)}(-\lambda,\eta)\right) \;. \tag{3.15}$$

The massive kernel of the Unitary Ensemble on the right hand side follows from the massive two-point function that we have already determined in eq. (3.9). Inspecting eq. (2.18) we see that it follows from

$$K_N^{(2N_f,\alpha)}(\lambda,\eta) \;=\; \frac{1}{N}\sqrt{-R_{N,conn}^{(2N_f,\alpha)}(\lambda,\eta)}\;, \tag{3.16}$$

where the connected part of the correlation function is defined as

$$R_{N,conn}^{(2N_f,\alpha)}(\lambda,\eta) \;\equiv\; R_N^{(2N_f,\alpha)}(\lambda,\eta) - R_N^{(2N_f,\alpha)}(\lambda)R_N^{(2N_f,\alpha)}(\eta)\;. \tag{3.17}$$

Using again eq. (2.18) we have finally also determined all massive correlation functions of the chiral ensemble. Our consideration establishes an alternative proof to the original one in [9], in a considerably simpler way. This is the gain we made by first solving the nonchiral ensemble. Let us stress that in contrast to [12] we have no restriction on the flavor number N_f.

Finally we mention that the identification of the potentials, eq. (3.14) is particularly useful when studying the multicritical point as it will be done in the next section. We can therefore restrict ourselves again to the unitary case.

4 New multicritical behavior at the origin

Multicritical points occur when the spectral density $\rho(\lambda)$ develops additional zeros inside (or at the edge of) the support of the eigenvalues. Since in this section we are interested in applications to the chiral phase transition in QCD, with the chiral condensate being proportional to $\rho(0)$, we restrict ourselves to additional zeros at the origin of the spectrum. The multicritical points can also be thought of the support merging together from two (or more) segments. In fact for the question of the order of the phase transition the free energy and its derivatives have to be calculated from both sides and compared. In general these transitions are of third order which also holds for additional zeros developing elsewhere on the support.

Before defining the new class of critical potentials let us briefly review what is known about multicritical models with polynomial potentials. Since we only study the behavior of the macroscopic density for the degree of criticality we can set again $N_f = \alpha = 0$ since it does not depend on these parameters. From eq. (2.31) we deduce that for polynomial potentials the density (2.33) consists of an even polynomial times a square root $\sqrt{c^2-\lambda^2}$. Expanding $\rho(\lambda)$ at the origin and tuning the

couplings for $\rho(0)$ to vanish we can only achieve that $\rho(\lambda = 0) \sim \lambda^{2m}$ for $m \in \mathbb{N}$. For the mth multicriticality we have to tune precisely m coupling constants. In [42] a set of minimal potentials was given. The corresponding mth multicritical correlation functions were determined in [17] and shown to be universal. We will find a similar situation here (although we will not touch the issue of universality), namely for a spectral density vanishing like a real power $\rho(\lambda = 0) \sim \lambda^{\kappa-1}$, with $2m - 2 < \kappa - 1 < 2m$ we have to tune m coupling constants.

To see this let us start with the simplest example, a Gaussian plus a single real positive power:

$$V(\lambda) = \frac{1}{2} g_2 \lambda^2 \ + \ g|\lambda|^\kappa \ . \tag{4.1}$$

Instead of the solution eq. (2.11) we use the equivalent representation for the integral, eq. (2.10):

$$\rho(\lambda) \ = \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \ + \ \frac{g\kappa H c^{\kappa-1}}{\pi(\kappa-1)} F\left(\frac{1}{2}, \frac{1}{2} - \frac{\kappa}{2}; \frac{3}{2} - \frac{\kappa}{2}; \frac{\lambda^2}{c^2}\right) \ + \ \frac{1}{2\pi} g_2 \sqrt{c^2 - \lambda^2}, \tag{4.2}$$

where we now have to restrict to $1 < \kappa \neq 3, 5, 7, \ldots$. The normalization condition eq. (2.12) which we repeat here for completeness fixes one of the coupling constants

$$1 \ = \ \frac{1}{4} g_2 c^2 + H g c^\kappa \ . \tag{4.3}$$

When expanding the density at the origin the Hypergeometric function as well as the square root both have an expansion in λ^2 and we obtain

$$\rho(\lambda) \ = \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \ + \ \frac{1}{2\pi}\left(\frac{2\kappa}{\kappa-1} g H c^{\kappa-1} \ + \ g_2 c\right) \ + \ \mathcal{O}(\lambda^2) \ . \tag{4.4}$$

By tuning the remaining coupling constant we can achieve $\rho(0) = 0$ through

$$0 \ = \ \frac{2\kappa}{\kappa-1} g H c^\kappa \ + \ g_2 c^2 \ . \tag{4.5}$$

However, only for $\kappa < 3$ the first term in eq. (4.4) with a real power is dominant and we have

$$\rho(\lambda = 0) \ \sim \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \ \lambda^{\kappa-1} \ + \ \mathcal{O}(\lambda^2) \quad \text{for} \ \ 1 < \kappa < 3 \ \ . \tag{4.6}$$

Consequently the potential eq. (4.1) together with the condition eq. (4.5) and the normalization eq. (4.3) defines our first new class (4.6) of multicritical models. A few comments are in order. First of all it is easy to see that equations (4.5) and (4.3) have a unique solution with $g_2 > 0, \ g < 0$ for $1 < \kappa < 2$ and with $g_2 < 0, \ g > 0$ for $2 < \kappa < 3$. In both cases the slope of $\rho(\lambda)$ in eq. (4.6) is positive (infinite or finite), as it should be. An example for each of such cases is given in Figure 1.

