Growing Random Networks with Fitness

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Three models of growing random networks with fitness dependent growth rates are analysed using the rate equations for the distribution of their connectivities. In the first model (A), a network is built by connecting incoming nodes to nodes of connectivity k and random additive fitness η, with rate $(k-1)+\eta$. For $\eta > 0$ we find the connectivity distribution is power law with exponent $\gamma = <\eta> +2$. In the second model (B), the network is built by connecting nodes to nodes of connectivity k, random additive fitness η and random multiplicative fitness ζ with rate $\zeta(k-1)+\eta$. This model also has a power law connectivity distribution, but with an exponent which depends on the multiplicative fitness at each node. In the third model (C), a directed graph is considered and is built by the addition of nodes and the creation of links. A node with fitness (α, β), i incoming links and j outgoing links gains a new incoming link with rate $\alpha(i+1)$, and a new outgoing link with rate $\beta(j+1)$. The distributions of the number of incoming and outgoing links both scale as power laws, with inverse logarithmic corrections.

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I. INTRODUCTION

Recently, there has been a considerable interest in the growth properties of human interaction networks such as the world wide web [1,2], the citation distribution of publications [3], the electrical distribution systems [4] and the social networks [5]. These networks all have very different physical forms, with different definitions for their nodes and links. However they appear to display considerable topological similarity, having connectivity distributions which behave as power laws. These distributions cannot be explained by traditional random graph theory, which is based on randomly connecting together a fixed number of nodes, and results in Poisson distributions for the connectivity [6,7].

Models of growing random graphs were first introduced by Barabási and Albert [4], who identified two important features that these graphs must possess in order to display power law distributed connectivities. These features are (i) networks grow by addition of new nodes and (ii) new nodes preferentially attach to highly connected nodes. Consideration of only these elements in [4] led to the conclusion that large networks can self-organize into a scale free state. Since then, many other models [8–17] have emerged to study various properties of these graphs such as aging [8,13], connectivity [11], inheritance [12], permanent deletion of links and nodes [16] and their effect on a growing network topology. The main conclusion of all these models is that incorporation of additional features changes the scaling behaviour of growing random networks. However it is still not understood why most of the empirical work observes power law exponents between 2 and 3, and the analytical work recovers exponents that range between 2 and ∞ [8,14]. Furthermore, some of the more detailed features of the networks have not yet been captured [2,19].

In this paper, based on an idea introduced by [10] we study the influence of quenched disorder which we call *fitness*, on the growth rates of networks. Similar ideas have been studied in other models, either through the initial attractiveness of a node [9] or the fitness of a site to compete for links [10]. However, our approach is somewhat different to these models. We use a rate equation approach [11] to generalize and solve three network models with different growth rates.

In Sec. II, we investigate the effect of additive randomness, while the effect of multiplicative randomness is analyzed in Sec. III. In Sec. IV, we assume the network is a directed graph [7,18–20] and both incoming and outgoing links are considered, to model the growth of the world wide web. We summarize our results and draw conclusions in the last section.

II. MODEL A

In this model, we consider a network where a fitness η, chosen from a probability distribution $f_A(\eta)$, is assigned to each node. The network is built by connecting incoming nodes to nodes of connectivity k and fitness η with rate $(k-1)+\eta$, that is to say, there is a linear preferential attachment to nodes with already high number of links and a high fitness η. This simply means that not all nodes which k existing links are equivalent because k does not enclose the full information about the popularity of a node. For instance, if a node is a web site, η could be a measure of the number of related TV commercials, or tube advertisements. Using the rate equation approach we describe the time evolution of the average number of nodes of connectivity k and fitness η, $N_k(\eta)$, as

$$\frac{\partial N_k(\eta)}{\partial t} = \frac{1}{M}\left[(k+\eta-2)N_{k-1}(\eta) - (k+\eta-1)N_k(\eta)\right] + \delta_{k1} f_A(\eta). \tag{1}$$

