Combination technique based second moment analysis for elliptic PDEs on random domains

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Abstract In this article, we propose the sparse grid combination technique for the second moment analysis of elliptic partial differential equations on random domains. By employing shape sensitivity analysis, we linearize the influence of the random domain perturbation on the solution. We derive deterministic partial differential equations to approximate the random solution's mean and its covariance with leading order in the amplitude of the random domain perturbation. The partial differential equation for the covariance is a tensor product Dirichlet problem which can efficiently be determined by Galerkin's method in the sparse tensor product space. We show that this Galerkin approximation coincides with the solution derived from the combination technique provided that the detail spaces in the related multiscale hierarchy are constructed with respect to Galerkin projections. This means that the combination technique does not impose an additional error in our construction. Numerical experiments quantify and qualify the proposed method.

1 Introduction

Various problems in science and engineering can be formulated as boundary value problems for an unknown function. In general, the numerical simulation is well understood provided that the input parameters are known exactly. In many applications, however, the input parameters are not known exactly. Especially, the treatment of uncertainties in the computational domain has become of growing interest, see e.g. [5, 23, 38, 41]. In this article, we consider the elliptic diffusion equation

$$-\operatorname{div}\big(\alpha\nabla u(\omega)\big) = f \text{ in } D(\omega), \quad u(\omega) = 0 \text{ on } \partial D(\omega), \tag{1}$$

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as a model problem where the underlying domain $D(\omega) \subset \mathbb{R}^d$ or respectively its boundary $\partial D(\omega)$ are random. For example, one might think of tolerances in the shape of products fabricated by line production, or shapes which stem from inverse problems, like for example tomography. Of course, besides a scalar diffusion coefficient $\alpha(\mathbf{x})$, one can also consider a diffusion matrix $\mathbf{A}(\mathbf{x})$, cf. [18]. Even so, the emphasis of our considerations will be laid on the case $\alpha(\mathbf{x}) \equiv 1$, that is the Poisson equation.

Besides the fictitious domain approach considered in [5], one might essentially distinguish two approaches: the *domain mapping method*, cf. [6, 21, 27, 38, 41], and the *perturbation method*. They result from a description of the random domain either in Lagrangian coordinates or in Eulerian coordinates, see e.g. [37]. The latter approach will be dealt with in this article.

The perturbation method starts with a prescribed perturbation field

$$\mathbf{V}(\omega)\colon \overline{D_{\text{ref}}} \to \mathbb{R}^d$$

for a fixed reference domain D_{ref} and uses a *shape Taylor expansion* with respect to this perturbation field to represent the solution to the model problem, see e.g. [19, 23]. In fact, as we will see later on, it is sufficient to know the perturbation field in a vicinity of ∂D_{ref}, i.e.

$$\mathbf{V}(\omega)\colon \partial D_{\text{ref}} \to \mathbb{R}^d.$$

This is a remarkable feature since it might in practice be much easier to obtain measurements from the outside of a work-piece to estimate the perturbation field $\mathbf{V}(\omega)$ rather than from its interior.

The starting point for our considerations will be the knowledge of an appropriate description of the the random field $\mathbf{V}(\omega)$. To that end, we assume that the random vector field is described in terms of its mean

$$\mathbb{E}[\mathbf{V}]\colon D_{\text{ref}} \to \mathbb{R}^d, \quad \mathbb{E}[\mathbf{V}](\mathbf{x}) = \big[\mathbb{E}[V_1](\mathbf{x}), \ldots, \mathbb{E}[V_d](\mathbf{x})\big]^\intercal$$

and its (matrix valued) covariance function

$$\operatorname{Cov}[\mathbf{V}]\colon D_{\text{ref}} \times D_{\text{ref}} \to \mathbb{R}^{d\times d}, \quad \operatorname{Cov}[\mathbf{V}](\mathbf{x},\mathbf{y}) = \begin{bmatrix} \operatorname{Cov}_{1,1}(\mathbf{x},\mathbf{y}) & \cdots & \operatorname{Cov}_{1,d}(\mathbf{x},\mathbf{y}) \\ \vdots & & \vdots \\ \operatorname{Cov}_{d,1}(\mathbf{x},\mathbf{y}) & \cdots & \operatorname{Cov}_{d,d}(\mathbf{x},\mathbf{y}) \end{bmatrix}.$$

For the considered perturbation method, this representation of the random vector field is already sufficient. Having the mean and the covariance of the random vector field at hand, we aim at approximating the corresponding statistics of the unknown random solution.

Making use of sensitivity analysis, we linearize the solution's nonlinear dependence on the random vector field $\mathbf{V}(\omega)$. Based on this, we derive deterministic equations, which compute, to leading order, the mean field and the covariance. In particular, the covariance solves a tensor product boundary value problem on the product domain $D_{\text{ref}} \times D_{\text{ref}}$. This linearization technique has already been applied to random diffusion coefficients or even to elliptic equations on random domains in [18, 22, 23]. In difference to these previous works, we do not explicitly use wavelets [23, 34, 35] or multilevel frames [18, 22] for the discretization in a sparse tensor product space. Instead, we define the complement spaces which enter the sparse tensor product construction by Galerkin projections. The Galerkin discretization leads then to a system of linear equations which decouples into subproblems with respect to full tensor product spaces of small size. These subproblems can be solved by standard multilevel finite element methods. In our particular realization, we need only the access to the stiffness matrix, the BPX preconditioner, cf. [3], and the sparse grid interpolant, cf. [4], of the covariance function of the random vector field under consideration. In this sense, our approach can be considered to be weakly intrusive. The resulting representation of the covariance is known as the *combination technique* [14]. Nevertheless, in difference to [14, 28, 32, 42], this representation is a consequence of the Galerkin method in the sparse tensor product space and is not an additional approximation step.

The rest of this article is structured as follows. In Section 2, we introduce the underlying framework. Here, we define random vector fields and the related Lebesgue-Bochner spaces. Moreover, we briefly refer to the Karhunen-Loève expansion of random vector fields. Section 3 is devoted to shape sensitivity analysis. Especially, the shape Taylor expansion is introduced here. In Section 4, we apply the shape Taylor expansion to our model problem and derive deterministic equations for the mean and the covariance. Section 5 deals with the approximation of tensor product Dirichlet problems. In Section 6, we present in detail the sparse grid combination technique for the solution of tensor product Dirichlet problems. The efficient implementation of the proposed method is non-trivial. Therefore, we think it is justified to dedicate Section 7 to this topic. Finally, in Section 8 we present our numerical results.

Throughout this article, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2 Preliminaries

The natural environment for the consideration of random vector fields are the so called *Lebesgue-Bochner spaces*. These spaces quantify the integrability of Banach space valued functions and have originally been introduced in [1]. In this section, we shall provide some facts and results on Lebesgue-Bochner spaces. For more details on this topic, we refer to [24].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ denote a complete and separable probability space with σ-algebra $\mathscr{F}$ and probability measure $\mathbb{P}$. Here, complete means that $\mathscr{F}$ contains all $\mathbb{P}$-null sets. The separability is e.g. obtained if $\mathscr{F}$ is countably generated, cf. [16, Theorem 40.B]. Furthermore, let $D_{\text{ref}} \subset \mathbb{R}^d$ denote a sufficiently smooth domain.

Definition 1. For $p \geqslant 0$, the *Lebesgue-Bochner space* $L^p_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big)$ consists of all equivalence classes of strongly $\mathbb{P}$-measurable maps $u\colon \Omega \to L^2(D_{\text{ref}};\mathbb{R}^d)$ with finite norm

$$\|u\|_{L^p_{\mathbb{P}}(\Omega;L^2(D_{\text{ref}};\mathbb{R}^d))} := \begin{cases} \left(\int_\Omega \|u(\omega,\cdot)\|^p_{L^2(D_{\text{ref}};\mathbb{R}^d)} \,\mathrm{d}\mathbb{P}\right)^{1/p}, & p < \infty \\ \operatorname*{ess\,sup}_{\omega\in\Omega} \|u(\omega,\cdot)\|_{L^2(D_{\text{ref}};\mathbb{R}^d)}, & p = \infty. \end{cases} \tag{2}$$

Two functions $u, v\colon \Omega \to L^2(D_{\text{ref}};\mathbb{R}^d)$ are identified if they coincide $\mathbb{P}$-almost everywhere, i.e. if $\mathbb{P}(\{u \neq v\}) = 0$. Moreover, the space $L^2(D_{\text{ref}};\mathbb{R}^d)$ is equipped with the inner product

$$(\mathbf{u},\mathbf{v})_{L^2(D_{\text{ref}};\mathbb{R}^d)} := \int_{D_{\text{ref}}} \langle \mathbf{u},\mathbf{v}\rangle \,\mathrm{d}\mathbf{x} \quad \text{for all } \mathbf{u},\mathbf{v} \in L^2(D_{\text{ref}};\mathbb{R}^d),$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $\mathbb{R}^d$.

In the definition, the term *strongly* $\mathbb{P}$*-measurable* refers to functions which are measurable in the classical sense and additionally essentially separable valued. The second condition is automatically met for functions $u\colon \Omega \to L^2(D_{\text{ref}};\mathbb{R}^d)$ which are measurable in the classical sense.

The spaces $L^p_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big)$ are for all $p \in [1,\infty]$ complete with respect to the norm defined in (2) and thus Banach spaces, see e.g. [24] for a proof of this statement. For $p = 1$, the space $L^1_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big)$ coincides with the space of *Bochner integrable* functions, cf. [9, Theorem 2.4]. It is moreover well known that $L^2_{\mathbb{P}}(\Omega)$ is separable if $(\Omega,\mathscr{F},\mathbb{P})$ is separable, cf. [16, Exercise 43.(1)]. Hence, for $p = 2$, the Bochner space $L^2_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big)$ is a separable Hilbert space equipped with the inner product

$$(u,v)_{L^2_{\mathbb{P}}(\Omega;L^2(D_{\text{ref}};\mathbb{R}^d))} := \int_\Omega \big(u(\omega,\cdot), v(\omega,\cdot)\big)_{L^2(D_{\text{ref}};\mathbb{R}^d)} \,\mathrm{d}\mathbb{P}.$$

In particular, it holds $L^2_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big) \cong L^2_{\mathbb{P}}(\Omega) \otimes L^2(D_{\text{ref}};\mathbb{R}^d)$.

