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The solution of a chiral random matrix model with complex eigenvalues

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Abstract

We describe in detail the solution of the extension of the chiral Gaussian unitary ensemble (chGUE) into the complex plane. The correlation functions of the model are first calculated for a finite number of N complex eigenvalues, where we exploit the existence of orthogonal Laguerre polynomials in the complex plane. When taking the large-N limit we derive new correlation functions in the case of weak and strong non-Hermiticity, thus describing the transition from the chGUE to a generalized Ginibre ensemble. We briefly discuss applications to the Dirac operator eigenvalue spectrum in quantum chromodynamics with non-vanishing chemical potential. This is an extended version of hep-th/0204068.

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1. Introduction

The application of random matrix models (RMMs) in a variety of physical systems has a long history [1]. Roughly speaking, the applications can be divided into two classes. On the one hand, the RMM can be used to describe local properties. The correlations, for example, of energy levels often show a universal behaviour on the distance of the mean level spacing, making description of the RMM possible. Choosing a RMM with the same global symmetries as the underlying Hamiltonian of the system fixes completely the form of the local spectral fluctuations. On the other hand, the RMM can be used as a combinatorial tool to count classes of graphs on random surfaces, which may then be interpreted as Feynman graphs of a corresponding quantum field theory. Our investigations here belong to the first class of applications on a local level, and the corresponding correlation functions are often called microscopic.

In most cases, RMMs have to be seen as effective models owing to their validity from symmetry considerations alone. However, in some cases the mapping to the theory one wishes

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to describe can be made exact, as for example in dynamical triangulations of random surfaces and its relation to quantum gravity [2] or in the application of the RMM to the Dirac operator spectrum in four-dimensional quantum chromodynamics (QCD) [3]. At the present stage, the RMM we propose here as a model for QCD with chemical potential is just an effective or phenomenological model. But a more precise mapping to QCD via chiral perturbation theory may also be possible.

In many applications, RMMs that have real eigenvalues are considered. All such possible models have been classified [4] and all the corresponding correlation functions are known (for a complete list of results and references see, for example, [5]). However, there are also physical situations where complex eigenvalues occur. For example, we mention localization in superconductors [6], dissipation and scattering in quantum chaos [7] or QCD with a non-vanishing chemical potential [8]. Other possible applications include the fractional quantum Hall effect as in [9] or a two-dimensional plasma of charged particles [10, 11]. A classification of all models with non-Hermitian matrices has been achieved more recently [12] and the number of possible cases exceeds by far those with real eigenvalues. But even when considering the simplest extension of the three RMM classes introduced by Dyson [13] much less is known. They consist of real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$) and quaternion self-dual matrices ($\beta = 4$) where the Dyson index β labels the ensemble by the power of the Jacobian diagonalizing the matrices, which is given by the Vandermonde determinant Δ. Due to their invariance, these ensembles are called Gaussian orthogonal, unitary or symplectic ensemble (GOE, GUE and GSE, respectively).

In a seminal work, the extension of these RMMs into the complex plane has been obtained by Ginibre [14]. He calculated the joint probability distribution function (JPDF) for $\beta = 2$ and 4 and partially for $\beta = 1$, as well as all microscopic correlation functions for $\beta = 2$. Since then progress has been slow. The full JPDF for $\beta = 1$ was calculated much more recently in [15] but, to date, the correlation functions are still lacking. The correlation functions for $\beta = 4$ are found in [16]. All these models are Gaussian ensembles with independently distributed matrix elements. More recently, generalized Gaussian ensembles have been considered, containing a parameter τ that permits us to interpolate between real and fully complex eigenvalues. The global density interpolating between the Wigner semi-circle on the real line and the constant density on a circular disc of Ginibre was found in [17] (see also [18] for numerical results of all three ensembles). Together with the introduction of the interpolating parameter τ the regime of so-called weak non-Hermiticity was discovered [19, 20] for $\beta = 2$ and 1. For $\beta = 2$ it interpolates smoothly between the known microscopic correlations of the GUE and those of Ginibre called the strong non-Hermitian limit [19]. The techniques to derive such results are either supersymmetry [19, 20] or orthogonal polynomials [21, 11] which also exist on the complex plane [9]. For $\beta = 4$, the τ-dependent correlations were discovered very recently in [22] in both the weak and strong limits. New correlation functions for $\beta = 2$ were also found in [23] where so-called Dirac mass terms were introduced in the application to three-dimensional QCD with complex eigenvalues.

From the RMM [24] which was first proposed to describe the QCD Dirac spectrum of real eigenvalues, the chiral Gaussian unitary ensemble (chGUE), we know that the model we are aiming for should belong to the class of chiral models. In the presence of a quark chemical potential, it is also known that the Dirac eigenvalues become complex [8]. We thus wish to define and solve such a chiral RMM with complex eigenvalues. The model is found by taking the known analogy between the GUE and the chGUE on the level of real eigenvalues and translating it into the complex plane. The starting point is the known complex GUE introduced in [9]. In analogy to the real case and from the finding of Hermite polynomials in the complex plane for the non-chiral complex GUE in [9], we expect and do find Laguerre

polynomials in the complex plane. This enables us to provide explicit solutions for a finite number N of eigenvalues as well as in the weakly and strongly non-Hermitian large-N limit.

The situation we wish to describe, QCD in the presence of a chemical potential μ, is quite delicate. The difficulties encountered when trying to carry out first-principles calculations by using Monte Carlo simulations on a lattice have been reviewed in [25]. These are due to the problem of convergence for a determinant of a Dirac operator with complex eigenvalues. Despite the fact that recent progress was made in [26], the full phase diagram with $\mu \neq 0$ remains unexplored so far. The analytical knowledge of microscopic correlation functions from the complex chiral RMM could thus be very useful, in view of its success in predicting real eigenvalue correlations.

The chemical potential problem has already been addressed in quite a few works using the RMM. Stephanov [8] has illuminated the failure of the quenched approximation. He also calculated the global eigenvalue density and its envelope as a function of μ. We use his results to determine our non-Hermiticity parameter τ as a function of μ for small values of μ. In [27] the temperature–density phase diagram of QCD with two light flavours was predicted using a RMM. A more refined model incorporating also vector and axial symmetries of the action was studied in [28]. Effective models for a phase transition with the spectral density vanishing to an arbitrary power were introduced in [29, 30]. Furthermore, the analyticity properties of the partition function [31], problems of convergence [32] as well as the global symmetry of such Dirac operators [18] were analysed using the RMM. Finally, first data for quenched simulations of QCD with $\mu \neq 0$ have been confronted with RMM predictions, finding a crossover from the Wigner–Dyson class over strongly non-Hermitian, Ginibre-type correlations to a Poisson distribution [33]. There the nearest-neighbour spacing of eigenvalues in the bulk from a non-chiral RMM were compared to the data. Here, we wish to provide analytical information on all microscopic correlation functions at the origin, which are important for chiral symmetry breaking [34], using a chiral eigenvalue model.

