

# Microscopic universality of complex matrix model correlation functions at weak non-Hermiticity

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## Abstract

The microscopic correlation functions of non-chiral random matrix models with complex eigenvalues are analyzed for a wide class of non-Gaussian measures. In the large- $N$  limit of weak non-Hermiticity, where  $N$  is the size of the complex matrices, we can prove that all  $k$ -point correlation functions including an arbitrary number of Dirac mass terms are universal close to the origin. To this aim we establish the universality of the asymptotics of orthogonal polynomials in the complex plane. The universality of the correlation functions then follows from that of the kernel of orthogonal polynomials and a mapping of massive to massless correlators.

# 1 Introduction

The Universality of correlation functions is one of the crucial properties of random matrix models. In any of their applications, where we refer to [1] for a review, the mapping between the underlying field theory and the matrix model as an effective model is only dictated by the global symmetries of the field theory. On the matrix model side the invariance under this symmetry, given for example by unitary transformations  $U(N)$ , does not entirely fix the matrix model action and allows for a large variety of terms. The statement of universality is that in general the quantities of interest such as matrix eigenvalue correlation functions are the same for a certain class of matrix model actions in the large- $N$  limit. The simplifying choice of for example a Gaussian therefore does not matter.

One has to distinguish between different kinds of universality. In the macroscopic large- $N$  limit the eigenvalues of the random matrices which for example correspond to energy levels are kept unscaled. In this limit the oscillatory behavior of the correlation functions is smoothed and all two- and higher  $k$ -point functions are universal, to all orders in an expansion in  $1/N^2$  [2]. Such universality results find applications in Quantum Gravity [2], condensed matter physics of conductors [3] or more recently in the correspondence between supersymmetric Yang-Mills theory and low-energy String Theory [4].

In this work we concentrate on a different universality called microscopic. Here, eigenvalues and correlation functions are rescaled with the size of the matrices  $N$ , magnifying a particular part of the energy spectrum. Microscopic universality states that correlation functions are the same for a class of matrix model actions when measured with respect to the same mean level spacing between the eigenvalues. The eigenvalue spectrum of random matrix models is typically compact, similar for example to lattice gauge theories where large momenta are cut off due to the finite lattice spacing. In the simplest case the matrix model spectral density given by the Wigner semi-circle on an interval. We distinguish between the origin, bulk and edge region of the spectrum where random matrix model universality was proven in [5], [6] and [7] respectively (for a review see [8]). The origin of the spectrum is particularly interesting for applications to chiral symmetry breaking in low-energy Quantum Chromodynamics (QCD) [9] because of the relation between the order parameter, the chiral condensate, and the spectral density of QCD Dirac operator eigenvalues at the origin [10]. Here, universality has been shown for a large class of polynomial [5] and non-polynomial potentials [11], at criticality [12] and in the presence of massive Dirac terms [13]. For classes with orthogonal and symplectic symmetry proofs including massless and massive Dirac terms have been given in [14] and [15] respectively.

All we have said so far is valid for random matrices with real eigenvalues only. However, more recently applications for models with complex eigenvalues have occurred as well. To give some examples we mention the fractional Quantum Hall Effect [16], two-dimensional charged plasmas [17, 18], QCD with chemical potential [19], as well as the inverse potential problem in integrable systems [20]. We note that the chiral ensembles also contain complex matrices [9, 2] but remain in the class of real eigenvalue models.

The number of different symmetry classes among complex matrix models is particularly rich [21]. Furthermore, one has to distinguish between the large- $N$  limit of strongly and weakly non-Hermitian matrices. While in the former case results date back to Ginibre [22] the latter limit has been defined rather recently [23] and correlation functions have been calculated in [24, 25, 26, 27]. The class of microscopic correlators in the weak non-Hermiticity limit plays an important rôle as it permits to interpolate between the correlations of real eigenvalues and those of complex eigenvalues at strong non-Hermiticity. Our studies are devoted to the microscopic origin scaling limit at weak non-Hermiticity, where up to date almost nothing was known about universality. The only exception is the microscopic spectral density which has been shown to be universal in [28] using supersymmetric techniques for ensembles of sparse matrices with independently distributed entries and finite second moment. While supersymmetric techniques become quite involved for higher correlators we will exploit the existence

of orthogonal polynomials in the complex plane [16, 27] and analyze their asymptotic properties. Due to their universality and the power of the method of orthogonal polynomials [29] we can derive the universality of all  $k$ -point functions. Along the way of our analysis we will encounter similarities between the complex matrix model and the two-matrix model with two independent matrices as being studied for example in [30].

We will restrict ourselves to the simplest model, the complex extension of the Hermitian one-matrix model as introduced and solved in [16, 24]. In [25] the same model including an arbitrary number of Dirac masses was solved in the limit of weak and strong<sup>1</sup> non-Hermiticity and its relevance for three dimensional QCD with chemical potential was pointed out. In order to show also microscopic universality of the chiral model with complex eigenvalues introduced and solved very recently in [27], we would need to know higher order  $1/N$  corrections to our method. The chiral model is relevant for QCD with chemical potential in four dimensions. Apart from its universality it would be very interesting to see if it is indeed in the same class of models as [19].

