Invariants of binary forms

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Acknowledgments

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Chapter 1

Introduction

We work over the field $\mathbb{C}$ of complex numbers. We denote by SL_2 the group of complex 2×2-matrices with determinant 1. Let $V_n = \mathbb{C}[x,y]_n$ be the SL_2-module of binary forms (homogeneous polynomials in x and y of degree n), on which SL_2 acts via

$$g \cdot f(v) = f(g^{-1}v),$$

for $g \in \mathrm{SL}_2$, $f \in \mathbb{C}[x,y]$ and $v \in \mathbb{C}^2$. The algebra of polynomial functions on V_n, denoted by $\mathcal{O}(V_n)$, is a polynomial ring in $n+1$ variables. The group SL_2 acts on $\mathcal{O}(V_n)$ via the action

$$g \cdot j(f) = j(g^{-1} \cdot f),$$

for $g \in \mathrm{SL}_2$, $j \in \mathcal{O}(V_n)$ and $f \in V_n$. An *invariant* of V_n is an element $j \in \mathcal{O}(V_n)$ such that $g \cdot j = j$ for all $g \in \mathrm{SL}_2$. The set of invariants of V_n forms the *algebra of invariants* $I := \mathcal{O}(V_n)^{\mathrm{SL}_2}$.

> "The theory of invariants originated in England about the middle of the nineteenth century as the genuine analytic instrument for describing configurations and their inner geometric relations in projective geometry. The functions and algebraic relations expressing them in terms of projective coordinates are to be invariant under all homogeneous linear transformations. "([Wey46, page 27])

In the nineteenth century mathematicians studied invariants of binary forms motivated by the philosophy that 'any' property of polynomials unaffected by linear transformations can be expressed by the vanishing of invariants. The polynomial $ax^2 + 2bx + c$ for example has a double root if and only if the invariant $b^2 - ac$ of $ax^2 + 2bxy + cy^2$ vanishes. Consider, as another example, the non-zero quartic $q(x,y) = ax^4 + bx^3y + \ldots + ey^4$; the solutions of the equation $q(x,y) = 0$ correspond to 4 points on the projective line. These four points form a harmonic division if and only if the invariant

$$j_3 = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

of the quartic q vanishes (cf. Dixmier [Dix90], p.42). The invariants of binary forms are related to the study of algebraic equations as well. For example, the binary form $q(x, y)$ of degree 5 has an invariant of degree 18 with a nice property: if this invariant vanishes, then the equation $q(x, 1) = 0$ can be solved by radicals (cf. Dixmier [Dix90], p.43).

Early in the 1800s it was known that the invariants of V_2 are spanned by the discriminant $b^2 - ac$ of $ax^2 + 2bxy + cy^2$, and that the invariants of V_3 are generated by the discriminant $(ad - bc)^2 - 4(ac - b^2)(bd - c^2)$ of $ax^3 + 3bx^2y + 3cxy^2 + dy^3$. About the binary forms $ax^4 + 4bx^3y + \ldots + ey^4$ of degree 4 it was known that the invariant j_3 defined above and $j_2 = ae - 4bd + 3c^2$ are algebraically independent and generate the invariants of V_4. The invariants j_2 and j_3 were discovered in the period 1840-1850 by Boole, Cayley, Eisenstein (cf. Dixmier [Dix90], p.41). Regarding the invariants of V_5, Hermite ([Her54]) proved in 1854 that they were generated by 4 irreducible invariants of degrees 4, 8, 12, and 18. Before that, only the invariants of degrees 4, 8, and 12 had been known, and for a long time people believed that all invariants of V_5 had degrees divisible by 4 (cf. Dixmier [Dix90], p.41).

Cayley [Cay56] claimed in 1856 that that the algebra I of invariants of V_n had no finite bases for the cases $n = 7$ and $n = 8$. However, twelve years later, in 1868, Gordan [Gor68] proved the contrary: he showed that I has a finite basis for all n.

> "Im 146sten Bande der Philosophical Transactions pag. 101 hat Herr Cayley sich mit der Frage beschäftigt, ob alle aus einer binären Form entstehenden Covarianten und Invarianten als ganze Functionen einer begrenzten Anzahl von Former mit numerischen Coefficienten darstellbar seien; er hat gezeigt, dass bei Formen zweiten, dritten und vierten Grades sich alles in der verlangten Weise ausdrücken lässt. Im Folgenden gebe ich für binäre Formen n^{ten} Grades ein endliches System von Covarianten und Invarianten an, von denen ich zeige, dass und wie alle aus der Form abgeleitete Formen sich als ganze rationale Functionen derselben mit numerischen Coefficienten darstellen lassen. Dieses für den allgemeinen Fall gegebene System ist immer zu gross und lässt sich in jedem besonderen Falle reduciren; für Formen fünften und sechsten Grades habe ich auch diese Reduction ausgeführt und ein möglichst kleines System von Grundformen geliefert." ([Gor68])

The study of invariants of binary forms was an important part of a new discipline, "die neue Algebra (the new Algebra)", as Clebsch named it in 1872. At that time Clebsch wrote that

> "die fundamentalen Untersuchungen von Gordan über die Endlichkeit der Formensysteme [...] eine Perspective in eine neue Classe tiefer und wichtiger Forschungen erröffnet".[Cle72]

About twenty years later, in 1890, Hilbert [Hil90] generalised the result of Gordan to a system of several homogeneous forms in a finite number of variables.

His proof was non-constructive and did not provide any tools to determine such finite bases. Hilbert 'only' proved that these finite bases existed, to which Gordan reacted with the famous exclamation ([Rei96]):

> *"Das ist nicht Mathematik. Das ist Theologie."*

Hilbert [Hil93] returned to the problem and in 1893 gave a proof which was this time constructive. Eventually Gordan appreciated Hilbert's new ideas, remarking

> *"I have convinced myself that Theology also has its merits."*[Rei96]

With his article from 1893 Hilbert opened a new chapter in mathematics: the article contains famous results such as "Hilbert's basis theorem" and "Hilbert's Nullstellensatz" (as they are known nowadays), without which it is hard to imagine the mathematics of today.

It is known, hence, since the nineteenth century, that the algebra I of invariants of binary forms of degree n is finitely generated over $\mathbb{C}$, i.e. there exist finitely many invariants $j_1, \ldots, j_r \in I$, such that $I = \mathbb{C}[j_1, \ldots, j_r]$. Nevertheless, finding the generators of I is, in general, a difficult problem. Two methods were developed in the nineteenth century: the *symbolic* method, and the *enumerative* method.

The symbolic method was developed by Aronhold and Clebsch in the middle of the nineteenth century. Called by Hermann Weyl "the great war-horse of nineteenth century invariant theory", the symbolic calculus allows the reduction of the computations with binary forms of degree n to the special cases of the nth power of a linear form $(\alpha_1 x + \alpha_2 y)^n$. The classics proved that the invariants of binary forms have symbolic representations as products of factors of type $[\alpha\beta]$, where $[\alpha\beta]$ stands for the determinant $\alpha_1\beta_2 - \alpha_2\beta_1$. The manipulation of invariants got simplified by representing them in succinct symbolic expressions. Kung & Rota [KR84] gave in 1984 "a rigorous and yet manageable account of the umbral or symbolic calculus" that was performed in the nineteenth century (see also Chap. 2.2). With the help of the symbolic method, Gordan [Gor68] proved, in 1868, that the covariants of a binary form f of degree n are generated by a finite number of iterated transvectants $f, (f, f)_d, (f, (f, f)_d)_e, \ldots$. For a modern interpretation of Gordan's algorithm we refer to Weyman [Wey93]. Gordan's proof was constructive and provided sets of generators for the invariants and covariants of V_6 ([Gor87]). Following Gordan's method, von Gall [Gal88] found in 1888 a set of generating covariants of V_7, but this set was not minimal. In two papers published in 1880, von Gall [Gal80] computed a set of generating covariants of V_8, but this set was again not minimal (he did found the correct number of the generating invariants of V_8). No sets of generating invariants of covariants of V_n for $n \geq 9$ were computed in the nineteenth century using Gordan's method.

The *enumerative method* was developed by Sylvester in the nineteenth century and aimed to find lower bounds for the number of generators of the invariants of binary forms. This method used the Poincaré series of the algebra I of invariants of binary forms and *tamisage* for finding these bounds. Tamisage meant

the following: suppose that the Poincaré series of I is $P(t) = 1 + a_2 t^2 + a_3 t^3 + \ldots$. We look for the smallest i such that the coefficient a_i is nonzero, as long as such an i exists. If $a_i > 0$, we put $m_i := a_i$, replace $P(t)$ by $P(t)(1 - t^i)^{m_i}$, and repeat the procedure. If $a_i < 0$, we stop. Undefined m_i are considered equal to 0. Sylvester claimed that the number of generators of I is at least $\sum_i m_i$ (more precisely, that the number of generators of I_i is at least m_i). For illustration, the Poincaré series of the algebra of invariants of binary forms of degree 10 is:

$$P(t) = 1 + t^2 + 2t^4 + 6t^6 + 12t^8 + 5t^9 + 24t^{10} + 13t^{11} + 52t^{12} + 33t^{13} +$$
$$97t^{14} + 80t^{15} + 177t^{16} + 160t^{17} + 319t^{18} + 301t^{19} + \cdots \text{(see Chap. 4.9)}$$

Using tamisage we obtain the following bounds m_i on the number of generators of invariants of degree i of V_{10} (the last row contains the actual number d_i of generators of degree i, obtained in Chap. 4.9):

i	2	4	6	8	9	10	11	12	13	14	15	16	17	18	19	21
m_i	1	1	4	5	5	8	8	12	15	13	19	5	3	0	0	0
d_i	1	1	4	5	5	8	8	12	15	13	19	5	5	1	2	2

Sylvester formulated a *postulate* as well, which stated that in a degree i one gets either a new generator of I, or *new* relations among the already found generators, but not both. This was not true: Hammond [Ham82] found a counterexample in 1882. Assisted by Franklin, Sylvester [Sy79b] used tamisage and his postulate for finding the number of generators of invariants of binary forms of degree up to 10. His predictions regarding the invariants of V_n were correct for $n \leq 6$ and for $n = 8$. For the remaining cases, Sylvester's numbers were correct up to degree 18 for $n = 7$ and for $n = 9$, and up to degree 16 for $n = 10$.

Both techniques were not powerful enough for solving complicated cases. Grace and Young noted in 1903:

> "Theoretically then Gordan's process gives an upper limit to the irreducible systems. The enumerative method [...] gives a lower limit to the system and when the two methods give the same result the irreducible set has been obtained. The results even when identical have to be received with some caution on account of the enormous labour involved. It may be recalled in fact in connection with the simultaneous system of a cubic and quartic (Gundelfinger, Math.Ann. Bd.iv.) that the two results originally agreed, but a revision of the generating function led to a reduction of the lower limit which it theoretically gives, and afterwards two forms included in the irreducible system as derived by the methods of Gordan and Clebsch were found to be reducible. The complete systems for the binary forms up to the octavic may be considered as accurately determined by the two methods combined."([GY03, page 131])

In reality, the cases $n = 7$ and $n = 8$ were surrounded by a shade of doubt until late in the 20th century. Gall's results [Gal88] regarding the generating invariants of V_7 were corrected by Dixmier & Lazard [DL86] in 1986. Shioda [Shi67]

proved in 1967 that the algebra of invariants of V_8 was generated by the nine invariants found by Gall in 1880 and also explicitly calculated the relations between these nine invariants. Regarding the covariants of V_7, Cröni [Crö02] and Bedratyuk [Bed09] indicated that Gall [Gal88] had six superfluous generatoring covariants in his set. However, they didn't prove that Gall's set was indeed a generating set for the covariants of V_7 (in particular they did not give an independent proof of the fact that there were no generators in higher degrees than the one indicated by Gall, but instead relied on Gall's results). Bedratyuk & Bedratiuk [BB08] showed in 2008 that the set of generating covariants of V_8 found by Gall [Gal80] contained a superfluous generator, confirming a result of Sylvester [Sy79b]. Again, they did not prove that no generators will appear in degrees higher than the one indicated by Gall, but relied on this information. In Chap. 5.7 and 5.8 we find sets of generating covariants of V_7, respectively of V_8, independently of von Gall's work. As for the case of the simultaneous system of a cubic and quartic, this was settled in 2012 by Brouwer and me [BP12] (see also Chap. 5.19).

Earlier work on the cases $n = 9$ and $n = 10$ was done by Cröni [Crö02], and, respectively by Hagedorn (unpublished). These last two cases were completed in 2010, when Brouwer and I [BP10a, BP10b] found sets of generators of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ for $n = 9$ and $n = 10$ (see also Chap. 4.8 and 4.9 in this thesis). The existence of a generator of degree 22 in the case $n = 9$ and the existence of a generator of degree 21 in the case $n = 10$ were new. No sets of generators of $\mathcal{O}(V_n)^{\mathrm{SL}_2}$ are known for $n \geq 11$. The bigger n is, the more difficult it is to find the generating invariants of V_n. The computations required for finding such sets of generators for the cases $n = 11$ and $n = 12$ are still too large.

An upper bound on the *degree* of the generating invariants of V_n is known from the nineteenth century, found by Camille Jordan [Jor76, Jor79]. He proved that the degree of the generating invariants of V_n was $\leq n^6$. About a century later, Popov [Pop81, Pop82] generalised this result and determined a bound on the degree of the generating invariants of a G-module, where G was any semi-simple group. However, applied to the particular case of the binary forms, Popov's result gives a weaker bound, compared to the one found by Jordan. In 2001 Derksen [Der01] improved Popov's result, but again, applied to the particular case of the binary forms, Derksen's bound was weaker than Jordan's bound.

Another direction that the classics followed in the nineteenth century was to find the generators of the invariants and of the covariants of several binary forms $V_{n_1} \oplus \ldots \oplus V_{n_p}$, with $p \geq 2$ and $n_i \geq 1$. Table 1.1 contains results of computations made in the nineteenth century, for some particular SL_2-modules (we use the notation mV_n for the direct sum $\bigoplus_{i=1}^{m} V_n$ of m copies of V_n). The references [Bes69, Ell95, Gor69, Gor87, GY03, Per87] give sets of generating invariants of $mV_1 \oplus nV_2$ in few particular cases, with small m and n. Gordan and Grace & Young ([Gor75, GY03]) gave an algorithm for computing the generators of the covariants of $V_2 \oplus V$. Gordan ([Gor69, Gor87]) gave good estimates on the number of generators of the invariants of nV_1 and of $V_1 \oplus nV_2$, with $n \geq 2$. Peano ([Pea82]) showed that the covariants of nV_3, with $n \geq 2$, are generated by those

module	# generators of invariants	# generators of covariants
V_3	1 ([Gor87])	4 ([Gor87])
V_4	2 ([Gor87])	5 ([Gor87])
V_5	4 ([Gor87])	23 ([Gor87])
V_6	5 ([Gor87])	26 ([Gor87])
V_7	$\underline{33}$ ([Gal88]), 30 ([DL86])	$\underline{153}$ ([Gal88]), $\underline{124}$ ([Sy79b]), 147 ([Crö02, Bed09], Chap. 5.7)
V_8	$\underline{12}$ ([Gal80]), 9 ([Gal80, Shi67])	$\underline{96},\underline{67},\underline{70}$ ([Gal80]) 69 ([Sy79b, BB08], Chap. 5.8)
$V_2 \oplus V_3$	5 ([Bes69, Gor87])	15 ([Bes69, Gor87])
$V_2 \oplus V_4$	6 ([GY03])	18 ([GY03])
$V_2 \oplus V_5$	29 ([Win80])	$\underline{94}$ ([Win80]), 92 (Chap. 5.16)
$V_2 \oplus V_6$	27 ([Gal74])	99 ([Gal74])
$2V_3$	7 ([Gor87, Pea82])	26 ([Gor87, Pea82])
$V_3 \oplus V_4$	20 ([Gun69])	$\underline{64}$ ([Gun69]), $\underline{61}$ ([Sy78b, Sy78c, Sy78d]), 63 ([BP12])
$2V_4$	8 ([You99])	28 ([You99])
$3V_3$	28 ([Gal94])	$\underline{98}$ ([Gal94]), 97 ([Sin05])
$3V_4$	25 ([You99])	103 ([You99])
$4V_4$	80 ([You99])	305 ([You99])
nV_1	$\binom{n}{2}$ ([Gor69])	$\binom{n+1}{2}$ ([Gor69])
nV_2	$\binom{n+1}{2} + \binom{n}{3}$ ([Gor87])	$n(n+1) + \binom{n}{3}$ ([Gor87])

Table 1.1: Cases treated in the 19th century (the underlined entries are results that in the first place turned out to be false and were later corrected)

of degree ≤ 6 and order ≤ 4. Young ([You99]) showed that the covariants of nV_4, with $n \geq 5$, are generated by those of degree ≤ 6 and order ≤ 6. These last results were confirmed and proved with modern methods by Kraft & Weyman ([KW99]) in 1999. They are extended in Chap. 5 of this thesis: we give sets of generating invariants for $mV_1 \oplus nV_2$, $mV_1 \oplus nV_3$, and $mV_1 \oplus nV_4$, with $m \geq 2$.

The difficulty of finding generators of the algebra I of invariants is captured by the *homological dimension* hd I of I. If r is the minimal number of generators of I, and m is the size of a system of parameters of I (set of algebraically

independent elements $P_1, \ldots, P_m \in I$ such that I is integral over $\mathbb{C}[P_1, \ldots, P_m]$), then m equals $n - 2$ for $n \geq 3$, and the homological dimension hd I equals $r - m$ ([Pop83, Corollary 1]). Popov [Pop83] classified in 1983 all the SL$_2$-modules $V_{n_1} \oplus \ldots \oplus V_{n_p}$ with hd $I \leq 10$ for a single binary form ($p = 1$) and hd $I \leq 3$ for several binary forms ($p > 1$). It turned out that all these cases were known classically, as cases in which one could easily find minimal sets of generating invariants. Brouwer and I [BP11] extended Popov's classification and determined for $p = 1$ the cases with hd $I \leq 100$ and for $p > 1$ the cases with hd $I \leq 15$ (see also Chap. 6 in this thesis). In the case of a single binary form, the homological dimension hd I rapidly increases if the degree is ≥ 9. The bigger the homological dimension hd I is, the harder it is to find generating invariants of I. The following table contains, for illustration, the values of hd I for binary forms of degree less than 14:

degree	1	2	3	4	5	6	7	8	9	10	11	12	13
hd I	0	0	0	0	1	1	25	3	85	98	≥ 149	≥ 103	≥ 491
r	1	1	1	2	4	5	30	9	92	106	≥ 158	≥ 113	≥ 502

With his book published in 1939 Weyl [Wey46] aimed to give "a modern introduction to the theory of invariants", translating the problem of finding the generating invariants of binary forms into the language of representation theory and the study of actions of the semisimple groups. He thought it was "high time for a rejuvenation of the classic invariant theory, which has fallen into an almost petrified state"([Wey46]). Parshall [Par90] talks in 1990 about a myth saying that

> "the death certificate of invariant theory effectively reads 15 February 1890. On that date, the twenty-eight-years-old David Hilbert signed off his paper Über die Theorie der algebraischen Formen [Hil90] and presented his proof of the so-called finite basis theorem to the readership of the mathematische Annalen, a theorem and proof that killed an entire area. Two and a half years later, he completed yet another invariant-theoretic work, entitled Über die vollen Invariantensysteme [Hil93] and put an end to any lingering hopes of the theory's resurrection. Thus, after fifty years of vigorous life, one of the nineteenth century's major areas of mathematical research abruptly ceased to exist."([Par90])

This myth originated probably from Weyl's words on Hilberts papers:

> "His papers (1890/92) mark a turning point in the history of invariant theory. He solves the main problems and thus almost kills the whole subject." ([Wey46, page 27])

But we should note the words "turning point" and "almost" in Weyl's quote, as Parshall [Par90] suggested. In the 1960's Mumford [Mum65] translated problems of invariant theory into the language of algebraic geometry. One of his

key insights was that one could analyse the geometry of group actions without actually knowing the invariants. For instance, he proved that a vector in a representation cannot be distinguished from zero by means of invariants if and only if there is a one-parameter subgroup sending the vector arbitrarily close to zero. The remarkable thing here is that a purely algebraic property (namely that all non constant homogeneous invariants vanish on a vector) can be checked in a purely geometric fashion (involving orbits of the vector under one-parameter subgroups). As a consequence, we know for instance that all invariants of binary forms of degree n vanish on the forms having a root of multiplicity $> \frac{n}{2}$ without any a priori information on the invariants themselves. Mumford's result generalises Hilbert's criterion mentioned in Chap. 2.7.1, which will be used throughout this thesis.

In the second half of the 20th century new techniques in commutative algebra became available that can be applied to invariant theory. These techniques are both of a theoretical nature, such as Hochster& Roberts's result that invariant rings of reductive group representations are Cohen Macaulay [HR74], and of an algorithmic nature, such as the use of Gröbner bases in invariant theory [DK00] or the computation of Poincaré series of invariant rings [Spr77, Bri82, Bro94]. Such algebraic and algorithmic techniques are the starting point of this thesis.

This thesis is organised as follows.

In Chapter 2 we introduce the definitions and notations that will be used in this thesis. In Chapter 3 we present the computational methods that we use for finding the generating invariants of SL_2-modules. In Chapter 4 we find the basic invariants of V_n for $n \in \{2, 3, \ldots, 10\}$, and give explicit systems of parameters in all these cases (Chap. 4.8 and 4.9 are joint work with Brouwer [BP10a, BP10b]). In Chapter 5 we review classical results regarding the invariants of $V_{n_1} \oplus \ldots \oplus V_{n_p}$, with $p \geq 2$. We correct a result of Winter [Win80] on the generating covariants of $V_2 \oplus V_5$ (see Chapter 5.16) and results of Gundelfinger [Gun69] and Sylvester [Sy78b, Sy78c, Sy78d] on the generating covariants of $V_3 \oplus V_4$ (see Chap. 5.19, joint work with Brouwer [BP12]). In Chapter 6 we classify the modules $V_{n_1} \oplus \ldots \oplus V_{n_p}$ whose algebras of invariants have a homological dimension ≤ 15, extending a result of Popov [Pop83] (joint work with Brouwer [BP12]).

Chapter 2

Invariant Theory

2.1 Definitions and notation

Recall from the introduction the definition of the group SL_2. Denote by V_n the space of binary forms of degree n.

Consider a rational, finite-dimensional SL_2-module V. Then there exist $(n_1, \ldots, n_p)$ such that $V \simeq V_{n_1} \oplus \ldots \oplus V_{n_p}$ as SL_2-modules (cf. [Spr77, 3.2.2]). The group SL_2 acts on the algebra of polynomial functions on $V_{n_1} \oplus \ldots \oplus V_{n_p}$ via
$$g \cdot j(f_1, \ldots, f_p) = j(g^{-1} \cdot f_1, \ldots, g^{-1} \cdot f_p),$$
where $g \in \mathrm{SL}_2$, $j \in \mathcal{O}(V_{n_1} \oplus \ldots \oplus V_{n_p})$ and $(f_1, \ldots, f_p) \in V_{n_1} \oplus \ldots \oplus V_{n_p}$.

Definition 2.1.1. Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. An *invariant* of V is an element j in the algebra $\mathcal{O}(V)$ of polynomial functions on V such that $g \cdot j = j$ for all $g \in \mathrm{SL}_2$. The set of invariants of V is denoted $\mathcal{O}(V)^{\mathrm{SL}_2}$.

Example 2.1.1. Consider the binary form $f = a_0 x^2 + 2a_1 xy + a_2 y^2 \in V_2$ of degree 2. The polynomial $j(f) = a_1^2 - a_0 a_2$ is an invariant of V_2.
Indeed, for $g = \left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix} \right) \in \mathrm{SL}_2$, $mq - np = 1$, we have $g \cdot j(f) = j(g^{-1} \cdot f)$ and

$$g^{-1} \cdot f = a_0(mx + ny)^2 + 2a_1(mx + ny)(px + qy) + a_2(px + qy)^2 =$$
$$= (a_0 m^2 + 2a_1 mp + a_2 p^2)x^2 + 2(a_0 mn + a_1 np + a_1 mq + a_2 pq)xy +$$
$$+ (a_0 n^2 + 2a_1 nq + a_2 q^2)y^2.$$

It follows that

$$j(g^{-1} \cdot f) = (a_0 mn + a_1 np + a_1 mq + a_2 pq)^2 - (a_0 m^2 + 2a_1 mp + a_2 p^2)(a_0 n^2 +$$
$$+ 2a_1 nq + a_2 q^2) = (a_1^2 - a_0 a_2)(mq - np)^2 = a_1^2 - a_0 a_2 = j(f),$$

hence $g \cdot j(f) = j(f)$ for all $g \in \mathrm{SL}_2$.
The invariant j has a geometric interpretation: $j(f) = 0$ if and only if f has a double root.

Example 2.1.2. Let $f, \ell \in V_2 \oplus V_1$, with $f = a_0 x^2 + 2a_1 xy + a_2 y^2 \in V_2$ and $\ell = b_0 x + b_1 y \in V_1$. The polynomial $j(f, \ell) = a_0 b_1^2 - 2a_1 b_0 b_1 + a_2 b_0^2$ is an invariant of $V_2 \oplus V_1$. Indeed, for $g = \left(\begin{smallmatrix} m & n \\ p & q \end{smallmatrix} \right) \in \mathrm{SL}_2$, $mq - np = 1$, we have $g \cdot j(f, \ell) = j(g^{-1} \cdot f, g^{-1} \cdot \ell)$ and

$$g^{-1} \cdot f = (a_0 m^2 + 2a_1 mp + a_2 p^2) x^2 + 2(a_0 mn + a_1 np + a_1 mq + a_2 pq) xy +$$
$$+ (a_0 n^2 + 2a_1 nq + a_2 q^2) y^2,$$
$$g^{-1} \cdot \ell = (b_0 m + b_1 p) x + (b_0 n + b_1 q) y.$$

It follows that

$$j(g^{-1} \cdot f, g^{-1} \cdot \ell) = (a_0 m^2 + 2a_1 mp + a_2 p^2)(b_0 n + b_1 q)^2 -$$
$$- 2(a_0 mn + a_1 np + a_1 mq + a_2 pq)(b_0 m + b_1 p)(b_0 n + b_1 q) +$$
$$+ (a_0 n^2 + 2a_1 nq + a_2 q^2)(b_0 m + b_1 p)^2 =$$
$$= (a_0 b_1^2 - 2a_1 b_0 b_1 + a_2 b_0^2)(mq - np)^2 = j(f, \ell),$$

hence $g \cdot j(f, \ell) = j(f, \ell)$ for all $g \in \mathrm{SL}_2$.

The invariant j has a geometric interpretation: $j(f, \ell) = 0$ if and only if ℓ and f have a common root.

Definition 2.1.2. Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. A *covariant* of V of order m and degree d of V is an SL_2-equivariant polynomial map $\phi : V \to V_m$ which is homogeneous of degree d. In other words, for all $g \in \mathrm{SL}_2$ we have $\phi(g \cdot v) = g \cdot \phi(v)$, where $v \in V$, and for all $t \in \mathbb{C}$ we have $\phi(tv) = t^d \phi(v)$, where $v \in V$. The set of covariants of V is denoted $\mathcal{C}(V)$.

Remark 2.1.1. *The covariants of V of order 0 are the homogeneous invariants of V. They form the homogeneous components of the ring $\mathcal{O}(V)^{\mathrm{SL}_2} = \bigoplus_d \mathcal{O}(V)_d^{\mathrm{SL}_2}$ of invariants of V, where $\mathcal{O}(V)_d^{\mathrm{SL}_2}$ are the invariants of V of degree d. The covariants form a doubly graded ring $\mathcal{C}(V) = \bigoplus_{d,e} \mathcal{C}(V)_{(d,e)}$, where $\mathcal{C}(V)_{(d,e)}$ are the covariants of V of degree d and order e.*

Definition 2.1.3. For a covariant of V_n of order m and degree d we define its *co-order* to be $(nd - m)/2$.

The main way to construct covariants is via *transvectants* (Überschiebungen). They are derived from the Clebsch-Gordan decomposition of the SL_2-module $V_m \otimes V_n$, with $m \geq n$:

$$V_m \otimes V_n \simeq V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{m-n} ([\text{KP96}, 9.1]).$$

This decomposition defines for each p, $0 \leq p \leq n$, an SL_2-equivariant linear map $V_m \otimes V_n \to V_{m+n-2p}$, denoted $(f, h) \mapsto (f, h)_p$, and called the *p-th transvectant*. It is given explicitly by the following formula:

$$(f, h) \mapsto (f, h)_p := \frac{(m-p)!(n-p)!}{m!n!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \frac{\partial^p f}{\partial x^{p-i} \partial y^i} \frac{\partial^p h}{\partial x^i \partial y^{p-i}} \quad (2.1)$$

(see [Olv99, Chap. 5]). The maps $(f, h) \mapsto (f, h)_p$ are clearly bilinear. Also, if $f = \ell_1^m$ and $h = \ell_2^n$, with $\ell_1 = a_0 x + a_1 y$ and $\ell_2 = b_0 x + b_1 y$, we have then

$$(\ell_1^m, \ell_2^n)_p = \ell_1^{m-p} \ell_2^{n-p} [\ell_1, \ell_2]^p \quad \text{where} \quad [\ell_1, \ell_2] := \det \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix} = a_0 b_1 - a_1 b_0.$$

Furthermore, if $g = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathrm{SL}_2$, with $mq - np = 1$, we have

$$(g \cdot \ell_1^m, g \cdot \ell_2^n)_p = (g \cdot \ell_1)^{m-p} (g \cdot \ell_2)^{n-p} [\ell_1, \ell_2]^p (mq - np)^p =$$
$$= (g \cdot \ell_1)^{m-p} (g \cdot \ell_2)^{n-p} [\ell_1, \ell_2]^p = g \cdot (\ell_1^m, \ell_2^n)_p.$$

Because V_m and V_n are linearly spanned by powers of linear forms, it follows that $(f, h) \mapsto (f, h)_p$ are SL_2-equivariant. They are also non-zero and as $V_{m+n}, V_{m+n-2}, \ldots, V_{m-n}$ are irreducible representations, it follows that

$$V_m \otimes V_n \to V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{m-n},$$

$$(f, h) \mapsto \sum_{p=0}^{n} (f, h)_p$$

is surjective. But $V_m \otimes V_n$ and $V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{m-n}$ have the same dimension, which implies that the map is actually a bijection.

Example 2.1.3. Let $f = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3$. The map

$$V_3 \to V_2,$$
$$f \mapsto (f, f)_2 = 2(a_0 a_2 - a_1^2) x^2 + 2(a_0 a_3 - a_1 a_2) xy + 2(a_1 a_3 - a_2^2) y^2 =$$
$$= \frac{1}{18} \Big[\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial^2 f}{\partial x \partial y} \Big)^2 \Big],$$

defines a covariant of V_3 of order 2 and degree 2. Note that the transvectant $(f, f)_2$ coincides, up to a constant, with the Hessian of f. This transvectant vanishes if and only if f is the 3th power of a linear form (see Proposition 2.7.2).

Remark 2.1.2. *The covariants of V can be identified with the invariants of $V_1 \oplus V$: we have $V_1 \oplus V \simeq V_1^* \oplus V$ as SL_2-representations and the algebra of covariants of V is isomorphic to $\mathcal{O}(V_1^* \oplus V)^{\mathrm{SL}_2}$ (see [Pro07, Chap. 15]). Each covariant ϕ of V of order m corresponds to the invariant of $V_1 \oplus V$ defined by the transvectant $(\phi(v), \ell^m)_m$, where $\ell \in V_1$.*

Notation agreement. Consider $f \in V_m$. One obvious covariant of V_m of degree 1 and order m is the identity map on V_m. From now by "the covariant f" we will mean the identity map on V_m.
Given two covariants $\phi_1 : V_m \to V_d$ and $\phi_2 : V_m \to V_e$ of V_m of orders d, respectively e, they define the covariants $\psi_p : V_m \to V_{d+e-p}$, $f \mapsto (\phi_1(f), \phi_2(f))_p$, with $0 \le p \le \min(d, e)$. By "the covariant $(\phi_1, \phi_2)_p$" we will mean the map $V_m \to V_{d+e-p}$, $f \mapsto (\phi_1(f), \phi_2(f))_p$.

Theorem 2.1.3. (Gordan [Gor68]) *Let $f \in V_n$. Then, the covariants of V_n are generated by a finite number of iterated transvectants*

$$f, (f,f)_p, (f,(f,f)_p)_q, \dots$$

In particular (see [Gor68, §2]), if C is a covariant of f of degree d, then C can be written as a linear combination of transvectants $(f, C_i)_{r_i}$, where C_i are covariants of f of degree $d-1$.

This gives a method for finding the generating covariants of f: suppose we know the generating covariants of f up to degree $d-1$. In order to find the generating covariants of degree d, we have to write down all transvectants $(f, C_{d-1})_r$, for suitable r, where C_{d-1} is a covariant of degree $d-1$, namely a generating covariant of degree $d-1$ or a product of total degree $d-1$ of generating covariants of lower degrees. Then we select out of this set the irreducible ones (we call a covariant C *reducible* if C is contained in the algebra generated by all covariants of degree $\leq \deg C$ and order $\leq \operatorname{ord} C$, where at least one of the inequalities is strict).

Lemma 2.1.4. *[KW99] Let $V = V_{n_1} \oplus \dots \oplus V_{n_p}$ be a representation of SL_2 and $C_1, \dots, C_r, C$ covariants of V, of orders $\operatorname{ord} C_i = e_i$. Then the transvectant $(C_1 \dots C_r, C)_k$ is reducible if there is a strict subset $S \subset \{1, 2, \dots, r\}$ and integers $k_i \leq e_i$ such that $k = \sum_{i \in S} k_i$.*

Proposition 2.1.5. *Let $f \in V_n$ and consider covariants $C_1, \dots, C_r$ of f, with $r \geq 2$. If the covariant $C = (C_1 \dots C_r, f)_k$ is irreducible, then $\operatorname{ord} C \leq n - r$. If n is even, then $\operatorname{ord} C \leq n - 2r + 2$.*

Proof. Denote $m_i = \operatorname{ord} C_i$. W.l.o.g. we can assume $m_1 \geq m_2 \geq \dots \geq m_r > 0$. From the definition of transvectants, k must be $\leq n$. From Lemma 2.1.4 we obtain:

$$m_1 + m_2 + \dots + m_{r-1} < k \leq m_1 + m_2 + \dots + m_r.$$

Then,

$$
\begin{aligned}
\operatorname{ord} C =& (m_1 + m_2 + \dots + m_r) + n - 2k \leq \\
&\leq (m_1 + m_2 + \dots + m_r) + n - 2(m_1 + m_2 + \dots + m_{r-1} + 1) = \\
&= n - (m_1 - m_r + (m_2 + \dots + m_{r-1}) + 2) \leq \\
&\leq n - r
\end{aligned}
$$

If n is even, then all m_i will be even as well and then

$$n - (m_1 - m_r + (m_2 + \dots + m_{r-1}) + 2) \leq n - 2r + 2.$$

$\square$

Example 2.1.4. Consider $f \in V_1$. Then, we have $(f,f)_p = 0$ for $p \neq 0$. The covariants of V_1 are generated by f.

Consider $f \in V_2$. The covariants of V_2 are generated by f and $(f, f)_2$; the invariants of V_2 are generated by $(f, f)_2$ (see Chap. 4.1).

Consider $f \in V_3$. The covariants of V_3 are generated by

$$f, (f, f)_2, (f, (f, f)_2)_1, ((f, f)_2, (f, f)_2)_2 = -(f, (f, (f, f)_2)_1)_3 \quad \text{(see Chap. 5.3.1)}.$$

The invariants of V_3 are generated by $((f, f)_2, (f, f)_2)_2$ (see Chap. 4.2).

Consider $f \in V_4$. The covariants of V_4 are generated by

$$f, (f, f)_2, (f, f)_4, (f, (f, f)_2)_1, (f, (f, f)_2)_4 \quad \text{(see Chap. 5.4.1)}.$$

The invariants of V_4 are generated by $(f, f)_4$ and $(f, (f, f)_2)_4$ (see Chap. 4.3).

For further examples see Chap. 4 and 5.

Hilbert [Hil90] generalised Gordan's result to a system of several homogeneous forms in a finite number of variables. Formulated for the particular case of the SL_2-module $V_{n_1} \oplus \ldots \oplus V_{n_p}$, Hilbert's result is:

Theorem 2.1.6. (Hilbert [Hil90]) *Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. The algebra of invariants of V is finitely generated, i.e. there exist finitely many invariants $j_1, \ldots, j_r \in \mathcal{O}(V)^{\mathrm{SL}_2}$ such that $\mathcal{O}(V)^{\mathrm{SL}_2} = \mathbb{C}[j_1, \ldots, j_r]$.*

Example 2.1.5. Let $f_1, f_2 \in V_3$. The invariants of $V_3 \oplus V_3$ are generated by

$$(f_1, f_2)_3, ((f_1, f_1)_2, (f_1, f_1)_2)_2, ((f_2, f_2)_2, (f_2, f_2)_2)_2, ((f_1, f_1)_2, (f_2, f_2)_2)_2,$$
$$((f_1, f_1)_2, (f_1, f_2)_2)_2, ((f_2, f_2)_2, (f_1, f_2)_2)_2, ((f_1, (f_1, f_2)_2)_2, (f_2, (f_1, f_2)_2)_2)_1$$

(see Chap. 5.3.2).

Let $f \in V_2$ and $g \in V_4$. The invariants of $V_2 \oplus V_4$ are generated by

$$(f^3, (g, (g, g)_2)_1)_6, (f^2, (g, g)_2)_4, (f^2, g)_4, (g, (g, g)_2)_4, (g, g)_4, (f, f)_2$$

(see Chap. 5.10).

For further examples see Chap. 5.

2.2 The symbolic method

The symbolic method was developed by Aronhold and Clebsch in the middle of the nineteenth century. The symbolic calculus permits the reduction of the computations with binary forms of degree n to the special cases of the nth power of a linear form $(\alpha_1 x + \alpha_2 y)^n$. The classics proved that the invariants of binary forms have symbolic representations as products of factors of type $[\alpha\beta]$, where $[\alpha\beta]$ stays for the determinant $\alpha_1\beta_2 - \alpha_2\beta_1$. The manipulation of invariants got simplified by representing them in succinct symbolic expressions.

Kung & Rota [KR84] gave in 1984 "a rigorous and yet manageable account of the umbral or symbolic calculus" that was performed in the nineteenth century. We introduce in this section the symbolic calculus, closely following the ideas of Kung & Rota [KR84].

Consider an alphabet $\mathcal{A} = \{\alpha, \beta, \ldots, \omega, u\}$ consisting of an infinite number of Greek letters and the Roman letter u. The letters in $\mathcal{A}$ are called *umbral letters*. To each Greek letter α and to u we associate two variables, α_1 and α_2, respectively u_1 and u_2. The ring of polynomials in the variables $\alpha_1, \alpha_2, \beta_1, \beta_2, \ldots, \omega_1, \omega_2, u_1, u_2$ is an infinite-dimensional vector space called the umbral space $\mathcal{U}$. We define a linear operator U from the umbral space $\mathcal{U}$ to the space $\mathbb{C}[A_0, A_1, \ldots, A_n, X, Y]$ of polynomials in the variables $A_0, \ldots, A_n, X, Y$ in the following way (we denote the image of an element $P \in \mathcal{U}$ under U by $\langle U \mid P \rangle$):

$$\langle U \mid \alpha_1^j \alpha_2^k \rangle = \begin{cases} A_k, & \text{if } j + k = n, \\ 0, & \text{if } j + k \neq n. \end{cases}$$
$$\langle U \mid u_1^i \rangle = (-Y)^i,$$
$$\langle U \mid u_2^j \rangle = X^j,$$
$$\langle U \mid \alpha_1^i \alpha_2^j \beta_1^k \beta_2^l \ldots u_1^p u_2^q \rangle = \langle U \mid \alpha_1^i \alpha_2^j \rangle \langle U \mid \beta_1^k \beta_2^l \rangle \ldots \langle U \mid u_1^p \rangle \langle U \mid u_2^q \rangle.$$

U is called the *umbral operator* associated to the space of binary forms of degree n. If $f = \sum_{i=0}^n \binom{n}{i} a_i x^{n-i} y^i$, we define the umbral functional $U(f)$, which is a linear map from $\mathcal{U}$ to $\mathbb{C}[x, y]$, as the composition of U with the homomorphism $\mathbb{C}[A_0, A_1, \ldots, A_n, X, Y] \to \mathbb{C}[x, y]$ determined by

$$A_0 \mapsto a_0, A_1 \mapsto a_1, \ldots, A_n \mapsto a_n, X \mapsto x, Y \mapsto y.$$

Every polynomial in the variables $A_0, \ldots, A_n, X, Y$ can be written as $\langle U \mid P(\alpha_1, \alpha_2, \ldots) \rangle$ for some polynomial $P(\alpha_1, \alpha_2, \ldots)$ in the umbral space $\mathcal{U}$. Indeed, we have

$$A_0^{d_0} A_1^{d_1} \ldots A_n^{d_n} X^{e_1} Y^{e_2} = \langle U \mid \underbrace{\alpha_1^n \alpha_2^0 \ldots \gamma_1^n \gamma_2^0}_{d_0 \text{ times}} \underbrace{\delta_1^{n-1} \delta_2^1 \ldots \epsilon_1^{n-1} \epsilon_2^1}_{d_1 \text{ times}} \ldots (-u_1)^{e_2} u_2^{e_1} \rangle,$$

where the umbral letters $\alpha, \ldots, \gamma, \delta, \ldots, \epsilon, \ldots$ are distinct. In general, the umbral representation of a polynomial is not unique.

We define now

$$[\alpha\beta] = \alpha_1 \beta_2 - \alpha_2 \beta_1 \quad \text{and} \quad [\alpha u] = \alpha_1 u_2 - \alpha_2 u_1.$$

We call a *bracket monomial* a nonconstant polynomial in $\mathcal{U}$ which can be written as a product of brackets, for example $[\alpha\beta][\alpha\delta]\ldots[\omega u]$. The *index* of a bracket monomial M is the number of brackets in M containing only Greek symbols. The *order* of a bracket monomial M is the number of brackets in M containing the Roman letter u. The *height* of a bracket monomial M is the total number of brackets in M.

A *bracket polynomial* is a linear combination of bracket monomials. The bracket polynomials form a subspace $\mathcal{B}$ of the umbral space $\mathcal{U}$.

Theorem 2.2.1. First Fundamental Theorem ([KR84, Theorem 3.1], [GY03, Chap. II]) *Consider U the umbral operator associated to the space of binary*

forms of degree n. The umbral evaluation $\langle U \mid P \rangle$ of a bracket polynomial of index g is a covariant of V_n of co-order g. Vice-versa, if C is a covariant of V_n of co-order g, then there exists a bracket polynomial P of index g such that $C = \langle U \mid P \rangle$.

Remark 2.2.2. *If $P = \sum P_i$ is an umbral representation of a covariant of V_n of degree d, order m, and co-order g, then the bracket monomials P_i share the following properties: in all P_i will occur exactly d Greek letters α such that the total degree of α_1 and α_2 in P_i is n; the order of P_i is m; the index of P_i is g.*

Example 2.2.1. Let $f = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3$. In Example 2.1.3 we saw that

$$C = 2(a_0 a_2 - a_1^2)x^2 + 2(a_0 a_3 - a_1 a_2)xy + 2(a_1 a_3 - a_2^2)y^2$$

is covariant of V_3 of order 2, degree 2 and co-order 2, which up to a constant coincides with the Hessian of f.

The bracket polynomial $P = [\alpha\beta]^2[\alpha u][\beta u]$ is an umbral representation of the covariant C. Indeed,

$$
\begin{aligned}
\langle U(f) \mid P \rangle = & \langle U \mid \alpha_2^3 \beta_1^2 \beta_2 u_1^2 - 2\alpha_1 \alpha_2^2 \beta_1 \beta_2^2 u_1^2 + \alpha_1^2 \alpha_2 \beta_2^3 u_1^2 - \alpha_2^3 \beta_1^3 u_1 u_2 + \\
& + \alpha_1 \alpha_2^2 \beta_1^2 \beta_2 u_1 u_2 + \alpha_1^2 \alpha_2 \beta_1 \beta_2^2 u_1 u_2 - \alpha_1^3 \beta_2^3 u_1 u_2 + \alpha_1 \alpha_2^2 \beta_1^3 u_2^2 - \\
& - 2\alpha_1^2 \alpha_2 \beta_1^2 \beta_2 u_2^2 + \alpha_1^3 \beta_1 \beta_2^2 u_2^2 \rangle = \\
= & a_3 a_1 y^2 - 2a_2^2 y^2 + a_1 a_3 y^2 - a_3 a_0 (-xy) + a_2 a_1 (-xy) + a_1 a_2 (-xy) - \\
& - a_0 a_3 (-xy) + a_2 a_0 x^2 - 2a_1^2 x^2 + a_0 a_2 x^2 = \\
= & 2(a_1 a_3 - a_2^2)y^2 + 2(a_0 a_3 - a_1 a_2)xy + 2(a_0 a_2 - a_1^2)x^2.
\end{aligned}
$$

Theorem 2.2.3. Second Fundamental Theorem ([KR84, Theorem 3.3]) *Let U be the umbral operator for binary forms of degree n and let P and Q be polynomials in the umbral space $\mathcal{U}$ such that $\langle U \mid P \rangle = \langle U \mid Q \rangle$. Then P can be obtained from Q by a sequence of of operations of the following kind:*

(1) *an application of the $\mathbb{C}$-algebra axioms in the polynomial algebra $\mathbb{C}[\alpha_1, \alpha_2, \beta_1, \beta_2, \ldots, u_1, u_2]$;*

(2) *adding a scalar multiple of a redundant monomial (a monomial in $\mathcal{U}$ containing a Greek letter γ such that the total degree of γ_1 and γ_2 is not equal to n or 0);*

(3) *replacing any monomial M by M', where M' is obtained from M by replacing all occurences of α_1 and α_2 for some Greek letter α appearing in M by the variables δ_1 and δ_2, where δ is not appearing in M;*

(4) *replacing any monomial in M by M', where M' is obtained from M by permuting the set of umbral letters occuring in M.*

The umbral notation can be extended to several binary forms $f_1, \ldots, f_r \in V_{n_1} \oplus \ldots \oplus V_{n_r}$ in the following way: we split the set of Greek letter in the alphabet $\mathcal{A}$ into r disjoint infinite subsets $\mathcal{A}_i$ and assign each letter in $\mathcal{A}_i$ to the form f_i. If two letters are assigned to the same form, they are called equivalent. We define now the umbral operator U as the linear operator from the umbral space $\mathcal{U}$ to the space $\mathbb{C}[A_{10}, A_{11}, \ldots, A_{1n_1}, \ldots, A_{r0}, A_{r1}, \ldots, A_{rn_r}, X, Y]$ of polynomials in the variables $A_{10}, A_{11}, \ldots, A_{1n_1}, \ldots, A_{r0}, A_{r1}, \ldots, A_{rn_r}, X, Y$ in the following way (we denote the image of an element $P \in \mathcal{U}$ under U by $\langle U \mid P \rangle$):

$$\langle U \mid \alpha_1^j \alpha_2^k \rangle = \begin{cases} A_{ik}, & \text{if } j + k = n_i, \\ 0, & \text{if } j + k \neq n_i. \end{cases}$$

$$\langle U \mid u_1^i \rangle = (-Y)^i,$$

$$\langle U \mid u_2^j \rangle = X^j,$$

$$\langle U \mid \alpha_1^i \alpha_2^j \beta_1^k \beta_2^l \ldots u_1^p u_2^q \rangle = \langle U \mid \alpha_1^i \alpha_2^j \rangle \langle U \mid \beta_1^k \beta_2^l \rangle \ldots \langle U \mid u_1^p \rangle \langle U \mid u_2^q \rangle.$$

(This last multiplication rule holds even if umbral letters are assigned to different forms.)

Theorem 2.2.4. ([You04, §3], Grace & Young [GY03, §265]) *Any covariant of $V_{n_1} \oplus \ldots \oplus V_{n_\delta}$ is represented by a bracket polynomial in the vector space spanned by bracket polynomials of the following three shapes*

(1) *bracket monomials of the form*

$$[\alpha_1 \alpha_2]^{\lambda_1} [\alpha_2 \alpha_3]^{\lambda_2} \ldots [\alpha_{\delta-1} \alpha_\delta]^{\lambda_{\delta-1}} [\alpha_1 u]^{n_1 - \lambda_1} [\alpha_2 u]^{n_2 - \lambda_1 - \lambda_2} \ldots [\alpha_\delta u]^{n_\delta - \lambda_{\delta-1}}$$

where $\lambda_1 \geq 2^{\delta-2}, \lambda_2 \geq 2^{\delta-3}, \ldots \lambda_{\delta-1} \geq 1$, and the arrangement of the letters $\alpha_1, \ldots, \alpha_\delta$ is beforehand fixed;

(2) *bracket monomials that have a factor of the form $[\alpha_i \alpha_j]^{\lambda} [\alpha_j \alpha_k]^{n_j - \lambda}$;*

(3) *products of covariants of lower total degree.*

2.3 Bounds on the degrees and orders of the generating covariants

We recall the following degree bound on generators for invariants of binary forms.

Proposition 2.3.1. (Jordan [Jor76, Jor79]) *Let $f \in V_n$. Then the generating covariants of f have degrees $< n^6$.*

This is the best known upper bound on the degree of the generating covariants of V_n, for general n. However, in particular cases, for small n, the gap between this bound and the highest degree of the generating covariants is still big: for example, the covariants of V_8 are generated by those of degree ≤ 10 (see Chap. 5.8), while Jordan's bound is 8^6.

Proposition 2.3.2. (Jordan [Jor79]) *Consider two sequences $\ell(n)$ and $\varphi(n)$ defined by:*

$$\ell(1) = 0, \quad \ell(2) = 1, \quad \ell(3) = 2,$$
$$\ell(2i+3) = \ell(2i+2) + 2[\frac{\ell(i+3)}{4}],$$
$$\ell(2i+2) = \ell(2i+1) + 2[\frac{\ell(i+2)+2}{4}],$$

and

$$\varphi(1) = 0, \quad \varphi(2) = 1, \quad \varphi(3) = 3,$$
$$\varphi(j) = \varphi(j-1) + \ell(j),$$

where $i \geq 1$ and $j \geq 4$.

Let $f \in V_n$ and consider δ the greatest integer such that $\ell(\delta) < \frac{n}{2}$. Then the generating covariants of f have orders $< \max\{n, 2n-2, n\delta - 2\varphi(\delta)\}$.

Proposition 2.3.3. (Grace & Young [GY03, §271]) *Let $f \in V_n$. Write $n = 2^r + n_1$ with maximal possible $r \in \mathbb{N}$ and define*

$$c_n = (r-1)2^r + n_1(r+1) + 2.$$

Then the generating covariants of f have orders $\leq c_n$.

Proof. We start by looking at the covariants of $V_{n_1} \oplus \ldots \oplus V_{n_\delta}$, with $n_1, \ldots, n_\delta \leq n$. If C is such a covariant, then, from Theorem 2.2.4, C can be symbolically expressed in terms of:

(1) covariants of the form

$$[\alpha_1\alpha_2]^{\lambda_1}[\alpha_2\alpha_3]^{\lambda_2}\ldots[\alpha_{\delta-1}\alpha_\delta]^{\lambda_{\delta-1}}[\alpha_1 u]^{n_1-\lambda_1}[\alpha_2 u]^{n_2-\lambda_1-\lambda_2}\ldots[\alpha_\delta u]^{n_\delta-\lambda_{\delta-1}}$$

where $\lambda_1 \geq 2^{\delta-2}, \lambda_2 \geq 2^{\delta-3}, \ldots \lambda_{\delta-1} \geq 1$, and the arrangement of the letters $\alpha_1, \ldots, \alpha_\delta$ is beforehand fixed;

(2) covariants which have a factor of the form $[\alpha_i\alpha_j]^{\lambda}[\alpha_j\alpha_k]^{n_j-\lambda}$;

(3) products of covariants of lower total degree.

Suppose C contains terms of second kind, containing a factor $[\alpha_i\alpha_j]^{\lambda}[\alpha_j\alpha_k]^{n_j-\lambda}$. Suppose w.l.o.g. $\lambda \geq n_j - \lambda$. Then this term can be obtained from a transvectant of the form

$$([\alpha_i\alpha_j]^{\lambda}[\alpha_i u]^{n_i-\lambda}[\alpha_j u]^{n_j-\lambda}, C')_w,$$

where the order of $[\alpha_i\alpha_j]^{\lambda}[\alpha_i u]^{n_i-\lambda}[\alpha_j u]^{n_j-\lambda}$ is $n_i + n_j - 2\lambda \leq n$. By introducing a new symbol for the covariant $[\alpha_i\alpha_j]^{\lambda}[\alpha_i u]^{n_i-\lambda}[\alpha_j u]^{n_j-\lambda}$, we reduce the degree of $([\alpha_i\alpha_j]^{\lambda}[\alpha_i u]^{n_i-\lambda}[\alpha_j u]^{n_j-\lambda}, C')_w$. In this way the terms of second kind appearing in C are reduced in degree. Now, we apply again Theorem 2.2.4 to the reduced covariants, hence they can be expressed in terms of covariants of the three different kinds. We reduce again the degree of the terms of second kind and so on until we are left with:

(i) covariants of the form

$$[\beta_1\beta_2]^{\lambda_1}[\beta_2\beta_3]^{\lambda_2}\ldots[\beta_{\delta'-1}\beta_{\delta'}]^{\lambda_{\delta'-1}}[\beta_1 u]^{n_1-\lambda_1}\ldots[\beta_{\delta'}u]^{n_{\delta'}-\lambda_{\delta'-1}},$$

where $\lambda_1 \geq 2^{\delta'-2}, \lambda_2 \geq 2^{\delta'-3},\ldots,\lambda_{\delta'-1} \geq 1$, and the symbols $\beta_1,\ldots,\beta_{\delta'}$ are assigned either to the original binary forms or to covariants of orders $\leq n$ of these binary forms;

(ii) products of covariants of lower total degree.

For this reason, the maximal possible order of a covariant of $V_{n_1} \oplus \ldots \oplus V_{n_\delta}$, with $n_1,\ldots,n_\delta \leq n$ is attained by a covariant of the form

$$[\alpha_1\alpha_2]^{\lambda_1}[\alpha_2\alpha_3]^{\lambda_2}\ldots[\alpha_{\epsilon-1}\alpha_\epsilon]^{\lambda_{\epsilon-1}}[\alpha_1 u]^{n_1-\lambda_1}[\alpha_2 u]^{n_2-\lambda_1-\lambda_2}\ldots[\alpha_\epsilon u]^{n_\epsilon-\lambda_{\epsilon-1}}$$

where $\lambda_1 \geq 2^{\epsilon-2}, \lambda_2 \geq 2^{\epsilon-3},\ldots\lambda_{\epsilon-1} \geq 1$. The order of this covariant is $\leq$

$$n\epsilon - (2^{\epsilon-1} + 2^{\epsilon-2} + \ldots + 2) = n\epsilon - 2^\epsilon + 2,$$

a maximum that is attained for $n_1 = n_2 = \ldots = n$ and $\lambda_1 = 2^{\epsilon-2}, \lambda_2 = 2^{\epsilon-3},\ldots,\lambda_{\epsilon-1} = 1$.

We look now at the maximum of

$$n, 2n - 2,\ldots,n\epsilon - 2^\epsilon + 2,\ldots$$

If $n = 2^r + n_1$ with maximal possible $r \in \mathbb{N}$, then the maximum is attained for $\epsilon = r + 1$. Hence, the order of a covariant of V_n will then be $\leq$

$$(r + 1)(2^r + n_1) - 2^{r+1} + 2 = (r - 1)2^r + n_1(r + 1) + 2.$$

$\square$

Remark 2.3.4. *Grace & Young [GY03, §271] claim that a generator of order c_n always appears if $n \neq 3$.*

In Table 2.1 we compare the bounds of Jordan and of Grace & Young on the orders of the generating covariants of V_n, with $n \leq 24$:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
J.	2	4	6	9	12	15	18	22	26	30	34	39	44	49	54	60	66	72	78	85	92	99	106
G.&Y.	2	4	6	9	12	15	18	22	26	30	34	38	42	46	50	55	60	65	70	75	80	85	90

Table 2.1: The bounds of Jordan and of Grace & Young

2.4 Cohen-Macaulayness

We start with recalling some properties of graded Cohen-Macaulay modules. Let $R = \oplus_{i \geq 0} R_i$ be a finitely generated graded $\mathbb{C}$-algebra with $\dim_{\mathbb{C}} R_i < \infty$. Let $M = \oplus_j M_j$ be a finitely generated graded R-module.

A sequence $r_1, \ldots, r_p \in R$ is a homogeneous system of parameters (hsop) for M if $M/(r_1, \ldots, r_p)M$ is finite dimensional over $\mathbb{C}$ and p is minimal with this property. We have $p = \dim M$ (see [Eis95, §10.1]). A sequence $r_1, \ldots, r_p \in R$ is a regular sequence on M if $(r_1, \ldots, r_p)M \neq M$ and for $i = 1, \ldots, p$, r_i is a non-zero divisor on $M/(r_1, \ldots, r_{i-1})M$. If $r_1, \ldots, r_p$ is a regular sequence on M, then M is a free module over $\mathbb{C}[r_1, \ldots, r_p]$ (see [Eis95, §18.4]). If M is Cohen Macaulay, then homogeneous systems of parameters are regular sequences.

If M is Cohen Macaulay and $r_1, \ldots, r_p \in R$ is a homogeneous system of parameters for M of degrees $d_1, \ldots, d_p$, denote by $e_1, \ldots, e_s$ the degrees of the generators of M as a $\mathbb{C}[r_1, \ldots, r_p]$-module, ordered increasingly. Then the Poincaré series of M is given by

$$P_M(t) = \frac{\sum_{i=1}^{s} t^{e_i}}{\prod_{j=1}^{p}(1 - t^{d_j})}.$$

This means that if we know the degrees of a homogeneous system of parameters for M and the Poincaré series $P_M(t)$, then we also get a bound on the degrees of the generators of M as a $\mathbb{C}[r_1, \ldots, r_p]$-module.

Theorem 2.4.1. *(Hochster & Roberts [HR74]) The ring $\mathcal{O}(V)^{\mathrm{SL}_2}$ of invariants is Cohen-Macaulay.*

For certain values of m, the $\mathcal{O}(V)^{\mathrm{SL}_2}$-module $\mathcal{C}(V)_m$ of covariants of V of order m are Cohen Macaulay as well. In the paper [VdB91] Michel van den Bergh gives a sufficient criterion for the Cohen-Macaulayness of a module of covariants of a stable representation V of a reductive group G, based on certain properties of the highest weight of irreducible representations of G. For $G = \mathrm{SL}_2$, consider the maximal torus $T = \{ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) \mid t \in \mathbb{C}^* \} \subset \mathrm{SL}_2$. The one-dimensional representations of T are of the form

$$\chi_i \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) := t^i, \text{ with } i \in \mathbb{Z},$$

and form the *character group* $X(T)$ of T, which is identified to $\mathbb{Z}$. An SL_2-module V can be decomposed with respect to T into

$$V = \bigoplus_{\chi \in \mathcal{X}(T)} V_\chi, \quad V_\chi := \{ v \in V \mid \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) \cdot v = \chi \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) v \text{ for all } t \in \mathbb{C}^* \}.$$

The characters $\chi \in X(T)$ such that $V_\chi \neq 0$ are called *weights* of V, the non-zero elements of V_χ are called *weight vectors*, and V_χ are called *weight spaces*. For example, the weights of the vector space V_n of binary forms of degree n are $-n + 2i$, corresponding to the weight vectors $x^{n-i}y^i$, with $0 \leq i \leq n$.

Given a one-parameter subgroup (1-PSG) $\lambda \in X(T)^* = \mathbb{Z}$,

$$\lambda : \quad \mathbb{C}^* \to \mathrm{SL}_2,$$
$$t \quad \mapsto \left(\begin{smallmatrix} t^\lambda & 0 \\ 0 & t^{-\lambda} \end{smallmatrix} \right),$$

define $\langle \lambda, \chi \rangle$ as

$$\chi \lambda(t) = t^{\langle \lambda, \chi \rangle}.$$

Let $\chi_1, \ldots, \chi_N$ be the weights of V, with multiplicities. For $\lambda \in X(T)^*$ set

$$I_\lambda := \{ j \in \{1, 2, \ldots, N\} \mid \langle \lambda, \chi_j \rangle \geq 0 \}.$$

For example, if $V = V_n$ and χ is the weight $-n + 2i$, with $0 \leq i \leq n$, then $\langle \lambda, \chi \rangle = -n\lambda + 2i\lambda$, and I_λ is the set of nonnegative weights of V_n if $\lambda > 0$.

Definition 2.4.1. Let $\lambda \in X(T)^*$. A character $\alpha \in X(T)$ is called *good* for (T, V, λ) if α is not of the form $\sum_j n_j \chi_j$, with $n_j \in \mathbb{Z}$, $n_j < 0$ for $j \in I_\lambda$ and $n_j \geq 0$ otherwise.
A character $\alpha \in X(T)$ is called *good* for (T, V) if it is good for (T, V, λ) for all $\lambda \neq 0$.

Definition 2.4.2. Consider χ the highest weight of an irreducible representation W of G. Then χ is called *good* for (G, V) if $\chi + \mu$ is good for (T, V) for every μ which is a sum of distinct roots of G.

In our case $G = \mathrm{SL}_2$ and $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. Consider m the highest weight of V_m (irreducible representation of SL_2). Then, m is good for (SL_2, V) if $m, m+2, m-2$ are good for (T, V), because the roots of SL_2 are 2 and -2.

Denote by $N = \sum_{i=1}^p \lfloor \frac{n_i+1}{2} \rfloor \lceil \frac{n_i+1}{2} \rceil$ the sum of nonnegative weights of V. From Definition 2.4.1, α is good for (T, V) if α is good for (T, V, λ) for all $\lambda \neq 0$. If λ is negative, then I_λ is the set of negative weights of V. If $\alpha < N$, then the coefficients of the positive weights in α can't be ≥ 0 and the coefficients of the negative weights in α can't be < 0, hence α is good for (T, V, λ). If λ is positive, then I_λ is the set of positive weights of V. If $\alpha > -N$, then the coefficients of the positive weights in α can't be < 0 and the coefficients of the negative weights in α can't be ≥ 0, hence α is good for (T, V, λ).

In conclusion, m is good for (SL_2, V) if $m < N - 2$.

Theorem 2.4.2. *([VdB91, Thm. 1.2 applied to $G = \mathrm{SL}_2$] Let V a representation of SL_2 of dimension > 2 which does not contain the trivial representation. If m is good for (SL_2, V), i.e. if $m < N - 2$, then the $\mathcal{O}(V)^{\mathrm{SL}_2}$-module $\mathcal{C}(V)_m$ of covariants of V of order m is Cohen Macaulay.*

For the case $V = V_n$ we obtain:

Proposition 2.4.3. *For $n \geq 2$, the $\mathcal{O}(V_n)^{\mathrm{SL}_2}$-module $\mathcal{C}(V_n)_m$ of covariants of V_n of order m is Cohen-Macaulay if $m < \lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil - 2$.*

Consider now $M = \mathcal{O}(V)^{\mathrm{SL}_2}$ and $d_1, \ldots, d_p$ the degrees of a homogeneous system of parameters of M. Then the Poincaré series of M can be written as

$$P_M(t) = \frac{\sum_{i=1}^s t^{e_i}}{\prod_{j=1}^p (1 - t^{d_j})},$$

where the sequence $e_0 := 0, e_1, \ldots, e_s$ is a palindrome, i.e. $e_i + e_{s-i} = e_s$ for all i and $e_s \leq \sum_{j=1}^p d_j - \dim(V /\!/ \mathrm{SL}_2)$ ([Kno89, Satz 4]). The Poincaré series of $\mathcal{O}(V)^{\mathrm{SL}_2}$ gives us some information about the degrees of the generators of the invariants of V. In the examples considered in this thesis we will use the following result:

Proposition 2.4.4. *Let V be a SL_2-module. Consider $d_1, \ldots, d_p$ the degrees of a homogeneous system of parameters of $\mathcal{O}(V)^{\mathrm{SL}_2}$ and the Poincaré series of $\mathcal{O}(V)^{\mathrm{SL}_2}$ given by*

$$P_{\mathcal{O}(V)^{\mathrm{SL}_2}}(t) = \frac{\sum_{i=1}^s t^{e_i}}{\prod_{j=1}^p (1 - t^{d_j})}.$$

Then the invariants of V are generated in degree $\leq \max(e_s, d_1, \ldots, d_p)$.

2.5   The Poincaré series

The Poincaré series of the algebra $\mathcal{O}(V)^{\mathrm{SL}_2}$ of invariants of V is defined as

$$P(t) = \sum_i \dim_{\mathbb{C}} \mathcal{O}(V)_i^{\mathrm{SL}_2} t^i.$$

For the case $V = V_n$, a closed expression of $P(t)$ as a rational function was given by Springer [Spr77]:

$$P(t) = \sum_{0 \leq j < \frac{n}{2}} (-1)^j \phi_{n-2j} \left(\frac{(1-t^2)t^{j(j+1)}}{(j, t^2)!(n-j, t^2)!} \right) \text{(cf. [BC79])},$$

where for $d \in \mathbb{N}$ the notation $(d, t)!$ stands for the polynomial function $(d, t)! = (1-t)(1-t^2) \ldots (1-t^d)$ and the operator ϕ_d transforms a rational function f in t to a rational function $\phi_d f$, with

$$(\phi_d f)(t^d) = \frac{1}{d} \sum_{j=1}^d f(e^{\frac{2\pi i j}{d}} t).$$

Example 2.5.1. Consider $n = 3$. Then:

$$P(t) = \phi_3 \left(\frac{1}{(1-t^4)(1-t^6)} \right) - \phi_1 \left(\frac{t^2}{(1-t^2)(1-t^4)} \right) =$$

$$= \frac{1}{(1-t^2)^2(1+t^2)} - \frac{t^2}{(1-t^2)(1-t^4)} = \frac{1}{1-t^4}.$$

Springer's formula was implemented by Brower & Cohen [BC79]. Littelman & Procesi [LP90] also used this formula for the computation of Poincaré series of binary forms of even degrees ≤ 36. The webpages [Bro09] list the results of the Poincaré series computations for $n \leq 30$.