We summarize some important facts about the Bochner integral from [24].

Theorem 1.

(a) The Bochner integral

$$\int_\Omega \cdot \,\mathrm{d}\mathbb{P}\colon \Omega \to L^2(D_{\text{ref}};\mathbb{R}^d)$$

is a linear map.

(b) For $u \in L^1_{\mathbb{P}}\big(\Omega; L^2(D_{\text{ref}};\mathbb{R}^d)\big)$ *it holds*

$$\left\| \int_A u(\omega,\cdot)\,\mathrm{d}\mathbb{P} \right\|_{L^2(D_{\text{ref}};\mathbb{R}^d)} \leqslant \int_A \|u(\omega,\cdot)\|_{L^2(D_{\text{ref}};\mathbb{R}^d)} \,\mathrm{d}\mathbb{P}$$

for all $A \in \mathscr{F}$.

(c) Let $\{u_n\}_n$ be a sequence of Bochner integrable functions with $\lim_{n\to\infty} u_n = u$ in $\mathbb{P}$-measure and g a Lebesgue integrable function on Ω such that $\|u_n\| \leqslant g$ $\mathbb{P}$-almost everywhere. Then, u is Bochner integrable and

$$\lim_{n\to\infty} \int_A u_n \, \mathrm{d}\mathbb{P} = \int_A u \, \mathrm{d}\mathbb{P}$$

for all $A \in \mathscr{F}$. Moreover, it holds

$$\lim_{n\to\infty} \int_\Omega \|u_n - u\|_{L^2(D_{\mathrm{ref}};\mathbb{R}^d)} \, \mathrm{d}\mathbb{P} = 0.$$

(d) Let $T\colon U \to \mathscr{B}$ be a closed linear operator for some Banach space $\mathscr{B}$ and $U \subseteq L^2(D_{\mathrm{ref}};\mathbb{R}^d)$. If u and Tu are Bochner integrable, then

$$T\left(\int_A u \, \mathrm{d}\mathbb{P}\right) = \int_A Tu \, \mathrm{d}\mathbb{P}$$

for all $A \in \mathscr{F}$.

Let the random vector field $\mathbf{V} \in L^2_{\mathbb{P}}\big(\Omega; L^2(D;\mathbb{R}^d)\big)$ be represented according to

$$\mathbf{V}(\omega,\mathbf{x}) = [V_1(\omega,\mathbf{x}),\ldots,V_d(\omega,\mathbf{x})]^\intercal.$$

Then, we can define the *mean* of $\mathbf{V}$ in terms of the Bochner integral

$$\mathbb{E}[\mathbf{V}](\mathbf{x}) := \int_\Omega \mathbf{V}(\omega,\mathbf{x}) \, \mathrm{d}\mathbb{P}(\Omega) \in L^2(D;\mathbb{R}^d).$$

Especially, it holds $\mathbb{E}[V_i](\mathbf{x}) = \int_\Omega V_i(\omega,\mathbf{x}) \, \mathrm{d}\mathbb{P}(\Omega)$. With respect to the *centered* random field

$$\mathbf{V}_0 = \mathbf{V} - \mathbb{E}[\mathbf{V}],$$

we introduce the (matrix valued) *covariance function* of $\mathbf{V}$ according to

$$\mathrm{Cov}[\mathbf{V}](\mathbf{x},\mathbf{y}) = [\mathrm{Cov}_{i,j}(\mathbf{x},\mathbf{y})]_{i,j=1}^d$$

with

$$\mathrm{Cov}_{i,j}(\mathbf{x},\mathbf{y}) = \mathbb{E}\big[V_{0,i}(\omega,\mathbf{x})V_{0,j}(\omega,\mathbf{y})\big]. \tag{3}$$

The boundedness of $\mathrm{Cov}_{i,j}(\mathbf{x},\mathbf{y})$ in $L^2(D_{\mathrm{ref}} \times D_{\mathrm{ref}})$ follows from the Cauchy-Schwarz inequality and the application of Fubini's theorem. Since $\mathrm{Cov}_{i,j}(\mathbf{x},\mathbf{y}) \in L^2(D_{\mathrm{ref}} \times D_{\mathrm{ref}})$ holds, we conclude $\mathrm{Cov}[\mathbf{V}](\mathbf{x},\mathbf{y}) \in L^2(D \times D;\mathbb{R}^{d\times d})$.

In order to make the random vector field $\mathbf{V}(\omega,\mathbf{x}) \in L^2_{\mathbb{P}}\big(\Omega;L^2(D;\mathbb{R}^d)\big)$ feasible for numerical computations, e.g. for a (quasi-) Monte Carlo method, we shall introduce its *Karhunen-Loève expansion*, cf. [26]. Since we may identify $L^2_{\mathbb{P}}\big(\Omega;L^2(D;\mathbb{R}^d)\big) \cong L^2_{\mathbb{P}}(\Omega) \otimes L^2(D;\mathbb{R}^d)$, one can show that $\mathbf{V}_0(\omega,\mathbf{x})$ exhibits the orthogonal decomposition

$$\mathbf{V}_0 = \sum_{i\in\mathscr{I}} \sigma_i X_i \otimes \varphi_i,$$

where $\{\varphi_i\}_{i\in\mathscr{I}} \subset L^2(D;\mathbb{R}^d)$ and $\{X_i\}_{i\in\mathscr{I}} \subset L^2_{\mathbb{P}}(\Omega)$ are orthonormal families. With respect to the canonical map

$$L^2_{\mathbb{P}}(\Omega)\otimes L^2(D;\mathbb{R}^d) \to L^2_{\mathbb{P}}\big(\Omega;L^2(D;\mathbb{R}^d)\big), \quad X\otimes\varphi \mapsto X(\omega)\varphi(\mathbf{x}),$$

we end up with the following

Definition 2. Let $\mathbf{V}(\omega,\mathbf{x})$ be a vector field in $L^2_{\mathbb{P}}\big(\Omega;L^2(D;\mathbb{R}^d)\big)$. The expansion

$$\mathbf{V}(\omega,\mathbf{x}) = \mathbb{E}[\mathbf{V}](\mathbf{x}) + \sum_{i\in\mathscr{I}} \sigma_i X_i(\omega)\varphi_i(\mathbf{x})$$

with $(X_i,X_j)_{L^2_{\mathbb{P}}(\Omega)} = \delta_{i,j}$ and $\mathbb{E}[X_i] = 0$ is called *Karhunen-Loève expansion* of $\mathbf{V}(\omega,\mathbf{x})$.

Remark 1. The knowledge of the random vector field $\mathbf{V}(\omega,\mathbf{x})$ is sufficient to compute the related Karhunen-Loève expansion. In practice, however, the random field is often only given in terms of its (empirical) mean $\mathbb{E}[\mathbf{V}]$ and its (empirical) covariance function $\operatorname{Cov}[\mathbf{V}]$. In this case, the orthogonal basis in $L^2_{\mathbb{P}}(\Omega)$ is only determined up to isometry. Therefore, for the use of e.g. a (quasi-) Monte Carlo method, the law of the random variables $\{X_i\}_{i\in\mathscr{I}}$ has to be approximated appropriately, for example by a *maximum likelihood estimate*, cf. [33]. This will be in contrast to the discretization in the perturbation method where we do not need to know the random variables' distribution at all.

Without loss of generality, we may assume that $\mathbb{E}[\mathbf{V}](\mathbf{x}) = \mathbf{x}$ is the identity mapping. Otherwise, we replace D_{ref} and φ_k by

$$\widetilde{D}_{\text{ref}} := \mathbb{E}[\mathbf{V}](D_{\text{ref}}) \quad \text{and} \quad \widetilde{\varphi}_k := \sqrt{\det(\mathbb{E}[\mathbf{V}]^{-1})'}\varphi_k \circ \mathbb{E}[\mathbf{V}]^{-1}.$$

3 Shape sensitivity analysis

In this section, we summarize results on shape sensitivity analysis for the Poisson equation

$$-\Delta u = f \text{ in } D_{\text{ref}}, \quad u = 0 \text{ on } \Gamma_{\text{ref}} := \partial D_{\text{ref}}. \tag{4}$$

For a more general framework and the details on this topic, we refer the reader to [8, 12, 37] and the references therein.

Assume that D_{ref} is of class C^2. This smoothness assumption guarantees the H^2-regularity of problem (4), cf. [37, Proposition 2.83]. Moreover, let $\mathbf{V}\in C^2(\mathbb{R}^d;\mathbb{R}^d)$ be a vector field. We may define the family of transformations $\{T_\varepsilon\}_{\varepsilon>0}$ by the perturbations of identity

$$T_\varepsilon(\mathbf{x}) = \operatorname{Id}(\mathbf{x}) + \varepsilon\mathbf{V}(\mathbf{x}).$$

Then, there exists an $\varepsilon_0 > 0$ such that the transformations T_ε are C^2-diffeomorphisms for all $\varepsilon \in [0,\varepsilon_0]$, cf. [36, Section 1.1]. The related family of domains will be denoted by $D_\varepsilon := T_\varepsilon(D_{\text{ref}})$. We shall consider the Poisson equation on these domains, i.e.

$$-\Delta u_\varepsilon = f \text{ in } D_\varepsilon, \quad u_\varepsilon = 0 \text{ on } \Gamma_\varepsilon := \partial D_\varepsilon. \tag{5}$$

Here, in order to guarantee the well-posedness of the equation, we assume that the right hand side is defined on the hold-all

$$\mathscr{D} = \bigcup_{0\leqslant\varepsilon\leqslant\varepsilon_0} D_\varepsilon.$$

Now, we have for the weak solution $u_\varepsilon \in H^s(D_\varepsilon)$ with $s \in [0,2]$ that

$$u^\varepsilon = u_\varepsilon \circ T_\varepsilon \in H^s(D_{\text{ref}})$$

for all $\varepsilon \in [0,\varepsilon_0]$, see e.g. [37]. Especially, we set $\overline{u} := u_0 \in H^s(D_{\text{ref}})$. Then, we may define the *material derivative* of u as in [37, Definition 2.71].