The article is organized as follows. In Section 2 we define our non-Gaussian complex matrix model together with all relevant quantities at finite- $N$ . In the following Section 3 a second order differential equation for the orthogonal polynomials in the complex plane is derived in order to determine their asymptotic behavior. This leads to the universality of all  $k$ -point correlation functions of complex eigenvalues in the microscopic origin scaling limit at weak non-Hermiticity in Section 4. Our conclusions and future prospects are presented in Section 5.

## 2 The model

The partition function of the matrix model we wish to study is defined as

$$\mathcal{Z}_N^{(2N_f)} \equiv \int dJ dJ^\dagger \prod_{f=1}^{N_f} \det[(J - im_f)(J^\dagger + im_f)] \exp \left[ -N \text{tr} V(J, J^\dagger) \right], \quad (2.1)$$

$$V(J, J^\dagger) \equiv \frac{1}{1 - \tau^2} \left( J J^\dagger - \frac{\tau}{2} (J^2 + J^{\dagger 2}) \right) + \frac{1}{2} \sum_{k=2}^d \frac{g_{2k}}{2k} (J^{2k} + J^{\dagger 2k}). \quad (2.2)$$

The complex  $N \times N$  matrix  $J$  is parameterized as

$$J \equiv H + \sqrt{\frac{1 - \tau}{1 + \tau}} iA, \quad \tau \in [0, 1], \quad (2.3)$$

where  $H$  and  $A$  are Hermitian. The potential  $V(J, J^\dagger)$  consists of a quadratic, Gaussian part depending on the parameter  $\tau$  as well as of an arbitrary but fixed number of higher even powers in  $J$  and  $J^\dagger$  of maximal degree  $2d$ . The Gaussian part is fixed by requiring a Gaussian weight for both matrices  $H$  and  $A$  at equal variance  $(1 + \tau)/(2N)$ . When enlarging the measure to study universality we have to make a choice which terms we allow. Instead of allowing for arbitrary even potentials for the matrices  $H$  and  $A$  respectively, inserting  $J$  and  $J^\dagger$  according to eq. (2.3), we have only introduced even powers of in  $J$  and  $J^\dagger$ . The reason why we have to exclude higher order powers of mixed terms,  $(JJ^\dagger)^{k \geq 2}$ , goes as follows. In order to write eq. (2.1) in terms of eigenvalues we have to do the following unitary transformation,  $J = U(Z + R)U^\dagger$  [24]. Here  $U$  is unitary,  $Z = \text{diag}(z_1, \dots, z_N)$  is the diagonal matrix

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<sup>1</sup>In the massless limit the one- and two-point function were already calculated in [17] at maximally strong non-Hermiticity, using the picture of a charged plasma. The arbitrary charge at the origin there corresponds to massless Dirac terms in [25].

of complex eigenvalues of  $J$ , and  $R$  is a strictly upper triangular matrix<sup>2</sup>. Since products of strictly upper triangular and diagonal matrices remain traceless, the  $R$ -dependence drops out in all powers  $\text{tr} J^{2k}$  and similarly in its Hermitian conjugate. In  $\text{tr}(JJ^\dagger) = \text{tr}(ZZ^\dagger + RR^\dagger)$  the  $R$ -dependence persists but can be integrated out because it is Gaussian. Adding higher powers  $\text{tr}(JJ^\dagger)^k$  mixed terms between  $Z$ ,  $Z^\dagger$  and  $RR^\dagger$  would occur and the matrix  $R$  could no longer be integrated out. This is why we restrict ourselves to the form of the potential eq. (2.2). A similar class of potentials has also been considered in [20]. In terms of complex eigenvalues  $z_{j=1,\dots,N}$  the partition function eq. (2.1) thus reads

$$\mathcal{Z}_N^{(2N_f)} = \int \prod_{j=1}^N \left( d^2 z_j \prod_{f=1}^{N_f} |z_j - im_f|^2 e^{-NV(z_j, z_j^*)} \right) |\Delta_N(z_1, \dots, z_N)|^2, \quad (2.4)$$

$$V(z, z^*) = \frac{1}{1 - \tau^2} \left( z z^* - \frac{\tau}{2} (z^2 + z^{*2}) \right) + \frac{1}{2} \sum_{k=2}^d \frac{g_{2k}}{2k} (z^{2k} + z^{*2k}), \quad (2.5)$$

with  $\Delta_N(z_1, \dots, z_N) = \prod_{k>l}^N (z_k - z_l)$  being the Vandermonde determinant. We have dropped the constants coming from the integration over the matrices  $U$  and  $R$ . The  $k$ -point correlation functions among the  $N$  eigenvalues  $z_j$  are defined in the usual way [29]:

$$\begin{aligned} R_N^{(2N_f)}(z_1, \dots, z_k) &\equiv \frac{N!}{(N-k)!} \frac{1}{\mathcal{Z}_N^{(2N_f)}} \\ &\times \int d^2 z_{k+1} \dots d^2 z_N \prod_{j=1}^N \left( \prod_{f=1}^{N_f} |z_j - im_f|^2 e^{-NV(z_j, z_j^*)} \right) |\Delta_N(z_1, \dots, z_N)|^2. \end{aligned} \quad (2.6)$$