The first term on the right hand side of Eq. (1) represents the increase in the number of sites with k links when a site with $k-1$ links gains a link. The second term expresses the loss of sites with k links when they gain a new link. The last term accounts for the continuous addition of nodes of connectivity 1 and fitness η with probability $f_A(\eta)$. The multiplicative factor M is defined by

$$M(t) = \sum_{k,\eta} (k + \eta - 1) N_k(\eta), \tag{2}$$

which ensures that the equation is properly normalized. Before going any further, let us make some remarks. First, to obtain a growing network, we need $k-1+\eta > 0$ for all k, so that $\eta > 0$, because from the definition of the model, each site is created with one link. Second, all sites associated with $\eta = 1$ have the simple linear preferential attachment of earlier models [4,11,14]. Finally, $f_A(\eta)$ can either be discrete or continuous.

We analyse the model from the rate equation starting with the moments of $N_k(\eta)$ defined by

$$M_{ij}(t) \equiv \sum_{k,\eta} k^i \eta^j N_k(\eta). \tag{3}$$

We can easily show that

$$\frac{\partial M_{00}}{\partial t} = 1, \quad \frac{\partial M_{10}}{\partial t} = 2 \quad \text{and} \quad \frac{\partial M_{01}}{\partial t} = < \eta > \tag{4}$$

where $< \eta >$ is the average value of the fitness. For large times, the initial values of the moments become irrelevant, so that we get

$$M(t) = M_{10}(t) + M_{01}(t) - M_{00}(t) = [< \eta > +1]t. \tag{5}$$

Similarly, it can be shown that $N_k(\eta, t)$ and all its moments grow linearly with time. Therefore, we can write $N_k(\eta, t) = t n_k(\eta)$ and $M(t) = mt$. The latter relation implies $m = < \eta > +1$, while we insert the former in Eq.(1) to obtain the recurrence relation

$$(k + \eta + m - 1) n_k(\eta) = (k + \eta - 2) n_{k-1}(\eta) + m \delta_{k1} f_A(\eta). \tag{6}$$

Solving Eq. (6), we obtain

$$n_k(\eta) = \frac{\Gamma(k + \eta - 1)}{\Gamma(k + \eta + m)} \frac{\Gamma(\eta + m)}{\Gamma(\eta)} m f_A(\eta). \tag{7}$$

In particular, the rate of change of the total number of links connected to the sites with fitness η is equal to

$$\sum_{k=1}^{\infty} k n_k(\eta) = \left(1 + \frac{\eta}{< \eta >}\right) f_A(\eta). \tag{8}$$

For large k, Eq. (7) is equivalent to

$$n_k(\eta) \sim k^{-(m+1)} \sim k^{-(<\eta>+2)}. \tag{9}$$

The distribution scales as a power law $n_k(\eta) \sim k^{-\gamma}$ with an exponent $\gamma = < \eta > +2$, which depends only on the average fitness $< \eta >$, and consequently is the same for every node. Hence, the introduction of an additive random fitness at each node, modifying the preferential attachment process, generates a power law connectivity distribution. The exponent of this power law is shifted by $< \eta > -1$ with respect to its value when the preferential attachment is simply linear. Of course, for $f_A(\eta) = \delta(\eta - 1)$,

$$n_k \sim k^{-3}, \tag{10}$$

which is, as expected, the result obtained in previous models without fitness [4,11,14].

III. MODEL B

In the previous section, we introduced a model where linear preferential attachment is decorated by a random additive process to construct an independent source of preferential attachment. However, even if it seems reasonable to assume that the attachment is proportional to the number of already existing links, there is no specific reason to assume that the coefficient of proportionally is the same for every node. In this section, we consider a network where each node is associated to a triplet (k, η, ζ). The network is built by adding a new node at each time step and connecting it to a node with random additive fitness η, random multiplicative fitness ζ and connectivity k with rate $\zeta(k-1) + \eta$. Where η and ζ are quenched variables, initially chosen from a probability distribution $f_B(\eta, \zeta)$. This model is a generalisation of that introduced in [10], and Model A is recovered when $f_B(\eta, \zeta) = f_A(\eta)\delta(\zeta - 1)$. The multiplicative fitness symbolizes the fact that, even if the growth rate is proportional to already existing links, there can exist different categories of nodes which attract new links at different rates.