Definition 3. The function $\dot{u}[\mathbf{V}] \in H^s(D_{\text{ref}})$ is called the strong (weak) *material derivative* of $u \in H^s(D_{\text{ref}})$ in the direction $\mathbf{V}$ if the strong (weak) limit

$$\dot{u}[\mathbf{V}] = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(u^\varepsilon - u)$$

exists.

The shape sensitivity analysis considered in this section is based on the notion of the *local shape derivative*. To this end, we consider for $\overline{u} \in H^s(D_{\text{ref}})$ and $u_\varepsilon \in H^s(D_\varepsilon)$ the expression

$$\frac{1}{\varepsilon}\big(u_\varepsilon(\mathbf{x}) - \overline{u}(\mathbf{x})\big).$$

Obviously, this expression is only meaningful if $\mathbf{x} \in D_\varepsilon \cap D_{\text{ref}}$. Nevertheless, according to [12, Remark 2.2.12], there exists an $\varepsilon(\mathbf{x},\mathbf{V})$ due to the regularity of T_ε such that $\mathbf{x} \in D_\varepsilon \cap D_{\text{ref}}$ for all $0 \leqslant \varepsilon \leqslant \varepsilon(\mathbf{x},\mathbf{V})$. Moreover, in order to define a meaningful functional analytic framework for the limit $\varepsilon \to 0$, one has to consider compact subsets $K \Subset D_{\text{ref}}$, cf. [36]. Hence, we have from [12, Definition 2.2.13] the following

Definition 4. For $K \Subset D_{\text{ref}}$, the function $\delta u[\mathbf{V}] \in H^s(K)$ is called the strong (weak) *local* $H^s(K)$ shape derivative of u in direction $\mathbf{V}$, if the strong (weak) $H^s(K)$ limit

$$\delta u[\mathbf{V}] = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(u_\varepsilon - \overline{u})$$

exists. It holds $\delta u \in H^s_{\text{loc}}(D_{\text{ref}})$ strongly (weakly) if the limit exists for arbitrary $K \Subset D_{\text{ref}}$.

Notice that the definition of $\delta u[\mathbf{V}]$ has no meaning on Γ_{ref} in general, cf. [12, Remark 2.2.14]. Nevertheless, since boundary values for $\dot{u}[\mathbf{V}]$ are obtained via the trace operator, cf. [37, Proposition 2.75], we may define the boundary values for $\delta u[\mathbf{V}]$ by employing the relation

$$\dot{u}[\mathbf{V}] = \delta u[\mathbf{V}] + \langle \nabla u, \mathbf{V} \rangle,$$

cf. [37, Definition 2.85]. Therefore, if $f \in H^1(\mathscr{D})$, the local shape derivative for the Poisson equation (5) satisfies the boundary value problem

$$\Delta \delta u = 0 \text{ in } D_{\text{ref}}, \quad \delta u = -\langle \mathbf{V}, \mathbf{n} \rangle \frac{\partial \overline{u}}{\partial \mathbf{n}} \text{ on } \Gamma_{\text{ref}}, \tag{6}$$

cf. [37, Proposition 3.1]. Here, $\mathbf{n}(\mathbf{x})$ denotes the outward normal at the boundary Γ_{ref}.

The representation (6) of $\delta u[\mathbf{V}]$ indicates that it is already sufficient to consider vector fields $\mathbf{V}$ which are compactly supported in a neighbourhood of Γ_{ref}, i.e. $\mathbf{V}\big|_K \equiv 0$ for all $K \Subset D_{\text{ref}}$, cf. [12, Remark 2.1.6]. More precisely, it holds for two perturbation fields $\mathbf{V}$ and $\tilde{\mathbf{V}}$ that

$$\delta u[\mathbf{V}] = \delta u[\tilde{\mathbf{V}}] \quad \text{if } \mathbf{V}\big|_{\Gamma_{\text{ref}}} = \tilde{\mathbf{V}}\big|_{\Gamma_{\text{ref}}},$$

cf. [37, Proposition 2.90]. For example, it is quite common to consider (normal) perturbations of the boundary, see e.g. [12, 25, 29, 30].

Having the local shape derivative of the solution u to (4) at hand, we can linearize the perturbed solution u_ε to (5) in a neighbourhood of D_{ref} in terms of a *shape Taylor expansion*, cf. [10, 11, 23, 31], according to

$$u_\varepsilon(\mathbf{x}) = \overline{u}(\mathbf{x}) + \varepsilon \delta u(\mathbf{x}) + \varepsilon^2 C(\mathbf{x}) \quad \text{for } \mathbf{x} \in K \Subset (D_{\text{ref}} \cap D_\varepsilon), \tag{7}$$

where the function $|C(\mathbf{x})| < \infty$ depends on the distance $\text{dist}(K, \Gamma_{\text{ref}})$ and the load f.

4 Approximation of mean and covariance

We shall go back to our model problem, the Poisson equation on a random domain:

$$-\Delta u(\omega, \mathbf{x}) = f(\mathbf{x}) \text{ in } D(\omega), \quad u(\omega, \mathbf{x}) = 0 \text{ on } \Gamma(\omega). \tag{8}$$

We assume that the random domain is described by a random vector field. This means, we have

$$D(\omega) := \mathbf{V}(\omega, D_{\text{ref}}).$$

In respect of the discussion in the end of Section 2, it is reasonable to assume that $\mathbf{V}$ is a perturbation of identity. More precisely, we assume that it holds

$$\mathbf{V}(\omega, \mathbf{x}) = \text{Id}(\mathbf{x}) + \mathbf{V}_0(\omega, \mathbf{x})$$

for a vector field $\mathbf{V}_0(\omega) \in C^2(D_{\text{ref}}; \mathbb{R}^d)$ for almost every $\omega \in \Omega$ with $\mathbb{E}[\mathbf{V}_0] = \mathbf{0}$. We shall further assume the uniformity condition $\|\mathbf{V}_0(\omega)\|_{C^2(\overline{D_{\text{ref}}}; \mathbb{R}^d)} \leqslant c$ for some $c < \infty$ and for almost every $\omega \in \Omega$. Then, in view of (7), the first-order shape Taylor expansion for the solution $u(\omega)$ to (8) with respect to the transformation

$$T_\varepsilon(\mathbf{x}) = \text{Id}(\mathbf{x}) + \varepsilon \mathbf{V}_0(\omega, \mathbf{x}), \tag{9}$$

is given by

$$u(\omega,\mathbf{x}) = \overline{u}(\mathbf{x}) + \varepsilon\delta u(\mathbf{x})[\mathbf{V}_0(\omega)] + \mathscr{O}(\varepsilon^2).$$

In this expansion, $\overline{u}$ is the solution to

$$-\Delta\overline{u} = f \text{ in } D_{\text{ref}}, \quad \overline{u} = 0 \text{ on } \Gamma_{\text{ref}} \tag{10}$$

while $\delta u[\mathbf{V}_0(\omega)]$ is the solution to

$$\Delta\delta u[\mathbf{V}_0(\omega)] = 0 \text{ in } D_{\text{ref}}, \quad \delta u[\mathbf{V}_0(\omega)] = -\langle\mathbf{V}_0(\omega),\mathbf{n}\rangle\frac{\partial\overline{u}}{\partial\mathbf{n}} \text{ on } \Gamma_{\text{ref}}. \tag{11}$$

As already pointed out in the end of the preceding section, it is sufficient to know $\mathbf{V}_0(\omega,\mathbf{x})$ only in a neighbourhood of the boundary Γ_{ref} of D_{ref}. This is in contrast to the domain mapping method where one always has to know the perturbation field for the whole domain D_{ref}.

In order to simplify the notation, we will write $\delta u(\omega)$ instead of $\delta u[\mathbf{V}_0(\omega)]$ in the sequel. Having the first-order shape Taylor expansion (9) of $u(\omega)$ at hand, we can approximate the related moments from it.

Theorem 2. *For $\varepsilon > 0$ sufficiently small, it holds for $K \Subset (D_{\text{ref}} \cap D_\varepsilon)$ that*

$$\mathbb{E}[u] = \overline{u} + \mathscr{O}(\varepsilon^2) \quad \text{in } K \tag{12}$$

with $\overline{u} \in H_0^1(D_{\text{ref}})$. The covariance of u satisfies

$$\operatorname{Cov}[u] = \varepsilon^2\operatorname{Cov}[\delta u] + \mathscr{O}(\varepsilon^3) \quad \text{in } K\times K \tag{13}$$

with the covariance $\operatorname{Cov}[\delta u] \in H^1(D_{\text{ref}})\otimes H^1(D_{\text{ref}})$. The covariance is given as the solution to the following boundary value problem

$$\begin{aligned}
(\Delta\otimes\Delta)\operatorname{Cov}[\delta u] &= 0 \quad \text{in } D_{\text{ref}}\times D_{\text{ref}},\\
(\Delta\otimes\gamma_0^{\text{int}})\operatorname{Cov}[\delta u] &= 0 \quad \text{in } D_{\text{ref}}\times \Gamma_{\text{ref}},\\
(\gamma_0^{\text{int}}\otimes\Delta)\operatorname{Cov}[\delta u] &= 0 \quad \text{in } \Gamma_{\text{ref}}\times D_{\text{ref}},\\
(\gamma_0^{\text{int}}\otimes\gamma_0^{\text{int}})\operatorname{Cov}[\delta u] &= \langle\mathbf{n}(\mathbf{x}),\operatorname{Cov}[\mathbf{V}]\mathbf{n}(\mathbf{y})\rangle\left(\frac{\partial\overline{u}}{\partial\mathbf{n}}\otimes\frac{\partial\overline{u}}{\partial\mathbf{n}}\right) \quad \text{on } \Gamma_{\text{ref}}\times\Gamma_{\text{ref}}.
\end{aligned} \tag{14}$$

Here, $\gamma_0^{\text{int}}\colon H^1(D_{\text{ref}}) \to H^{1/2}(\Gamma_{\text{ref}})$ denotes the interior trace operator.