The paper is organized as follows. In section 2 we introduce a chiral complex eigenvalue model. A justification is given as to why it is called chiral and why in particular it represents a complex extension of the chGUE. The identification of our parameter τ as a function of the chemical potential μ in QCD is made by a comparison with the RMM [8]. In section 3 we introduce the orthogonal polynomials belonging to our model. We show that they are given by the Laguerre polynomials in the complex plane and we give the exact solution for all correlators for a finite number of eigenvalues N. Sections 4 and 5 treat the large-N limit at weak and strong non-Hermiticity, respectively. New correlation functions are found in both cases by taking appropriate limits of the Laguerre polynomials. We illustrate our findings with a few examples in each case. We also prove that the weak correlations interpolate between the known correlations of the chGUE and those in the strong limit from section 5. This is done by letting the weak non-Hermiticity parameter α approach zero or infinity in sections 4 and 5 respectively, explicitly checking that the correlations coincide. Our conclusions are presented in section 6. A brief summary of our results has already been published in [35].

2. The model

We first define our model in terms of complex eigenvalues $z_{j=1,\ldots,N}$ and then explain in detail why it is the generalization of the chGUE in the complex plane. The partition function is defined as

$$\mathcal{Z}^{(a)}_{\text{chiral}}(\tau) \equiv \int \prod_{j=1}^{N} (\mathrm{d}z_j \, \mathrm{d}z_j^* w^{(a)}(z_j)) \left|\Delta_N\left(z_1^2, \ldots, z_N^2\right)\right|^2 \tag{2.1}$$

where we have introduced the following weight function

$$w^{(a)}(z) = |z|^{2a+1} \exp\left[-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2+z^{*2})\right)\right] \tag{2.2}$$

and $\Delta_N(z_1,\ldots,z_N) = \prod_{k>l}^{N}(z_k - z_l)$ is the Vandermonde determinant. The parameter $a > -1$ can be taken to be real. In the application to QCD it is an integer, $a = N_f + \nu$, where N_f labels the number of massless quark flavours and ν is the sector of topological charge. The parameter $\tau \in [0,1]$ controls the degree of non-Hermiticity. For $\tau = 1$ we return to the chGUE with real eigenvalues, as the weight function (2.2) then reduces to a δ-function in the imaginary part $\mathrm{Im}(z)$ times a Gaussian weight for the real part $\mathrm{Re}(z)$. On the other hand, for $\tau = 0$ the degree of non-Hermiticity is maximal, and we see, in the following, the role of τ in the weight of the real and imaginary parts.

The motivation for the definition of the model (2.1) comes from a comparison to the non-chiral model of complex eigenvalues, the complex extension of the GUE, introduced in [9, 19, 23]. Let us therefore first recall the relation between the GUE and chGUE for real eigenvalues. The chGUE is defined in terms of a complex matrix W of size $N \times (N+\nu)$ [24]

$$\begin{aligned}\mathcal{Z}^{(N_f,\nu)}_{\mathrm{chGUE}} &= \int \mathrm{d}W\, \mathrm{d}W^\dagger \det(W^\dagger W)^{N_f}\, \mathrm{e}^{-Ng\mathrm{tr}(W^\dagger W)} \\ &= \int_0^\infty \prod_{j=1}^{N}\left(\mathrm{d}\lambda_j \lambda_j^{N_f+\nu}\, \mathrm{e}^{-Ng\lambda_j}\right) |\Delta_N(\lambda_1,\ldots,\lambda_N)|^2 \\ &= \int_{-\infty}^\infty \prod_{j=1}^{N}\left(\mathrm{d}x_j |x_j|^{2(N_f+\nu)+1}\, \mathrm{e}^{-Ngx_j^2}\right) \left|\Delta_N\left(x_1^2,\ldots,x_N^2\right)\right|^2 .\end{aligned} \tag{2.3}$$

In the first step we have diagonalized and given the partition function in terms of the positive eigenvalues[2] λ_j of the matrix $W^\dagger W$, while in the second step we have simply substituted $\lambda_j = x_j^2$. Note the squares inside the Vandermonde due to this transformation. Now we compare to the GUE given in terms of a Hermitian matrix H of size $N \times N$:

$$\begin{aligned}\mathcal{Z}^{(2N_f)}_{\mathrm{GUE}} &= \int \mathrm{d}H \det(H)^{2N_f}\, \mathrm{e}^{-Ng\mathrm{tr}(H^2)} \\ &= \int_{-\infty}^\infty \prod_{j=1}^{N}\left(\mathrm{d}x_j x_j^{2N_f}\, \mathrm{e}^{-Ngx_j^2}\right) |\Delta_N(x_1,\ldots,x_N)|^2 .\end{aligned} \tag{2.4}$$

Both models can be solved using the method of orthogonal polynomials [16] and the corresponding sets are given by the generalized Laguerre polynomials for the chGUE and by the Hermite polynomials for the GUE (at $N_f = 0$, for $N_f > 0$ see [36]). Moreover, when taking the parameter $a = N_f + \nu$ to be real we can reconstruct the orthogonal polynomials of the chGUE on the full real line from the even subset of the orthogonal polynomials of the GUE. Because of the squares in the Vandermonde in equation (2.3) we only need even polynomials there. This relation also holds for more general, non-Gaussian weight functions (see, for example, [29]). For $a = -1/2$ this is nothing but the known relation between Hermite and Laguerre polynomials [37] (see equation (3.8)).

Let us now turn to the complex generalization of the GUE as introduced in [9, 19, 23]. The Hermitian matrix H is replaced by a complex matrix

$$J = H + \mathrm{i}\sqrt{\frac{1-\tau}{1+\tau}}A \tag{2.5}$$

[2] To be precise these are called singular values.

with $\tau \in [0, 1]$. The weight for both Hermitian and anti-Hermitian parts of J, the Hermitian matrices H and A, is chosen to be Gaussian with variance $(1+\tau)/2N$. We therefore obtain [19]

$$\begin{aligned}\mathcal{Z}^{(2N_f)}(\tau) &\equiv \int \mathrm{d}J\,\mathrm{d}J^\dagger \det(J^\dagger J)^{N_f} \exp\left[-\frac{N}{1-\tau^2}\mathrm{tr}\left(J^\dagger J - \frac{\tau}{2}(J^2+(J^\dagger)^2)\right)\right] \\ &= \int \prod_{j=1}^{N}\left(\mathrm{d}z_j\,\mathrm{d}z_j^*|z_j|^{2N_f}\exp\left[-\frac{N}{1-\tau^2}\left(|z_j|^2 - \frac{\tau}{2}\left(z_j^2+z_j^{*2}\right)\right)\right]\right) \\ &\quad \times |\Delta_N(z_1,\ldots,z_N)|^2. \end{aligned} \tag{2.6}$$

