Stretched exponentials and power laws in granular avalanching

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Abstract. We introduce a model for granular surface flow which exhibits both stretched exponential and power law avalanching over its parameter range. Two modes of transport are incorporated, a rolling layer consisting of individual particles and the overdamped, sliding motion of particle clusters. The crossover in behaviour observed in experiments on piles of rice is attributed to a change in the dominant mode of transport. We predict that power law avalanching will be observed whenever surface flow is dominated by clustered motion.

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Although the sandpile model was introduced some time ago [1], its relevance to real granular materials remains unclear. It predicts that a pile formed by adding particles one at a time to a flat surface with open boundaries will naturally evolve to a continuous phase transition, illustrating the concept of *self–organised criticality* or SOC. The signal that such a critical state has been reached is when the spectrum of avalanche sizes becomes scale-invariant; that is, *power law* in form. By contrast, experiments using sand and beads have shown that, if anything, the generic form for the avalanche spectrum is *stretched exponential* rather than power law [2]—[7]. Whatever else the sandpile model may be, it is clearly not a very good model for piles of sand.

A cursory observation of real sandpiles in action easily reveals the shortcomings of the model. Inertia has been neglected [8]—[10] and the particles are assumed to move purely by a series of toppling events, defined in terms of the relative instabilities of local regions of the pile's surface. In fact, real particles gain kinetic energy as they accelerate downslope, dislodging other particles from the static bulk which combine to form a *rolling layer*. An intuitively more plausible approach treats this rolling layer as being governed by a convective diffusion equation [11], subsequently likened to a fluid layer interacting with the solid bulk [12]. This seems to be more in line with the experiments, for instance in predicting hysteresis in the variation of the angle of repose.

The first round of experiments all involved roughly spherical particles that could easily roll under their own weight once activated. Recently, similar experiments have been performed using highly anisotropic particles instead, namely grains of rice [13]. For rounder grains the avalanche spectrum was again found to be stretched exponential, but for grains with a large aspect ratio a very different picture emerged. These grains did not often roll but tended to slide along the surface in coherent domains, hereafter referred to as *clusters* [24]—[16]. The corresponding avalanche spectrum was power law. This rekindled hopes that some granular systems might after all be SOC, and a number of modified sandpile models have now been devised [17]—[22]. However, these "ricepile" models lack a clear physical interpretation and fail to exhibit stretched exponential behaviour over any part of their parameter space. Furthermore, none of them have the same exponent for the avalanche spectrum as those obtained from the experiments. This has led to speculation about whether the power law really is a consequence of SOC behaviour [23].

In this paper, we present a picture for granular avalanching which closely follows the qualitative descriptions of surface transport made during the ricepile experiments [13]. A simple model is devised which describes the transport process on the level of clusters. Numerical simulations show that the avalanche spectrum is stretched exponential for a broad parameter range, but crosses over to a power law regime when the surface transport is principally in the form of clusters. The recovery of SOC behaviour is hence attributed to the overdamped motion of clusters as opposed to the inertial motion of individual grains.

2. A simple model for the dual transport mechanism

We now proceed to derive a model which incorprates both individual particle and clustered motion. Although clearly very simplified, this may be regarded as the minimal model for dual surface flow.

Only the clusters themselves will be explicitly represented; independent grains are assumed to form a rolling layer that will be implicitly incorporated into the rules of the model. To approximate the discrete nature of clusters, a lattice repesentation is adopted in which the width of the pile is given by the integer variable i, $1 \leq i \leq L$. The boundary at $i = 1$ is closed and clusters can only exit the system via the open boundary at $i = L$. On each site i are stacked N_i clusters of varying height dh_{ij}, $1 \leq j \leq N_i$, so the total height is

$$h_i = \sum_{j=1}^{N_i} \mathrm{dh}_{ij} \tag{1}$$

and the local slope is $z_i = h_i - h_{i+1}$ (it is customary in sandpile models to use positive slopes even though the height is decreasing in the direction of increasing i). This discrete representation is similar to that adopted by other sandpile models, except that here the blocks represent whole clusters rather than just single particles.