Proof. The equation for the mean is easily obtained by exploiting the linearity of the mean. It remains to show that

$$\mathbb{E}[\delta u] = 0.$$

By Theorem 1, we know that we may interchange the Bochner integral with the Laplace operator. Thus, from (11), we obtain the following boundary value problem for $\mathbb{E}[\delta u]$:

$$\Delta\mathbb{E}[\delta u] = 0 \text{ in } D_{\text{ref}}, \quad \mathbb{E}[\delta u] = -\mathbb{E}\left[\langle \mathbf{V}_0, \mathbf{n}\rangle \frac{\partial \overline{u}}{\partial \mathbf{n}}\right] \text{ on } \Gamma_{\text{ref}}.$$

By the linearity of the Bochner integral, the boundary condition can be written as

$$-\mathbb{E}\left[\langle \mathbf{V}_0, \mathbf{n}\rangle \frac{\partial \overline{u}}{\partial \mathbf{n}}\right] = -\langle \mathbb{E}[\mathbf{V}_0], \mathbf{n}\rangle \frac{\partial \overline{u}}{\partial \mathbf{n}} = 0$$

since $\mathbb{E}[\mathbf{V}_0] = \mathbf{0}$. Thus, $\mathbb{E}[\delta u]$ solves the Laplace equation with homogeneous boundary condition. From this, we infer $\mathbb{E}[\delta u] = 0$.

For the covariance $\operatorname{Cov}[u]$, we obtain

$$\begin{aligned}\operatorname{Cov}[u] &= \mathbb{E}\big[(u - \mathbb{E}[u]) \otimes (u - \mathbb{E}[u])\big] \\ &= \mathbb{E}\Big[\big(\overline{u} + \varepsilon\delta u(\omega) + \mathscr{O}(\varepsilon^2) - \mathbb{E}[u]\big) \otimes \big(\overline{u} + \varepsilon\delta u(\omega) + \mathscr{O}(\varepsilon^2) - \mathbb{E}[u]\big)\Big].\end{aligned}$$

Since we can estimate $\mathbb{E}[u] - \overline{u} = \mathscr{O}(\varepsilon^2)$ in K due to (12), we arrive at

$$\begin{aligned}\operatorname{Cov}[u] &= \mathbb{E}\Big[\big(\varepsilon\delta u(\omega) + \mathscr{O}(\varepsilon^2)\big) \otimes \big(\varepsilon\delta u(\omega) + \mathscr{O}(\varepsilon^2)\big)\Big] \\ &= \varepsilon^2\mathbb{E}[\delta u(\omega) \otimes \delta u(\omega)] + \mathscr{O}(\varepsilon^3).\end{aligned}$$

In view of $\operatorname{Cov}[\delta u] = \mathbb{E}[\delta u(\omega) \otimes \delta u(\omega)]$, we conclude (13). Finally, by tensorization of (11) and application of the mean together with Theorem 1, one infers that $\operatorname{Cov}[\delta u] \in H^1(D_{\text{ref}}) \otimes H^1(D_{\text{ref}})$ is given by (14). □

In the sequel, for $t \geqslant 0$, we set

$$H^t_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}}) := H^t(D_{\text{ref}}) \otimes H^t(D_{\text{ref}}),$$

$$H^t_{\text{mix}}(\Gamma_{\text{ref}} \times \Gamma_{\text{ref}}) := H^t(\Gamma_{\text{ref}}) \otimes H^t(\Gamma_{\text{ref}}).$$

Remark 2. The technique which we used to derive the approximation error for the covariance of u can straightforwardly be applied to obtain a similar result for the k-th moment, i.e.

$$\mathbb{E}\big[\underbrace{(u - \mathbb{E}[u]) \otimes \ldots \otimes (u - \mathbb{E}[u])}_{k\text{-times}}\big].$$

In this case, we end up with the expression

$$\mathbb{E}\big[\big(\varepsilon\delta u + \mathscr{O}(\varepsilon^2)\big) \otimes \ldots \otimes \big(\varepsilon\delta u + \mathscr{O}(\varepsilon^2)\big)\big] = \varepsilon^k\mathbb{E}[\delta u \otimes \ldots \otimes \delta u] + \mathscr{O}(\varepsilon^{k+1}),$$

where the constant depends exponentially on k, see also [7].

5 Discretization of tensor product Dirichlet problems

In the previous section, we have seen that we end up solving the tensor product Dirichlet problem (14) in order to approximate the covariance of the model problem's solution. The treatment of the non-homogenous tensor product Dirichlet boundary condition is non-trivial. Therefore, we think that it is justified to consider here the discretization by finite elements in detail.

We start with the discretization of univariate Dirichlet problems and then generalize the approach towards the tensor product case. We thus aim at solving the Dirichlet boundary value problem

$$\Delta u = 0 \text{ in } D_{\text{ref}}, \quad u = g \text{ on } \Gamma_{\text{ref}}. \tag{15}$$

By the inverse trace theorem, see e.g. [40], there exists an extension of $u_g \in H^1(D_{\text{ref}})$ with $\gamma_0^{\text{int}} u_g = g$ provided that $g \in H^{1/2}(\Gamma_{\text{ref}})$. Therefore, it remains to determine the function $u_0 = u - u_g \in H_0^1(D_{\text{ref}})$ such that there holds

$$a^D(u_0, v) = -a^D(u_g, v) \quad \text{for all } v \in H_0^1(D_{\text{ref}}). \tag{16}$$

Here and in the sequel, the elliptic bilinear form related to the Laplace operator is given by

$$a^D(u,v) := (\nabla u, \nabla v)_{L^2(D_{\text{ref}})} \quad \text{for } u,v \in H_0^1(D_{\text{ref}}).$$

The question arises how to numerically determine a suitable extension u_g of the Dirichlet data. We follow here the approach from [2], see also [13], where the extension is generated by means of an L^2-projection of the given boundary data. To that end, we introduce the nested sequence of finite element spaces

$$V_0 \subset V_1 \subset \cdots \subset V_J \subset H^1(D_{\text{ref}}).$$

Herein, given a uniform triangulation for D_{ref}, the space V_j corresponds to the space of continuous piecewise linear functions $\{\varphi_{j,k} \in V_j : k \in \mathscr{I}_j\}$. Of course, by performing obvious modifications, one can employ the presented framework also for higher order ansatz functions. Notice that we have $\dim V_j \eqsim 2^{dj}$. In the following, we distinguish between basis functions $\{\varphi_{j,k} \in V_j : k \in \mathscr{I}_j^D\}$ which are supported in the interior of the reference domain, i.e. $\varphi_{j,k}\big|_{\Gamma_{\text{ref}}} \equiv 0$, and boundary functions $\{\varphi_{j,k} \in V_j : k \in \mathscr{I}_j^\Gamma\}$ with $\varphi_{j,k}\big|_{\Gamma_{\text{ref}}} \not\equiv 0$. Notice that $\mathscr{I}_j = \mathscr{I}_j^D \cup \mathscr{I}_j^\Gamma$ and $\mathscr{I}_j^D \cap \mathscr{I}_j^\Gamma = \emptyset$. The related finite element spaces are then given by

$$V_j^D := \text{span}\{\varphi_{j,k} \in V_j : k \in \mathscr{I}_j^D\} \text{ and } V_j^\Gamma := \text{span}\{\varphi_{j,k}|_{\Gamma_{\text{ref}}} : \varphi_{j,k} \in V_j, k \in \mathscr{I}_j^\Gamma\}.$$

Moreover, we denote the L^2-inner product on Γ_{ref} by

$$a^\Gamma(u,v) := (u,v)_{L^2(\Gamma_{\text{ref}})} \quad \text{for } u,v \in L^2(\Gamma_{\text{ref}}).$$

Then, the L^2-orthogonal projection of the Dirichlet data is given by the solution to the following variational formulation:

$$\begin{aligned}&\text{Find } g_j \in V_j^\Gamma \text{ such that}\\ &\qquad a^\Gamma(g_j, v) = a^\Gamma(g, v) \quad \text{for all } v \in V_j^\Gamma.\end{aligned} \tag{17}$$

We are now prepared to formulate the Galerkin discretization of (16). To that end, we introduce the stiffness matrices

$$\mathbf{S}_j^\Lambda := \big[a^D(\varphi_{j,\ell}, \varphi_{j,k})\big]_{k\in\mathscr{I}_j^D, \ell\in\mathscr{I}_j^\Lambda}, \quad \Lambda \in \{D, \Gamma\} \tag{18}$$

and the mass matrices with respect to the boundary

$$\mathbf{G}_j := \big[a^\Gamma(\varphi_{j,\ell}, \varphi_{j,k})\big]_{k\in\mathscr{I}_j^\Gamma, \ell\in\mathscr{I}_{j'}^\Gamma}. \tag{19}$$

The related data vector reads

$$\mathbf{g}_j = \big[a^\Gamma(g, \varphi_{j,k})\big]_{k\in\mathscr{I}_j^\Gamma}.$$

In order to compute an approximate solution to this boundary value problem in the finite element space $V_J \subset H^1(D_{\text{ref}})$ for $J \in \mathbb{N}$, we make the ansatz

$$u_J = \sum_{k\in\mathscr{I}_J} u_{J,k}\varphi_{J,k} = \sum_{k\in\mathscr{I}_J^D} u_{J,k}\varphi_{J,k} + \sum_{k\in\mathscr{I}_J^\Gamma} u_{J,k}\varphi_{J,k} = u_J^D + u_J^\Gamma.$$

At first, we determine the boundary part $u_J^\Gamma \in H^1(D)$ such that

$$\mathbf{G}_J\mathbf{u}_J^\Gamma = \mathbf{g}_J. \tag{20}$$

Therefore, $u_J^\Gamma|_{\Gamma_{\text{ref}}}$ is the L^2-orthogonal projection of the Dirichlet datum g onto the discrete trace space V_J^Γ. Having u_J^Γ at hand, we can compute the domain part $u_J^D \in H_0^1(D)$ from

$$\mathbf{S}_J^D\mathbf{u}_J^D = -\mathbf{S}_J^\Gamma\mathbf{u}_J^\Gamma. \tag{21}$$

We use the conjugate gradient method to iteratively solve the systems of linear equations (20) and (21). Using a nested iteration, combined with the BPX-preconditioner, cf. [3], in case of (21), results in a linear over-all complexity, see [15]. Moreover, from [2, Theorem 1], we obtain the following convergence result.