While a closed form for the Poincaré series of V_n was found late in the twentieth century, one has a formula for computing $\dim_{\mathbb{C}} \mathcal{O}(V_n)_i^{\mathrm{SL}_2}$ since the nineteenth century. Cayley [Cay56] gave in 1856 an (unproved) formula for the number of linear independent covariants of V_n of degree i and order j. This formula was proved by Sylvester [Sy78a] in 1878. Formulated for the particular case of invariants, this formula states:

$$\dim_{\mathbb{C}} \mathcal{O}(V_n)_i^{\mathrm{SL}_2} = \begin{cases} (\frac{in}{2} : n, i) - ((\frac{in}{2} - 1) : n, i), & \text{if } ni \text{ is even,} \\ 0, & \text{if } ni \text{ is odd,} \end{cases}$$

where $(j : n, i)$ denotes the number of ways in which j can be written as the sum of i integers that are ≥ 0 and $\leq n$ (see [Sy78a]). Equivalently, Springer [Spr77] gave the formula:

$$\dim_{\mathbb{C}} \mathcal{O}(V_n)_i^{\mathrm{SL}_2} = \begin{cases} \gamma(n+i, i, \frac{ni}{2}) - \gamma(n+i, i, \frac{ni}{2} - 1), & \text{if } ni \text{ is even,} \\ 0, & \text{if } ni \text{ is odd,} \end{cases}$$

where $\gamma(d, i, j)$ is the coefficient of t^j in $\frac{(1-t^{d-i+1})...(1-t^d)}{(1-t)...(1-t^i)}$ when $0 < i \leq d$, respectively 1 if $i = j = 0$, respectively 0 if $i = 0$ and $j \geq 1$ or $0 < n < i$ (see [Spr77, Thm. 3.3.4]). It follows then that

$$\dim_{\mathbb{C}} \mathcal{O}(V_n)_i^{\mathrm{SL}_2} = \begin{cases} \text{coefficient of } t^{\frac{ni}{2}} \text{ in } \frac{(1-t^{n+1})...(1-t^{n+i})}{(1-t^2)...(1-t^i)}, & \text{if } ni \text{ is even,} \\ 0, & \text{if } ni \text{ is odd.} \end{cases} \quad (2.2)$$

Example 2.5.2. Let $n = 8$. We apply the formula 2.2: $\dim_{\mathbb{C}} \mathcal{O}(V_8)_2^{\mathrm{SL}_2} = 1$ because the coefficient of t^8 in

$$\frac{(1-t^9)(1-t^{10})}{1-t^2} = (1-t^9)(t^8 + t^6 + t^4 + t^2 + 1)$$

is 1. Also, $\dim_{\mathbb{C}} \mathcal{O}(V_8)_3^{\mathrm{SL}_2} = 1$ because the coefficient of t^{12} in

$$\frac{(1-t^9)(1-t^{10})(1-t^{11})}{(1-t^2)(1-t^3)} = (1-t^{11})(t^8 + t^6 + t^4 + t^2 + 1)(t^6 + t^3 + 1)$$

is 1. In a similar way $\dim_{\mathbb{C}} \mathcal{O}(V_8)_4^{\mathrm{SL}_2} = 2$, $\dim_{\mathbb{C}} \mathcal{O}(V_8)_5^{\mathrm{SL}_2} = 2$, $\dim_{\mathbb{C}} \mathcal{O}(V_8)_6^{\mathrm{SL}_2} = 4$, $\dim_{\mathbb{C}} \mathcal{O}(V_8)_7^{\mathrm{SL}_2} = 4$, and the Poincaré series of the algebra of invariants of V_8 is

$$P(t) = 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + \dots,$$

which can be written as well as

$$P(t) = \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)} ([\text{BC79}, \text{Results}]).$$

A closed expression of $P(t)$ for the general case was given by Brion [Bri82]. Bedratyuk [Bed10] found formulas for the computation of Poincaré series of the joint invariants and covariants of two binary forms. The webpages [Bro09] list some Poincaré series computations due to Bedratyuk and Brouwer.

2.5.1 A result of Broer and Poincaré series

In the paper [Bro94] Bram Broer presented a new method for calculating Poincaré series of the algebra of covariants of modules of covariants of G-modules, where G is a reductive group. His method consists of writing down a solvable system of linear equations between the Poincaré series of various modules of covariants, and then use linear algebra to solve this system and compute the Poincaré series.

In this section we adapt Broer's method to the case $G = \mathrm{SL}_2$ and the SL_2-modules $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$. Denote $N = \sum_{i=1}^{p} \lfloor \frac{n_i+1}{2} \rfloor \lceil \frac{n_i+1}{2} \rceil$ the sum of nonnegative weights of V. With this method we compute the Poincaré series of the $\mathcal{O}(V)^{\mathrm{SL}_2}$-module of covariants $\mathcal{C}(V)_m$ with $m \leq N - 2$.

Consider the standard maximal torus $T \subset \mathrm{SL}_2$. For each character $\mu \in X(T)$, define the $\mathbb{Z}[t]$-linear operator $\mathcal{B} : \mathbb{Z}[t, e, e^{-1}] \to \mathbb{Z}[t, e, e^{-1}]$ by

$$\mathcal{B}(e^\mu) = \begin{cases} e^\mu, & \text{if } \mu \geq 0, \\ 0, & \text{if } \mu = -1, \text{ ([Bro94, §2])} \\ -e^{-\mu-2}, & \text{if } \mu \leq -2. \end{cases}$$

Let $S = \{0, 1, \ldots, N - 2\}$. Collect in S' the weights of V that are ≤ 0, with multiplicities. For $\lambda, \mu \in S$ we define the matrix $(\alpha_\lambda^\mu)_{\lambda, \mu \in S}$ in the following way:

$$\mathcal{B}(e^\mu \prod_{i \in S'} (1 - e^i t)) =: \sum_{\lambda \in S} \alpha_\lambda^\mu e^\lambda (\text{[Bro94, §3]}).$$

Theorem 2.5.1. *([Bro94, Thm, 4.1 with $G = \mathrm{SL}_2$ and $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$]) With the notations above we have:*

(1) *the matrix $(\alpha_\lambda^\mu)_{\lambda, \mu \in S}$ is invertible as a matrix with coefficients in $\mathbb{Q}(t)$;*

(2) *if $\mathcal{P}$ is the column vector $(P_0(t), \ldots, P_{N-2}(t))$, where $P_i(t)$ is the Poincaré series of the $\mathcal{O}(V)^{\mathrm{SL}_2}$-module $\mathcal{C}(V)_i$, then*

$$\mathcal{P} = [(\alpha_\lambda^\mu)_{\lambda, \mu \in S}]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Example 2.5.3. Consider $V = V_4$. We have $N = 6$, $S' = \{-4, -2, 0\}$, and:

$\mu = 0:$ $\mathcal{B}(e^0(1 - e^{-4}t)(1 - e^{-2}t)(1 - t)) = (1 - t^2)e^0 - (t^2 - t)e^2 - (t^2 - t^3)e^4;$

$\mu = 1:$ $\mathcal{B}(e^1(1 - e^{-4}t)(1 - e^{-2}t)(1 - t)) = (1 - t^2)e^1 - (t^2 - t^3)e^3;$

$\mu = 2:$ $\mathcal{B}(e^2(1 - e^{-4}t)(1 - e^{-2}t)(1 - t)) = (1 - t - t^2 + t^3)e^2;$

$\mu = 3:$ $\mathcal{B}(e^3(1 - e^{-4}t)(1 - e^{-2}t)(1 - t)) = (1 - t)e^3 + (-t + t^3)e^1;$

$\mu = 4:$ $\mathcal{B}(e^4(1 - e^{-4}t)(1 - e^{-2}t)(1 - t)) = (1 - t)e^4 + (t^2 - t)e^2 + (-t + t^3)e^0.$

It follows that

$$
\begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ P_3(t) \\ P_4(t) \end{bmatrix} = \begin{bmatrix} 1 - t^2 & 0 & -t^2 + t & 0 & -t^2 + t^3 \\ 0 & 1 - t^2 & 0 & -t^2 + t^3 & 0 \\ 0 & 0 & (1 - t)(1 - t^2) & 0 & 0 \\ 0 & -t + t^3 & 0 & 1 - t & 0 \\ -t + t^3 & 0 & t^2 - t & 0 & 1 - t \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{(1 - t^2)(1 - t^3)} \\ 0 \\ 0 \\ 0 \\ \frac{t + t^2}{(1 - t^2)(1 - t^3)} \end{bmatrix}
$$

Hence $P_0(t) = \frac{1}{(1 - t^2)(1 - t^3)}$ and $P_4(t) = \frac{t + t^2}{(1 - t^2)(1 - t^3)}$.

Example 2.5.4. Consider $V = V_2 \oplus V_2$. We have $N = 4$, $S' = \{-2, -2, 0, 0\}$, and:

$\mu = 0:$ $\mathcal{B}(e^0(1 - e^{-2}t)^2(1 - t)^2) = 1 - 3t^2 - e^2t^2 + 2t^3 + 2e^2t^3 - e^2t^4;$

$\mu = 1:$ $\mathcal{B}(e^1(1 - e^{-2}t)^2(1 - t)^2) = e - 2et + 2et^3 - et^4;$

$\mu = 2:$ $\mathcal{B}(e^2(1 - e^{-2}t)^2(1 - t)^2) = e^2 - 2t - 2e^2t + 3t^2 + e^2t^2 - t^4.$

It follows that

$$
\begin{bmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} 1 - 3t^2 + 2t^3 & 0 & -t^2 + 2t^3 - t^4 \\ 0 & 1 - 2t + 2t^3 - t^4 & 0 \\ -2t + 3t^2 - t^4 & 0 & 1 - 2t + t^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{(1 - t^2)^3} \\ 0 \\ \frac{t(t + 2)}{(1 - t^2)^3} \end{bmatrix}
$$

Hence $P_0(t) = \frac{1}{(1 - t^2)^3}$ and $P_2(t) = \frac{t(t + 2)}{(1 - t^2)^3}$.

2.6 The quotient variety and nullforms

We define $V /\!/ \mathrm{SL}_2$ the affine variety corresponding to the ring of invariants $\mathcal{O}(V)^{\mathrm{SL}_2}$. We have

$$
\dim(V_n /\!/ \mathrm{SL}_2) = \begin{cases} 0, & \text{if } n \leq 1, \\ 1, & \text{if } n = 2, \\ n - 2, & \text{if } n \geq 3. \end{cases} \quad [\text{Bri96}, \text{Chap. 3}]
$$

One defines the *nullcone* $\mathcal{N}(V)$ of V as the fibre $\pi^{-1}(\pi(0))$ of the *quotient map* $\pi : V \to V//\operatorname{SL}_2$, dual to the inclusion $\mathcal{O}(V)^{\operatorname{SL}_2} \to \mathcal{O}(V)$ or, equivalently, as the set of elements of V on which all invariants of V of positive degree vanish.

One can show that $\mathcal{N}(V_n)$ coincides with the set of binary forms of degree n with a root of multiplicity $> \frac{n}{2}$ ([Bri96, Chap. 3]).

In a similar way, the nullcone $\mathcal{N}(V_{n_1} \oplus \ldots \oplus V_{n_p})$ is the set of binary forms $f_1, \ldots, f_p$ of degrees $n_1, \ldots, n_p$ with a common root that has multiplicity $> \frac{n_i}{2}$ in f_i for all $i = 1, \ldots, p$.

If the equations of the nullcone $\mathcal{N}(V_n)$ are known, one can find the equations of the nullcone $\mathcal{N}(mV_n)$ of m copies of V_n using *polarizations*. Consider an element j in the algebra $\mathcal{O}(V_n)$ of polynomial functions on V_n. For $m \geq 1$, $f_1, \ldots, f_m \in V_n$ and arbitrary parameters $t_1, \ldots, t_m$ we write

$$j(t_1 f_1 + t_2 f_2 + \ldots + t_m f_m) = \sum P_{i_1, \ldots, i_m} j(f_1, \ldots, f_m) \cdot t_1^{i_1} t_2^{i_2} \ldots t_m^{i_m}.$$

The regular functions $P_{i_1, \ldots, i_m} j$ defined on mV_n are called *polarizations* of j.

Example 2.6.1. Consider $f \in V_4$ and the invariant $j(f) = (f, f)_4$ of V_4. The polarizations of j defined on two copies of V_4 are $(f_1, f_1)_4$, $(f_1, f_2)_4$ and $(f_2, f_2)_4$, where $f_1, f_2 \in V_4$.

Theorem 2.6.1. [KW10, Theorem 4] *Consider the set V_n of binary forms of degree $n > 1$. Assume that the invariants $j_1, \ldots, j_p$ of V_n define the nullcone $\mathcal{N}(V_n)$, i.e. $\mathcal{N}(V_n) = \mathcal{V}(j_1, \ldots, j_p)$. Then the polarizations of the f_i's for any number m of copies of V_n define the nullcone $\mathcal{N}(mV_n)$ of mV_n.*

2.7 Finding a hsop

Finding a homogeneous systems of parameters (hsop) of $\mathcal{O}(V)^{\operatorname{SL}_2}$ or at least the degrees of such a system is the first step in finding the generators of the algebra of invariants of V, as we will see in Chap. 3.1. Recall that a system of parameters of $\mathcal{O}(V)^{\operatorname{SL}_2}$ is a set of algebraically independent elements $P_1, \ldots, P_m \in \mathcal{O}(V)^{\operatorname{SL}_2}$ such that $\mathcal{O}(V)^{\operatorname{SL}_2}$ is integral over $\mathbb{C}[P_1, \ldots, P_m]$, and $m = \dim(V//\operatorname{SL}_2)$.

One finds a hsop of the algebra of invariants of V by finding a set of invariants of the *right* size, that defines the nullcone of V (Hilbert's criterion 2.7.1).

One finds the degrees of a hsop of the algebra of invariants of V using Dixmier's criterion 2.7.5.

2.7.1 Hilbert's criterion

The following result, due to Hilbert [Hil93], gives a characterisation of homogeneous systems of parameters of $\mathcal{O}(V)^{\operatorname{SL}_2}$ as sets that define the nullcone of V ($\mathcal{V}(J)$ stands for the vanishing locus of the ideal J):

Proposition 2.7.1. (Hilbert [Hil93]) *Let V be a rational finite-dimensional SL_2-module, $I = \mathcal{O}(V)^{\operatorname{SL}_2}$, and $m = \dim(V//\operatorname{SL}_2)$. Consider a set $\{P_1, \ldots, P_m\}$ of homogeneous elements of I. The following are equivalent:*

(1) $\{P_1, \ldots, P_m\}$ *is a system of parameters of* I,

(2) $\mathcal{V}(P_1, \ldots, P_m) = \mathcal{N}(V)$.

Example 2.7.1. Consider $f \in V_4$. Denote $j_2 = (f,f)_4$ and $j_3 = (f,(f,f)_2)_4$. One can prove that if $j_2 = j_3 = 0$, then f has a root of multiplicity at least 3 (see Proposition 2.7.2). It follows that $\mathcal{V}(j_2, j_3) \subset \mathcal{N}(V_4)$. On the other hand, both j_2 and j_3 vanish on forms of degree 4 with a root of multiplicity at least 3, hence $\mathcal{V}(j_2, j_3) = \mathcal{N}(V_4)$. From Proposition 2.7.1 it follows that $\{j_2, j_3\}$ is a hsop of the algebra of invariants of V_4.

The following results will be used for finding sets of invariants that define the nullcone of V:

Proposition 2.7.2. (Weyman [Wey93]) *Let* $f \in V_n$. *If* $n > 4k - 4$ *and all* $(f,f)_{2k}$, $(f,f)_{2k+2}$, ... *vanish, then* f *has a root of multiplicity at least* $n-k+1$. *If* $n = 4k - 4$, *and* $((f,f)_{2k-2}, f)_n$, $(f,f)_{2k}$, $(f,f)_{2k+2}$, ... *vanish, then* f *has a root of multiplicity at least* $n - k + 1$.

Example 2.7.2. Consider $f \in V_6$. We apply Proposition 2.7.2.
If $(f,f)_2$, $(f,f)_4$, and $(f,f)_6$ vanish, then f has a root of multiplicity 6.
If $(f,f)_4$ and $(f,f)_6$ vanish, then f has a root of multiplicity at least 5.

Lemma 2.7.3. *Let* $f \in V_{2n+1}$, $n \geq 1$. *Denote* $c_1 = (f,f)_{2n}$ *and* $c_2 = (f,c_1)_2$. *If* $c_1 \neq 0$ *and* $(c_1, c_2) \in \mathcal{N}(V_2 \oplus V_{2n-1})$, *then* f *has a root of multiplicity at least* $n + 1$, *hence* f *is a nullform.*

Proof. Let $f = a_0 x^{2n+1} + (2n+1)a_1 x^{2n} y + \ldots + a_{2n+1} y^{2n+1}$.
If $(c_1, c_2) \in \mathcal{N}(V_2 \oplus V_{2n-1})$, then c_1 and c_2 have a common root that has multiplicity 2 in c_1, and multiplicity n in c_2. Without loss of generality we can suppose $c_1 = x^2$. Then:

$$c_2 = (f, x^2)_2 = b_0 a_2 x^{2n-1} + b_1 a_3 x^{2n-2} y + \ldots + b_{2n-1} a_{2n+1} y^{2n-1},$$

where b_i are non-zero coefficients. Because c_2 is divisible by x^n, we get $a_{n+2} = a_{n+3} = \ldots = a_{2n+1} = 0$. If we substitute these in $(f,f)_{2n}$, then the coefficient of y^2 becomes $(-1)^n \binom{2n}{n} a_{n+1}^2$. This coefficient must vanish, because we assumed $c_1 = x^2$. Therefore we get $a_{n+1} = 0$, which implies $x^{n+1} | f$. $\square$

Example 2.7.3. Consider $f \in V_5$. Denote $c_1 = (f,f)_4$, $c_2 = (f,c_1)_2$, and suppose $c_1 \neq 0$. We apply Lemma 2.7.3: if $(c_1, c_2) \in \mathcal{N}(V_2 \oplus V_3)$, then f has a root of multiplicity at least 3.

Lemma 2.7.4. *Let* $f \in V_{2n}$, $n \geq 2$. *Denote* $j_2 = (f,f)_{2n}$, $c_1 = (f,f)_{2n-2}$, *and* $c_2 = (f,c_1)_4$. *If* $c_1 \neq 0$, $j_2 = 0$, *and* $(c_1, c_2) \in \mathcal{N}(V_4 \oplus V_{2n-4})$, *then* f *has a root of multiplicity at least* $n + 1$, *hence* f *is a nullform.*

Proof. Let $f = a_0 x^{2n} + 2n a_1 x^{2n-1} y + \ldots + 2n a_{2n-1} x y^{2n-1} + a_{2n} y^{2n}$.
If $(c_1, c_2) \in \mathcal{N}(V_4 \oplus V_{2n-4})$, then c_1 and c_2 have a common root that has multiplicity 3 in c_1, and multiplicity $n - 1$ in c_2. Without loss of generality we consider the following two cases:

Case 1: $c_1 = x^4$. Then c_2 becomes:

$$c_2 = (f, x^4)_4 = b_0 a_4 x^{2n-4} + b_1 a_5 x^{2n-5} y + \ldots + b_{2n-4} a_{2n} y^{2n-4},$$

where b_i are non-zero coefficients. Because we assumed c_2 divisible by x^{n-1}, we get $a_{n+2} = a_{n+3} = \ldots = a_{2n} = 0$. If we substitute these in $(f, f)_{2n-2}$, then the coefficient of y^4 becomes $(-1)^{n-1} \binom{2n-2}{n-1} a_{n+1}^2$. Because we assumed $c_1 = x^4$, this coefficient must vanish, and hence $a_{n+1} = 0$. But then, the coefficient of $x^2 y^2$ in $(f, f)_{2n-2}$ becomes $(-2)^n \binom{2n-2}{n-1} \frac{n+1}{n} a_n^2$. Because of our assumption, this coefficient must again vanish, hence $a_n = 0$. It follows then that $x^{n+1} | f$.

Case 2: $c_1 = x^3 y$. Then c_2 becomes:

$$c_2 = (f, x^3 y)_4 = b_0 a_3 x^{2n-4} + b_1 a_5 x^{2n-5} y + \ldots + b_{2n-4} a_{2n-1} y^{2n-4},$$

where b_i are non-zero rational numbers depending on n. Because we assumed c_2 divisible by x^{n-1}, we get $a_{n+1} = a_{n+2} = \ldots = a_{2n-1} = 0$. From our assumptions, the invariant j_2 and the coefficient of $x^2 y^2$ in $(f, f)_{2n-2}$ are both equal to zero, which gives us the system:

$$\begin{cases} (-1)^n \binom{2n}{n} a_n^2 + 2a_0 a_{2n} = 0, \\ 2(-1)^{n-1} \binom{2n-2}{n-1} \frac{n+1}{n} a_n^2 + 2a_0 a_{2n} = 0. \end{cases}$$

The matrix of this system has the determinant equal to $12(-1)^n \binom{2n-2}{n-1}$, which is non-zero for all $n \geq 1$. We obtain, in consequence, $a_n = 0$. But then the coefficient of $x^3 y$ in $(f, f)_{2n-2}$ will also vanish, which contradicts our assumption $c_1 = x^3 y$. $\square$

Example 2.7.4. Consider $f \in V_6$. Denote $c_1 = (f, f)_4$, $c_2 = (f, c_1)_4$, and suppose $c_1 \neq 0$. We apply Lemma 2.7.4: if $(f, f)_6 = 0$ and if $(c_1, c_2) \in \mathcal{N}(V_4 \oplus V_2)$, then f has a root of multiplicity 4.

2.7.2 Dixmier's criterion

In some cases it is enough to have the degrees of a system of parameters in order to find the generating invariants. The following result, due to Dixmier [Dix85], can be used for finding the degrees of a hsop of I:

Proposition 2.7.5. (Dixmier [Dix85]) *Let G be a reductive group over $\mathbb{C}$, with a rational representation in a vector space V of finite dimension over $\mathbb{C}$. Let $\mathcal{O}(V)$ be the algebra of complex polynomials on V, $I := \mathcal{O}(V)^G$ the subalgebra of G-invariants, and I_d the subset of homogeneous polynomials of degree d in I. Let $m = \dim(V//G)$. Let $(d_1, \ldots, d_m)$ be a sequence of positive integers. Assume that for each subsequence $(j_1, \ldots, j_p)$ of $(d_1, \ldots, d_m)$, the subset of points of $V//G$, where all elements of all I_j, with $j \in \{j_1, \ldots, j_p\}$, vanish, has codimension at least p in $V//G$. Then I has a system of parameters of degrees $d_1, \ldots, d_m$.*

When applying this criterion it is convenient to have a notation for 'the codimension of the subset of $V//G$ defined by the vanishing of all invariants with degree in $\{j_1, \ldots, j_p\}$'. We use $[j_1, \ldots, j_p]$.

Note that for $e \geq 1$ an invariant g^e vanishes if and only if g vanishes. It follows that if $j_h | j_i$, $h \neq i$, then $[j_1, \ldots, j_p] = [j_1, \ldots, j_{h-1}, j_{h+1}, \ldots, j_p]$.

2.8 Schur modules

Consider an n-dimensional vector space W. In this paragraph we are looking at the m-fold tensor product $W^{\otimes d}$, on which the general linear group $\mathrm{GL}(W)$ and the symmetric group $\mathcal{S}_d$ are acting via:

$$g(w_1 \otimes \ldots \otimes w_d) := gw_1 \otimes \ldots \otimes gw_d, \quad \text{where } g \in \mathrm{GL}(W),$$
$$\sigma(w_1 \otimes \ldots \otimes w_d) := w_{\sigma^{-1}(1)} \otimes \ldots \otimes w_{\sigma^{-1}(d)}, \quad \text{where } \sigma \in \mathcal{S}_d,$$

extended linearly to the entire tensor product. There is a correspondence between the irreducible representations of $\mathrm{GL}(W)$ and the irreducible representations of $\mathcal{S}_d$, which was discovered by Schur. We first introduce the exterior and symmetric powers of W.

Exterior powers ([FH04, Appendix B.2]). The *exterior power* $\bigwedge^d W$ of the vector space W is the quotient space of $W^{\otimes d}$ by the subspace spanned by all $w_1 \otimes \ldots \otimes w_d - (-1)^{\mathrm{sgn}\,\sigma} w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(d)}$ with $\sigma \in \mathcal{S}_d$. We denote the coset of $w_1 \otimes \ldots \otimes w_d$ by $w_1 \wedge \ldots \wedge w_d$. Define $\bigwedge^0 W$ to be the ground field. If $\{b_i\}_{i=\overline{1,n}}$ is a basis for W, then $\{b_{i_1} \wedge \ldots \wedge b_{i_d} \mid i_1 < \ldots < i_d\}$ is a basis for $\bigwedge^d W$.

Symmetric powers ([FH04, Appendix B.2]). The *symmetric power* $S^d W$ of the vector space W is the quotient space of $W^{\otimes d}$ by the subspace spanned by all $w_1 \otimes \ldots \otimes w_d - w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(d)}$ with $\sigma \in \mathcal{S}_d$. We denote the coset of $w_1 \otimes \ldots \otimes w_d$ by $w_1 \cdot \ldots \cdot w_d$. Define $S^0 W$ to be the ground field. If $\{b_i\}_{i=\overline{1,n}}$ is a basis for W, then $\{b_1^{i_1} \cdot \ldots \cdot b_n^{i_n} \mid i_1 + \ldots + i_n = d\}$ is a basis for $S^d W$, hence we can see this space as the space of homogeneous polynomials of degree d in the variables b_i.

Schur modules ([FH04, Chap. 6]). Consider an integer $d \geq 1$. We call $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of d if $\lambda_1 \geq \ldots \lambda_k \geq 1$ and $d = \lambda_1 + \ldots + \lambda_k$. The number of irreducible representations of $\mathcal{S}_d$ coincides with the number of partitions of d ([FH04, Chap. 4]). One can associate to each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d a diagram of type
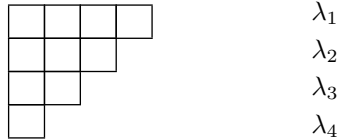
with λ_i boxes in the i-th row. For example, the diagram above corresponds to the partition $(4, 3, 2, 1)$ of 10. These kind of diagrams are called *Young diagrams*. The number of rows in the Young diagram of the partition λ of d is called the *height* of λ and denoted $|\lambda|$.

By numbering the boxes in a Young diagram by the integers $1, 2, \ldots, d$, we obtain a *tableau* of *shape* the given Young diagram. For example, with the Young diagram above we get

1	2	3	4
5	6	7	
8	9		
10			

Given a partition λ of d and a tableau of the Young diagram associated to λ, one can define the following two subgroups of $\mathcal{S}_d$:

$$P_\lambda = \{\sigma \in \mathcal{S}_d \mid \sigma \text{ preserves each row}\}$$
$$Q_\lambda = \{\sigma \in \mathcal{S}_d \mid \sigma \text{ preserves each column}\}$$

Each $\sigma \in \mathcal{S}_d$ is associated to basis element e_σ in the group algebra of $\mathcal{S}_d$ (the structure of algebra is given by $e_{\sigma_1} \cdot e_{\sigma_2} = e_{\sigma_1 \sigma_2}$). Define

$$c_\lambda = \left(\sum_{\sigma \in P_\lambda} e_\sigma \right) \cdot \left(\sum_{\sigma \in Q_\lambda} \text{sgn}(\sigma) e_\sigma \right), \quad \text{and} \quad S_\lambda W = \text{Im}(c_\lambda|_{W^{\otimes d}}).$$

The subspaces $S_\lambda W$ of $W^{\otimes d}$ obtained in this way are called *Schur modules*.

Example 2.8.1. If $\lambda = (2, 1)$ is a partition of 3, the tableau associated to it is

1	2
3	

We have $c_\lambda = (e_1 + e_{(12)}) \cdot (e_1 - e_{(13)}) = e_1 + e_{(12)} - e_{(13)} - e_{(312)}$ and $S_\lambda W$ is the subspace of $W^{\otimes 3}$ spanned by all

$$w_1 \otimes w_2 \otimes w_3 + w_2 \otimes w_1 \otimes w_3 - w_3 \otimes w_2 \otimes w_1 - w_3 \otimes w_1 \otimes w_2.$$

If $\lambda = (d)$, then $S_\lambda W$ is the subspace of $W^{\otimes d}$ spanned by $\sum_{\sigma \in \mathcal{S}_d} w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(d)}$, hence $S_\lambda W \cong S^d W$.
If $\lambda = (1, \ldots, 1)$ with $|\lambda| = d$, then $S_\lambda W$ is the subspace of $W^{\otimes d}$ spanned by all $\sum_{\sigma \in \mathcal{S}_d} \text{sgn}(\sigma) w_{\sigma(1)} \otimes \ldots \otimes w_{\sigma(d)}$, hence $S_\lambda W \cong \bigwedge^d W$.

Theorem 2.8.1. ([Pro07, Chap. 8.1]) *Consider a finite-dimensional complex vector space W with* $\dim W = n$. *Then:*

(1) The list of irreducible representations of $\text{GL}(W)$ *is*

$$S_\lambda W \otimes \left(\bigwedge^n W \right)^{\otimes k}, \quad |\lambda| \leq n - 1, \quad k \in \mathbb{Z}.$$

(2) The list of irreducible representations of $\mathrm{SL}(W)$ *is*

$$S_\lambda W, \quad |\lambda| \le n - 1.$$

Theorem 2.8.2. ([FH04, Chap. 6]) *Consider a finite-dimensional complex vector space* W *with* $\dim W = n$. *Then:*

(1) $S_\lambda W$ *is zero if the partition* λ *is of the form* $(\lambda_1, \ldots, \lambda_d)$, *with* $d > n$. *Otherwise*

$$\dim S_\lambda W = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{2.3}$$

In particular, $\dim S^d W = \binom{n+d-1}{d}$ *and* $\dim \bigwedge^d W = \binom{n}{d}$.

(2) *If* m_λ *is the dimension of the irreducible representation of* S_d *corresponding to* λ, *then*

$$W^{\otimes d} \simeq \bigoplus_\lambda m_\lambda S_\lambda W.$$

Example 2.8.2. We have the following decompositions:

$$W \otimes W = S^2 W \oplus \bigwedge^2 W, \quad W \otimes W \otimes W = S^3 W \oplus 2 S_{(2,1)} W \oplus \bigwedge^3 W,$$

(see [FH04, Chap. 6]). With the formula 2.3, $\dim S^3 W = \binom{n+2}{3}$, $\dim S_{(2,1)} W = 2\binom{n+1}{3}$, and $\dim \bigwedge^3 W = \binom{n}{3}$.

Theorem 2.8.3. ([FH04, Chap. 6]) *Consider two finite-dimensional complex vector spaces* W_1 *and* W_2. *Then,*

$$S^d(W_1 \oplus W_2) = \bigoplus_{a+b=d} (S^a W_1 \otimes S^b W_2);$$

$$\bigwedge^d(W_1 \oplus W_2) = \bigoplus_{a+b=d} (\bigwedge^a W_1 \otimes \bigwedge^b W_2);$$

$$S^d(W_1 \otimes W_2) = \bigoplus_{|\lambda|=d} S_\lambda W_1 \otimes S_\lambda W_2;$$

$$\bigwedge^d(W_1 \otimes W_2) = \bigoplus_{|\lambda|=d} S_\lambda W_1 \otimes S_{\lambda'} W_2,$$

where λ' *is the conjugate partition of* λ, *obtained by interchanging rows and columns in the Young diagram corresponding to* λ.

Example 2.8.3. If W_1 and W_2 are two finite-dimensional vector spaces,

$$S^2(W_1 \oplus W_2) = S^2 W_2 \oplus (W_1 \otimes W_2) \oplus S^2 W_1;$$

$$\bigwedge^2(W_1 \oplus W_2) = \bigwedge^2 W_2 \oplus (W_1 \otimes W_2) \oplus \bigwedge^2 W_1;$$

$$S^2(W_1 \otimes W_2) = (S^2 W_1 \otimes S^2 W_2) \oplus (\bigwedge^2 W_1 \otimes \bigwedge^2 W_2);$$

$$\bigwedge^2(W_1 \otimes W_2) = (S^2 W_1 \otimes \bigwedge^2 W_2) \oplus (\bigwedge^2 W_1 \otimes S^2 W_2).$$

Remark 2.8.4. *Consider the vector space V_n of binary forms of degree n. Then we identify V_n with $S^n W^*$, where $W = \mathbb{C}^2$. The modules $S^d V_n = S^d(S^n W^*)$ and $\bigwedge^d V_n = \bigwedge^d(S^n W^*)$ decompose into irreducible representations of SL_2. For example,*

$$S^4 V_6 = 2V_0 + 2V_4 + V_6 + 3V_8 + V_{10} + 3V_{12} + V_{14} + 2V_{16} + V_{18} + V_{20} + V_{24},$$

$$\bigwedge^4 V_6 = V_0 + V_4 + V_6 + V_8 + V_{12}.$$

(We computed these decompositions using the program LiE [LCL92].)

Proposition 2.8.5. *[Pro07, Chap. 15] The space of covariants of V_n of degree d and order e is the sum of all irreducible representations of $S^d V_n^*$ of type V_e. In particular, the dimension of the vector space of covariants of degree d and order e of V_n equals the multiplicity with which V_e appears in the decomposition of $S^d V_n^*$.*

Example 2.8.4. Consider $f \in V_3$. We have:

$$S^1(V_3) = V_3, \qquad\qquad S^2(V_3) = V_2 + V_6,$$
$$S^3(V_3) = V_3 + V_5 + V_9, \quad S^4(V_3) = V_0 + V_4 + V_6 + V_8 + V_{12}.$$

(We computed these decompositions using the program LiE [LCL92].)

In other words, the vector space of covariants of V_3 of degree 1 and order 3 has dimension 1 (spanned by f itself), the vector space of covariants of V_3 of degree 2 and order 2 has dimension 1 (spanned by $(f, f)_2$), the vector space of invariants of V_3 of degree 2 has dimension 1 (spanned by $((f, f)_2, (f, f)_2)_2)$, etc.

2.9 The type of the generating covariants

Denote by nV_m the direct sum $\bigoplus_{i=1}^{n} V_m$ of m copies of V_n. The space nV_m can be identified with the $GL_n \times SL_2$ module $\mathbb{C}^{n*} \otimes V_m$.

The covariants of nV_m are coupled with GL_n-subspaces of

$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_m) = \bigoplus_d S^d(\mathbb{C}^{n*} \otimes V_m) = \bigoplus_d \bigoplus_{|\lambda|=d} S_\lambda(\mathbb{C}^{n*}) \otimes S_\lambda(V_m)$$

Example 2.9.1. The generating covariants of nV_1 are the linear forms themselves $l_i \in V_1$, with $1 \le i \le n$, and the invariants $(l_i, l_j)_1$, with $1 \le i < j \le n$. $\mathcal{O}(\mathbb{C}^{n*} \otimes V_1)$ decomposes in the following way:

$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_1)_1 = \mathbb{C}^{n*} \otimes V_1,$$
$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_1)_2 = S^2\mathbb{C}^{n*} \otimes S^2 V_1 \oplus \bigwedge^2\mathbb{C}^{n*} \otimes \bigwedge^2 V_1 =$$
$$= S^2\mathbb{C}^{n*} \otimes V_2 \oplus \bigwedge^2\mathbb{C}^{n*} \otimes V_0.$$

(We used the program LiE [LCL92] to decompose $S^d V_1$ and $\bigwedge^d V_1$.)

The generating covariants of nV_1 of degree 1 and order 1 are of type $\mathbb{C}^{n*} \otimes V_1$, and the vector space of covariants of nV_1 of degree 1 and order 1 has dimension $n = \dim \mathbb{C}^{n*}$.

The generating invariants of nV_1 of degree 2 are of type $\bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, and the vector space of invariants of nV_1 of degree 2 has dimension $\binom{n}{2} = \dim \bigwedge^2 \mathbb{C}^{n*}$.

Chapter 3

Computational methods

3.1 The method

In this chapter we describe the computational approach to finding the generating invariants of SL$_2$-modules $V := V_{n_1} \oplus \ldots \oplus V_{n_p}$, with $n_i \geq 1$ and $p \geq 1$. Denote $I := \mathcal{O}(V)^{\mathrm{SL}_2}$ and by I_i the vector space of invariants of V of degree i. We start by giving a short description of our approach, followed in the second part of this chapter by examples and computational details.

The strategy used for finding the generating invariants of V combines mathematical results proved in the last two centuries. We start by determining an upper bound on the degree of the generating invariants of V: we find the degrees $d_1, \ldots, d_p$ of a homogeneous system of parameters of I and we compute the Poincaré series of I; then by Proposition 2.4.4 we obtain an upper bound on the degree of the generating invariants of V. Once we have this bound, called b, we construct a set of generators of I in the following way: we start with $i = 2$; if $\dim_{\mathbb{C}} I_i \neq 0$, then we look first at the set of invariants of degree i generated by invariants in smaller degrees, to see what part of I_i is known. If the known invariants don't span yet this vector space, we construct further invariants of degree i, until we have a set of invariants spanning I_i. Then, we increase i by 1 and repeat the procedure. We stop when i is greater than the upper bound b on the degree of the generating invariants.

As a remark, the hard part of the computational work involved in finding the generating invariants of V consists of finding a homogeneous system of parameters for the algebra of invariants I or the degrees of such a system (which gives us an upper bound b on the degree of the generators) and of investigating the existence of generators up to degree b. For example the generating invariants of V_9 have degrees ≤ 22, while the upper bound on the degree of the generating invariants is 66 (see Chap. 4.8). This means that we had to prove that no generators occured in degrees > 22 and ≤ 66, which required computing a basis of vector spaces of dimension at most $\dim \mathcal{O}(V_9)_{66}^{\mathrm{SL}_2} = 99074$. The entire computation was done in less than one month (see [BP10a]).

3.2   The computation of invariants and covariants

The main tool for computing invariants and covariants of binary forms are the transvectants. We implemented the formula 2.1 (Chap. 2.1), computing the p-th transvectant of two binary forms (in `Mathematica` [Wol10], see Chap. A.1 for details about implementation).

Example 3.2.1. For $f = a_0x^4 + 4a_1x^3y + \ldots a_4y^4 \in V_4$, we compute $(f,f)_4$, $(f,f)_2$, and $(f,(f,f)_2)_4$ (with `Mathematica` [Wol10]):

```
In[]:=ff=Table[a[4-i]Binomial[4,i],{i,0,4}]
Out[]={a[4],4a[3],6a[2],4a[1],a[0]}
    //this corresponds to the binary form
    x^4a[0]+4x^3ya[1]+6x^2y^2a[2]+4xy^3a[3]+y^4a[4]
In[]:=Transvectant[ff,ff,4]
Out[]={6a[2]^2-8a[1]a[3]+2a[0]a[4]}
In[]:=Transvectant[ff,ff,2]
Out[]={-2a[3]^2+2a[2]a[4],-4a[2]a[3]+4a[1]a[4],-6a[2]^2+4a[1]a[3]+2a[0]a[4],
 -4a[1]a[2]+4a[0]a[3],-2a[1]^2+2a[0]a[2]}
```

Hence:

$$(f,f)_2 = 6a_2^2 - 8a_1a_3 + 2a_0a_4,$$
$$(f,(f,f)_2)_4 = 2x^4(a_0a_2 - a_1^2) + 4x^3y(a_0a_3 - a_1a_2) + 2x^2y^2(2a_1a_3 + a_0a_4 - 3a_2^2) + 4xy^3(a_1a_4 - a_2a_3) + 2y^4(a_2a_4 - a_3^2).$$

3.3   The computation of the Poincaré series

We implemented an algorithm computing the Poincaré series of modules of covariants of $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, based on a result of Broer [Bro94] (see Chap. 2.5.1). The function `poincare[{n_1,...n_p}]` returns a closed expression as a rational function for the Poincaré series of the $\mathcal{O}(V)^{\mathrm{SL}_2}$-module of covariants $\mathcal{C}(V)_m$, with $m \leq \prod_{i=1}^p \lfloor \frac{n_i+1}{2} \rfloor \lceil \frac{n_i+1}{2} \rceil - 2$ (in `Mathematica` [Wol10], see Chap. A.1 for details about implementation).

Example 3.3.1. We compute the Poincaré series of

$$\mathcal{O}(V_2)^{\mathrm{SL}_2}, \mathcal{O}(V_3)^{\mathrm{SL}_2}, \mathcal{O}(V_4)^{\mathrm{SL}_2}, \mathcal{O}(V_2 \oplus V_2)^{\mathrm{SL}_2}, \mathcal{O}(V_1 \oplus V_3)^{\mathrm{SL}_2} :$$

(with `Mathematica` [Wol10])

```
In[]:= poincare[{2}]
Out[]= {1/(1 - t^2)}
In[]:= poincare[{3}]
Out[]= {1/(1 - t^4), 0, t^2/(1 - t^4)}
In[]:= poincare[{4}]
Out[]= {1/(1 - t^2 - t^3 + t^5), 0, 0, 0, t/((-1 + t)^2 (1 + t + t^2))}
In[]:= poincare[{2, 2}]
Out[]= {-1/(-1 + t^2)^3, 0, -t(2 + t)/(-1 + t^2)^3}
In[]:= poincare[{1, 3}]
Out[]= {(-1 + t^2 - t^4)/((-1 + t^2)^3 (1 + t^2)^2),-t/((-1 + t^2)^3 (1 + t^2)),
 -3t^2/((-1 + t^2)^3 (1 + t^2)^2),t(-1 - 2 t^2 + t^4)/((-1 + t^2)^3 (1 + t^2))}
```

Hence:

$$
\begin{aligned}
P_{\mathcal{O}(V_2)^{\mathrm{SL}_2}}(t) &= \frac{1}{1-t^2}, \\
P_{\mathcal{O}(V_3)^{\mathrm{SL}_2}}(t) &= \frac{1}{1-t^4}, \\
P_{\mathcal{C}(V_3)_2}(t) &= \frac{t^2}{1-t^4}, \\
P_{\mathcal{O}(V_4)^{\mathrm{SL}_2}}(t) &= \frac{1}{(1-t^2)(1-t^3)}, \\
P_{\mathcal{C}(V_4)_4}(t) &= \frac{t+t^2}{(1-t^2)(1-t^3)}, \\
P_{\mathcal{O}(V_2\oplus V_2)^{\mathrm{SL}_2}}(t) &= \frac{1}{(1-t^2)^3}, \\
P_{\mathcal{O}(V_1\oplus V_3)^{\mathrm{SL}_2}}(t) &= \frac{1-t^2+t^4}{(1-t^2)^3(1+t^2)^2} = \frac{1+t^6}{(1-t^4)^3}.
\end{aligned}
$$

The computation of $\dim_{\mathbb{C}} \mathcal{O}(V)_i^{\mathrm{SL}_2}$**.** These dimensions can be computed directly from the Poincaré series of V: the coefficient of t^i in the Poincaré series of V equals $\dim_{\mathbb{C}} \mathcal{O}(V)_i^{\mathrm{SL}_2}$.

Example 3.3.2. The Poincaré series of the algebra of invariants of V_8 is

$$
P_{\mathcal{O}(V_8)^{\mathrm{SL}_2}}(t) = \frac{1+t^8+t^9+t^{10}+t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)}.
$$

Then we have (with `Mathematica` [Wol10]):

```
In[]:=sp=poincare[{8}][[1]];
In[]:=Series[sp,{t,0,22}]
Out[]=1+t^2+t^3+2t^4+2t^5+4t^6+4t^7+7t^8+8t^9+12t^10+13t^11+20t^12+22t^13+31t^14+
    36t^15+47t^16+54t^17+71t^18+80t^19+102t^20+117t^21+144t^22+O[t]^23
```

Hence

$$
P_{\mathcal{O}(V_8)^{\mathrm{SL}_2}}(t) = 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + 7t^8 + \ldots
$$

and

$$
\begin{aligned}
\dim_{\mathbb{C}}\mathcal{O}(V_8)_2^{\mathrm{SL}_2} &= 1, \quad \dim_{\mathbb{C}}\mathcal{O}(V_8)_3^{\mathrm{SL}_2} = 1, \\
\dim_{\mathbb{C}}\mathcal{O}(V_8)_4^{\mathrm{SL}_2} &= 2, \quad \dim_{\mathbb{C}}\mathcal{O}(V_8)_5^{\mathrm{SL}_2} = 2, \\
\dim_{\mathbb{C}}\mathcal{O}(V_8)_6^{\mathrm{SL}_2} &= 4, \quad \ldots
\end{aligned}
$$

Another way of computing $\dim_{\mathbb{C}} \mathcal{O}(V_n)_i^{\mathrm{SL}_2}$ is using the formula 2.2 (Chap. 2.5). The function `CoefficientPoincare[k_,r_]` returns the dimension $\dim_{\mathbb{C}} \mathcal{O}(V_r)_k^{\mathrm{SL}_2}$ of the vector space of invariants of V_r of degree k (in `Mathematica` [Wol10], see Chap. A.1 for details about implementation).

Example 3.3.3. Consider $n = 8$. Then we have (with `Mathematica` [Wol10]):

```
In[]:= CoefficientPoincare[8, 4]
Out[]= 2
In[]:= CoefficientPoincare[8, 5]
Out[]= 2
In[]:= CoefficientPoincare[8, 6]
Out[]= 6
```

Hence

$$\dim_{\mathbb{C}} \mathcal{O}(V_8)_4^{\mathrm{SL}_2} = 2, \quad \dim_{\mathbb{C}} \mathcal{O}(V_8)_5^{\mathrm{SL}_2} = 2,$$
$$\dim_{\mathbb{C}} \mathcal{O}(V_8)_6^{\mathrm{SL}_2} = 4, \quad \ldots$$

Alternatively, $\dim_{\mathbb{C}} \mathcal{O}(V)_i^{\mathrm{SL}_2}$ equals the multiplicity of V_0 in the decomposition of the i-th symmetric power $S^i(V_n)$ as an SL_2-module (see Proposition 2.8.5). This decomposition can be computed using `LiE` [LCL92] or `Mathematica` [Wol10].

Example 3.3.4. We have $\dim_{\mathbb{C}} \mathcal{O}(V_8)_6^{\mathrm{SL}_2} = 4$ because the multiplicity of V_0 in the decomposition of $S^6(V_8)$ is 4.
With `LiE` [LCL92]:

```
>sym_tensor(6,[8],A1)
4X[0]+1X[2]+7X[4]+5X[6]+11X[8]+7X[10]+13X[12]+9X[14]+13X[16]+10X[18]+12X[20]+
8X[22]+11X[24]+7X[26]+8X[28]+5X[30]+6X[32]+3X[34]+4X[36]+2X[38]+2X[40]+1X[42]+
1X[44]+1X[48]
```

With `Mathematica` [Wol10]:

```
In[]:=SymTensor[8,6]
Out[]=4v[0]+v[2]+7v[4]+5v[6]+11v[8]+7v[10]+13v[12]+9v[14]+13v[16]+10v[18]+
12v[20]+8v[22]+11v[24]+7v[26]+8v[28]+5v[30]+6v[32]+3v[34]+4v[36]+2v[38]+2v[40]+
v[42]+v[44]+v[48]
```

3.4 Linearly independent invariants

One situation that will often occur is the following: given a set `listinv` of invariants of V of degrees `listdeg` and $i \in \mathbb{N}$, decide whether `listinv` spans the vector space I_i of invariants of V of degree i. We check this in the following way.

The first step is to write down all invariants of degree i spanned by the elements of `listinv`. Then we evaluate these invariants at $\dim_{\mathbb{C}} I_i$ random points in V, and calculate the rank of the resulting matrix. If the rank coincides with $\dim_{\mathbb{C}} I_i$, then `listinv` spans I_i.

For the case $V = V_n$ we implemented the function `Dim[n,i,listinv,listdeg]`, which does the following:

- computes and returns $\dim_{\mathbb{C}} I_i$;

- computes the set M of all monomials of degree i spanned by the elements of `listinv` of degree $< i$; returns the size of M;

- evaluates all the elements of M at $\dim_{\mathbb{C}} I_i$ random points in V; returns the rank of the resulting matrix (because the rank computations are made modulo a prime, we obtain a lower bound on the dimension of the subspace of I_i spanned by the elements of M);

- if `listinv` contains invariants of degree i, these invariants are added to M; then, all the elements of the new M are evaluated at $\dim_{\mathbb{C}} I_i$ random points in V; returns the rank of the resulting matrix (because the rank computations are made modulo a prime, we obtain a lower bound on the dimension of the subspace of I_i spanned by the elements of `listinv`).

Example 3.4.1. Let $f \in V_8$. Consider the invariants

$$j_2 = (f, f)_8, \qquad\qquad j_3 = (f, (f, f)_4)_8,$$
$$j_4 = ((f, f)_6, (f, f)_6)_4, \qquad j_6 = (((f, f)_6, (f, f)_6)_2, (f, f)_6)_4,$$

and `listinv`$= \{j_2, j_3, j_4\}$. Then the invariants of degree 6 spanned by `listinv` are $\{j_2^3, j_2 j_4, j_3^2\}$. After evaluating $\{j_2^3, j_2 j_4, j_3^2, j_6\}$ at 4 random points in V_8, we obtain a matrix of rank ≥ 4. On the other hand we know that $4 = \dim \mathcal{O}(V_8)_6^{\mathrm{SL}_2}$, from the Poincaré series of $\mathcal{O}(V_8)^{\mathrm{SL}_2}$, hence it follows that $\{j_2, j_3, j_4, j_6\}$ span the vector space of invariants of V_8 of degree 6 (with `Mathematica` [Wol10]):

```
In[]:=ff=Table[Binomial[8,i]*a[8-i],{i,0,8}]
Out[]={a[8],8a[7],28a[6],56a[5],70a[4],56a[3],28a[2],8a[1],a[0]}
In[]:=j2=Transvectant[ff,ff,8].{1};
In[]:=j3=Transvectant[ff,Transvectant[ff,ff,4],8].{1};
In[]:=j4=Transvectant[Transvectant[ff,ff,6],Transvectant[ff,ff,6],4].{1};
In[]:=cc1=Transvectant[ff,ff,6];cc2=Transvectant[ff,cc1,4];
In[]:=j6=Transvectant[cc1,Transvectant[cc1,cc1,2],4].{1};
In[]:=Dim[8,6,{j2,j3,j4,j6},{{2,1},{3,1},{4,1},{6,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_6 is 4
<<There are 3 invariants of degree 6 spanned by the input invariants
of degrees <6
<<The subspace of O(V_8)^SL2_6 spanned by the input invariants of
degrees <6 has dimension >=3
<<The subspace of O(V_8)^SL2_6 spanned by the input invariants of
degrees <=6 has dimension >=4
```

3.5 The example of the binary octavic

In this chapter we illustrate our computational methods by finding a set of generating invariants of the binary octavic. First we look for an upper bound on the degree of the generating invariants of the binary octavic. The algebra of invariants of V_8 has a homogeneous system of parameters of degrees $2, 3, 4, 5, 6, 7$ (see Proposition 4.7.1).
We compute the Poincaré series of this algebra (in `Mathematica` [Wol10]):

```
In[]:=sp=poincare[{8}][[1]];
In[]:=Simplify[sp*(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)]
Out[]=1+t^8+t^9+t^10+t^18
```

```
In[]:=Series[sp,{t,0,22}]
Out[]=1+t^2+t^3+2t^4+2t^5+4t^6+4t^7+7t^8+8t^9+12t^10+13t^11+20t^12+22t^13+31t^14+
  36t^15+47t^16+54t^17+71t^18+80t^19+102t^20+117t^21+144t^22+O[t]^23
```

Hence the Poincaré series can be written as:

$$P(t) = \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)}$$

and

$$P(t) = 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + 7t^8 + 8t^9 + 12t^{10} + \ldots + 71t^{18} + \ldots$$

The highest degree in which we have to look for generators is 18. We start with degree 2 and take $j_2 = (f, f)_8$, which spans the vector space of invariants of degree 2:

```
In[]:=ff=Table[Binomial[8,i]*a[8-i],{i,0,8}]
Out[]={a[8],8a[7],28a[6],56a[5],70a[4],56a[3],28a[2],8a[1],a[0]}
In[]:=j2=Transvectant[ff,ff,8].{1};
In[]:=Dim[8,2,{j2},{{2,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_2 is 1
<<There are 0 invariants of degree 2 spanned by the input invariants
of degrees <2
<<The subspace of O(V_8)^SL2_2 spanned by the input invariants of
degrees <2 has dimension >=0
<<The subspace of O(V_8)^SL2_2 spanned by the input invariants of
degrees <=2 has dimension >=1
```

In degree 3 we take $j_3 = (f, (f, f)_4)_8$, which spans the vector space of invariants of degree 3:

```
In[]:=j3=Transvectant[ff,Transvectant[ff,ff,4],8].{1};
In[]:=Dim[8,3,{j2,j3},{{2,1},{3,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_3 is 1
<<There are 0 invariants of degree 3 spanned by the input invariants
of degrees <3
<<The subspace of O(V_8)^SL2_3 spanned by the input invariants of
degrees <3 has dimension >=0
<<The subspace of O(V_8)^SL2_3 spanned by the input invariants of
degrees <=3 has dimension >=1
```

In degree 4 we take $j_4 = ((f, f)_6, (f, f)_6)_4$. We evaluate $\{j_4, j_2^2\}$ at $2 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_4^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 2. It follows that the vector space of invariants of degree 4 is spanned by $\{j_2, j_4\}$:

```
In[]:=j4=Transvectant[Transvectant[ff,ff,6],Transvectant[ff,ff,6],4].{1};
In[]:=Dim[8,4,{j2,j3,j4},{{2,1},{3,1},{4,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_4 is 2
<<There are 1 invariants of degree 4 spanned by the input invariants
of degrees <4
<<The subspace of O(V_8)^SL2_4 spanned by the input invariants of
degrees <4 has dimension >=1
<<The subspace of O(V_8)^SL2_4 spanned by the input invariants of
degrees <=4 has dimension >=2
```

Denote $c_1 = (f, f)_6$ and $c_2 = (f, c_1)_4$.

In degree 5 we take $j_5 = (c_1, c_2)_4$. We evaluate $\{j_5, j_2 j_3\}$ at $2 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_5^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 2. It follows that the vector space of invariants of degree 5 is spanned by $\{j_2, j_3, j_5\}$:

```
In[]:=cc1=Transvectant[ff,ff,6];cc2=Transvectant[ff,cc1,4];
In[]:=j5=Transvectant[cc1,cc2,4].{1};
In[]:=Dim[8,5,{j2,j3,j4,j5},{{2,1},{3,1},{4,1},{5,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_5 is 2
<<There are 1 invariants of degree 5 spanned by the input invariants
of degrees <5
<<The subspace of O(V_8)^SL2_5 spanned by the input invariants of
degrees <5 has dimension >=1
<<The subspace of O(V_8)^SL2_5 spanned by the input invariants of
degrees <=5 has dimension >=2
```

In degree 6 we take $j_6 = (c_1, (c_1, c_1)_2)_4$. We evaluate $\{j_6, j_2^3, j_2 j_4, j_3^2\}$ at $4 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_6^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 4. It follows that the vector space of invariants of degree 6 is spanned by $\{j_2, j_3, j_4, j_6\}$:

```
In[]:=j6=Transvectant[cc1,Transvectant[cc1,cc1,2],4].{1};
In[]:=Dim[8,6,{j2,j3,j4,j5,j6},{{2,1},{3,1},{4,1},{5,1},{6,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_6 is 4
<<There are 3 invariants of degree 6 spanned by the input invariants
of degrees <6
<<The subspace of O(V_8)^SL2_6 spanned by the input invariants of
degrees <6 has dimension >=3
<<The subspace of O(V_8)^SL2_6 spanned by the input invariants of
degrees <=6 has dimension >=4
```

In degree 7 we take $j_7 = (c_1, (c_1, c_2)_2)_4$. We evaluate $\{j_7, j_2^2 j_3, j_3 j_4, j_2 j_5\}$ at $4 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_7^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 4. It follows that the vector space of invariants of degree 7 is spanned by $\{j_2, j_3, j_4, j_5, j_7\}$:

```
In[]:=j7=Transvectant[cc1,Transvectant[cc1,cc2,2],4].{1};
In[]:=Dim[8,7,{j2,j3,j4,j5,j6,j7},{{2,1},{3,1},{4,1},{5,1},{6,1},{7,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_7 is 4
<<There are 3 invariants of degree 7 spanned by the input invariants
of degrees <7
<<The subspace of O(V_8)^SL2_7 spanned by the input invariants of
degrees <7 has dimension >=3
<<The subspace of O(V_8)^SL2_7 spanned by the input invariants of
degrees <=7 has dimension >=4
```

In degree 8 we take $j_8 = (c_2, (c_1, c_2)_2)_4$. We evaluate $\{j_8, j_2^4, j_2^2 j_4, j_2 j_6, j_2 j_3^2, j_3 j_5,$ $j_4^2\}$ at $7 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_8^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 7. It follows that the vector space of invariants of degree 8 is spanned by $\{j_2, j_3, j_4, j_5, j_6, j_8\}$:

```
In[]:=j8=Transvectant[cc2,Transvectant[cc1,cc2,2],4].{1};
In[]:=Dim[8,8,{j2,j3,j4,j5,j6,j7,j8},{{2,1},{3,1},{4,1},{5,1},{6,1},{7,1},{8,1}}]
```

```
Out[]=
<<The dimension of O(V_8)^SL2_8 is 7
<<There are 6 invariants of degree 8 spanned by the input invariants
of degrees <8
<<The subspace of O(V_8)^SL2_8 spanned by the input invariants of
degrees <8 has dimension >=6
<<The subspace of O(V_8)^SL2_8 spanned by the input invariants of
degrees <=8 has dimension >=7
```

In degree 9 we take $j_9 = (c_2, (c_2, c_2)_2)_4$. We evaluate $\{j_9, j_2 j_7, j_2^2 j_5, j_2^3 j_3, j_3 j_6, j_3^3,$ $j_4 j_5, j_2 j_3 j_4\}$ at $8 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_9^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 8. It follows that the vector space of invariants of degree 9 is spanned by $\{j_2, j_3, j_4, j_5, j_6, j_7, j_9\}$:

```
In[]:=j9=Transvectant[cc2,Transvectant[cc2,cc2,2],4].{1};
In[]:=Dim[8,9,{j2,j3,j4,j5,j6,j7,j8,j9},{{2,1},{3,1},{4,1},{5,1},{6,1},{7,1},
 {8,1},{9,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_9 is 8
<<There are 7 invariants of degree 9 spanned by the input invariants
of degrees <9
<<The subspace of O(V_8)^SL2_9 spanned by the input invariants of
degrees <9 has dimension >=7
<<The subspace of O(V_8)^SL2_9 spanned by the input invariants of
degrees <=9 has dimension >=8
```

In degree 10 we take $j_{10} = ((c_1, c_1)_2, (c_2, c_2)_2)_4$. Using invariants of degree ≤ 9 we construct a set of 11 monomials of degree 10. We evaluate these monomials, together with j_{10}, at $12 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_{10}^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 12. It follows that the vector space of invariants of degree 10 is spanned by $\{j_2, j_3, \ldots, j_{10}\}$:

```
In[]:=j10=Transvectant[Transvectant[cc2,cc2,2],Transvectant[cc1,cc1,2],4].{1};
In[]:=Dim[8,10,{j2,j3,j4,j5,j6,j7,j8,j9,j10},{{2,1},{3,1},{4,1},{5,1},{6,1},
 {7,1},{8,1},{9,1},{10,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_10 is 12
<<There are 11 invariants of degree 10 spanned by the input
invariants of degrees <10
<<The subspace of O(V_8)^SL2_10 spanned by the input invariants of
degrees <10 has dimension >=11
<<The subspace of O(V_8)^SL2_10 spanned by the input invariants of
degrees <=10 has dimension >=12
```

Using invariants of degree ≤ 10 we construct 73 monomials of degree 18. We evaluate these monomials at $71 = \dim_{\mathbb{C}} \mathcal{O}(V_8)_{18}^{\mathrm{SL}_2}$ random points in V_8 and obtain a matrix of rank 71. It follows that the vector space of invariants of degree 18 is spanned by $\{j_2, j_3, \ldots, j_{10}\}$:

```
In[]:=Dim[8,18,{j2,j3,j4,j5,j6,j7,j8,j9,j10},{{2,1},{3,1},{4,1},{5,1},{6,1},
 {7,1},{8,1},{9,1},{10,1}}]
Out[]=
<<The dimension of O(V_8)^SL2_18 is 71
<<There are 73 invariants of degree 18 spanned by the input
```

```
 invariants of degrees <=10
<<The subspace of O(V_8)^SL2_18 spanned by the input invariants of
 degrees <=10 has dimension >=71
```

Then, the generating invariants of V_8 are $\{j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9, j_{10}\}$.

Chapter 4

Invariants of binary forms

In this chapter we find the basic invariants of V_n for $n \in \{2, 3, \ldots, 10\}$, and give explicit systems of parameters in all these cases. The sections 4.8 and 4.9 are based on joint work with Brouwer [BP10a, BP10b]. The main result of this chapter is summarised in the following proposition.

Proposition 4.0.1. *Let $I = \mathcal{O}(V_n)^{\mathrm{SL}_2}$, with $n \in \{2, 3, \ldots, 10\}$. Denote by r the minimal number of generators of I, and by d_i the number of generators of degree i. Then we have:*

$n = 2:$ $r = 1,$

i	2
d_i	1

$n = 3:$ $r = 1,$

i	4
d_i	1

$n = 4:$ $r = 2,$

i	2	3
d_i	1	1

$n = 5:$ $r = 4,$

i	4	8	12	18
d_i	1	1	1	1

$n = 6:$ $r = 5,$

i	2	4	6	10	15
d_i	1	1	1	1	1

$n = 7:$ $r = 30,$

i	4	8	12	14	16	18	20	22	26	30
d_i	1	3	6	4	2	9	1	2	1	1

$n = 8:$ $r = 9,$

i	2	3	4	5	6	7	8	9	10
d_i	1	1	1	1	1	1	1	1	1

$n = 9:$ $r = 92,$

i	4	8	10	12	14	16	18	20	22
d_i	2	5	5	14	17	21	25	2	1

$n = 10:$ $r = 106,$

i	2	4	6	8	9	10	11	12	13	14	15	16	17	18	19	21
d_i	1	1	4	5	5	8	8	12	15	13	19	5	5	1	2	2

Moreover, I has a system of parameters (hsop) of the following degrees:

n	*hsop degrees*
2	2
3	4
4	$2, 3$
5	$4, 8, 12$
6	$2, 4, 6, 10$
7	$4, 8, 12, 12, 20$
8	$2, 3, 4, 5, 6, 7$
9	$4, 8, 10, 12, 12, 14, 16$
10	$2, 4, 6, 6, 8, 9, 10, 14$

4.1   The invariants of the binary quadratic

Let $f \in V_2$,

$$f = a_0 x^2 + 2a_1 xy + a_2 y^2.$$

and consider the following invariant of V_2:

$$j_2 = (f, f)_2 = 2(a_1^2 - a_0 a_2).$$

Proposition 4.1.1. *The set $\{j_2\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_2)^{\mathrm{SL}_2}$ of invariants of the binary quadratic.*

Proof. We have $\mathcal{N}(V_2) = \mathcal{V}(j_2)$, and, from Proposition 2.7.1, it follows that $\{j_2\}$ forms a system of parameters of $\mathcal{O}(V_2)^{\mathrm{SL}_2}$. □

The Poincaré series of the algebra $\mathcal{O}(V_2)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1}{1 - t^2} = 1 + t^2 + t^4 + \dots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 2. Therefore, the invariants of the binary quadratic are generated by $\{j_2\}$.

4.2   The invariants of the binary cubic

Let $f \in V_3$,

$$f = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3,$$

and consider the following covariants and invariants of V_3:

$$c_1 = (f, f)_2, \qquad j_4 = (c_1, c_1)_2.$$

Proposition 4.2.1. *The set $\{j_4\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_3)^{\mathrm{SL}_2}$ of invariants of the binary cubic.*

Proof. We have $\mathcal{N}(V_3) = \mathcal{V}(j_4)$: if $j_4 = 0$, then c_1 is a nullform. If $c_1 = 0$, then f has a triple root (Proposition 2.7.2), i.e. f is a nullform. If $c_1 \neq 0$ is a nullform, then f has a root of multiplicity 2 (Lemma 2.7.3), i.e. f is a nullform.

Then, from Proposition 2.7.1, it follows that $\{j_4\}$ forms a homogeneous system of parameters of $\mathcal{O}(V_3)^{\mathrm{SL}_2}$. □

The Poincaré series of the algebra $\mathcal{O}(V_3)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1}{1 - t^4} = 1 + t^4 + t^8 + \dots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 4. Therefore, the invariants of the binary cubic are generated by $\{j_4\}$.

4.3 The invariants of the binary quartic

Let $f \in V_4$,
$$f = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4,$$
and consider the following covariants and invariants of V_4:

$$c_1 = (f,f)_2, \quad j_2 = (f,f)_4, \quad j_3 = (f,c_1)_4.$$

Proposition 4.3.1. *The set $\{j_2, j_3\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_4)^{\mathrm{SL}_2}$ of invariants of the binary quartic.*

Proof. We have $\mathcal{V}(j_2, j_3) = \mathcal{N}(V_4)$: if $j_2 = j_3 = 0$, then f has a root of multiplicity 3 (see Proposition 2.7.2). From Proposition 2.7.1, it follows that $\{j_2, j_3\}$ forms a system of parameters of $\mathcal{O}(V_4)^{\mathrm{SL}_2}$. □

The Poincaré series of the algebra $\mathcal{O}(V_4)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1}{(1 - t^2)(1 - t^3)} = 1 + t^2 + t^3 + t^4 + \dots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 3. Therefore, the invariants of the binary quartic are generated by $\{j_2, j_3\}$.

4.4 The invariants of the binary quintic

Let $f \in V_5$,
$$f = a_0 x^5 + 5a_1 x^4 y + \dots + 5a_4 xy^4 + a_5 y^5,$$
and consider the following covariants of V_5:

$$c_1 = (f,f)_4, \qquad c_2 = (f,f)_2,$$
$$c_3 = (f,c_1)_2, \qquad c_4 = (c_3,c_3)_2,$$

and the following invariants of V_5:

$$j_4 = (c_1, c_1)_2, \qquad j_8 = (c_4, c_1)_2,$$
$$j_{12} = (c_4, c_4)_2.$$

Proposition 4.4.1. *The invariants j_4, j_8, j_{12} form a homogeneous system of parameters of the algebra $\mathcal{O}(V_5)^{\mathrm{SL}_2}$ of invariants of the binary quintic.*

Proof. First, we show that $\mathcal{N}(V_5) = \mathcal{V}(j_4, j_8, j_{12})$.
If $j_4 = 0$, then c_1 is a nullform. We distinguish the following two cases.

Case 1: $c_1 = 0$.
Then, from Proposition 2.7.2 it follows that f has a root of multiplicity 4, i.e. f is a nullform.

Case 2: $c_1 \neq 0$.
We show that $(c_1, c_3) \in \mathcal{N}(V_2 \oplus V_3)$: in Chap. 5.9 we prove that $\mathcal{N}(V_2 \oplus V_3)$ is the zero set of the invariants j_4, j_8, j_{12}, and j'_{12}, where $j'_{12} := (c_1^3, c_3^2)_6$. But, modulo j_4, we have
$$3j_{12} + j'_{12} = 0.$$

Therefore, $(c_1, c_3) \in \mathcal{N}(V_2 \oplus V_3)$, if $j_4 = j_8 = j_{12} = 0$. Then, from Lemma 2.7.3 it follows that f has a root of multiplicity 3, i.e. f is a nullform.

