Dimensional crossover in fragmentation

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October 30, 2006

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Abstract

Experiments in which thick clay plates and glass rods are fractured have revealed different behavior of fragment mass distribution function in the small and large fragment regions. In this paper we explain this behavior using non-extensive Tsallis statistics and show how the crossover between the two regions is caused by the change in the fragments' dimensionality during the fracture process: We obtain a physical criterion for the position of this crossover and an expression for the change in the power law exponent between the small and large fragment regions. These predictions are in good agreement with the experiments on thick clay plates.

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PACS number(s): 05.40.Fb, 24.60.-k

Experiments on plate fragmentation [1] have revealed two regions of behavior in the fragment size distribution function (FSDF). The crossover between the regions scales with the width of the plate. In these experiments (and also in [2] where an analogous behavior was observed in the breaking of glass rods) the transition from one kind of behavior to another is assigned to a dimensional crossover, as was also pointed out in [3]. It seems physically clear that a dimensional dependence exists in the fragmentation of objects. A long thin glass rod must reveal a different behavior for fracture than a "thick cylinder" of the same material, as the small fragments produced from the fracture of glass rods appear to be approximately cylindrical. Also, a plate should manifest a different behavior for fracture than a "three-dimensional" fragment for which all dimensions are similar.

In [3] it was shown that no crossover was present for fragmentation of three-dimensional objects. This was confirmed in [4] where experiments with falling mercury drops, which always preserve their spherical shape, were reported.

Dimensional dependence and multiscaling in fragmentation have received attention in [5, 6] with *ad hoc* models built to describe the dimensional dependence of FSDF and its multifractality.

On the other hand attempts have been made to obtain the FSDF starting from first principles, i.e., the maximum entropy principle [7, 8]. As the maximum entropy principle is universal, it has an almost unlimited range of applications, and consequently some of the properties of FSDF observed in experiments, such as scaling and multifractality, are expected to be revealed. However, no dimensional crossover was found in either [7] or [8]. In our opinion this is due to the assumption that not just the maximum entropy principle, but the formula for the Boltzmann-Gibbs (BG) entropy of a system, are valid for the description of fracture phenomena. The formula for the BG entropy is:

$$S = -k \sum_{i=1}^{W} p_i \log p_i,$$

where p_i is the probability of finding the system in the microscopic state i, k is Boltzmann's constant and W is the total number of microstates.

This formula has been shown to be restricted to the domain of validity of BG statistics, which seem to describe nature when the effective microscopic interactions and the microscopic memory are short ranged [9]. The process of shock fragmentation, specially when energies are high enough, leads to the existence of long range correlations between all parts of the object being fragmented. Then the use of the above formula to describe fragmentation processes seems to be inadequate. From here it can be concluded that the use of statistics able to describe long-range and long-memory interactions can be useful in the description of fracture processes.

In this work we report the deduction of the FSDF and its dimensional crossover from the maximum entropy principle using as a starting point Tsallis entropy [9]:

$$S_q = k \frac{1 - \int_0^\infty p^q(x) dx}{q - 1}. \quad (1)$$

The integral runs over all admissible values of the magnitude x and $p(x)dx$ is the probability of the system being in a state between x and $x + dx.k$ is the Boltzmann constant and q a real number (the entropic index).The Tsallis entropy S_q reduces to the BG entropy for $q = 1$ so recovering Boltzmann-Gibbs statistics.

Dealing with the case of fracture x could well describe the mass (volume) of the fragments. The statistics based on this entropy have been used to describe a number of non-equilibrium processes and phenomena for which BG statistics is not appropriate. (see [9]), although it seems to be still far from having shown all its capabilities.

Let us extremize $\frac{S_q}{k}$ with appropriate constraints. If we denote the volume of a fragment by V and some typical volume characteristic of the distribution by V_m, we can define a dimensionless volume $v = \frac{V}{V_m}$. The normalization condition reads

$$\int_0^\infty p(v)dv = 1. \tag{2}$$

The other condition to be imposed is mass conservation. But as the system is finite, this condition will lead to a very sharp decay in the asymptotic behavior of the fragment size distribution function (FSDF) for large sizes of the fragments. Consequently, we will impose a more general condition, like the "q-conservation" of the mass, in the form:

$$\int_0^\infty vp^q(v)dv = 1, \tag{3}$$

which reduces to the "classical" mass conservation when $q = 1$.