Theorem 3. *Let $g \in H^t(\Gamma_{\text{ref}})$ for $0 \leqslant t \leqslant 3/2$. Then, if $g_J \in V_J^\Gamma$ is given by* (17)*, the Galerkin solution u_J to* (15) *satisfies*

$$\|u - u_J\|_{L^2(D_{\text{ref}})} \lesssim 2^{-J(t+1/2)}\|g\|_{H^t(\Gamma_{\text{ref}})}.$$

Next, we deal with the tensor product boundary value problem (14) and discretize it in $V_J \otimes V_J$. We make the ansatz

$$
\begin{aligned}
\operatorname{Cov}[\delta u]_J &= \sum_{k\in\mathscr{I}_J}\sum_{k'\in\mathscr{I}_J} u_{J,k,k'}(\varphi_{J,k}\otimes\varphi_{J,k'}) \\
&= \operatorname{Cov}[\delta u]_J^{D,D} + \operatorname{Cov}[\delta u]_J^{D,\Gamma} + \operatorname{Cov}[\delta u]_J^{\Gamma,D} + \operatorname{Cov}[\delta u]_J^{\Gamma,\Gamma}
\end{aligned}
\tag{22}
$$

with

$$
\operatorname{Cov}[\delta u]_J^{\Lambda,\Lambda'} = \sum_{k\in\mathscr{I}_J^{\Lambda}}\sum_{k\in\mathscr{I}_J^{\Lambda'}} u_{J,k,k'}(\varphi_{J,k}\otimes\varphi_{J,k'}) \quad \text{for } \Lambda,\Lambda'\in\{D,\Gamma\}.
$$

In complete analogy to the non-tensor product case, we obtain the solution scheme

(1) Solve $(\mathbf{G}_J\otimes\mathbf{G}_J)\mathbf{u}_J^{\Gamma,\Gamma} = \mathbf{g}_J$.
(2) Solve $(\mathbf{G}_J\otimes\mathbf{S}_J^D)\mathbf{u}_J^{\Gamma,D} = -(\mathbf{G}_J\otimes\mathbf{S}_J^{\Gamma})\mathbf{u}_J^{\Gamma,\Gamma}$ and $(\mathbf{S}_J^D\otimes\mathbf{G}_J)\mathbf{u}_J^{D,\Gamma} = -(\mathbf{S}_J^{\Gamma}\otimes\mathbf{G}_J)\mathbf{u}_J^{\Gamma,\Gamma}$.
(3) Solve $(\mathbf{S}_J^D\otimes\mathbf{S}_J^D)\mathbf{u}_J^{D,D} = -(\mathbf{S}_J^{\Gamma}\otimes\mathbf{S}_J^{\Gamma})\mathbf{u}_J^{\Gamma,\Gamma} - (\mathbf{S}_{\Gamma}\otimes\mathbf{S}_J^D)\mathbf{u}_J^{\Gamma,D} - (\mathbf{S}_D\otimes\mathbf{S}_J^{\Gamma})\mathbf{u}_J^{D,\Gamma}$.

Herein, we set $\mathbf{u}_J^{\Lambda,\Lambda'} = [u_{J,k,k'}]_{k\in\mathscr{I}_J^{\Lambda},k'\in\mathscr{I}_J^{\Lambda'}}$ for $\Lambda,\Lambda'\in\{D,\Gamma\}$ and

$$
\mathbf{g}_J = \left[\left(\langle\mathbf{n}(\mathbf{x}),\operatorname{Cov}[\mathbf{V}]\mathbf{n}(\mathbf{y})\rangle\left(\frac{\partial\overline{u}}{\partial\mathbf{n}}\otimes\frac{\partial\overline{u}}{\partial\mathbf{n}}\right),\varphi_{J,k}\otimes\varphi_{J,k'}\right)_{L^2(\Gamma_{\mathrm{ref}}\times\Gamma_{\mathrm{ref}})}\right]_{k,k'\in\mathscr{I}_J^{\Gamma}}.
$$

The different tensor products of mass matrices and stiffness matrices in this formulation arise from the related tensor products of the bilinear forms $a^D(\cdot,\cdot)$ and $a^{\Gamma}(\cdot,\cdot)$. Namely, these are

$$
\begin{aligned}
a^{\Gamma,\Gamma}(u,v) &:= \big(u,v\big)_{L^2(\Gamma_{\mathrm{ref}}\times\Gamma_{\mathrm{ref}})} && \text{for } u,v\in L^2(\Gamma_{\mathrm{ref}})\otimes L^2(\Gamma_{\mathrm{ref}}),\\
a^{\Gamma,D}(u,v) &:= \big((\mathrm{Id}\otimes\nabla)u,(\mathrm{Id}\otimes\nabla)v\big)_{L^2(\Gamma_{\mathrm{ref}}\times D_{\mathrm{ref}})} && \text{for } u,v\in L^2(\Gamma_{\mathrm{ref}})\otimes H_0^1(D_{\mathrm{ref}}),\\
a^{D,\Gamma}(u,v) &:= \big((\nabla\otimes\mathrm{Id})u,(\nabla\otimes\mathrm{Id})v\big)_{L^2(D_{\mathrm{ref}}\times\Gamma_{\mathrm{ref}})} && \text{for } u,v\in H_0^1(D_{\mathrm{ref}})\otimes L^2(\Gamma_{\mathrm{ref}}),\\
a^{D,D}(u,v) &:= \big((\nabla\otimes\nabla)u,(\nabla\otimes\nabla)v\big)_{L^2(D_{\mathrm{ref}}\times D_{\mathrm{ref}})} && \text{for } u,v\in H_0^1(D_{\mathrm{ref}})\otimes H_0^1(D_{\mathrm{ref}}).
\end{aligned}
$$

For the approximation error of the Galerkin solution in $V_J\otimes V_J$, there holds a result similar to Theorem 3.

Theorem 4. *Let $g\in H^t_{\mathrm{mix}}(\Gamma_{\mathrm{ref}}\times\Gamma_{\mathrm{ref}})$ for $0\leqslant t\leqslant 3/2$. Then, if $g_J\in V_J^{\Gamma}\otimes V_J^{\Gamma}$ is the L^2-orthogonal projection of the Dirichlet data, the Galerkin solution u_J to the tensor product Dirichlet problem satisfies*

$$
\|u-u_J\|_{L^2(D_{\mathrm{ref}}\times D_{\mathrm{ref}})} \lesssim 2^{-J(t+1/2)}\|g\|_{H^t_{\mathrm{mix}}(\Gamma_{\mathrm{ref}}\times\Gamma_{\mathrm{ref}})}.
$$

Proof. By a tensor product argument, the proof of this theorem is obtained by summing up the uni-directional error estimates provided by Theorem 3. □

Unfortunately, the computational complexity of this approximation is of order $(\dim V_J)^2$, which may become prohibitive for increasing level J. A possibility to overcome this obstruction is given by the discretization in *sparse tensor product spaces*. In the following we shall focus on this approach.

6 Sparse second moment analysis

According to Section 4, to leading order, the mean of the solution of the random boundary value problem (8) satisfies the deterministic equation (10). This equation can be discretized straightforwardly by means of finite elements. The resulting system of linear equations may then be solved in optimal complexity e.g. by a multigrid solver. The solution of the tensor product structured problem (14) is a little more involved and requires another approach in order to maintain the overall complexity.

Instead of discretizing the tensor product boundary value problem (14) in the space $V_J \otimes V_J$, we consider here the discretization in the *sparse tensor product space*

$$\widehat{V_J \otimes V_J} := \sum_{j+j' \leqslant J} V_j \otimes V_{j'} = \sum_{j+j'=J} V_j \otimes V_{j'} \subset H^1_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}}). \tag{23}$$

For the dimension of the sparse tensor product space, we have

$$\dim \widehat{V_J \otimes V_J} \eqsim \dim V_J \log(\dim V_J)$$

instead of $(\dim V_J)^2$, which is the dimension of $V_J \otimes V_J$, cf. [4]. Thus, the dimension of the sparse tensor product space is substantially smaller than that of the full tensor product space.

The following lemma, proven in [35, 39], tells us that the approximation power in the sparse tensor product space is nearly as good as in the full tensor product space, provided that the given function has some extra regularity in terms of bounded mixed derivatives.

Lemma 1. *For $0 \leq t < 3/2$, $t \leq q \leq 2$ there holds the error estimate*

$$\inf_{\widehat{v}_J \in \widehat{V_J \otimes V_J}} \|v - \widehat{v}_J\|_{H^t_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}})} \lesssim \begin{cases} 2^{J(t-q)} \sqrt{J} \|v\|_{H^q_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}})}, & \text{if } q = 2, \\ 2^{J(t-q)} \|v\|_{H^q_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}})}, & \text{otherwise}, \end{cases}$$

provided that $v \in H^q_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}})$.

This lemma gives rise to an estimate for the Galerkin approximation $\widehat{\operatorname{Cor}[\delta u]}_J$ of (14) in the sparse tensor product space $\widehat{V_J \otimes V_J}$, see e.g. [18, Proposition 5]. We state it only for the case of piecewise linear finite elements as considered here.

Corollary 1. *The Galerkin approximate $\widehat{\operatorname{Cor}[\delta u]}_J \in \widehat{V_J \otimes V_J}$ to* (14) *satisfies the error estimate*

$$\big\| \operatorname{Cor}[\delta] - \widehat{\operatorname{Cor}[\delta u]}_J \big\|_{L^2(D_{\text{ref}} \times D_{\text{ref}})} \lesssim 2^{-2J} J \| \operatorname{Cor}[\delta u] \|_{H^2_{\text{mix}}(D_{\text{ref}} \times D_{\text{ref}})}$$

provided that the given data are sufficiently smooth.

The Galerkin discretization of (14) in the sparse tensor product space is now rather similar to the approach in [18], where *sparse multilevel frames*, cf. [22], have been employed for the discretization. We can considerably improve this approach by combining it with the idea from [20]: Instead of dealing with all combinations which occur in the discretization by a sparse frame for each of the four subproblems on $\Gamma_{\text{ref}} \times \Gamma_{\text{ref}}$, on $D_{\text{ref}} \times \Gamma_{\text{ref}}$, on $\Gamma_{\text{ref}} \times D_{\text{ref}}$ and in $D_{\text{ref}} \times D_{\text{ref}}$, we shall employ the *combination technique*, cf. [14]. Then, we have only to compute combinations of the solution on two consecutive levels instead of all combinations.