For averages over the matrix elements of J this corresponds to

$$\langle J_{kl}J_{lk}\rangle = \frac{\tau}{N} \qquad \langle J_{kl}J_{kl}^*\rangle = \frac{1}{N} \tag{2.7}$$

which illustrates the interpolating role of τ. For $\tau = 1$ we have $J_{lk} = J_{kl}^*$ and thus Hermitian matrices, whereas for $\tau = 0$ all J_{kl} are mutually independent corresponding to maximal non-Hermiticity. In the latter case, the model has already been defined and solved by Ginibre [14] where the orthogonal polynomials are monomials in the complex eigenvalues z_j of J. Here, we can study the transition between the GUE and the Ginibre ensemble and the corresponding polynomials have been found to be Hermite polynomials in the complex plane [9]. Using the same reasoning as in the mapping between the real eigenvalues model of the GUE equation (2.4) and the chGUE equation (2.3), equation (2.6) immediately leads to our model equation (2.1). In this way, we have solved the problem of how to extend the positive real eigenvalues in equation (2.3) into the complex plane. We may, of course, ask ourselves what kind of matrix representation equation (2.1) would correspond to. We would have to express equation (2.1) in terms of a different matrix $\tilde{J}$, probably with two different complex matrices now, in the form $\tilde{J} \sim W^\dagger W + \mathrm{i}V^\dagger V$. However, so far we have not been able to find such a representation.

Let us compare a different matrix model, which has been introduced by Stephanov [8] for QCD with non-vanishing chemical potential:

$$\mathcal{Z}^{(N_f,\nu)}(\mu) \equiv \int \mathrm{d}W\,\mathrm{d}W^\dagger \det\begin{pmatrix} 0 & \mathrm{i}W+\mu \\ \mathrm{i}W^\dagger+\mu & 0\end{pmatrix}^{N_f} \mathrm{e}^{-Ng\mathrm{tr}(W^\dagger W)}. \tag{2.8}$$

Here, the matrix inside the determinant replaces the QCD Dirac operator with the chemical potential added as $\gamma_0\mu$. The disadvantage of adding a constant matrix to the chiral random matrix is that when including mass terms the model can no longer be written in terms of eigenvalues of the matrix W. Our idea is to use the model equation (2.1) instead and to suppose that on the scale of the mean level spacing of complex Dirac operator eigenvalues the microscopic correlations are the same. A somewhat similar observation has been made in [38] where a chiral matrix model with temperature was considered. In order to capture the effect of the lowest Matsubara frequency, a constant matrix $\mathrm{i}\gamma_0 t$ was added to the Dirac matrix. An analysis in terms of eigenvalues of the matrix $W^\dagger W$ is no longer possible, although the Dirac eigenvalues remain real in this case. However, the microscopic correlations remain the same compared to the model at $t = 0$, when translating to the same mean level spacing [38]. To avoid misunderstandings let us stress that we do not claim that the correlations at $\mu = 0$ and $\mu \neq 0$ are the same but that models with and without an eigenvalue representation may share the same microscopic correlations. We conjecture that this is the case for the the matrix model (2.8) at $\mu \neq 0$ and our model equation (2.1) at $\tau < 1$ due to RMM universality.

In order to relate these two parameters we compare the macroscopic spectral densities of the two models. In our model, it can be read off from [17] since the pre-exponential factors in the weight equation (2.2) are subdominant in the macroscopic large-N limit,

$$\rho_\tau(z) = \frac{1}{\pi(1-\tau^2)} \qquad \text{if} \quad \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leqslant 1 \tag{2.9}$$

where $z = x + \mathrm{i}y$. The density is constant and bounded by an ellipse, where for $\tau = 0$ we recover the result of [14]. On the other hand, for equation (2.8) the macroscopic density has been calculated in [8]

$$\rho_\mu(z) = \frac{1}{4\pi}\left(\frac{\mu^2+y^2}{(\mu^2-y^2)^2} - 1\right) \tag{2.10}$$

where the eigenvalues are bounded by the following curve[3]

$$0 = y^4(x^2+4\mu^2) + y^2(1+2\mu^2(2-x^2-4\mu^2)) + \mu^4(x^2-4(1-\mu^2)). \tag{2.11}$$

The aim of the two models (2.1) and (2.8) is to describe the fluctuations of Dirac eigenvalues close to the origin, because of their relevance for chiral symmetry breaking from the Banks–Casher relation [34]. We therefore restrict ourselves to small values of μ since otherwise the determinant in equation (2.8) is completely dominated by μ. Another reason for small μ is that we wish to remain in the phase with broken chiral symmetry. For $\mu \geqslant 1$ the curve (2.11) splits into two and the vanishing of the macroscopic spectral density $\rho_\mu(0)$ indicates, through the Banks–Casher relation, that the chiral phase transition has taken place. If we solve the biquadratic equation (2.11) for the imaginary part y and expand in μ we obtain

$$y^2 = \mu^4(4-x^2) + \mathcal{O}(\mu^6). \tag{2.12}$$

Expanding the density equation (2.10) in $\mu \ll 1$ we can therefore neglect y^2 and finally arrive at

$$\rho_\mu(z) = \frac{1}{4\pi\mu^2} \qquad \text{if} \quad \frac{x^2}{4} + \frac{y^2}{4\mu^4} \leqslant 1. \tag{2.13}$$

Comparing equations (2.9) and (2.13) we identify

$$4\mu^2 \equiv (1-\tau^2) \tag{2.14}$$

valid for small chemical potential μ and τ close to unity meaning small non-Hermiticity. We have now identified all parameters in our model with the corresponding quantities in QCD. In contrast to the model (2.8) by Stephanov we will not be able to study the chiral phase transition since our $\rho_\tau(z)$ will remain constant for all $\tau \in [0, 1)$. On the other hand, we will be able to calculate all correlation functions, for finite N as well as in two different microscopic scaling limits in the next sections. This has not been achieved so far for the model (2.8), where only the macroscopic density is known [8].

We finally comment on the presence of the absolute value in the weight equation (2.2) which is chosen to make the partition function real. In the model (2.8) no absolute value is present and the Dirac determinant carries a complex phase. However, the action remains real and could be written in terms of eigenvalues of $W^\dagger W$. In our model the situation is reversed, the determinant only depends on the modulus while the action depends on the phase. We believe that due to such a nontrivial relation our model also captures the effect of QCD with nonzero quark flavours with the weight equation (2.2). Let us emphasize that the absolute value is also present for $N_f = 0$ (with or without topological charge ν) and that it is crucial for the generalization of the real eigenvalue model equation (2.3) to work.

[3] In our conventions, the Dirac eigenvalues are real for $\mu = 0$, as should have become clear by now. In comparison to [8] this means that we interchange real and imaginary parts, $x \leftrightarrow y$.

