

# Microscopic correlation functions for the QCD Dirac operator with chemical potential

G. Akemann

*Service de Physique Théorique, CEA/Saclay, F-91191 Gif-sur-Yvette Cedex, France*

A chiral random matrix model with complex eigenvalues is solved as an effective model for QCD with non-vanishing chemical potential. We derive new matrix model correlation functions which predict the local fluctuations of complex Dirac operator eigenvalues at zero virtuality. The parameter which measures the non-Hermiticity of the Dirac matrix is identified with the chemical potential. In the phase with broken chiral symmetry all spectral correlations of the Dirac eigenvalues are calculated as functions of quark flavors and chemical potential. The derivation uses the orthogonality of the Laguerre polynomials in the complex plane. Explicit results are given for any finite matrix size  $N$  as well in the large- $N$  limit for weak and strong non-Hermiticity.

PACS numbers: 12.38.Lg, 11.30.Rd, 05.40.-a

Random Matrix Models (RMM) have been a useful tool in Theoretical Physics for a long time. In many physical systems the local fluctuation properties for example of the energy levels are universal and can be successfully described by RMM, where we refer to [1] for a review. Although in the generic situation the Hamiltonian and other physical observables are Hermitian thus having real eigenvalues, there exist also important cases where complex eigenvalues occur. As examples we mention localization in superconductors [2], dissipation and scattering in Quantum Chaos [3] or Quantum Chromodynamics (QCD) with chemical potential [4].

Much less is known so far for spectral correlations of complex eigenvalues derived from RMM. Although the first results date back to Ginibre [5] where the correlations for the complex Unitary Ensemble labeled by the Dyson index  $\beta = 2$  were calculated, progress has been slow. The correlation functions of the ensemble with real non-symmetric matrices ( $\beta = 1$ ) are still unknown. Results for quaternion matrices ( $\beta = 4$ ) were obtained more recently in [6, 7] and the inclusion of Dirac mass terms for  $\beta = 2$  in [8]. Furthermore, it has been realized by works of Fyodorov and collaborators [9] that different regimes of complex eigenvalues exist, the weak and strong non-Hermiticity limit. In the present work we wish to extend the knowledge about complex matrix models to the so-called chiral models.

Chiral RMM of real eigenvalues have been introduced to describe the local fluctuation properties of the Dirac operator in QCD at the origin [10]. The low energy spectrum of the QCD Dirac operator is a very sensitive tool to study the phenomenon of chiral symmetry breaking [11]. The predictions of the different chiral RMM ensembles have been very successful in describing the dependence on the gauge group and its representation, the number of quark flavors and masses and the topology [12]. In particular the topology dependence has been very useful in comparison with new developments in lattice gauge theory, admitting to incorporate an exact chiral symmetry [13]. By now the field theoretic origin of the RMM description has also been well understood [14].

On the other hand lattice simulations in the presence of a chemical potential  $\mu$ , which renders the Dirac eigen-

values complex, remain extremely difficult, as reviewed in [15]. Although recent progress has been made [16] along the phase transition line the general phase diagram remains unexplored for  $\mu \neq 0$ . It is in this context that analytical knowledge of microscopic correlation functions from complex chiral RMM could be very useful, in view of its success in predicting real eigenvalue correlations.

Chiral RMM including a chemical potential have been already studied in several works [4, 17, 18, 19]. In [4] the nature of the quenched limit has been analyzed in such a model, and the global density of complex eigenvalues together with its boundary have been calculated as a function of  $\mu$ . The phase diagram of QCD in the temperature density plane has also been predicted with such a model [18]. Recently complex Dirac eigenvalues calculated on the lattice have been confronted to a complex matrix model on the microscopic scale given by the inverse volume in the bulk of the spectrum [19]. The nearest neighbor distribution along a given direction in the complex plane was considered and a transition from the Unitary to the Ginibre ensemble was observed at increasing  $\mu$ , ending in a Poisson distribution.