The rate equation for this model, which describes the time evolution of the average number of nodes with triplet (k, η, ζ), $N_k(\eta, \zeta)$, is given by

$$\begin{aligned}\frac{\partial N_k(\eta, \zeta)}{\partial t} = \frac{1}{M} &\left([\zeta(k-2) + \eta] N_{k-1}(\eta, \zeta)\right. \\ &\left. -[\zeta(k-1) + \eta] N_k(\eta, \zeta)\right) + \delta_{k1} f_B(\eta, \zeta).\end{aligned} \tag{11}$$

The terms on the right-hand side of this equation are analogous to those in Eq. (1), with the new preferential rates of growth. The normalization factor here is

$$M(t) = \sum_{k,\eta,\zeta} [\zeta(k-1) + \eta] N_k(\eta, \zeta). \tag{12}$$

To solve Eq. (11), we employ the same technique as in the previous section, defining the moments of $N_k(\eta, \zeta)$ by

$$M_{ijl} \equiv \sum_{k,\eta,\zeta} k^i \eta^j \zeta^l N_k(\eta, \zeta). \tag{13}$$

Looking at the lowest moments of $N_k(\eta,\zeta)$, we find

$$\frac{\partial M(t)}{\partial t} = \frac{1}{M}\sum_{k,\eta,\zeta} \zeta[\zeta(k-1)+\eta]N_k(\eta,\zeta) + <\eta>, \quad (14)$$

where $<\eta>$ is the average additive fitness. Again, it is easy to prove that $N_k(\eta,\zeta,t)$ and all its moments are linear functions of time. Hence, we define m and $n_k(\eta,\zeta)$ through $M(t) \equiv mt$ and $N_k(\eta,\zeta,t) \equiv tn_k(\eta,\zeta)$, respectively. We refer to m as the reduced moment from now on.

Eq. (14) implies that the reduced moment is a solution of

$$m = \frac{1}{m}\sum_{k,\eta,\zeta} \zeta[\zeta(k-1)+\eta]n_k(\eta,\zeta) + <\eta>. \quad (15)$$

From Eq. (11), we obtain

$$[\zeta(k-1)+\eta+m]n_k(\zeta,\eta) = [\zeta(k-2)+\eta]n_{k-1}(\zeta,\eta) + m\delta_{k1}f_B(\zeta,\eta). \quad (16)$$

The previous relation yields

$$n_k(\eta,\zeta) = \frac{\Gamma\left(k+\frac{\eta}{\zeta}-1\right)}{\Gamma\left(k+\frac{\eta+m}{\zeta}\right)}\frac{\Gamma\left(\frac{\eta+m}{\zeta}\right)}{\Gamma\left(\frac{\eta}{\zeta}\right)}\frac{m}{\zeta}f_B(\eta,\zeta). \quad (17)$$

When $k \to \infty$,

$$n_k(\eta,\zeta) \sim k^{-\left(1+\frac{m}{\zeta}\right)}. \quad (18)$$

The growth rate of the number of sites associated with a triplet (k,η,ζ), scales asymptotically as a power law $n_k \sim k^{-\gamma}$, with an exponent $\gamma = 1 + m/\zeta$, which depends on the fitness at a particular site. It means that, unlike the additive fitness, the multiplicative fitness generates multiscaling, with a different power law for each fitness.

To complete the solution of the model, we need to obtain an expression for the reduced moment, m. For this purpose, we introduce a generating function defined as

$$g(x,\eta,\zeta) \equiv \sum_{k=1}^{\infty} x^k n_k(\eta,\zeta). \quad (19)$$

Eq. (16) gives

$$g(1,\eta,\zeta) = f_B(\eta,\zeta) = \sum_{k=1}^{\infty} n_k(\eta,\zeta) \quad (20)$$

and

$$g'(1,\eta,\zeta) = \frac{\eta-\zeta+m}{m-\zeta}f_B(\eta,\zeta) = \sum_{k=1}^{\infty} kn_k(\eta,\zeta). \quad (21)$$