New clusters can emerge in regions over the surface where the rate of conversion from rolling to static particles is non-zero. Anisotropic particles quickly lose their kinetic energy via uneven rolling *etc.* and consequently the rolling layer is limited to the vicinity of the closed boundary. In this case new clusters only arise near to $i = 1$, as schematised in Fig. 1. By contrast particles that easily roll, such as the rounder grains of rice used in the experiments, gives rise to a rolling layer extends deep into the system, so clusters may emerge anywhere over the pile's surface. Note that clusters may emerge on sites $i > 1$ even though the individual particles are only added next to the closed boundary.

Clusters on the surface may move under perturbations from the rolling layer or the motion of adjacent clusters. To model this, a cluster at site i whose local slope z_i exceeds some threshold value $(z_c)_i$ becomes active and begins to move. Following Christensen *et al.*, $(z_c)_i$ is taken to be a site-dependent random variable that changes value after each sliding event [20]—[22]. This annealed disorder represents the disordered packing of granular media. In the model specified below we also incorporate a number of other forms of disorder, such as the size of clusters and the number of clusters that move at once. Although the critical exponents vary for small systems, we were unable to simulate large systems and so it is possible that all of these models belong to the same universality class of one dimensional sandpile models with annealed disorder [17]—[22].

A moving cluster is large compared to the individual particles that activated it, so its velocity will be low and hence its motion will be *overdamped*. This means it will only move one site before coming to rest. However, in our model there is a fixed probability p_{diss} per sliding event that the cluster disintegrates into its constituent particles and disperses into the rolling layer, which on this level of description is equivalent to
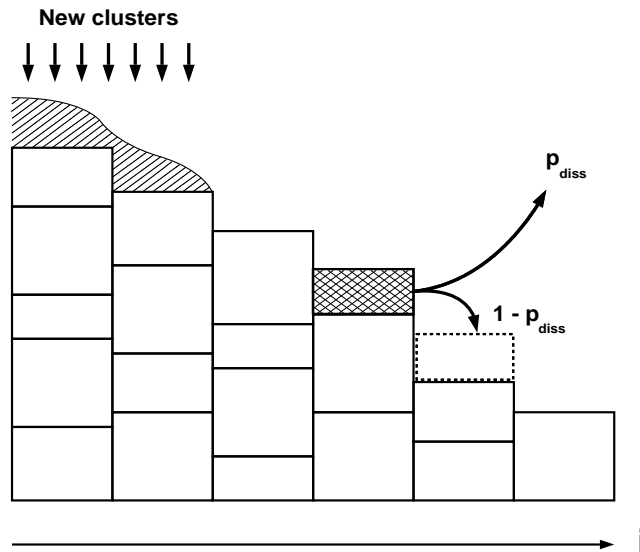
Figure 1. A schematic representation of the influx and movement of clusters. New clusters can only form on the surface in regions where the rolling layer (here denoted by the diagonally shaded area) is non–vanishing. An existing cluster begins sliding once the local slope exceeds some given threshold value. For instance, if the cross–hatched cluster in the diagram becomes activated, it either dissipates into the rolling layer with probability p_{diss} or moves one site to the right with probability $1 - p_{\text{diss}}$.

dissipation. For incompressible particles $p_{\text{diss}} = 1$, which may relate to the *fragility* of structures formed by such materials [25]. Conversely, $p_{\text{diss}} < 1$ for deformable materials, decreasing still further for elongated or jagged grains as a consequence of their increased contact surface area. Note that the dissipation of clusters into the rolling layer does *not* imply loss of mass conservation.

In summary, the model is specified as follows. Each site i is assigned a critical slope variable $(z_c)_i$. For every time step, the following procedure is performed.

(i) *Driving:* A site i is selected and a cluster of height dh_{ij} is added to the top of it.

(ii) *Check for stability:* Any site k whose local slope $z_k > (z_c)_k$ is marked for toppling.

(iii) *Toppling:* All of the unstable sites marked in step (ii) are toppled in parallel. For each k, the topmost n clusters are selected and moved to site $k+1$, or are removed from the system with probability p_{diss}.

(iv) *Annealed z_c:* Every site that toppled is assigned a new critical slope z_c.

(iv) *Avalanche:* Steps (ii)–(iv) are repeated until every site in the system is stable, then another cluster is added as per step (i).

The distributions for the random variables dh_{ij}, z_c, n and the site i where the clusters are added need to be specified in some manner that relates to the type of grain involved. This will be described in the following section.