For real $\kappa > 3$ the quadratic term in (4.4) is leading and we are back in the class $\rho(\lambda = 0) \sim \lambda^{2m}$ for $m = 1$ [17]. If we add a higher order term to the polynomial part of the potential, like a quartic power for example, we can still achieve that $\rho(0) = 0$. There is one free parameter left and we remain in the multicritical class of [17] unless we also cancel the $\mathcal{O}(\lambda^2)$ term. In the latter case, which we will give as a second example below we may arrive at higher criticality, $\rho(\lambda = 0) \sim \lambda^{\kappa-1}$, with $3 < \kappa < 5$. Before doing that let us remark that instead of eq. (4.1) we could have started with a potential $V(\lambda) = h_1|\lambda|^{\kappa_1} + h_2|\lambda|^{\kappa_2}$ instead of perturbing around the Gaussian. Going through the derivation again, where we now add two densities of the type eq. (2.10), we could again achieve that $\rho(0) = 0$. The leading order term of $\rho(\lambda)$ would then be to the power of $\min\{\kappa_1 - 1, \kappa_2 - 1, 2\}$. Hence in case of $1 < \kappa_i < 3$ for at least one of the κ_i we would be again in class (4.6), otherwise in class $m = 1$ of [17]. Adding higher order terms, polynomial or not, would then allow to obtain higher criticality as well.

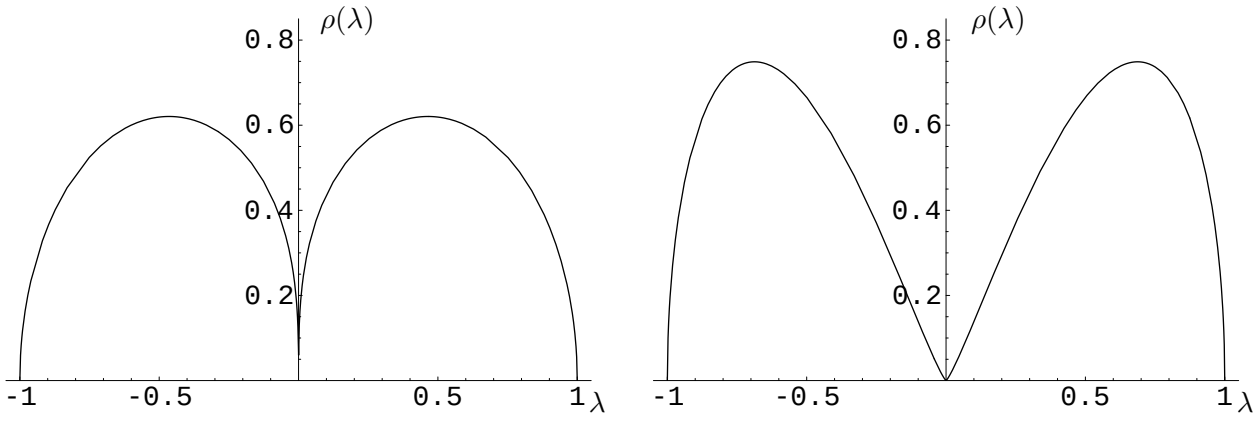

Figure 1: The multicritical density for $1 < \kappa = \frac{4}{3} < 2$ (left) and for $2 < \kappa = \sqrt{5} < 3$ (right).

In order to keep the algebra simple we make the minimal choice of one real power plus a polynomial part for the minimal critical potentials. As an example for the second multicritical class we thus add a quartic term to eq. (4.1) and start with

$$V(\lambda) = \frac{1}{2} g_2 \lambda^2 \ + \ \frac{1}{4} g_4 \lambda^4 \ + \ g|\lambda|^\kappa \ . \tag{4.7}$$

The density can be obtained using eqs. (2.33) and (2.31) or by simply adding the well known density of the purely quartic potential [36] to eq. (2.11) and normalizing properly. We obtain

$$\begin{aligned} \rho(\lambda) \ &= \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \ + \ \frac{g\kappa H c^{\kappa-1}}{\pi(\kappa-1)} F\left(\frac{1}{2}, \frac{1}{2} - \frac{\kappa}{2}; \frac{3}{2} - \frac{\kappa}{2}; \frac{\lambda^2}{c^2}\right) \\ &+ \ \frac{1}{2\pi}\left(g_2 + \frac{1}{2} g_4 c^2 + g_4 \lambda^2\right) \sqrt{c^2 - \lambda^2} \ , \end{aligned} \tag{4.8}$$

with the normalization condition

$$1 \ = \ \frac{1}{4} g_2 c^2 \ + \ \frac{3}{16} g_4 c^4 \ + \ H g c^\kappa \ . \tag{4.9}$$