Substituting in Eq. (15) leads to an implicit equation for m,

$$\int f_B(\eta,\zeta)\frac{\eta}{m-\zeta}\,d\eta\,d\zeta = 1 \quad (22)$$

which cannot be solved explicitly. We can define n_k, the connectivity distribution of the entire network, as

$$n_k \equiv \int f_C(\eta,\zeta)n_k(\eta,\zeta)d\eta d\zeta. \quad (23)$$

As an example, we consider $f_B(\eta,\zeta) = 1$, $0 \le \eta \le 1$ and $0 \le \zeta \le 1$. Solving Eq. (22) gives $m = 1/(1-e^{-2}) = 1.156$ and, integrating Eq. (17) over η and ζ within the chosen limits,

$$n_k = \int_0^1\int_0^1 k^{-\left(\frac{m}{\zeta}+1\right)}\frac{\Gamma\left(\frac{\eta+m}{\zeta}\right)}{\Gamma\left(\frac{\eta}{\zeta}\right)}\frac{m}{\zeta}\,d\eta d\zeta. \quad (24)$$

We find that the connectivity distribution in the asymptotic limit $k \to \infty$ is

$$n_k \sim \frac{1}{\ln k}k^{-(1+m)}. \quad (25)$$

This is simply a power law form multiplied with an inverse logarithmic correction and substitution of m yields

$$n_k \sim \frac{1}{\ln k}k^{-2.156}. \quad (26)$$

By using $f_B(\eta,\zeta) = \delta(\zeta-\eta)$ with $0 \le \zeta \le 1$, the solution obtained by [10] is recovered, which has the same functional form as Eq. (25) with a power law exponent $\gamma = 2.255$.

We can solve for the large k behaviour of the connectivity distribution for a number of different forms of the fitness. For instance, if

$$f_B(\eta,\zeta) = a\zeta^{a-1}\delta(\zeta-\eta) \quad (27)$$

and $0 \le \zeta \le 1$, $a > 0$, then the connectivity distribution behaves as

$$n_k \sim \frac{1}{\ln k}k^{-(m+1)} \quad (28)$$

as $k \to \infty$ and $m(a)$ satisfies

$$\int_0^1 \frac{a\zeta^a}{m-\zeta}d\zeta = 1. \quad (29)$$

Using this equation it is simple to show that as $a \to 0$, $m \to 1$ and as $a \to \infty$, $m \to 2$. In Fig. (1) this equation is solved numerically. We find that $1 \le m \le 2$, implying that the power law exponent is in the range of (2,3) which is in very good agreement with the experimental results [4,3,2].

Another example to consider is when

$$f_B(\eta,\zeta) = 6\zeta(1-\zeta)\delta(\zeta-\eta) \quad (30)$$

with $0 \le \zeta \le 1$. In this case the connectivity distribution takes the form

$$n_k \sim \frac{1}{(\ln k)^2}k^{-(m+1)}. \quad (31)$$

as $k \to \infty$ where $m = 1.550$.

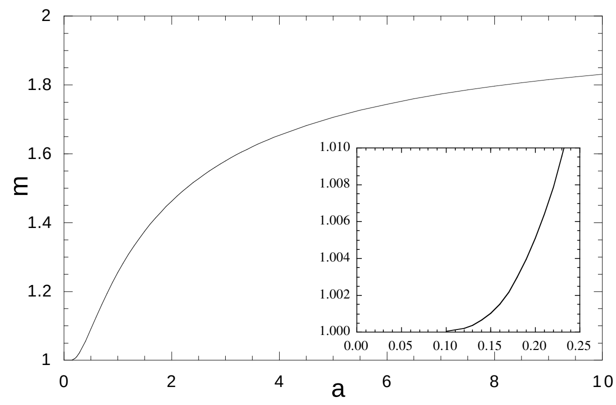

FIG. 1. m against a for Model B when the multiplicative fitness is distributed as a power law with exponent $a-1$. The inset is the region close to the origin.