3. Comparison with experimental data

It will now be demonstrated that this model exhibits the same crossover in behaviour as in the ricepile experiments. The parameter p_{diss} and the range of cluster addition depend upon the properties of the type of grain in question, in a manner to be described below. The other parameters used in the simulations were the same for both cases. The cluster heights dh_{ij} were drawn from the uniform distribution $[a, b]$. The threshold slopes z_c were uniformly distributed in $[2, 3]$, although other ranges were also considered with no qualitative change in behaviour observed. When a site becomes unstable the topmost n clusters are activated and start to move, where n is either fixed at 1 or randomly selected from $\{1, 2\}$. No significant changes in behaviour were observed for all reasonable parameter ranges, although deviations did occur in some extreme cases, for instance when $b \gg a$ or $a \to 0$.

To simulate a system comprising of the rounder grains that gave rise to a broad rolling layer but did not form stable clusters, new clusters are added to the surface uniformly over the range $1 \leq i \leq L$, and $p_{\text{diss}} \in (0.1, 0.6)$. A new cluster is not added until all of the sites in the system are stable. Numerical simulations measuring $P(E, L)$, the distribution of the potential energy E lost by the pile as the result of a single addition, shows good data collapse after the finite–size rescaling $P(E, L) = L^{-1}f(E/L)$, in accord with the experimental data. The scaling function $f(x)$ is a stretched exponential,

$$f(x) = A \exp\{-(x/x_0)^\gamma\} \tag{2}$$

with $0 < \gamma < 1$, where γ and x_0 depend upon the parameters. By varying p_{diss} we found it easy to obtain values close to those measured in the early experiments and the ricepile experiments [7, 13], as illustrated in Fig. 2.

The elongated grains of rice formed stable clusters that passed intact through the system, but the penetration of the rolling layer was limited. This translates into $p_{\text{diss}} = 0$ and new clusters only being added to sites i in a range such as $1 \leq i \leq L/10$ or $1 \leq i \leq L/20$. $P(E, L)$ obeys the same finite size scaling relation as before, but $f(x)$ is now flatter for small x and power law for large x, $f(x) \sim x^{-\alpha}$, as in Fig. 3. The exponent α depends upon the choice of parameters and the system size, typically taking values in the range 1.1 to 1.6. We were unable to obtain an exponent close to the experimental value, which was "just greater than 2" [13]. This might be due to some crucial dynamical effect which has been overlooked, although it may just be an artefact of the reduced dimensionality of the model. Since $\alpha < 2$ a finite–size cut–off is necessary and large systems are needed for a definite power law region to appear. However, measurements of the number of clusters that moved during each avalanche, rather than the change in potential energy, shows a clearer power law (with the same exponent α) for all system sizes. Coupled with the similarity to many sandpile models, this implies that the model is SOC in this region of parameter space.