We apply now Proposition 2.7.1: if $\mathcal{N}(V_5) = \mathcal{V}(j_4, j_8, j_{12})$, then $\{j_4, j_8, j_{12}\}$ forms a homogeneous system of parameters of $\mathcal{O}(V_5)^{\mathrm{SL}_2}$. $\qquad\square$

Proposition 4.4.2. *The algebra of invariants of the binary quintic is generated by 4 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	4	8	12	18
d_i	1	1	1	1

Proof. The Poincaré series of the algebra $\mathcal{O}(V_5)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^{18}}{(1 - t^4)(1 - t^8)(1 - t^{12})} =$$
$$= 1 + t^4 + 2t^8 + 3t^{12} + 4t^{16} + t^{18} + \dots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 18. The vector space of invariants of degree 4 is spanned by $\{j_4\}$. The vector space of invariants of degree 8 is spanned by $\{j_4^2, j_8\}$. The vector space of invariants of degree 12 is spanned by $\{j_4^3, j_4 j_8, j_{12}\}$. The vector space of invariants of degree 18 is spanned by $\{j_{18}\}$, where $j_{18} = (c_1^7, f(c_2, f)_1)_{14}$.

Then, the invariants of V_5 are generated by j_4, j_8, j_{12}, and j_{18}. $\qquad\square$

4.5 The invariants of the binary sextic

Let $f \in V_6$,

$$f = a_0 x^6 + 6a_1 x^5 y + \ldots + 6a_5 x y^5 + a_6 y^6,$$

and consider the following covariants and invariants of V_6:

$$
\begin{aligned}
c_1 &= (f, f)_4, & j_2 &= (f, f)_6, \\
c_2 &= (f, f)_2, & j_4 &= (c_1, c_1)_4, \\
c_3 &= (f, c_1)_4, & j_6 &= (c_4, c_1)_4, \\
c_4 &= (c_1, c_1)_2, & j_{10} &= (c_4, c_3^2)_4. \\
c_5 &= (f, c_1)_1.
\end{aligned}
$$

Proposition 4.5.1. *The set $\{j_2, j_4, j_6, j_{10}\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_6)^{\mathrm{SL}_2}$ of invariants of the binary sextic.*

Proof. First, we show that $\mathcal{N}(V_6) = \mathcal{V}(j_2, j_4, j_6, j_{10})$.
If $j_4 = j_6 = 0$, then c_1 is a nullform. We distinguish the following two cases.

Case 1: $c_1 = 0$.
In this case, because j_2 vanishes as well, from Proposition 2.7.2 it follows that f has a root of multiplicity 5, i.e. f is a nullform.

Case 2: $c_1 \neq 0$.
We show that $(c_1, c_3) \in \mathcal{N}(V_4 \oplus V_2)$: in Section 5.10 we prove that $\mathcal{N}(V_2 \oplus V_4)$ is the zero set of the invariants j_4, j_6, j_6', j_8, and j_{10}, where

$$j_6' = (c_3, c_3)_2, \qquad j_8 = (c_1, c_3^2)_4.$$

But, modulo j_2, we have $j_6' = 6j_6$. Also, an easy Gröbner basis computation shows that $j_8 \in (j_2, j_4, j_6, j_{10})$. Therefore, $(c_1, c_3) \in \mathcal{N}(V_4 \oplus V_2)$, if $j_2 = j_4 = j_6 = j_{10} = 0$. Then, from Lemma 2.7.4 it follows that f has a root of multiplicity 4, i.e. f is a nullform.

We apply now Proposition 2.7.1: if $\mathcal{N}(V_6) = \mathcal{V}(j_2, j_4, j_6, j_{10})$, then it follows that $\{j_2, j_4, j_6, j_{10}\}$ forms a homogeneous system of parameters of $\mathcal{O}(V_6)^{\mathrm{SL}_2}$. $\square$

Proposition 4.5.2. *The algebra of invariants of the binary sextic is generated by 5 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	4	6	10	15
d_i	1	1	1	1	1

Proof. The Poincaré series of the algebra $\mathcal{O}(V_6)^{\mathrm{SL}_2}$ is

$$
\begin{aligned}
P(t) &= \frac{1 + t^{15}}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})} = \\
&= 1 + t^2 + 2t^4 + 3t^6 + 4t^8 + 6t^{10} + 8t^{12} + 10t^{14} + t^{15} + \ldots
\end{aligned}
$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 15. The vector space of invariants of degree 2 is spanned by $\{j_2\}$. The vector space of invariants of degree 4 is spanned by $\{j_2^2, j_4\}$. The vector space of invariants of degree 6 is spanned by $\{j_2^3, j_2 j_4, j_6\}$. The vector space of invariants of degree 10 is spanned by $\{j_2^5, j_2^3 j_4, j_2^2 j_6, j_2 j_4^2, j_4 j_6, j_{10}\}$. The vector space of invariants of degree 15 is spanned by $\{j_{15}\}$, where $j_{15} = (c_3^4, c_5)_8$.

Then, the invariants of V_6 are generated by j_2, j_4, j_6, j_{10}, and j_{15}. $\qquad\square$

4.6 The invariants of the binary septic

Let $f \in V_7$,
$$f = a_0 x^7 + 7a_1 x^6 y + \ldots + 7a_6 xy^6 + a_7 y^7.$$
Consider the following covariants of V_7:

$$
\begin{aligned}
c_1 &= (f, f)_6, & c_2 &= (f, f)_4, \\
c_3 &= (f, f)_2, & c_4 &= (f, c_1)_2, \\
c_5 &= (c_2, c_2)_4, & c_6 &= (f, c_2)_5, \\
c_7 &= (c_4, c_4)_4, & c_8 &= (c_4, (c_4, c_4)_4)_2, \\
c_9 &= (c_6, c_6)_2, & c_{10} &= (c_5, c_6)_3,
\end{aligned}
$$

Lemma 4.6.1. *If $c_1 = 0$, $c_2 \neq 0$, and $c_2 \in \mathcal{N}(V_6)$, then f has a root of multiplicity at least 5.*

Proof. If c_2 is a nullform, then c_2 has a root of multiplicity at least 4. Without loss of generality we suppose $x^4 \mid c_2$. Also, by letting $\left(\begin{smallmatrix} 1 & 0 \\ \frac{a_6}{7a_7} & 1 \end{smallmatrix}\right)$ act on f, the coefficient of xy^6 in f becomes zero, and x^4 still divides c_2. We have $c_1 = 0$ and the coefficients of $x^3 y^3, x^2 y^4, xy^5, y^6$ in c_2 also vanish, which gives us the following equations:

$$
\begin{aligned}
10a_4^2 - 15a_3 a_5 - a_1 a_7 &= 0, & 3a_5^2 + a_3 a_7 &= 0, \\
5a_3 a_4 - 9a_2 a_5 - a_0 a_7 &= 0, & 2a_4 a_5 + a_2 a_7 &= 0, \\
10a_3^2 - 15a_2 a_4 + 6a_1 a_5 &= 0, & 5a_4^2 - 5a_3 a_5 + a_1 a_7 &= 0, \\
15a_3 a_4 - 21a_2 a_5 + a_0 a_7 &= 0.
\end{aligned}
$$

If $a_7 = 0$, it follows $a_5 = a_4 = a_3 = 0$, and then $x^5 \mid f$. If $a_7 \neq 0$, suppose $a_7 = 1$, which implies:

$$
\begin{aligned}
a_3 &= -3a_5^2, & a_2 &= -2a_4 a_5, \\
a_1 &= -5a_4^2 - 15a_5^3, & a_0 &= 3a_4 a_5^2,
\end{aligned}
$$

and $a_4^2 + 4a_5^3 = 0$. But then we have

$$c_2 = -90x^4 (-a_5 y^2 - a_4 xy + a_5^2 x^2)(a_4^2 + 4a_5^3) = 0,$$

which contradicts our assumption $c_2 \neq 0$. $\qquad\square$

Consider now the following invariants of V_7:

$$
\begin{aligned}
j_4 &= (c_1, c_1)_2, & j_{8,1} &= (c_7, c_1)_2, \\
j_{12,1} &= ((c_5, c_5)_2, c_5)_4, & j_{12,2} &= ((c_4, c_4)_2, c_1^3)_6, \\
j_{20} &= ([(c_2, c_5)_4]^2, (c_5, c_5)_2)_4.
\end{aligned}
$$

Proposition 4.6.2. *The set $\{j_4, j_{8,1}, j_{12,1}, j_{12,2}, j_{20}\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_7)^{\mathrm{SL}_2}$ of invariants of the binary septic.*

Proof. First we show that

$$
\mathcal{N}(V_7) = \mathcal{V}(j_4, j_{8,1}, j_{12,1}, j_{12,2}, j_{20}).
$$

If $j_4 = 0$, then c_1 is a nullform. We distinguish the following two cases.

Case 1: $c_1 = 0$.
In this case we have $(c_2, c_2)_6 \sim j_4 = 0$ (we use "$\sim$" for equalities up to a nonzero constant). A simple Gröbner basis computation shows that $(c_5, c_5)_4$ vanishes modulo the ideal generated by the coefficients of c_1. As $j_{12,1}$ and j_{20} also vanish, we obtain the equations of $\mathcal{N}_{V_6}$ (see Chap. 4.5), hence c_2 is a nullform. Then, from Lemma 4.6.1 it follows that f has a root of multiplicity 5, i.e. f is a nullform.

Case 2: $c_1 \neq 0$ and without loss of generality $c_1 = x^2$.
In this case we have $(c_1, c_4) \in \mathcal{N}(V_2 \oplus V_5)$: in Chap. 5.11 we show that j_4, $j_{8,1}$, and $j_{12,2}$, together with the following invariants, define the nullcone of $V_2 \oplus V_5$:

$$
\begin{aligned}
j_{12,3} &= (c_7, c_7)_2, & j_{16,1} &= (c_4^2, c_1^5)_{10}, \\
j_{24} &= ((c_8, c_8)_2, c_7)_2, & j_{36} &= ((c_8, c_8)_2, (c_8, c_8)_2)_2.
\end{aligned}
$$

We prove that $j_4 = j_{8,1} = j_{12,2} = j_{12,1} = j_{20} = 0$ implies that $j_{12,3} = j_{16,1} = j_{24} = j_{36} = 0$. By letting $\left(\begin{smallmatrix} 1 & 0 \\ \frac{a_6}{a_7} & 1 \end{smallmatrix} \right)$ act on f, we bring the coefficient of a_6 in f to zero, and x^2 still divides c_1. Then, we obtain (we use "$\sim$" for equalities up to a nonzero constant):

$$
j_{12,2} \sim a_5 a_7, \qquad\qquad j_{8,1} \sim 3a_5^2 + a_3 a_7.
$$

If $a_7 = 0$, then $a_5 = 0$ and, then, $j_{12,3} = j_{16,1} = j_{24} = j_{36} = 0$.
If $a_7 \neq 0$, then $a_5 = a_3 = 0$. Suppose $a_7 = 1$. In this case c_1 becomes

$$
c_1 \quad \sim \; -(10a_4^2 - a_1)y^2 + a_0 xy + 15a_2 a_4 x^2,
$$

and, because we supposed $x^2 \mid c_1$, we have $a_0 = 0$, $a_1 = 10a_4^2$ and $a_2, a_4 \neq 0$. Then we have:

$$
\begin{aligned}
j_{20} &\sim a_2^2(-779446800a_4^{20} - 4828896a_2^3 a_4^{15} + 67068a_2^6 a_4^{10} + \\
&\qquad + 252a_2^9 a_4^5 + a_2^{12}), \\
j_{12,1} &\sim a_4^4(7290a_4^{10} + 54a_2^3 a_4^5 + a_2^6).
\end{aligned}
$$

Because $a_2, a_4 \neq 0$, we obtain,

$$\begin{cases} -779446800a_4^{20} - 4828896a_2^3a_4^{15} + 67068a_2^6a_4^{10} + 252a_2^9a_4^5 + a_2^{12} = 0, \\ 7290a_4^{10} + 54a_2^3a_4^5 + a_2^6 = 0. \end{cases}$$

But a_4^{25} belongs to the ideal generated by this two polynomials, hence we get a contradiction with our assumption $c_1 \neq 0$.

Therefore, $(c_1, c_4) \in \mathcal{N}(V_2 \oplus V_5)$. Then, from Lemma 2.7.3 it follows that f has a root of multiplicity 4, i.e. f is a nullform.

We apply now Proposition 2.7.1: if $\mathcal{N}(V_7) = \mathcal{V}(j_4, j_{8,1}, j_{12,1}, j_{12,2}, j_{20})$, then it follows that $\{j_4, j_{8,1}, j_{12,1}, j_{12,2}, j_{20}\}$ forms a homogeneous system of parameters of $\mathcal{O}(V_7)^{\mathrm{SL}_2}$. □

Proposition 4.6.3. *The algebra of invariants of the binary septic is generated by 30 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	4	8	12	14	16	18	20	22	26	30
d_i	1	3	6	4	2	9	1	2	1	1

Proof. The Poincaré series of the algebra $\mathcal{O}(V_7)^{\mathrm{SL}_2}$ is:

$$\begin{aligned} P(t) &= \frac{a(t)}{(1-t^4)(1-t^8)(1-t^{12})^2(1-t^{20})} = \\ &= 1 + t^4 + 4t^8 + 10t^{12} + 4t^{14} + 18t^{16} + 13t^{18} + 35t^{20} + 26t^{22} + \\ &\quad + 62t^{24} + 52t^{26} + 97t^{28} + 92t^{30} + 153t^{32} + 144t^{34} + 229t^{36} + \\ &\quad + 223t^{38} + 325t^{40} + \ldots + 624t^{48} + \ldots \end{aligned}$$

with

$$\begin{aligned} a(t) &= 1 + 2t^8 + 4t^{12} + 4t^{14} + 5t^{16} + 9t^{18} + 6t^{20} + 9t^{22} + 8t^{24} + 9t^{26} + \\ &\quad + 6t^{28} + 9t^{30} + 5t^{32} + 4t^{34} + 4t^{36} + 2t^{40} + t^{48} \end{aligned}$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 48. Up to degree 30 we find the following 30 invariants:

degree	generators	
4	$j_4 = (c_1, c_1)_2,$	
8	$j_{8,1} = (c_7, c_1)_2,$	$j_{8,2} = (c_5, c_5)_4,$
	$j_{8,3} = (c_5, c_1^2)_4,$	
12	$j_{12,1} = ((c_5, c_5)_2, c_5)_4,$	$j_{12,2} = ((c_4, c_4)_2, c_1^3)_6,$
	$j_{12,3} = (c_7, c_7)_2,$	$j_{12,4} = (c_6^2, c_1^3)_6,$
	$j_{12,5} = ((c_9, c_2)_2, c_1^2)_4,$	$j_{12,6} = (c_9, c_9)_2,$
14	$j_{14,1} = (fc_6, c_1^5)_{10},$	$j_{14,2} = ((c_2, c_5)_1, c_1^4)_8,$
	$j_{14,3} = ((c_9, c_2)_1, c_1^3)_4,$	$j_{14,4} = (c_6c_{10}, c_1^2)_4,$
16	$j_{16,1} = (c_4^2, c_1^5)_{10},$	$j_{16,2} = (c_{10}^2, c_1)_2,$

degree	generators
18	$j_{18,1} = ((c_3, c_2)_1, c_1^7)_{14},$ $j_{18,2} = (f \cdot (c_2, c_6)_2, c_1^6)_{12},$ $j_{18,3} = (c_6 \cdot (f, c_2)_2, c_1^6)_{12},$ $j_{18,4} = (c_6 \cdot (c_2, c_6)_1, c_1^5)_{10},$ $j_{18,5} = (c_6 \cdot (c_5, c_6)_1, c_1^4)_8,$ $j_{18,6} = (c_6 \cdot (c_6, c_9)_1, c_1^3)_6,$ $j_{18,7} = ((c_2, c_6)_2 \cdot c_{10}, c_1^3)_6,$ $j_{18,8} = (((c_5, c_6)_1, c_{10})_1, c_1^2)_4,$ $j_{18,9} = (((c_6, (c_6, c_6)_2)_1, c_{10})_1, c_1)_2,$
20	$j_{20} = ([(c_2, c_5)_4]^2, (c_5, c_5)_2)_4,$
22	$j_{22,1} = (c_{10} \cdot (c_{10}, c_5)_1, c_1^2)_4,$ $j_{22,2} = (c_{10} \cdot (c_{10}, c_9)_1, c_1)_2,$
26	$j_{26} = (c_{10} \cdot (c_{10}, (c_{10}, c_6)_1)_1, c_1)_2,$
30	$j_{30} = ((c_6, c_9)_1, c_{10}^3)_3.$

Then we prove that no generators are needed in degrees 32, 34, 36, 40, and 48 by showing that for each $i \in \{32, 34, 36, 40, 48\}$ the vector space $\mathcal{O}(V_7)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 30 invariants found in degrees ≤ 30. For details about the computations see Chap. A.2. □

4.7   The invariants of the binary octavic

Let $f \in V_8$,

$$f = a_0 x^8 + 8a_1 x^7 y + \ldots + 8a_7 xy^7 + a_8 y^8.$$

Consider the following covariants of V_8:

$$c_1 = (f, f)_6, \qquad c_2 = (f, c_1)_4,$$
$$c_3 = (f, f)_4, \qquad c_4 = (f, f)_2,$$
$$c_5 = (c_1, c_1)_2, \qquad c_6 = (c_2, c_2)_2,$$

and the following invariants of V_8:

$$j_2 = (f, f)_8, \qquad j_3 = (f, c_3)_8,$$
$$j_4 = (c_1, c_1)_4, \qquad j_5 = (c_1, c_2)_4,$$
$$j_6 = (c_5, c_1)_4, \qquad j_7 = ((c_1, c_2)_2, c_1)_4.$$

Proposition 4.7.1. *The set $\{j_2, j_3, j_4, j_5, j_6, j_7\}$ forms a homogeneous system of parameters of the algebra $\mathcal{O}(V_8)^{\mathrm{SL}_2}$ of invariants of the binary octavic.*

Proof. First, we show that

$$\mathcal{N}(V_8) = \mathcal{V}(j_2, j_3, \ldots, j_7).$$

If $j_4 = j_6 = 0$, then c_1 is a nullform. We distinguish the following two cases.

Case 1: $c_1 = 0$.
In this case, because j_2 and j_3 vanish as well, it follows, from Proposition 2.7.2, that f has a root of multiplicity 6, i.e. f is a nullform.

Case 2: $c_1 \neq 0$.

In this case we have $(c_1, c_2) \in \mathcal{N}(V_4 \oplus V_4)$: from Proposition 5.4.4 it follows that the nullcone $\mathcal{N}(V_4 \oplus V_4)$ is generated by j_4, j_5, j_6, j_7, together with

$$j_6' = (c_2, c_2)_4, \qquad\qquad j_9 = (c_6, c_2)_4,$$
$$j_8 = ((c_1, c_2)_2, c_2)_4.$$

The vector space of invariants of degree 6 is spanned by j_2^3, j_3^2, $j_2 j_4$, and j_6. It follows, then, that j_6' is a linear combination of these invariants, hence $j_6' = 0$, if $j_2 = j_3 = j_6 = 0$.

   We introduce now the invariant $j_{10} = (c_5, c_6)_4$. Using the invariants of degrees ≤ 10, we built a list of forty-seven monomials of degree 16, each of them divisible by one of the invariants j_2, j_3, j_4, j_5, j_6, or j_7. We evaluated these monomials at $\dim_{\mathbb{C}} \mathcal{O}(V_8)_{16}^{\mathrm{SL}_2} = 47$ random points in V_8, and this resulted in a matrix of rank 47. It follows, then, that $j_8 = 0$ if $j_2 = j_3 = \ldots = j_7 = 0$. Using the invariants of degrees ≤ 10, we built a list of seventy-two monomials of degree 18, each of them divisible by one of the invariants j_2, j_3, j_4, j_5, j_6, j_7, or j_8. We evaluated these monomials at $\dim_{\mathbb{C}} \mathcal{O}(V_8)_{18}^{\mathrm{SL}_2} = 71$ random points in V_8, and this resulted in a matrix of rank 71. It follows, then, that $j_9 = 0$ if $j_2 = j_3 = \ldots = j_7 = 0$.

   Therefore, if $j_2 = j_3 = \ldots = j_7 = 0$, then $(c_1, c_2) \in \mathcal{N}(V_4 \oplus V_4)$. But then, from Lemma 2.7.4, it follows that f has a root of multiplicity 5, i.e. f is a nullform.

   We apply now Proposition 2.7.1: if $\mathcal{N}(V_8) = \mathcal{V}(j_2, j_3, \ldots, j_7)$, then the set $\{j_2, j_3, \ldots, j_7\}$ forms a homogeneous system of parameters of $\mathcal{O}(V_8)^{\mathrm{SL}_2}$. □

Proposition 4.7.2. *The algebra of invariants of the binary octavic is generated by 9 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7	8	9	10
d_i	1	1	1	1	1	1	1	1	1

Proof. The Poincaré series of the algebra $\mathcal{O}(V_8)^{\mathrm{SL}_2}$ is:

$$P(t) = \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)} =$$
$$= 1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 4t^7 + 7t^8 + 8t^9 + 12t^{10} + \ldots$$
$$+ \ldots 47t^{16} + 71t^{18} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 18. Up to degree 10 we found the following 9 invariants:

degree	generators
2	$j_2 = (f, f)_8,$
3	$j_3 = (f, c_3)_8,$
4	$j_4 = (c_1, c_1)_4,$
5	$j_5 = (c_1, c_2)_4,$

degree	generators
6	$j_6 = (c_5, c_5)_4,$
7	$j_7 = ((c_1, c_2)_2, c_1)_4,$
8	$j_8 = ((c_1, c_2)_2, c_2)_4,$
9	$j_9 = (c_6, c_2)_4,$
10	$j_{10} = (c_5, c_6)_4.$

In degree 18 there are no generators: we show that the vector space $\mathcal{O}(V_8)_{18}^{\mathrm{SL}_2}$ is spanned by monomials of degree 18 generated by the nine invariants found in degrees ≤ 10. The required computations can be seen in Chap. 3.5. $\qquad\square$

4.8   The invariants of the binary nonic

This chapter is based on joint work with Brouwer [BP10a].

Let $f \in V_9$,

$$f = a_0 x^9 + 9a_1 x^8 y + \ldots + 9a_8 xy^8 + a_9 y^9.$$

Consider the following covariants of V_9:

$$
\begin{aligned}
c_1 &= (f,f)_8, & c_2 &= (f,f)_6, \\
c_3 &= (f,f)_4, & c_4 &= (f,f)_2, \\
c_5 &= (f,c_1)_2, & c_6 &= (f,c_2)_6, \\
c_7 &= (c_2,c_2)_4, & c_8 &= (c_5,c_5)_6, \\
c_9 &= (c_5,c_5)_4, & c_{10} &= (c_5,c_8)_2, \\
c_{11} &= (c_9,c_9)_4, & c_{12} &= (c_9,c_{11})_4, \\
c_{13} &= (c_2,c_7)_4, & c_{14} &= (c_6,c_6)_2, \\
c_{15} &= (c_2,c_4)_6, & c_{16} &= (c_2,c_4)_4, \\
c_{17} &= (c_2,c_3)_6, & c_{18} &= (c_5,c_6)_3, \\
c_{19} &= (c_1,c_3)_2, & c_{20} &= (f,c_1)_2, \\
c_{21} &= (f,c_2)_2, & c_{22} &= (f,c_3)_6, \\
c_{23} &= (f,c_3)_8, & c_{24} &= (f,c_4)_8, \\
c_{25} &= (c_4,c_4)_{10}, & c_{26} &= (c_3,c_3)_6, \\
c_{27} &= (c_6^3,c_6)_3, & c_{28} &= (c_3,c_4)_{10}, \\
c_{29} &= (c_2,c_4)_2, & c_{30} &= (c_3,c_4)_8.
\end{aligned}
$$

Lemma 4.8.1. *If $c_1 = 0$, $c_2 \neq 0$, and $(c_2,c_3) \in \mathcal{N}(V_3 \oplus V_6)$, then f has a root of multiplicity at least 6.*

Proof. From $(c_2,c_3) \in \mathcal{N}(V_6 \oplus V_3)$ it follows that both c_2 and c_3 are nullforms and have a common root that has multiplicity 4 in c_2 and multiplicity 2 in c_3. Without loss of generality, we consider the following 3 cases: $c_2 = x^6$, $c_2 = x^5 y$, and $c_2 = x^4 y(x+y)$.

Case 1: $c_2 = x^6$. Then

$$c_3 = (f,x^6)_6 = a_9 y^3 + 3a_8 xy^2 + 3a_7 x^2 y + a_6 x^3,$$

and x^2 must divide c_3. We obtain $a_9 = a_8 = 0$, and substitute that in c_2 and c_1:

$$
\begin{aligned}
c_2 = (f, f)_6 \ &= (-20a_6^2 + 30a_5a_7)y^6 + (-30a_5a_6 + 54a_4a_7)xy^5 + \\
&\quad (-90a_5^2 + 114a_4a_6 - 12a_3a_7)x^2y^4 + \\
&\quad (-72a_4a_5 + 124a_3a_6 - 60a_2a_7)x^3y^3 + \\
&\quad (-90a_4^2 + 114a_3a_5 - 12a_2a_6 - 18a_1a_7)x^4y^2 + \\
&\quad (-30a_3a_4 + 54a_2a_5 - 30a_1a_6 + 6a_0a_7)x^5y + \\
&\quad (-20a_3^2 + 30a_2a_4 - 12a_1a_5 + 2a_0a_6)x^6, \\
c_1 = (f, f)_8 \ &= (70a_5^2 - 112a_4a_6 + 56a_3a_7)y^2 + \\
&\quad (28a_4a_5 - 56a_3a_6 + 40a_2a_7)xy + \\
&\quad (70a_4^2 - 112a_3a_5 + 56a_2a_6 - 16a_1a_7)x^2.
\end{aligned}
$$

Since we suppose $c_2 = x^6$ and $c_1 = 0$, the coefficients of x^iy^{6-i} in c_2, and of x^jy^{2-j} in c_1 are 0 for $0 \le i \le 5$ and $0 \le j \le 2$.

If $a_7 = 0$, then it follows that $a_6 = a_5 = a_4 = 0$, and then $x^6 \mid f$, so f will have a root of multiplicity 6. If $a_7 \neq 0$, then

$$
a_5 = \frac{2a_6^2}{3a_7}, \quad a_4 = \frac{10a_6^3}{27a_7^2}, \quad a_3 = \frac{5a_6^4}{27a_7^3},
$$
$$
a_2 = \frac{7a_6^5}{81a_7^4}, \quad a_1 = \frac{28a_6^6}{729a_7^5}, \quad a_0 = \frac{4a_6^7}{243a_7^6},
$$

but then we have $c_2 = 0$, contrary to the assumption.

Case 2: $c_2 = x^5y$. Then,

$$
c_3 = (f, c_2)_6 = -a_8y^3 - 3a_7xy^2 - 3a_6x^2y - a_5x^3,
$$

and x^2 must divide c_3. We obtain $a_8 = a_7 = 0$, and substitute this in c_2 and c_1:

$$
\begin{aligned}
c_2 = (f, f)_6 \ &= (-20a_6^2 + 2a_3a_9)y^6 + (-30a_5a_6 + 6a_2a_9)xy^5 + \\
&\quad (-90a_5^2 + 114a_4a_6 + 6a_1a_9)x^2y^4 + \cdots + \\
&\quad (-90a_4^2 + 114a_3a_5 - 12a_2a_6)x^4y^2 + \\
&\quad (-30a_3a_4 + 54a_2a_5 - 30a_1a_6)x^5y + \cdots \\
c_1 = (f, f)_8 \ &= (70a_5^2 - 112a_4a_6 + 2a_1a_9)y^2 + \cdots
\end{aligned}
$$

Since we supposed $c_2 = x^5y$ and $c_1 = 0$, the coefficient c of y^2 in c_1, and the coefficients d_i of x^iy^{6-i} in c_2 vanish for $0 \le i \le 4$, while $d_5 \neq 0$. Now

$$
5d_5a_9 = -75a_4d_0 + 45a_5d_1 - a_6(9c + 22d_2) = 0,
$$

so that $a_9 = 0$, and then also $a_6 = a_5 = a_4 = 0$, $d_5 = 0$, contradicting $d_5 \neq 0$.

Case 3: $c_2 = x^4 y(x+y)$. Then:

$$c_3 = (f, c_2)_6 = (a_7 - a_8)y^3 + 3(a_6 - a_7)xy^2 + 3(a_5 - a_6)x^2y + (a_4 - a_5)x^3,$$

and x^2 must divide c_3. We obtain $a_8 = a_7 = a_6$ which we replace in c_2 and c_1:

$$
\begin{aligned}
c_2 = (f, f)_6 = & -2(6a_4a_6 - 15a_5a_6 + 10a_6^2 - a_3a_9)y^6 - \\
& - 6(5a_3a_6 - 9a_4a_6 + 5a_5a_6 - a_2a_9)xy^5 - \\
& - 6(15a_5^2 + 3a_2a_6 + 2a_3a_6 - 19a_4a_6 - a_1a_9)x^2y^4 - \\
& - 2(36a_4a_5 - 3a_1a_6 + 30a_2a_6 - 62a_3a_6 - a_0a_9)x^3y^3 - \\
& - 6(15a_4^2 - 19a_3a_5 - a_0a_6 + 3a_1a_6 + 2a_2a_6)x^4y^2 - \\
& - 6(5a_3a_4 - 9a_2a_5 - a_0a_6 + 5a_1a_6)x^5y - \\
& - 2(10a_3^2 - 15a_2a_4 + 6a_1a_5 - a_0a_6)x^6, \\
c_1 = (f, f)_8 = & \, 2(35a_5^2 - 8a_2a_6 + 28a_3a_6 - 56a_4a_6 + a_1a_9)y^2 + \\
& \, 2(14a_4a_5 - 7a_1a_6 + 20a_2a_6 - 28a_3a_6 + a_0a_9)xy + \\
& \, 2(35a_4^2 - 56a_3a_5 + a_0a_6 - 8a_1a_6 + 28a_2a_6)x^2.
\end{aligned}
$$

As we supposed $c_2 = x^4 y(x+y)$ and $c_1 = 0$, the coefficients of y^6, xy^5, x^2y^4, x^3y^3, x^6 in c_2 and all coefficients of c_1 must vanish. We denote by J the ideal generated by these coefficients. Also, we denote by p_1, p_2 the coefficients of x^4y^2 and x^5y in c_2:

$$
\begin{aligned}
p_1 &= 15a_4^2 - 19a_3a_5 - a_0a_6 + 3a_1a_6 + 2a_2a_6, \\
p_2 &= 5a_3a_4 - 9a_2a_5 - a_0a_6 + 5a_1a_6.
\end{aligned}
$$

A Gröbner basis computation shows that p_1^4, $p_2^2 \in J$ so that p_1 and p_2 vanish, contradicting the assumption $c_2 = x^4 y(x+y)$. □

Consider now the following invariants of V_9:

$$
\begin{aligned}
j_{4,1} &= (c_1, c_1)_2, & j_{8,1} &= (c_2, c_6^2)_6, \\
j_{12,1} &= ((c_7, c_7)_2, c_7)_4, & j_{12,2} &= (c_9, c_1^3)_6, \\
j_{14,1} &= (c_2, c_{27})_3)_6, & j_{16,1} &= ((c_5, c_5)_2, c_1^5)_{10}, \\
j_{10,1} &= (((c_{25}, f)_6, c_{21})_5, c_2)_6.
\end{aligned}
$$

Proposition 4.8.2. *The seven invariants j_4, $j_{8,1}$, $j_{10,1}$, $j_{12,1}$, $j_{12,2}$, $j_{14,1}$, $j_{16,1}$ form a homogeneous system of parameters for the algebra $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ of invariants of the binary nonic.*

Note that Dixmier (1985) proved the following.

Proposition 4.8.3. (Dixmier [Dix85]) *The algebra $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ has a homogeneous system of parameters of degrees* 4, 8, 10, 12, 12, 14, 16.

Proposition 4.8.2 gives an explicit system of parameters for $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ with these degrees. Brouwer & Popoviciu ([BP10a]) proved the existence of systems of parameters for certain further sequences of degrees in the following.

Proposition 4.8.4. ([BP10a, Proposition 7.2]) *The algebra of invariants of V_9 has systems of parameters with each of the five sequences of degrees* 4, 8, 10, 12, 12, 14, 16 *and* 4, 4, 10, 12, 14, 16, 24 *and* 4, 4, 8, 12, 14, 16, 30 *and* 4, 4, 8, 10, 12, 16, 42 *and* 4, 4, 8, 10, 12, 14, 48.

Before proving Proposition 4.8.2, we give the following lemma.

Lemma 4.8.5. *Let $s \in V_6$, $c \in V_3$ be two non-zero binary forms. If both s and c are nullforms and if*

$$0 = ((s^2, s)_6, c^2)_6 = (((s, s)_2, s)_1, c^4)_{12} = (s, c^2)_6 = (s, (c, c)_2^3)_6 =$$
$$= (s, (c^3, c)_3)_6,$$

then $(s, c) \in \mathcal{N}(V_6 \oplus V_3)$.

Proof. Suppose that $(s, c) \notin \mathcal{N}(V_6 \oplus V_3)$. This means that s and c have no common root which has multiplicity 4 in s and multiplicity 2 in c. Without loss of generality we consider two cases:

$$\begin{cases} s = x^4(b_1 x^2 + b_2 xy + b_3 y^2), \\ c = y^3. \end{cases} \quad \text{and} \quad \begin{cases} s = x^4(b_1 x^2 + b_2 xy + b_3 y^2), \\ c = xy^2. \end{cases}$$

Case 1: $c = y^3$.
In this case we have ("$\sim$" is used for equalities up to a nonzero constant):

$$0 = ((s^2, s)_6, c^2)_6 \sim b_3^3,$$
$$0 = (((s, s)_2, s)_1, c^4)_{12} \sim b_2(5b_2^2 - 18b_1 b_3),$$
$$0 = (s, c^2)_6 = b_1,$$

and it follows that $b_1 = b_2 = b_3 = 0$, which implies $s = 0$, contradicting the assumption $s \neq 0$.

Case 2: $c = xy^2$.
In this case we have ("$\sim$" is used for equalities up to a nonzero constant):

$$0 = (s, (c, c)_2^3)_6 \sim b_1, \qquad 0 = (s, c^2)_6 \sim b_3,$$
$$0 = (s, (c^3, c)_3)_6 \sim b_2.$$

and it follows that $b_1 = b_2 = b_3 = 0$, which implies $s = 0$, contradicting the assumption $s \neq 0$. $\qquad\square$

Proof of Proposition 4.8.2. We introduce the invariants

$$
\begin{aligned}
j_{4,2} &= (c_2, c_2)_6, & j_{8,2} &= (c_7, c_7)_4, \\
j_{8,3} &= (c_8, c_1)_2, & j_{20,3} &= (c_{13}^2, (c_7, c_7)_2)_4, \\
j_{20,4} &= (c_5^2, c_1^7)_{14}.
\end{aligned}
$$

First we show that

$$
\mathcal{N}(V_9) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{8,2}, j_{8,3}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}, j_{20,3}, j_{20,4}).
$$

If $j_{4,1} = 0$, then $c_1 \in V_2$ is a nullform. Without loss of generality, we consider the following two cases: $c_1 = 0$ and $c_1 = x^2$.

Case 1: $c_1 = 0$.
If $j_{4,2} = j_{8,2} = j_{12,1} = j_{20,3} = 0$, then c_2 is a nullform.

If c_2 vanishes as well, then from Proposition 2.7.2 it follows that f has a root of multiplicity 7, i.e. f is a nullform.

If $c_2 \neq 0$, without loss of generality, we suppose $x^4 \mid c_2$. Modulo the ideal generated by the coefficients of c_1 and the coefficients of $x^3y^3, x^2y^4, xy^5, y^6$ in c_2 we have

$$
j_{8,1} = j_{12,3} = j_{12,4} = j_{18,1} = j_{20,1} = 0
$$

where

$$
\begin{aligned}
j_{12,3} &= (c_{14}, c_{14})_2, & j_{12,4} &= ((c_2^2, c_2)_6, c_6^2)_6, \\
j_{18,1} &= (((c_2, c_2)_2, c_2)_1, c_6^4)_{12}, & j_{20,1} &= (c_2, c_{14}^3)_6.
\end{aligned}
$$

It follows, then, from Lemma 4.8.5, that $(c_2, c_6) \in \mathcal{N}(V_6 \oplus V_3)$, and then, from Lemma 4.8.1, that f has a root of multiplicity 6, i.e. f is a nullform.

Case 2: $c_1 = x^2$.
Here we have (”$\sim$“ is used for equalities up to a nonzero constant):

$$
\begin{aligned}
0 &= j_{20,4} = a_9^2, & 0 &= j_{16,1} \sim a_8^2, \\
0 &= j_{12,2} \sim a_7^2, & 0 &= j_{8,3} \sim a_6^2.
\end{aligned}
$$

It follows that $a_9 = a_8 = a_7 = a_6 = 0$, and, if we combine this with $c_1 = x^2$, we get $a_5 = 0$ too, and then f is a nullform.

This shows that $\mathcal{N}(V_9) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{8,2}, j_{8,3}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}, j_{20,3}, j_{20,4})$.

We introduce now the following invariants:

$$
\begin{aligned}
j_{8,4} &= (c_7, c_1^2)_4, & j_{8,5} &= (c_7, c_{17})_4, \\
j_{10,2} &= (c_{18}, c_7)_4, & j_{10,4} &= ((c_6, c_{22})_3, (c_3, c_3)_8)_4, \\
j_{10,3} &= (c_{18}, c_1^2)_4, & j_{10,5} &= (((c_{26}, f)_6, c_{20})_3, c_2)_6.
\end{aligned}
$$

The invariants $j_{8,1}, j_{8,2}, \ldots, j_{8,5}$ are linearly independent and, together with $j_{4,1}^2$, $j_{4,2}^2$, and $j_{4,1}j_{4,2}$, span the vector space of invariants of degree 8 which is of dimension 8. (This can be seen, e.g., by a small computation in Mathematica.) In a similar way it can be seen that the vector space of invariants of degree 10 is spanned by $j_{10,1}, j_{10,2}, \ldots j_{10,5}$. Using invariants of degree ≤ 16 we built a list of 219 monomials of degree 20, each of them dividing one of the invariants $j_{4,1}$, $j_{4,2}$, $j_{8,1}, \ldots, j_{8,5}$, $j_{10,5}$, and $j_{10,1}$, to which we added

$$j_{20,1} = (c_{14}^3, c_2)_6, \quad j_{20,2} = ((c_{27}, c_2)_4, (c_{24}, c_{23})_3)_4.$$

Denote I be the algebra $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ of invariants, and I_i its i-th graded part. We evaluated the monomials at $\dim_{\mathbb{C}} I_{20} = 217$ random points in V_9, giving as result a matrix of (full) rank 217. Adding $j_{20,3}$, $j_{20,4}$, $j_{10,2}^2$, $j_{10,3}^2$, and $j_{10,4}^2$ to the list of monomials and repeating the evaluation step gave (of course) again matrices of rank 217. From the nullspaces of these matrices we obtained the relations

$$j_{20,3}, j_{20,4}, j_{10,2}^2, j_{10,3}^2, j_{10,4}^2 \in (j_{4,1}, j_{4,2}, j_{8,1}, j_{8,2}, \ldots, j_{8,5}, j_{10,1}, j_{10,5})$$

(that is, $j_{20,1}$ and $j_{20,2}$ are not needed to span the elements mentioned). Using invariants of degree ≤ 20 we built a list of 3561 monomials of degree 32, each of them dividing one of the invariants $j_{4,1}$, $j_{8,1}$, $j_{8,5}$, $j_{10,1}$, $j_{10,5}$, $j_{12,1}$, $j_{12,2}$, $j_{14,1}$, or $j_{16,1}$. We evaluated the monomials at $\dim_{\mathbb{C}} I_{32} = 2082$ random points in V_9, and this resulted in a matrix of rank 2082. The rank computations were made modulo 32003, but as we obtained the maximal rank, these monomials must span I_{32}. It follows that

$$j_{8,2}, j_{8,3}, j_{8,4}, j_{4,2} \in \sqrt{(j_{4,1}, j_{8,1}, j_{8,5}, j_{10,1}, j_{10,5}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1})},$$

and then we get

$$\mathcal{N}(V_9) = \mathcal{V}(j_{4,1}, j_{8,1}, j_{8,5}, j_{10,1}, j_{10,5}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}).$$

In the same way one can show that

$$\mathcal{N}(V_9) = \mathcal{V}(j_{4,2}, j_{8,1}, j_{8,5}, j_{10,1}, j_{10,5}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}).$$

It remains to remove two elements from one of these two sets of generators.

Let $H = (j_{4,1}, j_{8,1}, j_{10,1}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1})$. We computed $\dim_{\mathbb{C}} I_i \cap H$ for $i \leq 60$ and found $\dim_{\mathbb{C}} I_{60} \cap H = 59107 = \dim_{\mathbb{C}} I_{60}$, so that $I_{60} \subseteq H$. But then H contains powers of all invariants of degrees 4, 10, 20, so that in particular $j_{4,2}, j_{10,5} \in \sqrt{H}$. Now let $H' = (j_{4,1}, j_{4,2}, j_{8,1}, j_{10,1}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1})$. We computed $\dim_{\mathbb{C}} I_i \cap H'$ for $i \leq 40$ and found $\dim_{\mathbb{C}} I_{40} \cap H' = 6612 = \dim_{\mathbb{C}} I_{40}$, so that $I_{40} \subseteq H'$. But then H' contains powers of all invariants of degree 8, so that in particular $j_{8,5} \in \sqrt{H'}$. But then $\sqrt{H} = \sqrt{H'} = I$. Thus,

$$\mathcal{N}(V_9) = \mathcal{V}(j_{4,1}, j_{8,1}, j_{10,1}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}).$$

Then, from Proposition 2.7.1, it follows that the set

$$\{j_{4,1}, j_{8,1}, j_{10,1}, j_{12,1}, j_{12,2}, j_{14,1}, j_{16,1}\}$$

is a system of parameters of $\mathcal{O}(V_9)^{\mathrm{SL}_2}$.   □

Proposition 4.8.6. *The algebra of invariants of the binary nonic is generated by 92 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	4	8	10	12	14	16	18	20	22
d_i	2	5	5	14	17	21	25	2	1

Proof. The Poincaré series of $\mathcal{O}(V_9)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{a(t)}{(1-t^4)(1-t^8)(1-t^{10})(1-t^{12})^2(1-t^{14})(1-t^{16})}$$

with

$$\begin{aligned}
a(t) = {}& 1 + t^4 + 5t^8 + 4t^{10} + 17t^{12} + 20t^{14} + 47t^{16} + 61t^{18} + 97t^{20} + \\
& 120t^{22} + 165t^{24} + 189t^{26} + 223t^{28} + 241t^{30} + 254t^{32} + 254t^{34} + \\
& 241t^{36} + 223t^{38} + 189t^{40} + 165t^{42} + 120t^{44} + 97t^{46} + 61t^{48} + \\
& 47t^{50} + 20t^{52} + 17t^{54} + 4t^{56} + 5t^{58} + t^{62} + t^{66},
\end{aligned}$$

so that

$$\begin{aligned}
P(t) = {}& 1 + 2t^4 + 8t^8 + 5t^{10} + 28t^{12} + 27t^{14} + 84t^{16} + 99t^{18} + 217t^{20} + \\
& 273t^{22} + 506t^{24} + 647t^{26} + 1066t^{28} + 1367t^{30} + 2082t^{32} + 2649t^{34} + \\
& 3811t^{36} + 4796t^{38} + 6612t^{40} + 8228t^{42} + 10960t^{44} + 13483t^{46} + \\
& 17487t^{48} + 21274t^{50} + 26979t^{52} + 32490t^{54} + 40443t^{56} + 48242t^{58} + \\
& 59107t^{60} + 69885t^{62} + 84470t^{64} + 99074t^{66} + ...
\end{aligned}$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 66. Up to degree 22 we found the following generators:

deg	generators	
4	$j_{4,1} = (c_1, c_1)_2,$	$j_{4,2} = (c_2, c_2)_6,$
8	$j_{8,1} = (c_2, c_6^2)_6,$	$j_{8,2} = (c_7, c_7)_4,$
	$j_{8,3} = (c_8, c_1)_2,$	$j_{8,4} = (c_7, c_1^2)_4,$
	$j_{8,5} = (c_7, c_{17})_4,$	
10	$j_{10,1} = (((c_{25}, f)_6, c_{21})_5, c_2)_6,$	$j_{10,2} = (c_{18}, c_7)_4,$
	$j_{10,3} = (c_{18}, c_1^2)_4,$	$j_{10,4} = ((c_6, c_{22})_3, (c_3, c_3)_8)_4,$
	$j_{10,5} = (((c_{26}, f)_6, c_{20})_3, c_2)_6,$	
12	$j_{12,1} = ((c_7, c_7)_2, c_7)_4,$	$j_{12,2} = (c_9, c_1^3)_6,$
	$j_{12,3} = (c_{14}, c_{14})_2,$	$j_{12,4} = ((c_2^2, c_2)_6, c_6^2)_6,$
	$j_{12,5} = (c_8, c_8)_2,$	$j_{12,6} = (c_8, c_{13})_2,$
	$j_{12,7} = ((c_7, c_2)_2, (c_7, c_3)_4)_6,$	$j_{12,8} = ((c_7, c_3)_2, (c_7, c_4)_4)_{10},$
	$j_{12,9} = (c_{14}, c_8)_2,$	$j_{12,10} = (c_{14}, c_{13})_2,$
	$j_{12,11} = (c_6^2, (c_7, c_3)_4)_6,$	$j_{12,12} = (c_6 c_5, (c_7, c_3)_2)_{10},$
	$j_{12,13} = ((c_6, c_5)_2, (c_7, c_2)_2)_6,$	$j_{12,14} = ((c_8, c_4)_2, c_2^2)_{12},$

deg	generators	
14	$j_{14,1} = (c_2, c_{27})_3)_6,$	$j_{14,2} = (c_2, (c_{28}c_{17}, c_1c_2)_5)_6,$
	$j_{14,3} = (c_4, (c_{15}c_{19}, c_1c_4)_9)_{14},$	$j_{14,4} = (c_1, (c_{15}c_{19}, c_1c_4)_{15})_2,$
	$j_{14,5} = (c_2, (c_{15}c_{19}, c_1c_4)_{13})_6,$	$j_{14,6} = (c_3, (c_{15}c_{19}, c_1c_4)_{11})_{10},$
	$j_{14,7} = (c_1, (c_{28}c_{17}, c_1c_2)_7)_2,$	$j_{14,8} = (c_3, (c_{28}c_{17}, c_1c_2)_3)_{10},$
	$j_{14,9} = (c_4, (c_{28}c_{17}, c_1c_2)_1)_{14},$	$j_{14,10} = (((c_{16}, c_{19})_6, c_2)_5, c_1^2)_4,$
	$j_{14,11} = (((c_{16}, c_{19})_4, c_1)_1, c_2^2)_{12},$	$j_{14,12} = (((c_{16}, c_{19})_2, c_2)_1, c_3^2)_{20},$
	$j_{14,13} = ((c_{29}c_{19}, c_2)_1, c_4^2)_{28},$	$j_{14,14} = ((c_{16}, c_{19})_7, c_1^3)_6,$
	$j_{14,15} = ((c_{16}, c_{19})_1, c_2^3)_{18},$	$j_{14,16} = ((c_2c_4, c_1c_3)_1, c_3^3)_{30},$
	$j_{14,17} = ((c_{29}, c_3)_1, c_2^4)_{24},$	
16	$j_{16,1} = ((c_5, c_5)_2, c_1^5)_{10},$	$j_{16,2} = (c_7c_8, c_6^2)_6,$
	$j_{16,3} = ((c_1, c_4)_2, c_7^3)_{12},$	$j_{16,4} = ((c_2, c_3)_2, c_7^3)_{12},$
	$j_{16,5} = ((c_2, c_3)_2, c_6^4)_{12},$	$j_{16,6} = ((c_1, c_4)_2, c_6^4)_{12},$
	$j_{16,7} = (c_2c_1^4, c_5^2)_{14},$	$j_{16,8} = ((c_2c_3, c_1c_2)_8, c_1^4)_8,$
	$j_{16,9} = ((c_2c_3, c_3c_4)_{16}, c_1^4)_8,$	$j_{16,10} = ((c_4^2, c_2c_4)_{20}, c_1^4)_8,$
	$j_{16,11} = ((c_4^2, c_2c_4)_{20}, c_7^2)_8,$	$j_{16,12} = ((c_3c_4, c_2c_3)_{16}, c_7^2)_8,$
	$j_{16,13} = ((c_3c_2, c_2c_1)_8, c_7^2)_8,$	$j_{16,14} = (((c_1, c_3)_2, c_{16})_6, c_7^2)_8,$
	$j_{16,15} = (((c_1, c_3)_2, c_{16})_6, c_1^4)_8,$	$j_{16,16} = (((c_2, c_3)_4, c_{30})_4, c_1^4)_8,$
	$j_{16,17} = (((c_2, c_3)_4, c_{30})_4, c_7^2)_8,$	$j_{16,18} = ((c_{29}, c_{19})_8, c_7^2)_8,$
	$j_{16,19} = ((c_{29}, c_{19})_8, c_1^4)_8,$	$j_{16,20} = (c_{29}c_{19}, c_2^4)_{24},$
	$j_{16,21} = (c_1^2c_2^2, (c_{16}, c_{19})_2)_{16},$	
18	$j_{18,1} = (((c_2, c_2)_2, c_2)_1, c_6^4)_{12},$	$j_{18,2} = (c_8^2, c_{18})_4,$
	$j_{18,3} = ((c_7, c_1c_8)_2, c_{18})_4,$	$j_{18,4} = (c_7^2, (c_{30}, (c_3, (f, (f, c_2)_3)_6)_4)_4)_8,$
	$j_{18,5} = (c_1^2, ((c_{29}, c_3)_1, c_2^4)_{22})_4,$	$j_{18,6} = (c_1^2, ((c_2c_4, c_1c_3)_1, c_3^3)_{28})_4,$
	$j_{18,7} = (c_1^2, ((c_{16}, c_{19})_1, c_2^3)_{16})_4,$	$j_{18,8} = (c_1c_2, ((c_{16}, c_{19})_7, c_1^3)_2)_8,$
18	$j_{18,9} = (c_1^2, ((c_{29}c_{19}, c_2)_1, c_4^2)_{26})_4,$	$j_{18,10} = (c_1^2, (((c_{16}, c_{19})_2, c_2)_1, c_3^2)_{18})_4,$
	$j_{18,11} = (c_1^2, (((c_{16}, c_{19})_4, c_1)_1, c_2^2)_{10})_4,$	$j_{18,12} = (c_1c_2, ((c_{16}, c_{19})_6, c_2)_5 \cdot c_1^2)_8,$
	$j_{18,13} = (c_1^2, (c_4, (c_{28}c_{17}, c_1c_2)_1)_{12})_4,$	$j_{18,14} = (c_1^2, (c_3, (c_{28}c_{17}, c_1c_2)_3)_8)_4,$
	$j_{18,15} = (c_1^2, (c_2, (c_{15}c_{19}, c_1c_4)_{13})_4)_4,$	$j_{18,16} = (c_1c_2, (c_2, (c_{15}c_{19}, c_1c_4)_{13})_2)_8,$
	$j_{18,17} = (c_1^2, (c_4, (c_{15}c_{19}, c_1c_4)_9)_{12})_4,$	$j_{18,18} = (c_1^2, (c_2, (c_{28}c_{17}, c_1c_2)_5)_4)_4,$
	$j_{18,19} = (c_1c_2, (c_2, (c_{28}c_{17}, c_1c_2)_5)_2)_8,$	$j_{18,20} = (c_1c_2, (c_4, (c_{15}c_{19}, c_1c_4)_9)_{10})_8,$
	$j_{18,21} = (c_1c_2, (c_3, (c_{28}c_{17}, c_1c_2)_3)_6)_8,$	$j_{18,22} = (c_1c_2, (c_4, (c_{28}c_{17}, c_1c_2)_1)_{10})_8,$
	$j_{18,23} = (c_{14}^2, c_{18})_4,$	$j_{18,24} = (c_2c_3, (((c_{16}, c_{19})_2, c_2)_1, c_3^2)_{12})_{16},$
	$j_{18,25} = (c_2c_3, ((c_{29}c_{19}, c_2)_1, c_4^2)_{20})_{16},$	
20	$j_{20,1} = (c_{14}^3, c_2)_6,$	$j_{20,2} = ((c_{27}, c_2)_4, (c_{24}, c_{23})_3)_4,$
22	$j_{22} = (c_1^2, (c_2c_3, ((c_{29}c_{19}, c_2)_1, c_4^2)_{20})_{14})_4.$	

Then we prove that no generators are needed in degrees $24, 26, \ldots 58, 62, 66$ by showing that for each $i \in \{24, 26, \ldots 58, 62, 66\}$ the vector space $\mathcal{O}(V_9)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 92 invariants found in degrees ≤ 22. The required computations for $i \leq 32$ can be seen in Chap. A.3. The computations in degrees ≥ 34 were performed by A.E. Brouwer, with his own software ([BP10a]). □

4.9 The invariants of the binary decimic

This chapter is based on joint work with Brouwer [BP10b]

Consider $f \in V_{10}$,

$$f = a_0 x^{10} + 10a_1 x^9 y + \ldots + 10a_9 xy^9 + a_{10} y^{10},$$

and the following covariants

$$
\begin{aligned}
j_2 &= (f,f)_{10}, & c_1 &= (f,f)_8, \\
c_2 &= (f,f)_6, & c_3 &= (f,f)_4, \\
c_4 &= (f,f)_2, & c_5 &= (f,c_1)_4, \\
c_6 &= (f,c_2)_8, & c_7 &= (c_2,c_2)_6, \\
c_8 &= (c_5,c_5)_4, & c_9 &= (c_2,c_7)_4, \\
c_{10} &= (c_1,c_1)_2, & c_{11} &= (c_4,c_4)_{14}, \\
c_{12} &= (c_3,c_3)_{10}, & c_{13} &= (c_{10},c_1)_2, \\
c_{14} &= (c_8,c_5)_4, & c_{15} &= (c_2,c_2)_4, \\
c_{16} &= (c_5,c_5)_2, & c_{17} &= (c_4,c_2)_4, \\
c_{18} &= (c_4,c_2)_2, & c_{19} &= (c_5,c_1)_1, \\
c_{20} &= (c_5,c_3)_1, & c_{21} &= (c_5,c_2)_1, \\
c_{22} &= (c_1,c_2)_2, & c_{23} &= (c_1,c_4)_4, \\
c_{24} &= (c_7,c_9)_2, & c_{25} &= (c_7,c_7)_2.
\end{aligned}
$$

Lemma 4.9.1. *If $j_2 = 0$, $c_1 = 0$, $c_2 \neq 0$, and $c_2 \in \mathcal{N}_{V_8}$, then f has a root of multiplicity at least 7.*

Proof. The covariants c_1, c_2 and the invariant j_2 are:

$$j_2 = -252a_5^2 + 420a_4a_6 - 240a_3a_7 + 90a_2a_8 - 20a_1a_9 + 2a_0a_{10},$$

$$
\begin{aligned}
c_1 = &(70a_6^2 - 112a_5a_7 + 56a_4a_8 - 16a_3a_9 + 2a_2a_{10})y^4 + \\
&(56a_5a_6 - 112a_4a_7 + 80a_3a_8 - 28a_2a_9 + 4a_1a_{10})xy^3 + \\
&(168a_5^2 - 252a_4a_6 + 96a_3a_7 - 6a_2a_8 - 8a_1a_9 + 2a_0a_{10})x^2y^2 + \\
&(56a_4a_5 - 112a_3a_6 + 80a_2a_7 - 28a_1a_8 + 4a_0a_9)x^3y + \\
&(70a_4^2 - 112a_3a_5 + 56a_2a_6 - 16a_1a_7 + 2a_0a_8)x^4,
\end{aligned}
$$

$$
\begin{aligned}
c_2 = &(-20a_7^2 + 30a_6a_8 - 12a_5a_9 + 2a_4a_{10})y^8 + \\
&(-40a_6a_7 + 72a_5a_8 - 40a_4a_9 + 8a_3a_{10})xy^7 + \\
&(-140a_6^2 + 168a_5a_7 - 40a_3a_9 + 12a_2a_{10})x^2y^6 + \\
&(-168a_5a_6 + 280a_4a_7 - 120a_3a_8 + 8a_1a_{10})x^3y^5 + \\
&(-252a_5^2 + 280a_4a_6 + 40a_3a_7 - 90a_2a_8 + 20a_1a_9 + 2a_0a_{10})x^4y^4 + \\
&(-168a_4a_5 + 280a_3a_6 - 120a_2a_7 + 8a_0a_9)x^5y^3 + \\
&(-140a_4^2 + 168a_3a_5 - 40a_1a_7 + 12a_0a_8)x^6y^2 + \\
&(-40a_3a_4 + 72a_2a_5 - 40a_1a_6 + 8a_0a_7)x^7y + \\
&(-20a_3^2 + 30a_2a_4 - 12a_1a_5 + 2a_0a_6)x^8.
\end{aligned}
$$

Without loss of generality we suppose $x^5 \mid c_2$. We denote by J the ideal generated by j_2, the coefficients of c_1, and the coefficients of $x^4y^4, x^3y^5, \ldots, y^8$ in c_2. Denote also by p_1, p_2 and p_3 the coefficients of x^7y, x^6y^2 and x^5y^3, respectively, in c_2. We have

$$p_1^4, p_2^3, p_3^2 \in J,$$

which implies that $x^8 \mid c_2$.

Consider now the ideal J generated by j_2, the coefficients of c_1 and the coefficients of $x^7y, x^6y^2, \ldots, y^8$ in c_2. Denote by p_0 the coefficient of x^8 in c_2. We have $a_i p_0 \in J$ for $i = 10, 9, 8, 7, 6, 5, 4$. Because $c_2 \neq 0$ we find $a_{10} = \ldots = a_4 = 0$. This means that $x^7 \mid f$, so f will have a root of multiplicity 7. $\square$

Consider now the following invariants of V_{10}:

$$
\begin{aligned}
j_{4,1} &= (c_1, c_1)_4, & j_{6,1} &= (c_5, c_5)_6, \\
j_{6,2} &= (c_6, c_6)_2, & j_{8,1} &= (c_1, c_8)_4, \\
j_{9,1} &= (c_{19}, c_1^2)_8, & j_{10,1} &= (c_{16}, c_1^2)_8, \\
j_{14,1} &= (c_{25}, c_9)_4, & j_{14,2} &= (c_{10}^2, c_{16})_8.
\end{aligned}
$$

Proposition 4.9.2. *The eight invariants j_2, $j_{4,1}$, $j_{6,1}$, $j_{6,2}$, $j_{8,1}$, $j_{9,1}$, $j_{10,1}$, $j_{14,1} + j_{14,2}$ form a homogeneous system of parameters of the ring $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$ of invariants of the binary decimic.*

Proof. We introduce the following invariants:

$$j_{6,3} = (c_{15}, c_2)_8, \quad j_{6,4} = (c_{10}, c_1)_4, \quad j_{12,2} = (c_5^2, c_1^3)_{12}.$$

First we prove that

$$\mathcal{N}(V_{10}) = \mathcal{V}(j_2, j_{4,1}, j_{6,1}, j_{6,3}, j_{6,4}, j_{8,1}, j_{9,1}, j_{10,1}, j_{12,2}, j_{14,1}, j_{14,2}).$$

If $j_{4,1} = j_{6,4} = 0$, then c_1 is a nullform. Without loss of generality, we consider the following three cases.

Case 1: $c_1 = 0$.
In this case we have $c_2 \in \mathcal{N}(V_8)$: in Chap. 4.7 we proved that the nullcone of V_8 is the vanishing locus of the the invariants $j_{4,2}$, $j_{6,3}$, $j_{8,2}$, $j_{10,2}$, $j_{12,3}$, and $j_{14,1}$, where:

$$
\begin{aligned}
j_{4,2} &= (c_2, c_2)_8, & j_{10,2} &= (c_9, c_7)_4, \\
j_{8,2} &= (c_7, c_7)_4, & j_{12,3} &= (c_{25}, c_7)_4.
\end{aligned}
$$

We show that these invariants vanish: denote by $J = (j_2, c_1)$ the ideal generated by j_2 and the coefficients of c_1. Easy Gröbner basis computations show that $j_{4,2}, j_{8,2}, j_{10,2} \in J$ and $j_{12,3} \in (J, j_{6,3})$. Hence if $j_2 = j_{6,3} = c_1 = 0$, then $j_{4,2} = j_{8,2} = j_{10,2} = j_{12,3} = 0$. Because $j_{14,1}$ vanishes as well, it follows that c_2 is a nullform. Then, we apply Lemma 4.9.1, and obtain that f has a root of multiplicity 7, i.e. f is a nullform.

Case 2: $c_1 = x^4$. In this case we have ("$\sim$" is used for equalities up to a nonzero constant):

$$j_{12,2} \sim a_{10}^2,$$
$$j_{10,1} \sim -a_9^2 + a_8 a_{10},$$
$$j_{8,1} \sim 3a_8^2 - 4a_7 a_9 + a_6 a_{10},$$
$$j_{6,1} \sim -10a_7^2 + 15a_6 a_8 - 6a_5 a_9 + a_4 a_{10}.$$

If $j_{12,2} = j_{10,1} = j_{8,1} = j_{6,1} = 0$, then it follows that $a_{10} = \ldots = a_7 = 0$. If we substitute this in c_1, we obtain

$$c_1 = 70a_6^2 y^4 + 56a_5 a_6 xy^3 + (168a_5^2 - 252a_4 a_6)x^2 y^2 +$$
$$(56a_4 a_5 - 112a_3 a_6)x^3 y + (70a_4^2 - 112a_3 a_5 + 56a_2 a_6)x^4,$$

and, as we supposed $c_1 = x^4$, we get also $a_6 = a_5 = 0$, which implies that f is a nullform.

Case 3: $c_1 = x^3 y$. In this case we have ("$\sim$" is used for equalities up to a nonzero constant):

$$j_{9,1} \sim a_9,$$
$$j_{14,2} \sim a_7 a_9 - a_8^2,$$
$$j_{10,1} \sim -5a_7^2 + 2a_6 a_8 + 3a_5 a_9,$$
$$j_{6,1} \sim -10a_6^2 + 15a_5 a_7 - 6a_4 a_8 + a_3 a_9.$$

If $j_{9,1} = j_{14,2} = j_{10,1} = j_{6,1} = 0$, then $a_9 = \ldots = a_6 = 0$. If we substitute this in c_1 and j_2, we obtain:

$$c_1 = 2a_2 a_{10} y^4 + 4a_1 a_{10} xy^3 + (168a_5^2 + 2a_0 a_{10})x^2 y^2 + 56a_4 a_5 x^3 y +$$
$$+ (70a_4^2 - 112a_3 a_5)x^4,$$
$$j_2 = -252a_5^2 + 2a_0 a_{10}$$

From $168a_5^2 + 2a_0 a_{10} = -252a_5^2 + 2a_0 a_{10} = 0$ we find $a_5 = 0$, which contradicts $c_1 = x^3 y$.
Therefore, we have

$$\mathcal{N}(V_{10}) = \mathcal{V}(j_2, j_{4,1}, j_{6,1}, j_{6,3}, j_{6,4}, j_{8,1}, j_{9,1}, j_{10,1}, j_{12,2}, j_{14,1}, j_{14,2}).$$

So far, we defined the nullcone using 11 invariants, but we need a definition using 8 invariants. As a first step, replace the two invariants of degree 14 by a single one. Now for $f = x^2 y(2a_1 x^7 + 9a_8 y^7)$ all the invariants above defining $\mathcal{N}(V_{10})$ will vanish, except $j_{14,1}$. And for $f = y^3(120a_3 x^7 + a_{10} y^7)$ all the invariants above vanish, except $j_{14,2}$. That means that the single invariant of degree 14 cannot be either $j_{14,1}$ or $j_{14,2}$. However, it turns out that we can use $j_{14,1} + j_{14,2}$.

The final part of the construction of the system of parameters was done by computer. These computations were performed by A.E. Brouwer, with his own

software ([BP10b]). All computations were carried out in the ring R generated by the 106 invariants found in Proposition 4.9.3. Or, more precisely, in the quotient $Q = R/j_2R$, reduced mod p, where $p = 197$ (the different p has no significance), and a_4, a_7 and a_9 were taken to be zero. It was checked that the graded parts of the resulting ring have the expected dimension (for degree up to 54), so that no collapse occurred as a consequence of the reduction mod p or the substitution of variables.

The ideal generated in this ring by all invariants of degrees 4, 6, 8, 9, 10, 14 has full dimension 542 for its graded part of degree 24. We know that $\dim_{\mathbb{C}} \mathcal{O}(V_{10})^{\mathrm{SL}_2}_{24} = 1429$ and $\dim_{\mathbb{C}} \mathcal{O}(V_{10})^{\mathrm{SL}_2}_{22} = 887$ and multiplication by j_2 is an injection. Therefore we have $542 = \dim_{\mathbb{C}} \mathcal{O}(V_{10})^{\mathrm{SL}_2}_{24}/j_2\mathcal{O}(V_{10})^{\mathrm{SL}_2}_{22}$. It follows that the ideal generated by the invariants of degrees 4, 6, 8, 9, 10, 14, together with j_2, contains all of $\mathcal{O}(V_{10})^{\mathrm{SL}_2}_{24}$, so that no invariants of degree 12 are needed to define the nullcone (since their squares are in $\mathcal{O}(V_{10})^{\mathrm{SL}_2}_{24}$, and they themselves are in the radical).

With only $j_{14,1} + j_{14,2}$ instead of all invariants of degree 14 in the set of generators of the ideal, one finds full dimension 1148 for the graded part of degree 28, so this single invariant of degree 14 suffices.

With only $j_{10,1}$ instead of all invariants of degree 10, one finds full dimension 221 in degree 20, so this single invariant of degree 10 suffices.

With only $j_{9,1}$ instead of all invariants of degree 9, one finds full dimension 890 in degree 27, so this single invariant of degree 9 suffices.

With only $j_{8,1}$ instead of all invariants of degree 8, one finds full dimension 2279 in degree 32, so this single invariant of degree 8 suffices.