Expanding at $\lambda = 0$ and setting the terms of $\mathcal{O}(1)$ and $\mathcal{O}(\lambda^2)$ to zero we obtain

$$\rho(\lambda) \ \sim \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \ + \ \mathcal{O}(\lambda^4) \quad \text{for} \ \ 3 < \kappa < 5 \ \ . \tag{4.10}$$

with conditions

$$\begin{aligned} 0 \ &= \ \frac{2\kappa}{\kappa - 1} g H c^\kappa \ + \ g_2 c^2 \ + \ \frac{1}{2} g_4 c^4 \ , \\ 0 \ &= \ \frac{\kappa}{\kappa - 3} g H c^\kappa \ - \ \frac{1}{2} g_2 c^2 \ + \ \frac{3}{4} g_4 c^4 \ . \end{aligned} \tag{4.11}$$

After giving two explicit examples we now wish to come to the general mth multicritical density and potential. In order to have the first term in eq. (4.8) dominating for $2m - 2 < \kappa < 2m$ we have to add a polynomial potential with at least m coupling constants in order to cancel the first m terms in the expansion of the Hypergeometric function there. Generalizing the cases eq. (4.1) and eq. (4.7), hereafter we consider the potential

$$V(\lambda) \ = V_{pol}(\lambda) + g|\lambda|^\kappa = \ \sum_{j=1}^{m} \frac{g_{2j}}{2j} \lambda^{2j} \ + \ g|\lambda|^\kappa \ . \tag{4.12}$$

The general one-cut solution for the polynomial part is know to be [5]

$$\rho_{pol}(\lambda) = \frac{1}{2\pi}\sum_{j=1}^{m} g_{2j} \sum_{k=0}^{j-1} \left(\begin{array}{c} 2k \\ k \end{array} \right) \frac{c^{2k}}{2^{2k}} \lambda^{2j-2k-2} \sqrt{c^2-\lambda^2}\,, \tag{4.13}$$

which we rewrite as

$$\rho_{pol}(\lambda) \;=\; \sum_{j=0}^{m-1} a_j \frac{\lambda^{2j}}{c^{2j}} \sqrt{1-\frac{\lambda^2}{c^2}}\;, \tag{4.14}$$

with

$$a_j \;\equiv\; \frac{c\,2^{2j-1}}{\pi} \sum_{k=j}^{m-1} g_{2k+2} \left(\begin{array}{c} 2(k-j) \\ k-j \end{array} \right) \left(\frac{c}{2}\right)^{2k} \;, \quad j=0,\ldots,m-1\;. \tag{4.15}$$

The full density is obtained by adding the Freud-type density eq. (2.10) to eq. (4.14), and the normalization condition reads

$$1 \;=\; gHc^{\kappa} + c\pi \sum_{j=0}^{m-1} a_j \; \frac{(2j-1)!!}{2^{j+1}(j+1)!}\,, \tag{4.16}$$

with $(-1)!! \equiv 1$. When expanding the Hypergeometric function at $\lambda = 0$ we further obtain m equations by setting the terms $\mathcal{O}(1)$, ..., $\mathcal{O}(\lambda^{2m-2})$ to zero. We find[5]

$$a_j \;=\; \frac{g\kappa H c^{\kappa-1}}{\pi(1-\kappa)} \prod_{l=1}^{j} \frac{(2l-\kappa)}{(2l+1-\kappa)} \;\;,\; j=0,\ldots,m-1\;. \tag{4.17}$$

The mth multicritical density is thus reading

$$\begin{aligned} \rho(\lambda) \;&=\; \frac{g\kappa H c^{\kappa-1}}{\pi(\kappa-1)} \left(F\left(\frac{1}{2},\frac{1}{2}-\frac{\kappa}{2};\frac{3}{2}-\frac{\kappa}{2};\frac{\lambda^2}{c^2}\right) \;-\; \sum_{j=0}^{m-1}\prod_{l=1}^{j} \frac{(2l-\kappa)}{(2l+1-\kappa)}\; \frac{\lambda^{2j}}{c^{2j}} \sqrt{1-\frac{\lambda^2}{c^2}} \right) \\ &\quad + \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \;\;, \qquad\qquad 2m-1<\kappa<2m+1\;, \end{aligned} \tag{4.18}$$

and it behaves like $\rho(\lambda=0) \sim \lambda^{\kappa-1} + \mathcal{O}(\lambda^{2m})$. The critical values of the $m+1$ coupling constants g_{2j}^c and g^c in the potential eq. (4.12) can be obtained by iteratively solving the linear set of $m+1$ equations given by eq. (4.15) and eq. (4.16), which is already of triangular form. Actually there exists a much more compact form for eq. (4.18). It can be obtained by iteratively using the identity

$$\frac{(\kappa-2)}{(\kappa-3)}\; z^2\; F\left(\frac{1}{2},\frac{3}{2}-\frac{\kappa}{2};\frac{5}{2}-\frac{\kappa}{2};z^2\right) \;=\; F\left(\frac{1}{2},\frac{1}{2}-\frac{\kappa}{2};\frac{3}{2}-\frac{\kappa}{2};z^2\right) \;-\; \sqrt{1-z^2}\;, \tag{4.19}$$

which can be derived using standard formulas [39] . Going back to the first criticality $m=1$, in eq. (4.2) the square root and the Hypergeometric function can be put together using the identity (4.19) and eq. (4.5) (see also eq. (A.10)). At $m=2$ for eq. (4.8) we can do the same. The resulting Hypergeometric function can be simplified again together with the term $\lambda^2\sqrt{c^2-\lambda^2}$, using the identity (4.19) at $\kappa \to \kappa-2$. Proceeding by induction with the general form eq. (4.18) we arrive at our final result:

$$\rho(\lambda) = \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} + \frac{g\kappa H c^{\kappa-1}}{\pi(\kappa-1)} \frac{\lambda^{2m}}{c^{2m}} F\left(\frac{1}{2},\frac{1}{2}+m-\frac{\kappa}{2};\frac{3}{2}+m-\frac{\kappa}{2};\frac{\lambda^2}{c^2}\right) \prod_{l=1}^{m} \frac{(2l-\kappa)}{(2l+1-\kappa)}\,. \tag{4.20}$$

[5]The meaningless product $\prod_{l=1}^{0}$ which appears when $j=0$, is set equal to 1.