Christensen *et al.* performed a further set of experiments to measure the time taken for tracer particles (coloured grains of rice) to pass through the system [20]. Only the
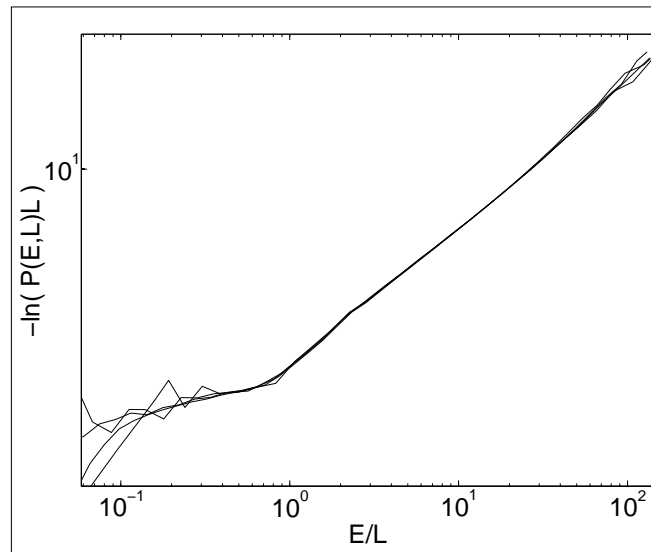
Figure 2. Finite–size scaling plot of $P(E, L)$ for system sizes $L = 100$, 250, 500 and 1000. New clusters were added uniformly over the entire system and $p_{\text{diss}} = 0.4$. If $f(x) = A \exp\{-(x/x_0)^\gamma\}$ then $-\ln f(x) = -\ln A + (x/x_0)^\gamma$, which will give a straight line on a log–log plot when x is large and the corrections due to $\ln A$ can be ignored. For the plots shown above, $\gamma \approx 0.43$ and $x_0 \approx 0.4$, in agreement with the values from the ricepile experiments, $\gamma = 0.43 \pm 0.03$ and $x_0 = 0.45 \pm 0.09$. The other parameters were $z_c \in [2, 3]$, $n \in \{1, 2\}$, and $\text{dh}_{ij} \in [0.5, 1.5]$. Units were chosen such that $mg = 1$.
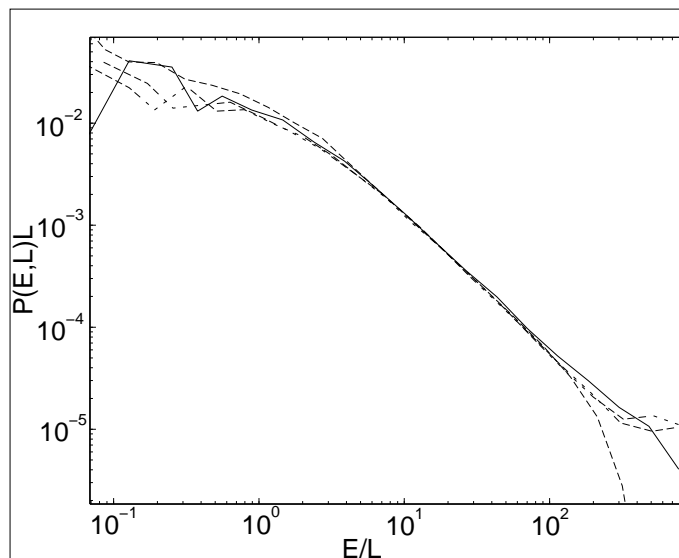

Figure 3. Finite–size scaling plot of $P(E, L)$ for four different system sizes, $L = 500$ (dotted line), $L = 1000$ (dashed-dotted line), $L = 1500$ (dashed line) and $L = 2000$ (solid line). New clusters were added uniformly over the range $1 \le i \le L/20$ and $p_{\text{diss}} = 0$. The straight line has a slope of approximately 1.30 ± 0.05. The other parameters were $z_c \in [2, 3]$, $n \in \{1, 2\}$ and $\text{dh}_{ij} \in [0.8, 1.2]$.
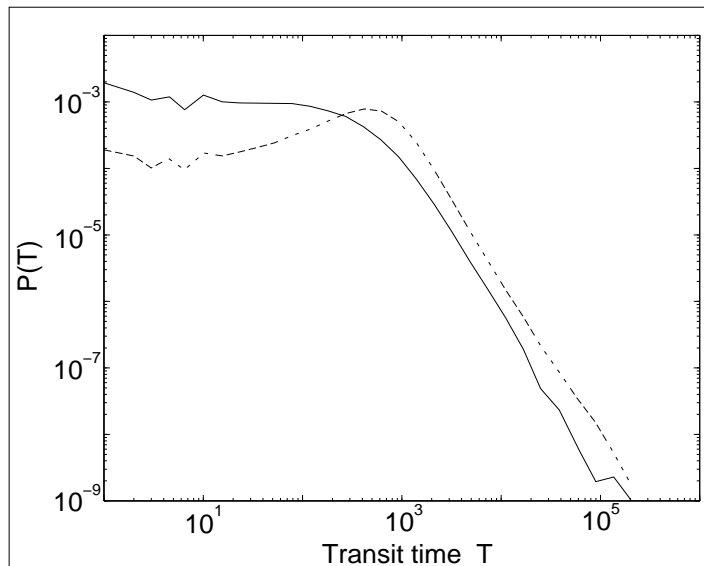
Figure 4. The distribution of transit times for clusters to move through the system, for the same parameter values as those given in Fig. 2 (solid line) and Fig. 3 (dashed line). The straight line has a slope of 2.3 ± 0.1. The system size was $L = 250$ in both cases.

elongated grains that gave the power law in the first set of experiments were used. They found that the distribution of transit times $P(t)$ was roughly constant for small t but crossed over to a power law for large t, $P(t) \sim t^{-\beta}$ with $\beta = 2.4 \pm 0.2$. Simulations of our model show the same behaviour with a similar exponent for *both* cases studied, as demonstrated in Fig. 4. Thus we conclude that this power law is *not* symbolic of a critical state and predict that the same exponent would be recovered if the experiments were repeated using the rounder grains.