Our aim here is to provide more detailed information by calculating all microscopic correlation functions in the complex plane, both in the limit of weak and strong non-Hermiticity. Our model gives a natural chiral extension of the ensemble treated in [8, 9]. Although our main motivation is the application to QCD lattice calculations with  $\mu \neq 0$  the new correlation functions we derive may find other applications as well, such as in the fractional Quantum Hall effect [20] or two-dimensional charged plasmas [21]. We restrict ourselves to work with a Gaussian chiral RMM. In particular we will not touch the issue of universality here although we expect that in analogy to [22] the same correlation functions hold for a more general weight function at large- $N$ .

We start our investigation by defining our model. The chiral random matrix partition function in terms of the complex eigenvalues  $z_{j=1,\dots,N}$  of a complex  $N \times N$  matrix

$J = H + i\sqrt{\frac{1-\tau}{1+\tau}}A$  is defined as

$$\mathcal{Z}_N^{(N_f, \nu)}(\tau) \equiv \int \prod_{j=1}^N dz_j dz_j^* w^{(a)}(z_j) |\Delta_N(z_1^2, \dots, z_N^2)|^2, \quad (1)$$

where we have introduced the weight function

$$w^{(a)}(z) = |z|^{2a+1} \exp \left[ -\frac{N}{1-\tau^2} \left( |z|^2 - \frac{\tau}{2}(z^2 + z^{*2}) \right) \right]. \quad (2)$$

It contains  $N_f$  massless quark flavors in a fixed sector of topological charge  $\nu$ , where  $a = N_f + \nu$ . We have taken the absolute value of the Dirac determinant but the weight depends on the real and imaginary part of the eigenvalues. The weight has been chosen to be Gaussian with variance  $(1+\tau)/2N$  for both the Hermitian and anti-Hermitian part of  $J$ ,  $H$  and  $A$  respectively. The parameter  $\tau \in [0, 1]$  controls the degree of non-Hermiticity and will be related to the chemical potential  $\mu$  in QCD below. The Jacobi determinant from the diagonalization of  $J$  [5] yields the Vandermonde determinant  $\Delta_N(z_1, \dots, z_N) = \prod_{k>l}^N (z_k - z_l)$ . The reason for the model eq. (1) being chiral can be seen as follows. For real eigenvalues the chiral ensemble [10] is usually defined on the real positive line,  $\int_0^\infty d\lambda \lambda^a \exp(-N\lambda)$ . By substituting  $\lambda = y^2$  it can be mapped to the full real axis,  $\int_{-\infty}^\infty dy |y|^{2a+1} \exp(-Ny^2)$ , where  $y$  corresponds to a real eigenvalue of the Dirac operator  $\mathcal{D}$  [23]. In the latter picture the continuation into the complex plane is straightforward, leading to the ensemble (1). The only difference to the massless non-chiral ensemble [8] is the additional power in  $|z|$ . Furthermore, the orthogonal polynomials for the ensemble (1) are given by Laguerre polynomials  $L_n^a(z^2)$  in the complex plane.

A chemical potential  $\mu$  is included in the QCD action by adding  $\mu\gamma_0$  to the Dirac operator  $\mathcal{D}$  and it renders its eigenvalues complex. Our aim is to describe the local fluctuation of small eigenvalues close to the origin, because of their importance for chiral symmetry breaking through the Banks-Casher relation [24]. For  $\mu$  not to completely dominate the Dirac determinant we will also restrict ourselves to small values of  $\mu$ .

In [4] a slightly different matrix model was introduced replacing  $\mathcal{D}$  by a chiral random matrix and keeping the additional term  $\mu\gamma_0$  explicitly. Here, we treat  $\mathcal{D} + \mu\gamma_0$  as a chiral, complex matrix instead. The advantage of our model is that we can directly calculate all microscopic correlation functions. We will take the model [4] to relate our parameter  $\tau$  to  $\mu$  by comparing to the macroscopic spectral density  $\rho(z)$  and its boundary calculated there as a function of  $\mu$ . In the limit of small  $\mu$  the spectral density of [4] becomes approximately constant,  $\rho(z) = 1/4\pi\mu^2$ , and it is bounded by an ellipse,  $x^2/4 + y^2/4\mu^2 = 1$ , where  $z = x + iy$ . This behavior can also be observed for lattice data with small  $\mu$  (e.g. in [19, 25]). The macroscopic density in our model can be read off from [26] since the Dirac determinants are

subdominant in the macroscopic large- $N$  limit:

$$\rho(z) = \frac{1}{\pi(1-\tau^2)}, \quad \text{if } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1. \quad (3)$$