Here the vanishing at $\lambda = 0$ is manifest. Notice that in our parameterization, the end-cut point c is a given positive real number: that means that in general one has at most a critical line in the space of the parameters of the model. This finishes our complete classification of new multicritical models.

We turn now to the microscopic scaling limit of the multicritical models. While the differential equation (3.1) together with (3.2) for the wave functions $\psi_n^{(\alpha)}(\lambda)$ still holds exactly for finite N the large-N limit will change. We first determine the appropriate microscopic scaling limit that corresponds to eq. (3.4) at criticality. For that purpose we look at the large-N version of the differential equation (3.3). Although this equation will acquire additional corrections at multicriticality, as it happens at multicriticality for polynomial potentials [17], it will be sufficient to derive the correct scaling limit. Assuming also the relation (2.33) to hold, which will again have to be corrected as in [17], implies $A(\lambda) \sim \lambda^{\kappa-1}$ at $\lambda = 0$. This leads to the following approximate equation at $\lambda \approx 0$

$$\begin{aligned} &\psi_N^{(\alpha)}(\lambda)'' - \frac{\kappa-1}{\lambda}\psi_N^{(\alpha)}(\lambda)' \; + \\ &\left(N^2\frac{g^2\kappa^2}{4}\tan^2\left[\frac{\kappa\pi}{2}\right]\lambda^{2\kappa-2}\left(1-\frac{\lambda^2}{c^2}\right) + \frac{(-1)^N\alpha-\alpha^2}{\lambda^2} + (-1)^N\frac{\alpha(\kappa-1)}{\lambda^2}\right)\psi_N(\lambda) \;=\; 0\;. \end{aligned} \tag{4.21}$$

It is easy to see that in order to obtain a nontrivial large-N limit we have to define

$$\xi \;=\; N^{1/\kappa}\lambda \tag{4.22}$$

as the appropriate microscopic scaling limit at criticality[6]. The naive differential equation obtained with the rescaling eq. (4.22) then reads

$$\psi_N^{(\alpha)}(\xi)'' - \frac{\kappa-1}{\xi}\psi_N^{(\alpha)}(\xi)' \; + \left(\frac{g^2\kappa^2}{4}\tan^2\left[\frac{\kappa\pi}{2}\right]\xi^{2\kappa-2} + \frac{(-1)^N\alpha\kappa-\alpha^2}{\xi^2}\right)\psi_N(\xi) \;=\; 0\;. \tag{4.23}$$

This equation is clearly no longer of Bessel type and thus the corresponding correlation functions following from it belong to a new class. However, it cannot be the correct differential equation because it is not invariant under the following symmetry on wave functions,

$$\psi_{2n}^{(\alpha+1)}(\lambda) \;=\; \psi_{2n+1}^{(\alpha)}(\lambda)\;. \tag{4.24}$$

This relation has been shown to hold in [17], independent of the fact whether the potential is polynomial or not. However, eq. (4.23) may hold in an approximate sense as in [17] when in analogy to there the condition $1 \ll \frac{g^2\kappa^2}{4}\tan^2\left[\frac{\kappa\pi}{2}\right]\xi^{2\kappa-2}$ is satisfied.

In order to derive the correct differential equation at multicriticality we would have to proceed as in [17] and analyze the corrections to eq. (3.3) from (3.1) which are no longer subdominant at criticality. However, because of the nonpolynomial part of the potential this is forbiddingly difficult. Following [17] we first would have to derive the auxiliary functions $A_n(\lambda)$ and $B_n(\lambda)$, eqs. (2.23) and (2.24) respectively, at finite n. Choosing an explicit example for the potential such as eq. (4.1) in the beginning of this section one would insert it into the definition (2.23) and try to evaluate it at finite n. The obvious way to do so would be to use the recursion relation eq. (2.20) and the orthogonality of the polynomials. We arrive at

$$A_n(\lambda) \;=\; Nr_n\left(g_2 + \kappa\int_{-\infty}^{\infty} dt\; w(t)\frac{t^\kappa-\lambda^\kappa}{t^2-\lambda^2}P_n^{(\alpha)}(t)^2\right)\;. \tag{4.25}$$

[6] It generalizes the scaling behavior found in [28] from a different matrix model: $\xi = N^{(2q+1)/(2q+2)}\lambda$, $q \in \mathbb{N}$. There, multicriticality is reached by adding and tuning an external matrix to a Gaussian model.