The analogue to the ansatz (22) for the Galerkin approximation in the sparse tensor product space reads

$$\begin{aligned}\widehat{\operatorname{Cov}[\delta u]}_J &= \sum_{j+j'\leqslant J} \sum_{k\in\mathscr{I}_j} \sum_{k'\in\mathscr{I}_{j'}} \widehat{u}_{j,j',k,k'}(\varphi_{j,k}\otimes\varphi_{j',k'}) \\ &= \widehat{\operatorname{Cov}[\delta u]}_J^{D,D} + \widehat{\operatorname{Cov}[\delta u]}_J^{D,\Gamma} + \widehat{\operatorname{Cov}[\delta u]}_J^{\Gamma,D} + \widehat{\operatorname{Cov}[\delta u]}_J^{\Gamma,\Gamma}\end{aligned} \tag{24}$$

with

$$\widehat{\operatorname{Cov}[\delta u]}_J^{\Lambda,\Lambda'} = \sum_{j+j'\leqslant J} \sum_{k\in\mathscr{I}_j^{\Lambda}} \sum_{k\in\mathscr{I}_{j'}^{\Lambda'}} \widehat{u}_{j,j',k,k'}(\varphi_{j,k}\otimes\varphi_{j',k'}) \in \widehat{V_J^{\Lambda}\otimes V_J^{\Lambda'}} \text{ for } \Lambda,\Lambda'\in\{D,\Gamma\}. \tag{25}$$

The basic idea of our approach is to define *detail spaces* with respect to Galerkin projections in order to remove the redundancy in the ansatz for the subproblems (25). We need thus the Galerkin projection $P_j\colon H_0^1(D_{\text{ref}}) \to V_j^D$, $w \mapsto P_j w$ defined via

$$\big(\nabla(w-P_j w),\nabla v_j\big)_{L^2(D_{\text{ref}})} = 0 \quad \text{for all } v_j \in V_j^D$$

and the L^2-orthogonal projection $Q_j\colon L^2(\Gamma_{\text{ref}}) \to V_j^{\Gamma}$, $w \mapsto Q_j w$, defined via

$$\big((w-Q_j w),v_j\big)_{L^2(\Gamma_{\text{ref}})} = 0 \quad \text{for all } v_j \in V_j^{\Gamma}.$$

Furthermore, we introduce the related *detail projections*

$$\Theta_j^D := P_j - P_{j-1}, \quad \text{where } P_{-1} := 0$$

and

$$\Theta_j^{\Gamma} := Q_j - Q_{j-1}, \quad \text{where } Q_{-1} := 0.$$

With the detail projections at hand, we define the related *detail spaces*

$$W_j^D := \Theta_j^D H_0^1(D_{\text{ref}}) = (P_j - P_{j-1}) H_0^1(D_{\text{ref}}) \subset V_j^D$$

and

$$W_j^{\Gamma} := \Theta_j^{\Gamma} L^2(\Gamma) = (Q_j - Q_{j-1}) L^2(\Gamma) \subset V_j^{\Gamma}.$$

Obviously, it holds $V_j^{\Lambda} = V_{j-1}^{\Lambda} \oplus W_j^{\Lambda}$ for $\Lambda \in \{D,\Gamma\}$. This gives rise to the decompositions

$$V_J^{\Lambda} = W_0^{\Lambda} \oplus W_1^{\Lambda} \oplus \ldots \oplus W_J^{\Lambda} \text{ for } \Lambda \in \{D,\Gamma\}.$$

Especially, these decompositions are orthogonal with respect to their defining inner products.

Lemma 2. *It holds*

$$(\nabla w_j, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} = 0 \quad \text{for } w_j \in W_j^D,\ w_{j'} \in W_{j'}^D \text{ and } j' \neq j$$

as well as

$$(w_j, w_{j'})_{L^2(D_{\mathrm{ref}})} = 0 \quad \text{for } w_j \in W_j^\Gamma,\ w_{j'} \in W_{j'}^\Gamma \text{ and } j \neq j'.$$

Proof. We show the assertion for the spaces W_j^D. The proof for the spaces W_j^Γ is analogous. Without loss of generality, let $j > j'$. Otherwise, due to the symmetry of the inner products, we may interchange the roles of j and j'. Let $w_j = \Theta_j v \in W_j^\Gamma$ for some $v \in H_0^1(D_{\mathrm{ref}})$ and $w_{j'} \in W_{j'}^\Gamma \subset V_{j'}^\Gamma$. Then, since $j-1 \geqslant j'$, we have that

$$(\nabla P_j v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} = (\nabla v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})}$$

and

$$(\nabla P_{j-1} v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} = (\nabla v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})}.$$

Thus, we obtain

$$\begin{aligned}(\nabla w_j, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} &= (\nabla P_j v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} - (\nabla P_{j-1} v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} \\ &= (\nabla v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} - (\nabla v, \nabla w_{j'})_{L^2(D_{\mathrm{ref}})} = 0.\end{aligned}$$

□

Now, we shall rewrite the sparse tensor product spaces given by (23) according to

$$\widehat{V_J^\Lambda \otimes V_J^{\Lambda'}} = \sum_{j+j'=J} V_j^\Lambda \otimes V_{j'}^{\Lambda'} = \sum_{j+j'=J} \left(\bigoplus_{i=0}^{j} W_i^\Lambda \right) \otimes V_{j'}^{\Lambda'} = \bigoplus_{j=0}^{J} W_j^\Lambda \otimes V_{J-j}^{\Lambda'}.$$

By exploiting the symmetry in this expression, we have also

$$\widehat{V_J^\Lambda \otimes V_J^{\Lambda'}} = \bigoplus_{j=0}^{J} W_j^\Lambda \otimes V_{J-j}^{\Lambda'} = \bigoplus_{j=0}^{J} V_j^\Lambda \otimes W_{J-j}^{\Lambda'}.$$

Thus, fixing a basis $\psi_{j,k} \in W_j^\Lambda$ for $\Lambda \in \{D, \Gamma\}$, we have for the subproblems (25) the formulation

$$\widehat{\mathrm{Cov}[\delta u]}_J^{\Lambda,\Lambda'} = \bigoplus_{j=0}^{J} \sum_{k \in \mathscr{I}_j^\Lambda} \sum_{k \in \mathscr{I}_{J-j}^{\Lambda'}} \widehat{u}_{j,J-j,k,k'} (\psi_{j,k} \otimes \varphi_{J-j,k'}) \quad \text{for } \Lambda, \Lambda' \in \{D, \Gamma\}. \tag{26}$$

Taking further into account the orthogonality described by Lemma 2, we can show that the computation of $\widehat{\mathrm{Cov}[\delta u]}_J^{\Lambda,\Lambda'}$ for $\Lambda,\Lambda' \in \{D,\Gamma\}$ decouples into independent subproblems.

Lemma 3. *Let $\Lambda,\Lambda' \in \{D,\Gamma\}$. For $\widehat{v}_j \in W_j^\Lambda \otimes V_{J-j}^{\Lambda'}$ and $\widehat{v}_{j'} \in W_{j'}^\Lambda \otimes V_{J-j'}^{\Lambda'}$, there holds*

$$a^{\Lambda,\Lambda'}(\widehat{v}_j,\widehat{v}_{j'}) = 0 \quad \text{if } j \neq j'.$$

Proof. We show the proof for the case $\Lambda = \Gamma$ and $\Lambda' = D$. The other cases are analogous, see also [20, Lemma 6]. Assume that

$$\widehat{v}_j = \sum_{i\in\mathscr{I}} \alpha_i(\psi_{j,i}\otimes\varphi_{J-j,i}) \quad \text{and} \quad \widehat{v}_{j'} = \sum_{i\in\mathscr{I}'} \beta_i(\psi_{j',i}\otimes\varphi_{J-j',i})$$

is a representation of $\widehat{v}_j \in W_j^\Gamma \otimes V_{J-j}^D$ and $\widehat{v}_{j'} \in W_{j'}^\Gamma \otimes V_{J-j'}^D$, respectively, for some finite index sets $\mathscr{I},\mathscr{I}' \subset \mathbb{N}$. Then, we obtain

$$\begin{aligned}
&a^{\Gamma,D}(\widehat{v}_j,\widehat{v}_{j'})\\
&= \bigg((\mathrm{Id}\otimes\nabla)\sum_{i\in\mathscr{I}} \alpha_i(\psi_{j,i}\otimes\varphi_{J-j,i}),(\mathrm{Id}\otimes\nabla)\sum_{i'\in\mathscr{I}'} \beta_i(\psi_{j',i'}\otimes\varphi_{J-j',i'})\bigg)_{L^2(\Gamma_{\mathrm{ref}}\times D_{\mathrm{ref}})}\\
&= \sum_{i\in\mathscr{I}}\sum_{i'\in\mathscr{I}'} \alpha_i\beta_{i'}(\psi_{j,i},\psi_{j',i'})_{L^2(\Gamma_{\mathrm{ref}})}(\nabla\varphi_{J-j,i},\nabla\varphi_{J-j',i'})_{L^2(D_{\mathrm{ref}})} = 0
\end{aligned}$$

whenever $j \neq j'$ due to Lemma 2. □

This lemma tells us that, given $\widehat{\mathrm{Cov}[\delta u]}_J^{\Gamma,\Gamma}$, the computation of $\widehat{\mathrm{Cov}[\delta u]}_J^{\Lambda,\Lambda'}$ for $\Lambda,\Lambda' \in \{D,\Gamma\}$ decouples into $J+1$ subproblems. It holds

$$\widehat{\mathrm{Cov}[\delta u]}_J^{\Lambda,\Lambda'} = \sum_{j=0}^{J} \widehat{v}_j,$$

where $\widehat{v}_j \in W_j^\Lambda \otimes V_{J-j}^{\Lambda'}$ is the solution to the following Galerkin formulation:

$$\begin{aligned}
&\text{Find } \widehat{v}_j \in W_j^\Lambda \otimes V_{J-j}^{\Lambda'} \text{ such that}\\
&\quad a^{\Lambda,\Lambda'}(\widehat{v}_j,\widehat{w}) = \mathrm{rhs}^{\Lambda,\Lambda'}(\widehat{w}) \quad \text{for all } \widehat{w} \in W_j^\Lambda \otimes V_{J-j}^{\Lambda'}.
\end{aligned}$$

Herein, we set

$$\mathrm{rhs}^{\Lambda,\Lambda'}(\widehat{w}) := \begin{cases} -a^{D,\Gamma}\big(\widehat{\mathrm{Cov}[\delta u]}_J^{\Gamma,\Gamma},\widehat{w}\big), & \Lambda=D,\ \Lambda'=\Gamma,\\ -a^{\Gamma,D}\big(\widehat{\mathrm{Cov}[\delta u]}_J^{\Gamma,\Gamma},\widehat{w}\big), & \Lambda=\Gamma,\ \Lambda'=D,\\ -a^{D,D}\big(\widehat{\mathrm{Cov}[\delta u]}_J^{D,\Gamma}+\widehat{\mathrm{Cov}[\delta u]}_J^{\Gamma,D}+\widehat{\mathrm{Cov}[\delta u]}_J^{\Gamma,\Gamma},\widehat{w}\big), & \Lambda=D,\ \Lambda'=D. \end{cases} \tag{27}$$

By taking into account the definition of the detail spaces, we end up with the final representation of the solution to (14) in the sparse tensor product space, which is known as the combination technique.