As it has already been pointed out in [15, 25] the correlation functions with and without the  $N_f$  quark masses are related. Absorbing the mass terms into the Vandermonde the following relation<sup>3</sup> has been derived [25]<sup>4</sup>:

$$R_N^{(2N_f)}(z_1, \dots, z_k) = \frac{R_{N+N_f}^{(0)}(z_1, \dots, z_k, im_1, \dots, im_{N_f})}{R_{N+N_f}^{(0)}(im_1, \dots, im_{N_f})}. \quad (2.7)$$

Consequently we only have to calculate the correlators at  $N_f = 0$  and prove their universality. The universality of the massive correlation functions then follows from eq. (2.7) at large- $N$ .

The  $k$ -point correlation functions eq. (2.6), massless or massive, can be obtained in the standard way [29] using orthogonal polynomials. These are defined in the complex plane as

$$\int d^2 z e^{-NV(z, z^*)} P_k(z) P_l(z^*) = \delta_{kl}, \quad (2.8)$$

were from now on we suppress the superscript ( $2N_f = 0$ ). The kernel of orthogonal polynomials is then given by

$$K_N(z_1, z_2^*) \equiv e^{-\frac{N}{2}(V(z_1, z_1^*) + V(z_2, z_2^*))} \sum_{l=0}^{N-1} P_l(z_1) P_l(z_2^*), \quad (2.9)$$

and the correlation functions follow from it [29]

$$R_N(z_1, \dots, z_k) = \det_{i,j=1,\dots,k} [K_N(z_i, z_j^*)]. \quad (2.10)$$

<sup>2</sup>We could also diagonalize  $J$  by a biunitary transformation,  $J = U\Lambda V^\dagger$ . This would diagonalize products  $(JJ^\dagger)^k$  but not  $J^2$  and  $J^{\dagger 2}$  in the Gaussian part.

<sup>3</sup>At finite- $N$  the relation holds exactly when the  $N$ -dependence in the weight is absorbed into the eigenvalues,  $e^{-V(z, z^*)}$ .

<sup>4</sup>In eq. (2.21) of ref. [25] an additional  $N$  is missing in the subscript of the denominator.

Already at finite- $N$  the orthogonal polynomials eq. (2.8) possess a matrix or eigenvalue representation for an arbitrary potential  $V(J, J^\dagger)$  eq. (2.2). For that purpose we introduce a parameter dependent partition function with matrices of smaller size  $n \times n$ , where  $n \leq N$ :

$$\mathcal{Z}_n(t) \equiv \int (dJdJ^\dagger)_{n \times n} \exp \left[ -\frac{n}{t} \text{tr} V(J, J^\dagger) \right] . \quad (2.11)$$

As one can easily convince oneself the following representation for orthogonal polynomials [31] also holds in the complex plane:

$$\tilde{P}_n(z) = \frac{1}{\mathcal{Z}_n(t = \frac{n}{N})} \int (dJdJ^\dagger)_{n \times n} \det(z - J) \exp \left[ -N \text{tr} V(J, J^\dagger) \right] = \langle \det(z - J) \rangle_{\mathcal{Z}_n(t = \frac{n}{N})} . \quad (2.12)$$

Here, they are normalized as  $\tilde{P}_n(z) = z^n + \dots$ . In order to obtain orthonormal polynomials as in the definition (2.8) we divide by their norm,  $h_n$ , to obtain

$$P_n(z) = h_n^{-\frac{1}{2}} \tilde{P}_n(z) = h_n^{-\frac{1}{2}} \langle \det(z - J) \rangle_{\mathcal{Z}_n(\frac{n}{N})} . \quad (2.13)$$

For the issue of universality we will also have to keep track of the  $(t = \frac{n}{N})$ -dependence of the polynomials in the asymptotic large- $n$  limit keeping  $t = \frac{n}{N}$  finite. We repeat that eqs. (2.8) – (2.13) also hold in the presence of  $2N_f$  masses in the measure.

The resolvent for the partition function eq. (2.11) is defined as

$$G_n(z; t) \equiv \frac{1}{\mathcal{Z}_n(t)} \int (dJdJ^\dagger)_{n \times n} \frac{1}{n} \text{tr} \frac{1}{z - J} \exp \left[ -\frac{n}{t} \text{tr} V(J, J^\dagger) \right] = \left\langle \frac{1}{n} \text{tr} \frac{1}{z - J} \right\rangle_{\mathcal{Z}_n(t)} . \quad (2.14)$$

It can also be expressed through the 1-point function, the spectral density, by

$$G_n(z; t) = \frac{1}{n} \int d^2w \frac{R_n(w)}{z - w} . \quad (2.15)$$

This representation is an alternative to calculate in particular the macroscopic large- $n$  spectral density from a saddle point analysis. We can deduce the following relation between density and resolvent in the complex plane which is different for real eigenvalues (see eq. (3.11) below):

$$\frac{1}{\pi} \partial_{z^*} G_n(z; t) = \frac{1}{n} R_n(z) . \quad (2.16)$$

### 3 The asymptotic of orthogonal polynomials

In this section we will derive a differential equation for the orthogonal polynomials from the representation eq. (2.13). The universality of the asymptotic polynomials and in consequence of the correlation functions then follow from it. The fact that we will take the large- $n$  limit at weak non-Hermiticity will allow us to borrow some results from the proof in the real case [5] despite the fact that the microscopic correlations spread out into the complex plane in our case. The polynomials defined in eq. (2.8) have parity symmetry because of the even potential chosen. We will therefore aim at a second order differential equation that closes on polynomials of equal parity.