That only leaves the invariants of degree 6. After some work it turned out that with only $j_{6,1}$ and $j_{6,2}$ one finds full dimension 37892 in degree 54, so these suffice, and we have constructed the promised homogeneous system of parameters. □

Proposition 4.9.3. *The algebra of invariants of the binary decimic is generated by 106 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	4	6	8	9	10	11	12	13	14	15	16	17	18	19	21
d_i	1	1	4	5	5	8	8	12	15	13	19	5	5	1	2	2

Proof. The Poincaré series of $\mathcal{O}(V_{10})^{\mathrm{SL}_2}$ is

$$P(t) = a(t)/(1 - t^2)(1 - t^4)(1 - t^6)^2(1 - t^8)(1 - t^9)(1 - t^{10})(1 - t^{14})$$

where

$$\begin{aligned}
a(t) = {} & 1 + 2t^6 + 4t^8 + 4t^9 + 7t^{10} + 8t^{11} + 15t^{12} + 15t^{13} + 20t^{14} + 27t^{15} + \\
& 29t^{16} + 35t^{17} + 40t^{18} + 44t^{19} + 47t^{20} + 55t^{21} + 52t^{22} + 57t^{23} + 56t^{24} + \\
& 57t^{25} + 52t^{26} + 55t^{27} + 47t^{28} + 44t^{29} + 40t^{30} + 35t^{31} + 29t^{32} + 27t^{33} + \\
& 20t^{34} + 15t^{35} + 15t^{36} + 8t^{37} + 7t^{38} + 4t^{39} + 4t^{40} + 2t^{42} + t^{48},
\end{aligned}$$

so that

$$P(t) = 1 + t^2 + 2t^4 + 6t^6 + 12t^8 + 5t^9 + 24t^{10} + 13t^{11} + 52t^{12} + 33t^{13} + 97t^{14} +$$
$$80t^{15} + 177t^{16} + 160t^{17} + 319t^{18} + 301t^{19} + 540t^{20} + 547t^{21} + 887t^{22} +$$
$$926t^{23} + 1429t^{24} + 1512t^{25} + 2219t^{26} + 2402t^{27} + 3367t^{28} + 3681t^{29} +$$
$$5015t^{30} + 5502t^{31} + 7294t^{32} + 8064t^{33} + 10419t^{34} + 11550t^{35} +$$
$$14664t^{36} + 16253t^{37} + 20287t^{38} + 22531t^{39} + 27682t^{40} + 30738t^{41} +$$
$$37319t^{42} + 41378t^{43} + 49671t^{44} + 55060t^{45} + 65390t^{46} + 72391t^{47} +$$
$$85250t^{48} + \cdots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generating invariants is 48. Up to degree 21 we found the following generators:

deg	generators	
2	$j_2 = (f, f)_{10}$,	
4	$j_4 = (c_1, c_1)_4$,	
6	$j_{6,1} = (c_5, c_5)_6$,	$j_{6,2} = (c_6, c_6)_2$,
	$j_{6,3} = (c_{15}, c_2)_8$,	$j_{6,4} = (c_{10}, c_1)_4$,
8	$j_{8,1} = (c_1, c_8)_4$,	$j_{8,2} = (c_7, c_7)_4$,
	$j_{8,3} = (c_{12}, c_7)_4$,	$j_{8,4} = (c_{10}, c_{11})_4$,
	$j_{8,5} = (c_{11}, c_{11})_4$,	
9	$j_{9,1} = (c_{19}, c_1^2)_8$,	$j_{9,2} = (c_{20}, c_2^2)_{16}$,
	$j_{9,3} = (c_{21}, c_1 c_2)_{16}$,	$j_{9,4} = ((c_5, c_4)_1, c_2 c_3)_{20}$,
	$j_{9,5} = ((c_6, c_4)_1, c_2^2)_{16}$,	
10	$j_{10,1} = (c_{16}, c_1^2)_8$,	$j_{10,2} = (c_9, c_7)_4$,
	$j_{10,3} = (c_7, c_6^2)_4$,	$j_{10,4} = (c_9, c_{12})_4$,
	$j_{10,5} = ((c_{23}, c_{22})_6, c_2)_8$,	$j_{10,6} = (c_8 c_5, f)_{10}$,
	$j_{10,7} = (((c_1, c_3)_2, c_{17})_{12}, c_1)_4$,	$j_{10,8} = (((c_1, c_3)_2, c_{18})_8, c_4)_{16}$,
11	$j_{11,1} = ((c_{19}, c_1^2)_4, c_2)_8$,	$j_{11,2} = ((c_{19}, c_1^2)_2, c_3)_{12}$,
	$j_{11,3} = (c_{19} c_1^2, c_4)_{16}$,	$j_{11,4} = ((c_{20}, c_2^2)_{14}, c_1)_4$,
	$j_{11,5} = ((c_{20}, c_2^2)_{12}, c_2)_8$,	$j_{11,6} = ((c_{20}, c_2^2)_{10}, c_3)_{12}$,
	$j_{11,7} = ((c_{20}, c_2^2)_8, c_4)_{16}$,	$j_{11,8} = ((c_{21}, c_1 c_2)_{10}, c_1)_4$,
12	$j_{12,1} = (c_8, c_8)_4$,	$j_{12,2} = (c_5^2, c_1^3)_{12}$,
	$j_{12,3} = (c_{25}, c_7)_4$,	$j_{12,4} = ((c_{16}, c_1^2)_6, c_1)_4$,
	$j_{12,5} = (c_{24}, c_1)_4$,	$j_{12,6} = ((c_{16}, c_1^2)_4, c_2)_8$,
	$j_{12,7} = ((c_{16}, c_1^2)_2, c_3)_{12}$,	$j_{12,8} = (c_{16} c_1^2, c_4)_{16}$,
	$j_{12,9} = ((c_7, c_6^2)_2, c_1)_4$,	$j_{12,10} = (c_7 c_6^2, c_2)_8$,
	$j_{12,11} = ((c_9, c_{12})_2, c_1)_4$,	$j_{12,12} = (((c_{23}, c_{22})_6, c_2)_6, c_1)_4$,
13	$j_{13,1} = (((c_{19}, c_1^2)_4, c_2)_6, c_1)_4$,	$j_{13,2} = (((c_{19}, c_1^2)_4, c_2)_4, c_2)_8$,
	$j_{13,3} = (((c_{19}, c_1^2)_4, c_2)_2, c_3)_{12}$,	$j_{13,4} = (((c_{19}, c_1^2)_2, c_3)_{10}, c_1)_4$,
	$j_{13,5} = (((c_{19}, c_1^2)_2, c_3)_8, c_2)_8$,	$j_{13,6} = (((c_{19}, c_1^2)_2, c_3)_6, c_3)_{12}$,
	$j_{13,7} = (((c_{20}, c_2^2)_{14}, c_1)_2, c_1)_4$,	$j_{13,8} = (c_1 \cdot (c_{20}, c_2^2)_{14}, c_2)_8$,
	$j_{13,9} = (((c_{20}, c_2^2)_{12}, c_2)_6, c_1)_4$,	$j_{13,10} = (((c_{20}, c_2^2)_{12}, c_2)_4, c_2)_8$,
	$j_{13,11} = (((c_{20}, c_2^2)_{12}, c_2)_2, c_3)_{12}$,	$j_{13,12} = (((c_{20}, c_2^2)_{10}, c_3)_{10}, c_1)_4$,
	$j_{13,13} = (((c_{20}, c_2^2)_{10}, c_3)_8, c_2)_8$,	$j_{13,14} = (((c_{20}, c_2^2)_{10}, c_3)_6, c_3)_{12}$,
	$j_{13,15} = (((c_{20}, c_2^2)_8, c_4)_{14}, c_1)_4$,	

deg	generators
14	$j_{14,1} = (c_{25}, c_9)_4,$ $\qquad\qquad\qquad\quad$ $j_{14,2} = (c_{10}^2, c_{16})_8,$ $j_{14,3} = ((c_8, c_8)_2, c_1)_4,$ $\qquad\qquad$ $j_{14,4} = (c_8^2, c_2)_8,$ $j_{14,5} = ((c_5^2, c_1^3)_{10}, c_1)_4,$ $\qquad\quad$ $j_{14,6} = ((c_5^2, c_1^3)_8, c_2)_8,$ $j_{14,7} = ((c_5^2, c_1^3)_6, c_3)_{12},$ $\qquad\quad$ $j_{14,8} = ((c_5^2, c_1^3)_4, c_4)_{16},$ $j_{14,9} = (c_1 \cdot (c_{16}, c_1^2)_6, c_2)_8,$ $\qquad$ $j_{14,10} = ((c_{24}, c_1)_2, c_1)_4,$ $j_{14,11} = (c_1 c_{24}, c_2)_8,$ $\qquad\qquad$ $j_{14,12} = (((c_{16}, c_1^2)_4, c_2)_6, c_1)_4,$ $j_{14,13} = (((c_{16}, c_1^2)_4, c_2)_4, c_2)_8,$
15	$j_{15,1} = ((((c_{19}, c_1^2)_4, c_2)_6, c_1)_2, c_1)_4,$ $\quad$ $j_{15,2} = (c_1 \cdot (c_{19}, c_1^2)_4, c_2)_6, c_2)_8,$ $j_{15,3} = (((((c_{19}, c_1^2)_4, c_2)_4, c_2)_6, c_1)_4,$ $\quad$ $j_{15,4} = ((((c_{19}, c_1^2)_4, c_2)_4, c_2)_4, c_2)_8,$ $j_{15,5} = ((((c_{19}, c_1^2)_4, c_2)_4, c_2)_2, c_3)_{12},$ $\quad$ $j_{15,6} = ((((c_{19}, c_1^2)_4, c_2)_2, c_3)_{10}, c_1)_4,$ $j_{15,7} = ((((c_{19}, c_1^2)_4, c_2)_2, c_3)_8, c_2)_8,$ $\quad$ $j_{15,8} = ((((c_{19}, c_1^2)_4, c_2)_2, c_3)_6, c_3)_{12},$ $j_{15,9} = ((((c_{19}, c_1^2)_2, c_3)_{10}, c_1)_2, c_1)_4,$ $\quad$ $j_{15,10} = (c_1 \cdot ((c_{19}, c_1^2)_2, c_3)_{10}, c_2)_8,$ $j_{15,11} = ((((c_{19}, c_1^2)_2, c_3)_8, c_2)_6, c_1)_4,$ $\quad$ $j_{15,12} = ((((c_{19}, c_1^2)_2, c_3)_8, c_2)_4, c_2)_8,$ $j_{15,13} = ((((c_{19}, c_1^2)_2, c_3)_8, c_2)_2, c_3)_{12},$ $\quad$ $j_{15,14} = (c_1 \cdot ((c_{20}, c_2^2)_{14}, c_1)_2, c_2)_8,$ $j_{15,15} = ((((c_{20}, c_2^2)_{12}, c_2)_6, c_1)_2, c_1)_4,$ $\quad$ $j_{15,16} = ((c_1 \cdot (c_{20}, c_2^2)_{14}, c_2)_6, c_1)_4,$ $j_{15,17} = ((c_1 \cdot (c_{20}, c_2^2)_{14}, c_2)_4, c_2)_8,$ $\quad$ $j_{15,18} = ((c_1 \cdot (c_{20}, c_2^2)_{14}, c_2)_2, c_3)_{12},$ $j_{15,19} = ((((c_{20}, c_2^2)_{12}, c_2)_2, c_3)_8, c_2)_8,$
16	$j_{16,1} = ((c_{25}, c_9)_2, c_1)_4,$ $\qquad\qquad$ $j_{16,2} = (c_9 c_{25}, c_2)_8,$ $j_{16,3} = (((c_8, c_8)_2, c_1)_2, c_1)_4,$ $\qquad$ $j_{16,4} = ((c_8^2, c_2)_6, c_1)_4,$ $j_{16,5} = (((c_5^2, c_1^3)_8, c_2)_6, c_1)_4,$
17	$j_{17,1} = ((c_1 \cdot ((c_{19}, c_1^2)_4, c_2)_6, c_2)_6, c_1)_4,$ $j_{17,2} = (((((c_{19}, c_1^2)_4, c_2)_4, c_2)_2, c_3)_{10}, c_1)_4,$ $j_{17,3} = (((((c_{19}, c_1^2)_4, c_2)_2, c_3)_{10}, c_1)_2, c_1)_4,$ $j_{17,4} = (((((c_{20}, c_2^2)_{12}, c_2)_2, c_3)_8, c_2)_4, c_2)_8,$ $j_{17,5} = (((c_1 \cdot (c_{20}, c_2^2)_{14}, c_2)_2, c_3)_8, c_2)_8,$
18	$j_{18} = (c_4 \cdot (c_5^2, c_1^3)_4, (f^2, f^2)_4)_{32},$
19	$j_{19,1} = (c_3 \cdot (((c_{19}, c_1^2)_4, c_2)_2, c_3)_6, (f^2, f^2)_8)_{24},$ $j_{19,2} = (c_{24} \cdot (f, (((f^2, c_4)_8, f)_5, f)_{10})_8, c_2)_8,$
21	$j_{21,1} = ((f, c_{25})_1, c_7^3)_{12},$ $\qquad\qquad$ $j_{21,2} = (((((f, c_{10}^3)_8, f)_4, f)_6, c_{19})_2, f)_{10}.$

Then we prove that no generators are needed in degrees $4, 6, 8, 9, \ldots, 40, 42, 48$, by showing that for each $i \in \{4, 6, 8, 9, \ldots, 40, 42, 48\}$ the vector space $\mathcal{O}(V_{10}^{\mathrm{SL}_2})_i$ is spanned by monomials of degree i generated by the 106 invariants found in degrees ≤ 21. The required computations for $i \leq 26$ can be seen in Chap. A.4. The computations in degrees ≥ 27 were performed by A.E. Brouwer, with his own software ([BP10b]). $\qquad\qquad\square$

Chapter 5

Invariants of several forms

In this chapter we review classical results regarding the invariants and the co-variants of $V_{n_1} \oplus \ldots \oplus V_{n_p}$, with $p \geq 2$.

Most of the cases that we treat in this chapter were considered in the nine-teenth century as well. We certify the "classical" results (i.e. obtained in the nineteenth century) with our computations and correct those results that turned out to be wrong. Table 5.1 contains results of computations made in the nineteenth century (the underlined entries are results that in the first place turned out to be false and were later corrected). In this chapter we concen-trate on these cases and correct a result of Winter [Win80] regarding the co-variants of $V_2 \oplus V_5$ (see Chap. 5.16) and results of Gundelfinger [Gun69] and Sylvester [Sy78b, Sy78c, Sy78d] regarding the covariants of $V_3 \oplus V_4$ (see Chap. 5.19; joint work with Brouwer [BP12]).

Further, we give a set of generating invariants of $mV_1 \oplus nV_2$ with $m \geq 2$ and $n \geq 3$ (see Chap. 5.2). Using results of Peano [Pea82], Young [You99] and Kraft & Weyman [KW99], we give a set of generating invariants of $mV_1 \oplus nV_3$ with $m \geq 2$ and $n \geq 2$ (see Chap. 5.3) and a set of generating invariants of $mV_1 \oplus nV_4$ with $m \geq 2$ and $n \geq 5$ (see Chap. 5.4).

For the cases considered in the coming sections, we give systems of parame-ters or, in some of them, only the degrees of a system of parameters, and a set of generators of their invariants. Table 5.2 gives an overview on the degrees of systems of parameters (hsop) in the cases that we consider. As a remark, not all these systems of parameters are multihomogeneous. In fact, such multiho-mogeneous hsop only exist in few cases:

Proposition 5.0.4. (Brion [Bri82]) *Consider V an SL_2-module and $\mathcal{O}(V)^{\mathrm{SL}_2}$ the algebra of invariants of V. There exist multihomogeneous systems of param-eters of $\mathcal{O}(V)^{\mathrm{SL}_2}$ iff V is one of the following modules: $2V_1$, $V_1 \oplus V_2$, $V_1 \oplus V_3$, $V_1 \oplus V_4$, $2V_2$, $V_2 \oplus V_3$, $V_2 \oplus V_4$, $2V_3$, $2V_4$, $3V_1$, $2V_1 \oplus V_2$, $V_1 \oplus 2V_2$, or $3V_2$.*

Whenever the classics looked for the generators of the covariants of $V \oplus W$, with V and W two SL_2-modules, they used the information they had about the

module	# generators of invariants	# generators of covariants
V_3	1 ([Gor87])	4 ([Gor87])
V_4	2 ([Gor87])	5 ([Gor87])
V_5	4 ([Gor87])	23 ([Gor87])
V_6	5 ([Gor87])	26 ([Gor87])
V_7	<u>33</u> ([Gal88]), 30 ([DL86])	<u>153</u> ([Gal88]), <u>124</u> ([Sy79b]), 147 ([Crö02, Bed09], Chap. 5.7)
V_8	<u>12</u> ([Gal80]), 9 ([Gal80, Shi67])	<u>96</u>,<u>67</u>,<u>70</u> ([Gal80]) 69 ([Sy79b, BB08], Chap. 5.8)
$V_2 \oplus V_3$	5 ([Bes69, Gor87])	15 ([Bes69, Gor87])
$V_2 \oplus V_4$	6 ([GY03])	18 ([GY03])
$V_2 \oplus V_5$	29 ([Win80])	<u>94</u> ([Win80]), 92 (Chap. 5.16)
$V_2 \oplus V_6$	27 ([Gal74])	99 ([Gal74])
$2V_3$	7 ([Gor87, Pea82])	26 ([Gor87, Pea82])
$V_3 \oplus V_4$	20 ([Gun69])	<u>64</u> ([Gun69]), <u>61</u> ([Sy78b, Sy78c, Sy78d]), 63 ([BP12], Chap. 5.19)
$2V_4$	8 ([You99])	28 ([You99])
$3V_3$	28 ([Gal94])	<u>98</u> ([Gal94]), 97 ([Sin05])
$3V_4$	25 ([You99])	103 ([You99])
$4V_4$	80 ([You99])	305 ([You99])
nV_1	$\binom{n}{2}$ ([Gor69])	$\binom{n+1}{2}$ ([Gor69])
nV_2	$\binom{n+1}{2} + \binom{n}{3}$ ([Gor87])	$n(n+1) + \binom{n}{3}$ ([Gor87])

Table 5.1: Cases treated in the 19th century

covariants of V and the covariants of W. The following result, due to Clebsch, plays an important role:

Proposition 5.0.5. *([Cle72, §54],[KW99, Proposition 7]) Let V and W be two SL_2-modules whose covariants are finitely generated. Then the covariants of $V \oplus W$ are also finitely generated. If $P_1, \ldots, P_r$ are the generators of the covariants of V, and $Q_1, \ldots, Q_s$ are the generators of the covariants of W, then a finite generating system can be chosen from the set of all transvectants $[P, Q]_l$, $l \geq 0$, where P is a monomial in the P_i's and Q a monomial in the Q_j's.*

Remark 5.0.6. *More precisely, a (finite) generating system for the covariants of $V \oplus W$ will be given by the non-irrelevant transvectants $[P, Q]_l$.*

(A transvectant $[P, Q]_l$ is called irrelevant *if there exist P_1, P_2, Q_1, Q_2 and l_1, l_2 such that $l = l_1 + l_2$, $P = P_1 \cdot P_2$, $Q = Q_1 \cdot Q_2$, and $l_1 \leq \operatorname{ord} P_1, \operatorname{ord} Q_1$, $l_2 \leq \operatorname{ord} P_2, \operatorname{ord} Q_2$)*

Gordan and Grace & Young ([Gor75, GY03]) gave an algorithm for computing

module	hsop degrees	module	hsop degrees
mV_1	$2(\ \times(2m-3))$	nV_2	$2(\times(3n-3))$
$V_1 \oplus V_3$	4,4,4	$V_1 \oplus V_4$	2,3,5,6
$V_1 \oplus V_5$	4,4,6,8,12	$V_1 \oplus V_6$	4,5,6,6,7,10
$V_2 \oplus V_3$	2,3,4,5	$V_2 \oplus V_4$	2,2,3,3,4
$V_2 \oplus V_5$	3,4,5,7,8,12	$V_2 \oplus V_6$	2,2,4,4,6,6,10
$V_3 \oplus V_4$	2,3,4,5,6,7	$2V_3$	2, 4($\times4$)
$2V_4$	2($\times3$), 3($\times4$)	$V_1 \oplus 2V_3$	4 ($\times7$)
$2V_1 \oplus V_3$	2, 4 ($\times4$)	$V_1 \oplus 2V_4$	2,2,2,3,3,5,5,6,6
$2V_1 \oplus V_4$	2,2,3,5,5,6,6	$V_1 \oplus V_2 \oplus V_3$	3,3,4,4,4,5
$V_1 \oplus V_2 \oplus V_4$	2,2,3,3,4,5,6	$V_1 \oplus V_3 \oplus V_4$	3,4,4,5,5,6,6,7

Table 5.2: Degrees of hsop

the generators of the covariants of $V_2 \oplus V$. Their result will be used as well in this chapter:

Proposition 5.0.7. (Gordan, Grace & Young [Gor75, GY03])
Consider $q \in V_2$ a quadratic form and V an SL_2-module whose covariants are generated by $\{C_1, \ldots, C_\lambda\}$. The set of generators of the covariants of $V_2 \oplus V$ will contain q, $(q,q)_2$, and the generators $C_1, \ldots, C_\lambda$. The rest of the generators belong to one of the following three classes:

$$(C_i, q^r)_{2r-1}, (C_i, q^r)_{2r}, (C_j C_k, q^r)_{2r},$$

where the orders of C_j and C_k are both odd.

Proposition 5.0.7 gives us a set of generators of the covariants of $V_2 \oplus V$. However, this set might not be minimal, as some of the transvectants belonging to these three classes could be reducible. Nevertheless, we obtain an upper bound on the degree of the generators of the covariants of $V_2 \oplus V$ by applying this algorithm.

5.1 The invariants of $mV_1 \oplus W$

Proposition 5.1.1. *([Cle72, §55], [GY03, §138A]) Consider W an SL_2-module with the generating covariants $C_1, \ldots, C_p$, of orders $n_1, \ldots, n_p$ respectively. Then, the generating covariants of $V_1 \oplus W$ are*

$$C_1, \ldots, C_p, \ell, \ \text{and} \ \{(C_i, \ell^\gamma)_\gamma\}_{1 \leq \gamma \leq n_i},$$

with $\ell \in V_1$ and $i = 1, p$.

Proof. We apply Proposition 5.0.5.

The transvectants of type $(C_1^{\alpha_1} \ldots C_p^{\alpha_p}, \ell^s)_\gamma$, with $s > \gamma$, will contain a factor ℓ, so they can't contribute to a generating set.

We look now at transvectants of type $(C_1^{\alpha_1} \ldots C_p^{\alpha_p}, \ell^\gamma)_\gamma$. Suppose there exist r, q such that $\alpha_r, \alpha_q \neq 0$. Then we can write

$$(C_1^{\alpha_1} \ldots C_p^{\alpha_p}, \ell^\gamma)_\gamma = (C_r \cdot C, \ell^{n_r} \ell^{\gamma - n_r})_\gamma,$$

with $n_r \leq \operatorname{ord} C_r, \operatorname{ord} \ell^{n_r}$ and $\gamma - n_r \leq \operatorname{ord} C, \operatorname{ord} \ell^{\gamma - n_r}$, hence the transvectants $(C_1^{\alpha_1} \ldots C_p^{\alpha_p}, \ell^\gamma)_\gamma$ are irrelevant in this case. In the same way, the transvectants $(C_i^{\alpha_i}, \ell^\gamma)_\gamma$, with $\alpha_i > 1$, are irrelevant as well, for all $i = 1, p$.

Therefore, the covariants of of $V_1 \oplus W$ are generated by $C_1, \ldots, C_p$, ℓ, and $\{(C_i, \ell^\gamma)_\gamma\}_{1 \leq \gamma \leq n_i}$ with $\ell \in V_1$ and $i = 1, p$. $\qquad\square$

Proposition 5.1.2. *Consider W an SL_2-module with the generating covariants $C_1, \ldots, C_p$, of orders $n_1, \ldots, n_p$ respectively. Consider $\ell_1, \ldots \ell_m \in V_1$. The generating invariants of $mV_1 \oplus W$ are the*

invariants of $W, \{(\ell_i, \ell_j)_1\}_{i \neq j}$ and $\{(C_i, \ell_1^{\gamma_1} \ldots \ell_m^{\gamma_m})_{n_i}\}_{\gamma_1 + \ldots \gamma_m = n_i}$.

Proof. If $m = 1$, we apply Proposition 5.1.1 and obtain that the invariants of $V_1 \oplus W$ are generated by $\{(C_i, \ell^{n_i})_{n_i}\}_{i=1,p}$, with $\ell \in V_1$ and $i = 1, p$ and by the invariants of W.

We look now at the case $m = 2$. Consider $\ell_1, \ell_2 \in V_1$. From Proposition 5.1.1 we know that the covariants of $V_1 \oplus W$ generated by $C_1, \ldots, C_p$, ℓ_1, and $\{(C_i, \ell_1^\gamma)_\gamma\}_{1 \leq \gamma \leq n_i}$. We apply again Proposition 5.1.1 for $V_1 \oplus (V_1 \oplus W)$ and obtain that the covariants of $2V_1 \oplus W$ are generated by

$$C_1, \ldots, C_p, \ell_1, \ell_2, (\ell_1, \ell_2)_1, \{(C_i, \ell_1^\gamma)_\gamma\}_{1 \leq \gamma \leq n_i}, \{(C_i, \ell_2^\gamma)_\gamma\}_{1 \leq \gamma \leq n_i},$$
$$\{((C_i, \ell_1^{\gamma_1})_{\gamma_1}, \ell_2^{\gamma_2})_{\gamma_2}\}_{\gamma_1 + \gamma_2 \leq n_i}.$$

But $((C_i, \ell_1^{\gamma_1})_{\gamma_1}, \ell_2^{\gamma_2})_{\gamma_2} = (C_i, \ell_1^{\gamma_1} \ell_2^{\gamma_2})_{\gamma_1 + \gamma_2}$, hence the invariants of $2V_1 \oplus W$ are generated by the invariants of W, $(\ell_1, \ell_2)_1$ and by $\{(C_i, \ell_1^{\gamma_1} \ell_2^{\gamma_2})_{n_i}\}_{\gamma_1 + \gamma_2 = n_i}$.

We proceed now by induction. Using the equality

$$((C_i, \ell_1^{\gamma_1} \ldots \ell_{m-1}^{\gamma_{m-1}})_{\gamma_1 + \ldots \gamma_{m-1}}, \ell_m^{\gamma_m})_{\gamma_m} = (C_i, \ell_1^{\gamma_1} \ldots \ell_m^{\gamma_m})_{\gamma_1 + \ldots \gamma_m},$$

it follows that the invariants of $mV_1 \oplus W$ are generated by $\{(C_i, \ell_1^{j_1} \ldots \ell_m^{j_m})_{n_i}\}_{i=1,p}$ with $j_1 + \ldots j_m = n_i$, by $\{(\ell_i, \ell_j)_1\}_{i \neq j}$, and by the invariants of W. $\qquad\square$

Example 5.1.1. The algebra of invariants of $2V_1 \oplus V_3$ is generated by 13 invariants. We can see this in the following way: the generating covariants of $c \in V_3$ are c, $(c, c)_2$, $(c, (c, c)_2)_1$, and $((c, c)_2, (c, c)_2)_2$. Then, the invariants of $(\ell, c) \in V_1 \oplus V_3$ are generated by $(c, \ell^3)_3$, $((c, c)_2, \ell^2)_2$, $((c, (c, c)_2)_1, \ell^3)_3$, and $((c, c)_2, (c, c)_2)_2$.

By polarization, the invariants of $(\ell_1, \ell_2, c) \in 2V_1 \oplus V_3$ are then generated by

$(c, \ell_1^3)_3$	$((c, c)_2, \ell_1^2)_2$	$((c, (c, c)_2)_1, \ell_1^3)_3$	$((c, c)_2, (c, c)_2)_2$	$(\ell_1, \ell_2)_1$
$(c, \ell_1^2 \ell_2)_3$	$((c, c)_2, \ell_1 \ell_2)_2$	$((c, (c, c)_2)_1, \ell_1^2 \ell_2)_3$		
$(c, \ell_1 \ell_2^2)_3$	$((c, c)_2, \ell_2^2)_2$	$((c, (c, c)_2)_1, \ell_1 \ell_2^2)_3$		
$(c, \ell_2^3)_3$		$((c, (c, c)_2)_1, \ell_2^3)_3$		

One finds in a similar way the following results (r stays for the number of the generating invariants):

module	r	module	r	module	r
$2V_1 \oplus V_3$	13	$2V_1 \oplus V_4$	20	$3V_1 \oplus V_3$	30
$2V_1 \oplus V_2 \oplus V_3$	35	$2V_1 \oplus 2V_4$	103	$3V_1 \oplus V_4$	63
$2V_1 \oplus V_2 \oplus V_4$	57				

Table 5.3: The number of the generating invariants

5.2 The invariants of $mV_1 \oplus nV_2$

The cases $2V_1$, $3V_1$, $4V_1$, V_2, $2V_2$, $3V_2$, $V_1 \oplus V_2$, $V_1 \oplus 2V_2$, $V_1 \oplus 3V_2$, $2V_1 \oplus V_2$, $2V_1 \oplus 2V_2$, and $3V_1 \oplus 2V_2$, are classically treated in the references [Bes69, Ell95, Gor69, Gor87, GY03, Per87]. Gordan and Kraft & Weyman [Gor87, KW99] proved that the covariants of nV_2 are generated by those of order ≤ 2 and degree ≤ 2.

Theorem 5.2.1. (Gordan, Kraft & Weyman [Gor87, KW99]) *Let* $q_1, \ldots, q_n \in V_2$, *with* $n \geq 3$. *The covariants of* nV_2 *are generated by*

1) *n covariants of type $\mathbb{C}^{n*} \otimes V_2$, of degree 1 and order 2; these are the forms themselves q_i, with $i = 1, \ldots, n$;*

2) *$\binom{n+1}{2}$ invariants of type $S^2 \mathbb{C}^{n*} \otimes V_0$, of degree 2; these are the transvectants $(q_i, q_j)_2$, with $i \leq j$;*

3) *$\binom{n}{2}$ covariants of type $\bigwedge^2 \mathbb{C}^{n*} \otimes V_2$, of degree 2 and order 2; these are the transvectants $(q_i, q_j)_1$, with $i < j$;*

4) *$\binom{n}{3}$ invariants of type $\bigwedge^3 \mathbb{C}^{n*} \otimes V_0$, of degree 3; these are $(q_i, (q_j, q_k)_1)_2$, with $i < j < k$;*

Theorem 5.2.2. *Let $\ell_1, \ldots, \ell_m \in V_1$ and $q_1, \ldots, q_n \in V_2$, with $m \geq 2, n \geq 3$. The invariants of $mV_1 \oplus nV_2$ are generated by*

1) *$\binom{m}{2}$ invariants of type $\bigwedge^2 \mathbb{C}^{m*} \otimes V_0$, of degree 2; these are the transvectants $(\ell_i, \ell_j)_1$, with $i < j$;*

2) *$\binom{n+1}{2}$ invariants of type $S^2 \mathbb{C}^{n*} \otimes V_0$, of degree 2; these are the transvectants $(q_i, q_j)_2$, with $i \leq j$;*

3) *$\binom{n}{3}$ invariants of type $\bigwedge^3 \mathbb{C}^{n*} \otimes V_0$, of degree 3; these are the transvectants $(q_i, (q_j, q_k)_1)_2$, with $i < j < k$;*

4) *$n\binom{m+1}{2}$ invariants of type $S^2 \mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0$, of degree 3; these are the transvectants $(q_k, \ell_i \ell_j)_2$, with $i \leq j$*

5) $\binom{m+1}{2}\binom{n}{2}$ *invariants of type* $S^2\mathbb{C}^{m*} \otimes \bigwedge^2\mathbb{C}^{n*} \otimes V_0$, *of degree 4; these are the transvectants* $((q_i, q_j)_1, \ell_k\ell_m)_2$, *with* $i < j$, $k \leq m$.

Proof of Theorem 5.2.1. The covariants of $q \in V_2$ are generated by q itself and the invariant $(q, q)_2$. We apply Proposition 5.0.7:

- if $(q_1, q_2) \in 2V_2$, then the covariants of $2V_2$ are generated by q_1, q_2, $(q_1, q_2)_1$, $(q_1, q_1)_2$, $(q_2, q_2)_2$, and $(q_1, q_2)_2$.

- if $(q_1, q_2, q_3) \in 3V_2$, then the covariants of $3V_2$ are generated by $\{q_i\}_{1 \leq i \leq 3}$, $\{(q_i, q_j)_2\}_{1 \leq i \leq j \leq 3}$, $\{(q_i, q_j)_1\}_{1 \leq i < j \leq 3}$, $\{(q_i, (q_j, q_k)_1)_1\}_{1 \leq i < j < k \leq 3}$, and respectively $\{(q_i, (q_j, q_k)_1)_2\}_{1 \leq i < j < k \leq 3}$.

But, for any three quadratic forms $q_1, q_2, q_3 \in V_2$ we have

$$((q_1, q_2)_1, q_3)_1 = \frac{1}{2}(q_2(q_1, q_3)_2 - q_1(q_2, q_3)_2).$$

Then, the generators $(q_i, (q_j, q_k)_1)_1$ are superfluous. Therefore, the covariants of $3V_2$ are generated by the following covariants: $\{q_i\}_{1 \leq i \leq 3}$, $\{(q_i, q_j)_1\}_{1 \leq i < j \leq 3}$, $\{(q_i, q_j)_2\}_{1 \leq i \leq j \leq 3}$, and $\{(q_i, (q_j, q_k)_1)_2\}_{1 \leq i < j < k \leq 3}$.

By induction, the covariants of nV_2 are generated by the following covariants: $\{q_i\}_{1 \leq i \leq n}$, $\{(q_i, q_j)_1\}_{1 \leq i < j \leq n}$, $\{(q_i, q_j)_2\}_{1 \leq i \leq j \leq n}$, and $\{(q_i, (q_j, q_k)_1)_2\}_{1 \leq i < j < k \leq n}$.

We look now at the type of the generating covariants. We have:

$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_2)_1 = \mathbb{C}^{n*} \otimes V_2,$$

$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_2)_2 = S^2\mathbb{C}^{n*} \otimes S^2V_2 \oplus \bigwedge^2\mathbb{C}^{n*} \otimes \bigwedge^2 V_2$$

$$= S^2\mathbb{C}^{n*} \otimes (V_4 \oplus V_0) \oplus \bigwedge^2\mathbb{C}^{n*} \otimes V_2,$$

$$\mathcal{O}(\mathbb{C}^{n*} \otimes V_2)_3 = S^3\mathbb{C}^{n*} \otimes S^3V_2 \oplus S_{(2,1)}\mathbb{C}^{n*} \otimes S_{(2,1)}V_2 \oplus \bigwedge^3\mathbb{C}^{n*} \otimes \bigwedge^3 V_2$$

$$= S^3\mathbb{C}^{n*} \otimes (V_6 \oplus V_2) \oplus S_{(2,1)}\mathbb{C}^{n*} \otimes (V_4 \oplus V_2) \oplus \bigwedge^3\mathbb{C}^{n*} \otimes V_0.$$

(We used the program `LiE` [LCL92] to obtain the decompositions of $S_\lambda V_2$.)
The generators $\{q_i\}$ correspond to $\mathbb{C}^{n*} \otimes V_2$. There are n such generators.
The generators $\{(q_i, q_j)_1\}$ correspond to $\bigwedge^2\mathbb{C}^{n*} \otimes V_2$. There are $\binom{n}{2} = \dim \bigwedge^2\mathbb{C}^{n*}$ such generators.
The generators $\{(q_i, q_j)_2\}$ correspond to $S^2\mathbb{C}^{n*} \otimes V_0$. There are $\binom{n+1}{2} = \dim S^2\mathbb{C}^{n*}$ such generators.
The generators $\{(q_i, (q_j, q_k)_1)_2\}$ correspond to $\bigwedge^3\mathbb{C}^{n*} \otimes V_0$. There are $\binom{n}{3} = \dim \bigwedge^3\mathbb{C}^{n*}$ such generators.
(We used Theorem 2.8.2 for computing these dimensions.) □

Proof of Theorem 5.2.2. We apply Proposition 5.0.5 with $V = mV_1$ and $W = nV_2$. The covariants of mV_1 are generated by the ℓ_i themselves, and the invariants $(\ell_i, \ell_j)_1$ for $i < j$. The covariants of nV_2 are generated by the q_i themselves, the covariants $(q_i, q_j)_1$ for $i \leq j$, and the invariants $(q_i, q_j)_2$ for $i \leq j$ and $(q_i, (q_j, q_k)_1)_2$ for $i < j < k$.

Then, the invariants of $mV_1 \oplus nV_2$ will be generated by the generators of $\mathcal{O}(mV_1)^{\mathrm{SL}_2}$, the generators of $\mathcal{O}(nV_2)^{\mathrm{SL}_2}$, the invariants $(q_k, \ell_i\ell_j)_2$ for $i \leq j$, of degree 3, and the invariants $((q_i, q_j)_1, \ell_k\ell_m)_2$ for $i < j$, $k \leq m$, of degree 4. Given that ($\sim$ is used for equalities up to a nonzero constant)

$$(q_1 \ldots q_p, \ell_1 \ldots \ell_{2p})_{2p} \sim \sum (q_1, \ell_{i_1}\ell_{i_2})_2 \ldots (q_p, \ell_{i_{2p-1}}\ell_{i_{2p}})_2,$$

there are no other irreducible invariants.

We look now at the type of the generating covariants.
Denote $W = \mathbb{C}^{m*} \otimes V_1 \oplus \mathbb{C}^{n*} \otimes V_2$. Then:

$$\mathcal{O}(W)_2 = S^2(\mathbb{C}^{m*} \otimes V_1) \oplus \mathbb{C}^{m*} \otimes V_1 \otimes \mathbb{C}^{n*} \otimes V_2 \oplus S^2(\mathbb{C}^{n*} \otimes V_2) =$$

$$= \ldots \oplus \bigwedge\nolimits^2 \mathbb{C}^{m*} \otimes V_0 \oplus \ldots \oplus S^2\mathbb{C}^{n*} \otimes (V_4 \oplus V_0) \oplus \ldots,$$

$$\mathcal{O}(W)_3 = S^3(\mathbb{C}^{m*} \otimes V_1) \oplus S^2(\mathbb{C}^{m*} \otimes V_1) \otimes \mathbb{C}^{n*} \otimes V_2 \oplus \mathbb{C}^{m*} \otimes V_1 \otimes$$

$$\otimes S^2(\mathbb{C}^{n*} \otimes V_2) \oplus S^3(\mathbb{C}^{n*} \otimes V_2) =$$

$$= \ldots \oplus S^2\mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes (V_4 \oplus V_0) \oplus \ldots \bigwedge\nolimits^3 \mathbb{C}^{n*} \otimes V_0 \oplus \ldots$$

$$\mathcal{O}(W)_4 = S^4(\mathbb{C}^{m*} \otimes V_1) \oplus S^3(\mathbb{C}^{m*} \otimes V_1) \otimes \mathbb{C}^{n*} \otimes V_2 \oplus S^2(\mathbb{C}^{m*} \otimes V_1) \otimes$$

$$\otimes S^2(\mathbb{C}^{n*} \otimes V_2) \oplus \mathbb{C}^{m*} \otimes V_1 \otimes S^3(\mathbb{C}^{n*} \otimes V_2) \oplus S^4(\mathbb{C}^{n*} \otimes V_2) =$$

$$= \ldots \oplus S^2\mathbb{C}^{m*} \otimes \bigwedge\nolimits^2 \mathbb{C}^{n*} \otimes (V_4 \oplus V_0) \oplus \ldots$$

(We used the program LiE [LCL92] to obtain the decompositions of $S_\lambda V_2$ and of $S_\lambda V_1$.)
The generators $\{(\ell_i, \ell_j)_1\}$ correspond to $\bigwedge^2 \mathbb{C}^{m*} \otimes V_0$. There are $\binom{m}{2} = \dim \bigwedge^2 \mathbb{C}^{m*}$ such generators.
The generators $\{(q_i, q_j)_2\}$ correspond to $S^2\mathbb{C}^{n*} \otimes V_0$. There are $\binom{n+1}{2} = \dim S^2\mathbb{C}^{n*}$ such generators.
The generators $\{(q_i, (q_j, q_k)_1)_2\}$ correspond to $\bigwedge^3 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{n}{3} = \dim \bigwedge^3 \mathbb{C}^{n*}$ such generators.
The generators $(q_k, \ell_i\ell_j)_2$ correspond to $S^2\mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0$. There are $n\binom{m+1}{2} = \dim(S^2\mathbb{C}^{m*} \otimes \mathbb{C}^{n*})$ such generators.
The generators $((q_i, q_j)_1, \ell_k\ell_m)_2$ correspond to $S^2\mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{m+1}{2}\binom{n}{2} = \dim(S^2\mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*})$ such generators.
(We used Theorem 2.8.2 for computing these dimensions.) $\square$

The following result proved by Brouwer ([BP11]) gives the degrees of a system of parameters of the algebra $\mathcal{O}(mV_1 \oplus nV_2)^{\mathrm{SL}_2}$:

Proposition 5.2.3. *Let $V = mV_1 \oplus nV_2$, and let $I = \mathcal{O}(V)^{\mathrm{SL}_2}$.*

If $2n + 1 \geq m$, then I has a system of parameters consisting of $m + 2n - 2$ invariants of degree 2 and $m + n - 1$ invariants of degree 3.

If $2n + 1 < m$, then I has a system of parameters consisting of $m + 2n - 2$ invariants of degree 2 and $3n$ invariants of degree 3, and $m - 2n - 1$ invariants of degree 6.

If $n = 0$, then I has a system of parameters consisting of $2m - 3$ invariants of degree 2.

If $m = 0$, then I has a system of parameters consisting of $3n - 3$ invariants of degree 2.

5.3 The invariants of $mV_1 \oplus nV_3$

The generating invariants and covariants of V_3, $2V_3$ [Pea82, Gor87, Ell95, GY03], and of $V_1 \oplus V_3$ [Ell95] can be found in the classical literature. Von Gall found 28 generating invariants of $3V_3$ and respectively 98 generating covariants of $3V_3$ [Gal94]. His result was corrected later by Sinigallia, who found 97 generating covariants of $3V_3$ [Sin05]. Peano and Kraft & Weyman [Pea82, KW99] proved that the covariants of nV_3 are generated by those of order ≤ 4 and degree ≤ 6.

Theorem 5.3.1. (Peano, Kraft & Weyman [Pea82, KW99]) *Consider the cubics $c_1, \ldots, c_n \in V_3$, with $n \geq 2$. The generators of the covariants of nV_3 are of the following types:*

1) *n covariants of type $\mathbb{C}^{n*} \otimes V_3$, of degree 1; these are the cubics $c_i \in V_3$ themselves;*

2) *$\binom{n}{2}$ covariants of type $\bigwedge^2 \mathbb{C}^{n*} \otimes V_4$, of degree 2 and order 4; these are given by the transvectants $(c_i, c_j)_1$, with $i \neq j$;*

3) *$\binom{n+1}{2}$ covariants of type $S^2 \mathbb{C}^{n*} \otimes V_2$, of degree 2 and order 2; these are given by the transvectants $(c_i, c_j)_2$;*

4) *$\binom{n}{2}$ invariants of type $\bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, of degree 2; these are given by the transvectants $(c_i, c_j)_3$, with $i \neq j$;*

5) *$\binom{n+2}{3}$ covariants of type $S^3 \mathbb{C}^{n*} \otimes V_3$, of degree 3 and order 3; these are given by the transvectants $(c_i, (c_j, c_k)_2)_1$;*

6) *$2\binom{n+1}{3}$ covariants of type $S_{(2,1)} \mathbb{C}^{n*} \otimes V_1$, of degree 3 and order 1; these are given by the transvectants $(c_i, (c_j, c_k)_2)_2$;*

7) *$\binom{n+3}{4}$ invariants of type $S^4 \mathbb{C}^{n*} \otimes V_0$, of degree 4; these are given by the transvectants $((c_i, c_j)_2, (c_k, c_l)_2)_2$;*

8) *$3\binom{n+2}{4}$ covariants of type $S_{(3,1)} \mathbb{C}^{n*} \otimes V_2$, of degree 4 and order 2; these are given by the transvectants $((c_i, c_j)_2, (c_k, c_l)_2)_1$;*

9) *$4\binom{n+3}{5}$ covariants of type $S_{(4,1)} \mathbb{C}^{n*} \otimes V_1$, of degree 5 and order 1; these are given by the transvectants $((c_i, c_j)_2, (c_k, (c_l, c_m)_2)_2)_1$.*

10) *$\frac{1}{4}\binom{n+1}{3}\binom{n+2}{3}$ invariants of type $S_{(3,3)} \mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $((c_i, (c_j, c_k)_2)_2, (c_l, (c_m, c_n)_2)_2)_1$.*

(We used Theorem 2.8.2 for computing these dimensions.)

Remark 5.3.2. *Peano [Pea82] estimated correctly the number of the generating covariants of nV_3, except the number of those in degree 6. In an article from 1899, Young [You99] mentions a result "due to Peano", which states that the generating set of covariants of nV_3 contains $\binom{n+4}{6}$ invariants of degree 6. However, the correct number of these generators is $\frac{1}{4}\binom{n+1}{3}\binom{n+2}{3} = \dim S_{(3,3)}\mathbb{C}^{n*}$.*

Theorem 5.3.3. *Let $\ell_1, \ldots, \ell_m \in V_1$ and $c_1, \ldots, c_n \in V_3$, with $m \geq 2, n \geq 2$. The invariants of $mV_1 \oplus nV_3$ are generated by*

1) *$\binom{m}{2}$ invariants of type $\bigwedge^2 \mathbb{C}^{m*} \otimes V_0$, of degree 2; these are given by the transvectants $(\ell_i, \ell_j)_1$, with $i \neq j$;*

2) *$\binom{n}{2}$ invariants of type $\bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, of degree 2; these are given by the transvectants $(c_i, c_j)_3$, with $i \neq j$;*

3) *$n\binom{m+2}{3}$ invariants of type $S^3\mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0$, of degree 4; these are given by the transvectants $\{(c_i, \ell_j\ell_k\ell_l)\}$;*

4) *$\binom{m+1}{2}\binom{n+1}{2}$ invariants of type $S^2\mathbb{C}^{m*} \otimes S^2\mathbb{C}^{n*} \otimes V_0$, of degree 4; these are given by the transvectants $\{((c_i, c_j)_2, \ell_k\ell_l)_2\}$;*

5) *$2m\binom{n+1}{3}$ invariants of type $\mathbb{C}^{m*} \otimes S_{(2,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 4; these are given by the transvectants $((c_i, (c_j, c_k)_2)_2, \ell_l)_1$;*

6) *$\binom{n+3}{4}$ invariants of type $S^4\mathbb{C}^{n*} \otimes V_0$, of degree 4; these are given by the transvectants $((c_i, c_j)_2, (c_k, c_l)_2)_2$;*

7) *$\binom{m+3}{4}\binom{n}{2}$ invariants of type $S^4\mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $((c_i, c_j)_1, \ell_k\ell_l\ell_p\ell_q)_4$;*

8) *$\binom{m+2}{3}\binom{n+2}{3}$ invariants of type $S^3\mathbb{C}^{m*} \otimes S^3\mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $((c_i, (c_j, c_k)_2)_1, \ell_l\ell_p\ell_q)_3$;*

9) *$3\binom{m+1}{2}\binom{n+2}{4}$ invariants of type $S^2\mathbb{C}^{m*} \otimes S_{(3,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $(((c_i, c_j)_2, (c_k, c_l)_2)_1, \ell_p\ell_q)_2$;*

10) *$4m\binom{n+3}{5}$ invariants of type $\mathbb{C}^{m*} \otimes S_{(4,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $(((c_i, c_j)_2, (c_k, (c_l, c_p)_2)_2)_1, \ell_q)_1$;*

11) *$\frac{1}{4}\binom{n+1}{3}\binom{n+2}{3}$ invariants of type $S_{(3,3)}\mathbb{C}^{n*} \otimes V_0$, of degree 6; these are given by the transvectants $((c_i, (c_j, c_k)_2)_2, (c_l, (c_m, c_n)_2)_2)_1$.*

Proof. We obtain the invariants of $V_1 \oplus nV_3$ in the following way: we replace each covariant C of nV_3 of order $s > 0$ by the invariant $(C, \ell^s)_s$. These invariants together with the invariants of nV_3 will generate the invariants of $V_1 \oplus nV_3$. The invariants of $mV_1 \oplus nV_3$ are then obtained by polarisation (see Proposition 5.1.2).

   We give the type of the generating covariants.

Denote $W = \mathbb{C}^{m*} \otimes V_1 \oplus \mathbb{C}^{n*} \otimes V_3$. Then:

$$\begin{aligned}
\mathcal{O}(W)_2 =& S^2(\mathbb{C}^{m*} \otimes V_1) \oplus (\mathbb{C}^{m*} \otimes V_1) \otimes (\mathbb{C}^{n*} \otimes V_3) \oplus S^2(\mathbb{C}^{n*} \otimes V_3) = \\
=& \ldots \oplus \bigwedge{}^2 \mathbb{C}^{m*} \otimes V_0 \oplus \ldots \oplus \bigwedge{}^2 \mathbb{C}^{n*} \otimes (V_4 \oplus V_0) \oplus \ldots, \\
\mathcal{O}(W)_4 =& S^4(\mathbb{C}^{m*} \otimes V_1) \oplus S^3(\mathbb{C}^{m*} \otimes V_1) \otimes (\mathbb{C}^{n*} \otimes V_3) \oplus S^2(\mathbb{C}^{m*} \otimes V_1) \\
& \otimes S^2(\mathbb{C}^{n*} \otimes V_3) \oplus (\mathbb{C}^{m*} \otimes V_1) \otimes S^3(\mathbb{C}^{n*} \otimes V_3) \oplus S^4(\mathbb{C}^{n*} \otimes V_3) \\
=& \ldots \oplus S^3 \mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \oplus S^2 \mathbb{C}^{m*} \otimes S^2 \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \\
& \oplus \mathbb{C}^{m*} \otimes S_{(2,1)} \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \oplus S^4 \mathbb{C}^{n*} \otimes V_0 \oplus \ldots, \\
\mathcal{O}(W)_6 =& S^6(\mathbb{C}^{m*} \otimes V_1) \oplus S^5(\mathbb{C}^{m*} \otimes V_1) \otimes (\mathbb{C}^{n*} \otimes V_3) \oplus S^4(\mathbb{C}^{m*} \otimes V_1) \\
& \otimes S^2(\mathbb{C}^{n*} \otimes V_3) \oplus S^3(\mathbb{C}^{m*} \otimes V_1) \otimes S^3(\mathbb{C}^{n*} \otimes V_3) \\
& \oplus S^2(\mathbb{C}^{m*} \otimes V_1) \otimes S^4(\mathbb{C}^{n*} \otimes V_3) \oplus (\mathbb{C}^{m*} \otimes V_1) \otimes S^5(\mathbb{C}^{n*} \otimes V_3) \\
& \oplus S^6(\mathbb{C}^{n*} \otimes V_3) = \ldots \oplus S^4 \mathbb{C}^{m*} \otimes \bigwedge{}^2 \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \\
& \oplus S^3 \mathbb{C}^{m*} \otimes S^3 \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \oplus S^2 \mathbb{C}^{m*} \otimes S_{(3,1)} \mathbb{C}^{n*} \otimes V_0 \oplus \ldots \\
& \oplus \mathbb{C}^{m*} \otimes S_{(4,1)} \mathbb{C}^{n*} \otimes V_0 \oplus \ldots S_{(3,3)} \mathbb{C}^{n*} \otimes V_0 \oplus \ldots
\end{aligned}$$

(We used the program LiE [LCL92] to obtain the decompositions of $S_\lambda V_3$ and of $S_\lambda V_1$.)

The generators $\{(\ell_i, \ell_j)_1\}$ correspond to $\bigwedge^2 \mathbb{C}^{m*} \otimes V_0$. There are $\binom{m}{2} = \dim \bigwedge^2 \mathbb{C}^{m*}$ such generators.

The generators $\{(c_i, c_j)_3\}$ correspond to $\bigwedge^2 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{n}{2} = \dim \bigwedge^2 \mathbb{C}^{n*}$ such generators.

The generators $\{(c_i, \ell_j \ell_k \ell_l)_3\}$ correspond to $S^3 \mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0$. There are $n\binom{m+2}{3} = \dim S^3 \mathbb{C}^{m*} \otimes \mathbb{C}^{n*}$ such generators.

The generators $\{((c_i, c_j)_2, \ell_k \ell_l)_2\}$ correspond to $S^2 \mathbb{C}^{m*} \otimes S^2 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{m+1}{2}\binom{n+1}{2} = \dim S^2 \mathbb{C}^{m*} \otimes S^2 \mathbb{C}^{n*}$ such generators.

The generators $((c_i, (c_j, c_k)_2)_2, \ell_l)_1$ correspond to $\mathbb{C}^{m*} \otimes S_{(2,1)} \mathbb{C}^{n*} \otimes V_0$. There are $2m\binom{n+1}{3} = \dim \mathbb{C}^{m*} \otimes S_{(2,1)} \mathbb{C}^{n*}$ such generators.

The generators $((c_i, c_j)_2, (c_k, c_l)_2)_2$ correspond to $S^4 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{n+3}{4} = \dim S^4 \mathbb{C}^{n*}$ such generators.

The generators $((c_i, c_j)_1, \ell_k \ell_l \ell_p \ell_q)_4$ correspond to $S^4 \mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{m+3}{4}\binom{n}{2} = \dim S^4 \mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*}$ such generators.

The generators $((c_i, (c_j, c_k)_2)_1, \ell_l \ell_p \ell_q)_3$ correspond to $S^3 \mathbb{C}^{m*} \otimes S^3 \mathbb{C}^{n*} \otimes V_0$. There are $\binom{m+2}{3}\binom{n+2}{3} = \dim S^3 \mathbb{C}^{m*} \otimes S^3 \mathbb{C}^{n*}$ such generators.

The generators $(((c_i, c_j)_2, (c_k, c_l)_2)_1, \ell_p \ell_q)_2$ correspond to $S^2 \mathbb{C}^{m*} \otimes S_{(3,1)} \mathbb{C}^{n*} \otimes V_0$. There are $3\binom{m+1}{2}\binom{n+2}{4} = \dim S^2 \mathbb{C}^{m*} \otimes S_{(3,1)} \mathbb{C}^{n*}$ such generators.

The generators $(((c_i, c_j)_2, (c_k, (c_l, c_p)_2)_2)_1, \ell_q)_1$ correspond to $\mathbb{C}^{m*} \otimes S_{(4,1)} \mathbb{C}^{n*} \otimes V_0$. There are $4m\binom{n+3}{5} = \dim \mathbb{C}^{m*} \otimes S_{(4,1)} \mathbb{C}^{n*}$ such generators.

The generators $((c_i, (c_j, c_k)_2)_2, (c_l, (c_m, c_n)_2)_2)_1$ correspond to $S_{(3,3)} \mathbb{C}^{n*} \otimes V_0$. There are $\frac{1}{4}\binom{n+1}{3}\binom{n+2}{3} = \dim S_{(3,3)} \mathbb{C}^{n*}$ such generators.

(We used Theorem 2.8.2 for computing these dimensions.) $\qquad \square$

5.3.1 The invariants of $V_1 \oplus V_3$

Proposition 5.3.4. *The algebra of invariants of $V_1 \oplus V_3$ has systems of parameters with each of the two sequences of degrees 4, 4, 4 and 4, 4, 6.*

Proof. Let $\ell \in V_1$ and $c \in V_3$,

$$\ell = a_0 x + a_1 y, \qquad c = b_0 x^3 + 3b_1 x^2 y + 3b_2 xy^2 + b_3 y^3,$$

and consider the following invariants:

$$j_{4,1} = ((c,c)_2, (c,c)_2)_2, \qquad j_{4,2} = ((c,c)_2, \ell^2)_2,$$
$$j_{4,3} = (c, \ell^3)_3, \qquad j_6 = ((c, (c,c)_2)_1, \ell^3)_3.$$

First we show that $\mathcal{N}(V_1 \oplus V_3) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{4,3}) = \mathcal{V}(j_{4,1}, j_{4,3}, j_6)$.
Indeed, if $j_{4,1} = 0$, then c is a nullform, and without loss of generality we can suppose $x^2 \mid c$, i.e. $b_3 = b_2 = 0$. Then

$$j_{4,2} \sim a_1^2 b_1^2, \qquad j_6 \sim a_1^3 b_1^3,$$
$$j_{4,3} = a_1^2 (a_1 b_0 - 3a_0 b_1),$$

where $\sim$ denotes equalities up to a nonzero constant. If $a_1 = 0$, then $\ell \sim x$ and it follows that $(\ell, c) \in \mathcal{N}(V_1 \oplus V_3)$. If $a_1 \neq 0$, then we get $b_1 = b_0 = 0$, and again $(\ell, c) \in \mathcal{N}(V_1 \oplus V_3)$.
We proved that $\mathcal{N}(V_1 \oplus V_3) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{4,3}) = \mathcal{V}(j_{4,1}, j_{4,3}, j_6)$. From Proposition 2.7.1 it follows that $\{j_{4,1}, j_{4,2}, j_{4,3}\}$ and $\{j_{4,1}, j_{4,3}, j_6\}$ are two systems of parameters of $\mathcal{O}(V_1 \oplus V_3)^{\mathrm{SL}_2}$. $\square$

Let $\ell \in V_1$ and $c \in V_3$. The generating invariants of $V_1 \oplus V_3$ are:

deg	generators	
4	$((c,c)_2, (c,c)_2)_2$	$((c,c)_2, \ell^2)_2$
	$(c, \ell^3)_3$	
6	$((c, (c,c)_2)_1, \ell^3)_3$	

5.3.2 The invariants of $2V_3$

Let $c_1, c_2 \in V_3$.

Proposition 5.3.5. *The algebra of invariants of $2V_3$ has systems of parameters of degrees 2, 4, 4, 4, 4.*

Proof. We apply Dixmier's criterion (Proposition 2.7.5). First we show that $[4] = 5$. Indeed, if all invariants of degree 4 vanish, then $(c_1, c_2) \in \mathcal{N}(2V_3)$: these invariants are the polarizations of $((c,c)_2, (c,c)_2)_2$ with $c \in V_3$. Because $((c,c)_2, (c,c)_2)_2$ defines the nullcone of $\mathcal{N}(V_3)$, it follows that its polarizations define the nullcone of $2V_3$ (see Theorem 2.6.1). Hence $[4] = 5$. Because $[2] \geq 1$, if follows that $\mathcal{O}(2V_3)^{\mathrm{SL}_2}$ has a system of parameters of degrees $2, 4, 4, 4, 4$. $\square$

The generating invariants of $2V_3$ are:

deg	generators	
2	$(c_1, c_2)_3$	
4	$((c_1, c_1)_2, (c_1, c_1)_2)_2$	$((c_2, c_2)_2, (c_2, c_2)_2)_2$
	$((c_1, c_1)_2, (c_2, c_2)_2)_2$	$((c_1, c_1)_2, (c_1, c_2)_2)_2$
	$((c_2, c_2)_2, (c_1, c_2)_2)_2$	
6	$((c_1, (c_1, c_2)_2)_2, (c_2, (c_1, c_2)_2)_2)_1$	

5.3.3 The invariants of $V_1 \oplus 2V_3$

Let $\ell \in V_1$ and $c_1, c_2 \in V_3$.

Proposition 5.3.6. *The algebra of invariants of $V_1 \oplus 2V_3$ has systems of parameters with each of the two sequences of degrees $2(\times 1), 4(\times 6)$ and $4(\times 7)$ respectively.*

Proof. We apply Dixmier's criterion (Proposition 2.7.5). From Propositions 5.3.4 and 5.3.5 it follows that if all invariants of degree 4 vanish, then $(\ell, c_1, c_2) \in \mathcal{N}(V_1 \oplus 2V_3)$. Then, we have $[4] = 7$, which implies that $\mathcal{O}(V_1 \oplus 2V_3)^{\mathrm{SL}_2}$ has a system of parameters of degrees $4(\times 7)$. Because $[2] \geq 1$, if follows that there is also a system of parameters of $\mathcal{O}(V_1 \oplus 2V_3)^{\mathrm{SL}_2}$ of degrees $2(\times 1), 4(\times 6)$. $\square$

The generating invariants of $V_1 \oplus 2V_3$ are:

deg	generators	
2	$(c_1, c_2)_3$	
4	$((c_1, c_1)_2, (c_1, c_1)_2)_2$	$((c_2, c_2)_2, (c_2, c_2)_2)_2$
	$((c_1, c_1)_2, (c_2, c_2)_2)_2$	$((c_1, c_1)_2, (c_1, c_2)_2)_2$
	$((c_2, c_2)_2, (c_1, c_2)_2)_2$	$((c_1, c_1)_2, \ell^2)_2$
	$((c_2, c_2)_2, \ell^2)_2$	$((c_1, c_2)_2, \ell^2)_2$
	$(c_1, \ell^3)_3$	$(c_2, \ell^3)_3$
	$((c_1, (c_1, c_2)_2)_2, \ell)_1$	$((c_2, (c_1, c_2)_2)_2, \ell)_1$
6	$((c_1, (c_1, c_2)_2)_2, (c_2, (c_1, c_2)_2)_2)_1$	$((c_1, (c_1, c_1)_2)_1, \ell^3)_3$
	$((c_1, (c_2, c_2)_2)_1, \ell^3)_3$	$((c_1, (c_1, c_2)_2)_1, \ell^3)_3$
	$((c_2, (c_2, c_2)_2)_1, \ell^3)_3$	$((c_1, c_1)_2, (c_2, c_2)_2)_1, \ell^2)_2$
	$((c_1, c_1)_2, (c_1, c_2)_2)_1, \ell^2)_2$	$((c_2, c_2)_2, (c_1, c_2)_2)_1, \ell^2)_2$
	$((c_1, c_2)_1, \ell^4)_4$	$(((c_1, c_1)_2, (c_1, (c_2, c_2)_2)_2)_1, \ell)_1$
	$(((c_1, c_1)_2, (c_1, (c_1, c_2)_2)_2)_1, \ell)_1$	$(((c_2, c_2)_2, (c_1, (c_1, c_2)_2)_2)_1, \ell)_1$
	$(((c_2, c_2)_2, (c_1, (c_2, c_2)_2)_2)_1, \ell)_1$	

5.3.4 The invariants of $2V_1 \oplus V_3$

Proposition 5.3.7. *The algebra of invariants of $2V_1 \oplus V_3$ has systems of parameters of degrees 2, 4, 4, 4, 4.*

Proof. Let $\ell_1, \ell_2 \in V_1$ and $c \in V_3$,

$$\ell_1 = a_0 x + a_1 y, \quad c = c_0 x^3 + 3c_1 x^2 y + 3c_2 xy^2 + c_3 y^3,$$
$$\ell_2 = b_0 x + b_1 y.$$

and consider the following invariants:

$$
\begin{aligned}
j_2 \;&= (\ell_1, \ell_2)_2, & j_{4,1} &= ((c,c)_2, (c,c)_2)_2, \\
j_{4,2} &= ((c,c)_2, \ell_1^2)_2, & j_{4,3} &= ((c,c)_2, \ell_2^2)_2, \\
j_{4,4} &= (c, \ell_1^3)_3, & j_{4,5} &= (c, \ell_2^3)_3, \\
j_{4,6} &= ((c,c)_2, \ell_1 \ell_2)_2, & j_{4,7} &= (c, \ell_1^2 \ell_2)_3, \\
j_{4,8} &= (c, \ell_1 \ell_2^2)_3.
\end{aligned}
$$

We prove that $\{j_2, j_{4,1}, j_{4,2} + j_{4,3}, j_{4,4} + j_{4,6}, j_{4,5} + j_{4,6}\}$ forms a system of parameters of the algebra of invariants of $2V_1 \oplus V_3$.
First we show that

$$\mathcal{N}(2V_1 \oplus V_3) = \mathcal{V}(j_2, j_{4,1}, j_{4,2} + j_{4,3}, j_{4,4} + j_{4,6}, j_{4,5} + j_{4,6}).$$

We consider the following cases:

Case 1: $\ell_1 = 0$.
In this case $j_{4,1} = j_{4,3} = j_{4,5} = 0$ implies that $(\ell_2, c) \in \mathcal{N}(V_1 \oplus V_3)$, see Proposition 5.3.4.

Case 2: $\ell_2 = 0$.
In this case $j_{4,1} = j_{4,2} = j_{4,4} = 0$ implies that $(\ell_1, c) \in \mathcal{N}(V_1 \oplus V_3)$ (the proof is similar to Case 1).

Case 3: $\ell_1 = \ell_2 = 0$.
In this case $j_{4,1} = 0$ implies that $c \in \mathcal{N}(V_3)$.

Case 4: $c = 0$.
In this case $j_2 = 0$ implies that $(\ell_1, \ell_2) \in \mathcal{N}(2V_1)$.

Case 5: $\ell_1, \ell_2, c \neq 0$.
In this case $j_{4,1} = 0$ implies that c is a nullform. Without loss of generality we suppose $x^2 \mid c$, i.e. $c_2 = c_3 = 0$. Also, $j_2 = 0$ implies that ℓ_1 and ℓ_2 have a common root. Suppose that this root is y, i.e. $a_0 = b_0 = 0$. Then:

$$
\begin{aligned}
j_{4,2} + j_{4,3} &\sim c_1^2 (a_1^2 + b_1^2), \\
j_{4,4} + j_{4,6} &= a_1 (a_1^2 c_0 - 2b_1 c_1^2), \\
j_{4,5} + j_{4,6} &= b_1 (b_1^2 c_0 - 2a_1 c_1^2),
\end{aligned}
$$

If $c_1 = 0$, it follows either $c_0 = 0$, or $a_1 = b_1 = 0$, which both contradict the assumption $\ell_1, \ell_2, c \neq 0$. If $c_1 \neq 0$, then we get $a_1 = -b_1$, which together with $a_1^2 + b_1^2 = 0$ implies $a_1 = b_1 = 0$, and contradicts the assumption $\ell_1, \ell_2 \neq 0$.

   Therefore, $\mathcal{N}(2V_1 \oplus V_3) = \mathcal{V}(j_2, j_{4,1}, j_{4,2} + j_{4,3}, j_{4,4} + j_{4,6}, j_{4,5} + j_{4,6})$. From Proposition 2.7.1 it follows that $\{j_2, j_{4,1}, j_{4,2} + j_{4,3}, j_{4,4} + j_{4,6}, j_{4,5} + j_{4,6}\}$ is a system of parameters of $\mathcal{O}(2V_1 \oplus V_3)^{\mathrm{SL}_2}$. $\qquad\square$

Let $\ell_1, \ell_2 \in V_1$ and $c \in V_3$. The generating invariants of $2V_1 \oplus V_3$ are:

deg	generators	
2	$(\ell_1, \ell_2)_3$	
4	$((c,c)_2, (c,c)_2)_2$	$((c,c)_2, \ell_1^2)_2$
	$((c,c)_2, \ell_2^2)_2$	$((c,c)_2, \ell_1\ell_2)_2$
	$(c, \ell_1^3)_3$	$(c, \ell_2^3)_3$
	$(c, \ell_1^2\ell_2)_3$	$(c, \ell_1\ell_2^2)_3$
6	$((c, (c,c)_2)_1, \ell_1^3)_3$	$((c, (c,c)_2)_1, \ell_1^2\ell_2)_3$
	$((c, (c,c)_2)_1, \ell_1\ell_2^2)_3$	$((c, (c,c)_2)_1, \ell_2^3)_3$

5.4 The invariants of $mV_1 \oplus nV_4$

The generating invariants and covariants of $2V_4$, $3V_4$ [Ber77, You99] and of $V_1 \oplus V_4$ [Ell95] can be found in the classical literature. Young and Kraft & Weyman [You99, KW99] proved that the covariants of nV_4 are generated by those of order ≤ 6 and degree ≤ 6.