3. The solution from orthogonal polynomials

We now turn to the solution of our model for a finite number of eigenvalues N, where we use the powerful method of orthogonal polynomials [16]. In the two following sections we then take the large-N limit. The orthogonal polynomials in the complex plane we are searching for are defined with respect to the weight function equation (2.2)

$$\int \mathrm{d}z\,\mathrm{d}z^* w^{(a)}(z) P_k^{(a)}(z) P_l^{(a)}(z^*) = \delta_{kl}. \tag{3.1}$$

All k-point correlation functions defined as

$$\rho_N^{(a)}(z_1,\ldots,z_k) \equiv \mathcal{Z}_{\mathrm{chiral}}^{(a)}(\tau)^{-1} \frac{N!}{(N-k)!} \int \prod_{j=k+1}^{N} \mathrm{d}z_j\,\mathrm{d}z_j^* \prod_{l=1}^{N} w^{(a)}(z_l) \left|\Delta_N\left(z_1^2,\ldots,z_N^2\right)\right|^2 \tag{3.2}$$

can be expressed in terms of the kernel of orthogonal polynomials

$$K_N^{(a)}(z_1,z_2^*) \equiv [w^{(a)}(z_1)w^{(a)}(z_2^*)]^{\frac{1}{2}} \sum_{k=0}^{N-1} P_k^{(a)}(z_1) P_k^{(a)}(z_2^*). \tag{3.3}$$

Due to the invariance of the determinant the monomial powers inside the Vandermonde Δ_N in equation (2.1) can be replaced by the orthogonal polynomials and we arrive at [16]

$$\rho_N^{(a)}(z_1,\ldots,z_k) = \det_{1\leqslant i,j\leqslant k}\left[K_N^{(a)}(z_i,z_j^*)\right]. \tag{3.4}$$

In equation (2.1) only even powers of z_j occur inside the Vandermonde Δ_N and we thus only need even polynomials. From the known relation between Hermite and Laguerre polynomials [37] for $a = -1/2$ we expect to obtain Laguerre polynomials of the squared argument, $P_k^{(a)}(z) \sim L_k^a(z^2)$, and we find that this is indeed the case (see equation (3.15)).

In order to derive the orthogonality of the Laguerre polynomials in the complex plane, we define the normalization integral

$$\begin{aligned} f^{(a)}(\tau) &\equiv \int \mathrm{d}z\,\mathrm{d}z^* w^{(a)}(z) \\ &= N^{-a-\frac{3}{2}}\pi\Gamma\left(a+\frac{3}{2}\right)(1-\tau^2)^{\frac{a}{2}+\frac{3}{4}} P_{a+\frac{1}{2}}\left(\frac{1}{\sqrt{1-\tau^2}}\right) \end{aligned} \tag{3.5}$$

where $P_{a+\frac{1}{2}}(x)$ is the Legendre function. While the expression given is exact for finite and infinite N, it simplifies in the weak non-Hermiticity limit in section 4 where $1-\tau^2$ is rescaled with N. Following the idea of [9] we use the invariance of the integral equation (3.5) under reparametrization to make the generating function of the Laguerre polynomials appear. Performing a change of variables $z \to \mathrm{e}^{\mathrm{i}\varphi} z$ inside the integral equation (3.5) we obtain from equation (2.2)

$$\begin{aligned} f^{(a)}(\tau) &= \int \mathrm{d}z\,\mathrm{d}z^* |z|^{2a+1} \exp\left[-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(\mathrm{e}^{\mathrm{i}2\varphi}z^2 + \mathrm{e}^{-\mathrm{i}2\varphi}z^{*2})\right)\right] \\ &\iff \quad 1 = \left\langle \exp\left[\frac{(\mathrm{e}^{\mathrm{i}2\varphi}-1)\tau^2}{1-\tau^2}\left(\frac{Nz^2}{2\tau}\right)\right] \exp\left[\frac{(\mathrm{e}^{-\mathrm{i}2\varphi}-1)\tau^2}{1-\tau^2}\left(\frac{Nz^{*2}}{2\tau}\right)\right]\right\rangle \end{aligned} \tag{3.6}$$

where the average is taken with respect to the weight function equation (2.2) and we have divided by $f^{(a)}(\tau)$ in order to normalize. We now wish to obtain the following generating functional for the Laguerre polynomials, which is also valid for complex arguments Z,

$$(1-u)^{-a-1}\exp\left[\frac{u}{u-1}Z^2\right]=\sum_{k=0}^{\infty}L_k^a(Z^2)u^k \qquad |u|<1 \tag{3.7}$$

and similarly for complex conjugate arguments. The split into variables u and Z^2 is not unique in equation (3.6). We use the known results [9] for the Hermite polynomials He_n of the model equation (2.6) to read off the argument in the case $a=-1/2$ using [39]

$$\mathrm{He}_{2k}\left(\sqrt{\frac{N}{\tau}}z\right)=(-1)^k 2^k k! L_k^{-\frac{1}{2}}\left(\frac{Nz^2}{2\tau}\right). \tag{3.8}$$

We therefore choose $Z^2=\frac{Nz^2}{2\tau}$ and

$$\frac{u}{u-1}\equiv\frac{(\mathrm{e}^{\mathrm{i}2\varphi}-1)\tau^2}{1-\tau^2} \tag{3.9}$$

which can be written as

$$u=\frac{\tau^2(1-\mathrm{e}^{\mathrm{i}2\varphi})}{1-\tau^2\,\mathrm{e}^{\mathrm{i}2\varphi}}. \tag{3.10}$$

This is complex and satisfies $|u|<1$. We now multiply equation (3.6) with $[(1-u)(1-u^*)]^{-a-1}$ to obtain

$$[(1-u)(1-u^*)]^{-a-1}=\sum_{k,l=0}^{\infty}\left\langle L_k^a\left(\frac{Nz^2}{2\tau}\right)L_l^a\left(\frac{Nz^{*2}}{2\tau}\right)\right\rangle u^k u^{*l} \tag{3.11}$$

where we have inserted the generating functional equation (3.7). Using the fact that

$$uu^*=\frac{4\tau^4\sin^2\varphi}{(1-\tau^2)^2+4\tau^2\sin^2\varphi} \tag{3.12}$$

together with the definition (3.9) it is easy to see that the left-hand side of equation (3.11) only depends on the combination uu^* and we obtain

$$[(1-u)(1-u^*)]^{-a-1}=[1-\tau^{-2}uu^*]^{-a-1}=\sum_{k=0}^{\infty}\frac{\Gamma(a+1+k)}{\Gamma(a+1)k!}\frac{(uu^*)^k}{\tau^{2k}} \tag{3.13}$$

which converges due to $\tau^{-2}uu^*<1$ from equation (3.12). Consequently also the right-hand side of equation (3.11) depends only on the combination uu^* and thus the Laguerre polynomials have to be orthogonal. By comparing coefficients of equations (3.13) and (3.11) we arrive at

$$\left\langle L_k^a\left(\frac{Nz^2}{2\tau}\right)L_l^a\left(\frac{Nz^{*2}}{2\tau}\right)\right\rangle=\delta_{kl}\frac{\Gamma(a+1+k)}{\Gamma(a+1)k!}\frac{1}{\tau^{2k}}. \tag{3.14}$$