We start by taking the derivative of  $P_n(z)$  from eq.(2.13)

$$\begin{aligned} \partial_z P_n(z) &= nh_n^{-\frac{1}{2}} \left\langle \frac{1}{n} \text{tr} \frac{1}{z - J} \det(z - J) \right\rangle_{\mathcal{Z}_n(\frac{n}{N})} \\ &= nG_n \left( z; t = \frac{n}{N} \right) P_n(z) + nh_n^{-\frac{1}{2}} \left\langle \frac{1}{n} \text{tr} \frac{1}{z - J} \det(z - J) \right\rangle_{\mathcal{Z}_n(\frac{n}{N})}^{com} , \end{aligned} \quad (3.1)$$

where we have introduced the connected part (*com*) of the expectation value. At large- $n$  expectation values are known to factorize and the second term will be suppressed by  $\frac{1}{n^2}$  with respect to the first. We could in principle also determine the asymptotic of  $P_n(z)$  from the above equation as it has been previously done in [6] in the real case. However, in order to obtain the correct behavior, sine and cosine as it will turn out, we would have to know also subleading corrections in  $N$  to fix their parity as in the real case [6]. Furthermore it is known that for orthogonal polynomials of real eigenvalues an exact, finite- $n$  differential equation exists [8] which only closes on the polynomials as an equation of second order. Taking one more derivative we arrive at

$$\begin{aligned} \partial_z^2 P_n(z) &= n \partial_z G_n \left( z; \frac{n}{N} \right) P_n(z) + n^2 G_n \left( z; \frac{n}{N} \right)^2 P_n(z) \\ &+ n G_n \left( z; \frac{n}{N} \right) h_n^{-\frac{1}{2}} \left\langle \frac{1}{n} \text{tr} \frac{1}{z-J} \det(z-J) \right\rangle_{Z_n(\frac{n}{N})}^{com} + n h_n^{-\frac{1}{2}} \partial_z \left\langle \frac{1}{n} \text{tr} \frac{1}{z-J} \det(z-J) \right\rangle_{Z_n(\frac{n}{N})}^{com} \end{aligned} \quad (3.2)$$

which is exact at finite  $n$ . At large- $n$  the connected expectation values in the second line are suppressed by  $\frac{1}{n^2}$  as follows from general counting arguments in matrix models with unitary invariance. Nevertheless the last term is of  $\mathcal{O}(1)$  since the derivative may produce another factor of  $n$ . We thus end up with the following second order differential equation:

$$\partial_z^2 P_n(z) = \left[ n^2 G_n \left( z; \frac{n}{N} \right)^2 + n \partial_z G_n \left( z; \frac{n}{N} \right) + \mathcal{O}(1) \right] P_n(z). \quad (3.3)$$

We now proceed exactly as in the universality proofs for real eigenvalues. The coefficient functions in front of the polynomials in eq. (3.3) are smoothed and thus replaced by their values in the macroscopic large- $n$  limit. The precise meaning of smoothing is defined as follows. The orthogonal polynomials  $P_n(z)$  obey a so-called recursion relation where  $zP_{n-1}(z)$  is reexpressed in terms of linear combinations of polynomials  $P_l(z)$  for  $l \leq n$ . On the real line a three step recursion relation holds for arbitrary polynomial potentials, which is no longer true in the complex plane. Here, the degree of the recursion explicitly depends on the degree  $2d$  of the potential eq. (2.5), similarly to the two-matrix model [30]. Smoothing is defined by the assumption that the recursion coefficients  $r_l$  approach a single function in the variable  $t = \frac{l}{N}$ . Let us give an example. For the Gaussian potential given by  $V(z, z^*) = \frac{1}{1-\tau^2} (zz^* - \frac{\tau}{2}(z^2 + z^{*2}))$  the orthogonal polynomials are known to be Hermite polynomials in the complex plane [16],  $P_n(z) = \sqrt{\frac{N}{\pi n!}} (1-\tau^2)^{-\frac{1}{4}} \tau^{\frac{n}{2}} He_n(z\sqrt{N/\tau})$ , and they obey the following recursion relation<sup>5</sup>

$$z P_{n-1}^{Gauss}(z) = \sqrt{\frac{n}{N}} P_n^{Gauss}(z) + \sqrt{\frac{n-1}{N}} \tau P_{n-2}^{Gauss}(z). \quad (3.4)$$