We therefore identify

$$4\mu^2 \equiv (1-\tau^2), \quad (4)$$

valid for small chemical potential and  $\tau$  close to unity meaning small non-Hermiticity. For large values of  $\mu$  the eigenvalue density on the lattice is no longer constant and develops a hole in the middle (see e.g. [19, 25]). We will also see such a hole develop in the microscopic correlations in the limit of strong non-Hermiticity, as shown in Fig. 2 below.

After having identified all parameters in our model eq. (1) we turn to its solution using the powerful method of orthogonal polynomials [6]. We only state the results and refer to [27] for details. All eigenvalue correlation functions are first given for a finite number of eigenvalues  $N$  and then in two different large- $N$  limits corresponding to weak and strong non-Hermiticity. The orthogonal polynomials in the complex plane are defined as

$$\int dz dz^* w^{(a)}(z) P_k^{(a)}(z) P_l^{(a)}(z^*) = \delta_{kl}. \quad (5)$$

Following standard techniques [6] the knowledge of the kernel of orthogonal polynomials

$$K_N^{(a)}(z_1, z_2^*) \equiv [w^{(a)}(z_1)w^{(a)}(z_2^*)]^{1/2} \sum_{k=0}^{N-1} P_k^{(a)}(z_1)P_k^{(a)}(z_2^*) \quad (6)$$

allows to calculate all  $k$ -point correlation functions

$$\rho_N^{(a)}(z_1, \dots, z_k) = \det_{1 \leq i, j \leq k} \left[ K_N^{(a)}(z_i, z_j^*) \right]. \quad (7)$$

The result for the orthogonal polynomials eq. (5) is given in terms of Laguerre polynomials

$$P_k^{(a)}(z) \equiv \left[ \binom{a+k}{k} f^{(a)}(\tau) \right]^{-1/2} (-\tau)^k L_k^a \left( \frac{Nz^2}{2\tau} \right), \quad (8)$$

with the normalization integral

$$\begin{aligned} f^{(a)}(\tau) &\equiv \int dz dz^* w^{(a)}(z) \\ &= N^{-a-\frac{3}{2}} \pi \Gamma \left( a + \frac{3}{2} \right) (1-\tau^2)^{\frac{a}{2}+\frac{3}{4}} P_{a+\frac{3}{2}} \left( \frac{1}{\sqrt{1-\tau^2}} \right) \end{aligned} \quad (9)$$

and  $P_{a+\frac{3}{2}}(x)$  being the Legendre function. All  $k$ -point correlation functions then follow by inserting the polynomials into eqs. (6) and (7). In our results the parameter  $a = N_f + \nu$  can be kept real (with  $a > -1$ ). For example we can set  $a = -\frac{1}{2}$  as a check, recovering the even subset of the Hermite polynomials in the complex plane [20].

Since we did not find eq. (8) in the literature we briefly sketch its derivation. Performing a change of variables  $z \rightarrow e^{i\varphi} z$  in the normalization integral eq. (9) we obtain

$$1 = \left\langle \exp \left[ \frac{u}{u-1} \left( \frac{Nz^2}{2\tau} \right) \right] \exp \left[ \frac{u^*}{u^*-1} \left( \frac{Nz^{*2}}{2\tau} \right) \right] \right\rangle, \quad (10)$$

with

$$u \equiv \frac{\tau^2(1 - e^{2i\varphi})}{(1 - \tau^2 e^{2i\varphi})} \quad (11)$$

and the average taken with respect to  $w^{(a)}(z)$ . Multiplying both sides of eq. (10) with  $((1-u)(1-u^*))^{-a-1}$  and recognizing the generating functional of the Laguerre polynomials we obtain the desired orthogonality relation, given properly normalized in eq. (8).