The orthogonal polynomials from equation (3.1) are thus finally given as

$$P_k^{(a)}(z)\equiv\left(\frac{\Gamma(a+1+k)}{\Gamma(a+1)k!}f^{(a)}(\tau)\right)^{-\frac{1}{2}}(-\tau)^k L_k^a\left(\frac{Nz^2}{2\tau}\right). \tag{3.15}$$

The phase factor $(-1)^k$, which can be chosen arbitrarily, is taken such that the relation (3.8) is reproduced at $a=-1/2$. All k-point correlation functions are therefore completely

determined inserting the polynomials $P_k^{(a)}(z)$ into equations (3.3) and (3.4). We only give the expression for the kernel at finite-N:

$$K_N^{(a)}(z_1, z_2^*) = [w^{(a)}(z_1)w^{(a)}(z_2^*)]^{\frac{1}{2}} \frac{\Gamma(a+1)}{f^{(a)}(\tau)} \sum_{k=0}^{N-1} \frac{k!}{\Gamma(a+1+k)} \tau^{2k} L_k^a\left(\frac{Nz_1^2}{2\tau}\right) L_k^a\left(\frac{Nz_2^{*2}}{2\tau}\right). \tag{3.16}$$

Let us point out that, although we have checked our results by comparing with the orthogonal Hermite polynomial of the model (2.6) at $a = -1/2$, we cannot do the same check with the correlation functions. This is because in the model (2.6) the odd polynomials also contribute to the correlation functions.

In the following sections, we investigate the large-N limit. For this purpose usually a different form of the kernel equation (3.3) is given, which only contains the polynomials of order N and $N-1$ and is derived from the Christoffel–Darboux identity. However, due to the τ dependence of the arguments in equation (3.15) this identity no longer holds in the complex plane[4]. The modified Christoffel–Darboux identity now contains polynomials of all orders, $0, \ldots, N$, similar to the two-matrix model (see, for example, [40]). The large-N limit is no longer governed by the asymptotic polynomials alone and we have to use special properties of the Laguerre polynomials to achieve our goal in the next sections.

Before we continue, let us explain why we did not include Dirac quark masses into our model, in contrast to [23]. In the non-chiral model, mass terms of the form $\prod_f^{N_f} |z - \mathrm{i}m_f|^2$ were included in the weight function. The corresponding correlation functions were calculated by relating them to massless correlations, using the fact that the masses can be produced from a larger Vandermonde [41, 23]. Using the same technique in our model equation (2.1) we would only be able to treat additional terms of the form $\prod_f^{N_f} \left(z^2+m_f^2\right)\left(z^{*2}+m_f^2\right)$ in the weight (2.2). While these terms might be interesting for other applications, they are not of the right form for QCD mass terms of a complex Dirac matrix. For this reason we refrain from deriving such correlation functions here.

4. The weak non-Hermiticity limit

The large-N limit at weak non-Hermiticity has been introduced in [19] and is defined as follows. We take the limit $\tau \to 1$ that leads to Hermitian matrices at the same time as $N \to \infty$ such that the combination

$$\lim_{N\to\infty} N(1-\tau^2) \equiv \alpha^2 \tag{4.1}$$

is kept finite. In this limit, the macroscopic density equation (2.9) depends on real eigenvalues only. It becomes the Wigner semi-circle, $\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}$, for the Gaussian weight we have chosen, with support $[-2, 2]$ on the real line only. However, the properly rescaled microscopic correlations remain complex and are different from those of real eigenvalues as well as from the strong non-Hermiticity limit where $\tau < 1$. Because of the identification equation (2.14) we have made between τ and μ, equation (4.1) invokes to send also $\mu \to 0$ as $N \to \infty$ while keeping

$$4N\mu^2 = \alpha^2 \tag{4.2}$$

fixed[5]. In other words the weak non-Hermiticity parameter α^2 directly measures the chemical potential in the microscopic scaling limit. The rescaling of μ is similar to that of the quark

[4] I wish to thank E Kanzieper for pointing out this fact to me.

[5] In [23] an attempt was made to relate μ to α by comparing the macroscopic density (2.10) to the average of microscopic eigenvalues and a wrong power was obtained.

masses m_f, however with a different power in N. The same rescaling (4.2) appears in chiral perturbation theory replacing N by the volume when analysing the isospin chemical potential in quenched QCD (see, for example, [42]). Such a rescaling makes sure that the eigenvalues in the Dirac determinant are not completely dominated by the masses or the chemical potential (see equation (2.8)). Namely in the weakly non-Hermitian limit at the origin we also rescale the complex eigenvalues keeping

$$N(\mathrm{Re}\, z + \mathrm{i}\, \mathrm{Im}\, z) = Nz \equiv \xi. \tag{4.3}$$

The matrix size N corresponds to the lattice volume. It has been found in [32] that in the RMM equation (2.8) the numerical effort to obtain convergence grows exponentially with $N\mu^2$ for $N_f > 0$. Keeping the product fixed could possibly make simulations, and thus a comparison to data, more feasible.

Equations (4.1)–(4.3) define our microscopic origin scaling limit in the complex plane at weak non-Hermiticity. The kernel equation (3.3) and correlation function equation (3.4) also have to be rescaled with the mean level spacing $1/N$ of the eigenvalues. In the definition of the correlation functions for each variable z_j we integrate over its real and imaginary parts and we obtain one power $1/N$ from each integration. The microscopic kernel and correlation functions are therefore defined as follows in the weak non-Hermiticity limit:

$$\begin{aligned} K_S^{(a)}(\xi_1, \xi_2^*) &\equiv \lim_{N\to\infty} \frac{1}{N^2} K_N^{(a)}\left(\frac{\xi_1}{N}, \frac{\xi_2^*}{N}\right) \\ \rho_S^{(a)}(\xi_1, \ldots, \xi_k) &\equiv \lim_{N\to\infty} \frac{1}{N^{2k}} \rho_N^{(a)}\left(\frac{\xi_1}{N}, \ldots, \frac{\xi_k}{N}\right). \end{aligned} \tag{4.4}$$

When looking at the kernel at finite N equation (3.16) and the appropriate rescaling of the eigenvalues, equation (4.3), we observe that we are missing a power in N inside the argument of the Laguerre polynomials. The asymptotic of the Laguerre polynomials

$$\lim_{k\to\infty} k^{-a} L_k^a(Z^2) = (kZ^2)^{-\frac{a}{2}} J_a(2\sqrt{kZ^2}) \tag{4.5}$$

suggests that we introduce a variable $t \equiv \frac{k}{N}$ and thus replace the sum in equation (3.16) by an integral, $\sum_{k=0}^{N-1} \to N \int_0^1 \mathrm{d}t$. In doing so, we write everything in terms of the variables t and ξ_j. Under the integral we can replace everywhere the Laguerre polynomials by J Bessel functions using equation (4.5) since finite $t = \frac{k}{N} \in (0, 1]$ means that k has to become large and the point $t = 0$ is of measure zero.