The integral can only be evaluated at large-N using the result eq. (2.30) and not at finite n as we need. Let us point out a second difficulty in the finite-N analysis of the string equation (2.21) which determines the recursion coefficients r_n as a functions of the coupling constants. Taking the large-n limit and imposing $r_{n\pm1} \to r_n$ we easily obtain

$$\frac{n}{N} \;=\; g_2 r_n^2 \;+\; gH(2r_n)^\kappa \;, \tag{4.26}$$

using eq. (2.30). At $n = N$ we obtain the normalization condition eq. (4.3) using $2r_N = c$, as we have mentioned already in the beginning. However, it is unclear how to derive the finite-n version of eq. (4.26) in terms of $r_{n\pm1}$, $r_{n\pm2}$, etc. It may happen that the string equation contains all recursion coefficients from r_N down to r_0.

Let us summarize our findings for the multicritical points at the origin obtained so far. We have shown that new classes of multicritical models exist, with $\rho(\lambda) \sim \lambda^{\kappa-1}$ for all real κ with $2m-1 < \kappa < 2m+1$ and $m \in \mathbb{N}$, giving explicit examples. We have shown that at criticality the eigenvalues scale with $N^{1/\kappa}$ (eq. (4.22)) and we have given an approximate differential equation for the wave functions, eq. (4.23). Because of the relations (3.13) and (3.14) to the chiral ensemble all the results obtained in this section immediately translate to the chiral matrix model eq. (3.10). In the next section we will investigate if the nonpolynomial potentials also lead to new critical exponents in the large-N scaling limit at the edge of the spectrum.

5 Recovering usual multicriticality at the spectrum edge

In this section we investigate the multicritical points at the endpoint c of the spectrum. This is particularly interesting since it is known from matrix models with polynomial potentials that here the free energy displays critical behavior, $\mathcal{F} \sim (g - g_c)^{2-\gamma_{str}}$, with the critical exponents $\gamma_{str} = -\frac{1}{m}$ coinciding with the minimal models [25]. One might expect that by adding a nonpolynomial part to the potential these exponents could be altered, as has been already suggested in [23]. In particular one could hope that because of the presence of a real parameter, $\kappa > 1$, these exponents could even be nonrational, thus representing a one-matrix model representation of such conformal field theories, supposedly again a nonunitary one. However, we will find that this is not true in our case. We will first expand the density around the endpoint $\lambda = c$ and tune to criticality. It turns out that additional zeros occur precisely in the same way as in the polynomial case, with $\rho(\lambda) \sim (c^2 - \lambda^2)^{\frac{3}{2}}$ in the simplest case. We also calculate the free energy and its derivatives and find the same critical exponent as in [25].

In order to be most explicit we have chosen to work again with the simplest example, a Gaussian plus a real power, eq. (4.1). Since we find the same phenomenon as for the corresponding critical polynomial potential we do not expect changes when adding higher order terms and tuning to higher criticality.

We start by expanding the density eq. (4.2) for the potential $V(\lambda) = \frac{1}{2}g_2\lambda^2 + g|\lambda|^\kappa$, eq. (4.1), around the endpoint of support c. We find that it has an expansion in powers $(c^2 - \lambda^2)^{\frac{1}{2}}$ which immediately follows from eq. (2.8) for its Freud part:

$$\rho(\lambda) \;=\; \frac{1}{2\pi}\left(g_2 + 2\kappa g c^{\kappa-2} H\right)(c^2-\lambda^2)^{\frac{1}{2}} \;+\; \mathcal{O}\left((c^2-\lambda^2)^{\frac{3}{2}}\right) \quad \text{at} \quad \lambda^2 \approx c^2 \;. \tag{5.1}$$

By requiring

$$0 \;=\; g_2 \;+\; 2\kappa g c^{\kappa-2} H \;, \tag{5.2}$$

we obtain an additional zero at the edge of the spectrum. Together with the normalization condition for the density, eq. (2.12) $1 = \frac{1}{4}g_2 c^2 + Hgc^\kappa$, this fixes both coupling constants g_2 and g. We thus

have for the critical endpoint c_c in terms of g_2:

$$c_c^2 \;\equiv\; \frac{4\kappa}{g_2(\kappa-2)} \;. \tag{5.3}$$

The fact that the density vanishes with the same power as the density of the critical quartic potential, does not suffice to conclude that the critical exponents of the corresponding free energies are the same. The free energy is given as an integral over the full support and thus may very well behave differently. We use here the saddle-point approach to compute the free energy recalling some known facts. First, we calculate the free energy in general and then switch to the critical point (of its derivatives) which turns out to coincide with eq. (5.2).

The planar free energy is given by inserting the solution $\rho(\lambda)$ of the saddle-point equation (2.4) into eq. (2.3). Using that $\rho(\lambda)$ satisfies (2.4) (here we put the normalization to unity, $a = 1$) and eliminating the Lagrange multiplier K we obtain

$$\mathcal{F} \;=\; \frac{1}{2}\int_\sigma d\lambda\rho(\lambda)V(\lambda) - \int_\sigma d\lambda\,\rho(\lambda)\ln|\lambda| \;. \tag{5.4}$$