Theorem 5.4.1. (Young, Kraft & Weyman [You99, KW99]) *The generating covariants of nV_4, $n \geq 2$, are:*

1) *n covariants of type $\mathbb{C}^{n*} \otimes V_4$, of degree 1 and order 4; these are the quartics $q_i \in V_4$ themselves;*

2) *$\binom{n+1}{2}$ invariants of type $S^2\mathbb{C}^{n*} \otimes V_0$, of degree 2; these are given by the transvectants $(q_i, q_j)_4$;*

3) *$\binom{n}{2}$ covariants of type $\bigwedge^2\mathbb{C}^{n*} \otimes V_2$, of degree 2 and order 2; these are given by the transvectants $(q_i, q_j)_3$, with $i \neq j$;*

4) *$\binom{n+1}{2}$ covariants of type $S^2\mathbb{C}^{n*} \otimes V_4$, of degree 2 and order 4; these are given by the transvectants $(q_i, q_j)_2$;*

5) *$\binom{n}{2}$ covariants of type $\bigwedge^2\mathbb{C}^{n*} \otimes V_6$, of degree 2 and order 6; these are given by the transvectants $(q_i, q_j)_1$, with $i \neq j$;*

6) *$\binom{n+2}{3}$ invariants of type $S^3\mathbb{C}^{n*} \otimes V_0$, of degree 3; these are given by the transvectants $(q_i, (q_j, q_k)_2)_4$;*

7) *$\binom{n}{3} + 2\binom{n+1}{3}$ covariants of type $(S_{(2,1)}\mathbb{C}^{n*} \oplus \bigwedge^3\mathbb{C}^{n*}) \otimes V_2$, of degree 3 and order 2;*

8) *$2\binom{n+1}{3}$ covariants of type $S_{(2,1)}\mathbb{C}^{n*} \otimes V_4$, of degree 3 and order 4;*

9) *$\binom{n+2}{3}$ covariants of type $S^3\mathbb{C}^{n*} \otimes V_6$, of degree 3 and order 6;*

10) *$\binom{n+1}{4} + \binom{n+2}{4}$ invariants of type $S_{(2,2)}\mathbb{C}^{n*} \otimes V_0$, of degree 4;*

11) *$3\binom{n+2}{4} + 3\binom{n+1}{4}$ covariants of type $(S_{(3,1)}\mathbb{C}^{n*} \oplus S_{(2,1,1)}\mathbb{C}^{n*}) \otimes V_2$, of degree 4 and order 2;*

12) $3\binom{n+1}{4}$ covariants of type $S_{(2,1,1)}\mathbb{C}^{n*} \otimes V_4$, of degree 4 and order 4;

13) $\binom{n}{5} + 2\binom{n+1}{5} + 3\binom{n+2}{5}$ invariants of type $(S_{(2,2,1)}\mathbb{C}^{n*} \oplus \bigwedge^5 \mathbb{C}^{n*}) \otimes V_0$, of degree 5;

14) $3\binom{n+2}{5} + 2\binom{n+3}{5}$ covariants of type $S_{(3,2)}\mathbb{C}^{n*} \otimes V_2$, of degree 5 and order 2;

15) $10\binom{n+2}{6}$ invariants of type $S_{(3,1,1,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 6.

(We used Theorem 2.8.2 for computing these dimensions.)

Theorem 5.4.2. *The generating invariants of $mV_1 \oplus nV_4$, $m \geq 2$, $n \geq 2$, are:*

1) $\binom{m}{2}$ invariants of type $\bigwedge^2 \mathbb{C}^{m*} \otimes V_0$, of degree 2;

2) $\binom{n+1}{2}$ invariants of type $S^2 \mathbb{C}^{n*} \otimes V_0$, of degree 2;

3) $\binom{n+2}{3}$ invariants of type $S^3 \mathbb{C}^{n*} \otimes V_0$, of degree 3;

4) $\binom{m+1}{2}\binom{n}{2}$ invariants of type $S^2 \mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, of degree 4;

5) $\binom{n+1}{4} + \binom{n+2}{4}$ invariants of type $S_{(2,2)}\mathbb{C}^{n*} \otimes V_0$, of degree 4;

6) $n\binom{m+3}{4}$ invariants of type $S^4 \mathbb{C}^{m*} \otimes \mathbb{C}^{n*} \otimes V_0$, of degree 5;

7) $\binom{m+1}{2}[\binom{n}{3} + 2\binom{n+1}{3}]$ invariants of type $S^2 \mathbb{C}^{m*} \otimes (S_{(2,1)}\mathbb{C}^{n*} \oplus \bigwedge^3 \mathbb{C}^{n*}) \otimes V_0$, of degree 5;

8) $\binom{n}{5} + 2\binom{n+1}{5} + 3\binom{n+2}{5}$ invariants of type $(S_{(2,2,1)}\mathbb{C}^{n*} \oplus \bigwedge^5 \mathbb{C}^{n*}) \otimes V_0$, of degree 5;

9) $\binom{m+3}{4}\binom{n+1}{2}$ invariants of type $S^4 \mathbb{C}^{m*} \otimes S^2 \mathbb{C}^{n*} \otimes V_0$, of degree 6;

10) $3\binom{m+1}{2}(\binom{n+2}{4} + \binom{n+1}{4})$ invariants of type $S^2 \mathbb{C}^{m*} \otimes (S_{(3,1)}\mathbb{C}^{n*} \oplus S_{(2,1,1)}) \otimes V_0$, of degree 6;

11) $10\binom{n+2}{6}$ invariants of type $S_{(3,1,1,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 6;

12) $2\binom{m+3}{4}\binom{n+1}{3}$ invariants of type $S^4 \mathbb{C}^{m*} \otimes S_{(2,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 7;

13) $\binom{m+1}{2}[3\binom{n+2}{5} + 2\binom{n+3}{5}]$ covariants of type $S^2 \mathbb{C}^{m*} \otimes S_{(3,2)}\mathbb{C}^{n*} \otimes V_0$, of degree 7;

14) $\binom{m+5}{6}\binom{n}{2}$ invariants of type $S^6 \mathbb{C}^{m*} \otimes \bigwedge^2 \mathbb{C}^{n*} \otimes V_0$, of degree 8;

15) $3\binom{m+3}{4}\binom{n+1}{4}$ invariants of type $S^4 \mathbb{C}^{m*} \otimes S_{(2,1,1)}\mathbb{C}^{n*} \otimes V_0$, of degree 8;

16) $\binom{m+5}{6}\binom{n+2}{3}$ invariants of type $S^6 \mathbb{C}^{m*} \otimes S^3 \mathbb{C}^{n*} \otimes V_6$, of degree 9.

(We used Theorem 2.8.2 for computing these dimensions.)

Proof. We obtain the invariants of $V_1 \oplus nV_4$ in the following way: we replace each covariant C of nV_4 of order $s > 1$ by the invariant $(C, \ell^s)_s$. These invariants together with the invariants of nV_4 will generate the invariants of $V_1 \oplus nV_4$. By polarization we obtain now the invariants of $mV_1 \oplus nV_4$ (see Proposition 5.1.2). $\square$

5.4.1 The invariants of $V_1 \oplus V_4$

Proposition 5.4.3. *The algebra invariants of $V_1 \oplus V_4$ has systems of parameters of degrees 2, 3, 5, 6.*

Proof. Let $\ell \in V_1$ and $q \in V_4$,

$$\ell = a_0 x + a_1 y, \qquad q = b_0 x^4 + 4b_1 x^3 y + \ldots 4b_3 xy^3 + b_4 y^4,$$

and consider the following invariants:

$$j_2 = (q, q)_4, \qquad\qquad j_3 = (q, (q, q)_2)_4,$$
$$j_5 = (q, \ell^4)_4, \qquad\qquad j_6 = ((q, q)_2, \ell^4)_4.$$

We first show that

$$\mathcal{N}(V_1 \oplus V_4) = \mathcal{V}(j_2, j_3, j_5, j_6).$$

Indeed, if $j_2 = j_3 = 0$, then q is a nullform, and without loss of generality we can suppose $x^3 \mid q$, i.e. $b_2 = b_3 = b_4 = 0$. Then $j_6 \sim a_1^4 b_1^2$, where $\sim$ denotes equalities up to a nonzero constant. If $a_1 = 0$, then $\ell \sim x$ and it follows that $(\ell, q) \in \mathcal{N}(V_1 \oplus V_4)$. If $a_1 \neq 0$, then we have $b_1 = 0$, which implies $j_5 \sim a_1^4 b_0$, and we obtain $b_0 = 0$. Hence $q = 0$ and, again, $(\ell, q) \in \mathcal{N}(V_1 \oplus V_4)$.

We proved that $\mathcal{N}(V_1 \oplus V_4) = \mathcal{V}(j_2, j_3, j_5, j_6)$. From Proposition 2.7.1 it follows that $\{j_2, j_3, j_5, j_6\}$ is a system of parameters of $\mathcal{O}(V_1 \oplus V_4)^{\mathrm{SL}_2}$. $\square$

Let $\ell \in V_1$ and $q \in V_4$. The generating invariants of $V_1 \oplus V_4$ are:

deg	generators
2	$(q, q)_4$
3	$(q, (q, q)_2)_4$
5	$(q, \ell^4)_4$
6	$((q, q)_2, \ell^4)_4$
9	$((q, (q, q)_2)_1, \ell^6)_6$

5.4.2 The invariants of $2V_4$

Proposition 5.4.4. *The algebra of invariants of $2V_4$ has systems of parameters of degrees $2(\times 3), 3(\times 4)$.*

Proof. Let $q_1, q_2 \in V_4$. Consider the following invariants:

$$
\begin{aligned}
j_{2,1} &= (q_1, q_1)_4, & j_{2,2} &= (q_2, q_2)_4, \\
j_{2,3} &= (q_1, q_2)_4, & j_{3,1} &= (q_1, (q_1, q_1)_2)_4, \\
j_{3,2} &= (q_2, (q_2, q_2)_2)_4, & j_{3,3} &= (q_1, (q_1, q_2)_2)_4, \\
j_{3,4} &= (q_2, (q_1, q_2)_2)_4.
\end{aligned}
$$

In Chap. 4.3 we saw that $(q, q)_4$ and $(q, (q, q)_2)_4$, with $q \in V_4$, define the nullcone of V_4. Theorem 2.6.1 implies that the polarizations of these two invariants define the nullcone $\mathcal{N}(2V_4)$, hence $\mathcal{N}(2V_4) = \mathcal{V}(j_{2,1}, \ldots, j_{2,3}, j_{3,1}, \ldots, j_{3,4})$. Then, from Proposition 2.7.1, it follows that $\{j_{2,1}, \ldots, j_{2,3}, j_{3,1}, \ldots, j_{3,4}\}$ is a system of parameters of $\mathcal{O}(2V_4)^{\mathrm{SL}_2}$. $\qquad\square$

Let $q_1, q_2 \in V_4$. The generating invariants of $2V_4$ are:

deg	generators	
2	$(q_1, q_1)_4$	$(q_2, q_2)_4$
	$(q_1, q_2)_4$	
3	$(q_1, (q_1, q_1)_2)_4$	$(q_2, (q_2, q_2)_2)_4$
	$(q_1, (q_1, q_2)_2)_4$	$(q_2, (q_1, q_2)_2)_4$
4	$((q_1, q_1)_2, (q_2, q_2)_2)_4$	

5.4.3 The invariants of $V_1 \oplus 2V_4$

Proposition 5.4.5. *The algebra of invariants of invariants of $V_1 \oplus 2V_4$ has systems of parameters of degrees $2(\times 3), 3(\times 2), 5(\times 2), 6(\times 2)$.*

Proof. Let $l \in V_1$, and $q_1, q_2 \in V_4$. Consider the following invariants:

$$
\begin{aligned}
j_{2,1} &= (q_1, q_1)_4, & j_{2,2} &= (q_2, q_2)_4, \\
j_{2,3} &= (q_1, q_2)_4, & j_{3,1} &= (q_1, (q_1, q_1)_2)_4, \\
j_{3,2} &= (q_2, (q_2, q_2)_2)_4, & j_{3,3} &= (q_1, (q_1, q_2)_2)_4, \\
j_{3,4} &= (q_2, (q_1, q_2)_2)_4, & j_{5,1} &= (q_1, \ell^4)_4, \\
j_{5,2} &= (q_2, \ell^4)_4, & j_{6,1} &= ((q_1, q_1)_2, \ell^4)_4, \\
j_{6,2} &= ((q_2, q_2)_2, \ell^4)_4.
\end{aligned}
$$

Propositions 5.4.3 and 5.4.4 imply that if $j_{2,1} = \ldots = j_{2,3} = j_{3,1} = \ldots = j_{3,4} = j_{5,1} = j_{5,2} = j_{6,1} = j_{6,2}$, then $(\ell, q_1, q_2) \in \mathcal{N}(V_1 \oplus 2V_4)$. A Gröbner basis computations shows that $j_{3,1}^3$ and $j_{3,2}^3$ are in the ideal generated by $j_{2,1}, j_{2,2}, j_{2,3}, j_{3,3}, j_{3,4}, j_{5,1}, j_{5,2}, j_{6,1} + j_{3,2}^2, j_{6,2} + j_{3,1}^2$, which implies that

$$
\mathcal{N}(V_1 \oplus 2V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_{2,3}, j_{3,3}, j_{3,4}, j_{5,1}, j_{5,2}, j_{6,1} + j_{3,2}^2, j_{6,2} + j_{3,1}^2).
$$

From Proposition 2.7.1, it follows that these invariants form a system of parameters of $\mathcal{O}(V_1 \oplus 2V_4)^{\mathrm{SL}_2}$. $\qquad\square$

Let $q_1, q_2 \in V_4$ and $\ell \in V_1$. The generating invariants of $V_1 \oplus 2V_4$ are:

deg	generators	
2	$(q_1, q_1)_4$	$(q_2, q_2)_4$
	$(q_1, q_2)_4$	
3	$(q_1, (q_1, q_1)_2)_4$	$(q_2, (q_2, q_2)_2)_4$
	$(q_1, (q_1, q_2)_2)_4$	$(q_2, (q_1, q_2)_2)_4$
4	$((q_1, q_1)_2, (q_2, q_2)_2)_4$	$((q_1, q_2)_3, \ell^2)_2$
5	$(q_1, \ell^4)_4$	$(q_2, \ell^4)_4$
	$((q_1, (q_1, q_2)_1)_4, \ell^2)_2$	$((q_2, (q_1, q_2)_1)_4, \ell^2)_2$
6	$((q_1, q_1)_2, \ell^4)_4$	$((q_2, q_2)_2, \ell^4)_4$
	$((q_1, q_2)_2, \ell^4)_4$	$(((q_1, q_1)_2, (q_2, q_2)_2)_3, \ell^2)_2$
	$(((q_1, q_1)_2, (q_1, q_2)_2)_3, \ell^2)_2$	$(((q_1, q_2)_2, (q_2, q_2)_2)_3, \ell^2)_2$
7	$((q_1, (q_1, q_2)_2)_2, \ell^4)_4$	$((q_2, (q_1, q_2)_2)_2, \ell^4)_4$
	$(((q_1, (q_1, q_2)_2)_2, (q_1, q_2)_2)_3, \ell^2)_2$	$(((q_2, (q_1, q_2)_2)_2, (q_1, q_2)_2)_3, \ell^2)_2$
8	$((q_1, q_2)_1, \ell^6)_6$	
9	$((q_1, (q_1, q_1)_2)_1, \ell^6)_6$	$((q_2, (q_2, q_2)_2)_1, \ell^6)_6$
	$((q_1, (q_1, q_2)_2)_1, \ell^6)_6$	$((q_2, (q_1, q_2)_2)_1, \ell^6)_6$

5.4.4 The invariants of $2V_1 \oplus V_4$

Proposition 5.4.6. *The algebra of invariants of $2V_1 \oplus V_4$ has systems of parameters of degrees 2, 3, 5, 5, 6, 6.*

Proof. Let $\ell_1, \ell_2 \in V_1$ and $q \in V_4$. Consider the following invariants:

$$\begin{aligned} j_{2,1} &= (q, q)_4, & j_{2,2} &= (\ell_1, \ell_2)_1, \\ j_3 &= (q, (q, q)_2)_4, & j_{5,1} &= (q, \ell_1^4)_4, \\ j_{5,2} &= (q, \ell_2^4)_4, & j_{6,1} &= ((q, q)_2, \ell_1^4)_4, \\ j_{6,2} &= ((q, q)_2, \ell_2^4)_4. \end{aligned}$$

We prove that $\{j_{2,1} + j_{2,2}, j_3, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2}\}$ forms a system of parameters of the algebra of invariants $2V_1 \oplus V_4$. First we show that

$$\mathcal{N}(2V_1 \oplus V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_3, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2}).$$

Indeed, $j_{2,2} = 0$ implies that $(\ell_1, \ell_2) \in \mathcal{N}(2V_1)$. Also, $j_{2,1} = j_3 = j_{5,1} = j_{5,2} = j_{6,1} = j_{6,2} = 0$ implies that $(\ell_1, q) \in \mathcal{N}(V_1 \oplus V_4)$ and $(\ell_2, q) \in \mathcal{N}(V_1 \oplus V_4)$, see Proposition 5.4.3. Therefore, $\mathcal{N}(2V_1 \oplus V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_3, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2})$. A Gröbner basis computation shows that $j_{2,1}^{10}, j_{2,2}^{10}$ are in the ideal generated by $j_{2,1} + j_{2,2}, j_3, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2}$. Then, we have

$$\mathcal{N}(2V_1 \oplus V_4) = \mathcal{V}(j_{2,1} + j_{2,2}, j_3, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2}).$$

From Proposition 2.7.1 it follows that these invariants form a system of parameters of $\mathcal{O}(2V_1 \oplus V_4)^{\mathrm{SL}_2}$. $\qquad\square$

Let $\ell_1, \ell_2 \in V_1$ and $q \in V_4$. The generating invariants of $V_1 \oplus 2V_4$ are:

deg	generators	
2	$(q,q)_4$	$(\ell_1, \ell_2)_1$
3	$(q, (q,q)_2)_4$	
5	$(q, \ell_1^4)_4$	$(q, \ell_2^4)_4$
	$(q, \ell_1^3 \ell_2)_4$	$(q, \ell_1^2 \ell_2^2)_4$
	$(q, \ell_1 \ell_2^3)_4$	
6	$((q,q)_2, \ell_1^4)_4$	$((q,q)_2, \ell_2^4)_4$
	$((q,q)_2, \ell_1^3 \ell_2)_4$	$((q,q)_2, \ell_1^2 \ell_2^2)_4$
	$((q,q)_2, \ell_1 \ell_2^3)_4$	
9	$((q, (q,q)_2)_1, \ell_1^6)_6$	$((q, (q,q)_2)_1, \ell_1^5 \ell_2)_6$
	$((q, (q,q)_2)_1, \ell_1^4 \ell_2^2)_6$	$((q, (q,q)_2)_1, \ell_1^3 \ell_2^3)_6$
	$((q, (q,q)_2)_1, \ell_1^2 \ell_2^4)_6$	$((q, (q,q)_2)_1, \ell_1 \ell_2^5)_6$
	$((q, (q,q)_2)_1, \ell_2^6)_6$	

5.5 The invariants of $V_1 \oplus V_5$

This case is treated in [Gor87, GY03].
Let $\ell \in V_1$ and $q \in V_5$,

$$\ell = a_0 x + a_1 y, \qquad q = b_0 x^5 + 5b_1 x^4 y + \ldots 5b_4 xy^4 + b_5 y^5.$$

Consider the covariants $c_1 = (q,q)_4$, $c_2 = (q,q)_2$, $c_3 = (q, c_1)_2$, and the following invariants:

$$\begin{aligned}
j_{4,1} &= (c_1, c_1)_2, & j_{4,2} &= (c_1, \ell^2)_2, \\
j_{6,1} &= (q, \ell^5)_5, & j_{8,1} &= (c_1, (c_3, c_3)_2)_5, \\
j_{12,1} &= ((c_3, c_3)_2, (c_3, c_3)_2)_2, & j_{12,2} &= ((q, c_2)_1, \ell^9)_9.
\end{aligned}$$

Proposition 5.5.1. *The set $\{j_{4,1}, j_{4,2}, j_{6,1}, j_{8,1}, j_{12,1} + j_{12,2}\}$ forms a system of parameters of the ring of invariants of $V_1 \oplus V_5$.*

Proof. We show that $\mathcal{N}(V_1 \oplus V_5) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{6,1}, j_{8,1}, j_{12,1} + j_{12,2})$. We consider the following cases:

Case 1: $\ell = 0$.
In this case we have $j_{4,1} = j_{8,1} = j_{12,1} = 0$, which implies that $q \in \mathcal{N}(V_5)$ (see Chap. 4.4).

Case 2: $\ell \neq 0$.
If $j_{4,1} = 0$, then c_1 is a nullform.
   Subcase 2.1: $c_1 = 0$.
In this case we can suppose without loss of generality that $a_1 = 0$. Then we have $j_{6,1} \sim a_0^5 b_5$, where $\sim$ denotes equalities up to a nonzero constant. Because

$\ell \neq 0$, we obtain $b_5 = 0$. But then we have $j_{12,1} + j_{12,2} \sim a_0^9 b_4^3$, which implies $b_4 = 0$. Then we have

$$0 = c_1 = (6b_2^2 - 8b_1 b_3)x^2 + 8b_2 b_3 xy + 6b_3^2 y^2,$$

which implies $b_3 = b_2 = 0$. Therefore we have $\ell \sim x$ and $x^4 \mid q$, i.e. $(\ell, q) \in \mathcal{N}(V_1 \oplus V_5)$.

Subcase 2.2: $c_1 \neq 0$.
In this case we can suppose without loss of generality that $c_1 = x^2$. Then, we have $j_{4,2} = a_1^2$, which implies $a_1 = 0$. Then we obtain $j_{6,1} \sim a_0^5 b_5$, which implies $b_5 = 0$. Then we have $j_{8,1} \sim b_4^2$, which implies $b_4 = 0$. Then we have

$$x^2 = c_1 = (6b_2^2 - 8b_1 b_3)x^2 + 8b_2 b_3 xy + 6b_3^2 y^2,$$

which implies $b_3 = 0$. Therefore, we have $\ell \sim x$ and $x^3 \mid q$, i.e. $(l, q) \in \mathcal{N}(V_1 \oplus V_5)$.
We proved that $\mathcal{N}(V_1 \oplus V_5) = \mathcal{V}(j_{4,1}, j_{4,2}, j_{6,1}, j_{8,1}, j_{12,1} + j_{12,2})$. From Proposition 2.7.1 it follows that $\{j_{4,1}, j_{4,2}, j_{6,1}, j_{8,1}, j_{12,1} + j_{12,2}\}$ forms a system of parameters of $\mathcal{O}(V_1 \oplus V_5)^{\mathrm{SL}_2}$. □

Proposition 5.5.2. *The algebra of invariants of $V_1 \oplus V_5$ is generated by 23 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	4	6	8	10	12	14	18
d_i	2	3	7	3	6	1	1

Proof. The Poincaré series of $\mathcal{O}(V_1 \oplus V_5)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{a(t)}{(1-t^4)^2(1-t^6)(1-t^8)(1-t^{12})}$$

with

$$a(t) = 1 + 2t^6 + 6t^8 + 3t^{10} + 7t^{12} + 7t^{14} + 3t^{16} + 6t^{18} + 2t^{20} + t^{26}$$

so that

$$P(t) = 1 + 2t^4 + 3t^6 + 10t^8 + 9t^{10} + 29t^{12} + 31t^{14} + 61t^{16} + 75t^{18} +$$
$$+ 125t^{20} + 144t^{22} + 229t^{24} + 264t^{26} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 26. Up to degree 18 we find the following 23 invariants:

deg	generators	
4	$(c_1,c_1)_2$	$(c_1,\ell^2)_2$
6	$(q,\ell^5)_5$	$(c_3,\ell^3)_3$
	$((c_1,c_3)_2,\ell)_1$	
8	$(c_1,(c_3,c_3)_2)_5$	$(c_2,\ell^6)_6$
	$((c_2,q)_3,\ell^5)_5$	$((c_1,c_3)_1,\ell^3)_3$
	$((c_1,c_2)_2,\ell^4)_4$	$((c_1^2,c_2)_4,\ell^2)_2$
	$((c_1^3,q)_5,\ell)_1$	
10	$((c_1,c_2)_1,\ell^6)_6$	$((c_1^2,c_2)_3,\ell^4)_4$
	$((c_1^3,c_2)_5,\ell^2)_2$	
12	$((c_3,c_3)_2,(c_3,c_3)_2)_2$	$((q,c_2)_1,\ell^9)_9$
	$(((q,c_2)_1,c_1)_2,\ell^7)_7$	$(((q,c_2)_1,c_1^2)_4,\ell^5)_5$
	$(((q,c_2)_1,c_1^3)_6,\ell^3)_3$	$(((q,c_2)_1,c_1^4)_8,\ell)_1$
14	$(((q,c_2)_1,c_1^5)_9,\ell)_1$	
18	$(c_1^7,q\cdot(q,c_2)_1)_{14}$	

Then we prove that no generators are needed in degrees 20 and 26 by showing that for each $i \in \{20,26\}$ the vector space $\mathcal{O}(V_1\oplus V_5)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 23 invariants found in degrees ≤ 18. For details about the computations see Chap. A.5. □

5.6 The invariants of $V_1 \oplus V_6$

This case is treated in [Gor87, GY03].
Let $\ell \in V_1$ and $s \in V_6$,

$$\ell = a_0 x + a_1 y, \qquad s = b_0 x^6 + 6b_1 x^5 y + \ldots 6b_5 x y^5 + b_6 y^6.$$

Consider the covariants $c_1 = (s,s)_4$, $c_2 = (s,s)_2$, $c_3 = (s,c_1)_4$, $c_4 = (c_1,c_1)_2$, $c_5 = (s,c_1)_1$, and the following invariants:

$$
\begin{aligned}
j_{2,1} &= (s,s)_6, & j_{4,1} &= (c_1,c_1)_4, \\
j_{5,1} &= (c_3,\ell^2)_2, & j_{6,1} &= (c_1,c_4)_4, \\
j_{6,2} &= (c_1,\ell^4)_4, & j_{7,1} &= (s,\ell^6)_6, \\
j_{10,1} &= (c_4,c_3^2)_4, & j_{10,2} &= (c_2,\ell^8)_8.
\end{aligned}
$$

Proposition 5.6.1. *The set $\{j_{4,1}, j_{5,1}, j_{6,1}, j_{6,2}+j_{21}^3, j_{7,1}, j_{10,1}+j_{10,2}\}$ forms a system of parameters of the ring of invariants of $V_1 \oplus V_6$.*

Proof. We first show that

$$\mathcal{N}(V_1 \oplus V_6) = \mathcal{V}(j_{2,1}, j_{4,1}, j_{6,1}, j_{6,2}, j_{7,1}, j_{10,1}, j_{10,2}).$$

Indeed, if $j_{2,1} = j_{4,1} = j_{6,1} = j_{10,1} = 0$, then s is a nullform (see Chap. 4.5).

Without loss of generality suppose $b_6 = b_5 = b_4 = b_3 = 0$. Then, we have:

$$
\begin{aligned}
j_{6,2} &\sim a_1^4 b_2^2, \\
j_{7,1} &\sim a_1^4 (a_1^2 b_0 - 6a_0 a_1 b_1 + 15a_0^2 b_2), \\
j_{10,2} &\sim a_1^6 (a_1^2 b_1^2 - a_1^2 b_0 b_2 - 4a_0 a_1 b_1 b_2 + 10a_0^2 b_2^2),
\end{aligned}
$$

where $\sim$ denotes equalities up to a non-zero constant. If $a_1 = 0$, then $(\ell, s) \in \mathcal{N}(V_1 \oplus V_6)$. If $a_1 \neq 0$, then $b_2 = b_1 = b_0 = 0$, i.e. $s = 0$ and $(\ell, s) \in \mathcal{N}(V_1 \oplus V_6)$. Now we show that

$$
\mathcal{N}(V_1 \oplus V_6) = \mathcal{V}(j_{4,1}, j_{5,1}, j_{6,1}, j_{6,2} + j_{2,1}^3, j_{7,1}, j_{10,1} + j_{10,2}).
$$

Indeed, if $j_{4,1} = j_{6,1} = 0$, then c_1 is a nullform. We consider the following cases:

Case 1: $c_1 = 0$.
In this case we have $j_{6,2} = j_{10,2} = 0$, which implies $j_{2,1} = j_{10,2} = 0$. We obtain $j_{2,1} = j_{4,1} = j_{6,1} = j_{6,2} = j_{7,1} = j_{10,1} = j_{10,2} = 0$, and this implies $(\ell, s) \in \mathcal{N}(V_1 \oplus V_6)$.

Case 2: $c_1 = x^4$.
In this case we have

$$
\begin{aligned}
x^4 = c_1 &= 2(3b_2^2 - 4b_1 b_3 + b_0 b_4)x^4 + 4(2b_2 b_3 - 3b_1 b_4 + 4b_0 b_5)x^3 y + \\
&\quad + 2(8b_3^2 - 9b_2 b_4 + b_0 b_6)x^2 y^2 + 4(2b_3 b_4 - 3b_2 b_5 + b_1 b_6)xy^3 + \\
&\quad + 2(3b_4^2 - 4b_3 b_5 + b_2 b_6)y^4.
\end{aligned}
$$

A Gröbner basis computation shows that b_6, b_5, b_4, b_3 are in the ideal generated by $6b_2^2 - 8b_1 b_3 + 2b_0 b_4 - 1$, $2b_2 b_3 - 3b_1 b_4 + 4b_0 b_5$, $8b_3^2 - 9b_2 b_4 + b_0 b_6$, $2b_3 b_4 - 3b_2 b_5 + b_1 b_6$, and $3b_4^2 - 4b_3 b_5 + b_2 b_6$. If $b_6 = b_5 = b_4 = b_3 = 0$, then we obtain $j_{6,2} + j_{2,1}^3 = a_1^4$, so a_1 will vanish as well. Then, $x \mid \ell$ and $x^4 \mid s$, and therefore $(\ell, s) \in \mathcal{N}(V_1 \oplus V_6)$.

Case 3: $c_1 = x^3 y$.
In this case we have

$$
\begin{aligned}
x^3 y = c_1 &= 2(3b_2^2 - 4b_1 b_3 + b_0 b_4)x^4 + 4(2b_2 b_3 - 3b_1 b_4 + 4b_0 b_5)x^3 y + \\
&\quad + 2(8b_3^2 - 9b_2 b_4 + b_0 b_6)x^2 y^2 + 4(2b_3 b_4 - 3b_2 b_5 + b_1 b_6)xy^3 + \\
&\quad + 2(3b_4^2 - 4b_3 b_5 + b_2 b_6)y^4.
\end{aligned}
$$

A Gröbner basis computation shows that $b_6, b_4, b_3, b_2, -1 + 16b_0 b_5$ are in the ideal generated by $3b_2^2 - 4b_1 b_3 + b_0 b_4$, $4(2b_2 b_3 - 3b_1 b_4 + 4b_0 b_5) - 1$, $8b_3^2 - 9b_2 b_4 + b_0 b_6$, $2b_3 b_4 - 3b_2 b_5 + b_1 b_6$, and $3b_4^2 - 4b_3 b_5 + b_2 b_6$. If $b_6 = b_4 = b_3 = b_2 = 0$, then we obtain $j_{5,1} \sim a_0^2 b_5$. Because $b_5 \neq 0$, we have $a_0 = 0$. Then, $j_{7,1} \sim b_1^3 b_5^3$, which implies $b_1 = 0$. But then, $j_{10,1} + j_{10,2} \sim b_5^2$, which leads to a contradiction.

We proved that $\mathcal{N}(V_1 \oplus V_6) = \mathcal{V}(j_{4,1}, j_{5,1}, j_{6,1}, j_{6,2} + j_{2,1}^3, j_{7,1}, j_{10,1} + j_{10,2})$. From Proposition 2.7.1 it follows that $\{j_{4,1}, j_{5,1}, j_{6,1}, j_{6,2} + j_{21}^3, j_{7,1}, j_{10,1} + j_{10,2}\}$ forms a system of parameters of the ring of invariants of $V_1 \oplus V_6$. $\qquad\square$

Proposition 5.6.2. *The algebra of invariants of $V_1 \oplus V_6$ is generated by 26 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	4	5	6	7	8	9	10	11	12	13	14	15
d_i	1	1	1	2	2	1	3	4	2	3	2	2	2

Proof. The Poincaré series of $\mathcal{O}(V_1 \oplus V_6)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{a(t)}{(1 - t^4)(1 - t^5)(1 - t^6)^2(1 - t^7)(1 - t^{10})}$$

with

$$\begin{aligned}
a(t) = {} & 1 + t^2 + t^4 + t^6 + t^7 + 2t^8 + 4t^9 + 5t^{10} + 6t^{11} + 8t^{12} + 7t^{13} + 9t^{14} + 9t^{15} + \\
& + 7t^{16} + 8t^{17} + 6t^{18} + 5t^{19} + 4t^{20} + 2t^{21} + t^{22} + t^{23} + t^{25} + t^{27} + t^{29}
\end{aligned}$$

so that

$$\begin{aligned}
P(t) = {} & 1 + t^2 + 2t^4 + t^5 + 4t^6 + 3t^7 + 6t^8 + 7t^9 + 13t^{10} + 13t^{11} + 23t^{12} + 23t^{13} + \\
& + 38t^{14} + 43t^{15} + 61t^{16} + 69t^{17} + 94t^{18} + 108t^{19} + 143t^{20} + 164t^{21} + \\
& + 209t^{22} + 236t^{23} + \ldots + 339t^{25} + \ldots + 472t^{27} + \ldots + 641t^{29} + \ldots
\end{aligned}$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 29. Up to degree 15 we find the following 26 invariants:

deg	generators	
2	$(s, s)_6$	
4	$(c_1, c_1)_4$	
5	$(c_3, \ell^2)_2$	
6	$(c_1, c_4)_4$	$(c_1, \ell^4)_4$
7	$(s, \ell^6)_6$	$((c_1, c_3)_2, l^2)_2$
8	$((s, c_3)_2, l^4)_4$	
9	$((s, c_1)_2, \ell^6)_6$	$((c_1, c_3)_1, \ell^4)_4$
	$((s, c_3^2)_4, \ell^2)_2$	
10	$(c_4, c_3^2)_4$	$(c_2, \ell^8)_8$
	$((s, c_3)_1, \ell^6)_6$	$((c_1, c_3^2)_3, \ell^2)_2$
11	$(c_5, \ell^8)_8$	$((s, c_3^2)_3, \ell^4)_4$
12	$(((s, c_1)_2, c_3)_1, \ell^6)_6$	$(((s, c_1)_1, c_3)_2, \ell^6)_6$
	$((s, c_3^3)_5, \ell^2)_2$	
13	$((c_2, c_3)_1, \ell^8)_8$	$((c_5, c_3^2)_4, \ell^4)_4$
14	$((c_1, c_2)_1, l^{10})_{10}$	$((c_5, c_3^3)_6, l^2)_2$
15	$(c_5, c_3^4)_8$	$((s, c_2)_1, \ell^{12})_{12}$

Then we prove that no generators are needed in degrees $16, \ldots, 23, 25, 27, 29$ by showing that for each $i \in \{16, \ldots, 23, 25, 27, 29\}$ the vector space $\mathcal{O}(V_1 \oplus V_6)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 26 invariants found in degrees ≤ 15. For details about the computations see Chap. A.6. $\qquad\square$

5.7 The covariants of V_7

In 1888 von Gall [Gal88] computed a set of 153 generating covariants of V_7, of order ≤ 15 and degree ≤ 30. This set was not minimal: in 1879 Sylvester [Sy79b] found 124 generators; about ten years ago Cröni [Crö02] found 147 generators and this result was (independently) confirmed by Bedratyuk [Bed09] seven years later. However, both Cröni and Bedratyuk relied in their proofs on Gall's statement, saying that no generating covariant occurs in degree > 30.

In this chapter we show that V_7 has 147 generating covariants of order ≤ 15 and degree ≤ 30. We prove, independently of von Gall's work, that the maximal degree of the generators is indeed 30. We are working with the covariants mentioned by Gall [Gal88] in his paper, after removing the superfluous ones. Tables 5.4 and 5.5 show the orders and the degrees of these covariants. We use the following notations:

$$
\begin{array}{llll}
l = (f,f)_6, & \kappa = (f,f)_4, & H = (f,f)_2, & r = (f,\kappa)_5, & \epsilon = (f,\kappa)_2, \\
\gamma = (f,\kappa)_1, & T = (f,H)_1, & p = (\kappa,\kappa)_4, & \Delta = (\kappa,\kappa)_2, & \zeta = (H,\kappa)_1, \\
g = (r,\kappa)_2, & \eta = (r,\kappa)_1, & \tau = (r,r)_2, & \beta = (p,\kappa)_1, & \alpha = (p,r)_3, \\
\vartheta = (p,r)_1, & \mu = (\tau,\kappa)_2, & \lambda = (\tau,\kappa)_1, & Q = (r,\tau)_1, & \nu = (r,\alpha)_1, \\
\varphi = (p,\alpha)_1, & \rho = (\tau,\alpha)_1, & \psi = (\vartheta,\alpha)_1, & \sigma = (Q,\alpha)_1, & \omega = (\nu,\alpha)_1.
\end{array}
$$

	0	1	2	3	4	5	6
1	-	-	-	-	-	-	-
2	-	l	-	-	-	κ	-
3	-	-	-	r	-	$(f,l)_2$	-
4	$(l,l)_2$	-	-	-	$p,(\kappa,l)_2$	-	$(\kappa,l)_1$
5	-	$(r,l)_2$	-	$(r,l)_1$ $(f,l^2)_4$	-	g $(f,l^2)_3$	-
6	-	-	τ $(\kappa,l^2)_4$ $(p,l)_2$	-	$(\kappa,l^2)_3$ $(p,l)_1$	-	$(H,l^2)_4$ $(\Delta,l)_2$
7	-	α $(f,l^3)_6$ $(r,l^2)_3$	-	$(f,l^3)_5$ $(g,l)_2$	-	ϑ $(\epsilon,l^2)_4$ $(\eta,l)_2$ $(g,l)_1$	-
8	$(\kappa,l^3)_6$ $(p,l^2)_4$ $(\tau,l)_2$	-	$(\kappa,l^3)_5$ $(p,l^2)_3$ $(\tau,l)_1$	-	μ $(H,l^3)_6$ $(\Delta,l^2)_4$	-	λ $(H,l^3)_5$ $(\beta,l)_2$
9	-	$(f,l^4)_7$ $(g,l^2)_4$ $(\alpha,l)_1$	-	Q $(\epsilon,l^3)_6$ $(\eta,l^2)_4$ $(g,l^2)_3$ $(\vartheta,l)_2$	-	$(\gamma,l^3)_6$ $(\epsilon,l^3)_5$	-
10	-	-	$\nu,(H,l^4)_8$ $(\Delta,l^3)_6,(\mu,l)_2$	-	$(H,l^4)_7,(\beta,l^2)_4$ $(\lambda,l)_2,(\mu,l)_1$	-	-
11	-	$(\epsilon,l^4)_8,(\mu,l^3)_6$ $(g,l^3)_5,(\vartheta,l^2)_4$ $(Q,l)_2$	-	$(\gamma,l^4)_8,(\epsilon,l^4)_7$ φ	-	-	-
12	$(\tau,\tau)_2,(H,l^5)_{10}$ $(\Delta,l^4)_8,(r^2,l^3)_6$ $(\mu,l^2)_4,(\nu,l)_2$	-	$(H,l^5)_9,(\Delta,l^4)_7$ $(\beta,l^3)_6,(\lambda,l^2)_4$ $(\mu,l^2)_3,(\nu,l)_1$	-	-	-	$(\zeta,l^4)_8$
13	-	$\rho,(\gamma,l^5)_{10}$ $(\epsilon,l^5)_9,(\eta,l^4)_7$ $(\vartheta,l^3)_5,(Q,l^2)_3$	-	$(\varphi,l)_1$	-	$(T,l^5)_{10}$	-

	0	1	2	3	4	5	6
13		$(\varphi, l)_2$					
14	$(rf, l^5)_{10}, (\beta, l^4)_8$ $(\lambda, l^3)_6, (r\alpha, l^2)_4$	-	-	-	$\psi, (\zeta, l^5)_{10}$	-	-
15	-	$(\gamma, l^6)_{11}$ $(\varphi, l^2)_3, (\rho, l)_1$	-	$(T, l^6)_{12}$	-	-	-
16	$(f^2, l^7)_{14}$ $(\alpha^2, l)_2$	-	$\sigma, (\psi, l)_2$ $(\zeta, l^6)_{12}$	-	-	-	-
17	-	$\omega, (T, l^7)_{14}$	-	-	-	-	-
18	$(\zeta, l^7)_{14}, (\sigma, l)_2$ $(fg, l^6)_{12}, (\psi, l^2)_4$ $(\epsilon r, l^6)_{12}, (g\alpha, l^3)_6$ $(r\vartheta, l^4)_8, (rQ, l^3)_6$ $(r\eta, l^5)_{10}$	-	-	-	-	-	-
19	-	$(\omega, l)_1$	-	-	-	-	-
20	$(\tau, \alpha^2)_2$	-	-	-	-	-	-
22	$(\varphi\alpha, l^2)_4$ $(\rho\alpha, l)_2$	-	-	-	-	-	-
23	-	$(\sigma, \alpha)_1$	-	-	-	-	-
24	-	-	-	-	-	-	-
...							
25	-	-	-	-	-	-	-
26	$(\alpha\omega, l)_2$	-	-	-	-	-	-
27	-	-	-	-	-	-	-
...							
29	-	-	-	-	-	-	-
30	$((\sigma, \alpha)_1, \alpha)_1$	-	-	-	-	-	-

Table 5.4: The generating covariants of V_7 of orders ≤ 7

	7	8	9	10	11	12	13	14	15
1	f	-	-	-	-	-	-	-	-
2	-	-	-	H	-	-	-	-	-
3	$(f, l)_1$	-	ϵ	-	γ	-	-	-	T
4	-	$\Delta, (H, l)_2$	-	$(H, l)_1$	-	-	-	ζ	-
5	η $(\epsilon, l)_2$	-	$(\epsilon, l)_1$ $(\gamma, l)_2$	-	-	-	$(T, l)_2$	-	-
6	-	β $(H, l^2)_3$	-	-	-	$(\zeta, l)_2$	-	-	-
7	$(\gamma, l^2)_4, (\epsilon, l^2)_3$	-	-	-	$(T, l^2)_4$	-	-	-	-
8	-	-	-	$(\zeta, l^2)_4$	-	-	-	-	-
9	-	-	$(T, l^3)_6$	-	-	-	-	-	-
10	-	$(\zeta, l^3)_6$	-	-	-	-	-	-	-
11	-	-	-	-	-	-	-	-	-
...									
31	-	-	-	-	-	-	-	-	-

Table 5.5: The generating covariants of V_7 of orders ≥ 7

Proposition 5.7.1. *The generating covariants of V_7 are the 147 covariants contained in Tables 5.4 and 5.5.*

Proof. The generating covariants of V_7 have order ≤ 15 (this follows from Proposition 2.3.3).

First we show that the covariants from Tables 5.4 and 5.5 generate the vector space of covariants of order m and degree d with $m \leq 15$ and $d \leq 24$

(with Mathematica, for details about the computations see Chap. A.7).

The invariants of V_7 are generated by those of degree ≤ 30 (see Chap. A.7). If there are generators of degree > 24 and order ≥ 1, then they are of the form $C = (C_1 \ldots C_r, f)_k$, where $C_1, C_2, \ldots, C_r$ are covariants of lower degrees. If this transvectant is irreducible, from Proposition 2.1.5 it follows that

(1) $r = 1$, or

(2) $r = 2$ and $\operatorname{ord} C \leq 5$, or

(3) $r = 3$ and $\operatorname{ord} C \leq 4$, or

(4) $r = 4$ and $\operatorname{ord} C \leq 3$, or

(5) $r = 5$ and $\operatorname{ord} C \leq 2$, or

(6) $r = 6$ and $\operatorname{ord} C \leq 1$.

We first find sets of generators for the covariants of V_7 of order m, with $m \in \{1, 2, \ldots, 5\}$. These five modules of covariants (over the invariant ring of V_7) are Cohen-Macaulay, from Proposition 2.4.3. Then, from their Poincaré series, using Proposition 2.4.4, we know the highest degree in which we have to look for generators. Here are the Poincaré series of these modules (we already know from Chapter 4.6 that $\mathcal{O}(V_7)^{\mathrm{SL}_2}$ has a homogeneous system of parameters of degrees $4, 8, 12, 12, 20$):

order 1: $\dfrac{p_1(t)}{(1 - t^4)(1 - t^8)(1 - t^{12})^2(1 - t^{20})}$, where

$p_1(t) = t^5 + 3t^7 + 3t^9 + 5t^{11} + 9t^{13} + 9t^{15} + 12t^{17} + 15t^{19} + 14t^{21} + 17t^{23} + 17t^{25} +$

$+ 14t^{27} + 15t^{29} + 12t^{31} + 9t^{33} + 9t^{35} + 5t^{37} + 3t^{39} + 3t^{41} + t^{43}.$

order 2: $\dfrac{p_2(t)}{(1 - t^4)(1 - t^8)(1 - t^{12})^2(1 - t^{20})}$, where

$p_2(t) = t^2 + 3t^6 + 3t^8 + 7t^{10} + 9t^{12} + 13t^{14} + 18t^{16} + 19t^{18} + 23t^{20} + 23t^{22} + 26t^{24} +$

$+ 23t^{26} + 23t^{28} + 19t^{30} + 18t^{32} + 13t^{34} + 9t^{36} + 7t^{38} + 3t^{40} + 3t^{42} + t^{46}.$

order 3: $\dfrac{p_3(t)}{(1 - t^4)(1 - t^8)(1 - t^{12})^2(1 - t^{20})}$, where

$p_3(t) = t^3 + 2t^5 + 3t^7 + 8t^9 + 11t^{11} + 14t^{13} + 21t^{15} + 24t^{17} + 28t^{19} + 32t^{21} + 32t^{23} +$

$+ 32t^{25} + 32t^{27} + 28t^{29} + 24t^{31} + 21t^{33} + 14t^{35} + 11t^{37} + 8t^{39} + 3t^{41} + 2t^{43} + t^{45}.$

order 4: $\dfrac{p_4(t)}{(1 - t^4)(1 - t^8)(1 - t^{12})^2(1 - t^{20})}$, where

$p_4(t) = 3t^4 + 2t^6 + 7t^8 + 11t^{10} + 16t^{12} + 22t^{14} + 28t^{16} + 33t^{18} + 37t^{20} + 42t^{22} +$

$+ 38t^{24} + 42t^{26} + 37t^{28} + 33t^{30} + 28t^{32} + 22t^{34} + 16t^{36} + 11t^{38} + 7t^{40} + 2t^{42} + 3t^{44}.$

order 5: $\dfrac{p_5(t)}{(1 - t^4)(1 - t^8)(1 - t^{12})^2(1 - t^{20})}$, where

$p_5(t) = t^3 + 3t^5 + 6t^7 + 10t^9 + 16t^{11} + 23t^{13} + 30t^{15} + 37t^{17} + 42t^{19} + 47t^{21} + 49t^{23} +$

$+ 49t^{25} + 47t^{27} + 42t^{29} + 37t^{31} + 30t^{33} + 23t^{35} + 16t^{37} + 10t^{39} + 6t^{41} + 3t^{43} + t^{45}.$

Then $\mathcal{C}(V_7)_1$ is generated in degree ≤ 43, $\mathcal{C}(V_7)_2$ is generated in degree ≤ 46, $\mathcal{C}(V_7)_3$ is generated in degree ≤ 45, $\mathcal{C}(V_7)_4$ is generated in degree ≤ 44, and $\mathcal{C}(V_7)_5$ is generated in degree ≤ 45. We prove that the covariants from Tables 5.4 and 5.5 generate the vector space of covariants of V_7 of order 1 and degree d with $d \leq 43$, the vector space of covariants of V_7 of order 2 and degree d with $d \leq 46$, the vector space of covariants of V_7 of order 3 and degree d with $d \leq 45$, the vector space of covariants of V_7 of order 4 and degree d with $d \leq 44$, and the vector space of covariants of V_7 of order 5 and degree d with $d \leq 45$ (with Mathematica, for details about the computations see Chap. A.7).

Finally, there is no irreducible covariant of the form $(C, f)_k$ of degree ≥ 24: there are no generators in degree 24 and any other irreducible covariant $(C_1 \ldots C_r, f)_k$, with $r > 1$, should have order ≤ 5 and there are none in degree ≥ 24. This concludes our proof. $\qquad\square$

5.8 The covariants of V_8

In two papers published in 1880, von Gall [Gal80] computed a set of 70 generating covariants of V_8, of order ≤ 18 and degree ≤ 12. This set was not minimal: Sylvester [Sy79b] found 69 generating covariants in 1879; few years ago Cröni [Crö02] and Bedratyuk & Bedratiuk [BB08] independently found 69 generating covariants of V_8. However, both of them relied in their proofs on Gall's statement saying that no generating covariant occurs in degree ≥ 13.

In this chapter we show that V_8 has indeed 69 generating covariants of orders ≤ 18 and degree ≤ 12. We prove, independently of Gall's work, that no generator has degree ≥ 13. Table 5.6 contains the generators.

	0	2	4	6	8	10	12	14	18
1	-	-	-	-	f	-	-	-	-
2	$(f,f)_8$	-	$k=(f,f)_6$	-	$i=(f,f)_4$	-	$h=(f,f)_2$	-	-
3	$(f,i)_8$	-	$(f,k)_4$	$(f,k)_3$	$(f,k)_2$	$(f,k)_1$	$p=(f,i)_2$	$\theta=(f,i)_1$	$(f,h)_1$
4	$(k,k)_4$	-	$\Delta=(k,k)_2$ $(i,k)_4$	$(i,k)_3$	$(i,k)_2$	$(i,k)_1$ $(h,k)_3$	$(h,k)_2$	$(h,k)_1$	$(h,i)_1$
5	$(f,k^2)_8$	$((f,k)_4,k)_3$	$((f,k)_4,k)_2$ $(f,\Delta)_4$	$((f,k)_4,k)_1$ $(f,\Delta)_3$	$(f,\Delta)_2$	$(f,\Delta)_1$ $(p,k)_3$ $(\theta,k)_4$	-	$(p,k)_1$	-
6	$(i,k^2)_8$	$((i,k)_4,k)_3$	$((i,k)_4,k)_2$ $(i,\Delta)_4$	$((i,k)_4,k)_1$ $(i,\Delta)_3$ $\tau=(k,\Delta)_1$	$(i,\Delta)_2$	$(i,\Delta)_1$	-	-	-
7	$((f,k)_4,\Delta)_4$	$((f,k)_4,\Delta)_3$ $(f,\tau)_6$	$((f,k)_4,\Delta)_2$ $(f,\tau)_5$	$((f,k)_4,\Delta)_1$ $(f,\tau)_4$ $(\theta,k^2)_8$	-	-	-	-	-
8	$((i,k)_4,\Delta)_4$	$((i,k)_4,\Delta)_3$ $(i,\tau)_6$	$((i,k)_4,\Delta)_2$ $(i,\tau)_5$	$((i,k)_4,\Delta)_1$ $(i,\tau)_4$	-	-	-	-	-
9	$(f,\Delta^2)_8$	$((f,k)_4,\tau)_4$ $((f,\Delta)_4,\Delta)_3$ $((\theta,k)_4,k^2)_8$	$((f,\Delta)_4,\Delta)_2$	-	-	-	-	-	-
10	$(i,\Delta^2)_8$	$((i,k)_4,\tau)_4$ $((i,\Delta)_4,\Delta)_3$	-	-	-	-	-	-	-
11	-	$((f,\Delta)_4,\tau)_4$ $(((\theta,k)_4,k^2)_8,k)_2$	-	-	-	-	-	-	-
12	-	$((i,\Delta)_4,\tau)_4$	-	-	-	-	-	-	-
13	-	-	-	-	-	-	-	-	-

Table 5.6: The generating covariants of V_8

Proposition 5.8.1. *The generating covariants of V_8 are the covariants contained in Table 5.6.*

Proof. The generating covariants of V_8 have order ≤ 18 (this follows from Proposition 2.3.3).

First we show that the covariants from Table 5.6 generate the vector space of covariants of order m and degree d with $m \leq 18$ and $d \leq 13$ (with Mathematica, for details about the computations see Chap. A.8).

If there are generators of degree > 13, then they are of the form $C = (C_1 \dots C_r, f)_k$, where $C_1, C_2, \dots, C_r$ are covariants of lower degrees. If this transvectant is irreducible, from Proposition 2.1.5 it follows that

(1) $r = 4$ and $\mathrm{ord}\, C \leq 2$, or

(2) $r = 3$ and $\mathrm{ord}\, C \leq 4$, or

(3) $r = 2$ and $\mathrm{ord}\, C \leq 6$, or

(4) $r = 1$, or

(5) C is an invariant.

The invariants of V_8 are generated by those of degree ≤ 10 (see Chap. 4.7). This settles case (5).

We consider now the cases (1), (2) and (3). The modules of covariants of V_8 of order m, with $m \in \{2, 4, 6\}$ are Cohen-Macaulay (as modules over the invariant ring of V_8), from Proposition 2.4.3. Then, from their Poincaré series, using Proposition 2.4.4, we determine the highest degree in which we have to look for generators. Here are the Poincaré series of these modules (we already know from Chapter 4.7 that $\mathcal{O}(V_8)^{\mathrm{SL}_2}$ has a homogeneous system of parameters of degrees $2, 3, 4, 5, 6, 7$):

order 2: $\dfrac{t^5 + t^6 + 2t^7 + 2t^8 + 3t^9 + 2t^{10} + 2t^{11} + t^{12} + t^{13}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)}$,

order 4: $\dfrac{p(t)}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)}$, where

$p(t) = t^2 + t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + t^9 + 2t^{10} + 2t^{11} + 2t^{12} + 2t^{13} + 2t^{14} + t^{15} + t^{16}$,

order 6: $\dfrac{q(t)}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)}$, where

$q(t) = t^3 + t^4 + 2t^5 + 3t^6 + 4t^7 + 4t^8 + 5t^9 + 4t^{10} + 4t^{11} + 3t^{12} + 2t^{13} + t^{14} + t^{15}$.

Then $\mathcal{C}(V_8)_2$ is generated in degree ≤ 13, $\mathcal{C}(V_8)_4$ is generated in degree ≤ 16, and $\mathcal{C}(V_8)_6$ is generated in degree ≤ 15. We already know that no generators occur in degree 13, hence there are no generating covariants of V_8 of order 2 and degree ≥ 13. In Chapter A.8 we show that the covariants from Table 5.6 generate the vector space of covariants of V_8 of order 4 and degree d with $d \leq 16$ and the vector space of covariants of V_8 of order 6 and degree d with $d \leq 15$ (with Mathematica, for details about the computations see Chap. A.8).

Finally, there is no irreducible covariant of the form $(C, f)_k$ of degree $\geq$ 13: there are no generators in degree 13 and any other irreducible covariant $(C_1 \ldots C_r, f)_k$, with $r > 1$, should have order ≤ 6 and there are none in degree ≥ 13. This concludes our proof. $\qquad \square$

5.9 The invariants of $V_2 \oplus V_3$

This case is treated in [Bes69, Ell95, Gor87, GY03].
Let $q \in V_2$ and $c \in V_3$,

$$q = a_0 x^2 + 2a_1 xy + a_2 y^2, \qquad c = b_0 x^3 + 3b_1 x^2 y + 3b_2 xy^2 + b_3 y^3,$$

and consider the following invariants:

$$j_2 = (q, q)_2, \qquad\qquad j_4 = ((c, c)_2, (c, c)_2)_2,$$
$$j_3 = ((c, c)_2, q)_2, \qquad\qquad j_5 = (c^2, q^3)_6.$$

Proposition 5.9.1. *The invariants j_2, j_3, j_4, j_5 form a system of parameters of the ring of invariants of $V_2 \oplus V_3$.*

Proof. First we show that $\mathcal{N}(V_2 \oplus V_3) = \mathcal{V}(j_2, j_3, j_4, j_5)$.
Indeed, if $j_2 = j_4 = 0$, then q and c are nullforms. Suppose $q, c \neq 0$ and $(q, c) \notin \mathcal{N}(V_2 \oplus V_3)$. Without loss of generality we can then suppose $q = x^2$ and $y^2 \mid c$. This implies $j_3 = -2b_2^2$ and $j_5 = b_3^2$. But then $j_3 = j_5 = 0$ will imply $c = 0$, which contradicts our assumption $c \neq 0$.
We proved that $\mathcal{N}(V_2 \oplus V_3) = \mathcal{V}(j_2, j_3, j_4, j_5)$. From Proposition 2.7.1 it follows that $\{j_2, j_3, j_4, j_5\}$ forms a system of parameters of $\mathcal{O}(V_2 \oplus V_3)^{\mathrm{SL}_2}$. $\qquad \square$

Proposition 5.9.2. *The algebra of invariants of $V_2 \oplus V_3$ is generated by 5 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	7
d_i	1	1	1	1	1

Proof. The Poincaré series of $\mathcal{O}(V_2 \oplus V_3)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^7}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)} =$$
$$= 1 + t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 4t^7 + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 7. We find the following 5 invariants:

deg	generators
2	$(q, q)_2$
3	$((c, c)_2, q)_2$
4	$((c, c)_2, (c, c)_2)_2$
5	$(c^2, q^3)_6$
7	$(q^3, c(c, (c, c)_2)_1)_6$

For details about the computations see Chap. A.9. □

5.10   The invariants of $V_2 \oplus V_4$

This case is treated in [GY03]

Let $q \in V_2$ and $f \in V_4$,

$$q = a_0 x^2 + 2a_1 xy + a_2 y^2,$$
$$f = b_0 x^4 + 4b_1 x^3 y + \ldots + 4b_3 xy^3 + b_4 y^4.$$

Consider the following invariants:

$$j_{2,1} = (q,q)_2, \qquad\qquad j_{2,2} = (f,f)_4,$$
$$j_{3,1} = ((f,f)_2, f)_4, \qquad j_{3,2} = (f, q^2)_4,$$
$$j_4 \;= ((f,f)_2, q^2)_4.$$

Proposition 5.10.1. *The invariants $j_{2,1}, j_{2,2}, j_{3,1}, j_{3,2}, j_4$ form a system of parameters of the ring of invariants of $V_2 \oplus V_4$.*

Proof. We first show that $\mathcal{N}(V_2 \oplus V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_{3,1}, j_{3,2}, j_4)$.

Indeed, if $j_{2,1} = j_{2,2} = j_{3,1} = 0$, then q and f are nullforms. Suppose $q, f \neq 0$ and $(q,f) \notin \mathcal{N}(V_2 \oplus V_4)$. Without loss of generality we can then suppose $q = x^2$ and $y^3 \mid c$. This implies $j_{3,2} = b_4$ and $j_4 = -2b_3^2$. But then $j_{3,2} = j_4 = 0$ will imply $f = 0$, which contradicts our assumption $f \neq 0$.

We proved that $\mathcal{N}(V_2 \oplus V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_{3,1}, j_{3,2}, j_4)$. From Proposition 2.7.1 it follows that $\{j_{2,1}, j_{2,2}, j_{3,1}, j_{3,2}, j_4\}$ is a system of parameters of $\mathcal{O}(V_2 \oplus V_4)^{\mathrm{SL}_2}$. □

Proposition 5.10.2. *The algebra of invariants of $V_2 \oplus V_4$ is generated by 6 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	6
d_i	2	2	1	1

Proof. The Poincaré series of $\mathcal{O}(V_2 \oplus V_4)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^6}{(1 - t^2)^2 (1 - t^3)^2 (1 - t^4)} = 1 + 2t^2 + 2t^3 + 4t^4 + 4t^5 + 10t^6 + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 6. We find the following 6 invariants:

deg	generators	
2	$(q,q)_2$	$(f,f)_4$
3	$((f,f)_2, f)_4$	$(f, q^2)_4$
4	$((f,f)_2, q^2)_4$	
6	$(q^3, (f, (f,f)_2)_1)_6$	

For details about the computations see Chap. A.10. □

5.11 The invariants of $V_2 \oplus V_5$

This case is treated in [Win80].
Let $q \in V_2$ and $f \in V_5$,

$$q = a_0 x^2 + 2a_1 xy + a_2 y^2,$$
$$f = b_0 x^5 + 5b_1 x^4 y + \ldots + 5b_4 xy^4 + b_5 y^5.$$

Consider the covariants $c_1 = (f, f)_4$, $c_2 = (f, c_1)_2$, $c_3 = (f, f)_2$, $c_4 = (f, c_1)_1$, $c_5 = (f, c_3)_1$, and the invariants:

$$
\begin{aligned}
j_2 \quad &= (q, q)_2, & j_3 \quad &= (c_1, q)_2, \\
j_4 \quad &= (c_1, c_1)_2, & j_5 \quad &= (c_3, q^3)_6, \\
j_{7,1} \quad &= (f^2, q^5)_{10}, & j_{8,1} &= ((c_2, c_2)_2, c_1)_2, \\
j_{12,1} &= ((c_2, c_2)_2, (c_2, c_2)_2)_2.
\end{aligned}
$$

Proposition 5.11.1. *The set $\{j_3, j_4 + j_2^2, j_5, j_{7,1}, j_{8,1} - j_2^4, j_{12,1}\}$ forms a system of parameters of the ring of invariants of $V_2 \oplus V_5$.*

Proof. We first show that $\mathcal{N}(V_2 \oplus V_5) = \mathcal{V}(j_3, j_2^2 + j_4, j_5, j_{7,1}, j_{8,1} - j_2^4, j_{12,1})$. We consider the following cases:

Case 1: $q = 0$.
In this case we have $j_4 = j_{8,1} = j_{12,1} = 0$, and this implies that f is a nullform (see Proposition 4.4.1), i.e. $(q, f) \in \mathcal{N}(V_2 \oplus V_5)$.

Case 2: $q \neq 0$ and $q \in \mathcal{N}(V_2)$.
In this case we have $j_2 = 0$, and then we get $j_4 = j_{8,1} = j_{12,1} = 0$, hence f is a nullform as well (see Proposition 4.4.1). Suppose, without loss of generality, that $x^2 \mid q$ and $y^3 \mid f$, i.e. $a_2 = a_1 = b_0 = b_1 = b_2 = 0$. Then we have:

$$j_3 \sim a_0 b_3^2, \qquad j_5 \sim a_0^3(b_4^2 - b_3 b_5), \qquad j_{7,1} \sim a_0^3 b_5^2,$$

where $\sim$ denotes equalities up to a nonzero constant. Because $q \neq 0$, we obtain $f = 0$. Then we have $(q, f) \in \mathcal{N}(V_2 \oplus V_5)$.

Case 3: $q \neq 0$ and $q \notin \mathcal{N}(V_2)$.
In this case we can suppose without loss of generality that $xy \mid q$. We obtain:

$$
\begin{aligned}
j_3 \quad &\sim a_1(2b_2 b_3 - 3b_1 b_4 + b_0 b_5), \\
j_5 \quad &\sim a_1^3(8b_2 b_3 - 7b_1 b_4 - b_0 b_5), \\
j_{7,1} &\sim a_1^5(100 b_2 b_3 + 25 b_1 b_4 + b_0 b_5),
\end{aligned}
$$

Because we supposed $q \neq 0$, we get $b_2 b_3 = b_1 b_4 = b_0 b_5 = 0$. If $b_5 = b_4 = b_3 = 0$, then f is a nullform, and then $j_4 = j_{8,1} = j_{12,1} = 0$, which implies $j_2 = 0$. This contradicts our assumption $q \notin \mathcal{N}(V_2)$. If $b_5 = b_4 = b_2 = 0$, then we have

$$
\begin{aligned}
j_4 + j_2^2 \quad &\sim a_1^4 - 24 b_1 b_3^3, \\
j_{8,1} - j_2^4 &\sim a_1^8 - 9 b_1^2 b_3^6,
\end{aligned}
$$

which implies $a_1 = 0$. This contradicts the assumption $q \neq 0$. If $b_5 = b_1 = b_2 = 0$, then we have

$$
\begin{aligned}
j_4 + j_2^2 \ &\sim a_1^4 + 6b_0 b_3^2 b_4, \\
j_{8,1} - j_2^4 &\sim a_1^8 - 9b_0^2 b_3^4 b_4^2 + b_0^3 b_4^5, \\
j_{12,1} \qquad &\sim b_0^2 b_3^2 (27b_3^8 - 14b_0 b_3^4 b_4^4 + 3b_0^2 b_4^6),
\end{aligned}
$$

and $b_0^2 b_3^{14}$ lies in the ideal generated by these three polynomials. Then, we get $a_1 = 0$, hence $q = 0$, which contradicts the assumption $q \neq 0$. If $b_5 = b_1 = b_3 = 0$, then we have

$$
\begin{aligned}
j_4 + j_2^2 \ &\sim a_1^4 - 24b_2^3 b_4 - 8b_0 b_2 b_4^2, \\
j_{8,1} - j_2^4 &\sim a_1^8 - 9b_2^6 b_4^2 + 19b_0 b_2^4 b_4^3 - 11b_0^2 b_2^2 b_4^4 + b_0^3 b_4^5, \\
j_{12,1} \qquad &\sim b_2 b_4^3 (b_2^2 - b_0 b_4)^4,
\end{aligned}
$$

and $b_2^{13} b_4^3$ lies in the ideal generated by these three polynomials. Then, we get $a_1 = 0$, hence $q = 0$, which contradicts the assumption $q \neq 0$.