In the following, we separately give the asymptotic of each factor in equation (3.16) before stating the final answer. The weight functions already contain the right powers of N in the exponential and using equations (4.1) and (4.3) we obtain

$$\lim_{N\to\infty} w^{(a)}(z)^{\frac{1}{2}} = N^{-a-\frac{1}{2}} |\xi|^{a+\frac{1}{2}} \exp\left[-\frac{1}{\alpha^2}(\mathrm{Im}\, \xi)^2\right]. \tag{4.6}$$

To calculate the weak limit of the normalization integral $f^{(a)}(\tau)$ in equation (3.5) we need to know the asymptotic of the Legendre functions, given by [43]

$$\lim_{\tau\to 1} P_{a+\frac{1}{2}}\left(\frac{1}{\sqrt{1-\tau^2}}\right) = \frac{2^{a+\frac{1}{2}}\Gamma(a+1)}{\sqrt{\pi}\Gamma(a+\frac{3}{2})}(1-\tau^2)^{-\frac{a}{2}-\frac{1}{4}}. \tag{4.7}$$

This leads to

$$\lim_{\tau\to 1} f^{(a)}(\tau) = \sqrt{2\pi\alpha^2} N^{-a-2} 2^a \Gamma(a+1). \tag{4.8}$$

Under the integral we first replace

$$\lim_{k\to\infty} \frac{k!}{\Gamma(a+1+k)} = N^{-a} t^{-a}. \tag{4.9}$$

Rewriting equation (4.1) as $\tau^2 = 1 - \alpha^2/N$ the power in τ turns into an exponential:

$$\lim_{N\to\infty} \left(\tau^{2k} = \mathrm{e}^{k\ln(\tau^2)}\right) = \mathrm{e}^{-t\alpha^2}. \tag{4.10}$$

Finally the Laguerre polynomials are replaced as

$$\lim_{k,N\to\infty} L_k^a\left(\frac{Nz^2}{2\tau}\right) = N^a(2t)^{\frac{a}{2}}\xi^{-a}J_a(\sqrt{2t}\xi). \tag{4.11}$$

Putting together equations (4.6)–(4.11) all powers of N cancel as they should and we obtain

$$K_S^{(a)}(\xi_1,\xi_2^*) = \frac{|\xi_1\xi_2^*|^{\frac{1}{2}}}{\sqrt{2\pi\alpha^2}}\frac{|\xi_1\xi_2^*|^a}{(\xi_1\xi_2^*)^a}\,\mathrm{e}^{-\frac{1}{\alpha^2}\left((\mathrm{Im}\xi_1)^2+(\mathrm{Im}\xi_2^*)^2\right)}\int_0^1 \mathrm{d}t\,\mathrm{e}^{-\alpha^2 t}J_a(\sqrt{2t}\xi_1)J_a(\sqrt{2t}\xi_2^*). \tag{4.12}$$

When inserting the kernel into the determinant equation (3.4) for the correlators, all factors apart from the integral can be taken out. In particular, the second factor cancels due to the occurrence of ξ_j^{-a} and ξ_j^{*-a} in each column and row. We arrive at the microscopic, weakly non-Hermitian correlation functions:

$$\rho_S^{(a)}(\xi_1,\ldots,\xi_k) = \prod_{l=1}^{k}\left(\frac{|\xi_l|}{\sqrt{2\pi\alpha^2}}\,\mathrm{e}^{-\frac{2}{\alpha^2}(\mathrm{Im}\xi_l)^2}\right)\det_{1\leqslant i,j\leqslant k}\left[\int_0^1 \mathrm{d}t\,\mathrm{e}^{-\alpha^2 t}J_a(\sqrt{2t}\xi_i)J_a(\sqrt{2t}\xi_j^*)\right]. \tag{4.13}$$

These are clearly different from the correlations of the non-chiral model equation (2.6) as calculated for $a = 0$ in [19] and for $a = N_f$ massless flavours in [23]. There, the product of two J Bessel functions is replaced by a single cosine, $\cos(\sqrt{2t}(\xi_i - \xi_j^*))$, which corresponds to a half-integer Bessel $J_{-\frac{1}{2}}$. As an important check we can compare the correlation functions of real eigenvalues [24] of the chGUE equation (2.3), which have been shown to be universal [44]. We take the Hermitian limit by sending $\alpha^2 \to 0$ which corresponds to $\tau = 1$. The exponential pre-factors become δ-functions for the imaginary parts of the eigenvalues, as they should. The integral inside the determinant then makes the universal Bessel kernel (of squared arguments) appear, reproducing the known correlation functions of [24]:

$$\int_0^1 \mathrm{d}t\,J_a(\sqrt{2t}x_1)J_a(\sqrt{2t}x_2) = \sqrt{2}\frac{x_1J_{a+1}(\sqrt{2}x_1)J_a(\sqrt{2}x_2) - x_2J_{a+1}(\sqrt{2}x_2)J_a(\sqrt{2}x_1)}{x_1^2 - x_2^2}. \tag{4.14}$$

To give an example for equation (4.13) we show the microscopic density in the weak limit for two values of a in figure 1. For one quark flavour ($a = 1$) the level repulsion at the origin is stronger, as expected. The plot continues symmetrically from the given first quadrant into the full complex plane by reflecting along the real and imaginary axis. In cutting along the positive real axis we can see the oscillations familiar to the correlations of real eigenvalues. Each maximum corresponds to the location of a single eigenvalue in the complex plane. We also show the α-dependence of the quenched microscopic density in figure 2. For increasing α and thus for increasing rescaled chemical potential μ, the eigenvalues spread further into the complex plane. In particular, the density also builds up along the imaginary axis. As we see in the next section, in the limit $\alpha \to \infty$ we obtain the microscopic correlations at strong non-Hermiticity to be defined later. The limiting picture to be compared with is figure 3 with the quenched density on the left.

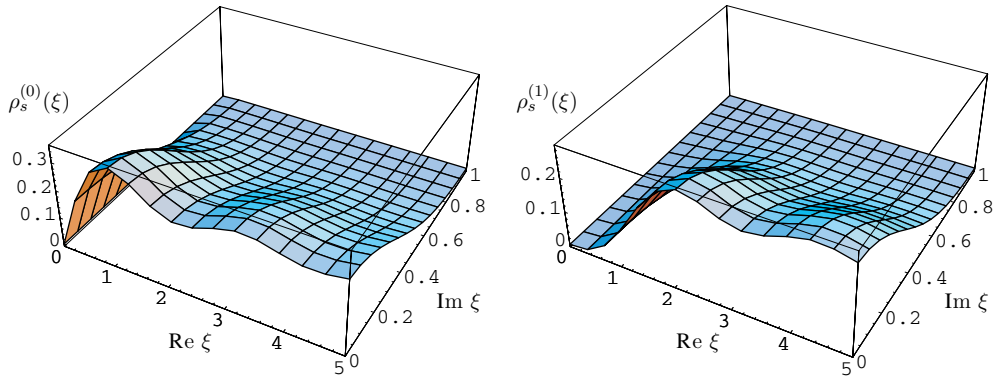

Figure 1. The microscopic density at $\alpha^2 = 0.6$ for $a = 0$ (left) and $a = 1$ (right) massless flavours.