After giving the exact solution for finite  $N$  we turn to the large- $N$  limit. We first consider the weak non-Hermiticity limit. Following [9] we take the limit  $\tau \rightarrow 1$  such that the combination

$$\lim_{N \rightarrow \infty} N(1 - \tau^2) \equiv \alpha^2 = 4N\mu^2 \quad (12)$$

is kept fixed. Because of the identification eq. (4) we consequently also rescale  $\mu$  going to zero when  $N \rightarrow \infty$ . In other words the weak non-Hermiticity parameter  $\alpha^2$  directly measures the chemical potential in the microscopic scaling limit. Such a rescaling is similar to that of the quark masses [28]. It has been already mentioned in [17] that in a RMM the numerical effort to obtain convergence grows exponentially with  $N\mu^2$ . Keeping it fixed here should make a comparison to data feasible.

Furthermore, we also rescale the complex eigenvalues keeping

$$N(\Re z + i\Im z) = Nz \equiv \xi, \quad (13)$$

fixed. The matrix size  $N$  corresponds to the volume on the lattice. This defines our microscopic origin scaling limit in the complex plane. The kernel eq. (6) and correlators eq. (7) also have to be rescaled with the mean level spacing  $1/N$  of the eigenvalues.

In order to obtain the microscopic kernel from eq. (6) we replace the sum by an integral,  $\sum_{k=0}^{N-1} \rightarrow N \int_0^1 dt$ , where  $t = \frac{k}{N}$ , and use the asymptotic limit of the Laguerre polynomials to finally arrive at

$$K_S^{(a)}(\xi_1, \xi_2^*) = \frac{|\xi_1 \xi_2^*|^{a+\frac{1}{2}}}{\sqrt{2\pi\alpha^2} (\xi_1 \xi_2^*)^a} e^{-\frac{1}{\alpha^2} ((\Im \xi_1)^2 + (\Im \xi_2^*)^2)} \times \int_0^1 dt e^{-\alpha^2 t} J_a(\sqrt{2t}\xi_1) J_a(\sqrt{2t}\xi_2^*). \quad (14)$$

The microscopic, weakly non-Hermitian correlation functions obtained from eq. (7) are our first main result:

$$\rho_S^{(a)}(\xi_1, \dots, \xi_k) = \prod_{l=1}^k \left( \frac{|\xi_l|}{\sqrt{2\pi\alpha^2}} e^{-\frac{2}{\alpha^2} (\Im \xi_l)^2} \right) \times \det_{1 \leq i, j \leq k} \left[ \int_0^1 dt e^{-\alpha^2 t} J_a(\sqrt{2t}\xi_i) J_a(\sqrt{2t}\xi_j^*) \right]. \quad (15)$$

As an important check in the Hermitian limit  $\alpha^2 \rightarrow 0$  corresponding to  $\tau = 1$  the universal correlations [10, 22] of the chiral RMM with real eigenvalues are reproduced.

To give an example for eq. (15) we have depicted the quenched microscopic density in part of the complex plane in Fig 1. The other directions follow from symmetry. The oscillations known from the real case [10]

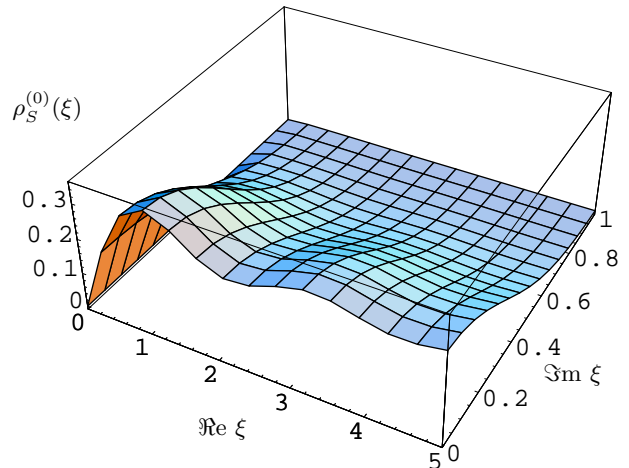


FIG. 1: The quenched microscopic density for  $\alpha^2 = 0.6$

spread into the complex plane, indicating the locations of the individual eigenvalues.