IV. MODEL C

In the previous two models, the links were undirected and the number of links and nodes were equal, which is not a good model for some growing networks such as the www. In this section, a directed network is built by node and link addition. At each time step, with probability p, a new node is added and with probability $q = 1 - p$, a new directed link is created between two nodes. A node with fitness (α, β), i incoming links and j outgoing links, will gain a new incoming link with rate $\alpha(i+1)$ and a new outgoing link with rate $\beta(j+1)$. Then, the connectivity distribution $N_{ij}(\alpha, \beta)$, the average number of nodes with i incoming and j outgoing links, evolves as

$$\begin{aligned}\frac{\partial N_{ij}}{\partial t}(\alpha,\beta) = {} & \frac{q\alpha}{M_1}\left[iN_{i-1j}(\alpha,\beta) - (i+1)N_{ij}(\alpha,\beta)\right] \\ & + \frac{q\beta}{M_2}\left[jN_{ij-1}(\alpha,\beta) - (j+1)N_{ij}(\alpha,\beta)\right] \\ & + p\,\delta_{i0}\delta_{j0}\,f_C(\alpha,\beta). \end{aligned} \tag{32}$$

The first term in the first square brackets represents the increase of N_{ij} nodes when nodes with $i-1$ incoming and j outgoing links, gain an incoming link and the second term represents the corresponding loss. The second square brackets contain the analogous terms for outgoing links and the last term ensures the continuous addition of new nodes with fitness α, β with probability $f_C(\alpha,\beta)$. M_1 and M_2 are the normalization factors, given by

$$M_1 = \sum_{ij\alpha\beta}(i+1)\alpha N_{ij}(\alpha,\beta) \quad \text{and} \tag{33}$$

$$M_2 = \sum_{ij\alpha\beta}(j+1)\beta N_{ij}(\alpha,\beta). \tag{34}$$

From the definition of the model, one has

$$\sum_{ij\alpha\beta} N_{ij}(\alpha,\beta) = pt, \tag{35}$$

which simply states that nodes are added with probability p. We also have

$$\sum_{ij\alpha\beta} \alpha N_{ij}(\alpha,\beta) = p < \alpha > t \tag{36}$$

$$\sum_{ij\alpha\beta} \beta N_{ij}(\alpha,\beta) = p < \beta > t. \tag{37}$$

We define $n_{ij}(\alpha,\beta)$ through $N_{ij}(\alpha,\beta,t) \equiv tn_{ij}$, where from now on we drop the explicit (α,β) dependence to ease the notation. Also, we can define the reduced moments m_1 and m_2 by $M_1(t) \equiv tm_1$ and $M_2(t) \equiv tm_2$. Hence we have

$$m_1 = \frac{q}{m_1}\sum_{ij\alpha\beta}\alpha^2(i+1)n_{ij} + p < \alpha > \tag{38}$$

and similarly

$$m_2 = \frac{q}{m_2}\sum_{ij\alpha\beta}\beta^2(j+1)n_{ij} + p < \beta > . \tag{39}$$

From Eq. (32), we obtain

$$\begin{aligned} & n_{ij}[m_1 m_2 + m_1 q\beta(j+1) + m_2 q\alpha(i+1)] \\ & = m_2\, q\, \alpha\, i n_{i-1j} + m_1\, q\, \beta\, j n_{ij-1} \\ & + m_1\, m_2\, p\, \delta_{i0}\delta_{j0}\, f_C(\alpha,\beta). \end{aligned} \tag{40}$$

Now, we consider the incoming link distribution

$$g_i = \sum_{j=0}^{\infty} n_{ij} \tag{41}$$

and the outgoing link distribution

$$h_j = \sum_{i=0}^{\infty} n_{ij}. \tag{42}$$

From Eq. (40), the recurrence relations

$$\left(i + 1 + \frac{m_1}{q\alpha}\right) g_i = i g_{i-1} + \frac{m_1 p}{q\alpha}\delta_{i0} f_C(\alpha,\beta) \tag{43}$$

and

$$\left(j + 1 + \frac{m_2}{q\beta}\right) h_j = j h_{j-1} + \frac{m_2 p}{q\beta}\delta_{j0} f_C(\alpha,\beta) \tag{44}$$

are obtained. Solving these gives the incoming and the outgoing links distributions

$$g_i = \frac{\Gamma(i+1)\Gamma(\frac{m_1}{q\alpha}+1)}{\Gamma(\frac{m_1}{q\alpha}+i+2)}\frac{m_1 p}{q\alpha} f_C(\alpha,\beta) \tag{45}$$

and

$$h_j = \frac{\Gamma(j+1)\Gamma(\frac{m_2}{q\beta}+1)}{\Gamma(\frac{m_2}{q\beta}+j+2)}\frac{m_2 p}{q\beta} f_C(\alpha,\beta). \tag{46}$$