Obviously the recursion coefficients  $r_n = n/N$  approach a smooth limit,  $r_{n, n\pm 1, \pm 2, \dots} \rightarrow r(t) = t$ . Taking this limit we obtain the following expression for the Gaussian resolvent outside the support  $\sigma$  of the complex eigenvalues

$$G^{Gauss}(z; t) = \frac{1}{t} \left( \frac{z}{2\tau} - \sqrt{\frac{z^2}{4\tau^2} - \frac{1}{\tau}} \right), \quad z \in \mathbb{C} \setminus \sigma, \quad (3.5)$$

which coincides with [23] at  $t = 1$ . Here, we have used the fact that the resolvent eq. (2.15) can be expressed through the kernel via eq. (2.10). Expanding the pole and smoothing the non-vanishing even powers,  $\int d^2z e^{-nV/t} z^{2k} P_l(z) P_l(z^*) = (\tau t)^k \binom{2k}{k}$ , where we can follow the calculation for real eigenvalues given e.g. in [8], we arrive at eq. (3.5). It is the macroscopic resolvent outside the support as can be

<sup>5</sup>From eq. (3.4) one can see that the Christoffel-Darboux formula for the kernel eq. (2.9) no longer holds for  $\tau < 1$ .

seen from the vanishing of eq. (2.16),  $\partial_{z^*} G^{Gauss}(z; t) = 0$ . For our potential eq. (2.5) the recursion relation (3.4) extends down to  $P_{n-2d+1}(z)$ . We assume that the recursion coefficients have a smooth limit and that the procedure leads to the macroscopic resolvent,  $G(z, t)$ , and its derivative  $\partial_z G(z, t)$  in eq. (3.3). Because of the complicated form of the recursion relation for our general potential we use another derivation of the macroscopic resolvent using loop equations [2]. Exploiting the invariance of the partition function eq. (2.11) under the change of variables  $J \rightarrow J + \epsilon/(z - J)$  and its Hermitian conjugate (*H.c.*) leads to the following equation, after requiring  $\partial_\epsilon \mathcal{Z}(t)|_{\epsilon=0} = 0$ :

$$0 = \left\langle \left( \text{tr} \frac{1}{z - J} \right)^2 \right\rangle_{\mathcal{Z}(t)} - \left\langle \frac{n}{t} \text{tr} \left[ (\partial_J V(J, J^\dagger)) \frac{1}{z - J} \right] \right\rangle_{\mathcal{Z}(t)} + H.c. \quad (3.6)$$

It contains mixed terms as  $J^\dagger 2k-1/(z - J)$  which cannot be written solely in terms of eigenvalues, as explained in Section 2. This mixing can be partially removed for the Gaussian part of the potential with the help of the identity

$$0 = \left\langle \frac{n}{t} \text{tr} \left[ (\partial_J V(J, J^\dagger)) \frac{1}{z^* - J^\dagger} \right] \right\rangle_{\mathcal{Z}(t)} + H.c. \quad (3.7)$$

It follows from the invariance of  $\mathcal{Z}(t)$  under  $J \rightarrow J + \epsilon/(z^* - J^\dagger)$ . Adding  $\tau$  times eq. (3.6) to eq. (3.7) leads to

$$0 = \tau \left\langle \left( \text{tr} \frac{1}{z - J} \right)^2 \right\rangle_{\mathcal{Z}(t)} - \left\langle \frac{n}{t} \text{tr} \left[ \left( J + \frac{1}{2} \sum_{k=2}^d g_{2k} (\tau J^{2k-1} + J^\dagger 2k-1) \right) \frac{1}{z - J} \right] \right\rangle_{\mathcal{Z}(t)} + H.c. \quad (3.8)$$

from which we easily recover eq. (3.5) at large- $n$  when restricting us to the Gaussian. We cannot deduce the macroscopic resolvent  $G(z, t)$  from eq. (3.8) for a generic potential as it still depends on the matrix  $R$  from the diagonalization of  $J$ . To disentangle  $J$  and  $J^\dagger$  we would need more identities of the type eq. (3.7). However, eq. (3.8) will be sufficient to obtain the macroscopic resolvent in the weak non-Hermiticity limit. It is defined by taking  $\tau \rightarrow 1$ , as in the Hermitian limit, but with keeping the following product fixed:

$$N(1 - \tau^2) \equiv \alpha^2 \quad (3.9)$$

Whereas the appropriately rescaled microscopic quantities (see definition (3.16)) will depend on complex scaling variables and on the parameter  $\alpha$ , the macroscopic quantities are projected onto the real axis because of  $\tau \rightarrow 1$ . The smoothed resolvent  $G(z; t)$  or the macroscopic spectral density,  $\rho(z) \equiv \lim_{N \rightarrow \infty} N^{-1} R_N(z)$ , only depend on  $x = \Re z$  since macroscopically  $\Im z$  vanishes. This can be explicitly seen from eq. (3.5) where no  $\alpha$  dependence occurs. Furthermore, from the defining relation between the microscopic and macroscopic density at the origin we have

$$\lim_{|\xi| \rightarrow \infty} \rho_S(\xi) = \rho(0) \quad (3.10)$$

Taking the results for a Gaussian potential from [25] which depend on  $\xi, \alpha$  and the masses, it can be explicitly seen that the macroscopic density  $\rho(0)$  at weak non-Hermiticity is only non-vanishing when taking the limit  $|\xi| \rightarrow \infty$  along the real line<sup>6</sup>. Therefore the support of the macroscopic density collapses from a two dimensional set to an interval, e.g.  $\sigma = [-2, 2]$  in the Gaussian case (see eq.