We now turn to the strong non-Hermiticity limit. In this limit  $\tau \in [0, 1)$  and consequently also  $\mu$  from eq. (4) is kept fixed in the large- $N$  limit. The eigenvalues are now rescaled with the square root of the volume [8, 9],

$$\sqrt{N}(\Re z + i\Im z) = \sqrt{N}z \equiv \xi, \quad (16)$$

defining our microscopic origin limit at strong non-Hermiticity. In this limit the infinite sum in the kernel eq. (6) can be evaluated using standard formulas for Laguerre polynomials and we obtain

$$K_S^{(a)}(\xi_1, \xi_2^*) = \frac{2^a \Gamma(a+1)}{N^{a+\frac{3}{2}} f^{(a)}(\tau)(1-\tau^2)} \frac{|\xi_1 \xi_2^*|^{a+\frac{1}{2}}}{(\xi_1 \xi_2^*)^a} \times e^{\frac{-1}{2(1-\tau^2)} (|\xi_1|^2 + |\xi_2^*|^2 - \frac{\tau}{2} (\xi_1^{*2} - \xi_1^2 + \xi_2^2 - \xi_2^{*2}))} I_a \left( \frac{\xi_1 \xi_2^*}{1-\tau^2} \right). \quad (17)$$

The correlation functions in the strong limit then read

$$\rho_S^{(a)}(\xi_1, \dots, \xi_k) \sim \prod_{l=1}^k |\xi_l| e^{\frac{-1}{2(1-\tau^2)} |\xi_l|^2} \det_{1 \leq i, j \leq k} \left[ I_a \left( \frac{\xi_i \xi_j^*}{1-\tau^2} \right) \right] \quad (18)$$

where we have suppressed the normalization constant. They can also be obtained from the correlators eq. (15) in the weak limit by taking the limit  $\alpha \rightarrow \infty$  there and identifying  $\alpha^2 = 1 - \tau^2$ . They differ from ref. [21] due to the different interaction term in eq. (1). As an example for eq. (18) we give the quenched microscopic density in Fig. 2. The microscopic spectral density develops a

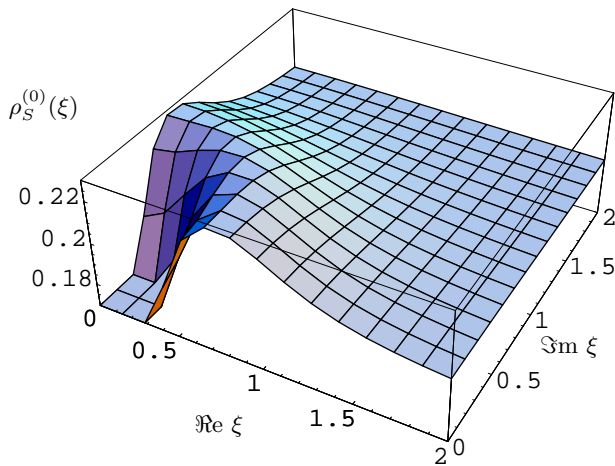


FIG. 2: The quenched microscopic density for  $\tau = 0.5$

hole at the origin and becomes flat for large values. This has indeed been observed in quenched lattice data for in-

termediate values of the chemical potential [19, 25]. At large  $\mu$  the eigenvalues form a thin ring. The qualitative picture of the microscopic density in Fig. 2 remains unchanged in the presence of massless flavors  $N_f$ , although the barrier present in the figure gets flattened out.

In summary we have introduced a new class of a chiral RMM having complex eigenvalues. All  $k$ -point correlation functions have been calculated explicitly at finite- $N$  as well as in the limits of weak and strong non-Hermiticity. The parameter  $\tau$  that governs the non-Hermiticity has been related to the chemical potential  $\mu$  as it occurs in the QCD Dirac operator. While the microscopic density shows qualitative features of Dirac operator eigenvalues calculated in lattice QCD a quantitative comparison with data remains to be done.