In the asymptotic limit both distributions are power laws; as $i \to \infty$, $g_i \sim i^{-\gamma_{in}}$ with $\gamma_{in} = (1 + m_1/q\alpha)$ and for $j \to \infty$, $h_j \sim j^{-\gamma_{out}}$ with $\gamma_{out} = (1 + m_2/q\beta)$.

The appearance of both multiplicative fitnesses α and β in the exponents of the power laws, reflects the fact that growing networks such as the www are evolving on the basis of competition. The fitnesses here, can be thought of as a measure of attractiveness of the content of a web page. This means that within a particular commercial sector on the web, such as search engines, e-mail account providers, the software design, films, music and specific information, the fittest competitors have managed to gather millions of registered users in a very short span of time.

To express these exponents numerically we will find implicit equations for m_1 and m_2. Therefore, we use a generating function defined as

$$g(x, y, \alpha, \beta) = \sum_{i,j=0}^{\infty} x^i y^j n_{ij} \tag{47}$$

to find equations for m_1 and m_2. We have

$$g(1, 1) = p f_C(\alpha, \beta) \tag{48}$$

and

$$\left.\frac{\partial g}{\partial x}\right|_{x=y=1} = \frac{pq\alpha}{m_1 - q\alpha} f_C(\alpha, \beta). \tag{49}$$

Hence

$$\sum_{i,j=0}^{\infty} i n_{ij} = \frac{pq\alpha}{m_1 - q\alpha} f_C(\alpha, \beta) \tag{50}$$

and by an identical method

$$\sum_{i,j=0}^{\infty} j n_{ij} = \frac{pq\beta}{m_2 - q\beta} f_C(\alpha, \beta). \tag{51}$$

Substitution of the above relations into Eq. (38) and Eq. (39) gives implicit equations for m_1 and m_2,

$$p \sum_{\alpha,\beta} \frac{\alpha f_C(\alpha, \beta)}{m_1 - q\alpha} = 1 \tag{52}$$

and

$$p \sum_{\alpha,\beta} \frac{\beta f_C(\alpha, \beta)}{m_2 - q\beta} = 1, \tag{53}$$

respectively. The summations run over all possible values of α and β and can be replaced by integrations for continuous distributions.

First, we consider a general case; if

$$f_C(\alpha, \beta) = f_C(\beta, \alpha) \tag{54}$$

then the distribution of incoming and outgoing links is the same, $g_i = h_i$.

As with Model B, we will consider two particular nontrivial distributions of the fitness. For power law fitnesses

$$f_C(\alpha, \beta) = ab\alpha^{a-1}\beta^{b-1} \tag{55}$$

we find that the distribution of incoming links is given by

$$g_i \sim \frac{1}{\ln i} i^{-(1+\frac{m_1}{q})} \tag{56}$$

and analogously for outgoing links

$$h_j \sim \frac{1}{\ln j} j^{-(1+\frac{m_2}{q})} \tag{57}$$

for large i and j. The parameter m_1 is a function of both a and p and m_2 is a function of b and p. It is a simple matter to show that $m_1(a) = g(a)$ and $m_2(b) = g(b)$ where $g(c)$ satisfies

$$p \int_0^1 \frac{c\, x^c}{g(c) - qx} dx = 1. \tag{58}$$

Consequently, as $a \to 0$, $m_1 \to q$ and as $a \to \infty$, $m_1 \to 1$. Thus by picking a appropriately, the power law in the distribution of incoming links Eq. (56) can have an exponent with any value between 2 and $1+1/q$. A similar situation occurs with $m_2(b)$.