Theorem 5. *Given* $\widehat{\operatorname{Cov}[\delta u]}_J^{\Gamma,\Gamma}$*, the computation of* $\widehat{\operatorname{Cov}[\delta u]}_J^{\Lambda,\Lambda'}$ *for* $\Lambda,\Lambda' \in \{D,\Gamma\}$ *decouples as follows. It holds*

$$\widehat{\operatorname{Cov}[\delta u]}_J^{\Lambda,\Lambda'} = \sum_{j=0}^{J} p_{j,J-j} - p_{j-1,J-j}, \tag{28}$$

where $p_{j,J-j} \in V_j^{\Lambda} \otimes V_{J-j}^{\Lambda'}$ *and* $p_{j-1,J-j} \in V_{j-1}^{\Lambda} \otimes V_{J-j}^{\Lambda'}$ *satisfy the following subproblems which are defined relative to full tensor product spaces:*

$$\begin{aligned} &\textit{Find } p_{j,j'} \in V_j^{\Lambda} \otimes V_{j'}^{\Lambda'} \textit{ such that} \\ &\qquad a^{\Lambda,\Lambda'}(p_{j,j'},q_{j,j'}) = \operatorname{rhs}^{\Lambda,\Lambda'}(q_{j,j'}) \quad \textit{for all } q_{j,j'} \in V_j^{\Lambda} \otimes V_{j'}^{\Lambda'}. \end{aligned}$$

Here, the right hand side is given according to (27)*.*

Proof. The proof of this theorem is a consequence of the previous lemma together with the definition of the detail spaces W_j^{Λ} for $\Lambda \in \{D,\Gamma\}$. □

7 Numerical implementation

Our numerical realization heavily relies on the sparse frame discretization of the model problem as presented in [18]. Nevertheless, in contrast to this work, we make here use of the fact, that we already obtain a sparse tensor product representation of the solution if we have the representations in the spaces $V_j \otimes V_{J-j}$ and $V_{j-1} \otimes V_{J-j}$. This means that it is sufficient to compute the diagonal $(j, J-j)$ for $j = 0,\ldots,J$ and the subdiagonal $(j, J-j-1)$ for $j = 0,\ldots,J-1$ of a sparse frame representation. Moreover, each block in this representation corresponds to the solution of a tensor product subproblem as stated in Theorem 5. The corresponding right hand sides are obtained by means of the matrix-vector product in the frame representation. Therefore, in this context, the combination technique can be considered as an improved solver for the approach presented in [18], which results in a remarkable speed-up. In the sequel, we describe this approach in detail.

We start by discretizing the Dirichlet data. The proceeding is as considered in [17]. Setting $\mathscr{J}_0 := \mathscr{I}_0^{\Gamma}$ and $\mathscr{J}_j := \mathscr{I}_j^{\Gamma} \setminus \mathscr{I}_{j-1}^{\Gamma}$ for $j > 0$, the hierarchical basis in $\operatorname{span}\{\varphi_{j,k} \in V_j : k \in \mathscr{I}_j^{\Gamma}\}$ is given by $\bigcup_{j=0}^{J} \{\varphi_{j,k}\}_{k \in \mathscr{J}_j}$. We replace the normal part of the covariance by its piecewise linear sparse grid interpolant, cf. [4],

$$\langle \mathbf{n}(\mathbf{x}), \operatorname{Cov}[\mathbf{V}]\mathbf{n}(\mathbf{y})\rangle \approx \left(\sum_{j+j' \le J} \sum_{k \in \mathscr{J}_j} \sum_{k' \in \mathscr{J}_{j'}} \gamma_{(j,k),(j',k')} (\varphi_{j,k} \otimes \varphi_{j',k'}) \right)\Bigg|_{\Gamma_{\mathrm{ref}} \times \Gamma_{\mathrm{ref}}}.$$

Thus, the coefficient vector $\mathbf{g}_{j,j'}$ of the Dirichlet data becomes

$$\mathbf{g}_{j,j'} = \sum_{\ell+\ell' \leq J} (\mathbf{B}_{j,\ell} \otimes \mathbf{B}_{j',\ell'})[\gamma_{(\ell,k),(\ell',k')}]_{k\in\mathscr{J}_j,k'\in\mathscr{J}_{j'}}, \tag{29}$$

where the matrices $\mathbf{B}_{j,j'}$ are given by

$$\mathbf{B}_{j,j'} = \left[a^{\Gamma}\left(\frac{\partial\overline{u}}{\partial\mathbf{n}}\varphi_{j,k},\varphi_{j',k'}\right)\right]_{k\in\mathscr{I}_j,k'\in\mathscr{J}_{j'}}, \quad 0 \leq j,j' \leq J.$$

The expression (29) can be evaluated in optimal complexity by applying the matrix-vector multiplication from [43]. Nevertheless, for the sake of an easier implementation, we employ here the matrix-vector multiplication from [22], which is optimal up to logarithmic factors. In particular, by using prolongations and restrictions, the matrices $\mathbf{B}_{j,j'}$ have to be provided only for the case $j = j'$. Thus, having all right hand sides at hand, we can solve next

$$(\mathbf{G}_j \otimes \mathbf{G}_{j'})\mathbf{p}_{j,j'}^{\Gamma,\Gamma} = \mathbf{g}_{j,j'}$$

for all indices satisfying $j' = J - j$ or $j' = J - j - 1$. With these coefficients, we determine the right hand sides for the problems on $D_{\text{ref}} \times \Gamma_{\text{ref}}$ and $\Gamma_{\text{ref}} \times D_{\text{ref}}$ according to

$$\mathbf{f}_{j,j'}^{D,\Gamma} = -\sum_{\ell+\ell'\leq J}(\mathbf{S}_{j,\ell}^{\Gamma}\otimes\mathbf{G}_{j',\ell'})\mathbf{p}_{\ell,\ell'}^{\Gamma,\Gamma} \quad \text{and} \quad \mathbf{f}_{j,j'}^{\Gamma,D} = -\sum_{\ell+\ell'\leq J}(\mathbf{G}_{j,\ell}\otimes\mathbf{S}_{j',\ell'}^{\Gamma})\mathbf{p}_{\ell,\ell'}^{\Gamma,\Gamma},$$

where the matrices $\mathbf{S}_{j,j'}^{\Gamma}$ and $\mathbf{G}_{j,j'}$ are given by

$$\left.\begin{aligned}\mathbf{S}_{j,j'}^{\Gamma} &= \left[a^{D}(\varphi_{j',\ell},\varphi_{j,k})\right]_{k\in\mathscr{I}_j^{D},\ell\in\mathscr{I}_{j'}^{\Gamma}}\\ \mathbf{G}_{j,j'} &= \left[a^{\Gamma}(\varphi_{j',\ell},\varphi_{j,k})\right]_{k\in\mathscr{I}_j^{\Gamma},\ell\in\mathscr{I}_{j'}^{\Gamma}}\end{aligned}\right\} \quad 0 \leq j,j' \leq J.$$

Notice that we have $\mathbf{S}_{j,j}^{\Gamma} = \mathbf{S}_j^{\Gamma}$ and $\mathbf{G}_{j,j} = \mathbf{G}_j$, cf. (18) and (19). Now, we can solve

$$(\mathbf{S}_j^{D}\otimes\mathbf{G}_{j'})\mathbf{p}_{j,j'}^{D,\Gamma} = \mathbf{f}_{j,j'}^{D,\Gamma} \quad \text{and} \quad (\mathbf{G}_j\otimes\mathbf{S}_{j'}^{D})\mathbf{p}_{j,j'}^{\Gamma,D} = \mathbf{f}_{j,j'}^{\Gamma,D}$$

for all indices satisfying $j' = J - j$ or $j' = J - j - 1$.

From the solutions $\mathbf{p}_{j,j'}^{D,\Gamma}$ and $\mathbf{p}_{j,j'}^{\Gamma,D}$, we can finally determine the right hand sides

$$\mathbf{f}_{j,j'}^{D,D} = -\sum_{\ell+\ell'\leq J}(\mathbf{S}_{j,\ell}^{\Gamma}\otimes\mathbf{S}_{j',\ell'}^{\Gamma})\mathbf{p}_{\ell,\ell'}^{\Gamma,\Gamma} + (\mathbf{S}_{j,\ell}^{D}\otimes\mathbf{S}_{j',\ell'}^{\Gamma})\mathbf{p}_{\ell,\ell'}^{D,\Gamma} + (\mathbf{S}_{j,\ell}^{\Gamma}\otimes\mathbf{S}_{j',\ell'}^{D})\mathbf{p}_{\ell,\ell'}^{\Gamma,D},$$

where the matrices $\mathbf{S}_{j,j'}^{D}$ are given by

$$\mathbf{S}_{j,j'}^{D} = \left[a^{D}(\varphi_{j',\ell},\varphi_{j,k})\right]_{k\in\mathscr{I}_j^{D},\ell\in\mathscr{I}_{j'}^{D}}, \quad 0 \leq j,j' \leq J.$$

It remains to compute the solutions to

$$(\mathbf{S}_j^D \otimes \mathbf{S}_{j'}^D)\mathbf{p}_{j,j'}^{D,D} = \mathbf{f}_{j,j'}^{D,D}$$

for all indices satisfying $j' = J - j$ or $j' = J - j - 1$.