Therefore, we have $\mathcal{N}(V_2 \oplus V_5) = \mathcal{V}(j_3, j_2^2 + j_4, j_5, j_{7,1}, j_{8,1} - j_2^4, j_{12,1})$. From Proposition 2.7.1 it follows that $\{j_3, j_2^2 + j_4, j_5, j_{7,1}, j_{8,1} - j_2^4, j_{12,1}\}$ forms a system of parameters of $\mathcal{O}(V_2 \oplus V_5)^{\mathrm{SL}_2}$.   □

Proposition 5.11.2. *The algebra of invariants of $V_2 \oplus V_5$ is generated by 29 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
d_i	1	1	1	1	1	3	3	3	2	3	3	2	1	1	1	1	1

Proof. The Poincaré series of $\mathcal{O}(V_2 \oplus V_5)^{\mathrm{SL}_2}$ is

$$
P(t) = \frac{a(t)}{(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^7)(1 - t^8)(1 - t^{12})}
$$

with

$$
\begin{aligned}
a(t) = \ &1 + t^2 + t^4 + 2t^6 + 2t^7 + 4t^8 + 5t^9 + 6t^{10} + 8t^{11} + 8t^{12} + 10t^{13} + 8t^{14} + \\
&+ 10t^{15} + 8t^{16} + 10t^{17} + 8t^{18} + 8t^{19} + 6t^{20} + 5t^{21} + 4t^{22} + 2t^{23} + 2t^{24} + \\
&+ t^{26} + t^{28} + t^{30}
\end{aligned}
$$

so that

$$
\begin{aligned}
P(t) = \ &1 + t^2 + t^3 + 2t^4 + 2t^5 + 4t^6 + 6t^7 + 9t^8 + 12t^9 + 17t^{10} + 24t^{11} + 33t^{12} + \\
&+ 42t^{13} + 56t^{14} + 73t^{15} + 94t^{16} + 117t^{17} + 148t^{18} + 183t^{19} + 226t^{20} + \\
&+ 274t^{21} + 332t^{22} + 400t^{23} + 479t^{24} + 566t^{25} + 669t^{26} + 787t^{27} + 920t^{28} + \\
&+ 1067t^{29} + 1238t^{30} + \ldots
\end{aligned}
$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 30. Up to degree 18 we find the following 29 invariants:

deg	generators	
2	$(q,q)_2$	
3	$(c_1,q)_2$	
4	$(c_1,c_1)_2$	
5	$(c_3,q^3)_6$	
6	$((c_1,c_3)_2,q^2)_4$	
7	$(f^2,q^5)_{10}$	$((f,c_4)_2,q^3)_6$
	$((c_1^2,c_3)_4,q)_2$	
8	$((c_2,c_2)_2,c_1)_2$	$((c_1^2,c_3)_3,q^2)_4$
	$(fc_2,q^4)_8$	
9	$(f\cdot(c_1^2,f)_4,q^3)_6$	$((c_1^3,c_3)_5,q)_2$
	$(q^5,fc_4)_{10}$	
10	$(c_2\cdot(c_1^2,f)_4,q^2)_4$	$(c_2c_4,q^4)_8$
11	$(fc_5,q^7)_{14}$	$(f\cdot(c_1^3,f)_5,q^3)_6$
	$([(c_1^2,f)_4]^2,q)_2$	
12	$((c_2,c_2)_2,(c_2,c_2)_2)_2$	$(c_2\cdot(c_1^3,f)_5,q^2)_4$
	$(f\cdot(c_1,c_5)_2,q^6)_{12})_6$	
13	$((c_1^3,f)_5\cdot(c_1^2,f)_4,q)_2$	$(f\cdot(c_1^2,c_5)_4,q^5)_{10}$
14	$(f\cdot(c_1^3,c_5)_6,q^4)_8$	
15	$(f\cdot(c_1^4,c_5)_8,q^3)_6$	
16	$(c_2\cdot(c_1^4,c_5)_8,q^2)_4$	
17	$((c_1^2,f)_4\cdot(c_1^4,c_5)_8,q)_2$	
18	$(c_1^7,fc_5)_{14}$	

Then we prove that no generators are needed in degrees $19,\dots,24,26,28,30$ by showing that for each $i\in\{19,\dots,24,26,28,30\}$ the vector space $\mathcal{O}(V_2\oplus V_5)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 29 invariants found in degrees ≤ 18. For details about the computations see Chap. A.11. $\qquad\square$

5.12 The invariants of $V_2\oplus V_6$

This case is treated in [Gal74].
Let $q\in V_2$ and $s\in V_6$,

$$q=a_0x^2+2a_1xy+a_2y^2,$$
$$s=b_0x^6+6b_1x^5y+\dots+6b_5xy^5+b_6y^6.$$

Consider the covariants $c_1=(s,s)_4$, $c_2=(s,s)_2$, $c_3=(s,c_1)_4$, $c_4=(c_1,c_1)_2$, $c_5=(s,c_1)_1$, and the invariants:

$$j_{2,1}=(s,s)_6, \qquad j_{2,2}=(q,q)_2, \qquad j_{4,1}=(c_1,c_1)_4,$$
$$j_{4,2}=(c_1,q^2)_4, \qquad j_{4,3}=(s,q^3)_6, \qquad j_{4,4}=(c_3,q)_2,$$
$$j_{6,1}=(c_1,c_4)_4, \qquad j_{6,2}=(c_2,q^4)_8, \qquad j_{10}=(c_4,c_3^2)_4.$$

Proposition 5.12.1. *The set* $\{j_{2,1}, j_{2,2}, j_{4,1} + j_{4,2} - 2j_{4,4}, j_{4,2} - j_{4,3} + j_{4,4}, j_{6,1},$ $j_{6,2}, j_{10}\}$ *forms a system of parameters of the ring of invariants of* $V_2 \oplus V_6$.

Proof. We first show that

$$\mathcal{N}(V_2 \oplus V_6) = \mathcal{V}(j_{2,1}, j_{2,2}, j_{4,1} + j_{4,2} - 2j_{4,4}, j_{4,2} - j_{4,3} + j_{4,4}, j_{6,1}, j_{6,2}, j_{10}).$$

If $j_{2,2} = 0$, then q is a nullform. We distinguish the following two cases.

Case 1: $q = 0$. In this case we obtain $j_{2,1} = j_{4,1} = j_{6,1} = j_{10} = 0$, which implies, from Chap. 4.5, that s is a nullform, and then $(q, s) \in \mathcal{N}(V_2 \oplus V_6)$.

Case 2: $q \neq 0$. In this case we can suppose without loss of generality that $q = x^2$.
If $b_6 = 0$, then $j_{6,2} \sim b_5^2$, where $\sim$ denotes equalities up to a nonzero constant. It follows that $b_5 = 0$. Then, $j_{4,2} - j_{4,3} + j_{4,4} + \frac{2}{5}b_4 j_{2,1} \sim b_4^2$, and it follows that $b_4 = 0$. Then, $j_{2,1} \sim b_3^2$, and it follows that $b_3 = 0$. We obtain that $(q, s) \in \mathcal{N}(V_2 \oplus V_6)$.
If $b_6 \neq 0$, then without loss of generality we can suppose $b_6 = 1$, and then $j_{6,2} \sim b_4 - b_5^5$, hence $b_4 = b_5^5$. We replace this in $j_{2,1}, j_{4,1} + j_{4,2} - 2j_{4,4}, j_{4,2} - j_{4,3} + j_{4,4}, j_{6,1}, j_{10}$, and, using `Singular` [DGPS], we show that the Gröbner basis of the ideal generated by these polynomials is 1, which leads to a contradiction. We proved that $\mathcal{N}(V_2 \oplus V_6)$ is the vanishing locus of the ideal generated by the invariants $j_{2,1}, j_{2,2}, j_{4,1} + j_{4,2} - 2j_{4,4}, j_{4,2} - j_{4,3} + j_{4,4}, j_{6,1}, j_{6,2}, j_{10}$. From Proposition 2.7.1 it follows that these invariants form a system of parameters of $\mathcal{O}(V_2 \oplus V_6)^{\mathrm{SL}_2}$. $\qquad\square$

Proposition 5.12.2. *The algebra of invariants of* $V_2 \oplus V_6$ *is generated by 27 invariants. The nonzero numbers* d_i *of basic invariants of degree* i *are*

i	2	4	6	7	8	9	10	11	13	15
d_i	2	4	5	3	1	7	1	2	1	1

Proof. The Poincaré series of $\mathcal{O}(V_2 \oplus V_6)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{a(t)}{(1 - t^2)^2 (1 - t^4)^2 (1 - t^6)^2 (1 - t^{10})}$$

with

$$a(t) = 1 + 2t^4 + 3t^6 + 3t^7 + 4t^8 + 7t^9 + 4t^{10} + 7t^{11} + 6t^{12} + 7t^{13} + 4t^{14} +$$
$$+ 7t^{15} + 4t^{16} + 3t^{17} + 3t^{18} + 2t^{20} + t^{24}$$

so that

$$P(t) = 1 + 2t^2 + 7t^4 + 17t^6 + 3t^7 + 38t^8 + 13t^9 + 78t^{10} + 36t^{11} + 151t^{12} +$$
$$+ 86t^{13} + 271t^{14} + 180t^{15} + 469t^{16} + 341t^{17} + 774t^{18} + 607t^{19} + 1233t^{20} +$$
$$+ 2857t^{24} + \dots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 24. Up to degree 15 we find the following 27 invariants:

deg	generators	
2	$(s,s)_6$	$(q,q)_2$
4	$(c_1,c_1)_4$	$(c_1,q^2)_4$
	$(s,q^3)_6$	$(c_3,q)_2$
6	$(c_1,c_4)_4$	$(c_2,q^4)_8$
	$((c_1,c_3)_2,q)_2$	$((s,c_1)_2,q^3)_6$
	$((s,c_3)_2,q^2)_4$	
7	$((c_1,c_3)_1,q^2)_4$	$((s,c_3)_1,q^3)_6$
	$((s,c_1)_1,q^4)_8$	
8	$((s,c_3^2)_4,q)_2$	
9	$((c_1,c_3^2)_3,q)_2$	$((s,c_3^2)_3,q^2)_4$
	$(((s,c_1)_2,c_3)_1,q^3)_6$	$(((s,c_1)_1,c_3)_2,q^3)_6$
	$((c_2,c_3)_1,q^4)_8$	$((c_2,c_1)_1,q^5)_{10}$
	$((s,c_2)_1,q^6)_{12}$	
10	$(c_4,c_3^2)_4$	
11	$((s,c_3^3)_5,q)_2$	$(((s,c_1)_1,c_3^2)_4,q^2)_4$
13	$(((s,c_1)_1,c_3^3)_6,q)_2$	
15	$(c_5,c_3^4)_8$	

Then we prove that no generators are needed in degrees $16,17,18,20,24$ by showing that for each $i \in \{16,17,18,20,24\}$ the vector space $\mathcal{O}(V_2 \oplus V_6)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 27 invariants found in degrees ≤ 15. For details about the computations see Chap. A.12. $\qquad \square$

5.13   The invariants of $V_3 \oplus V_4$

This case is treated in [Gun69].
Let $c \in V_3$, and $q \in V_4$,

$$c = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3,$$
$$q = b_0 x^4 + 4b_1 x^3 y + 6b_2 x^2 y^2 + 4b_3 xy^3 + b_4 y^4.$$

Consider the covariants $c_1 = (c,c)_2$, $c_2 = (q,q)_2$, and the following invariants:

$$
\begin{aligned}
j_2 &= (q,q)_4, & j_3 &= (c_2,q)_4, \\
j_4 &= (c_1,c_1)_2, & j_{5,1} &= ((q,c_2)_1,c^2)_6, \\
j_{5,2} &= ((q,c^2)_2,c^2)_6, & j_{6,1} &= (c_1^2,c_2)_4, \\
j_{7,1} &= (c^4,q^3)_{12}.
\end{aligned}
$$

Proposition 5.13.1. *The invariants $j_2, j_3, j_4, j_{5,1}+j_{5,2}, j_{6,1}, j_{7,1}$ form a system of parameters of the algebra of invariants of $V_3 \oplus V_4$.*

Proof. We first show that $\mathcal{N}(V_3 \oplus V_4) = \mathcal{V}(j_2, j_3, j_4, j_{5,1} + j_{5,2}, j_{6,1}, j_{7,1})$.
Indeed, if $j_2 = j_3 = j_4 = 0$, then c and q are nullforms. If c or q vanish identically, then $(c, q) \in \mathcal{N}(V_3 \oplus V_4)$.
If $c, q \neq 0$, suppose without loss of generality, that $x^2 \mid c$ and $y^3 \mid q$, i.e. $a_3 = a_2 = b_0 = b_1 = b_2 = 0$. Then, $j_{61} \sim a_1 b_3$, where $\sim$ denotes equalities up to non-zero constants. If $a_1 = 0$, then $j_{51} + j_{52} \sim a_0 b_3$, and $j_7 \sim a_0 b_4$, which implies either $c = 0$, or $q = 0$, and contradicts the assumption $c, q \neq 0$. If $b_3 = 0$, then $j_{51} + j_{52} \sim a_1 b_4$, and $j_7 \sim a_0 b_4$, which implies either $c = 0$, or $q = 0$, and contradicts again the assumption $c, q \neq 0$.
We proved that $\mathcal{N}(V_3 \oplus V_4) = \mathcal{V}(j_2, j_3, j_4, j_{5,1} + j_{5,2}, j_{6,1}, j_{7,1})$. From Proposition 2.7.1 it follows that $\{j_2, j_3, j_4, j_{5,1} + j_{5,2}, j_{6,1}, j_{7,1}\}$ forms a system of parameters of $\mathcal{O}(V_3 \oplus V_4)^{\mathrm{SL}_2}$. $\qquad \square$

Proposition 5.13.2. *The algebra of invariants of $V_3 \oplus V_4$ is generated by 20 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7	8	9	10	11
d_i	1	1	1	2	2	3	3	4	2	1

Proof. The Poincaré series of $\mathcal{O}(V_3 \oplus V_4)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^5 + t^6 + 2t^7 + 3t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12} + t^{13} + t^{18}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^7)} =$$
$$= 1 + t^2 + t^3 + 2t^4 + 3t^5 + 5t^6 + 7t^7 + 11t^8 + 16t^9 + 22t^{10} + 30t^{11} +$$
$$+ 42t^{12} + 55t^{13} + \ldots + 193t^{18} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 18. Up to degree 11 we find the following 20 invariants:

deg	generators	
2	$(q, q)_4$	
3	$(c_2, q)_4$	
4	$(c_1, c_1)_2$	
5	$((q, c_2)_1, c^2)_6$	$((q, c^2)_2, c^2)_6$
6	$(c_1^2, c_2)_4$	$(((q, c_1)_2, c^2)_2, q)_4$
7	$(c^4, q^3)_{12}$	$(((q, c_1)_2, c^2)_2, c_2)_4$
	$((q, c_2)_1, c(c, c_1)_1)_6$	
8	$(((q, c_1)_2, c(c, c_1)_1)_2, q)_4$	$(c^4, c_2 q^2)_{12}$
	$(((c_1, c_2)_2, c^2)_2, c_2)_4$	
9	$(((c_1, c_2)_2, c(c, c_1)_1)_2, q)_4$	$(c^4, q c_2^2)_{12}$
	$(c^3(c, c_1)_1, q^3)_{12}$	$(((q, c_1)_2, c(c, c_1)_1)_2, c_2)_4$
10	$(((c_1, c_2)_2, c(c, c_1)_1)_2, c_2)_4$	$((c^3(c, c_1)_1, q^3)_{10}, q)_4$
11	$(((c^3(c, c_1)_1, q^3)_{10}, q)_2, q)_4$	

Then we prove that no generators are needed in degrees $12, 13, 18$ by showing that for each $i \in \{12, 13, 18\}$ the vector space $\mathcal{O}(V_3 \oplus V_4)^{\mathrm{SL}_2}_i$ is spanned by

monomials of degree i generated by the 20 invariants found in degrees ≤ 11. For details about the computations see Chap. A.13. $\qquad\square$

5.14 The invariants of $V_1 \oplus V_2 \oplus V_3$

This case is classically treated in [Bes69, Ell95, Gor87, GY03].
Let $\ell \in V_1$, $q \in V_2$, and $c \in V_3$,

$$\ell = a_0 x + a_1 y,$$
$$q = b_0 x^2 + 2b_1 xy + b_2 y^2,$$
$$c = c_0 x^3 + 3c_1 x^2 y + 3c_2 xy^2 + c_3 y^3.$$

Consider the covariants $c_1 = (c, c)_2$, $c_2 = (c, c_1)_1$, and the following invariants:

$$
\begin{aligned}
j_2 &= (q, q)_2, & j_{3,1} &= (q, \ell^2)_2, \\
j_{3,2} &= (q, c_1)_2, & j_{4,1} &= (c_1, c_1)_2, \\
j_{4,2} &= (c, \ell^3)_3, & j_{4,3} &= (c_1, \ell^2)_2, \\
j_{5,1} &= (c^2, q^3)_6, & j_{5,2} &= (((c, \ell)_1, q)_1, \ell^2)_2,
\end{aligned}
$$

Proposition 5.14.1. *The set $\{j_{3,1}, j_{3,2}, j_{4,1}, j_{4,2}+j_2^2, j_{4,3}-j_2^2, j_{5,1}+j_{5,2}\}$ forms a system of parameters of the algebra of invariants of $V_1 \oplus V_2 \oplus V_3$.*

Proof. First we show that

$$\mathcal{N}(V_1 \oplus V_2 \oplus V_3) = \mathcal{V}(j_{3,1}, j_{3,2}, j_{4,1}, j_{4,2} + j_2^2, j_{4,3} - j_2^2, j_{5,1} + j_{5,2}).$$

We distinguish the following four cases:

Case 1: $\ell = 0$.
In this case we obtain $j_2 = j_{4,1} = j_{3,2} = j_{5,1} = 0$, which implies $(q, c) \in \mathcal{N}(V_2 \oplus V_3)$ (see Proposition 5.9.1).

Case 2: $q = 0$.
In this case we obtain $j_{4,1} = j_{4,2} = j_{4,3} = 0$, which implies $(\ell, c) \in \mathcal{N}(V_1 \oplus V_3)$ (see Proposition 5.3.4).

Case 3: $c = 0$.
In this case we obtain $j_2 = j_{3,1} = 0$, which implies $(\ell, q) \in \mathcal{N}(V_1 \oplus V_2)$.

Case 4: $\ell, c, q \neq 0$.
In this case we have $j_{4,1} = 0$, which implies that c is a nullform. Without loss of generality we suppose that $x^2 \mid c$, i.e. $c_2 = c_3 = 0$. Then, $j_{3,2} \sim b_2 c_1^2$, where $\sim$ denotes equalities up to a nonzero constant.
If $c_1 = 0$, then we have $j_{4,3} = 0$, hence $j_2 = j_{4,2} = 0$. But $j_{4,2} \sim a_1^3 c_0$, which implies $a_1 = 0$, as $c \neq 0$. Then, $j_{5,1} + j_{5,2} \sim b_2^3 c_0^2$, which implies $b_2 = 0$, as $c \neq 0$. Then, $j_2 \sim b_1^2$, which implies $b_1 = 0$. But then $\ell \sim x$, $x^2 \mid c$, and $q \sim x^2$, therefore $(\ell, c, q) \in \mathcal{N}(V_1 \oplus V_2 \oplus V_3)$.

If $b_2 = 0$, then $j_{3,1} = a_1(a_1b_0 - 2a_0b_1)$. If $a_1 = 0$, then $j_{4,2} + j_2^2 \sim b_1^4$, which implies $b_1 = 0$. Hence, we obtain $(\ell, c, q) \in \mathcal{N}(V_1 \oplus V_2 \oplus V_3)$. If $a_1 \neq 0$, then $a_1b_0 - 2a_0b_1 = 0$, and then we obtain $j_{5,1} + j_{5,2} = a_1^2 b_1(a_1c_0 - 3a_0c_1)$. If $b_1 = 0$, then it follows $b_0 = 0$ as well, and we get a contradiction with the assumption $q \neq 0$. If $a_1c_0 - 3a_0c_1 = 0$, then $j_{4,2} + j_2^2 \sim b_1^2$, hence $b_1 = 0$, and we get again a contradiction.

We proved that the invariants $j_{3,1}, j_{3,2}, j_{4,1}, j_{4,2} + j_2^2, j_{4,3} - j_2^2, j_{5,1} + j_{5,2}$ define the nullcone of $V_1 \oplus V_2 \oplus V_3$. From Proposition 2.7.1 it follows that $\{j_{3,1}, j_{3,2}, j_{4,1}, j_{4,2} + j_2^2, j_{4,3} - j_2^2, j_{5,1} + j_{5,2}\}$ forms a system of parameters of the algebra of invariants of $V_1 \oplus V_2 \oplus V_3$. □

Proposition 5.14.2. *The algebra of invariants of $V_1 \oplus V_2 \oplus V_3$ is generated by 15 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7
d_i	1	3	4	4	2	1

Proof. The Poincaré series of $\mathcal{O}(V_1 \oplus V_2 \oplus V_3)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^2 + t^3 + 2t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 2t^{10} + t^{11} + t^{12} + t^{14}}{(1 - t^3)^2(1 - t^4)^3(1 - t^5)} =$$
$$= 1 + t^2 + 3t^3 + 5t^4 + 7t^5 + 13t^6 + 20t^7 + 31t^8 + 44t^9 + 63t^{10} +$$
$$+ 88t^{11} + 123t^{12} + \ldots + 213t^{14} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 14. Up to degree 7 we find the following 15 invariants:

deg	generators	
2	$(q, q)_2$	
3	$(q, \ell^2)_2$	$(q, c_1)_2$
	$(c, q\ell)_3$	
4	$(c_1, c_1)_2$	$(c, \ell^3)_3$
	$(c_1, \ell^2)_2,$	$(c\ell, q^2)_4$
5	$(c^2, q^3)_6$	$(((c, \ell)_1, q)_1, \ell^2)_2$
	$(q, (c_2, \ell)_1)_2$	$((c_1, q)_1, \ell^2)_2$
6	$(\ell c, (\ell c, \ell c)_2)_4$	$((c_1, q)_1, (q\ell, c)_2)_2$
7	$(q^3, c_2c)_6$	

Then we prove that no generators are needed in degrees $8, \ldots, 12, 14$ by showing that for each $i \in \{8, \ldots, 12, 14\}$ the vector space $\mathcal{O}(V_1 \oplus V_2 \oplus V_3)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 15 invariants found in degrees ≤ 7. For details about the computations see Chap. A.14. □

5.15 The invariants of $V_1 \oplus V_2 \oplus V_4$

Let $\ell \in V_1$, $q \in V_2$, and $f \in V_4$,

$$l = a_0 x + a_1 y,$$
$$q = b_0 x^2 + 2b_1 xy + b_2 y^2,$$
$$f = c_0 x^4 + 4c_1 x^3 y + \ldots + 4c_3 xy^3 + c_4 y^4.$$

Consider the covariants $c_1 = (f, f)_2$, $c_2 = (f, c_1)_1$, $c_3 = (q, f)_2$, and the following invariants:

$$j_{2,1} = (q, q)_2, \qquad j_{2,2} = (f, f)_4,$$
$$j_{3,1} = (f, c_1)_4, \qquad j_{3,2} = (f, q^2)_4,$$
$$j_{3,3} = (q, \ell^2)_2, \qquad j_{4,1} = (c_1, q^2)_4,$$
$$j_{5,1} = (f, \ell^4)_4 \qquad j_{6,1} = (c_1, \ell^4)_4.$$

Proposition 5.15.1. *The set $\{j_{2,1}, j_{2,2}, j_{3,1}+j_{3,2}, j_{3,3}-j_{3,1}, j_{4,1}, j_{5,1}, j_{6,1}\}$ forms a system of parameters of the algebra of invariants of $V_1 \oplus V_2 \oplus V_4$.*

Proof. We first show that

$$\mathcal{N}(V_1 \oplus V_2 \oplus V_4) = \mathcal{V}(j_{2,1}, j_{2,2}, j_{3,1} + j_{3,2}, j_{3,3} - j_{3,1}, j_{4,1}, j_{5,1}, j_{6,1})$$

Consider the following three cases.

Case 1: $\ell = 0$.
In this case we obtain $j_{2,1} = j_{2,2} = j_{3,1} = j_{3,2} = j_{4,1} = 0$, which implies $(q, f) \in \mathcal{N}(V_2 \oplus V_4)$ (see Proposition 5.10.1).

Case 2: $\ell \neq 0$.
Without loss of generality we suppose $l \sim x$, i.e. $a_1 = 0$. This implies:

$$j_{5,1} \sim a_0^4 c_4, \quad j_{6,1} \sim a_0^4 (c_3^2 - c_2 c_4),$$

hence $c_4 = c_3 = 0$. But then, $j_{2,2} \sim c_2^2$, which implies $c_2 = 0$. Therefore $x^3 \mid f$. In this case $0 = j_{3,3} \sim a_0^2 b_2$, which implies $b_2 = 0$. Then $0 = j_{2,1} \sim b_1^2$, which implies $b_1 = 0$ as well. Therefore $x^2 \mid q$. Then $(\ell, q, f) \in \mathcal{N}(V_1 \oplus V_2 \oplus V_4)$.
We proved that the invariants $j_{2,1}, j_{2,2}, j_{3,1} + j_{3,2}, j_{3,3} - j_{3,1}, j_{4,1}, j_{5,1}, j_{6,1}$ define the nullcone of $V_1 \oplus V_2 \oplus V_4$. From Proposition 2.7.1 it follows that $j_{2,1}, j_{2,2}, j_{3,1} + j_{3,2}, j_{3,3} - j_{3,1}, j_{4,1}, j_{5,1}, j_{6,1}$ form a system of parameters of the algebra of invariants of $V_1 \oplus V_2 \oplus V_4$. $\square$

Proposition 5.15.2. *The algebra of invariants of $V_1 \oplus V_2 \oplus V_4$ is generated by 18 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7	8	9
d_i	2	3	2	3	4	2	1	1

Proof. The Poincaré series of $\mathcal{O}(V_1 \oplus V_2 \oplus V_4)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{1 + t^3 + t^4 + 2t^5 + 4t^6 + 3t^7 + 3t^8 + 4t^9 + 2t^{10} + t^{11} + t^{12} + t^{15}}{(1-t^2)^2(1-t^3)^2(1-t^4)(1-t^5)(1-t^6)} =$$
$$= 1 + 2t^2 + 3t^3 + 5t^4 + 9t^5 + 18t^6 + 23t^7 + 43t^8 + 63t^9 + 93t^{10}+$$
$$+ 136t^{11} + 200t^{12} + \ldots + 510t^{15} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 15. Up to degree 9 we find the following 18 invariants:

deg	generators	
2	$(q,q)_2$	$(f,f)_4$
3	$(f,c_1)_4$	$(f,q^2)_4$
	$(q,\ell^2)_2$	
4	$(c_1,q^2)_4$	$(c_3,\ell^2)_2$
5	$(f,\ell^4)_4$	$(c_1,q\ell^2)_4$
	$((q,\ell^2)_1,c_3)_2$	
6	$(c_1,\ell^4)_4$	$(c_2,q^3)_6$
	$((q,\ell^2)_1,(q,c_1)_2)_2$	$((q,f)_1,\ell^4)_4$
7	$((q,c_1)_1,\ell^4)_4$	$((c_2,q^2)_4,\ell^2)_2$
8	$(((q,c_1)_2,\ell)_1,\ell^4)_4$	
9	$(c_2,\ell^6)_6$	

Then we prove that no generators are needed in degrees $10, 11, 12, 15$ by showing that for each $i \in \{10, 11, 12, 15\}$ the vector space $\mathcal{O}(V_1 \oplus V_2 \oplus V_4)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 18 invariants found in degrees ≤ 9. For details about the computations see Chap. A.15. □

5.16 The invariants of $V_1 \oplus V_2 \oplus V_5$

In a paper from 1880 Winter [Win80] found 94 generating covariants of $V_2 \oplus V_5$. We prove that the actual number of generators is 92.

The algebra of invariants of $V_2 \oplus V_5$ is generated by 29 invariants, see Chap. 5.11. Let $q \in V_2$, $f \in V_5$, and $c_1 = (f,f)_4$, $c_2 = (f,c_1)_2$, $c_3 = (f,f)_2$, $c_4 = (f,c_1)_1$, $c_6 = (f,c_3)_1$, $c_7 = (c_1,c_3)_2$, $c_8 = (c_1,c_3)_1$, $c_9 = (c_6,c_1)_2$, $c_{10} = (c_6,c_1^2)_4$, $c_{11} = (c_1^2,c_3)_3$. Using Proposition 5.0.7 we obtain a not minimal set of generating covariants of $V_2 \oplus V_5$, shown in Table 5.7 (we excluded the invariants).

deg	order							
	1	2	3	4	5	6	7	9
1	-	q	-	-	f	-	-	-
2	-	c_1	$(f,q)_2$	-	$(f,q)_1$	c_3	-	-
3	$(f,q^2)_4$	$(c_1,q)_1$	c_2, $(f,q^2)_3$	$(c_3,q)_2$	c_4	$(c_3,q)_1$	-	c_6
4	$(c_2,q)_2$	$(c_3,q^2)_4$	$(c_4,q)_2$	c_7	$(c_4,q)_1$	c_8	$(c_6,q)_2$	$(c_6,q)_1$
	$(f,q^3)_5$		$(c_2,q)_1$	$(c_3,q^2)_3$				

deg	order							
	1	2	3	4	5	6	7	9
5	$(c_1,c_2)_2$ $(c_4,q^2)_4$ $(c_2,q^2)_3$	$(c_7,q)_2$ $(c_3,q^3)_5$	$(c_1,c_2)_1$ $\underline{(c_4,q^2)_3}$	$(c_8,q)_2$ $(c_7,q)_1$	$(c_6,q^2)_4$	$\underline{(c_8,q)_1}$	c_9 $\underline{(c_6,q^2)_3}$	-
6	$((c_1,c_2)_1,q)_2$ $(c_4,q^3)_5$ $((c_1,c_2)_2,q)_1$	$(c_1^2,c_3)_4$ $(c_8,q^2)_4$ $(c_7,q^2)_3$	$(c_6,q^3)_6$ $\underline{((c_1,c_2)_1,q)_1}$	c_{11} $(c_8,q^2)_3$	$(c_9,q)_2$ $(c_6,q^3)_5$	-	$\underline{(c_9,q)_1}$	-
7	$(c_1^3,f)_5$ $(c_6,q^4)_8$ $((c_1,c_2)_1,q^2)_3$	$(c_{11},q)_2$ $\underline{(c_8,q^3)_5}$ $((c_1^2,c_3)_4,q)_1$	$(c_9,q^2)_4$ $\underline{(c_6,q^4)_7}$	$\underline{(c_{11},q)_1}$	c_{10} $(c_9,q^2)_3$	-	-	-
8	$(c_9,q^3)_6$ $(c_6,q^5)_9$ $((c_1^3,f)_5,q)_1$	$(c_1^3,c_3)_5$ $\underline{(c_{11},q^2)_3}$	$(c_{10},q)_2$ $\underline{(c_9,q^3)_5}$	-	$\underline{(c_{10},q)_1}$	-	-	-
9	$(c_{10},q^2)_4$ $(c_9,q^4)_7$	$\underline{((c_1^3,c_3)_5,q)_1}$	$(c_6,c_1^3)_6$ $(c_{10},q^2)_3$	-	-	-	-	-
10	$((c_6,c_1^3)_6,q)_2$ $(c_{10},q^3)_5$	-	$\underline{((c_6,c_1^3)_6,q)_1}$	-	-	-	-	-
11	$(c_6,c_1^4)_8$ $((c_6,c_1^3)_6,q^2)_3$	-	-	-	-	-	-	-
12	$((c_6,c_1^4)_8,q)_1$	-	-	-	-	-	-	-
13	$(c_6,c_1^5)_9$	-	-	-	-	-	-	-
14	$((c_6,c_1^5)_9,q)_1$	-	-	-	-	-	-	-

Table 5.7: A (not minimal) set of generating covariants of $V_2 \oplus V_5$, of ord ≥ 1

Each covariant C of $V_2 \oplus V_5$ of order s corresponds to the invariant $(C,l^s)_s$ of $V_1 \oplus V_2 \oplus V_5$. We obtain in this way a (not minimal) set of generators for the invariants of $V_1 \oplus V_2 \oplus V_5$: the (known) invariants of $V_2 \oplus V_5$, of degrees ≤ 18 (see Chap. 5.11), and the set of invariants derived from Table 5.7, of degrees ≤ 15. For each $i \leq 18$ we look now for a set of invariants that spans the vector space of invariants of $V_1 \oplus V_2 \oplus V_5$ of degree i. The dimensions of these vector spaces are known from the Poincaré series of the algebra of invariants of $V_1 \oplus V_2 \oplus V_5$:

$$P(t) = t^2 + 2t^3 + 4t^4 + 7t^5 + 15t^6 + 25t^7 + 45t^8 + 70t^9 + 112t^{10} + 175t^{11} +$$
$$+ 270t^{12} + 385t^{13} + 566t^{14} + 800t^{15} + \ldots$$

After isolating the superfluous generators (underlined in Table 5.7), we concluded that the generating invariants of $V_1 \oplus V_2 \oplus V_5$ had the following degrees (i denotes the degree of the generators and d_i the number of generators that are needed in degree i):

i	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
d_i	29	1	2	3	5	8	12	14	12	9	8	9	3	2	1	1	1	1

For details about the computations see Chap. A.16.

It follows that the algebra of invariants of $V_1 \oplus V_2 \oplus V_5$ has 92 generators. Therefore the algebra of covariants of $V_2 \oplus V_5$ has 92 generators as well.

5.17 The invariants of $2V_2 \oplus V_3$

Denote our binary forms $q_1, q_2 \in V_2$, and $f \in V_4$:

$$q_1 = a_0 x^2 + 2a_1 xy + a_2 y^2,$$
$$q_2 = b_0 x^2 + 2b_1 xy + b_2 y^2,$$
$$c \ = c_0 x^3 + 3c_1 x^2 y + 3c_2 xy^2 + c_3 y^3.$$

Consider the covariants $c_1 = (c, c)_2$, $c_2 = (c, c_1)_1$, and the following invariants:

$$j_{2,1} = (q_1, q_1)_2, \qquad\qquad j_{2,2} = (q_1, q_2)_2,$$
$$j_{2,3} = (q_2, q_2)_2, \qquad\qquad j_{3,1} = (c_1, q_1)_2,$$
$$j_{3,2} = (c_1, q_2)_2, \qquad\qquad j_{4,1} = (c_1, c_1)_2,$$
$$j_{5,1} = (c^2, q_1^3)_2, \qquad\qquad j_{5,2} = (c^2, q_2^3)_2,$$
$$j_{6,1} = ((q_1, c)_2 \cdot (q_1^2, c)_3, q_2)_2.$$

Proposition 5.17.1. *The set* $\{j_{2,1}, j_{2,3}, j_{3,1} + j_{3,2}, j_{4,1}, j_{5,1}, j_{5,2}, j_{6,1} + j_{2,2}^3 - j_{3,1}^2\}$
forms a system of parameters of the algebra of invariants of $2V_2 \oplus V_3$.

Proof. We first show that

$$\mathcal{N}(2V_2 \oplus V_3) = \mathcal{V}(j_{2,1}, j_{2,3}, j_{3,1} + j_{3,2}, j_{4,1}, j_{5,1}, j_{5,2}, j_{6,1} + j_{2,2}^3 - j_{3,1}^2).$$

Consider the following cases:

Case 1: $q_1 = 0$.
In this case we have $j_{2,3} = j_{3,2} = j_{4,1} = j_{5,2} = 0$, which implies $(q_2, c) \in \mathcal{N}(V_2 \oplus V_3)$ (see Proposition 5.9.1).

Case 2: $q_2 = 0$.
In this case we have $j_{2,1} = j_{3,1} = j_{4,1} = j_{5,1} = 0$, which implies $(q_1, c) \in \mathcal{N}(V_2 \oplus V_3)$ (see Proposition 5.9.1).

Case 3: $c = 0$.
In this case we have $j_{2,1} = j_{2,2} = j_{2,3} = 0$, which implies $(q_1, q_2) \in \mathcal{N}(2V_2)$.

Case 4: $q_1, q_2, c \neq 0$.
In this case we have $j_{4,1} = 0$, which implies that c is a nullform. Also, $j_{2,1} = 0$ implies that q_1 is a nullform. If $(q_1, q_2, c) \notin \mathcal{N}(2V_2 \oplus V_3)$, then, without loss of generality, we can suppose that $x^2 \mid c$, i.e. $c_2 = c_3 = 0$, and $y^2 \mid q_1$, i.e. $a_0 = a_1 = 0$. Then, $j_{5,1} = a_2^3 c_0^2$, which implies $c_0 = 0$, as $q_1 \neq 0$. Then, $j_{3,1} + j_{3,2} \sim c_1^2(a_2 + b_2)$, where $\sim$ denotes equalities up to a nonzero constant. This implies $b_2 = -a_2$, as $c \neq 0$. Then, we have

$$j_{2,3} \sim b_1^2 + a_2 b_0,$$
$$j_{5,2} \sim a_2 c_1^2 (4b_1^2 - a_2 b_0),$$

which implies $b_0 = b_1 = 0$, as $c, q_1 \neq 0$. But then, $j_{6,1} + j_{2,2}^3 - j_{3,1}^2 \sim a_2^2 c_1^4$, which leads to a contradiction with the assumption $c, q_1 \neq 0$.

We proved that the invariants $j_{2,1}, j_{2,3}, j_{3,1} + j_{3,2}, j_{4,1}, j_{5,1}, j_{5,2}, j_{6,1} + j_{2,2}^3 - j_{3,1}^2$ define the nullcone of $2V_2 \oplus V_3$. From Proposition 2.7.1 it follows that $\{j_{2,1}, j_{2,3}, j_{3,1} + j_{3,2}, j_{4,1}, j_{5,1}, j_{5,2}, j_{6,1} + j_{2,2}^3 - j_{3,1}^2\}$ forms a system of parameters of the ring of invariants of $2V_2 \oplus V_3$. □

Proposition 5.17.2. *The algebra of invariants of $2V_2 \oplus V_3$ is generated by 18 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7
d_i	3	2	2	4	3	4

Proof. The Poincaré series of $\mathcal{O}(2V_2 \oplus V_3)^{\mathrm{SL}_2}$ is

$$P(t) = \frac{a(t)}{(1 - t^2)^2 (1 - t^3)(1 - t^4)(1 - t^5)^2(1 - t^6)}$$

with

$$a(t) = 1 + t^2 + t^3 + 2t^4 + 3t^5 + 5t^6 + 8t^7 + 7t^8 + 7t^9 + 8t^{10} + 5t^{11} +$$
$$+ 3t^{12} + 2t^{13} + t^{14} + t^{15} + t^{17},$$

so that

$$P(t) = 1 + 3t^2 + 2t^3 + 8t^4 + 10t^5 + 22t^6 + 32t^7 + 55t^8 + 80t^9 + 128t^{10} +$$
$$+ 178t^{11} + 268t^{12} + 362t^{13} + 515t^{14} + 686t^{15} + \ldots + 1218t^{17} + \ldots$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 17. Up to degree 7 we find the following 18 invariants:

deg	generators	
2	$(q_1, q_1)_2$	$(q_1, q_2)_2$
	$(q_2, q_2)_2$	
3	$(c_1, q_1)_2$	$(c_1, q_2)_2$
4	$(c_1, c_1)_2$	$((q_1, c)_1, q_2)_2$
5	$(c^2, q_1^3)_2$	$(c^2, q_2^3)_2$
	$(q_1 q_2^2, c^2)_6$	$(q_1^2 q_2, c^2)_6$
6	$((q_1, c)_2 \cdot (q_1^2, c)_3, q_2)_2$	$((q_1, c)_2 \cdot (q_1, c)_1, q_2^2)_4$
	$(c \cdot (q_1, c)_1, q_2^3)_6$	
7	$(q_1^3, cc_2)_6$	$(q_2^3, cc_2)_6$
	$(q_1 q_2^2, cc_2)_6$	$(q_1^2 q_2, cc_2)_6$

Then we prove that no generators are needed in degrees $8, \ldots, 15, 17$ by showing that for each $i \in \{8, \ldots, 15, 17\}$ the vector space $\mathcal{O}(2V_2 \oplus V_3)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 18 invariants found in degrees ≤ 7. For details about the computations see Chap. A.17. □

5.18 The invariants of $2V_2 \oplus V_4$

Denote our binary forms $q_1, q_2 \in V_2$, and $f \in V_4$. Consider the covariants $c_1 = (f, f)_2$, $c_2 = (f, c_1)_1$, $c_3 = (q_1, f)_2$, $c_4 = (q_2, f)_2$, and the following invariants:

$$
\begin{aligned}
j_{2,1} &= (q_1, q_1)_2, & j_{2,2} &= (q_2, q_2)_2, \\
j_{2,3} &= (q_1, q_2)_2, & j_{2,4} &= (f, f)_4, \\
j_{3,1} &= (f, c_1)_4, & j_{3,2} &= (f, q_1^2)_4, \\
j_{3,3} &= (f, q_2^2)_4, & j_{4,1} &= (c_1, q_1^2)_4, \\
j_{4,2} &= (c_1, q_2^2)_4, &
\end{aligned}
$$

Proposition 5.18.1. *The set $\{j_{2,1} + j_{2,3},\ j_{2,2} + j_{2,4},\ j_{2,1} - j_{2,4},\ j_{3,1},\ j_{3,2},\ j_{3,3},\ j_{4,1},\ and\ j_{4,2}\}$ forms a system of parameters of the algebra of invariants of $2V_2 \oplus V_4$.*

Proof. We first prove that

$$
\mathcal{N}(2V_2 \oplus V_4) = \mathcal{V}(j_{2,1} \ldots, j_{2,4}, j_{3,1}, \ldots, j_{3,3}, j_{4,1}, j_{4,2}).
$$

Indeed, $j_{2,1} = j_{2,2} = j_{2,3} = 0$ implies that $(q_1, q_2) \in \mathcal{N}(2V_2)$, and $j_{2,1} = j_{2,2} = j_{2,4} = j_{3,1} = j_{3,2} = j_{3,3} = j_{4,1} = j_{4,2} = 0$ implies that (q_1, f) and (q_2, f) are both in the nullcone $\mathcal{N}(V_2 \oplus V_4)$ (see Proposition 5.10.1). Then, it follows that $(q_1, q_2, f) \in \mathcal{N}(2V_2 \oplus V_4)$.
Using `Singular` [DGPS], we found that $j_{2,1}^7, \ldots, j_{2,4}^7$ are in the ideal generated by $j_{2,1} + j_{2,3}, j_{2,2} + j_{2,4}, j_{2,1} - j_{2,4}, j_{3,1}, j_{3,2}, j_{3,3}, j_{4,1}$, and $j_{4,2}$. It follows that these invariants generate the nullcone of $2V_2 \oplus V_4$, and then, from Proposition 2.7.1, it follows that they form a system of parameters of $\mathcal{O}(2V_2 \oplus V_4)^{\mathrm{SL}_2}$. $\square$

Proposition 5.18.2. *The algebra of invariants of $2V_2 \oplus V_4$ is generated by 19 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6
d_i	4	4	5	2	4

Proof. The Poincaré series of $\mathcal{O}(2V_2 \oplus V_4)^{\mathrm{SL}_2}$ is

$$
\begin{aligned}
P(t) &= \frac{1 + t^2 + t^3 + 4t^4 + 3t^5 + 6t^6 + 3t^7 + 4t^8 + t^9 + t^{10} + t^{12}}{(1 - t^2)^3 (1 - t^3)^3 (1 - t^4)^2} = \\
&= 1 + 4t^2 + 4t^3 + 15t^4 + 18t^5 + 53t^6 + 65t^7 + 148t^8 + 198t^9 + \\
&\quad + 371t^{10} + \ldots + 853t^{12} + \ldots
\end{aligned}
$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 12. Up to degree 6 we find the following 19 invariants:

deg	generators	
2	$(q_1, q_1)_2$ $(q_1, q_2)_2$	$(q_2, q_2)_2$ $(f, f)_4$
3	$(f, c_1)_4$ $(f, q_2^2)_4$	$(f, q_1^2)_4$ $(q_1 q_2, f)_4$
4	$(c_1, q_1^2)_4$ $(c_1, q_1 q_2)_4$ $((c_4, q_2)_1, q_1)_2$	$(c_1, q_2^2)_4$ $((c_3, q_1)_1, q_2)_2$
5	$((c_1, q_1 q_2)_3, q_1)_2$	$((c_1, q_1 q_2)_3, q_2)_2$
6	$(c_2, q_1^3)_6$ $(c_2, q_1^2 q_2)_6$	$(c_2, q_2^3)_6$ $(c_2, q_1 q_2^2)_6$

Then we prove that no generators are needed in degrees $7, \ldots, 10, 12$ by showing that for each $i \in \{7, \ldots, 10, 12\}$ the vector space $\mathcal{O}(2V_2 \oplus V_4)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 19 invariants found in degrees ≤ 6. For details about the computations see Chap. A.18. □

5.19 The invariants of $V_1 \oplus V_3 \oplus V_4$

This chapter is based on joint work with Brouwer [BP12].

In a paper from 1869 Gundelfinger [Gun69] found 64 generating covariants of $V_3 \oplus V_4$. In a series of papers from 1878 Sylvester [Sy78b, Sy78c, Sy78d] claimed that there were only 61 generating covariants for $V_3 \oplus V_4$. We prove that the actual number of generators is 63.

We identify the covariants of $V_3 \oplus V_4$ with the invariants of $V_1 \oplus V_3 \oplus V_4$ and show that a minimal set of generators for the algebra of invariants of this module has size 63. Denote our binary forms $\ell \in V_1$, $c \in V_3$, and $q \in V_4$:

$$\ell = c_0 x + c_1 y,$$
$$c = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3,$$
$$q = b_0 x^4 + 4b_1 x^3 y + 6b_2 x^2 y^2 + 4b_3 xy^3 + b_4 y^4.$$

Consider the covariants $c_1 = (c, c)_2$, $c_2 = (q, q)_2$, $c_3 = (c, q)_2$, and the following invariants:

$$
\begin{aligned}
j_2 \;&= (q, q)_4, & j_{3,1} &= (c_2, q)_4, \\
j_{4,1} &= (c_1, c_1)_2, & j_{4,2} &= (\ell c, \ell c)_4, \\
j_{4,3} &= (c, \ell^3)_3, & j_{5,1} &= ((q, c_2)_1, c^2)_6, \\
j_{5,2} &= ((q, c^2)_2, c^2)_6, & j_{5,3} &= (q, \ell^4)_4, \\
j_{6,1} &= (c_1^2, c_2)_4, & j_{6,2} &= ((\ell c, \ell c)_2, \ell c)_4, \\
j_{6,3} &= (c_2, \ell^4)_4, & j_{7,1} &= (c^4, q^3)_{12}.
\end{aligned}
$$

Proposition 5.19.1. *With the notations above, the invariants*

$$p_1 = j_{3,1}, \qquad p_2 = j_{4,1} + j_2^2, \qquad p_3 = j_{4,2} + j_{4,3} - j_2^2, \qquad p_4 = j_{5,1} + j_{5,2},$$
$$p_5 = j_{5,3}, \qquad p_6 = j_{6,1} + j_{6,2}, \qquad p_7 = j_{6,3}, \qquad\qquad p_8 = j_{7,1},$$

form a system of parameters of the algebra of invariants of $V_1 \oplus V_3 \oplus V_4$.

Proof. We show that $\mathcal{V}(p_1, \ldots, p_8) = \mathcal{N}(V_1 \oplus V_3 \oplus V_4)$. Consider three cases.

Case 1: $q = 0$.
In this case, the vanishing of $p_1, \ldots, p_8$ reduces to $j_{4,1} = j_{4,2} + j_{4,3} = j_{6,2} = 0$, which implies that $(\ell, c) \in \mathcal{N}(V_1 \oplus V_3)$ (see Proposition 5.3.4).

Case 2: $\ell = 0$.
In this case, the vanishing of $p_1, \ldots, p_8$ reduces to $j_2 = j_{3,1} = j_{4,1} = j_{5,1} + k_{5,2} = j_{6,1} = j_{7,1} = 0$, which implies that $(c, q) \in \mathcal{N}(V_3 \oplus V_4)$ (see Proposition 5.13.1).

Case 3: $q, \ell \neq 0$.
In this case, $p_5 = 0$ implies that q and ℓ have a common root (up to a constant, p_5 is the resultant of q and ℓ). Without loss of generality, we can suppose that the common factor of q and ℓ is x, i.e., $c_1 = b_4 = 0$ and $c_0 \neq 0$. Then $p_7 \sim b_3^2 c_0^4$, which implies $b_3 = 0$. Then $p_1 \sim b_2^3$, which implies $b_2 = 0$. Then a_3 becomes a factor of p_8. If $a_3 = 0$, then $p_3 \sim a_2^2 c_0^2$, which implies $a_2 = 0$, and then $(\ell, c, q) \in \mathcal{N}(V_1 \oplus V_3 \oplus V_4)$. If $a_3 \neq 0$, we may take $a_3 = c_0 = 1$. Now

$$p_3 \sim 3a_2^2 - 3a_1 - 2,$$

and it follows that $a_1 = a_2^2 - \frac{2}{3}$. Then

$$p_6 \sim 27a_2^3 - 54a_2 - 27a_0 - 256b_1^2,$$

and it follows that $a_0 = a_2^3 - 2a_2 - \frac{256}{27}b_1^2$. Then

$$p_4 \sim 36b_0 - 144a_2 b_1 - 949b_1^3,$$

and it follows that $b_0 = 4a_2 b_1 + \frac{949}{36}b_1^3$. Then

$$p_2 \sim 27 - 2048b_1^4,$$
$$p_8 \sim b_1^5(33205248 - 4273351745b_1^4).$$

But $p_2 = p_8 = 0$ has no solution. This settles Case 3.

By Proposition 2.7.1, it follows that these eight invariants form a hsop of the ring of invariants of $V_1 \oplus V_3 \oplus V_4$. $\square$

Proposition 5.19.2. *The algebra of invariants of $V_1 \oplus V_3 \oplus V_4$ is generated by 63 invariants. The nonzero numbers d_i of basic invariants of degree i are*

i	2	3	4	5	6	7	8	9	10	11
d_i	1	2	4	8	10	13	11	10	3	1

Proof. The Poincaré series of $\mathcal{O}(V_1 \oplus V_3 \oplus V_4)^{\mathrm{SL}_2}$ is

$$\begin{aligned}
P(t) = 1 &+ t^2 + 2t^3 + 5t^4 + 10t^5 + 18t^6 + 31t^7 + 55t^8 + 92t^9 + \\
&144t^{10} + 223t^{11} + 341t^{12} + 499t^{13} + 725t^{14} + 1031t^{15} + \\
&1436t^{16} + 1978t^{17} + 2685t^{18} + 3592t^{19} + 4761t^{20} + 6235t^{21} + \\
&8078t^{22} + 10379t^{23} + 13226t^{24} + 16698t^{25} + 20937t^{26} + \\
&26069t^{27} + 32230t^{28} + 39614t^{29} + 48401t^{30} + ... = \\
= &\frac{a(t)}{(1-t^3)(1-t^4)^2(1-t^5)^2(1-t^6)^2(1-t^7)}
\end{aligned}$$

where

$$\begin{aligned}
a(t) = 1 &+ t^2 + t^3 + 3t^4 + 7t^5 + 12t^6 + 21t^7 + 32t^8 + 47t^9 + \\
&58t^{10} + 72t^{11} + 83t^{12} + 89t^{13} + 94t^{14} + 94t^{15} + 89t^{16} + \\
&83t^{17} + 72t^{18} + 58t^{19} + 47t^{20} + 32t^{21} + 21t^{22} + 12t^{23} + \\
&7t^{24} + 3t^{25} + t^{26} + t^{27} + t^{29}
\end{aligned}$$

We apply the strategy described in Chap. 3.1. From the Poincaré series, the maximal degree in which we have to look for generators is 29. Up to degree 11 we find the following 63 invariants:

deg	generators	
2	$(q,q)_4$	
3	$(c_2,q)_4$	$(q,\ell c)_4$
4	$(c_1,c_1)_2$	$(\ell c,\ell c)_4$
	$(c,\ell^3)_3$	$(c_2,\ell c)_4$
5	$((q,c_2)_1,c^2)_6$	$((q,c^2)_2,c^2)_6$
	$(q,\ell^4)_4$	$(q\ell^2,c^2)_6$
	$((q,c_1)_2,\ell^2)_2$	$(c_3,\ell^3)_3$
	$(\ell \cdot (c,c_1)_1,q)_4$	$((\ell,c_1)_1,(q,c)_3)_1$
6	$(c_1^2,c_2)_4$	$((\ell c,\ell c)_2,\ell c)_4$
	$(c_2,\ell^4)_4$	$(c_2,\ell \cdot (c,c_1)_1)_4$
	$(((c_1,q)_2,c^2)_2,q)_4$	$(c_2,c \cdot (\ell,c_1)_1)_4$
	$(\ell^2 c_2,c^2)_6$	$((c_2,c_1)_2,\ell^2)_2$
	$((c,c_2)_2,\ell^3)_3$	$((c^2,q^2)_6,(c,\ell)_1)_2$
7	$(c^4,q^3)_{12}$	$(((c_1,q)_2,c^2)_2,c_2)_4$
	$((c_2,q)_1,c \cdot (c,c_1)_1)_6$	$((c_1,\ell)_1,(q,(c_1,c)_1)_3)_1$
	$((c^2,qc_2)_6,(c,\ell)_1)_2$	$(((c_1,c)_1,q)_2,\ell^3)_3$
	$(q\ell^2,c \cdot (c_1,c)_1)_6$	$((q,(c_1,\ell^2)_1)_2,\ell^2)_2$
	$(\ell^5,(c,q)_1)_5$	$(((c_3,(q,\ell)_1)_2,c_3)_1,c)_3$
	$(((c_3,c)_2,(q,\ell)_1)_1,(q,\ell)_1)_3$	$(\ell^2(q,c_1)_1,c^2)_6$
	$((c_3,(q,\ell)_1)_2,(q,\ell^2)_2)_2$	

deg	generators	
8	$((c^2, q^2)_6, ((c, c_1)_1, \ell)_1)_2$	$(ql \cdot (c, q)_3, c \cdot (c, c_1)_1)_6$
	$(((c_1, q)_2, c \cdot (c, c_1)_1)_2, q)_4$	$(c^4, c_2 q^2)_{12}$
	$(((c_2, c_1)_2, c^2)_2, c_2)_4$	$((c^2, c_2 q)_6, (c_3, \ell)_1)_2$
	$((c_1, \ell)_1, (q, (c_1, (q, \ell)_1)_1)_3)_1$	$(((c_1, c_3)_1, q)_2, \ell^3)_3$
	$(((c_1, (q, \ell)_1)_1, q)_2, \ell^3)_3$	$(q\ell^2, c \cdot (c_1, c_3)_1)_6$
	$((q, ((c, (q, \ell)_1)_2, \ell^2)_1)_2, \ell^2)_2$	
9	$(((c_2, c_1)_2, c \cdot (c, c_1)_1)_2, q)_4$	$(c^4, qc_2^2)_{12}$
	$(c^3(c, c_1)_1, q^3)_{12}$	$(((q, c_1)_2, c \cdot (c, c_1)_1)_2, c_2)_4$
	$(((q, c_1)_2, c \cdot ((q, \ell)_1, c_1)_1)_2, q)_4$	$((c^2, cc_2 \ell)_6, (c_3, \ell)_1)_2$
	$((c^2, cc_2 \ell)_6, ((q, \ell)_1, \ell)_1)_2$	$((((q, (q, c\ell)_2)_2, c_1)_2, c^2)_2, q)_4$
	$(((q, \ell)_1, c_2)_1, \ell^5)_5$	$(((q, c_2)_1, c^2)_4, \ell^4)_4$
10	$(((c_2, c_1)_2, c \cdot (c, c_1)_1)_2, c_2)_4$	$(((c^3(c, c_1)_1, q^3)_{10}, q)_4$
	$((c, (c, (c, (c, (q, c_2)_1)_3)_1)_3 \cdot (c, q)_2)_3, \ell)_1$	
11	$(((c^3(c, c_1)_1, q^3)_{10}, q)_2, q)_4$	

Then we prove that no generators are needed in degrees $12, \ldots, 27, 29$ by showing that for each $i \in \{12, \ldots, 27, 29\}$ the vector space $\mathcal{O}(V_1 \oplus V_3 \oplus V_4)_i^{\mathrm{SL}_2}$ is spanned by monomials of degree i generated by the 63 invariants of degrees ≤ 11.

The computations in degrees ≥ 16 were performed by A.E. Brouwer, with his own software ([BP12]). The computations in degrees < 16 are in Chap. A.19.

$\square$

Chapter 6

Homological dimension

6.1 The main results

This chapter is based on joint work with Brouwer [BP11].

Consider V an SL_2-module and $I := \mathcal{O}(V)^{SL_2}$ the algebra of polynomial functions on V invariant under the action of SL_2. The algebra I is finitely generated ([Hil90]), i.e. there exist a finite number of invariants $j_1, \ldots, j_r$ of V such that $I = \mathbb{C}[j_1, \ldots, j_r]$. Denote by r the minimal number of generators of I and by m the size of a system of parameters of I (set of algebraically independent elements $P_1, \ldots, P_m$ of I, such that I is integral over $\mathbb{C}[P_1, \ldots, P_m]$). Then m equals $\dim V//G$, and the homological dimension $\operatorname{hd} I$ of I equals $r - m$ (see [Pop83, Corollary 1]).

Popov [Pop83] classified the modules V with the property that $\operatorname{hd} I \leq 3$ (see Table 6.1), and noticed that all of these were known classically.

V	$\operatorname{hd} I$
V_1, V_2, V_3, V_4, $2V_1$, $V_1 \oplus V_2$, $2V_2$, $3V_1$	0
V_5, V_6, $V_1 \oplus V_3$, $V_1 \oplus V_4$, $V_2 \oplus V_3$, $V_2 \oplus V_4$, $2V_4$ $2V_1 \oplus V_2$, $V_1 \oplus 2V_2$, $3V_2$, $4V_1$	1
$2V_3$	2
V_8, $5V_1$	3

Table 6.1: Popov's classification of SL_2-modules with small $\operatorname{hd} I$

We extend Popov's classification by finding the modules V with the property that $4 \leq \operatorname{hd} I \leq 15$. The two main results of this chapter are:

Theorem 6.1.1. *Let $I := \mathcal{O}(V_n)^{SL_2}$ and suppose that $\operatorname{hd} I \leq 100$. Then $n \leq 10$.*

Theorem 6.1.2. *Let* $I := \mathcal{O}(V)^{\mathrm{SL}_2}$ *where* $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$, *and suppose that* $4 \leq \mathrm{hd}\, I \leq 15$. *Then we have one of the following:*

$n_1,\ldots,n_p$	hd	m	hsop degrees	r	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
$1,1,1,2$	4	6	2 (3×), 3 (3×)	10	4	6								
$1,2,2,2$	5	8	2 (5×), 3 (3×)	13	6	4	3							
$2,2,2,2$	5	9	2 (9×)	14	10	4								
$1,1,2,2$	6	7	2 (4×), 3 (3×)	13	4	6	3							
1 (6×)	6	9	2 (9×)	15	15									
$1,1,3$	8	5	2, 4 (4×)	13	1		8		4					
$1,2,3$	9	6	$3,3,4,4,4,5$	15	1	3	4	4	2	1				
$1,1,1,1,2$	9	8	2 (4×), 3 (3×), 6	17	7	10								
1 (7×)	10	11	2 (11×)	21	21									
$1,2,4$	11	7	$2,2,3,3,4,5,6$	18	2	3	2	3	4	2	1	1		
$2,2,3$	11	7	$2,2,3,4,5,5,6$	18	3	2	2	4	3	4				
$2,2,4$	11	8	$2,2,2,3,3,3,4,4$	19	4	4	5	2	4					
$1,2,2,2,2$	13	11	2 (7×), 3 (4×)	24	10	8	6							
$2,2,2,2,2$	13	12	2 (12×)	25	15	10								
$4,4,4$	13	12	2 (6×), 3 (6×)	25	6	10	6	3						
$1,1,4$	14	6	$2,3,5,5,6,6$	20	2	1		5	5			7		
$3,4$	14	6	$2,3,4,5,6,7$	20	1	1	1	2	2	3	3	4	2	1
1 (8×)	15	13	2 (13×)	28	28									
$1,1,1,2,2$	15	9	2 (5×), 3 (4×)	24	6	12	6							

Here V *has a minimal set of generators of size* r, *with* d_i *generators of degree* i $(2 \leq i \leq 11)$. *The size of any homogeneous system of parameters (hsop) is* m, *and the degrees for one particular such system are as given. The column* hd *gives* $\mathrm{hd}\, I$.

6.2 Bounds on $\mathrm{hd}\, \mathcal{O}(V)^G$

Consider a group G acting on a set V.

If $v \in V$, then $Gv = \{g \cdot v \mid g \in G\}$ is called the *orbit* of v, and $G_v = \{g \in G \mid g \cdot v = v\}$ is called the *stabilizer* of v. If the orbit Gv of a point $v \in V$ is closed, then the stabilizer G_v is reductive and the tangent space of V at v decomposes into the direct sum of the tangent space of Gv at v and a complementary G_v-module N_v. The corresponding representation (G_v, N_v) is called slice-representation (cf. [Pop83]).

If $S \subset G$, then $N_G(S) = \{g \in G \mid gSg^{-1} = S\}$ is called the *normalizer* of S, and $Z_G(S) = \{g \in G \mid gsg^{-1} = s$ for all $s \in S\}$ is called the *centralizer* of S.

Consider the group $T = \left\{ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) \mid t \in \mathbb{C}^* \right\} \subset \mathrm{SL}_2$. Then the normalizer of T is
$$N_{\mathrm{SL}_2}(T) = \left\{ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & t \\ -t^{-1} & 0 \end{smallmatrix} \right) \mid t \in \mathbb{C}^* \right\}.$$

Lemma 6.2.1. ([Pop83, Lemma 1]) *Consider* $v = x^n y^n \in V_{2n}$, $n > 0$. *Then:*

a) $\mathrm{SL}_2 v$ *is closed in* V_{2n}.

b) $\mathrm{SL}_{2v} = \begin{cases} T & \text{if } n = 1 \text{ modulo } 2, \\ N_{\mathrm{SL}_2}(T) & \text{if } n = 0 \text{ modulo } 2. \end{cases}$

Proposition 6.2.2. ([Pop83, Proposition 4]) *Consider G a reductive group of rank 1 and V a G-module. Consider T a maximal torus of G and suppose that p, n, r are the dimensions of the T-weight subspaces of V which have positive, negative, zero weight respectively.*

If $pn > 0$, then $\operatorname{hd}\mathcal{O}(V)^G \geq (p-1)(n-1)$.

Proposition 6.2.3. ([Pop83, Proposition 4]) *Consider G a reductive group of dimension 1, T a maximal torus of G, with $Z_G(T)$ diagonalizable and with $Z_G(T) \neq G$. Consider and V a G-module, and V^+, V^-, V^0 the sum of those T-weight subspaces of V which have positive, negative, and zero weight respectively.*

Then, there exist bases of $Z_G(T)$-weight vectors $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ in $(V^+)^*$ and in $(V^-)^*$ respectively, such that the T-weights of x_i and y_i are inverse of each other, and $g\langle x_i \rangle = \langle y_i \rangle$ and $g\langle y_i \rangle = \langle x_i \rangle$, if $g \notin Z_G(T)$.

If $p > 0$, then, $\operatorname{hd}\mathcal{O}(V)^G \geq \frac{(p-1)(p-2)}{2}$.

Proposition 6.2.4. ([Pop83, Propositions 6 and 7])

a) If $n \geq 2$, then $\operatorname{hd}\mathcal{O}(V_{2n})^{\mathrm{SL}_2} \geq \begin{cases} (n-2)^2 & \text{if } n \text{ is odd,} \\ \frac{(n-2)(n-3)}{2} & \text{if } n \text{ is even.} \end{cases}$

b) If $n \geq 3$, then $\operatorname{hd}\mathcal{O}(V_{2n-1})^{\mathrm{SL}_2} \geq n^2 - 2n - 2$.

c) If $\operatorname{hd}\mathcal{O}(V_{2n})^{\mathrm{SL}_2} = d$ and $n \geq 2$, then $n \leq \begin{cases} \frac{\sqrt{8d+1}+5}{2} & \text{if } n \text{ is even,} \\ \sqrt{d}+2 & \text{if } n \text{ is odd.} \end{cases}$

d) If $\operatorname{hd}\mathcal{O}(V_{2n-1})^{\mathrm{SL}_2} = d$ and $n \geq 3$, then $n \leq \sqrt{d+3}+1$.

Theorem 6.2.5. ([Pop83, Theorem 2]) *(A monotony theorem.) Consider G a reductive group and V a G-module. We have the following:*

a) If $v \in V$ and Gv is closed in V. Denote by (G_v, N_v) the slice-representation. Then $\operatorname{hd}\mathcal{O}(N_v)^{G_v} \leq \operatorname{hd}\mathcal{O}(V)^G$.

b) If $V = W_1 \oplus \ldots \oplus W_p$, then $\sum_{i=1}^{p} \operatorname{hd}\mathcal{O}(W_i)^G \leq \operatorname{hd}\mathcal{O}(V)^G$.

c) If W is a submodule of V, then $\operatorname{hd}\mathcal{O}(W)^G \leq \operatorname{hd}\mathcal{O}(V)^G$.