Using the method of Lagrange multipliers we construct the function:

$$L(p;\alpha_1;\alpha_2) = S_q - \alpha_1 \int_0^\infty p(v)dv + \alpha_2 \int_0^\infty p^q(v)vdv \tag{4}$$

The Lagrange multipliers α_1 and α_2 are determined by the constraints. The extremization of $L(p; \alpha_1; \alpha_2)$ leads to:

$$p(v)dv = C(1 + (q-1)\alpha_2 v)^{-\frac{1}{q-1}} dv \qquad (5)$$

where the constant C is given by

$$C = [\frac{q-1}{q}\alpha_1]^{\frac{1}{q-1}}.$$

This is a FSDF expressed as a function of the volume of the fragments. It is valid for $1 < q < 2$. The FSDF given by Eq. 5 can describe satisfactorily the behavior for "small" and "large" fragment sizes. Indeed, we can apply this equation to the fragmentation of plates described in [1].

As the object to be broken has mainly a two-dimensional shape, large fragments show a mass scaling with the surface of the basis of the plate, so that if the width of the plate in units of a characteristic length of the system is Δ then the mass (volume) scales with the linear dimensions as $v \sim \Delta l^2$. This must be taken into account when calculating the element of volume dv. Then, for large fragments we have

$$p_+(l)dl = Cl[1 + (q-1)\alpha_2 l^2 \Delta)]^{-\frac{1}{q-1}} dl. \qquad (6)$$

The small fragments do not resemble plates but volumetric objects. This means that the volume of the fragment scales as l^3 and we have for the distribution of small fragments:

$$p_-(l)dl = Cl^2[1 + (q-1)\alpha_2 l^3]^{-\frac{1}{q-1}} dl, \qquad (7)$$

From 6 and 7 we may deduce the asymptotic behavior to obtain the slope β for small and large fragments, with the notation that the distribution function has the asymptotic behavior $p(l) \sim l^{-\beta}$ as in [1]. Designating the slope for large (small) fragments by $\beta_+(\beta_-)$ respectively, we obtain:

$$\frac{\beta_+}{\beta_-} = \frac{3-q}{5-2q}, \tag{8}$$

from where the values of $\frac{\beta_+}{\beta_-}$ can be calculated restricting q to its range of validity $1 < q < 2$. This ratio lies in the interval $\frac{2}{3} < \frac{\beta_+}{\beta_-} < 1$.

These values should be regarded as a coarse grained estimate, since to obtain them we have postulated a very definite scaling of the mass with the dimensionality, although it is clear that this dependence should be no more than approximate, since the objects are not exactly "one" or "two-dimensional". Yet it will be seen that this simple model is good enough to describe the behavior of the FSDF.

From [1] we may obtain that all the values of the ratio $\frac{\beta_+}{\beta_-}$ in the reported experiment satisfy this condition, going from .67 to .79. This is illustrated in table 1

plate	β_-	β_+	$\frac{\beta_+}{\beta_-}$
2	1.62	1.19	0.73
3	1.5	1.17	0.78
4	1.67	1.12	0.67
5	1.5	1.9	0.79

Following the same reasoning we could predict the value of $\frac{\beta_+}{\beta_-}$ for the experiments reported in [2]. In this case, as reported there, it is the cumulative number which has slope β. In that case, respecting the notation in [2], the behavior of the distribution function $p(l)$ should be $p(l) \sim l^{-\beta-1}$. We predict that for the breaking of rods, where the crossover is from one-dimensional to three-dimensional objects the value of the ratio should be around 3 irrespective of q. New experimental results in this case would be very welcome to investigate our predictions.

From the present formulation we can evaluate the order of magnitude of the crossover length with the assumption that the crossover occurs in the transition region of scaling of the mass with the dimension, i.e., the first point where $p_+(l)$ and $p_-(l)$ become equal.

Then the crossover dimension is $l \sim \Delta$ for plates and $l \sim S^{\frac{1}{2}}$ for rods, being S the area of the basis of the rod. This criterion, which is very acceptable physically, is directly obtained in this formulation.

So, the usefulness of Tsallis entropy to describe processes of fragmentation has been tested in this work, where the FSDF for small and large fragments have been obtained for fragmentation involving a change in the geometry (dimensionality) of the fragments. We showed analytically that the crossover detected in the experiments can be obtained when the explicit scaling of the mass of the fragments with the dimensionality is considered. In this respect we have established that, as was pointed in [1], the slope is determined by local rather than by global features of the original object. Once again, the geometry is shown to be an important factor in this phenomenon.

We point out again that non-extensive statistics seem to have very much applications in non-equilibrium phenomena, a number of which are yet to be investigated.

This work was partially supported by the “Alma Mater” contest, Havana University. One of us (O.S) is grateful to the Department of Mathematical Sciences of Brunel University for kind hospitality and the Royal Society, London for financial support.

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1 Table caption

Table 1: Comparison of the results from [1]with predictions from 8. Observe that all values of $\frac{\beta_+}{\beta_-}$ lie in the predicted range $\frac{2}{3} < \frac{\beta_+}{\beta_-} < 1$.