The fitnesses for incoming and outgoing links can be more strongly coupled together. An example of this is

$$f_C(\alpha, \beta) = \begin{cases} 2 & \alpha > \beta, \\ 0 & \beta > \alpha \end{cases} \tag{59}$$

where we find that for large i and j, g_i has the same form as Eq. (56) and

$$h_j \sim \frac{1}{(\ln j)^2} j^{-(1+\frac{m_2}{q})}. \tag{60}$$

The stronger coupling between the fitnesses is reflected in the different functional forms of the probability distributions for incoming and outgoing links.

V. DISCUSSION AND CONCLUSIONS

We have studied three growing network models with the consideration of two key elements in mind; (i) networks are continuously growing, (ii) the attachment process is preferential. In the first model in Sec. II, we found that the introduction of random additive fitness (quenched disorder) η at each node modified the preferential attachment process and the generated network had a power law connectivity distribution with an exponent $\gamma = <\eta> + 1$, where only the average value of the fitness is of importance. However, introduction of both random additive fitness η and random multiplicative fitness ζ in Sec. III, led to a scale free network where the exponent $\gamma = 1+m/\zeta$ depends on the fitness ζ at each node. When the fitnesses were distributed with a power law distribution between 0 and 1, the connectivity distribution of the whole system was power law with a logarithmic correction, with the value of the exponent in the power law between 2 and 3.

In Sec. IV we studied a directed graph which was allowed to form loops in an attempt to model a different class of growing random graphs. The incoming and the outgoing link distributions exhibit power law forms with exponents corresponding to $\gamma_{in} = (1 + m_1/q\alpha)$ and $\gamma_{out} = (1 + m_2/q\beta)$ depending upon the values of the fitness at each site. Choosing a particular fitness distribution and calculating the connectivity distribution for the whole system often results in power laws mediated by logarithmic corrections. We gave two examples of such behaviour.

There are a great many examples of random growing graphs in science, social science, technology and biology. Only a fraction of these systems have been characterised experimentally. Whilst the systems studied in this paper are not as theoretically appealing as those with pure power law forms, it seems likely that some of the real random growing networks will be described by models of this particular type.

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[1] R. Albert, H. Jeong and A.-L. Barabási, Nature **401**, 130 (1999).
[2] A. Broder, R. Kumar, F. Maghul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins and J. Wiener, Computer Networks **33**, 309 (2000).
[3] S. Redner, Eur. Phys. J. B **4**, 131 (1998).
[4] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[5] M. E. J. Newman, *cond-mat/0011144*.
[6] B. Bollobás, *Random Graphs* (Academic Press, London, 1985).
[7] M. E. J. Newman, S. H. Strogatz and D. J. Watts, *cond-mat/0007235*.
[8] S.N. Dorogovtsev and J.F.F. Mendes, Phys. Rev. E **62**, 1842 (2000).
[9] S.N. Dorogovtsev, J.F.F. Mendes and A. N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[10] G. Bianconi and A.-L. Barabási, *cond-mat/0011029*.
[11] P. L. Krapivsky, S. Redner and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[12] S.N. Dorogovtsev, J.F.F. Mendes and A. N. Samukhin, *cond-mat/0011077*.
[13] L. A. N. Amaral, A. Scala, M. Barthélémy and H. E. Stanley, Proc. Nat. Acad. Sci. USA **97**, 11149 (2000).
[14] P. L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).
[15] S.N. Dorogovtsev, J.F.F. Mendes and A. N. Samukhin, *cond-mat/0011115*.
[16] S.N. Dorogovtsev and J.F.F. Mendes, Phys. Rev. E **63**, 056125 (2001).
[17] G. Bianconi and A.-L. Barabási, *cond-mat/0011224*.
[18] B. Tadić, Physica A **293**, 273 (2001).
[19] P. L. Krapivsky, G. J. Rodgers and S. Redner, Phys. Rev. Lett. **86**, 5401 (2001).
[20] G. J. Rodgers and K. Darby-Dowman, submitted to Eur. Phys. J. B.