Choosing our specific example eq. (4.1) we insert the density in the form of eq. (2.11) into eq. (5.4). The following rewriting of integrals,

$$\int_{-c}^{c} d\lambda \int_{|\frac{\lambda}{c}|}^{1} ds \;=\; \int_0^1 ds \int_{-s}^{s} \frac{d\lambda}{c} \;, \tag{5.5}$$

turns out to be particularly useful and the remaining integrals are straightforward. We obtain

$$\mathcal{F} \;=\; \frac{3}{2\kappa} \;-\; \frac{1}{2}\ln\left[\frac{c^2}{4}\right] \;+\; \frac{(\kappa-2)(\kappa+1)}{4\kappa(\kappa+2)}\,g_2c^2 \;-\; \frac{(\kappa-2)^2}{64\kappa(\kappa+2)}\,g_2^2c^4 \;. \tag{5.6}$$

In order to find the critical point and exponent of the free energy we have to study its derivatives with respect to some coupling constant. Since we do not yet impose criticality only one of the coupling constants g_2 and g is fixed by the normalization condition eq. (2.12). We choose to keep the Gaussian coupling g_2 as fixed parameter and vary g and thus c as a function of g. Its derivative can be obtained from the normalization condition, eq. (2.12), keeping g_2 fixed:

$$\begin{aligned}\frac{\partial c^2}{\partial g} \;&=\; \frac{-4Hc^\kappa}{g_2 + 2\kappa H g c^{\kappa-2}} \;,\\ &=\; \frac{8Hc^{\kappa+2}}{g_2(\kappa-2)}\frac{1}{(c^2-c_c^2)} \;,\end{aligned} \tag{5.7}$$

where we have used again eq. (2.12) as well as the definition eq. (5.3). The derivative has a pole precisely at the critical value c_c. The first derivative of the free energy thus reads

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial g} \;&=\; \frac{-4Hc^\kappa}{g_2 + 2\kappa H g c^{\kappa-2}}\left(-\frac{1}{2c^2} \;+\; \frac{(\kappa-2)(\kappa+1)}{4\kappa(\kappa+2)}\,g_2 \;+\; \frac{(\kappa-2)^2}{32\kappa(\kappa+2)}\,g_2^2c^2\right),\\ &=\; \frac{Hc^\kappa}{4\kappa(\kappa+2)}\left(4\kappa + (2-\kappa)g_2c^2 + 8\right) \;,\end{aligned} \tag{5.8}$$

where the denominator from eq. (5.7) has been canceled due to eq. (2.12). Obviously $\frac{\partial}{\partial g}\mathcal{F}$ is analytic everywhere for $\kappa > 1$ and at the critical point for the density eq. (5.2) we can just set $c = c_c$. The second derivative can be calculated in a similar way and we obtain

$$\frac{\partial^2\mathcal{F}}{\partial g^2} \;=\; \frac{-H^2c^{2\kappa}}{\kappa} \;, \tag{5.9}$$

which is again analytic everywhere. For the third derivative we finally obtain

$$\frac{\partial^3 \mathcal{F}}{\partial g^3} \;=\; \frac{8H^3 c^{3\kappa}}{g_2(\kappa-2)} \frac{1}{(c^2-c_c^2)} \;, \tag{5.10}$$

which has a singularity at $c = c_c$ (and at $\kappa = 2$ which we can obviously exclude). This is precisely where the spectral density develops an additional zero. We have so far obtained that the transition is of third order at c_c given by eq. (5.2) and thus that the corresponding exponent γ_{str} is negative. In order to determine its value we still have to find the exponent ε, with which the denominator vanishes in $g - g_c$.

Since we cannot directly solve the transcendental equation (2.12) for c we will expand it around the critical point c_c. We make the ansatz

$$c^2 \;=\; c_c^2 + \Lambda (g-g_c)^{\varepsilon} + \mathcal{O}(g-g_c)^{\varepsilon+1} \;, \tag{5.11}$$

where the critical coupling g_c follows from eqs. (2.12) and (5.2) to be

$$g_c \equiv \; \frac{-2c_c^{-\kappa}}{H(\kappa-2)} \;. \tag{5.12}$$

Inserting the ansatz (5.11) into eq. (2.12) and expanding we find

$$0 \;=\; H g_c \frac{\kappa}{8} (\kappa-2) \Lambda^2 c_c^{\kappa-4} (g-g_c)^{2\varepsilon} + H c_c^{\kappa} (g-g_c) \;+\; \mathcal{O}(g-g_c)^{\varepsilon+1} \;, \tag{5.13}$$

which only has a solution for

$$\varepsilon \;=\; \frac{1}{2} \;. \tag{5.14}$$

We can also determine the proportionality constant

$$\Lambda^2 \;=\; \frac{4H}{\kappa} c_c^{\kappa+4} \;, \tag{5.15}$$

and we finally obtain

$$c^2 \;=\; c_c^2 \left(1 - \sqrt{4H\kappa^{-1} c_c^{\kappa}} \; (g-g_c)^{\frac{1}{2}} + \mathcal{O}(g-g_c)^{\frac{3}{2}} \right) \;. \tag{5.16}$$

Therefore it follows from eq. (5.10) that

$$\frac{\partial^3 \mathcal{F}}{\partial g^3} \sim (g-g_c)^{-\frac{1}{2}} \tag{5.17}$$

and thus $\gamma_{str} = -\frac{1}{2}$ as for the first multicritical point of polynomial potentials [25].