Appropriate tensorization of the BPX-preconditioner, cf. [3], yields an asymptotically optimal preconditioning for each of the preceding linear systems, cf. [22, Theorem 7]. Consequently, the computational complexity for their solution is linear, which means it is of the order $\mathscr{O}(2^{(j+j')d})$. Moreover, the right hand sides $\mathbf{f}_{j,j'}^{\Lambda,\Lambda'}$ for $\Lambda,\Lambda' \in \{D,\Gamma\}$ can be computed by the algorithm proposed in [43] with an effort of $\mathscr{O}(J2^{dJ})$. We thus obtain the following result:

Theorem 6. *The cost of computing the Galerkin solution* $\widehat{\mathrm{Cor}[\delta u]}_J$ *via the expansion* (28) *is of optimal order* $\mathscr{O}(J2^{dJ})$.

Proof. For each $0 \le j \le J$ and $\Lambda,\Lambda' \in \{D,\Gamma\}$, the cost to determine $p_{j,J-j}^{\Lambda,\Lambda'}$ and $p_{j-1,J-j}^{\Lambda,\Lambda'}$ is of order $\mathscr{O}(2^{dJ})$. Summing over j yields immediately the assertion. □

8 Numerical results

To demonstrate the described method, we consider an analytical example on the one hand and a stochastic example on the other hand. In the latter, for a given random domain perturbation described by the random vector field $\mathbf{V}$, we compute the approximate mean $\overline{u}$ in accordance with (10) and the approximate covariance $\mathrm{Cov}[\delta u]$ in accordance with (14). All computations are carried out on a computing server with two Intel(R) Xeon(R) X5550 CPUs with a clock rate of 2.67GHz and 48GB of main memory. The computations have been performed single-threaded, i.e. on a single core.

8.1 An analytical example

In this analytical example, we want to validate the convergence rates of the combination technique for the sparse tensor product solution of tensor product Dirichlet problems. To that end, consider the tensor product boundary value problem

$$\begin{aligned}
(\Delta \otimes \Delta)u &= 0 && \text{in } D_{\mathrm{ref}} \times D_{\mathrm{ref}},\\
(\Delta \otimes \gamma_0^{\mathrm{int}})u &= 0 && \text{in } D_{\mathrm{ref}} \times \Gamma_{\mathrm{ref}},\\
(\gamma_0^{\mathrm{int}} \otimes \Delta)u &= 0 && \text{in } \Gamma_{\mathrm{ref}} \times D_{\mathrm{ref}},\\
(\gamma_0^{\mathrm{int}} \otimes \gamma_0^{\mathrm{int}})u &= g_1 \otimes g_2 && \text{on } \Gamma_{\mathrm{ref}} \times \Gamma_{\mathrm{ref}},
\end{aligned} \tag{30}$$

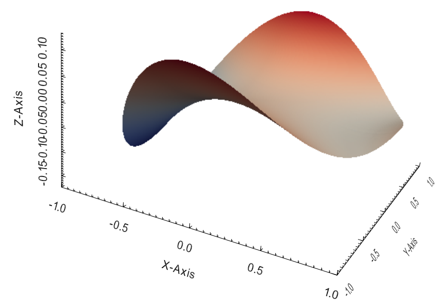

Fig. 1 Trace $u|_{\mathbf{x}=\mathbf{y}}$ of the solution u to (30).

where $D_{\text{ref}} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$ is the two-dimensional unit disk. We choose g_1 and g_2 to be the traces of harmonic functions. More precisely, we set

$$g_1(\mathbf{x}) = x_1^2 - x_2^2 \quad \text{and} \quad g_2(\mathbf{x}) = -\frac{1}{2\pi} \log\left(\sqrt{(x_1-2)^2 + (x_2-2)^2}\right) \quad \text{for } \mathbf{x} \in \Gamma_{\text{ref}}.$$

Then, the solution u is simply given by the product

$$u(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}(x_1^2 - x_2^2)\log\left(\sqrt{(y_1-2)^2 + (y_2-2)^2}\right).$$

A visualization of the trace $u|_{\mathbf{x}=\mathbf{y}}$ of this function is found in Figure 1.

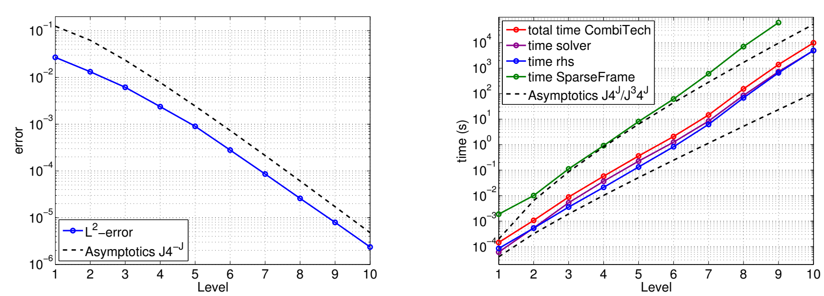

Fig. 2 Relative L^2-error (left) and computation times (right) of the combination technique in case of the analytic example.

The convergence plot on the left of Figure 2 shows that the relative L^2-error, indicated by the blue line, exhibits almost the convergence rate predicted in Corollary 1, indicated by the black dashed line. On level 10, there are about 2.1 million degrees of freedom in each variable, which is, up to a logarithmic factor, the number of degrees of freedom appearing in the discretization by the combination technique.

Vice versa, a full tensor product discretization on this level would result in about $4.4 \cdot 10^{12}$ degrees of freedom, which is no more feasible.

The plot on the right hand side of Figure 2 depicts the related computational times. For comparison, we have added here the computational times for the sparse tensor product frame discretization from [18]. The related curve is indicated in green. The computational time consumed by the combination technique is represented by the red curve. Notice that we have set up both methods such that they provide similar accuracies for the approximation of the solution. From level 3 to 9, the combination technique is in average a factor 30 faster than the frame discretization, where the speed-up is growing when the level increases. Nevertheless, it seems that, from level 7 on, both methods do not achieve the theoretical rate of $J^3 4^J$ anymore.

We present in the plot on the right hand side of Figure 2 also the time consumed for exclusively computing the appropriate right hand sides for the combination technique, indicated by the blue line. As can be seen, on the higher levels, this computation takes nearly half of the total computational time. A potential improvement could thus be made by using the matrix-vector product from [43]. Finally, we have plotted the time which is needed for exclusively solving the linear systems by the tensor product solver. Here, it seems that we have the optimal behavior of order $J4^J$ up to level 7. Then, also this rate deteriorates.

8.2 The Poisson equation on the random unit disc

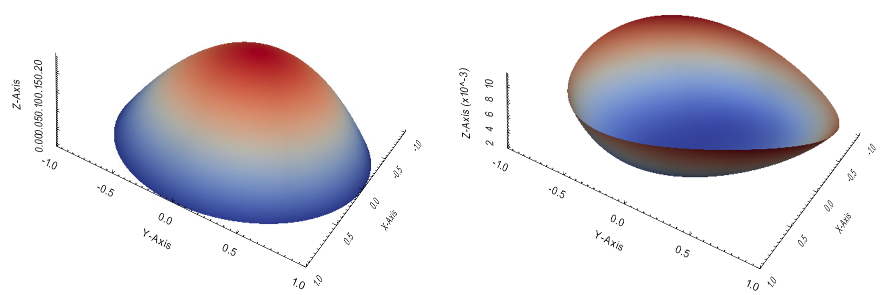

Fig. 3 Solution $\overline{u}$ (left) and variance $\mathbb{V}[\delta u]$ (right) on the unit disc.

For this example, we consider $D_{\text{ref}} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$ as reference domain and the load is set to $f(\mathbf{x}) \equiv 1$. The random vector field $\mathbf{V}$ is provided by its mean $\mathbb{E}[\mathbf{V}](\mathbf{x}) = \mathbf{x}$ and its covariance function

$$\operatorname{Cov}[\mathbf{V}](\mathbf{x},\mathbf{y}) = \frac{\varepsilon^2}{125} \begin{bmatrix} 5\exp(-4\|\mathbf{x}-\mathbf{y}\|_2^2) & \exp(-0.1\|2\mathbf{x}-\mathbf{y}\|_2^2) \\ \exp(-0.1\|\mathbf{x}-2\mathbf{y}\|_2^2) & 5\exp(-\|\mathbf{x}-\mathbf{y}\|_2^2) \end{bmatrix}.$$

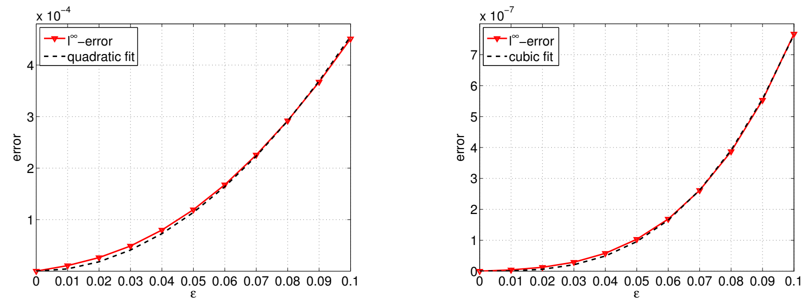

Fig. 4 Error in $\mathbb{E}[u]$ (left) and $\mathbb{V}[u]$ (right) for increasing values of ε on K.

In Figure 3, a visualization of the solution $\overline{u}$ to (10) (left) and the variance $\mathbb{V}[\delta u]$ of the solution to (14) (right) is depicted. In order to validate the computational method, we consider a reference solution computed with a quasi-Monte Carlo method based on Halton points. To that end, we have solved the Poisson equation on 10^4 realizations of the random parameter on level $J = 7$ (this corresponds to 65536 finite elements). The solutions obtained have then been interpolated on a mesh on level $J = 5$ (this corresponds to 4096 finite elements) for the compactum $K = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 \leqslant 0.8\}$. For the combination technique, we set $J = 7$ for the computation of the mean and $J = 9$ for the computation of the variance. The related error plots for combination technique with respect to different values of ε are shown in Figure 4, where we used the ℓ^∞-norm to measure the error. As can be seen, the error in the mean, found on the left hand side of the figure, exhibits exactly the expected quadratic behavior, whereas the error in the variance, found on the right hand side of the figure, shows exactly a cubic rate.

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