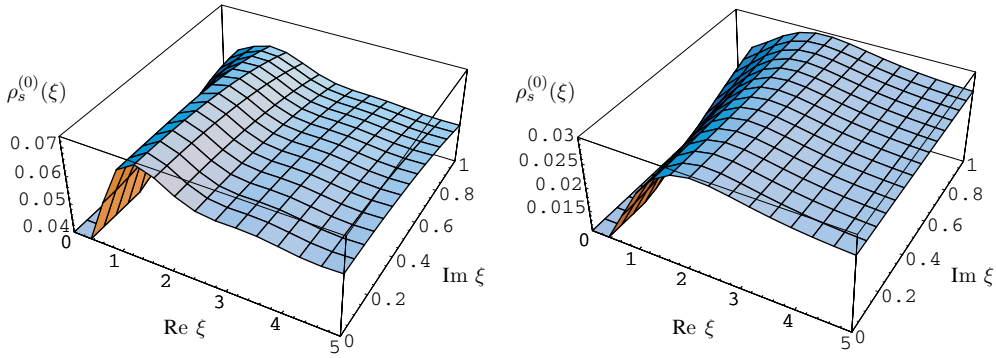

Figure 2. The quenched microscopic density ($a = 0$) at $\alpha^2 = 1.6$ (left) and $\alpha = 2.5$ (right).

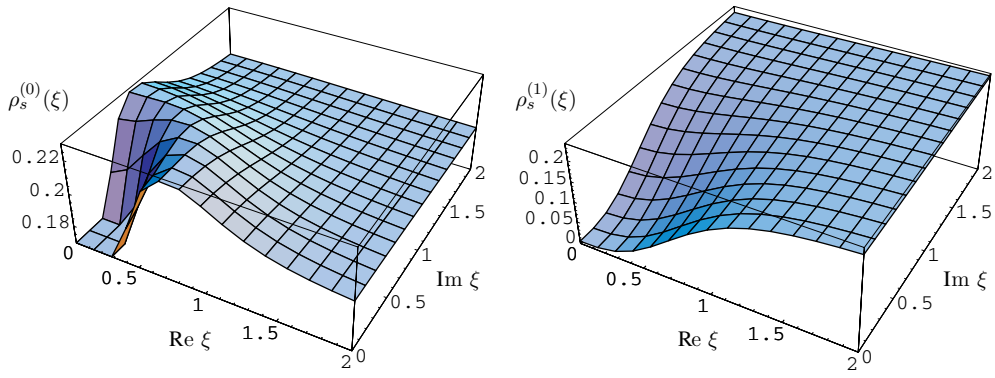

Figure 3. The microscopic density at $\tau = 0.5$ for $a = 0$ (left) and $a = 1$ (right) massless flavours.

5. The strong non-Hermiticity limit

We now turn to the strong non-Hermiticity limit. In this limit $\tau \in [0, 1)$ is kept fixed in the large-N limit and consequently also μ from equation (2.14) is kept finite and un-scaled,

in contrast to equation (4.2). The eigenvalues are now rescaled differently, namely with the square root of N [19, 23],

$$\sqrt{N}(\mathrm{Re}\, z + \mathrm{i}\,\mathrm{Im}\, z) = \sqrt{N} z \equiv \xi. \tag{5.1}$$

This defines our microscopic origin limit at strong non-Hermiticity. In consequence we also have to change the rescaling of the microscopic kernel and correlation functions accordingly:

$$\begin{aligned} K_S^{(a)}(\xi_1, \xi_2^*) &\equiv \lim_{N\to\infty} \frac{1}{N} K_N^{(a)}\left(\frac{\xi_1}{\sqrt{N}}, \frac{\xi_2^*}{\sqrt{N}}\right) \\ \rho_S^{(a)}(\xi_1, \ldots, \xi_k) &\equiv \lim_{N\to\infty} \frac{1}{N^k} \rho_N^{(a)}\left(\frac{\xi_1}{\sqrt{N}}, \ldots, \frac{\xi_k}{\sqrt{N}}\right). \end{aligned} \tag{5.2}$$

In the strong scaling limit equation (5.1), the argument of the Laguerre polynomials in equation (3.16) can be rewritten in terms of the scaling variable ξ alone. In particular, we can no longer use the asymptotic equation (4.5) for the Laguerre polynomials. Fortunately there is an identity for an infinite sum of Laguerre polynomials at hand [37]

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{k!}{\Gamma(a+1+k)} \tau^{2k} L_k^a\left(Z_1^2\right) L_k^a\left(Z_2^{*2}\right) \\ &\qquad = \frac{\left(Z_1^2 Z_2^{*2} \tau^2\right)^{-\frac{a}{2}}}{1-\tau^2} \exp\left[\frac{-\tau^2}{1-\tau^2}\left(Z_1^2 + Z_2^{*2}\right)\right] I_a\left(2\frac{\sqrt{Z_1^2 Z_2^{*2}\tau^2}}{1-\tau^2}\right) \end{aligned} \tag{5.3}$$

which is the analogue of Mehlers formula for the Hermite polynomials used in [19] to derive the strong correlation functions. We can thus immediately read off the large-N limit of the kernel equation (3.16) and obtain

$$\begin{aligned} K_S^{(a)}(\xi_1, \xi_2^*) = &\frac{2^a \Gamma(a+1)}{\pi \Gamma\left(a+\frac{3}{2}\right)(1-\tau^2)^{\frac{a}{2}+\frac{7}{4}} P_{a+\frac{1}{2}}\left(\frac{1}{\sqrt{1-\tau^2}}\right)} \frac{|\xi_1 \xi_2^*|^{a+\frac{1}{2}}}{(\xi_1 \xi_2^*)^a} \\ &\times \exp\left[\frac{-1}{2(1-\tau^2)}\left(|\xi_1|^2 + |\xi_2^*|^2 - \frac{\tau}{2}\left(\xi_1^{*2} - \xi_1^2 + \xi_2^2 - \xi_2^{*2}\right)\right)\right] I_a\left(\frac{\xi_1 \xi_2^*}{1-\tau^2}\right). \end{aligned} \tag{5.4}$$

Here we have inserted the explicit expression for the normalization integral equation (3.5). After taking out factors from the determinant, the correlation functions again simplify considerably and our final result reads

$$\begin{aligned} \rho_S^{(a)}(\xi_1, \ldots, \xi_k) = &\prod_{l=1}^{k} \left(\frac{2^a \Gamma(a+1)|\xi_l|}{\pi \Gamma\left(a+\frac{3}{2}\right)(1-\tau^2)^{\frac{a}{2}+\frac{7}{4}} P_{a+\frac{1}{2}}\left(\frac{1}{\sqrt{1-\tau^2}}\right)} \mathrm{e}^{-\frac{1}{1-\tau^2}|\xi_l|^2}\right) \\ &\times \det_{1\leqslant i,j\leqslant k}\left[I_a\left(\frac{\xi_i \xi_j^*}{1-\tau^2}\right)\right]. \end{aligned} \tag{5.5}$$

In [10] correlation functions for a charged two-dimensional plasma with fractional charges at the origin have been obtained, in what we call the strong non-Hermiticity limit at $\tau = 0$. However, since the interaction term induced by the Vandermonde determinant of squared arguments in our model equation (2.1) is different from that of [10], the correlation functions obtained there also differ from ours.