Acknowledgments: I wish to thank P. Di Francesco, P. Forrester and E. Kanzieper for enjoyable discussions and correspondence. This work was supported by the European network on “Discrete Random Geometries” HPRN-CT-1999-00161 (EUROGRID).

- 
- [1] T. Guhr, A. Müller-Groeling and H.A. Weidenmüller, Phys. Rep. **299** (1998) 190.
- [2] N. Hatano and D.R. Nelson, Phys. Rev. Lett. **77** (1996) 570.
- [3] R. Grobe, F. Haake and H.-J. Sommers, Phys. Rev. Lett. **61** (1988) 1899; K.B. Efetov, Phys. Rev. Lett. **79** (1997) 491; Y.V. Fyodorov and B.A. Khoruzhenko, Phys. Rev. Lett. **83** (1999) 65.
- [4] M.A. Stephanov, Phys. Rev. Lett. **76** (1996) 4472.
- [5] J. Ginibre, J. Math. Phys. **6** (1965) 440.
- [6] M.L. Mehta, *Random Matrices*, Academic Press, London 1991.
- [7] E. Kanzieper, Eigenvalue correlations in Ginibre’s non-Hermitian random matrices at  $\beta=4$ , cond-mat/0109287.
- [8] G. Akemann, Phys. Rev. **D64** (2001) 114021.
- [9] Y.V. Fyodorov, B.A. Khoruzhenko and H.-J. Sommers, Phys. Lett. **A226** (1997) 46; Phys. Rev. Lett. **79** (1997) 557.
- [10] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306; J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70** (1993) 3852; J. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531.
- [11] J.J.M. Verbaarschot and T. Wettig, Ann. Rev. Nucl. Part. Sci. **50**, (2000) 343.
- [12] M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, Phys. Rev. Lett. **80** (1998) 1146; R.G. Edwards, U.M. Heller, J. Kiskis and R. Narayanan, Phys. Rev. Lett. **82** (1999) 4188; M. Göckeler, H. Hehl, P.E.L. Rakow, A. Schäfer and T. Wettig, Phys. Rev. **D59** (1999) 94503; R.G. Edwards, U.M. Heller and R. Narayanan, Phys. Rev. **D60** (1999) 77502; P.H. Damgaard, U.M. Heller and A. Krasnitz, Phys. Lett. **B445** (1999) 366; G. Akemann and E. Kanzieper, Phys. Rev. Lett. **85** (2000) 1174.
- [13] F. Niedermayer, Nucl. Phys. Proc. Suppl. **73** (1999) 105, and references therein.
- [14] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B540** (1999) 317; P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B547** (1999) 305; D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B560** (1999) 259.
- [15] S. Chandrasekharan, Nucl. Phys. Proc. Suppl. **94** (2001) 71; F. Karsch, Nucl. Phys. Proc. Suppl. **83** (2000) 14.
- [16] Z. Fodor and S.D. Katz, hep-lat/0104001; JHEP 03 (2002) 014.
- [17] M.A. Halasz, A.D. Jackson and J.J.M. Verbaarschot, Phys. Rev. **D56** (1997) 5140.
- [18] M.A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov and J.J.M. Verbaarschot, Phys. Rev. **D58** (1998) 096007.
- [19] H. Markum, R. Pullirsch and T. Wettig, Phys. Rev. Lett. **83** (1999) 484.
- [20] P. Di Francesco, M. Gaudin, C. Itzykson and F. Lesage, Int. J. Mod. Phys. **A9** (1994) 4257.
- [21] B. Jancovici, Molec. Phys. **42** (1984) 1251.
- [22] G. Akemann, P.H. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B487** (1997) 721.
- [23] In our conventions Dirac eigenvalues at  $\mu = 0$  are real.
- [24] T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.
- [25] I. Barbour, N.E. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone and H.W. Wyld, Nucl. Phys. **B275** (1986) 296.
- [26] H.J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, Phys. Rev. Lett. **60** (1988) 1895.
- [27] G. Akemann, hep-th/0204246.
- [28] P.H. Damgaard, Phys. Lett. **B425** (1998) 151.