One can use the Poincaré series of the algebra I of invariants of an SL_2-module V to obtain a lower bound on the number of generators of I, and hence on the homological dimension of this algebra. The following result was proved by Brower [BP12]:

Proposition 6.2.6. (Brower [BP12]) *Table 6.2 contains lower bounds on the homological dimension of the algebra of invariants of several SL_2-modules. In the table are listed the modules, the Poincaré series of the algebra I of invariants of these modules, and lower bounds on the number of generators r of I and on $\operatorname{hd} I$:*

module	Poincaré series	$r \geq$	hd $I \geq$
V_{11}	$1 + 2t^4 + 13t^8 + 13t^{10} + 73t^{12} + 110t^{14} + \ldots$	158	149
V_{12}	$1 + t^2 + t^3 + 3t^4 + 3t^5 + 8t^6 + 10t^7 + 20t^8 +$		
	$28t^9 + 52t^{10} + 73t^{11} + 127t^{12} + 181t^{13} + \ldots$	113	103
V_{13}	$1 + 2t^4 + 22t^8 + 33t^{10} + 181t^{12} + 375t^{14} + \ldots$	502	491
V_{14}	$1 + t^2 + 3t^4 + 10t^6 + 4t^7 + 31t^8 + 27t^9 +$		
	$97t^{10} + 110t^{11} + \ldots$	182	170
V_{15}	$1 + 3t^4 + t^6 + 36t^8 + 80t^{10} + 418t^{12} + \ldots$	425	412
V_{16}	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 13t^6 + 18t^7 + 47t^8 +$		
	$84t^9 + 177t^{10} + \ldots$	198	184
V_{18}	$1 + t^2 + 4t^4 + t^5 + 16t^6 + 13t^7 + 71t^8 + 99t^9 +$	161	145
V_{20}	$1 + t^2 + t^3 + 4t^4 + 5t^5 + 20t^6 + 35t^7 + 102t^8 +$	123	105
V_{22}	$1 + t^2 + 4t^4 + t^5 + 24t^6 + 26t^7 + 144t^8 + \ldots$	164	144
V_{24}	$1 + t^2 + t^3 + 5t^4 + 7t^5 + 29t^6 + 62t^7 + 201t^8 +$	242	220
V_{28}	$1 + t^2 + t^3 + 5t^4 + 8t^5 + 40t^6 + 97t^7 + 365t^8 +$	440	414
V_{32}	$1 + t^2 + t^3 + 6t^4 + 10t^5 + 54t^6 + 153t^7 + \ldots$	201	171
$V_2 \oplus V_8$	$1 + 2t^2 + t^3 + 5t^4 + 5t^5 + 15t^6 + 17t^7 +$		
	$41t^8 + 54t^9 + 108t^{10} + \ldots$	35	26
$V_3 \oplus V_8$	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 9t^6 + 16t^7 + 30t^8 + \ldots$	37	27
$V_4 \oplus V_8$	$1 + 2t^2 + 4t^3 + 8t^4 + 16t^5 + 35t^6 + 60t^7 + \ldots$	42	31
$V_5 \oplus V_8$	$1 + t^2 + t^3 + 3t^4 + 6t^5 + 15t^6 + 31t^7 + \ldots$	43	31
$V_6 \oplus V_8$	$1 + 2t^2 + 2t^3 + 10t^4 + 14t^5 + 46t^6 + 82t^7 + \ldots$	88	75
$V_1 \oplus 2V_3$	$1 + t^2 + 13t^4 + 26t^6 + \ldots$	26	19
$V_2 \oplus 2V_3$	$1 + 2t^2 + 3t^3 + 9t^4 + 12t^5 + 26t^6 + 44t^7 + \ldots$	26	18
$V_1 \oplus 2V_2 \oplus V_3$	$1 + 3t^2 + 6t^3 + 15t^4 + 30t^5 + 65t^6 + \ldots$	34	25
$V_2 \oplus V_3 \oplus V_4$	$1 + 2t^2 + 3t^3 + 7t^4 + 14t^5 + 29t^6 + 52t^7 + \ldots$	43	34
$V_1 \oplus 2V_2 \oplus V_4$	$1 + 4t^2 + 6t^3 + 18t^4 + 33t^5 + \ldots$	27	17
$3V_2 \oplus V_4$	$1 + 7t^2 + 8t^3 + 42t^4 + 64t^5 + \ldots$	37	26
$2V_3 \oplus V_4$	$1 + 2t^2 + 2t^3 + 9t^4 + 16t^5 + 37t^6 + 71t^7 + \ldots$	69	59
$V_3 \oplus 2V_4$	$1 + 3t^2 + 4t^3 + 10t^4 + 22t^5 + 49t^6 + 96t^7 + \ldots$	45	34
$V_3 \oplus V_5$	$1 + 6t^4 + 7t^6 + 36t^8 + \ldots$	28	21
$V_4 \oplus V_5$	$1 + t^2 + t^3 + 2t^4 + 4t^5 + 8t^6 + 12t^7 + 22t^8 +$		
	$37t^9 + 56t^{10} + \ldots$	59	51
$2V_5$	$1 + t^2 + 7t^4 + 14t^6 + 72t^8 + 168t^{10} + \ldots$	105	96
$V_3 \oplus V_6$	$1 + t^2 + t^3 + 3t^4 + 4t^5 + 8t^6 + 12t^7 + 21t^8 + \ldots$	24	16
$V_4 \oplus V_6$	$1 + 2t^2 + 2t^3 + 7t^4 + 8t^5 + 24t^6 + 31t^7 + 68t^8 +$	33	24
$V_5 \oplus V_6$	$1 + t^2 + t^3 + 3t^4 + 5t^5 + 12t^6 + 22t^7 + \ldots$	31	21
$2V_6$	$1 + 3t^2 + 12t^4 + 6t^5 + 44t^6 + 40t^7 + 150t^8 + \ldots$	29	18

Table 6.2: Bounds from the Poincaré series

6.3   The proofs of the main results

Proof of Theorem 6.1.1. Consider the SL$_2$-module V_n and denote by I the algebra of invariants of V_n. We want to determine n such that hd $I \leq 100$.

By Proposition 6.2.4, if n is even and $\operatorname{hd} I \leq 100$, then $n \leq 24$ or $n \in \{28, 32\}$. By [Kac83, p. 106], if n is odd, then $r \geq p(n-2) + \phi(n-2) - 1$, where $p()$ is the partition function and $\phi()$ is Euler's totient function. It follows that $\operatorname{hd} I \geq 168$ for odd $n \geq 17$. For $n \leq 10$, $\operatorname{hd} I$ takes the following values:

n	1	2	3	4	5	6	7	8	9	10
$\operatorname{hd} I$	0	0	0	0	1	1	25	3	85	98

(This follows from Proposition 4.0.1.)

For the remaining cases we have $\operatorname{hd} I \geq 103$ from Table 6.2. This proves Theorem 6.1.1. □

Proof of Theorem 6.1.2. Consider $V = V_{n_1} \oplus \ldots \oplus V_{n_p}$ with $n_i \geq 1$ for all i and $p > 1$. Let $I := \mathcal{O}(V)^{\mathrm{SL}_2}$ be the algebra of invariants of V. We want to determine V such that $\operatorname{hd} I \leq 15$.

We follow Popov's proof ([Pop83, Theorem 4]), in which he determined V with $\operatorname{hd} I \leq 3$.

By the monotony theorem 6.2.5 it follows that if $V = W \oplus W'$, then $\operatorname{hd} I \geq \operatorname{hd} \mathcal{O}(W)^{\mathrm{SL}_2} + \operatorname{hd} \mathcal{O}(W')^{\mathrm{SL}_2}$. Therefore, all n_i belong to $\{1, 2, 3, 4, 5, 6, 8\}$, and direct summands W have $\operatorname{hd} \mathcal{O}(W)^{\mathrm{SL}_2} \leq 15$.

If all n_i are either 1 or 2, so that $V = mV_1 \oplus nV_2$, then we have explicit formulas for the number of generators of I and for the size of a system of parameters of I (see Chap. 5.2). Therefore we have the following table:

$n \backslash m$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	1	3	6	10	15	21
1	0	0	1	4	9	16	25			
2	0	1	6	15	28					
3	1	5	16	34						
4	5	13	32							
5	13	26								
6	26									

We investigate the remaining possibilities.

Case 1: One of the n_i, say n_1, is equal to 8.

Let $v = x^4 y^4 \in V_8$. From Lemma 6.2.1 it follows that the orbit $\mathrm{SL}_2 \cdot v$ is closed and $\mathrm{SL}_{2v} = N_{\mathrm{SL}_2}(T)$. We apply Proposition 6.2.3 for $G = \mathrm{SL}_{2v}$ and for the slice-representation (SL_{2v}, N_v) of SL_{2v} for the module V.

We obtain $\operatorname{hd} \mathcal{O}(N_v)^{\mathrm{SL}_{2v}} \geq \frac{(b-1)(b-2)}{2}$, where $b = \dim N_v^+$. By the monotony theorem 6.2.5 it follows that $\operatorname{hd} I \geq \frac{(b-1)(b-2)}{2}$. We determine b. We have:

$$(\mathrm{SL}_{2v}, N_v) = (\mathrm{SL}_{2v}, N_{v, V_8}) \oplus (\mathrm{SL}_{2v}, V_{n_2} \oplus \ldots \oplus V_{n_p}).$$

Now $\dim(V_{n_2} \oplus \ldots \oplus V_{n_p})^+ = [\frac{n_2+1}{2}] + \ldots + [\frac{n_p+1}{2}]$, where $[\alpha]$ is the maximal integer $\leq \alpha$. The positive weights of $N_{x^n y^n}$ are $4, 6, \ldots, 2n$, hence $\dim N_{v, V_8}^+ = 3$. Therefore,

$$b = 3 + [\frac{n_2 + 1}{2}] + \ldots + [\frac{n_p + 1}{2}],$$

and hd $I \geq \frac{(b-1)(b-2)}{2}$. If $b \geq 8$, then hd $I \geq 21$. We check the cases with $b \leq 7$. By monotony it suffices to look at $V_m \oplus V_8$ for $m = 1, 2, 3, 4, 5, 6$. If $V = V_1 \oplus V_8$ then I is the algebra of covariants of V_8, generated by 69 elements ([BB08]), and hd $I = 61$. In the other cases hd $I \geq 26$ by Table 6.2.

Case 2: One of the n_i, say n_1, is equal to 2.

Let $v = xy \in V_2$. From Lemma 6.2.1 it follows that the orbit $SL_2 \cdot v$ is closed and $SL_{2v} = T$. We apply Proposition 6.2.2 for $G = SL_{2v}$ and for the slice-representation (SL_{2v}, N_v) of SL_{2v} for the module V.
We obtain hd $\mathcal{O}(N_v)^{SL_{2v}} \geq (c-1)^2$, where $c = \dim N_v^+$. By the monotony theorem 6.2.5 it follows that hd $I \geq (c-1)^2$. We determine c. We have:

$$(SL_{2v}, N_v) = (SL_{2v}, N_{v,V_2}) \oplus (SL_{2v}, V_{n_2} \oplus \ldots \oplus V_{n_p}).$$

Now $\dim(V_{n_2} \oplus \ldots \oplus V_{n_p})^+ = [\frac{n_2+1}{2}] + \ldots + [\frac{n_p+1}{2}]$, where $[\alpha]$ is the maximal integer $\leq \alpha$. The positive weights of $N_{x^n y^n}$ are $4, 6, \ldots, 2n$, hence $\dim N_{v,V_2}^+ = 0$. Therefore,
$$c = [\frac{n_2+1}{2}] + \ldots + [\frac{n_p+1}{2}],$$

and hd $I \geq (c-1)^2$. Since $n_i > 2$ for some i, we have $c \geq 2$. We have hd $I \geq 16$ for $c \geq 5$. We therefore check the cases $c \in \{2, 3, 4\}$.

If $c = 2$, then V is $V_2 \oplus V_3$ or $V_2 \oplus V_4$ and hd $I = 1$ (this follows from Chap. 5.9 and 5.10).

If $c = 3$, then V is one of $V_1 \oplus V_2 \oplus V_3$, $V_1 \oplus V_2 \oplus V_4$, $2V_2 \oplus V_3$, $2V_2 \oplus V_4$, $V_2 \oplus V_5$ or $V_2 \oplus V_6$. In these six cases one has hd $I = 9, 11, 11, 11, 23, 20$ (this follows from Table 5.1).

If $c = 4$, then by monotony and the above V does not have a direct summand V_5 or V_6, so that V is one of $V_2 \oplus 2V_3$, $V_2 \oplus V_3 \oplus V_4$, $V_2 \oplus 2V_4$, $2V_1 \oplus V_2 \oplus V_3$, $2V_1 \oplus V_2 \oplus V_4$, $V_1 \oplus 2V_2 \oplus V_3$, $V_1 \oplus 2V_2 \oplus V_4$, $3V_2 \oplus V_3$, $3V_2 \oplus V_4$. If V is $2V_1 \oplus V_2 \oplus V_3$ or $2V_1 \oplus V_2 \oplus V_4$, then hd $I = 27$ or 48 by Table 5.3. Explicit generation of invariants for $V_2 \oplus 2V_4$ and $3V_2 \oplus V_3$ shows that $r \geq 29, 49$ so that hd $I \geq 19, 39$ in these cases. By Table 6.2 hd $I \geq 17$ in the remaining five cases.

Case 3: All of the n_i equal 1, 3, 4, 5 or 6.
If V is $V_1 \oplus V_3$, $V_1 \oplus V_4$, $2V_3$, $V_3 \oplus V_4$, $2V_4$, $V_1 \oplus V_5$, or $V_1 \oplus V_6$, then hd I equals 1, 1, 2, 14, 1, 18, 20, respectively (this follows from Table 5.1). If V is $V_3 \oplus V_5$, $V_4 \oplus V_5$, $2V_5$, $V_3 \oplus V_6$, $V_4 \oplus V_6$, $V_5 \oplus V_6$, $2V_6$, $2V_3 \oplus V_4$ or $V_3 \oplus 2V_4$, then hd $I \geq 16$ by Table 6.2.

If V is $2V_1 \oplus V_3$, $2V_1 \oplus V_4$, $V_1 \oplus 2V_3$, $V_1 \oplus V_3 \oplus V_4$, $V_1 \oplus 2V_4$, $3V_3$, $3V_4$, $3V_1 \oplus V_3$, $3V_1 \oplus V_4$, $4V_4$, then hd I equals 8, 14, 19, 55, 19, 19, 13, 23, 55, 63, respectively (this follows from Tables 5.1 and 5.3). By monotony we are done.

This finishes the determination of the V with hd $I \leq 15$.

$\square$

Appendix A

Computations

A.1 Implemented functions

The function `Transvectant[ll1,ll2,r]` implemented in `Mathematica` [Wol10]
returns the coefficient list of `(ff1,ff2)_r`, where `ff1` is a binary form with the
coefficient list `ll1` and `ff2` is a binary form with the coefficient list `ll2`:

```
<<Combinatorica`
Transvectant[ll1_,ll2_,rr_]:=Module[{w1,w2,yy,xx,tt,zz},
w1[nn_]:=Table[yy^(nn-i)*xx^i,{i,0,nn}];
w2[mm_]:=Table[tt^(mm-i)*zz^i,{i,0,mm}];
Coefficient[ Expand[((Length[ll1]-1-rr)!*(Length[ll2]-1-rr)!*
 Sum[(-1)^i*Binomial[rr,i]*
 D[D[ll1.w1[Length[ll1]-1],{xx,rr-i}],{yy,i}]*
 D[D[ll2.w2[Length[ll2]-1],{tt,rr-i}],{zz,i}],{i,0,rr}])/
 ((Length[ll1]-1)!*(Length[ll2]-1)!)/.{zz->xx,tt->yy}]/.{yy->1},xx,#]
 &/@Range[0,Length[ll1]+Length[ll2]-2*(rr+1)]];
```

Example:

```
In[]:=ff=Table[Binomial[3,i]*a[3-i],{i,0,3}]
Out[]={a[3],3a[2],3a[1],a[0]}
In[]:=Transvectant[ff,ff,2]
Out[]={-2a[2]^2+2a[1]a[3],-2a[1]a[2]+2a[0]a[3],-2a[1]^2+2a[0]a[2]}
```

The function `TransvectantPoly[ff1,ff2,r]` implemented in `Mathematica` [Wol10]
returns the transvectant `(ff1,ff2)_r`, where `ff1` and `ff2` are two binary forms:

```
<<Combinatorica`
TransvectantPoly[ff1_,ff2_,rr_]:=Module[{w1,w2,yy,xx,tt,zz,LL,pp,ll1,ll2},
ll1=CoefficientList[ff1/.y->1,x];
ll2=CoefficientList[ff2/.y->1,x];
w1[nn_]:=Table[yy^(nn-i)*xx^i,{i,0,nn}];
w2[mm_]:=Table[tt^(mm-i)*zz^i,{i,0,mm}];
LL=Coefficient[Expand[((Length[ll1]-1-rr)!*(Length[ll2]-1-rr)!*
 Sum[(-1)^i*Binomial[rr,i]*
 D[D[ll1.w1[Length[ll1]-1],{xx,rr-i}],{yy,i}]*
 D[D[ll2.w2[Length[ll2]-1],{tt,rr-i}],{zz,i}],{i,0,rr}])/
```

```
 ((Length[ll1]-1)!*(Length[ll2]-1)!)/.{zz->xx,tt->yy}]/.{yy->1},xx,#]
 &/@Range[0,Length[ll1]+Length[ll2]-2*(rr+1)];
pp=Length[LL];
If[pp==1,LL.{1},LL.Table[y^(pp-1-i)*x^i,{i,0,pp-1}]]];
```

Example:

```
In[]:=ff=a[0]x^3+3a[1]x^2y+3a[2]xy^2+a[3]y^3;
In[]:=TransvectantPoly[ff,ff,2]
Out[]=x^2(-2a[1]^2+2a[0]a[2])+ xy(-2a[1]a[2]+2a[0]a[3])+y^2(-2a[2]^2+2a[1]a[3])
```

The function `prod[lf_,lg_]` implemented in `Mathematica` [Wol10] returns the coefficient list of fg, where f is a binary form with the coefficient list `lf` and g is a binary form with the coefficient list `lg`:

```
<<Combinatorica`
prod[lf_, lg_] := Transvectant[lf, lg, 0]
```

Example:

```
In[]:=f={a[1],a[0]};
In[]:=g={b[2],2b[1],b[0]};
In[]:=prod[f,g]
Out[]={a[1]b[2],2a[1]b[1]+a[0]b[2],a[1]b[0]+2a[0]b[1],a[0]b[0]}
```

The function `exp[lf_,m_]` implemented in `Mathematica` [Wol10] returns the coefficient list of f^m, where f is a binary form with the coeffient list `lf`:

```
<<Combinatorica`
exp[lf_,2]:=prod[lf,lf];
exp[lf_,m_]:=Transvectant[ff,exp[ff,m-1],0]
```

Example:

```
In[]:=f={a[1],a[0]};
In[]:=exp[f,4]
Out[]={a[1]^4,4a[0]a[1]^3,6a[0]^2a[1]^2,4a[0]^3a[1],a[0]^4}
```

The function `poincare[listdeg_]` returns for listdeg=$\{n_1, \ldots, n_p\}$ the Poincaré series of the modules of covariants of $V_{n_1} \oplus \ldots \oplus V_{n_p}$ of order $\leq \prod_{i=1}^{p} \lfloor \frac{n_i+1}{2} \rfloor \lceil \frac{n_i+1}{2} \rceil - 2$ (in `Mathematica` [Wol10]):

```
<< Combinatorica`
poincare[listdeg_] := Module[{N,pp,qq,rr,LL,rules,matr,n,ww},
 N =Sum[Ceiling[(listdeg[[i]]+1)/2]*Floor[(listdeg[[i]]+1)/2],
      {i,1,Length[listdeg]}]-2;
 n =Sum[listdeg[[i]], {i, 1, Length[listdeg]}];
 matr = {};
 ww = Sort[Flatten[
     Table[Table[i,{i,-listdeg[[j]],0,2}],{j,1,Length[listdeg]}],1]];
 For[mu = 0, mu < N + 1, mu++,
  pp = Expand[e^mu*Product[1-te^ww[[i]],{i,1,Length[ww]}]];
  rules = Table[e^(-i)->-e^(i - 2),{i,2,4N}];
  qq = pp /. rules /. e^(-1)->0 /. e^(-2)->-e^0;
  rr = PadRight[CoefficientList[qq,e],N+1];
  matr = Flatten[{matr,{rr}},1]];
 LL = PadRight[{1}, N + 1];
 Simplify[Inverse[matr].LL]]
```

Example:

```
In[]:= poincare[{2, 3}]
Out[]= {(1-t+t^2-t^3+t^4-t^5+t^6)/(1-t-t^3+t^6+t^7-t^10-t^12+t^13),
  t^2/(1-t-t^2+t^4+t^7-t^9-t^10+t^11),
  t(1-t+t^2))/((-1+t)^4(1+2t+3t^2+4t^3+4t^4+3t^5+2t^6+t^7)),
  (t+t^3)/(1-t-t^2+t^4+t^7-t^9-t^10+t^11),
  (t^2+t^4-t^5)/((-1+t)^4(1+2t+3t^2+3t^3+2t^4+t^5))}
```

The function `CoefficientPoincare[k_,r_]` returns the dimension of the vector space of invariants of V_r of degree k (in `Mathematica` [Wol10]):

```
CoefficientPoincare[k_,r_]:=Module[{x,p},p=kr/2;
 If[EvenQ[k r]==True,Coefficient[Series[
   Product[(1-x^i),{i,k+1,k+r}]/Product[(1-x^i),{i,2,r}],{x,0,p + 1}],x,p],
   Print["There are no invariants of degree ",r,"!"]]]
```

Example:

```
In[]:= CoefficientPoincare[10, 8]
Out[]= 12
In[]:= CoefficientPoincare[10, 10]
Out[]= 24
```

The function `SymTensor[k,r]` returns the decomposition of the k-th symmetric power $S^k(V_r)$ of V_r as an SL$_2$-module (`Mathematica` [Wol10]):

```
SymTensor[k_,r_]:=Module[{x},Sum[Coefficient[Series[
 Product[(1-x^i),{i,k+1,k+r}]/Product[(1-x^i),{i,2,r}],{x,0,p+1}],x,p]*v[kr-2p],
 {p,0,IntegerPart[kr/2]}]];
```

Example:

```
In[]:=SymTensor[10, 8]
Out[]=12v[0]+10v[2]+32v[4]+30v[6]+51v[8]+48v[10]+66v[12]+61v[14]+77v[16]
  +70v[18]+83v[20]+74v[22]+84v[24]+74v[26]+80v[28]+69v[30]+74v[32]+62v[34]
  +64v[36]+53v[38]+54v[40]+43v[42]+43v[44]+33v[46]+33v[48]+25v[50]+24v[52]
  +17v[54]+17v[56]+11v[58]+11v[60]+7v[62]+7v[64]+4v[66]+4v[68]+2v[70]+2v[72]
  +v[74]+v[76]+v[80]
```

The function `Dim[n,N,listinv,listdeg]` takes as input `n`, a list `listinv` of invariants of V_n, and a list `listdeg` of the degrees of the elements of `listinv`, together with their multiplicities, and returns:

- the dimension of the vector space of invariants of V_n of degree `N`;

- the size of the vector space of invariants of degree `N` spanned by the elements of `listinv` of degree $<$`N`.

- a lower bound for the dimension of the subspace of invariants of degrees `N` spanned by the elements of `listinv` of degree $<$`N`.

- a lower bound for the dimension of the subspace of invariants of degree `N` spanned by the elements of `listinv` of degree $\leq$`N` (in the case `listinv` contains elements of degree `N`).

This has been implemented in `Mathematica` [Wol10] as follows:

```
<<Combinatorica`
Dim[n_,N_,listinv_,listdeg_]:=
Module[{old,inv,matr1,r1,matr2,r2,invariants,jj,listdeg1,p,invariantsN,MM,var,ss},
var=Variables[listinv[[1]]];ss=CoefficientPoincare[n,N];
If[ss==0,Print["There are no invariants of degree",N,"!"],
MM=Table[listinv/.Table[var[[j]]->Random[Integer,{-10,10}],{j,1,Length[var]}],
 {i,1,ss+1}];
If[Dimensions[MM][[1]]<ss,Print["<<The evaluation matrix is too small"],
listdeg1=Table[listdeg[[i]][[1]],{i,1,Length[listdeg]}];
invariants=Table[Table[jj[listdeg[[i]][[1]]][[j],{j,1,listdeg[[i]][[2]]}],{i,1,
 Length[listdeg]}];
inv=Flatten[invariants];old=listmonomials[N,invariants,listdeg1];
matr1=Table[old/.Table[inv[[j]]->MM[[i]][[j]],{j,1,Length[inv]}],{i,1,ss+1}];
r1=If[old=={},0,MatrixRank[matr1,Modulus->32003]];
p=0;For[i=1,i<Length[listdeg]+1,i++,If[listdeg[[i]][[1]]==N,p=i]];
If[p=!=0,invariantsN=invariants[[p]]];
Print["<<The dimension of O(V_",n,")^SL2_",N,"is",ss];
Print["<<There are",Length[old],"invariants of degree",N,
"spanned by the given invariants of degrees<",N];
Print["<<The subspace of O(V_",n,")^SL2_",N,"spanned by the given invariants
 of degrees<",N,"has","dimension>=",r1];
matr2=Table[Union[old,invariantsN]/.Table[inv[[j]]->MM[[i]][[j]],
 {j,1,Length[inv]}],{i,1,ss+1}];r2=MatrixRank[matr2,Modulus->32003];
Print["<<The subspace of O(V_",n,")^SL2_",N,"spanned by the given
  invariants of degrees<=",N,"has dimension >=",r2],
Print["<<The dimension of O(V_",n,")^SL2_",N,"is",ss];
Print["<<There are",Length[old],"invariants of degree",N,"spanned by
  the given invariants of degrees<=",listdeg[[Length[listdeg]]][[1]]];
Print["<<The subspace of O(V_",n,")^SL2_",N,"spanned by the given invariants
 of degrees<=",listdeg[[Length[listdeg]]][[1]],"has","dimension>=",r1];]]]]
```

Example:

```
In[]:=ff=Table[Binomial[3,i]*a[3-i],{i,0,3}];
 j4=Transvectant[Transvectant[ff,ff,2],Transvectant[ff,ff,2],2].{1};
In[]:=Dim[3,4,{j4},{{4,1}}]
 <<The dimension of O(V_3)^SL2_4 is 1
 <<There are 0 invariants of degree 4 spanned by the given
 invariants of degree <4
 <<The subspace of O(V_3)^SL2_4 spanned by the given invariants
 of degree <4 has dimension >=0
 <<The subspace of O(V_3)^SL2_4 spanned by the given invariants
 of degree <=4 has dimension >=1
In[]:= Dim[3,8,{j4},{{4,1}}]
 <<The dimension of O(V_3)^SL2_8 is 1
 <<There are 1 invariants of degree 8 spanned by the given
 invariants of degree <8
 <<The subspace of O(V_3)^SL2_8 spanned by the given invariants
 of degree <8 has dimension >=1
```

The function `Dim2[n,N,matr,listdeg]` takes as input `n`, a matrix `matr`, which contains the evaluations of a "input" set of invariants of V_n, and a list `listdeg` of the degrees of these invariants, together with their multiplicities, and returns:

- the dimension of the vector space of invariants of V_n of degree `N`;

- the size of the vector space of invariants of degrees N spanned by the input invariants of degrees <N.

- a lower bound for the dimension of the subspace of invariants of degrees N spanned by the input invariants of degrees <N.

- a lower bound for the dimension of the subspace of invariants of degrees N spanned by the input invariants of degrees ≤N.

This has been implemented in Mathematica [Wol10] as follows:

```
<<Combinatorica'
Dim2[n_,N_,MM_,listdeg_]:=
Module[{old,inv,matr1,r1,matr2,r2,invariants,jj,listdeg1,p,invariantsN,ee},
ee=Coefficient[SymTensor[n,N],v[0]];
If[Dimensions[MM][[1]]<ee,Print["<<Theevaluationmatrixistoosmall"],
 listdeg1=Table[listdeg[[i]][[1]],{i,1,Length[listdeg]}];
 invariants=Table[Table[jj[listdeg[[i]][[1]]][j],{j,1,listdeg[[i]][[2]]}],
              {i,1,Length[listdeg]}];
 inv=Flatten[invariants];old=listmonomials[N,invariants,listdeg1];
 matr1=Table[old/.Table[inv[[j]]->MM[[i]][[j]],{j,1,Length[inv]}],{i,1,ee}];
 r1=If[old=={},0,MatrixRank[matr1,Modulus->32003]];
 Print["<<The dimension of O(V_",n,")^SL2_",N,"is",ee];
 Print["<<The set of invariants of degree",N,"spanned by the given
   invariants of degrees <",N,"has size",Length[old]];
Print["<<The subspace of O(V_",n,")^SL2_",N,"spanned by the given
   invariants of degrees <",N,"has","dimension >=",r1];
p=0;For[i=1,i<Length[listdeg]+1,i++,If[listdeg[[i]][[1]]==N,p=i]];
If[p=!=0,invariantsN=invariants[[p]]];
matr2=Table[Union[old,invariantsN]/.Table[inv[[j]]->MM[[i]][[j]],
   {j,1,Length[inv]}],{i,1,ee}];
r2=MatrixRank[matr2,Modulus->32003];
Print["<<The subspace of O(V_",n,")^SL2_",N,"spanned by the given
   invariants of degrees <=",N,"has dimension >=",r2]]]]
```

Example:

```
In[]:=ff=Table[Binomial[3,i]*a[3-i],{i,0,3}];
In[]:=matr={};For[i=1,i<10,i++,
 j4=Transvectant[Transvectant[ff,ff,2],Transvectant[ff,ff,2],2].{1}/.
    Table[a[j]->Random[Integer,{-10,10}],{j,0,3}];matr=Append[matr,{j4}]];
In[]:Dimensions[matr]
Out[]={9,1}
In[]:=Dim2[3,4,matr,{{4,1}}]
 <<The dimension of O(V_3)^SL2_4 is 1
 <<The set of invariants of degree 4 spanned by the given
 invariants of degree <4 has size 0
 <<The subspace of O(V_3)^SL2_4 spanned by the given invariants
 of degree <4 has dimension >=0
 <<The subspace of O(V_3)^SL2_4 spanned by the given invariants
 of degree <=4 has dimension >=1
```

A.2   The invariants of the binary septic

Here are the Mathematica computations supporting the proof of Proposition 4.6.3. We first compute the invariants found in Proposition 4.6.3:

```
In[]:=Timing[ff=Table[Binomial[7,i]*a[7-i],{i,0,7}];
cc1=Transvectant[ff,ff,6]; cc2=Transvectant[ff,ff,4];
cc4=Transvectant[ff,cc1,2]; cc3=Transvectant[ff,ff,2];
cc5=Transvectant[cc2,cc2,4]; cc6=Transvectant[ff,cc2,5];
alpha=Transvectant[cc6,cc5,3];j4=Transvectant[cc1,cc1,2].{1};
j81=Transvectant[Transvectant[cc4,cc4,4],cc1,2].{1};
j82=Transvectant[cc5,cc5,4].{1};j83=Transvectant[cc5,exp[cc1,2],4].{1};
j121=Transvectant[Transvectant[cc5,cc5,2],cc5,4].{1};
j122=Transvectant[Transvectant[cc4,cc4,2],exp[cc1,3],6].{1};
j123=Transvectant[Transvectant[cc4,cc4,4],Transvectant[cc4,cc4,4],2].{1};
j124=Transvectant[exp[cc6,2],exp[cc1,3],6].{1};
j125=Transvectant[Transvectant[Transvectant[cc6,cc6,2],cc2,2],exp[cc1,2],4].{1};
j126=Transvectant[Transvectant[cc6,cc6,2],Transvectant[cc6,cc6,2],2].{1};
j141=Transvectant[prod[cc6,ff],exp[cc1,5],10].{1};
j142=Transvectant[Transvectant[cc2,cc5,1],exp[cc1,4],8].{1};
j143=Transvectant[Transvectant[Transvectant[cc6,cc6,2],cc2,1],exp[cc1,3],6].{1};
j144=Transvectant[prod[cc6,alpha],exp[cc1,2],4].{1};
j161=Transvectant[exp[cc4,2],exp[cc1,5],10].{1};
j162=Transvectant[exp[alpha,2],cc1,2].{1};
j181=Transvectant[Transvectant[cc3,cc2,1],exp[cc1,7],14].{1};
j182=Transvectant[prod[ff,Transvectant[cc2,cc6,2]],exp[cc1,6],12].{1};
j183=Transvectant[prod[cc6,Transvectant[cc2,ff,2]],exp[cc1,6],12].{1};
j184=Transvectant[prod[cc6,Transvectant[cc2,cc6,1]],exp[cc1,5],10].{1};
j185=Transvectant[prod[cc6,Transvectant[cc6,cc5,1]],exp[cc1,4],8].{1};
j186=Transvectant[prod[cc6,Transvectant[cc6,Transvectant[cc6,cc6,2],1]],
 exp[cc1,3],6].{1};
j187=Transvectant[prod[Transvectant[cc2,cc6,2],alpha],exp[cc1,3],6].{1};
j188=Transvectant[Transvectant[Transvectant[cc6,cc5,1],alpha,1],exp[cc1,2],4].{1};
j189=Transvectant[Transvectant[Transvectant[cc6,
 Transvectant[cc6,cc6,2],1],alpha,1],cc1,2].{1};
j20=Transvectant[Transvectant[cc5,cc5,2],exp[Transvectant[cc2,cc5,4],2],4].{1};
j221=Transvectant[prod[alpha,Transvectant[alpha,cc5,1]],exp[cc1,2],4].{1};
j222=Transvectant[cc1,prod[alpha,Transvectant[alpha,
 Transvectant[cc6,cc6,2],1]],2].{1};
j26=Transvectant[cc1,prod[alpha,Transvectant[alpha,
 Transvectant[alpha,cc6,1],1]],2].{1};
j30=Transvectant[Transvectant[cc6,Transvectant[cc6,cc6,2],1],exp[alpha,3],3].{1};]
Out[]={1466.9,Null} // the computation took 1466.9 seconds
```

Here are the rank computations:

```
In[]:=listinv={j4,j81,j82,j83,j121,j122,j123,j124,j125,j126,j141,j142,j143,j144,
 j161,j162,j181,j182,j183,j184,j185,j186,j187,j188,j189,j20,j221,j222,j26,j30};
In[]:=listdeg={{4,1},{8,3},{12,6},{14,4},{16,2},{18,9},{20,1},{22,2},{26,1},{30,1}};
In[]:=For[l=1,l<49,l++,pp=Timing[Dim[7,l,listinv,listdeg]];
 Print["Computation time: ",pp[[1]]];Print["=============="]]
Out[]=
There are no invariants of degree 1!
Computation time: 0.384024
==============
There are no invariants of degree 2!
Computation time: 0.396025
==============
There are no invariants of degree 3!
Computation time: 1.21608
==============
<<The dimension of O(V_7)^SL2_4 is 1
<<There are 0 invariants of degree 4 spanned by the given invariants
```

```
of degrees <4
<<The subspace of O(V_7)^SL2_4 spanned by the given invariants of
degrees <4 has dimension >=0
<<The subspace of O(V_7)^SL2_4 spanned by the given invariants of
degrees <=4 has dimension >=1
Computation time: 19.7492
==============
There are no invariants of degree 5!
Computation time: 0.400025
==============
There are no invariants of degree 6!
Computation time: 0.408025
==============
There are no invariants of degree 7!
Computation time: 1.21608
==============
<<The dimension of O(V_7)^SL2_8 is 4
<<There are 1 invariants of degree 8 spanned by the given invariants
of degrees <8
<<The subspace of O(V_7)^SL2_8 spanned by the given invariants of
degrees <8 has dimension >=1
<<The subspace of O(V_7)^SL2_8 spanned by the given invariants of
degrees <=8 has dimension >=4
Computation time: 46.0509
==============
There are no invariants of degree 9!
Computation time: 0.396024
==============
There are no invariants of degree 10!
Computation time: 0.408026
==============
There are no invariants of degree 11!
Computation time: 1.24808
==============
<<The dimension of O(V_7)^SL2_12 is 10
<<There are 4 invariants of degree 12 spanned by the given
invariants of degrees <12
<<The subspace of O(V_7)^SL2_12 spanned by the given invariants of
degrees <12 has dimension >=4
<<The subspace of O(V_7)^SL2_12 spanned by the given invariants of
degrees <=12 has dimension >=10
Computation time: 99.4462
==============
There are no invariants of degree 13!
Computation time: 0.416026
==============
<<The dimension of O(V_7)^SL2_14 is 4
<<There are 0 invariants of degree 14 spanned by the given
invariants of degrees <14
<<The subspace of O(V_7)^SL2_14 spanned by the given invariants of
degrees <14 has dimension >=0
<<The subspace of O(V_7)^SL2_14 spanned by the given invariants of
degrees <=14 has dimension >=4
Computation time: 45.2508
==============
There are no invariants of degree 15!
Computation time: 0.396025
```

```
==============
<<The dimension of O(V_7)^SL2_16 is 18
<<There are 16 invariants of degree 16 spanned by the given
invariants of degrees <16
<<The subspace of O(V_7)^SL2_16 spanned by the given invariants of
degrees <16 has dimension >=16
<<The subspace of O(V_7)^SL2_16 spanned by the given invariants of
degrees <=16 has dimension >=18
Computation time: 173.035
==============
There are no invariants of degree 17!
Computation time: 0.396025
==============
<<The dimension of O(V_7)^SL2_18 is 13
<<There are 4 invariants of degree 18 spanned by the given
invariants of degrees <18
<<The subspace of O(V_7)^SL2_18 spanned by the given invariants of
degrees <18 has dimension >=4
<<The subspace of O(V_7)^SL2_18 spanned by the given invariants of
degrees <=18 has dimension >=13
Computation time: 127.624
==============
There are no invariants of degree 19!
Computation time: 0.396025
==============
<<The dimension of O(V_7)^SL2_20 is 35
<<There are 36 invariants of degree 20 spanned by the given
invariants of degrees <20
<<The subspace of O(V_7)^SL2_20 spanned by the given invariants of
degrees <20 has dimension >=34
<<The subspace of O(V_7)^SL2_20 spanned by the given invariants of
degrees <=20 has dimension >=35
Computation time: 320.972
==============
There are no invariants of degree 21!
Computation time: 0.400025
==============
<<The dimension of O(V_7)^SL2_22 is 26
<<There are 25 invariants of degree 22 spanned by the given
invariants of degrees <22
<<The subspace of O(V_7)^SL2_22 spanned by the given invariants of
degrees <22 has dimension >=24
<<The subspace of O(V_7)^SL2_22 spanned by the given invariants of
degrees <=22 has dimension >=26
Computation time: 247.499
==============
There are no invariants of degree 23!
Computation time: 0.392025
==============
<<The dimension of O(V_7)^SL2_24 is 62
<<There are 74 invariants of degree 24 spanned by the given
invariants of degrees <24
<<The subspace of O(V_7)^SL2_24 spanned by the given invariants of
degrees <24 has dimension >=62
Computation time: 569.244
==============
There are no invariants of degree 25!
```

```
Computation time: 0.392025
==============
<<The dimension of O(V_7)^SL2_26 is 52
<<There are 78 invariants of degree 26 spanned by the given
invariants of degrees <26
<<The subspace of O(V_7)^SL2_26 spanned by the given invariants of
degrees <26 has dimension >=51
<<The subspace of O(V_7)^SL2_26 spanned by the given invariants of
degrees <=26 has dimension >=52
Computation time: 478.346
==============
==============
<<The dimension of O(V_7)^SL2_28 is 97
<<There are 135 invariants of degree 28 spanned by the given
invariants of degrees <28
<<The subspace of O(V_7)^SL2_28 spanned by the given invariants of
degrees <28 has dimension >=97
Computation time: 884.471
==============
==============
<<The dimension of O(V_7)^SL2_30 is 92
<<There are 171 invariants of degree 30 spanned by the given
invariants of degrees <30
<<The subspace of O(V_7)^SL2_30 spanned by the given invariants of
degrees <30 has dimension >=91
<<The subspace of O(V_7)^SL2_30 spanned by the given invariants of
degrees <=30 has dimension >=92
Computation time: 832.364
==============
==============
<<The dimension of O(V_7)^SL2_32 is 153
<<There are 270 invariants of degree 32 spanned by the given
invariants of degrees <32
<<The subspace of O(V_7)^SL2_32 spanned by the given invariants of
degrees <32 has dimension >=153
Computation time: 1392.58
==============
==============
<<The dimension of O(V_7)^SL2_34 is 144
<<There are 335 invariants of degree 34 spanned by the given
invariants of degrees <34
<<The subspace of O(V_7)^SL2_34 spanned by the given invariants of
degrees <34 has dimension >=144
Computation time: 1304.78
==============
==============
<<The dimension of O(V_7)^SL2_36 is 229
<<There are 513 invariants of degree 36 spanned by the given
invariants of degrees <36
<<The subspace of O(V_7)^SL2_36 spanned by the given invariants of
degrees <36 has dimension >=229
Computation time: 2080.95
==============
==============
<<The dimension of O(V_7)^SL2_38 is 223
<<There are 679 invariants of degree 38 spanned by the given
invariants of degrees <38
```

```
<<The subspace of O(V_7)^SL2_38 spanned by the given invariants of
degrees <38 has dimension >=223
Computation time: 2015.41
==============
==============
<<The dimension of O(V_7)^SL2_40 is 325
<<There are 940 invariants of degree 40 spanned by the given
invariants of degrees <40
<<The subspace of O(V_7)^SL2_40 spanned by the given invariants of
degrees <40 has dimension >=325
Computation time: 2927.34
==============
==============
<<The dimension of O(V_7)^SL2_48 is 624
<<There are 3262 invariants of degree 48 spanned by the given
invariants of degrees <48
<<The subspace of O(V_7)^SL2_48 spanned by the given invariants of
degrees <48 has dimension >=624
Computation time: 5661.69
```

A.3   The invariants of the binary nonic

Here are the computations supporting the proof of Proposition 4.8.6. We first write a matrix MM with 4000 evaluations at random points in V_9 of the 92 invariants found in Proposition 4.8.6:

```
In[]:=MM={}; For[p=1,p<4001,p++,
ff=Table[Binomial[9,i]*a[9-i],{i,0,9}]/.{
 a[9]->Random[Integer,{-10,10}],a[8]->Random[Integer,{-10,10}],
 a[7]->Random[Integer,{-10,10}],a[6]->Random[Integer,{-10,10}],
 a[5]->Random[Integer,{-10,10}],a[4]->Random[Integer,{-10,10}],
 a[3]->Random[Integer,{-10,10}],a[2]->Random[Integer,{-10,10}],
 a[1]->Random[Integer,{-10,10}],a[0]->Random[Integer,{-10,10}]}};
cc1=Transvectant[ff,ff,8];cc2=Transvectant[ff,ff,6];
cc3=Transvectant[ff,ff,4];cc4=Transvectant[ff,ff,2];
cc5=Transvectant[ff,cc1,2];cc6=Transvectant[ff,cc2,6];
cc7=Transvectant[cc2,cc2,4];cc8=Transvectant[cc5,cc5,6];
cc9=Transvectant[cc5,cc5,4];cc10=Transvectant[cc5,cc8,2];
cc11=Transvectant[cc9,cc9,4];cc12=Transvectant[cc9,cc11,4];
cc13=Transvectant[cc2,cc7,4];cc14=Transvectant[cc6,cc6,2];
cc15=Transvectant[cc2,cc4,6];cc16=Transvectant[cc2,cc4,4];
cc17=Transvectant[cc2,cc3,6];cc18=Transvectant[cc5,cc6,3];
cc19=Transvectant[cc1,cc3,2];cc20=Transvectant[ff,cc1,2];
cc21=Transvectant[ff,cc2,2];cc22=Transvectant[ff,cc3,6];
cc23=Transvectant[ff,cc3,8];cc24=Transvectant[ff,cc4,8];
cc25=Transvectant[cc4,cc4,10];cc26=Transvectant[cc3,cc3,6];
cc27=Transvectant[exp[cc6,3],cc6,3];cc28=Transvectant[cc3,cc4,10];
cc29=Transvectant[cc2,cc4,2];cc30=Transvectant[cc3,cc4,8];
j41=Transvectant[cc1,cc1,2].{1};j42=Transvectant[cc2,cc2,6].{1};
j82=Transvectant[cc7,cc7,4].{1};j83=Transvectant[cc8,cc1,2].{1};
j81=Transvectant[cc2,exp[cc6,2],6].{1};
j84=Transvectant[cc7,exp[cc1,2],4].{1};
j85=Transvectant[cc7,cc17,4].{1};j102=Transvectant[cc18,cc7,4].{1};
j103=Transvectant[cc18,exp[cc1,2],4].{1};
j104=Transvectant[Transvectant[cc6,cc22,3],Transvectant[cc3,cc3,8],4].{1};
```

```
j105=Transvectant[Transvectant[Transvectant[cc26,ff,6],cc20,3],cc2,6].{1};
j101=Transvectant[Transvectant[Transvectant[cc25,ff,6],cc21,5],cc2,6].{1};
j121=Transvectant[Transvectant[cc7,cc7,2],cc7,4].{1};
j122=Transvectant[cc9,exp[cc1,3],6].{1};j123=Transvectant[cc14,cc14,2].{1};
j124=Transvectant[Transvectant[exp[cc2,2],cc2,6],exp[cc6,2],6].{1};
j125=Transvectant[cc8,cc8,2].{1};j126=Transvectant[cc8,cc13,2].{1};
j127=Transvectant[Transvectant[cc7,cc2,2],Transvectant[cc7,cc3,4],6].{1};
j128=Transvectant[Transvectant[cc7,cc3,2],Transvectant[cc7,cc4,4],10].{1};
j129=Transvectant[cc14,cc8,2].{1};j1210=Transvectant[cc14,cc13,2].{1};
j1211=Transvectant[exp[cc6,2],Transvectant[cc7,cc3,4],6].{1};
j1212=Transvectant[prod[cc6,cc5],Transvectant[cc7,cc3,2],10].{1};
j1213=Transvectant[Transvectant[cc6,cc5,2],Transvectant[cc7,cc2,2],6].{1};
j1214=Transvectant[Transvectant[cc8,cc4,2],exp[cc2,2],12].{1};
j141=Transvectant[cc2,cc27,6].{1};
j142=Transvectant[cc2,Transvectant[prod[cc28,cc17],prod[cc1,cc2],5],6].{1};
j143=Transvectant[cc4,Transvectant[prod[cc15,cc19],prod[cc1,cc4],9],14].{1};
j144=Transvectant[cc1,Transvectant[prod[cc15,cc19],prod[cc1,cc4],15],2].{1};
j145=Transvectant[cc2,Transvectant[prod[cc15,cc19],prod[cc1,cc4],13],6].{1};
j146=Transvectant[cc3,Transvectant[prod[cc15,cc19],prod[cc1,cc4],11],10].{1};
j147=Transvectant[cc1,Transvectant[prod[Transvectant[cc3,cc4,10],cc17],
 prod[cc1,cc2],7],2].{1};
j148=Transvectant[cc3, Transvectant[prod[cc28,cc17],prod[cc1,cc2],3],10].{1};
j149=Transvectant[cc4,Transvectant[prod[cc28,cc17],prod[cc1,cc2],1],14].{1};
j1410=Transvectant[Transvectant[Transvectant[cc16,cc19,6],cc2,5],
 exp[cc1,2],4].{1};
j1411=Transvectant[Transvectant[Transvectant[cc16,cc19,4],cc1,1],
 exp[cc2,2],12].{1};
j1412=Transvectant[Transvectant[Transvectant[cc16,cc19,2],cc2,1],
 exp[cc3,2],20].{1};
j1413=Transvectant[Transvectant[prod[cc29,cc19],cc2,1],exp[cc4,2],28].{1};
j1414=Transvectant[Transvectant[cc16,cc19,7],exp[cc1,3],6].{1};
j1415=Transvectant[Transvectant[cc16,cc19,1],exp[cc2,3],18].{1};
j1416=Transvectant[Transvectant[prod[cc2,cc4],prod[cc1,cc3],1],exp[cc3,3],30].{1};
j1417=Transvectant[Transvectant[cc29,cc3,1],exp[cc2,4],24].{1};
j161=Transvectant[Transvectant[cc5,cc5,2],exp[cc1,5],10].{1};
j162=Transvectant[prod[cc7,cc8],exp[cc6,2],6].{1};
j163=Transvectant[exp[cc7,3],Transvectant[cc1,cc4,2],12].{1};
j164=Transvectant[exp[cc7,3],Transvectant[cc2,cc3,2],12].{1};
j165=Transvectant[exp[cc6,4],Transvectant[cc2,cc3,2],12].{1};
j166=Transvectant[exp[cc6,4],Transvectant[cc1,cc4,2],12].{1};
j167=Transvectant[prod[exp[cc1,4],cc2],exp[cc5,2],14].{1};
j168=Transvectant[exp[cc1,4],Transvectant[prod[cc2,cc3],prod[cc2,cc1],8],8].{1};
j169=Transvectant[exp[cc1,4],Transvectant[prod[cc2,cc3],prod[cc3,cc4],16],8].{1};
j1610=Transvectant[exp[cc1,4],Transvectant[exp[cc4,2],prod[cc2,cc4],20],8].{1};
j1611=Transvectant[exp[cc7,2],Transvectant[exp[cc4,2],prod[cc2,cc4],20],8].{1};
j1612=Transvectant[exp[cc7,2],Transvectant[prod[cc3,cc4],prod[cc2,cc3],16],8].{1};
j1613=Transvectant[exp[cc7,2],Transvectant[prod[cc3,cc2],prod[cc2,cc1],8],8].{1};
j1614=Transvectant[exp[cc7,2], Transvectant[cc16,Transvectant[cc3,cc1,2],6],8].{1};
j1615=Transvectant[exp[cc1,4],Transvectant[cc16,Transvectant[cc3,cc1,2],6],8].{1};
j1616=Transvectant[exp[cc1,4],Transvectant[cc30,Transvectant[cc3,cc2,4],4],8].{1};
j1617=Transvectant[exp[cc7,2],Transvectant[cc30,Transvectant[cc3,cc2,4],4],8].{1};
j1618=Transvectant[exp[cc7,2],Transvectant[cc29,cc19,8],8].{1};
j1619=Transvectant[exp[cc1,4],Transvectant[cc29,cc19,8],8].{1};
j1620=Transvectant[exp[cc2,4],prod[cc29,cc19],24].{1};
j1621=Transvectant[prod[exp[cc2,2],exp[cc1,2]],Transvectant[cc16,cc19,2],16].{1};
j181=Transvectant[Transvectant[Transvectant[cc2,cc2,2],cc2,1],exp[cc6,4],12].{1};
j182=Transvectant[exp[cc8,2],cc18,4].{1};
```

```
j183=Transvectant[Transvectant[cc7,prod[cc1,cc8],2],cc18,4].{1};
j184=Transvectant[exp[cc7,2],Transvectant[cc30,Transvectant[cc3,
 Transvectant[ff,Transvectant[ff,cc2,3],6],4],4],8].{1};
j185=Transvectant[exp[cc1,2],Transvectant[Transvectant[cc29,cc3,1],
 exp[cc2,4],22],4].{1};
j186=Transvectant[exp[cc1,2],Transvectant[Transvectant[
 prod[cc2,cc4],prod[cc1,cc3],1],exp[cc3,3],28],4].{1};
j187=Transvectant[exp[cc1,2],Transvectant[
 Transvectant[cc16,cc19,1],exp[cc2,3],16],4].{1};
j188=Transvectant[prod[cc1,cc2],Transvectant[
 Transvectant[cc16,cc19,7],exp[cc1,3],2],8].{1};
j189=Transvectant[exp[cc1,2],Transvectant[
 Transvectant[prod[cc29,cc19],cc2,1],exp[cc4,2],26],4].{1};
j1810=Transvectant[exp[cc1,2],Transvectant[Transvectant[
 Transvectant[cc16,cc19,2],cc2,1],exp[cc3,2],18],4].{1};
j1811=Transvectant[exp[cc1,2],Transvectant[Transvectant[
 Transvectant[cc16,cc19,4],cc1,1],exp[cc2,2],10],4].{1};
j1812=Transvectant[prod[cc1,cc2],prod[Transvectant[
 Transvectant[cc16,cc19,6],cc2,5],exp[cc1,2]],8].{1};
j1813=Transvectant[exp[cc1,2],Transvectant[cc4,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],1],12],4].{1};
j1814=Transvectant[exp[cc1,2],Transvectant[cc3,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],3],8],4].{1};
j1815=Transvectant[exp[cc1,2],Transvectant[cc2,
 Transvectant[prod[cc15,cc19],prod[cc1,cc4],13],4],4].{1};
j1816=Transvectant[prod[cc1,cc2],Transvectant[cc2,
 Transvectant[prod[cc15,cc19],prod[cc1,cc4],13],2],8].{1};
j1817=Transvectant[exp[cc1,2],Transvectant[cc4,
 Transvectant[prod[cc15,cc19],prod[cc1,cc4],9],12],4].{1};
j1818=Transvectant[exp[cc1,2],Transvectant[cc2,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],5],4],4].{1};
j1819=Transvectant[prod[cc1,cc2],Transvectant[cc2,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],5],2],8].{1};
j1820=Transvectant[prod[cc1,cc2],Transvectant[cc4,
 Transvectant[prod[cc15,cc19],prod[cc1,cc4],9],10],8].{1};
j1821=Transvectant[prod[cc1,cc2],Transvectant[cc3,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],3],6],8].{1};
j1822=Transvectant[prod[cc1,cc2],Transvectant[cc4,
 Transvectant[prod[cc28,cc17],prod[cc1,cc2],1],10],8].{1};
j1823=Transvectant[exp[cc14,2],cc18,4].{1};
j1824=Transvectant[prod[cc2,cc3],Transvectant[Transvectant[
 Transvectant[cc16,cc19,2],cc2,1],exp[cc3,2],12],16].{1};
j1825=Transvectant[prod[cc2,cc3],Transvectant[
 Transvectant[prod[cc29,cc19],cc2,1],exp[cc4,2],20],16].{1};
j201=Transvectant[cc2,prod[exp[cc14,2],cc14],6].{1};
j202=Transvectant[Transvectant[cc2,cc27,4],Transvectant[cc24,cc23,3],4].{1};
j203=Transvectant[exp[cc13,2],Transvectant[cc7,cc7,2],4].{1};
j204=Transvectant[exp[cc5,2],exp[cc1,7],14].{1};
j22=Transvectant[exp[cc1,2],Transvectant[prod[cc2,cc3],Transvectant[
 Transvectant[prod[cc29,cc19],cc2,1],exp[cc4,2],20],14],4].{1};
MM=Append[MM,{j41,j42,j81,j82,j83,j84,j85,j101,j102,j103,j104,j105,j121,j122,
j123,j124,j125,j126,j127,j128,j129,j1210,j1211,j1212,j1213,j1214,j141,j142,
j143,j144,j145,j146,j147,j148,j149,j1410,j1411,j1412,j1413,j1414,j1415,j1416,
j1417,j161,j162,j163,j164,j165,j166,j167,j168,j169,j1610,j1611,j1612,j1613,
j1614,j1615,j1616,j1617,j1618,j1619,j1620,j1621,j181,j182,j183,j184,j185,j186,
j187,j188,j189,j1810,j1811,j1812,j1813,j1814,j1815,j1816,j1817,j1818,j1819,
j1820, j1821,j1822,j1823,j1824,j1825,j201,j202,j203,j204,j22}]]
```

Here are the rank computations:

```
In[]:=listdeg={{4,2},{8,5},{10,5},{12,14},{14,17},{16,21},{18,25},{20,2},{22,1}};
In[]:=For[l = 1,l<31,l++,pp = Timing[Dim2[9,l,MM,listdeg]];
 Print["Computation time: ",pp[[1]]]; Print["=============="]]
Out[]=
There are no invariants of degree 1!
Computation time: 0.
==============
There are no invariants of degree 2!
Computation time: 0.004001
==============
There are no invariants of degree 3!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_4 is 2
<<The set of invariants of degree 4 spanned by the given invariants
of degrees <4 has size 0
<<The subspace of O(V_9)^SL2_4 spanned by the given invariants of
degrees <4 has dimension >=0
<<The subspace of O(V_9)^SL2_4 spanned by the given invariants of
degrees <=4 has dimension >=2
Computation time: 0.008001
==============
There are no invariants of degree 5!
Computation time: 0.
==============
There are no invariants of degree 6!
Computation time: 0.008001
==============
There are no invariants of degree 7!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_8 is 8
<<The set of invariants of degree 8 spanned by the given invariants
of degrees <8 has size 3
<<The subspace of O(V_9)^SL2_8 spanned by the given invariants of
degrees <8 has dimension >=3
<<The subspace of O(V_9)^SL2_8 spanned by the given invariants of
degrees <=8 has dimension >=8
Computation time: 0.024002
==============
There are no invariants of degree 9!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_10 is 5
<<The set of invariants of degree 10 spanned by the given invariants
of degrees <10 has size 0
<<The subspace of O(V_9)^SL2_10 spanned by the given invariants of
degrees <10 has dimension >=0
<<The subspace of O(V_9)^SL2_10 spanned by the given invariants of
degrees <=10 has dimension >=5
Computation time: 0.028002
==============
There are no invariants of degree 11!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_12 is 28
```

```
<<The set of invariants of degree 12 spanned by the given invariants
of degrees <12 has size 14
<<The subspace of O(V_9)^SL2_12 spanned by the given invariants of
degrees <12 has dimension >=14
<<The subspace of O(V_9)^SL2_12 spanned by the given invariants of
degrees <=12 has dimension >=28
Computation time: 0.060003
==============
There are no invariants of degree 13!
Computation time: 0.004
==============
<<The dimension of O(V_9)^SL2_14 is 27
<<The set of invariants of degree 14 spanned by the given invariants
of degrees <14 has size 10
<<The subspace of O(V_9)^SL2_14 spanned by the given invariants of
degrees <14 has dimension >=10
<<The subspace of O(V_9)^SL2_14 spanned by the given invariants of
degrees <=14 has dimension >=27
Computation time: 0.064005
==============
There are no invariants of degree 15!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_16 is 84
<<The set of invariants of degree 16 spanned by the given invariants
of degrees <16 has size 63
<<The subspace of O(V_9)^SL2_16 spanned by the given invariants of
degrees <16 has dimension >=63
<<The subspace of O(V_9)^SL2_16 spanned by the given invariants of
degrees <=16 has dimension >=84
Computation time: 0.296018
==============
There are no invariants of degree 17!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_18 is 99
<<The set of invariants of degree 18 spanned by the given invariants
of degrees <18 has size 74
<<The subspace of O(V_9)^SL2_18 spanned by the given invariants of
degrees <18 has dimension >=74
<<The subspace of O(V_9)^SL2_18 spanned by the given invariants of
degrees <=18 has dimension >=99
Computation time: 0.428027
==============
There are no invariants of degree 19!
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_20 is 217
<<The set of invariants of degree 20 spanned by the given invariants
of degrees <20 has size 225
<<The subspace of O(V_9)^SL2_20 spanned by the given invariants of
degrees <20 has dimension >=215
<<The subspace of O(V_9)^SL2_20 spanned by the given invariants of
degrees <=20 has dimension >=217
Computation time: 2.28014
==============
There are no invariants of degree 21!
```

```
Computation time: 0.
==============
<<The dimension of O(V_9)^SL2_22 is 273
<<The set of invariants of degree 22 spanned by the given invariants
of degrees <22 has size 326
<<The subspace of O(V_9)^SL2_22 spanned by the given invariants of
degrees <22 has dimension >=272
<<The subspace of O(V_9)^SL2_22 spanned by the given invariants of
degrees <=22 has dimension >=273
Computation time: 5.63635
=============
==============
<<The dimension of O(V_9)^SL2_24 is 506
<<The set of invariants of degree 24 spanned by the given invariants
of degrees <24 has size 700
<<The subspace of O(V_9)^SL2_24 spanned by the given invariants of
degrees <24 has dimension >=506
Computation time: 9.88062
==============
==============
<<The dimension of O(V_9)^SL2_26 is 647
<<The set of invariants of degree 26 spanned by the given invariants
of degrees <26 has size 1098
<<The subspace of O(V_9)^SL2_26 spanned by the given invariants of
degrees <26 has dimension >=647
Computation time: 21.7054
=============
==============
<<The dimension of O(V_9)^SL2_28 is 1066
<<The set of invariants of degree 28 spanned by the given invariants
of degrees <28 has size 2040
<<The subspace of O(V_9)^SL2_28 spanned by the given invariants of
degrees <28 has dimension >=1066
Computation time: 74.5487
==============
==============
<<The dimension of O(V_9)^SL2_30 is 1367
<<The set of invariants of degree 30 spanned by the given invariants
of degrees <30 has size 3231
<<The subspace of O(V_9)^SL2_30 spanned by the given invariants of
degrees <30 has dimension >=1367
Computation time: 173.235
==============
==============
<<The dimension of O(V_9)^SL2_32 is 2082
<<The set of invariants of degree 32 spanned by the given invariants
of degrees <32 has size 5614
<<The subspace of O(V_9)^SL2_32 spanned by the given invariants of
degrees <32 has dimension >=2082
Computation time: 563.463
```

A.4   The invariants of the binary decimic

Here are the computations supporting the proof of Proposition 4.9.3. We first write a matrix MM with 3000 evaluations at random points in V_{10} of the 106

generating invariants from Proposition 4.9.3:

```
Timing[MM={};For[p=1,p<3001,p++,ff=Table[Binomial[10,i]*a[10-i],{i,0,10}]]/.
 Table[a[i]->Random[Integer,{-4,4}],{i,0,10}]];
cc1=Transvectant[ff,ff,8];cc2=Transvectant[ff,ff,6];
cc3=Transvectant[ff,ff,4];cc4=Transvectant[ff,ff,2];
cc5=Transvectant[ff,cc1,4];cc6=Transvectant[ff,cc2,8];
cc7=Transvectant[cc2,cc2,6];cc8=Transvectant[cc5,cc5,4];
cc9=Transvectant[cc2,cc7,4];cc10=Transvectant[cc1,cc1,2];
cc11=Transvectant[cc4,cc4,14];cc12=Transvectant[cc3,cc3,10];
cc13=Transvectant[cc10,cc1,2];cc14=Transvectant[cc8,cc5,4];
cc15=Transvectant[cc2,cc2,4];cc16=Transvectant[cc5,cc5,2];
cc17=Transvectant[cc4,cc2,4];cc18=Transvectant[cc4,cc2,2];
cc19=Transvectant[cc5,cc1,1];cc20=Transvectant[cc5,cc3,1];
cc21=Transvectant[cc5,cc2,1];cc22=Transvectant[cc1,cc2,2];
cc23=Transvectant[cc1,cc4,4];cc24=Transvectant[cc7,cc9,2];
cc25=Transvectant[cc7,cc7,2];j2=Transvectant[ff,ff,10].{1};
j4=Transvectant[cc1,cc1,4].{1};j61=Transvectant[cc5,cc5,6].{1};
j62=Transvectant[cc10,cc1,4].{1};j63=Transvectant[cc15,cc2,8].{1};
j64=Transvectant[cc6,cc6,2].{1};j81=Transvectant[cc8,cc1,4].{1};
j82=Transvectant[cc7,cc7,4].{1};j83=Transvectant[cc12,cc7,4].{1};
j84=Transvectant[cc10,cc11,4].{1};j85=Transvectant[cc11,cc11,4].{1};
j91=Transvectant[cc19,exp[cc1,2],8].{1};j92=Transvectant[cc20,exp[cc2,2],16].{1};
j93=Transvectant[cc21,prod[cc2,cc1],12].{1};
j94=Transvectant[Transvectant[cc5,cc4,1],prod[cc2,cc3],20].{1};
j95=Transvectant[Transvectant[cc6,cc4,1],exp[cc2,2],16].{1};
j101=Transvectant[cc16,exp[cc1,2],8].{1};j102=Transvectant[cc9,cc7,4].{1};
j103=Transvectant[cc7,exp[cc6,2],4].{1};j104=Transvectant[cc9,cc12,4].{1};
j105=Transvectant[Transvectant[cc23,cc22,6],cc2,8].{1};
j106=Transvectant[prod[cc8,cc5],ff,10].{1};
j107=Transvectant[Transvectant[Transvectant[cc1,cc3,2],cc17,12],cc1,4].{1};
j108=Transvectant[Transvectant[Transvectant[cc1,cc3,2],cc18,8],cc4,16].{1};
j111=Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,8].{1};
j112=Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,12].{1};
j113=Transvectant[prod[cc19,exp[cc1,2]],cc4,16].{1};
j114=Transvectant[Transvectant[cc20,exp[cc2,2],14],cc1,4].{1};
j115=Transvectant[Transvectant[cc20,exp[cc2,2],12],cc2,8].{1};
j116=Transvectant[Transvectant[cc20,exp[cc2,2],10],cc3,12].{1};
j117=Transvectant[Transvectant[cc20,exp[cc2,2],8],cc4,16].{1};
j118=Transvectant[Transvectant[cc21,prod[cc2,cc1],10],cc1,4].{1};
j121=Transvectant[cc8,cc8,4].{1};j122=Transvectant[exp[cc5,2],exp[cc1,3],12].{1};
j123=Transvectant[cc25,cc7,4].{1};
j124=Transvectant[Transvectant[cc16,exp[cc1,2],6],cc1,4].{1};
j125=Transvectant[cc24,cc1,4].{1};
j126=Transvectant[Transvectant[cc16,exp[cc1,2],4],cc2,8].{1};
j127=Transvectant[Transvectant[cc16,exp[cc1,2],2],cc3,12].{1};
j128=Transvectant[prod[cc16,exp[cc1,2]],cc4,16].{1};
j129=Transvectant[Transvectant[cc7,exp[cc6,2],2],cc1,4].{1};
j1210=Transvectant[prod[cc7,exp[cc6,2]],cc2,8].{1};
j1211=Transvectant[Transvectant[cc9,cc12,2],cc1,4].{1};
j1212=Transvectant[Transvectant[Transvectant[cc23,cc22,6],cc2,6],cc1,4].{1};
j131=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,6],cc1,4].{1};
j132=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,4],cc2,8].{1};
j133=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,2],cc3,12].{1};
j134=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,10],cc1,4].{1};
j135=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,8],cc2,8].{1};
j136=Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,6],cc3,12].{1};
j137=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],14],cc1,2],cc1,4].{1};
```

```
j138=Transvectant[prod[Transvectant[cc20,exp[cc2,2],14],cc1],cc2,8].{1};
j139=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],12],cc2,6],cc1,4].{1};
j1310=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],12],cc2,4],cc2,8].{1};
j1311=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],12],cc2,2],cc3,12].{1};
j1312=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],10],cc3,10],cc1,4].{1};
j1313=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],10],cc3,8],cc2,8].{1};
j1314=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],10],cc3,6],cc3,12].{1};
j1315=Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],8],cc4,14],cc1,4].{1};
j141=Transvectant[cc25,cc9,4].{1};j142=Transvectant[exp[cc10,2],cc16,8].{1};
j143=Transvectant[Transvectant[cc8,cc8,2],cc1,4].{1};
j144=Transvectant[exp[cc8,2],cc2,8].{1};
j145=Transvectant[Transvectant[exp[cc5,2],exp[cc1,3],10],cc1,4].{1};
j146=Transvectant[Transvectant[exp[cc5,2],exp[cc1,3],8],cc2,8].{1};
j147=Transvectant[Transvectant[exp[cc5,2],exp[cc1,3],6],cc3,12].{1};
j148=Transvectant[Transvectant[exp[cc5,2],exp[cc1,3],4],cc4,16].{1};
j149=Transvectant[prod[Transvectant[cc16,exp[cc1,2],6],cc1],cc2,8].{1};
j1410=Transvectant[Transvectant[cc24,cc1,2],cc1,4].{1};
j1411=Transvectant[prod[cc24,cc1],cc2,8].{1};
j1412=Transvectant[Transvectant[Transvectant[cc16,exp[cc1,2],4],cc2,6],cc1,4].{1};
j1413=Transvectant[Transvectant[Transvectant[cc16,exp[cc1,2],4],cc2,4],cc2,8].{1};
j151=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,6],
 cc1,2],cc1,4].{1};
j152=Transvectant[prod[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,6],cc1],
  cc2,8].{1};
j153=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,4],
 cc2,6],cc1,4].{1};
j154=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,4],
 cc2,4],cc2,8].{1};
j155=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,4],
 cc2,2],cc3,12].{1};
j156=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,2],
 cc3,10],cc1,4].{1};
j157=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,2],
 cc3,8],cc2,8].{1};
j158=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],cc2,2],
 cc3,6],cc3,12].{1};
j159=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,10],
 cc1,2],cc1,4].{1};
j1510=Transvectant[prod[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,10],cc1],
 cc2,8].{1};
j1511=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,8],
 cc2,6],cc1,4].{1};
j1512=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,8],
 cc2,4],cc2,8].{1};
j1513=Transvectant[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],2],cc3,8],
 cc2,2],cc3,12].{1};
j1514=Transvectant[prod[Transvectant[Transvectant[cc20,exp[cc2,2],14],cc1,2],cc1],
 cc2,8].{1};
j1515=Transvectant[Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],12],cc2,6],
 cc1,2],cc1,4].{1};
j1516=Transvectant[Transvectant[prod[Transvectant[cc20,exp[cc2,2],14],cc1],cc2,6],
 cc1,4].{1};
j1517=Transvectant[Transvectant[prod[Transvectant[cc20,exp[cc2,2],14],cc1],cc2,4],
cc2,8].{1};
j1518=Transvectant[Transvectant[prod[Transvectant[cc20,exp[cc2,2],14],cc1],cc2,
2],cc3,12].{1};
j1519=Transvectant[Transvectant[Transvectant[Transvectant[cc20,exp[cc2,2],12],
```

```
 cc2,2],cc3,8],cc2,8].{1};
j161=Transvectant[Transvectant[cc25,cc9,2],cc1,4].{1};
j162=Transvectant[prod[cc25,cc9],cc2,8].{1};
j163=Transvectant[Transvectant[Transvectant[cc8,cc8,2],cc1,2],cc1,4].{1};
j164=Transvectant[Transvectant[exp[cc8,2],cc2,6],cc1,4].{1};
j165=Transvectant[Transvectant[Transvectant[exp[cc5,2],exp[cc1,3],8],cc2,6],
 cc1,4].{1};
j171=Transvectant[Transvectant[prod[Transvectant[Transvectant[cc19,exp[cc1,2],4],
 cc2,6],cc1],cc2,6],cc1,4].{1};
j172=Transvectant[Transvectant[Transvectant[Transvectant[Transvectant[cc19,
 exp[cc1,2],4],cc2,4],cc2,2],cc3,10],cc1,4].{1};
j173=Transvectant[Transvectant[Transvectant[Transvectant[Transvectant[cc19,
 exp[cc1,2],4],cc2,2],cc3,10],cc1,2],cc1,4].{1};
j174=Transvectant[Transvectant[Transvectant[Transvectant[Transvectant[cc20,
 exp[cc2,2],12],cc2,2],cc3,8],cc2,4],cc2,8].{1};
j175=Transvectant[Transvectant[Transvectant[prod[Transvectant[cc20,exp[cc2,2],14],
 cc1],cc2,2],cc3,8],cc2,8].{1};
j18=Transvectant[prod[Transvectant[exp[cc5,2],exp[cc1,3],4],cc4],Transvectant[
 exp[ff,2],exp[ff,2],4],32].{1};
j191=Transvectant[prod[Transvectant[Transvectant[Transvectant[cc19,exp[cc1,2],4],
 cc2,2],cc3,6],cc3],Transvectant[exp[ff,2],exp[ff,2],8],24].{1};
j192=Transvectant[prod[cc24,Transvectant[ff,Transvectant[Transvectant[ff,
 Transvectant[exp[ff,2],cc4,8],5],ff,10],8]],cc2,8].{1};
j211=Transvectant[Transvectant[cc25,ff,1],exp[cc7,3],12].{1};
j212=Transvectant[Transvectant[Transvectant[Transvectant[Transvectant[
 exp[cc10,3],ff,8],ff,4],ff,6],cc19,2],ff,10].{1};
MM=Append[MM,{j2,j4,j61,j62,j63,j64,j81,j82,j83,j84,j85,j91,j92,j93,j94,j95,j101,
j102,j103,j104,j105,j106,j107,j108,j111,j112,j113,j114,j115,j116,j117,j118,j121,
j122,j123,j124,j125,j126,j127,j128,j129,j1210,j1211,j1212,j131,j132,j133,j134,j135,
j136,j137,j138,j139,j1310,j1311,j1312,j1313,j1314,j1315,j141,j142,j143,j144,j145,
j146,j147,j148,j149,j1410,j1411,j1412,j1413,j151,j152,j153,j154,j155,j156,j157,j158,
j159,j1510,j1511,j1512,j1513,j1514,j1515,j1516,j1517,j1518,j1519,j161,j162,j163,j164,
j165,j171,j172,j173,j174,j175,j18,j191,j192,j211,j212}]]]
```