A consistency check for the correlation function equation (5.5) can be obtained from the weakly non-Hermitian correlations equation (4.13) by taking the limit $\alpha \to \infty$. We evaluate

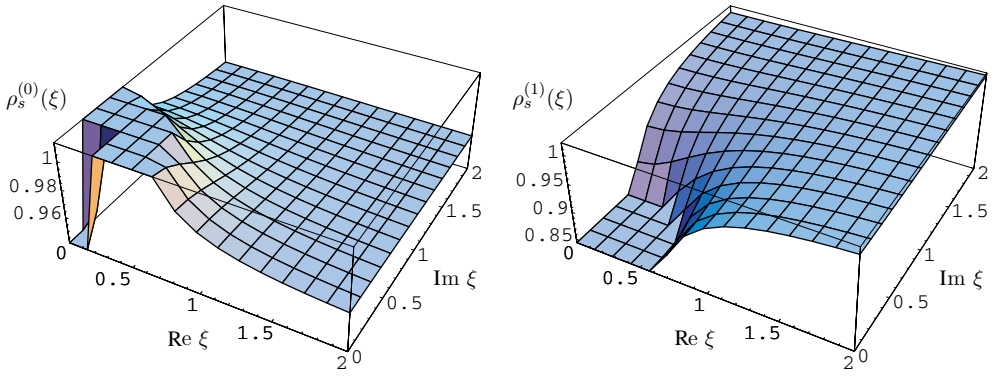

Figure 4. The microscopic density at $\tau \approx 0.9$ for $a = 0$ (left) and $a = 1$ (right) massless flavours.

the integral inside the determinant as follows

$$\lim_{\alpha^2\to\infty} \alpha^2 \int_0^1 \mathrm{d}t\, \mathrm{e}^{-\alpha^2 t} J_a(\sqrt{2t}\xi_i) J_a(\sqrt{2t}\xi_j^*) = \lim_{\alpha^2\to\infty} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s} J_a\left(\frac{\sqrt{2s}}{\alpha}\xi_i\right) J_a\left(\frac{\sqrt{2s}}{\alpha}\xi_j^*\right)$$
$$= \lim_{\alpha^2\to\infty} \exp\left[-\frac{1}{2\alpha^2}\left(\xi_1^2 + \xi_2^{*2}\right)\right] I_a\left(\frac{\xi_1\xi_2^*}{\alpha^2}\right). \tag{5.6}$$

Here the eigenvalues ξ_i are rescaled according to the weak non-Hermiticity equation (4.3). When identifying $\alpha^2 = N(1-\tau^2)$ we recover the scaling appropriate in the strong non-Hermiticity equation (5.1). Taking the exponential out of the determinant in equation (4.13) we obtain the strong correlation equation (5.5). In order to match the normalizations we also have to expand the Legendre function for large argument, according to equation (4.7). We thus have control over the full parameter range starting from real eigenvalues via weakly to strongly non-Hermitian correlation functions.

As an example, for equation (5.5) we depict the microscopic density in figure 3 for zero and one massless flavour. The value for $\tau = 0.5$ corresponds to $\mu = \sqrt{3}/4 \approx 0.43$ in our units, according to equation (2.14). The microscopic spectral density develops a hole at the origin and becomes flat for large values. This has indeed been observed in quenched lattice data for intermediate values of the chemical potential [33]. For one massless flavour ($a = 1$) the level repulsion at the origin is again stronger and the barrier becomes flattened. The picture is very much reminiscent of the microscopic density with massless flavours of the non-chiral model [23] in the strong limit (compared to figure 3 in [23]). When getting closer to the Hermitian limit by raising $\tau \nearrow 1$ or equivalently lowering $\mu \searrow 0$ the repulsion at the origin becomes much more pronounced. In figure 4 we have depicted the same densities as in figure 3 but for $\tau = \sqrt{21}/5 \approx 0.9$ corresponding to $\mu = 0.2$. Both values for μ in figures 3 and 4 are in the broken phase, compared to [8]. In [8], the phase transition occurs at a critical value $\mu_c = 1$ as follows from equation (2.11). Even if we take the extreme case of maximal non-Hermiticity which corresponds to $\tau = 0$ in our model we remain inside the phase with broken chiral symmetry. This can be seen when assuming that the relation (2.14) is still approximately valid, leading to $\mu(\tau = 0) = \frac{1}{2}$.

6. Conclusions

We have introduced and solved a new class of chiral RMM with complex eigenvalues, which corresponds to the extension of the chGUE into the complex plane. It has been shown that the orthogonal polynomials of our model are given by the Laguerre polynomials in the complex plane. Consequently all k-point correlation functions can be given exactly for a finite number of eigenvalues N in terms of the kernel of these polynomials. We have then investigated two different large-N limits with weak and strong non-Hermiticity. In both cases, explicit results have been given for all correlation functions in terms of the two different limiting kernels. Furthermore, we can prove that the results for weak non-Hermiticity interpolate between the known correlation functions of real eigenvalues of the chGUE and those of strong non-Hermiticity, taking the limit of the weak non-Hermiticity parameter $\alpha \to 0$ or ∞, respectively.

As an application of our model we have proposed that it describes the correlations of QCD Dirac eigenvalues close to the origin in the presence of a small chemical potential μ. For this purpose we related our parameter τ that measures the non-Hermiticity with the chemical potential as it occurs in QCD. In the identification we compared another matrix model that incorporates the symmetries of the Dirac operator with $\mu \neq 0$ more closely. It remains to compare our predictions to lattice data in four-dimensional QCD yet to be generated and to see if the success of the RMM in predicting real eigenvalues extends to the complex plane. It would be also very interesting to find other applications too, in particular given the interpretation of the complex eigenvalues as positions of charges distributed in a two-dimensional plane.

So far, we have only defined our model as an eigenvalue model and obtained its chiral interpretation by analogy with the corresponding real eigenvalue model, the chGUE, and its non-chiral counterpart, the GUE. It would be very interesting to find an explicit matrix representation of our model, in particular in view of the definition of Dirac mass terms and the corresponding solution.

The structure of RMM with complex eigenvalues seems to be much richer as the number of symmetry classes is much larger than those of real eigenvalue models. While not all of these models may be accessible to the technique of orthogonal polynomials presented here, at least the chiral versions of the non-Hermitian or non-symmetric matrices with the Dyson index $\beta = 4$ or $\beta = 1$, respectively, could be worth investigating. The fact that our chiral model with $\beta = 2$ again contains the classical polynomials of Laguerre type may be of help.

Acknowledgments

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