In summary we have shown that for our prototype example of a Gaussian plus a real power exactly the same exponent of $\mathcal{F}$ and number of zeros of the density at the endpoint c occur, when the potential is tuned to criticality. The question is if the same is true for higher critical points, which can be obtained by adding higher powers to the potential. From eq. (2.8) it is clear that the spectral density of a polynomial plus a Freud type potential always has an expansion in powers of $\sqrt{c^2-\lambda^2}$ and that thus the accumulation of zeros at $\lambda = c$ happens the same way as in the polynomial case. For the free energy however there is no other way to calculate its exponent than in the tedious analysis as presented above, by taking derivatives and expanding the solution of the normalization condition. The method of orthogonal polynomials [1] typically used to determine the exponents of $\mathcal{F}$ for arbitrary criticality m does not easily apply here since we do not have an explicit form of the string equation

(2.21) at hand for finite n. But as our example above was arbitrarily chosen (for simplicity) we do not expect that new critical exponents occur for higher multicriticality compared to [25].

Since we have calculated the free energy for arbitrary couplings we may also ask about the order of the phase transition for the critical points of Section 4 at the origin. The free energy is analytic everywhere away from the critical endpoint c_c eq. (5.3) which differs from the value it takes for criticality at the origin (see eq. (4.5)). However, the fact that $\mathcal{F}$ is thus analytic there does not imply a smooth crossover. In principle we would have to compare to the free energy and its derivatives computed on the two-cut side at the transition [43]. Our findings in Appendix A make this a very delicate task.

6 Conclusions

We have introduced a new class of one-matrix models which contain nonpolynomial powers in the potential. These models do not seem to have an immediate graphical interpretation in terms of Feynman diagrams. However, they still share properties of models with polynomial potentials such as critical exponents related to Quantum Gravity, while in other aspects they show new features: a spectral density vanishing proportional to an arbitrary real power at the origin and a new scaling behavior of the eigenvalues.

We have first determined the macroscopic large-N spectral density using saddle-point and orthogonal polynomial techniques, giving explicit examples such as the simplest nonpolynomial extension of a Gaussian matrix model. When taking the microscopic scaling limit at the origin we could prove that the know massless and massive universality classes for both the chiral and nonchiral ensembles were maintained, with the spectral density $\rho(0)$ providing the universal parameter.

After tuning the coupling constants of the potential to make $\rho(0)$ vanish we found a set of new classes of multicritical matrix models with $\rho(\lambda = 0) \sim \lambda^{\kappa-1}$ for arbitrary real $\kappa > 1$. Our analysis completes the study of such critical models, where previously only multicritical models with a density vanishing as an even power were known. Similarly to the findings there for higher critical points additional couplings have to be introduced and tuned. Our results fall into classes with $2m - 1 < \kappa < 2m + 1$ where $m \in \mathbb{N}$. For each class of criticality a minimal set of m coupling constants has to be tuned and we explicitly gave a realization of such potentials. The scaling behavior for such multicritical densities was determined and can be tested on the Lattice in applications to the chiral phase transition.

We would like to emphasize the generality of the multicritical models we have considered. Our results also apply to the case when the critical exponents of the underlying physical model differ from the mean field values as we can incorporate arbitrary real positive exponents. It would be thus very interesting to calculate exactly the correlation functions for such models as a function of the real parameter κ. Here, we could only derive an approximate differential equation for the asymptotic wavefunctions, which enter the kernel of orthogonal polynomials and thus determine all correlation functions. An exact differential equation is currently known only for even integer values of κ. Another open question left for further investigation is the order of the corresponding phase transition in our model. While the continuity of κ at even integer values suggests a third order transition the infinite slope of the first critical density with $1 < \kappa < 2$ may still indicate a lower order transition.

To this aim we have also analyzed the possible critical points of the planar free energy in our model. Nonanalytic parts in its derivatives only occurred at the edge of the spectrum. For a nonanalyticity to occur at the origin we would have to compare with the derivatives of the free energy coming from the two-cut side, which we have not been able to determine so far. When studying criticality by the accumulation of extra zeros of the density at the spectrum edge we recovered the critical exponents known from polynomial matrix models. A first hope of finding new classes of possibly real

valued exponents was not fulfilled. Such models would represent nonrational conformal field theories. Obviously more complicated nonpolynomial terms have to be added to the one-matrix model potential to achieve such goal. This could be thought of as a generalization of the known two-matrix models which do allow for representations of rational theories, with one matrix formally integrated out.

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A The two-cut solution

In this Appendix we give the solution for nonpolynomial potentials when the support of the eigenvalue density splits into two pieces. Since we restrict ourselves to the case of even potentials we will not have to enter the subtleties encountered for such multi-cut solutions where non-perturbative contributions [45] may change the meanfield solution [46]. Our derivation follows [47] (see also [35]) and since the formalism has been presented in detail in Section 2 we shall remain brief here. We will calculate only the smoothed macroscopic density as well as the free energy. As has been noted in [47] the higher point correlation functions do no longer possess a smooth large-N limit.