<sup>6</sup>Eq. (3.10) also holds at strong non-Hermiticity. There  $\rho(0)$  is non-vanishing when taking the limit  $|\xi| \rightarrow \infty$  along all directions, indicating that the support extends into the complex plane.

(3.5)). For the macroscopic quantities on the real line the relation (2.16) is no longer valid. From the loop equation (3.8) at  $\tau = 1$  and large- $n$  we have<sup>7</sup>

$$G(x \pm i\epsilon, t) = \frac{1}{2t} V'(x) \mp i\pi\rho_t(x), \quad x \in \sigma \subset \mathbb{R}, \quad (3.11)$$

where

$$V(x) \equiv \sum_{k=1}^d \frac{g_{2k}}{2k} x^{2k}, \quad g_2 = 1, \quad (3.12)$$

$$\rho_t(x) \equiv \frac{1}{2\pi} \sum_{j=1}^m \frac{g_{2j}}{t} \sum_{k=0}^{j-1} \binom{2k}{k} r(t)^k x^{2(j-k-1)} \sqrt{4r(t) - x^2}. \quad (3.13)$$

All quantities are now real valued functions depending on  $x = \Re z$  only, with the above eqs. being familiar from the Hermitian one-matrix model (e.g. in [5]). The quantity  $r(t)$  is the limiting function of the recursion coefficients  $r_n$  at large- $n$  in the three step recursion relation generalizing eq. (3.4). It is related to the endpoint  $c$  of the eigenvalue support,  $\sigma = [-c, c]$ , through  $c^2 = 4r(t)$ . The determining equation following from the normalization condition  $\int_{\sigma} dx \rho_t(x) = 1$  is reading

$$\frac{1}{2} \sum_{k=1}^m \frac{g_{2k}}{t} \binom{2k}{k} r(t)^k = 1. \quad (3.14)$$

With all ingredients supplied we can finally take the microscopic scaling limit of the asymptotic differential equation (3.3) at the origin. It is defined as

$$Nz = N(\Re z + i\Im z) \equiv \xi. \quad (3.15)$$

In this limit the complex eigenvalues  $z$  approach zero while  $\xi$  is kept fixed. The rescaling of the microscopic correlators is given by

$$\rho_S(\xi_1, \dots, \xi_k) \equiv \lim_{N \rightarrow \infty} N^{-2k} R_N(N^{-1}\xi_1, \dots, N^{-1}\xi_k). \quad (3.16)$$

We take the smoothed eq. (3.3) only depending on the macroscopic resolvent  $G(z, t)$ . The latter is projected to the real axis, due to the weak non-Hermiticity limit eq. (3.9). At the same time we perform the microscopic limit (3.15), rescaling the differential equation by  $N^{-2}$  to obtain

$$\partial_{\xi}^2 P_t(\xi) = t^2 G(0 \pm i\epsilon; t)^2 P_t(\xi). \quad (3.17)$$

Here we have already removed the  $\epsilon$  from the argument of the polynomials since they are analytic. The subscript  $t$  indicates that they explicitly depend on  $t = \frac{n}{N} \in (0, 1]$  which is kept finite at large- $N$ . Taking the sum of eq. (3.17) for both signs and using eq. (3.11) we finally arrive at

$$\partial_{\xi}^2 P_t(\xi) = -t^2 \pi^2 \rho_t(0)^2 P_t(\xi), \quad (3.18)$$

where the contribution from the potential  $V'(0)$  vanishes. The following form for the asymptotic polynomials thus holds:

$$\lim_{n, N \rightarrow \infty; \tau \rightarrow 1} P_n \left( z = \frac{\xi}{N} \right) \rightarrow P_t(\xi) = \begin{cases} f(t) \cos(t\pi\rho_t(0)\xi) & n \text{ even} \\ g(t) \sin(t\pi\rho_t(0)\xi) & n \text{ odd} \end{cases}, \quad t = \frac{n}{N}, \quad (3.19)$$

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<sup>7</sup>Since the resolvent is no longer defined on the support  $\sigma$  of real eigenvalues we add a small imaginary part  $\pm i\epsilon$ .



where we have used their parity to select the appropriate solution. The normalization constants  $f(t)$  and  $g(t)$  still have to be determined. Our universal parameter  $\rho_t(0)$  is the  $t$ -dependent macroscopic spectral density for the partition function eq. (2.11) with potential  $V(x)/t$  from eq. (3.12). At  $t = 1$  it coincides with the macroscopic density  $\rho(0)$  of our model eq. (2.1) in the Hermitian limit (times a delta-function in  $\Im m z$ ). We now introduce the following abbreviation,

$$\pi t \rho_t(0) \equiv u(t) , \quad (3.20)$$

identifying the function  $u(t) = \int_0^t ds / (2\sqrt{r(s)})$  as it occurs in the universality proof of ref. [5]. The asymptotics of orthogonal polynomials in eq. (3.19) is thus the same as that of the polynomials on the real line [5], replacing the argument by a complex scaling variable  $\xi$ . However, the unknown coefficients  $f(t)$  and  $g(t)$  may still depend on the potential  $V(x)$  in a complicated fashion. In particular, they also depend on the weak non-Hermiticity parameter  $\alpha$  which is not present for real eigenvalues. Another difference is that because of the lack of a Christoffel-Darboux formula the correlation functions from eq. (2.10) do not only depend on  $P_t(\xi)$  at  $t = 1$  but on an integral over  $t$  [24, 25]. It is therefore a non-trivial task to show that the correlation functions of a Gaussian and our potential eq. (2.5) agree.