4. Summary

In summary, we have argued that the crossover in behaviour observed in the ricepile experiments can be attributed to a change in the dominant mode of transport, from the inertial motion of the rounder grains to the overdamped motion of clusters of the elongated grains. A simple model was introduced in which blocks represent coherent clusters of particles. Not only does this explain why inertial effects may be ignored, since clusters naturally move in an overdamped fashion, but also allows for the dissipation of blocks in a physically plausible manner. This may be a important mechanism for the emergence of the stretched exponential behaviour.

If this hypothesis is correct, then power law avalanching should also be observed for any material in which surface flow is dominated by clustered motion. It would be interesting to see if this prediction might be verified experimentally. We have recently become aware of experiments on watered soil ridges where regions of the surface moved as whole regions [26]. Power law behaviour was observed, which would appear to be in

accord with our hypothesis.

References

[1] P. Bak, C. Tang and K. Wiesenfeld 1988 *Phys. Rev. A* **38** 364
[2] H. M. Jaeger, C. Liu and S. R. Nagel 1989 *Phys. Rev. Lett.* **27** 40
[3] M. Bretz, J. B. Cunningham, P. L. Kurczynski and F. Nori 1992 *Phys. Rev. Lett.* **69** 2431
[4] J. Rosendahl, M. Vekić and J. Kelly 1993 *Phys. Rev. E* **47** 1401
[5] G. A. Held, D. H. Solina II, D. T. Keane, W. J. Haag, P. H. Horn and G. Grinstein 1990 *Phys. Rev. Lett.* **65** 1120
[6] S. R. Nagel 1992 *Rev. Mod. Phys.* **64** 321
[7] J. Feder 1995 *Fractals* **3** 431
[8] J. Krug, J. E. S. Socolar and G. Grinstein 1992 *Phys. Rev. A* **46** R4479
[9] C. P. C. Prado and Z. Olami 1992 *Phys. Rev. A* **45** 665
[10] D. A. Head and G. J. Rodgers 1997 *Phys. Rev. E* **55** 2573
[11] J.-P. Bouchaud, M. E. Cates, J. R. Prakash and S. F. Edwards 1995 *Phys. Rev. Lett.* **74** 1982
[12] H. M. Jaeger, S. R. Nagel and R. P. Behringer 1996 *Rev. Mod. Phys.* **68** 1259
[13] V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T. Jøssang and P. Meakin 1996 *Nature* **379** 49
[14] A. Mehta 1992 *Physica A* **186** 121
[15] G. C. Barker and A. Mehta 1993 *Phys. Rev. E* **47** 184
[16] A. Mehta in *Granular Matter: An Interdisciplinary Approach*, edited by A. Mehta 1994 (New York: Springer-Verlag)
[17] L. A. N. Amaral and K. B. Lauritsen 1996 *Physica A* **231** 608
[18] L. A. N. Amaral and K. B. Lauritsen 1996 *Phys. Rev. E* **54** R4512
[19] M. Markošovà, M. H. Jensen, K. B. Lauritsen and K. Sneppen 1997 *Phys. Rev. E* **55** R2085
[20] K. Christensen, Á. Corral, V. Frette, J. Feder and T. Jøssang 1996 *Phys. Rev. Lett.* **77** 107
[21] M. Paczuski and S. Boettcher 1996 *Phys. Rev. Lett.* **77** 111
[22] M. Boguñá and Á. Corral 1997 *Phys. Rev. Lett.* **78** 4950
[23] M. E. J. Newman and K. Sneppen 1996 *Phys. Rev. E* **54** 6226
[24] G. C. Barker and A. Mehta 1996 *Phys. Rev. E* **53** 5704
[25] M. E. Cates, J. Wittmer, J.-P. Bouchaud and P. Claudin 1997 *preprint cond-mat/9803197*
[26] E. Somfai, A. Czirók and T. Vicsek 1994 *J. Phys.A* **27** L757