The main difference to the single support solution is that the recursion coefficients r_n from eq. (2.20) approach two different rather than a single smooth function:

$$r_N \;=\; \frac{1}{2}\left(c \;-\; (-1)^N d\right) \;, \tag{A.1}$$

where c and d are the endpoints of the support $\sigma = [-c,-d]\cup[d,c]$ with $0 < d < c$. Under the condition eq. (A.1) with support σ it has been shown in [35] that the wave functions will approach the following "smoothed" limits:

$$\overline{\psi_N^{(\alpha)}(\lambda)^2} \;=\; \frac{1}{\pi}\frac{|\lambda|}{\sqrt{(c^2-\lambda^2)(\lambda^2-d^2)}}\;\theta(c^2-\lambda^2)\theta(\lambda^2-d^2)\;, \tag{A.2}$$

$$\overline{\psi_N^{(\alpha)}(\lambda)\psi_{N-1}^{(\alpha)}(\lambda)} \;=\; \frac{1}{r_N\pi}\frac{\lambda^2-(-1)^N c\,d}{\sqrt{(c^2-\lambda^2)}(\lambda^2-d^2)}\;\theta(c^2-\lambda^2)\theta(\lambda^2-d^2)\;. \tag{A.3}$$

Similar to eq. (2.30) no restrictions on the potential have been made in [35]. The auxiliary functions eqs. (2.23) and (2.24) can thus easily be obtained:

$$A(\lambda) \;\equiv\; \frac{1}{N}\overline{A_N(\lambda)} \;=\; \frac{r_N}{\pi}\int_d^c dt\;\frac{tV'(t)-\lambda V'(\lambda)}{t^2-\lambda^2}\;\frac{t}{\sqrt{(c^2-t^2)(t^2-d^2)}}\;, \tag{A.4}$$

$$B(\lambda) \;\equiv\; \frac{1}{N}\overline{B_N(\lambda)} \;=\; \frac{1}{\pi}\int_d^c dt\;\frac{\lambda V'(t)-tV'(\lambda)}{t^2-\lambda^2}\;\frac{t^2-(-1)^N c\,d}{\sqrt{(c^2-t^2)(t^2-d^2)}}\;. \tag{A.5}$$

Note the alternating sign which is present in both quantities due to eq. (A.1). We have used already the symmetry with respect to the origin by writing intervals over the positive part of σ only. Inserting these quantities into eq. (2.29) which is exact for finite N we arrive at the following large-N expression for the smoothed, macroscopic density:

$$\rho(\lambda) \;=\; \frac{1}{\pi^2}\;\int_d^c dt\;\frac{\lambda V'(t)-tV'(\lambda)}{t^2-\lambda^2}\sqrt{\frac{(c^2-\lambda^2)(\lambda^2-d^2)}{(c^2-t^2)(t^2-d^2)}}\;. \tag{A.6}$$

The normalization condition for the density can be read off from the string equations [35] analogous to eq. (2.21):

$$\begin{aligned} 1 &= \frac{1}{\pi}\int_d^c dt \frac{t^2 V'(t)}{\sqrt{(c^2-t^2)(t^2-d^2)}} \ , \\ 0 &= \frac{1}{\pi}\int_d^c dt \frac{V'(t)}{\sqrt{(c^2-t^2)(t^2-d^2)}} \ . \end{aligned} \tag{A.7}$$

To make contact to Section 4 we give the two-cut result for the same simple example, the potential

$$V(\lambda) = \frac{1}{2} g_2 \lambda^2 \ + \ g|\lambda|^\kappa \ . \tag{A.8}$$

Inserting it into the general result eq. (A.6) we see that the Gaussian terms cancels and we obtain

$$\rho(\lambda) \ = \ \frac{\kappa g}{\pi^2} \ \int_d^c dt \ \frac{t^{\kappa-2} - \lambda^{\kappa-2}}{t^2-\lambda^2} \ \lambda t \ \sqrt{\frac{(c^2-\lambda^2)(\lambda^2-d^2)}{(c^2-t^2)(t^2-d^2)}} \ . \tag{A.9}$$

It is very interesting to compare this result at the transition point, when the two intervals of σ merge, with the corresponding density eq. (4.2) with a single interval support at criticality. By sending $d \to 0$ it is easy to see that eq. (A.9) leads exactly to λ^2 times a density of Freud type eq. (2.10) at shifted $\kappa \to \kappa - 2$ (see also eq. (2.36)). We thus arrive at

$$\rho(\lambda)|_{d=0} \ = \ \frac{g\kappa}{2\pi} \tan\left[\frac{\kappa\pi}{2}\right] \lambda^{\kappa-1} \ + \ \frac{g\kappa(\kappa-2)Hc^{\kappa-3}}{\pi(\kappa-1)(\kappa-3)} \lambda^2 \ F\left(\frac{1}{2}, \frac{3}{2}-\frac{\kappa}{2}; \frac{5}{2}-\frac{\kappa}{2}; \frac{\lambda^2}{c^2}\right), \tag{A.10}$$

where we have again chosen non-integer $1 < \kappa < 3$. It is easy to see that the normalization conditions eq. (A.7) matches with the normalization condition eq. (4.3) for the single cut density together with the multicriticality condition eq. (4.5). Moreover the two densities eqs. (A.10) and (4.2) agree exactly at multicriticality (4.5), which follows from the general form eq. (4.20) for $m = 1$. As an immediate consequence on the transition point the two-cut free energy following from eq. (5.4) is identical to eq. (5.6) together with eq. (4.5) which has been calculated coming from the one-cut side.

In order to calculate also derivatives of the free energy in the two-cut case we would have to first solve the integral in eq. (A.9) in order to be able to determine $\mathcal{F}$ from eq. (5.4). These integrals are not simple elliptic integrals as encountered in [46], and the presence of the nonpolynomial terms makes such a task fairly difficult.

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