## 4 Universality of all correlation functions

As a first step we calculate the prefactors  $f(t)$  and  $g(t)$  of the asymptotic polynomials eq. (3.19) by taking the microscopic limit (3.15) of the orthogonality relation (2.8). By keeping  $t = \frac{k}{N}$  and  $t' = \frac{l}{N}$  fixed at large- $N$  the Kronecker  $\delta_{kl}$  becomes a delta-function. On the left hand side we have to introduce microscopic variables replacing  $z = \frac{\xi}{N}$ . In the potential eq. (2.5) terms of higher order than quadratic are suppressed in the limits (3.9) and (3.15), leading to  $\exp[-\frac{2}{\alpha^2}(\Im m \xi)^2]$ . We arrive at

$$\int d\Re \xi d\Im \xi e^{-\frac{2}{\alpha^2}(\Im m \xi)^2} P_t(\xi) P_{t'}(\xi^*) = \frac{1}{N} \delta(t - t') . \quad (4.1)$$

Inserting the asymptotic form of the polynomials (3.19) we first perform the integral over the real part,

$$\int_{-\infty}^{\infty} d\Re \xi \left\{ \begin{array}{c} \cos(u(t)\xi) \cos(u(t')\xi^*) \\ \sin(u(t)\xi) \sin(u(t')\xi^*) \end{array} \right\} = [\delta(u(t) - u(t')) + \delta(u(t) + u(t'))] \pi \cosh(2u(t)\Im m \xi) . \quad (4.2)$$

The second term can be dropped because of the positivity of the spectral density and thus of  $u(t)$ . The equation for the prefactors then reads

$$\frac{1}{N^2} \frac{\pi}{u(t)'} \delta(t - t') \left\{ \begin{array}{c} |f(t)|^2 \\ |g(t)|^2 \end{array} \right\} \int_{-\infty}^{\infty} d\Im \xi e^{-\frac{2}{\alpha^2}(\Im m \xi)^2} \cosh(2u(t)\Im m \xi) = \frac{1}{N} \delta(t - t') , \quad (4.3)$$

where the derivative  $u(t)'$  occurs. It leads to the following result

$$f(t) = g(t) = \left[ \sqrt{2} N u(t)' \alpha^{-1} \pi^{-\frac{3}{2}} \right]^{\frac{1}{2}} e^{-\frac{\alpha^2}{4} u(t)^2} . \quad (4.4)$$

The constant phase of the two functions which is arbitrary has been set to unity. The final result for the asymptotics of the polynomials is reading

$$\lim_{n, N \rightarrow \infty; \tau \rightarrow 1} P_n \left( z = \frac{\xi}{N} \right) = \left[ \sqrt{2} N u(t)' \alpha^{-1} \pi^{-\frac{3}{2}} \right]^{\frac{1}{2}} e^{-\frac{\alpha^2}{4} u(t)^2} \begin{cases} \cos(u(t)\xi) & n \text{ even} \\ \sin(u(t)\xi) & n \text{ odd} \end{cases} , \quad t = \frac{n}{N} . \quad (4.5)$$

The evaluation of the microscopic kernel defined as  $K_S(\xi_1, \xi_2^*) \equiv \lim_{N \rightarrow \infty} N^{-2} K_N(\xi_1/N, \xi_2^*/N)$  is now straight forward:

$$\begin{aligned}
K_S(\xi_1, \xi_2^*) &= e^{-\frac{1}{\alpha^2}((\Im m \xi_1)^2 + (\Im m \xi_2^*)^2)} \frac{2}{\alpha\pi} \int_0^1 \frac{dt}{\sqrt{2\pi}} u(t)' e^{-\frac{\alpha^2}{2}u(t)^2} [\cos(u(t)\xi_1) \cos(u(t)\xi_2^*) \\
&\quad + \sin(u(t)\xi_1) \sin(u(t)\xi_2^*)] \\
&= e^{-\frac{1}{\alpha^2}((\Im m \xi_1)^2 + (\Im m \xi_2^*)^2)} \frac{2}{\alpha\pi} \int_0^{\pi\rho(0)} \frac{du}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}u^2} \cos(u(\xi_1 - \xi_2^*)) . \tag{4.6}
\end{aligned}$$

Here, we have replaced the sum in eq. (2.9) by an integral,  $\sum_{i=0}^{N-1} = N \int_0^1 dt$  and we have substituted  $t \rightarrow u(t)$ . From eq. (3.20)  $u(0)$  vanishes at the boundary. Our result eq. (4.6) exactly coincides with the microscopic kernel calculated in [24] for a Gaussian potential and is thus universal. The known universal sine-kernel [6] is reproduced in the limit  $\alpha \rightarrow 0$ . The universal parameter is the macroscopic density of the Hermitian model  $\rho(0)$  at the origin and it resumes the dependence on all the coupling constants in our potential eq. (2.5) via eq. (3.13) at  $t = 1$ . The derivation presented here is an alternative to the one given in [28] for a Gaussian only, where an integral representation of the Hermite polynomials was used. A generalization of such a representation is not easy to obtain for our general potential considered.

The universality of the microscopic correlation functions at weak non-Hermiticity follows from eq. (4.6) by simply inserting eq. (2.10) into the definition (3.16). In taking the microscopic large- $N$  limit of eq. (2.7) it trivially carries over to the massive correlation functions as well, as it has already been pointed out in [25]. For explicit expressions of the massive correlation functions we also refer to [25].

In [24] the microscopic correlation functions at weak non-Hermiticity away from the origin have also been calculated for a Gaussian. It is not easy to translate our analysis away from the origin. Already in the Gaussian case the leading order coefficient of the asymptotic expansion of the Hermite polynomials is not sufficient to reproduce [24]. Since already in the Gaussian case we need the full finite- $n$  differential equation for the asymptotics of the Hermite polynomials the question of universality in the weak non-Hermiticity limit away from the origin is beyond the scope of this article.

Let us add two more remarks. As we have mentioned in Section 3 the whole formalism of orthogonal polynomials also holds in the presence of mass terms in the weight function. A natural question is why we could not directly derive the orthogonal polynomials including masses from the very same differential equation (3.3). The reason is that the mass terms in eq. (2.1) are given as a product of determinants which at large- $N$  would also factorize and thus completely drop out of the representation eq. (2.12). However, we know from the real case [13] that the massive and massless orthogonal polynomials are different. Therefore our leading order analysis in eq. (3.3) does not suffice in that case and we have circumvented this problem by the use of relation (2.7). Our second remark concerns the universality of the chiral complex matrix model containing Laguerre polynomials in the complex plane, as very recently introduced in [27]. The modification inside the partition function eq. (2.1) to obtain these orthogonal polynomials can be simply achieved by adding a term  $-(2a+1) \ln |z|/N$  to the potential in eq. (2.8) (and by squaring the arguments of the Vandermonde). However, going through the derivation we find that in eq. (3.3) this additional term is of the same order  $\mathcal{O}(1)$  as those which we neglected. Unfortunately we know from the analysis of the real eigenvalue case as reviewed in [8] that terms of this order are important and do contribute. Our method is therefore not easily applicable to the chiral case.

## 5 Conclusions

We have shown that universality in random matrix models holds also for correlation functions of eigenvalues in the complex plane. The proof we have presented holds in the limit of weakly non-Hermitian matrices where the parameter  $\alpha$  measuring the non-Hermiticity times the size of the matrices  $N$  is kept fixed when  $N$  goes to infinity. Furthermore, we have restricted ourselves to the microscopic limit at the origin where complex eigenvalues at zero are magnified. This limit is particularly important in the application to QCD as the spectral density (of Dirac eigenvalues) directly measures the chiral condensate, the order parameter of chiral symmetry breaking.

We have shown universality by analyzing the asymptotics of orthogonal polynomials in the complex plane using factorization in the large- $N$  limit. This led us to a universal differential equation exactly as in the case of real eigenvalues. The universality of the kernel of polynomials and thus of all correlation functions then followed. The inclusion of an arbitrary number of Dirac mass terms was done by expressing massive through massless correlators.

Our results present only a first step towards the universality of all complex eigenvalues correlations. This is not only due to the fact that the number of symmetry classes is by far larger than that of matrix models with real eigenvalues. There exist also two fundamentally different large- $N$  limits for matrices with complex eigenvalues: the limit of weak and strong non-Hermiticity where we have only investigated the former. The limit of weak non-Hermiticity is particularly important as it smoothly connects the known classes of real and strongly non-Hermitian eigenvalue correlations, by taking the limits  $\alpha \rightarrow 0$  and  $\alpha \rightarrow \infty$  respectively. These two classes would otherwise be completely disjoint. In fact by taking the limit  $\alpha \rightarrow \infty$  of our universal, weakly non-Hermitian correlators we have a good argument in favor of universality at strong non-Hermiticity as well.

Apart from the two classes of non-Hermiticity different correlations may also be found in other regions of the spectrum, in the bulk away from the origin and at the edge. Another question is that of macroscopic universality, which holds for all connected two- and higher  $k$ -point correlation functions of real eigenvalues. Such finding may have important consequences in phenomena such as wetting in two dimensions related to the inverse potential problem for analytic curves in the complex plane.

However, we consider universality of chiral complex matrix models, as very recently introduced by the author, as most urgent. Such models are aimed to describe the microscopic fluctuations of Dirac operator eigenvalues in four dimensional QCD which are complex due to the presence of a chemical potential. To answer that question we need to know subleading corrections in  $N$  to the differential equation of polynomials. We leave this task for future plans.

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