Here are the rank computations:

```
In[]:=listdeg={{2,1},{4,1},{6,4},{8,5},{9,5},{10,8},{11,8},{12,12},{13,15},{14,13},
 {15,19},{16,5},{17,5},{18,1},{19,2},{21,2}};
In[]:=For[l=1,l<27,l++,pp=Timing[Dim2[10,l,MM,listdeg]];
 Print["Computation time: ",pp[[1]]];Print["=============="]]
Out[]=
There are no invariants of degree 1!
Computation time: 0.044003
==============
<<The dimension of O(V_10)^SL2_2 is 1
<<The set of invariants of degree 2 spanned by the given invariants
of degrees <2 has size 0
<<The subspace of O(V_10)^SL2_2 spanned by the given invariants of
degrees <2 has dimension >=0
<<The subspace of O(V_10)^SL2_2 spanned by the given invariants of
degrees <=2 has dimension >=1
Computation time: 0.008001
==============
There are no invariants of degree 3!
Computation time: 0.004
==============
<<The dimension of O(V_10)^SL2_4 is 2
<<The set of invariants of degree 4 spanned by the given invariants
```

```
of degrees <4 has size 1
<<The subspace of O(V_10)^SL2_4 spanned by the given invariants of
degrees <4 has dimension >=1
<<The subspace of O(V_10)^SL2_4 spanned by the given invariants of
degrees <=4 has dimension >=2
Computation time: 0.008
==============
There are no invariants of degree 5!
Computation time: 0.008
==============
<<The dimension of O(V_10)^SL2_6 is 6
<<The set of invariants of degree 6 spanned by the given invariants
of degrees <6 has size 2
<<The subspace of O(V_10)^SL2_6 spanned by the given invariants of
degrees <6 has dimension >=2
<<The subspace of O(V_10)^SL2_6 spanned by the given invariants of
degrees <=6 has dimension >=6
Computation time: 0.020002
==============
There are no invariants of degree 7!
Computation time: 0.016001
==============
<<The dimension of O(V_10)^SL2_8 is 12
<<The set of invariants of degree 8 spanned by the given invariants
of degrees <8 has size 7
<The subspace of O(V_10)^SL2_8 spanned by the given invariants of
degrees <8 has dimension >=7
<<The subspace of O(V_10)^SL2_8 spanned by the given invariants of
degrees <=8 has dimension >=12
Computation time: 0.032001
==============
<<The dimension of O(V_10)^SL2_9 is 5
<<The set of invariants of degree 9 spanned by the given invariants
of degrees <9 has size 0
<<The subspace of O(V_10)^SL2_9 spanned by the given invariants of
degrees <9 has dimension >=0
<<The subspace of O(V_10)^SL2_9 spanned by the given invariants of
degrees <=9 has dimension >=5
Computation time: 0.028003
==============
<<The dimension of O(V_10)^SL2_10 is 24
<<The set of invariants of degree 10 spanned by the given invariants
of degrees <10 has size 16
<<The subspace of O(V_10)^SL2_10 spanned by the given invariants of
degrees <10 has dimension >=16
<<The subspace of O(V_10)^SL2_10 spanned by the given invariants of
degrees <=10 has dimension >=24
Computation time: 0.064004
==============
<<The dimension of O(V_10)^SL2_11 is 13
<<The set of invariants of degree 11 spanned by the given invariants
of degrees <11 has size 5
<<The subspace of O(V_10)^SL2_11 spanned by the given invariants of
degrees <11 has dimension >=5
<<The subspace of O(V_10)^SL2_11 spanned by the given invariants of
degrees <=11 has dimension >=13
Computation time: 0.044002
```

```
==============
<<The dimension of O(V_10)^SL2_12 is 52
<<The set of invariants of degree 12 spanned by the given invariants
of degrees <12 has size 40
<<The subspace of O(V_10)^SL2_12 spanned by the given invariants of
degrees <12 has dimension >=40
<<The subspace of O(V_10)^SL2_12 spanned by the given invariants of
degrees <=12 has dimension >=52
Computation time: 0.16401
==============
<<The dimension of O(V_10)^SL2_13 is 33
<<The set of invariants of degree 13 spanned by the given invariants
of degrees <13 has size 18
<<The subspace of O(V_10)^SL2_13 spanned by the given invariants of
degrees <13 has dimension >=18
<<The subspace of O(V_10)^SL2_13 spanned by the given invariants of
degrees <=13 has dimension >=33
Computation time: 0.092006
==============
<<The dimension of O(V_10)^SL2_14 is 97
<<The set of invariants of degree 14 spanned by the given invariants
of degrees <14 has size 84
<<The subspace of O(V_10)^SL2_14 spanned by the given invariants of
degrees <14 has dimension >=84
<<The subspace of O(V_10)^SL2_14 spanned by the given invariants of
degrees <=14 has dimension >=97
Computation time: 0.488031
==============
<<The dimension of O(V_10)^SL2_15 is 80
<<The set of invariants of degree 15 spanned by the given invariants
of degrees <15 has size 61
<<The subspace of O(V_10)^SL2_15 spanned by the given invariants of
degrees <15 has dimension >=61
<<The subspace of O(V_10)^SL2_15 spanned by the given invariants of
degrees <=15 has dimension >=80
Computation time: 0.348021
==============
<<The dimension of O(V_10)^SL2_16 is 177
<<The set of invariants of degree 16 spanned by the given invariants
of degrees <16 has size 172
<<The subspace of O(V_10)^SL2_16 spanned by the given invariants of
degrees <16 has dimension >=172
<<The subspace of O(V_10)^SL2_16 spanned by the given invariants of
degrees <=16 has dimension >=177
Computation time: 1.6081
==============
<<The dimension of O(V_10)^SL2_17 is 160
<<The set of invariants of degree 17 spanned by the given invariants
of degrees <17 has size 157
<<The subspace of O(V_10)^SL2_17 spanned by the given invariants of
degrees <17 has dimension >=155
<<The subspace of O(V_10)^SL2_17 spanned by the given invariants of
degrees <=17 has dimension >=160
Computation time: 1.38409
==============
<<The dimension of O(V_10)^SL2_18 is 319
<<The set of invariants of degree 18 spanned by the given invariants
```

of degrees <18 has size 345
<<The subspace of O(V_10)^SL2_18 spanned by the given invariants of
degrees <18 has dimension >=318
<<The subspace of O(V_10)^SL2_18 spanned by the given invariants of
degrees <=18 has dimension >=319
Computation time: 7.14045
==============
<<The dimension of O(V_10)^SL2_19 is 301
<<The set of invariants of degree 19 spanned by the given invariants
of degrees <19 has size 349
<<The subspace of O(V_10)^SL2_19 spanned by the given invariants of
degrees <19 has dimension >=299
<<The subspace of O(V_10)^SL2_19 spanned by the given invariants of
degrees <=19 has dimension >=301
Computation time: 6.64441
==============
<<The dimension of O(V_10)^SL2_20 is 540
<<The set of invariants of degree 20 spanned by the given invariants
of degrees <20 has size 664
<<The subspace of O(V_10)^SL2_20 spanned by the given invariants of
degrees <20 has dimension >=540
Computation time: 11.3727
==============
<<The dimension of O(V_10)^SL2_21 is 547
<<The set of invariants of degree 21 spanned by the given invariants
of degrees <21 has size 758
<<The subspace of O(V_10)^SL2_21 spanned by the given invariants of
degrees <21 has dimension >=545
<<The subspace of O(V_10)^SL2_21 spanned by the given invariants of
degrees <=21 has dimension >=547
Computation time: 26.2616
==============
<<The dimension of O(V_10)^SL2_22 is 887
<<The set of invariants of degree 22 spanned by the given invariants
of degrees <22 has size 1265
<<The subspace of O(V_10)^SL2_22 spanned by the given invariants of
degrees <22 has dimension >=887
Computation time: 39.1344
==============
<<The dimension of O(V_10)^SL2_23 is 926
<<The set of invariants of degree 23 spanned by the given invariants
of degrees <23 has size 1525
<<The subspace of O(V_10)^SL2_23 spanned by the given invariants of
degrees <23 has dimension >=926
Computation time: 52.5793
==============
<<The dimension of O(V_10)^SL2_24 is 1429
<<The set of invariants of degree 24 spanned by the given invariants
of degrees <24 has size 2419
<<The subspace of O(V_10)^SL2_24 spanned by the given invariants of
degrees <24 has dimension >=1429
Computation time: 141.369
==============
<<The dimension of O(V_10)^SL2_25 is 1512
<<The set of invariants of degree 25 spanned by the given invariants
of degrees <25 has size 2973
<<The subspace of O(V_10)^SL2_25 spanned by the given invariants of

```
degrees <25 has dimension >=1512
Computation time: 192.716
==============
<<The dimension of O(V_10)^SL2_26 is 2219
<<The set of invariants of degree 26 spanned by the given invariants
of degrees <26 has size 4487
<<The subspace of O(V_10)^SL2_26 spanned by the given invariants of
degrees <26 has dimension >=2219
Computation time: 505.208
==============
```

A.5   The invariants of $V_1 \oplus V_5$

Here are the Mathematica computations supporting the proof of Proposition . We compute first the invariants $j_{4,1}, j_{4,2}, j_{6,1}, \ldots, j_{6,3}, j_{8,1}, \ldots j_{8,7}, j_{10,1}, j_{10,2}, j_{10,3}, j_{12,1}, \ldots, j_{12,6}, j_{14,1}, j_{18,1}$:

```
ll=Table[a[1-i]Binomial[1,i],{i,0,1}];
qq=Table[b[5-i]Binomial[5,i],{i,0,5}];
cc1=Transvectant[qq,qq,4];cc2=Transvectant[qq,qq,2];
cc3=Transvectant[qq,cc1,2];j41=Transvectant[cc1,cc1,2].{1};
j42=Transvectant[cc1,exp[ll,2],2].{1};j61=Transvectant[qq,exp[ll,5],5].{1};
j62=Transvectant[cc3,exp[ll,3],3].{1};
j63=Transvectant[Transvectant[cc1,cc3,2],ll,1].{1};
j81=Transvectant[cc1,Transvectant[cc3,cc3,2],2].{1};
j82=Transvectant[cc2,exp[ll,6],6].{1};
j83=Transvectant[Transvectant[cc2,qq,3],exp[ll,5],5].{1};
j84=Transvectant[Transvectant[cc1,cc3,1],exp[ll,3],3].{1};
j85=Transvectant[Transvectant[cc1,cc2,2],exp[ll,4],4].{1};
j86=Transvectant[Transvectant[exp[cc1,2],cc2,4],exp[ll,2],2].{1};
j87=Transvectant[Transvectant[exp[cc1,3],qq,5],ll,1].{1};
j101=Transvectant[Transvectant[cc1,cc2,1],exp[ll,6],6].{1};
j102=Transvectant[Transvectant[exp[cc1,2],cc2,3],exp[ll,4],4].{1};
j103=Transvectant[Transvectant[exp[cc1,3],cc2,5],exp[ll,2],2].{1};
j121=Transvectant[Transvectant[cc3,cc3,2],Transvectant[cc3,cc3,2],2].{1};
j122=Transvectant[Transvectant[qq,cc2,1],exp[ll,9],9].{1};
j123=Transvectant[Transvectant[Transvectant[qq,cc2,1],cc1,2],exp[ll,7],7].{1};
j124=Transvectant[
Transvectant[Transvectant[qq,cc2,1],exp[cc1,2],4],exp[ll,5],5].{1};
j125=Transvectant[
Transvectant[Transvectant[qq,cc2,1],exp[cc1,3],6],exp[ll,3],3].{1};
j126=Transvectant[
Transvectant[Transvectant[qq,cc2,1],exp[cc1,4],8],ll,1].{1};
j141=Transvectant[
Transvectant[Transvectant[qq,cc2,1],exp[cc1,5],9],ll,1].{1};
j181=Transvectant[exp[cc1,7],prod[qq,Transvectant[qq,cc2,1]],14].{1};
```

Here are the rank computations:

```
In[]:=gens={{j41,j42},{j61,j62,j63},{j81,j82,j83,j84,j85,j86,j87},
{j101,j102,j103},{j121,j122,j123,j124,j125,j126},{j141},{j181}};
degrees={4,6,8,10,12,14,18};
=== computations in degree 4 ===
In[]:=mon={j41,j42};
In[]:=mat=Table[mon/.
```

```
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,4}];
MatrixRank[mat]
Out[]=2
=== computations in degree 6 ===
In[]:=mon={j61,j62,j63};
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,4}];
MatrixRank[mat]
Out[]=3
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j81,j82,j83,j84,j85,j86,j87}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,11}];
MatrixRank[mat]
Out[]=10
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=6
In[]:=mon=Union[mon1,{j101,j102,j103}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,10}];
MatrixRank[mat]
Out[]=9
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=24
In[]:=mon=Union[mon1,{j121,j122,j123,j124,j125,j126}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,30}];
MatrixRank[mat]
Out[]=29
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=36
In[]:=mon=Union[mon1,{j141}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,32}];
MatrixRank[mat]
Out[]=31
=== computations in degree 16 ===
In[]:=mon1=listmonomials[16,gens,degrees];Length[mon1]
Out[]=87
In[]:=Timing[
mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,62}];
MatrixRank[mat]]
Out[]={9.79661,61}    //the rank is 61;
                      //the computation took 9.7 seconds
=== computations in degree 18 ===
```

```
In[]:=mon1=listmonomials[18,gens,degrees];Length[mon1]
Out[]=114
In[]:=mon=Union[mon1,{j181}];
In[]:=Timing[
mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,76}];
MatrixRank[mat]]
Out[]={24.5055,75}   //the rank is 75;
                     //the computation took 24.5 seconds
=== computations in degree 20 ===
In[]:=mon1=listmonomials[20,gens,degrees];Length[mon1]
Out[]=237
In[]:=Timing[
mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,126}];
MatrixRank[mat,Modulus->32003]]
Out[]={77.9809,125}   //the rank is 125;
                      //the computation took 77.9 seconds
=== computations in degree 26 ===
In[]:=mon1=listmonomials[26,gens,degrees];Length[mon1]
Out[]=842
In[]:=Timing[
mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,265}];
MatrixRank[mat,Modulus->32003]]
Out[]={810.687,264}   //the rank is 264;
                      //the computation took 810.6 seconds
```

A.6   The invariants of $V_1 \oplus V_6$

Here are the Mathematica computations supporting the proof of Proposition
5.6.2. First we compute the invariants $j_{2,1}, j_{4,1}, j_{5,1}, j_{6,1}, j_{6,2}, j_{7,1}, j_{7,2}, j_{8,1}, j_{9,1}$
$\dots j_{9,3}, j_{10,1}, \dots, j_{10,4}, j_{11,1}, j_{11,2}, j_{12,1}, j_{12,3}, j_{13,1}, j_{13,2}, j_{14,1}, j_{14,2}, j_{15,1}, j_{15,2}$:

```
ll=Table[a[1-i]Binomial[1,i],{i,0,1}];
ss=Table[b[6-i]Binomial[6,i],{i,0,6}];
cc1=Transvectant[ss,ss,4];cc2=Transvectant[ss,ss,2];
cc3=Transvectant[ss,cc1,4];cc4=Transvectant[cc1,cc1,2];
cc5=Transvectant[ss,cc1,1];j21=Transvectant[ss,ss,6].{1};
j41=Transvectant[cc1,cc1,4].{1};j51=Transvectant[cc3,exp[ll,2],2].{1};
j61=Transvectant[cc1,cc4,4].{1};j62=Transvectant[cc1,exp[ll,4],4].{1};
j71=Transvectant[ss,exp[ll,6],6].{1};
j72=Transvectant[Transvectant[cc1,cc3,2],exp[ll,2],2].{1};
j81=Transvectant[Transvectant[ss,cc3,2],exp[ll,4],4].{1};
j91=Transvectant[Transvectant[ss,cc1,2],exp[ll,6],6].{1};
j92=Transvectant[Transvectant[cc1,cc3,1],exp[ll,4],4].{1};
j93=Transvectant[Transvectant[ss,exp[cc3,2],4],exp[ll,2],2].{1};
j101=Transvectant[cc4,exp[cc3,2],4].{1};
j102=Transvectant[cc2,exp[ll,8],8].{1};
j103=Transvectant[Transvectant[ss,cc3,1],exp[ll,6],6].{1};
j104=Transvectant[Transvectant[cc1,exp[cc3,2],3],exp[ll,2],2].{1};
j111=Transvectant[cc5,exp[ll,8],8].{1};
j112=Transvectant[Transvectant[ss,exp[cc3,2],3],exp[ll,4],4].{1};
j121=Transvectant[Transvectant[Transvectant[ss,cc1,2],cc3,1],
 exp[ll,6],6].{1};
```

```
j122=Transvectant[Transvectant[Transvectant[ss,cc1,1],cc3,2],
 exp[ll,6],6].{1};
j123=Transvectant[Transvectant[ss,exp[cc3,3],5],exp[ll,2],2].{1};
j131=Transvectant[Transvectant[cc2,cc3,1],exp[ll,8],8].{1};
j132=Transvectant[Transvectant[cc5,exp[cc3,2],4],exp[ll,4],4].{1};
j141=Transvectant[Transvectant[cc1,cc2,1],exp[ll,10],10].{1};
j142=Transvectant[Transvectant[cc5,exp[cc3,3],6],exp[ll,2],2].{1};
j151=Transvectant[cc5,exp[cc3,4],8].{1};
j152=Transvectant[Transvectant[ss,cc2,1],exp[ll,12],12].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j41},{j51},{j61,j62},{j71,j72},{j81},{j91,j92,j93},
{j101,j102,j103,j104},{j111,j112},{j121,j122,j123},{j131,j132},
{j141,j142},{j151,j152}};
degrees={2,4,5,6,7,8,9,10,11,12,13,14,15};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,3}];
MatrixRank[mat]
Out[]=2
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=2
In[]:=mon=Union[mon1,{j61,j62}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,5}];
MatrixRank[mat]
Out[]=4
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j71,j72}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,4}];
MatrixRank[mat]
Out[]=3
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=5
In[]:=mon=Union[mon1,{j81}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,7}];
MatrixRank[mat]
Out[]=6
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=4
In[]:=mon=Union[mon1,{j91,j92,j93}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
```

```
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,8}];
MatrixRank[mat]
Out[]=7
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=9
In[]:=mon=Union[mon1,{j101,j102,j103,j104}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,14}];
MatrixRank[mat]
Out[]=13
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=11
In[]:=mon=Union[mon1,{j111,j112}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,14}];
MatrixRank[mat]
Out[]=13
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=20
In[]:=mon=Union[mon1,{j121,j122,j123}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,24}];
MatrixRank[mat]
Out[]=23
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=22
In[]:=mon=Union[mon1,{j131,j132}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,24}];
MatrixRank[mat]
Out[]=23
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=38
In[]:=mon=Union[mon1,{j141,j142}];
In[]:=Timing[mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,39}];
MatrixRank[mat]]
Out[134]={4.97631,38}}    //the rank is 38;
                          //the computation took 4.9 seconds
=== computations in degree 15 ===
In[]:=mon1=listmonomials[15,gens,degrees];Length[mon1]
Out[]=43
In[]:=mon=Union[mon1,{j151,j152}];
In[]:=Timing[mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,44}];
MatrixRank[mat]]
```

```
Out[]={5.90037,43}   //the rank is 43;
                      //the computation took 5.9 seconds
=== computations in degree 16 ===
In[]:=mon1=listmonomials[16,gens,degrees];Length[mon1]
Out[]=69
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,62}];
MatrixRank[mat]]
Out[]={15.253,61}   //the rank is 61;
                      //the computation took 15.2 seconds
=== computations in degree 17 ===
In[]:=mon1=listmonomials[17,gens,degrees];Length[mon1]
Out[]=79
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,70}];
MatrixRank[mat,Modulus->32003]]
Out[]={18.2731,69}   //the rank is 69;
                      //the computation took 18.2 seconds
=== computations in degree 18 ===
In[]:=mon1=listmonomials[18,gens,degrees];Length[mon1]
Out[]=117
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,95}];
MatrixRank[mat,Modulus->32003]]
Out[]={44.0708,94}   //the rank is 94;
                      //the computation took 44.0 seconds
=== computations in degree 19 ===
In[]:=mon1=listmonomials[19,gens,degrees];Length[mon1]
Out[]=140
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,109}];
MatrixRank[mat,Modulus->32003]]
Out[]={66.5602,108}  //the rank is 108;
                      //the computation took 66.5 seconds
=== computations in degree 20 ===
In[]:=mon1=listmonomials[20,gens,degrees];Length[mon1]
Out[]=197
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,144}];
MatrixRank[mat,Modulus->32003]]
Out[]={128.156,143}  //the rank is 143;
                      //the computation took 128.1 seconds
=== computations in degree 21 ===
In[]:=mon1=listmonomials[21,gens,degrees];Length[mon1]
Out[]=237
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,165}];
MatrixRank[mat,Modulus->32003]]
Out[]={207.713,164}  //the rank is 164;
                      //the computation took 207.7 seconds
=== computations in degree 22 ===
```

```
In[]:=mon1=listmonomials[22,gens,degrees];Length[mon1]
Out[]=324
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,210}];
MatrixRank[mat,Modulus->32003]]
Out[]={380.392,209}  //the rank is 209;
                     //the computation took 380.3 seconds
=== computations in degree 23 ===
In[]:=mon1=listmonomials[23,gens,degrees];Length[mon1]
Out[]=390
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,237}];
MatrixRank[mat,Modulus->32003]]
Out[]={478.538,236}  //the rank is 236;
                     //the computation took 478.5 seconds
=== computations in degree 25 ===
In[]:=mon1=listmonomials[25,gens,degrees];Length[mon1]
Out[]=639
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,340}];
MatrixRank[mat,Modulus->32003]]
Out[]={1304.81,339}  //the rank is 339;
                     //the computation took 1304.8 seconds
=== computations in degree 27 ===
In[]:=mon1=listmonomials[27,gens,degrees];Length[mon1]
Out[]=1027
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,473}];
MatrixRank[mat,Modulus->32003]]
Out[]={3144.84,472}  //the rank is 472;
                     //the computation took 3144.8 seconds
=== computations in degree 29 ===
In[]:=mon1=listmonomials[29,gens,degrees];Length[mon1]
Out[]=1629
In[]:=Timing[mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,642}];
MatrixRank[mat,Modulus->32003]]
Out[]={7260.85,641}
```

A.7 The covariants of V_7

First we create a matrix with the evaluations of the 147 covariants from Tables at 2750 random integers.

Next we show that the 147 covariants generate $\mathcal{C}(V_7)_{d,m}$ with $d \leq 23$ and $m \leq 15$. For each $d \leq 24$ and $m \leq 15$ we do the following: we construct all possible monomials of degree d and order m spanned by the 147 covariants and evaluate them at random integers (using the matrix we just created). These monomials will span $\mathcal{C}(V_7)_{d,m}$ if their evaluation matrix has the rank equal to the dimension of $\mathcal{C}(V_7)_{d,m}$. We will use a function that we implemented and

called `mon{deg,ord}` which returns a list of all covariants degree `deg` and order `ord` as monomials of degree `deg` spanned by the above 147 covariants.

```
In[]:=listdeg={{1,7},{2,2},{2,6},{2,10},{3,3},{3,5},{3,7},{3,9},{3,11},
{3,15},{4,0},{4,4},{4,4},{4,6},{4,8},{4,8},{4,10},{4,14},{5,1},{5,3},
{5,3},{5,5},{5,5},{5,7},{5,7},{5,9},{5,9},{5,13},{6,2},{6,2},{6,2},
{6,4},{6,4},{6,6},{6,6},{6,8},{6,8},{6,12},{7,1},{7,1},{7,1},{7,3},
{7,3},{7,5},{7,5},{7,5},{7,5},{7,7},{7,7},{7,11},{8,0},{8,0},{8,0},
{8,2},{8,2},{8,2},{8,4},{8,4},{8,4},{8,6},{8,6},{8,6},{8,10},{9,1},
{9,1},{9,1},{9,3},{9,3},{9,3},{9,3},{9,3},{9,5},{9,5},{9,9},{10,2},
{10,2},{10,2},{10,2},{10,4},{10,4},{10,4},{10,4},{10,8},{11,1},
{11,1},{11,1},{11,1},{11,1},{11,3},{11,3},{11,3},{11,7},{12,0},{12,0},
{12,0},{12,0},{12,0},{12,0},{12,2},{12,2},{12,2},{12,2},{12,2},{12,2},
{12,6},{13,1},{13,1},{13,1},{13,1},{13,1},{13,1},{13,1},{13,3},{13,5},
{14,0},{14,0},{14,0},{14,0},{14,4},{14,4},{15,1},{15,1},{15,1},{15,3},
{16,0},{16,0},{16,2},{16,2},{16,2},{17,1},{17,1},{18,0},{18,0},{18,0},
{18,0},{18,0},{18,0},{18,0},{18,0},{19,1},{20,0},{22,0},{22,0},
{23,1},{26,0},{30,0}};
In[]:=listgen={FF,LL,CHI,HH,RR,C35,C37,Epsilon,Gamm,TT,Inv4,PP,C44,C46,
Delta,C48,C410,Zet,C51,C531,C532,GG,C55,ETA,C57,C591,C592,C513,Tau,
C621,C622,C641,C642,C661,C662,Bet,C68,C612,Alpha,C711,C712,C731,C732,
Theta,C751,C752,C753,C771,C772,Cov711,C801,C802,C803,C821,C822,C823,MU,
C841,C842,Lambda,C861,C862,C810,C911,C912,C913,QQ,C931,C932,C933,C934,
C951,C952,C99,NU,C1021,C1022,C1023,C1041,C1042,C1043,C1044,C108,C1111,
C1112,C1113,C1114,C1115,PHI,C1131,C1132,C117,Rbig,C1201,C1202,C1203,
C1204,C1205,C1221,C1222,C1223,C1224,C1225,C1226,C126,RO,C1311,C1312,
C1313,C1314,C1315,C1316,C133,C135,C1401,C1402,C1403,C1404,PSI,C1441,
C1511,C1512,C1513,C153,C1601,C1602,Sigma,C1621,C1622,Omega,C1711,C1801,
C1802,C1803,C1804,C1805,C1806,C1807,C1808,C1809,C191,C201,C2201,C2202,
C231,C260,C300};
In[]:=mon[md_,j_]:=
 mon[md,j]=If[md=={0,0},1,If[md[[1]]<0||md[[2]]<0,{},
      Flatten[Table[listgen[[i]]mon[md-listdeg[[i]],i],{i,1,j}]]]]
```

Here we show that the 147 covariants generate $\mathcal{C}(V_7)_{d,m}$ with $d \leq 24$ and $m \leq 15$:

```
In[]:= Timing[
 For[deg=2,deg<24,deg++,ord=1;
  While[ord<16,
   symm=SymTensor[7,deg]/.Table[v[i]->0,{i,16,1000}];
   kk=Coefficient[symm,v[ord]]+1;
   If[kk!=1,
    Print[{deg,ord},": the vector space of covs has dimension",kk-1];
    ss=mon[{deg,ord},147];
    If[ss!={},
     tt=Intersection[ss,listgen];
     If[tt!={},
      ss1=Complement[ss,tt];
      eval1 = Table[ss1 /.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
        {j,1,Length[Matr[[k]]]}],{k,1,kk}];
      rk1=If[ss1=={},0,MatrixRank[eval1,Modulus->32003]];
      Print["-- the subpace spanned by covs of degree < ", deg,
       " has dim >=", rk1];
      Print["-- there are ", Length[tt], " gens of degree ", deg];
      eval=Table[ss/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
        {j,1,Length[Matr[[k]]]}],{k,1,kk}];
```

```
          rk=MatrixRank[eval,Modulus->32003];
          Print["-- the subpace spanned by covs of degree <= ", deg,
           " has dim >=", rk],
          If[Length[ss]>3000,DIM=kk-1;
           dd1=RandomSample[ss,kk+10];
           dd2=Complement[ss,dd1];
           eval=Table[dd1/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
             {j,1,Length[Matr[[k]]]}],{k,1,kk}];
           eval1=ColumnEchelon[eval];
           rk1=MatrixRank[eval1,Modulus->32003];
           rk=rk1;
           While[rk!=DIM,
            AA=Table[RandomInteger[{-10,10}],{i,1,50}];
            uu=RandomSample[dd2,50].AA;
            evaluu=Table[uu/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
              {j,1,Length[Matr[[k]]]}],{k,1,kk}];
            eval=Table[Join[eval1[[i]],{evaluu[[i]]}],{i,1,kk}];
            eval=Mod[eval,32003];
            rk=MatrixRank[eval,Modulus->32003];
            If[rk>rk1,eval1=eval;rk1=rk]];
           Print["-- the subpace spanned by covs of degree < ", deg,
            " has dim >=", rk],
           eval=Table[ss/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
             {j,1,Length[Matr[[k]]]}],{k,1,kk}];
           rk=MatrixRank[eval,Modulus->32003];
           Print["-- the subpace spanned by covs of degree < ", deg,
            " has dim >=", rk]]]]; ord++]
 ]]
{2,2}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,6}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,10}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,14}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=1
{3,3}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,5}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,7}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,9}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 3 has dim >=1
-- there are 1 gens of degree 3
```

```
-- the subpace spanned by covs of degree <= 3 has dim >=2
{3,11}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,13}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=1
{3,15}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{4,4}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 2 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=3
{4,6}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 4 has dim >=0
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=1
{4,8}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 2 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=3
{4,10}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{4,12}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 4 has dim >=3
{4,14}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{5,1}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 5 has dim >=0
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=1
{5,3}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 5 has dim >=0
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=2
{5,5}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 5 has dim >=1
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=3
{5,7}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 5 has dim >=2
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=4
{5,9}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 5 has dim >=2
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=4
{5,11}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 5 has dim >=5
{5,13}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 5 has dim >=3
```

```
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=4
{5,15}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 5 has dim >=5
{6,2}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 6 has dim >=1
-- there are 3 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=4
{6,4}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 6 has dim >=0
-- there are 2 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=2
{6,6}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 6 has dim >=5
-- there are 2 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=7
{6,8}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 6 has dim >=3
-- there are 2 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=5
{6,10}: the vector space of covs has dimension 8
-- the subpace spanned by covs of degree < 6 has dim >=8
{6,12}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 6 has dim >=6
-- there are 1 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=7
{6,14}: the vector space of covs has dimension 9
-- the subpace spanned by covs of degree < 6 has dim >=9
{7,1}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 7 has dim >=0
-- there are 3 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=3
{7,3}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 7 has dim >=2
-- there are 2 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=4
{7,5}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 7 has dim >=3
-- there are 4 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=7
{7,7}: the vector space of covs has dimension 9
-- the subpace spanned by covs of degree < 7 has dim >=7
-- there are 2 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=9
{7,9}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 7 has dim >=10
{7,11}: the vector space of covs has dimension 11
-- the subpace spanned by covs of degree < 7 has dim >=10
-- there are 1 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=11
{7,13}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 7 has dim >=13
{7,15}: the vector space of covs has dimension 12
-- the subpace spanned by covs of degree < 7 has dim >=12
{8,2}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 8 has dim >=0
-- there are 3 gens of degree 8
```

```
-- the subpace spanned by covs of degree <= 8 has dim >=3
{8,4}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 8 has dim >=7
-- there are 3 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=10
{8,6}: the vector space of covs has dimension 9
-- the subpace spanned by covs of degree < 8 has dim >=6
-- there are 3 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=9
{8,8}: the vector space of covs has dimension 16
-- the subpace spanned by covs of degree < 8 has dim >=16
{8,10}: the vector space of covs has dimension 14
-- the subpace spanned by covs of degree < 8 has dim >=13
-- there are 1 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=14
{8,12}: the vector space of covs has dimension 19
-- the subpace spanned by covs of degree < 8 has dim >=19
{8,14}: the vector space of covs has dimension 17
-- the subpace spanned by covs of degree < 8 has dim >=17
{9,1}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 9 has dim >=1
-- there are 3 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=4
{9,3}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 9 has dim >=5
-- there are 5 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=10
{9,5}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 9 has dim >=11
-- there are 2 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=13
{9,7}: the vector space of covs has dimension 17
-- the subpace spanned by covs of degree < 9 has dim >=17
{9,9}: the vector space of covs has dimension 21
-- the subpace spanned by covs of degree < 9 has dim >=20
-- there are 1 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=21
{9,11}: the vector space of covs has dimension 24
-- the subpace spanned by covs of degree < 9 has dim >=24
{9,13}: the vector space of covs has dimension 25
-- the subpace spanned by covs of degree < 9 has dim >=25
{9,15}: the vector space of covs has dimension 29
-- the subpace spanned by covs of degree < 9 has dim >=29
{10,2}: the vector space of covs has dimension 12
-- the subpace spanned by covs of degree < 10 has dim >=8
-- there are 4 gens of degree 10
-- the subpace spanned by covs of degree <= 10 has dim >=12
{10,4}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 10 has dim >=9
-- there are 4 gens of degree 10
-- the subpace spanned by covs of degree <= 10 has dim >=13
{10,6}: the vector space of covs has dimension 23
-- the subpace spanned by covs of degree < 10 has dim >=23
{10,8}: the vector space of covs has dimension 23
-- the subpace spanned by covs of degree < 10 has dim >=22
-- there are 1 gens of degree 10
-- the subpace spanned by covs of degree <= 10 has dim >=23
```

```
{10,10}: the vector space of covs has dimension 34
-- the subpace spanned by covs of degree < 10 has dim >=34
{10,12}: the vector space of covs has dimension 31
-- the subpace spanned by covs of degree < 10 has dim >=31
{10,14}: the vector space of covs has dimension 40
-- the subpace spanned by covs of degree < 10 has dim >=40
{11,1}: the vector space of covs has dimension 8
-- the subpace spanned by covs of degree < 11 has dim >=3
-- there are 5 gens of degree 11
-- the subpace spanned by covs of degree <= 11 has dim >=8
{11,3}: the vector space of covs has dimension 16
-- the subpace spanned by covs of degree < 11 has dim >=13
-- there are 3 gens of degree 11
-- the subpace spanned by covs of degree <= 11 has dim >=16
{11,5}: the vector space of covs has dimension 24
-- the subpace spanned by covs of degree < 11 has dim >=24
{11,7}: the vector space of covs has dimension 31
-- the subpace spanned by covs of degree < 11 has dim >=30
-- there are 1 gens of degree 11
-- the subpace spanned by covs of degree <= 11 has dim >=31
{11,9}: the vector space of covs has dimension 38
-- the subpace spanned by covs of degree < 11 has dim >=38
{11,11}: the vector space of covs has dimension 43
-- the subpace spanned by covs of degree < 11 has dim >=43
{11,13}: the vector space of covs has dimension 49
-- the subpace spanned by covs of degree < 11 has dim >=49
{11,15}: the vector space of covs has dimension 52
-- the subpace spanned by covs of degree < 11 has dim >=52
{12,2}: the vector space of covs has dimension 12
-- the subpace spanned by covs of degree < 12 has dim >=6
-- there are 6 gens of degree 12
-- the subpace spanned by covs of degree <= 12 has dim >=12
{12,4}: the vector space of covs has dimension 29
-- the subpace spanned by covs of degree < 12 has dim >=29
{12,6}: the vector space of covs has dimension 33
-- the subpace spanned by covs of degree < 12 has dim >=32
-- there are 1 gens of degree 12
-- the subpace spanned by covs of degree <= 12 has dim >=33
{12,8}: the vector space of covs has dimension 48
-- the subpace spanned by covs of degree < 12 has dim >=48
{12,10}: the vector space of covs has dimension 49
-- the subpace spanned by covs of degree < 12 has dim >=49
{12,12}: the vector space of covs has dimension 65
-- the subpace spanned by covs of degree < 12 has dim >=65
{12,14}: the vector space of covs has dimension 64
-- the subpace spanned by covs of degree < 12 has dim >=64
{13,1}: the vector space of covs has dimension 14
-- the subpace spanned by covs of degree < 13 has dim >=7
-- there are 7 gens of degree 13
-- the subpace spanned by covs of degree <= 13 has dim >=14
{13,3}: the vector space of covs has dimension 26
-- the subpace spanned by covs of degree < 13 has dim >=25
-- there are 1 gens of degree 13
-- the subpace spanned by covs of degree <= 13 has dim >=26
{13,5}: the vector space of covs has dimension 39
-- the subpace spanned by covs of degree < 13 has dim >=38
-- there are 1 gens of degree 13
```

```
-- the subpace spanned by covs of degree <= 13 has dim >=39
{13,7}: the vector space of covs has dimension 53
-- the subpace spanned by covs of degree < 13 has dim >=53
{13,9}: the vector space of covs has dimension 62
-- the subpace spanned by covs of degree < 13 has dim >=62
{13,11}: the vector space of covs has dimension 74
-- the subpace spanned by covs of degree < 13 has dim >=74
{13,13}: the vector space of covs has dimension 83
-- the subpace spanned by covs of degree < 13 has dim >=83
{13,15}: the vector space of covs has dimension 91
-- the subpace spanned by covs of degree < 13 has dim >=91
{14,2}: the vector space of covs has dimension 30
-- the subpace spanned by covs of degree < 14 has dim >=30
{14,4}: the vector space of covs has dimension 37
-- the subpace spanned by covs of degree < 14 has dim >=35
-- there are 2 gens of degree 14
-- the subpace spanned by covs of degree <= 14 has dim >=37
{14,6}: the vector space of covs has dimension 62
-- the subpace spanned by covs of degree < 14 has dim >=62
{14,8}: the vector space of covs has dimension 68
-- the subpace spanned by covs of degree < 14 has dim >=68
{14,10}: the vector space of covs has dimension 91
-- the subpace spanned by covs of degree < 14 has dim >=91
{14,12}: the vector space of covs has dimension 95
-- the subpace spanned by covs of degree < 14 has dim >=95
{14,14}: the vector space of covs has dimension 116
-- the subpace spanned by covs of degree < 14 has dim >=116
{15,1}: the vector space of covs has dimension 20
-- the subpace spanned by covs of degree < 15 has dim >=17
-- there are 3 gens of degree 15
-- the subpace spanned by covs of degree <= 15 has dim >=20
{15,3}: the vector space of covs has dimension 42
-- the subpace spanned by covs of degree < 15 has dim >=41
-- there are 1 gens of degree 15
-- the subpace spanned by covs of degree <= 15 has dim >=42
{15,5}: the vector space of covs has dimension 62
-- the subpace spanned by covs of degree < 15 has dim >=62
{15,7}: the vector space of covs has dimension 80
-- the subpace spanned by covs of degree < 15 has dim >=80
{15,9}: the vector space of covs has dimension 101
-- the subpace spanned by covs of degree < 15 has dim >=101
{15,11}: the vector space of covs has dimension 116
-- the subpace spanned by covs of degree < 15 has dim >=116
{15,13}: the vector space of covs has dimension 132
-- the subpace spanned by covs of degree < 15 has dim >=132
{15,15}: the vector space of covs has dimension 147
-- the subpace spanned by covs of degree < 15 has dim >=147
{16,2}: the vector space of covs has dimension 33
-- the subpace spanned by covs of degree < 16 has dim >=30
-- there are 3 gens of degree 16
-- the subpace spanned by covs of degree <= 16 has dim >=33
{16,4}: the vector space of covs has dimension 70
-- the subpace spanned by covs of degree < 16 has dim >=70
{16,6}: the vector space of covs has dimension 81
-- the subpace spanned by covs of degree < 16 has dim >=81
{16,8}: the vector space of covs has dimension 117
-- the subpace spanned by covs of degree < 16 has dim >=117
```

```
{16,10}: the vector space of covs has dimension 129
-- the subpace spanned by covs of degree < 16 has dim >=129
{16,12}: the vector space of covs has dimension 159
-- the subpace spanned by covs of degree < 16 has dim >=159
{16,14}: the vector space of covs has dimension 168
-- the subpace spanned by covs of degree < 16 has dim >=168
{17,1}: the vector space of covs has dimension 31
-- the subpace spanned by covs of degree < 17 has dim >=29
-- there are 2 gens of degree 17
-- the subpace spanned by covs of degree <= 17 has dim >=31
{17,3}: the vector space of covs has dimension 62
-- the subpace spanned by covs of degree < 17 has dim >=62
{17,5}: the vector space of covs has dimension 92
-- the subpace spanned by covs of degree < 17 has dim >=92
{17,7}: the vector space of covs has dimension 122
-- the subpace spanned by covs of degree < 17 has dim >=122
{17,9}: the vector space of covs has dimension 149
-- the subpace spanned by covs of degree < 17 has dim >=149
{17,11}: the vector space of covs has dimension 177
-- the subpace spanned by covs of degree < 17 has dim >=177
{17,13}: the vector space of covs has dimension 200
-- the subpace spanned by covs of degree < 17 has dim >=200
{17,15}: the vector space of covs has dimension 225
-- the subpace spanned by covs of degree < 17 has dim >=225
{18,2}: the vector space of covs has dimension 63
-- the subpace spanned by covs of degree < 18 has dim >=63
{18,4}: the vector space of covs has dimension 85
-- the subpace spanned by covs of degree < 18 has dim >=85
{18,6}: the vector space of covs has dimension 137
-- the subpace spanned by covs of degree < 18 has dim >=137
{18,8}: the vector space of covs has dimension 157
-- the subpace spanned by covs of degree < 18 has dim >=157
{18,10}: the vector space of covs has dimension 203
-- the subpace spanned by covs of degree < 18 has dim >=203
{18,12}: the vector space of covs has dimension 223
-- the subpace spanned by covs of degree < 18 has dim >=223
{18,14}: the vector space of covs has dimension 265
-- the subpace spanned by covs of degree < 18 has dim >=265
{19,1}: the vector space of covs has dimension 46
-- the subpace spanned by covs of degree < 19 has dim >=45
-- there are 1 gens of degree 19
-- the subpace spanned by covs of degree <= 19 has dim >=46
{19,3}: the vector space of covs has dimension 88
-- the subpace spanned by covs of degree < 19 has dim >=88
{19,5}: the vector space of covs has dimension 133
-- the subpace spanned by covs of degree < 19 has dim >=133
{19,7}: the vector space of covs has dimension 176
-- the subpace spanned by covs of degree < 19 has dim >=176
{19,9}: the vector space of covs has dimension 216
-- the subpace spanned by covs of degree < 19 has dim >=216
{19,11}: the vector space of covs has dimension 255
-- the subpace spanned by covs of degree < 19 has dim >=255
{19,13}: the vector space of covs has dimension 295
-- the subpace spanned by covs of degree < 19 has dim >=295
{19,15}: the vector space of covs has dimension 326
-- the subpace spanned by covs of degree < 19 has dim >=326
{20,2}: the vector space of covs has dimension 71
```

```
-- the subpace spanned by covs of degree < 20 has dim >=71
{20,4}: the vector space of covs has dimension 140
-- the subpace spanned by covs of degree < 20 has dim >=140
{20,6}: the vector space of covs has dimension 175
-- the subpace spanned by covs of degree < 20 has dim >=175
{20,8}: the vector space of covs has dimension 241
-- the subpace spanned by covs of degree < 20 has dim >=241
{20,10}: the vector space of covs has dimension 273
-- the subpace spanned by covs of degree < 20 has dim >=273
{20,12}: the vector space of covs has dimension 335
-- the subpace spanned by covs of degree < 20 has dim >=335
{20,14}: the vector space of covs has dimension 363
-- the subpace spanned by covs of degree < 20 has dim >=363
{21,1}: the vector space of covs has dimension 61
-- the subpace spanned by covs of degree < 21 has dim >=61
{21,3}: the vector space of covs has dimension 126
-- the subpace spanned by covs of degree < 21 has dim >=126
{21,5}: the vector space of covs has dimension 185
-- the subpace spanned by covs of degree < 21 has dim >=185
{21,7}: the vector space of covs has dimension 245
-- the subpace spanned by covs of degree < 21 has dim >=245
{21,9}: the vector space of covs has dimension 304
-- the subpace spanned by covs of degree < 21 has dim >=304
{21,11}: the vector space of covs has dimension 360
-- the subpace spanned by covs of degree < 21 has dim >=360
{21,13}: the vector space of covs has dimension 411
-- the subpace spanned by covs of degree < 21 has dim >=411
{21,15}: the vector space of covs has dimension 466
-- the subpace spanned by covs of degree < 21 has dim >=466
{22,2}: the vector space of covs has dimension 119
-- the subpace spanned by covs of degree < 22 has dim >=119
{22,4}: the vector space of covs has dimension 173
-- the subpace spanned by covs of degree < 22 has dim >=173
{22,6}: the vector space of covs has dimension 261
-- the subpace spanned by covs of degree < 22 has dim >=261
{22,8}: the vector space of covs has dimension 312
-- the subpace spanned by covs of degree < 22 has dim >=312
{22,10}: the vector space of covs has dimension 400
-- the subpace spanned by covs of degree < 22 has dim >=400
{22,12}: the vector space of covs has dimension 443
-- the subpace spanned by covs of degree < 22 has dim >=443
{22,14}: the vector space of covs has dimension 525
-- the subpace spanned by covs of degree < 22 has dim >=525
{23,1}: the vector space of covs has dimension 85
-- the subpace spanned by covs of degree < 23 has dim >=84
-- there are 1 gens of degree 23
-- the subpace spanned by covs of degree <= 23 has dim >=85
{23,3}: the vector space of covs has dimension 169
-- the subpace spanned by covs of degree < 23 has dim >=169
{23,5}: the vector space of covs has dimension 253
-- the subpace spanned by covs of degree < 23 has dim >=253
{23,7}: the vector space of covs has dimension 334
-- the subpace spanned by covs of degree < 23 has dim >=334
{23,9}: the vector space of covs has dimension 415
-- the subpace spanned by covs of degree < 23 has dim >=415
{23,11}: the vector space of covs has dimension 491
-- the subpace spanned by covs of degree < 23 has dim >=491
```

```
{23,13}: the vector space of covs has dimension 567
-- the subpace spanned by covs of degree < 23 has dim >=567
{23,15}: the vector space of covs has dimension 636
-- the subpace spanned by covs of degree < 23 has dim >=636
{24,2}: the vector space of covs has dimension 136
-- the subpace spanned by covs of degree < 24 has dim >=136
{24,4}: the vector space of covs has dimension 254
-- the subpace spanned by covs of degree < 24 has dim >=254
{24,6}: the vector space of covs has dimension 330
-- the subpace spanned by covs of degree < 24 has dim >=330
{24,8}: the vector space of covs has dimension 445
-- the subpace spanned by covs of degree < 24 has dim >=445
{24,10}: the vector space of covs has dimension 513
-- the subpace spanned by covs of degree < 24 has dim >=513
{24,12}: the vector space of covs has dimension 626
-- the subpace spanned by covs of degree < 24 has dim >=626
{24,14}: the vector space of covs has dimension 688
-- the subpace spanned by covs of degree < 24 has dim >=688
Out[]= {1453.54, Null}
```

Here we prove that the covariants from Tables 5.4 and 5.5 generate the
vector space of covariants of V_7 of order 1 and degree d with $d \leq 43$, the vector
space of covariants of V_7 of order 2 and degree d with $d \leq 46$, the vector space
of covariants of V_7 of order 3 and degree d with $d \leq 45$, the vector space of
covariants of V_7 of order 4 and degree d with $d \leq 44$, and the vector space of
covariants of V_7 of order 5 and degree d with $d \leq 45$:

```
In[]:=listdeg2 = {{1, 43}, {2, 46}, {3, 45}, {4, 44}, {5, 45}};
In[]:=Timing[
 For[zz=1,zz<6,zz++,
  ord=listdeg2[[zz]][[1]];
  For[deg = 25, deg < listdeg2[[zz]][[2]] + 1, deg++,
   symm = SymTensor[7, deg] /. Table[v[i] -> 0, {i, 16, 1000}];
   kk = Coefficient[symm, v[ord]] + 1;
   If[kk != 1,
    Print[{deg,ord},": the vector space of covs has dimension",kk-1];
    ss = mon[{deg, ord}, 147];
    If[ss != {},
     DIM = kk - 1;
     dd1 = RandomSample[ss, kk + 10];
     dd2 = Complement[ss, dd1];
     eval=Table[dd1/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
       {j,1,Length[Matr[[k]]]}],{k,1,kk}];
     eval1= ColumnEchelon[eval];
     rk1= MatrixRank[eval1, Modulus -> 32003];
     rk= rk1;
     While[rk != DIM,
      AA = Table[RandomInteger[{-10, 10}], {i, 1, 50}];
      uu = RandomSample[dd2, 50].AA;
      evaluu= Table[uu/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
        {j,1,Length[Matr[[k]]]}],{k,1,kk}];
      eval= Table[Join[eval1[[i]], {evaluu[[i]]}], {i, 1, kk}];
      eval= Mod[eval, 32003];
      rk= MatrixRank[eval, Modulus -> 32003];
      If[rk > rk1, eval1 = eval; rk1 = rk]];
     Print["-- the subpace spanned by covs of degree < ", deg,
```

```
      " has dim >=", rk]
    ]]]]
```
{25,1}: the vector space of covs has dimension 114
-- the subpace spanned by covs of degree < 25 has dim >=114
{27,1}: the vector space of covs has dimension 146
-- the subpace spanned by covs of degree < 27 has dim >=146
{29,1}: the vector space of covs has dimension 189
-- the subpace spanned by covs of degree < 29 has dim >=189
{31,1}: the vector space of covs has dimension 241
-- the subpace spanned by covs of degree < 31 has dim >=241
{33,1}: the vector space of covs has dimension 297
-- the subpace spanned by covs of degree < 33 has dim >=297
{35,1}: the vector space of covs has dimension 369
-- the subpace spanned by covs of degree < 35 has dim >=369
{37,1}: the vector space of covs has dimension 452
-- the subpace spanned by covs of degree < 37 has dim >=452
{39,1}: the vector space of covs has dimension 543
-- the subpace spanned by covs of degree < 39 has dim >=543
{41,1}: the vector space of covs has dimension 653
-- the subpace spanned by covs of degree < 41 has dim >=653
{43,1}: the vector space of covs has dimension 780
-- the subpace spanned by covs of degree < 43 has dim >=780
{26,2}: the vector space of covs has dimension 207
-- the subpace spanned by covs of degree < 26 has dim >=207
{28,2}: the vector space of covs has dimension 236
-- the subpace spanned by covs of degree < 28 has dim >=236
{30,2}: the vector space of covs has dimension 336
-- the subpace spanned by covs of degree < 30 has dim >=336
{32,2}: the vector space of covs has dimension 383
-- the subpace spanned by covs of degree < 32 has dim >=383
{34,2}: the vector space of covs has dimension 519
-- the subpace spanned by covs of degree < 34 has dim >=519
{36,2}: the vector space of covs has dimension 589
-- the subpace spanned by covs of degree < 36 has dim >=589
{38,2}: the vector space of covs has dimension 769
-- the subpace spanned by covs of degree < 38 has dim >=769
{40,2}: the vector space of covs has dimension 868
-- the subpace spanned by covs of degree < 40 has dim >=868
{42,2}: the vector space of covs has dimension 1100
-- the subpace spanned by covs of degree < 42 has dim >=1100
{44,2}: the vector space of covs has dimension 1235
-- the subpace spanned by covs of degree < 44 has dim >=1235
{46,2}: the vector space of covs has dimension 1529
-- the subpace spanned by covs of degree < 46 has dim >=1529
{25,3}: the vector space of covs has dimension 224
-- the subpace spanned by covs of degree < 25 has dim >=224
{27,3}: the vector space of covs has dimension 295
-- the subpace spanned by covs of degree < 27 has dim >=295
{29,3}: the vector space of covs has dimension 378
-- the subpace spanned by covs of degree < 29 has dim >=378
{31,3}: the vector space of covs has dimension 476
-- the subpace spanned by covs of degree < 31 has dim >=476
{33,3}: the vector space of covs has dimension 599
-- the subpace spanned by covs of degree < 33 has dim >=599
{35,3}: the vector space of covs has dimension 736
-- the subpace spanned by covs of degree < 35 has dim >=736
{37,3}: the vector space of covs has dimension 898

```
-- the subpace spanned by covs of degree < 37 has dim >=898
{39,3}: the vector space of covs has dimension 1090
-- the subpace spanned by covs of degree < 39 has dim >=1090
{41,3}: the vector space of covs has dimension 1306
-- the subpace spanned by covs of degree < 41 has dim >=1306
{43,3}: the vector space of covs has dimension 1551
-- the subpace spanned by covs of degree < 43 has dim >=1551
{45,3}: the vector space of covs has dimension 1838
-- the subpace spanned by covs of degree < 45 has dim >=1838
{26,4}: the vector space of covs has dimension 309
-- the subpace spanned by covs of degree < 26 has dim >=309
{28,4}: the vector space of covs has dimension 433
-- the subpace spanned by covs of degree < 28 has dim >=433
{30,4}: the vector space of covs has dimension 513
-- the subpace spanned by covs of degree < 30 has dim >=513
{32,4}: the vector space of covs has dimension 686
-- the subpace spanned by covs of degree < 32 has dim >=686
{34,4}: the vector space of covs has dimension 810
-- the subpace spanned by covs of degree < 34 has dim >=810
{36,4}: the vector space of covs has dimension 1039
-- the subpace spanned by covs of degree < 36 has dim >=1039
{38,4}: the vector space of covs has dimension 1212
-- the subpace spanned by covs of degree < 38 has dim >=1212
{40,4}: the vector space of covs has dimension 1520
-- the subpace spanned by covs of degree < 40 has dim >=1520
{42,4}: the vector space of covs has dimension 1748
-- the subpace spanned by covs of degree < 42 has dim >=1748
{44,4}: the vector space of covs has dimension 2144
-- the subpace spanned by covs of degree < 44 has dim >=2144
{25,5}: the vector space of covs has dimension 336
-- the subpace spanned by covs of degree < 25 has dim >=336
{27,5}: the vector space of covs has dimension 440
-- the subpace spanned by covs of degree < 27 has dim >=440
{29,5}: the vector space of covs has dimension 564
-- the subpace spanned by covs of degree < 29 has dim >=564
{31,5}: the vector space of covs has dimension 715
-- the subpace spanned by covs of degree < 31 has dim >=715
{33,5}: the vector space of covs has dimension 892
-- the subpace spanned by covs of degree < 33 has dim >=892
{35,5}: the vector space of covs has dimension 1103
-- the subpace spanned by covs of degree < 35 has dim >=1103
{37,5}: the vector space of covs has dimension 1346
-- the subpace spanned by covs of degree < 37 has dim >=1346
{39,5}: the vector space of covs has dimension 1630
-- the subpace spanned by covs of degree < 39 has dim >=1630
{41,5}: the vector space of covs has dimension 1954
-- the subpace spanned by covs of degree < 41 has dim >=1954
{43,5}: the vector space of covs has dimension 2327
-- the subpace spanned by covs of degree < 43 has dim >=2327
{45,5}: the vector space of covs has dimension 2748
-- the subpace spanned by covs of degree < 45 has dim >=2748
Out[]= {132186., Null}
```

A.8 The covariants of V_8

First we create a matrix with the evaluations of the 69 covariants from Table 5.6 at 250 random integers:

```
In[]:=MatrixEvaluations[N_]:=Module[{FF,Inv2,KK,II,HH,Inv3,Fk,F3,F2,F1,PP,Teta,
  TT,Inv4,Delta,Ik,I3,I2,I1,H3,H2,H1,SS,Inv5,FK3,FK2,Fdelta,FK1,F3rd,F2nd,
  F1st,PP3,Tetak,PP1,Inv6,Ik3,Ik2,Idelta,Ik1,I3rd,Tau,I2nd,I1st,Inv7,Fk3rd,
  Ftau6,Fk2nd,Ftau5,Fk1st,Ftau4,Tetakk,Inv8,Ik3rd,Itau6,Ik2nd,Itau5,Ik1st,
  Itau4,Inv9,Fktau4,Fdelta3rd,Tetakkk,Fdelta2nd,Inv10,Iktau4,Idelta3rd,
  Fdeltatau4,Last,Ideltatau4},
MM={};
For[i=1,i<N+1,i++,FF=Table[Binomial[8,i]*a[8-i],{i,0,8}]]/.
Table[a[j]->RandomInteger[{-10,10}],{j,0,8}]];
Inv2=Transvectant[FF,FF,8];KK=Transvectant[FF,FF,6];
II=Transvectant[FF,FF,4];HH=Transvectant[FF,FF,2];
Inv3=Transvectant[FF,II,8];Fk=Transvectant[FF,KK,4];
F3=Transvectant[FF,KK,3];F2=Transvectant[FF,KK,2];
F1=Transvectant[FF,KK,1];PP=Transvectant[FF,II,2];
Teta=Transvectant[FF,II,1];TT=Transvectant[FF,HH,1];
Inv4=Transvectant[KK,KK,4];Delta=Transvectant[KK,KK,2];
Ik=Transvectant[II,KK,4];I3=Transvectant[II,KK,3];
I2=Transvectant[II,KK,2];I1=Transvectant[II,KK,1];
H3=Transvectant[HH,KK,3];H2=Transvectant[HH,KK,2];
H1=Transvectant[HH,KK,1];SS=Transvectant[HH,II,1];
Inv5=Transvectant[FF,exp[KK,2],8];FK3=Transvectant[Fk,KK,3];
FK2=Transvectant[Fk,KK,2];Fdelta=Transvectant[FF,Delta,4];
FK1=Transvectant[Fk,KK,1];F3rd=Transvectant[FF,Delta,3];
F2nd=Transvectant[FF,Delta,2];F1st=Transvectant[FF,Delta,1];
PP3=Transvectant[PP,KK,3];Tetak=Transvectant[Teta,KK,4];
PP1=Transvectant[PP,KK,1];Inv6=Transvectant[II,exp[KK,2],8];
Ik3=Transvectant[Ik,KK,3];Ik2=Transvectant[Ik,KK,2];
Idelta=Transvectant[II,Delta,4];Ik1=Transvectant[Ik,KK,1];
I3rd=Transvectant[II,Delta,3];Tau=Transvectant[KK,Delta,1];
I2nd=Transvectant[II,Delta,2];I1st=Transvectant[II,Delta,1];
Inv7=Transvectant[Fk,Delta,4];Fk3rd=Transvectant[Fk,Delta,3];
Ftau6=Transvectant[FF,Tau,6];Fk2nd=Transvectant[Fk,Delta,2];
Ftau5=Transvectant[FF,Tau,5];Fk1st=Transvectant[Fk,Delta,1];
Ftau4=Transvectant[FF,Tau,4];Tetakk=Transvectant[Teta,exp[KK,2],8];
Inv8=Transvectant[Ik,Delta,4];Ik3rd=Transvectant[Ik,Delta,3];
Itau6=Transvectant[II,Tau,6];Ik2nd=Transvectant[Ik,Delta,2];
Itau5=Transvectant[II,Tau,5];Ik1st=Transvectant[Ik,Delta,1];
Itau4=Transvectant[II,Tau,4];Inv9=Transvectant[FF,exp[Delta,2],8];
Fktau4=Transvectant[Fk,Tau,4];Fdelta3rd=Transvectant[Fdelta,Delta,3];
Tetakkk=Transvectant[Tetak,exp[KK,2],8];
Fdelta2nd=Transvectant[Fdelta,Delta,2];
Inv10=Transvectant[II,exp[Delta,2],8];Iktau4=Transvectant[Ik,Tau,4];
Idelta3rd=Transvectant[Idelta,Delta,3];
Fdeltatau4=Transvectant[Fdelta,Tau,4];Last=Transvectant[Tetakkk,KK,2];
Ideltatau4=Transvectant[Idelta,Tau,4];
MM=Append[MM,{FF,Inv2,KK,II,HH,Inv3,Fk,F3,F2,F1,PP,Teta,TT,Inv4,Delta,
  Ik,I3,I2,I1,H3,H2,H1,SS,Inv5,FK3,FK2,Fdelta,FK1,F3rd,F2nd,F1st,PP3,
  Tetak,PP1,Inv6,Ik3,Ik2,Idelta,Ik1,I3rd,Tau,I2nd,I1st,Inv7,Fk3rd,Ftau6,
  Fk2nd,Ftau5,Fk1st,Ftau4,Tetakk,Inv8,Ik3rd,Itau6,Ik2nd,Itau5,Ik1st,Itau4,
  Inv9,Fktau4,Fdelta3rd,Tetakkk,Fdelta2nd,Inv10,Iktau4,Idelta3rd,Fdeltatau4,
  Last,Ideltatau4}]]]
In[]:=Timing[MatrixEvaluations[250];Matr=MM;]
```

```
Out[]={107.624000,Null}
```

Next we show that the 69 covariants generate $\mathcal{C}(V_8)_{d,m}$ with $d \leq 13$ and $m \leq 18$. For each $d \leq 13$ and $m \leq 18$ we do the following: we construct all possible monomials of degree d and order m spanned by the 69 covariants and evaluate them at random integers (using the matrix we just created). These monomials will span $\mathcal{C}(V_8)_{d,m}$ if their evaluation matrix has the rank equal to the dimension of $\mathcal{C}(V_8)_{d,m}$. We will use a function that we implemented and called mon{deg,ord} which returns a list of all covariants degree deg and order ord as monomials of degree deg spanned by the above 69 covariants.

```
In[]:=listdeg={{1,8},{2,0},{2,4},{2,8},{2,12},{3,0},{3,4},{3,6},{3,8},
{3,10},{3,12},{3,14},{3,18},{4,0},{4,4},{4,4},{4,6},{4,8},{4,10},
{4,10},{4,12},{4,14},{4,18},{5,0},{5,2},{5,4},{5,4},{5,6},{5,8},
{5,10},{5,10},{5,10},{5,14},{6,0},{6,2},{6,4},{6,4},{6,6},{6,6},{6,6},
{6,8},{6,10},{7,0},{7,2},{7,2},{7,4},{7,4},{7,6},{7,6},{7,6},{8,0},
{8,2},{8,2},{8,4},{8,4},{8,6},{8,6},{9,0},{9,2},{9,2},{9,2},{9,4},{10,0},
{10,2},{10,2},{11,2},{11,2},{12,2}};
In[]:=listgen={FF,Inv2,KK,II,HH,Inv3,Fk,F3,F2,F1,PP,Teta,TT,Inv4,Delta,Ik,
I3,I2,I1,H3,H2,H1,SS,Inv5,FK3,FK2,Fdelta,FK1,F3rd,F2nd,F1st,PP3,Tetak,PP1,
Inv6,Ik3,Ik2,Idelta,Ik1,I3rd,Tau,I2nd,I1st,Inv7,Fk3rd,Ftau8,Fk2nd,Ftau5,
Fk1st,Ftau4,Tetakk,Inv8,Ik3rd,Itau6,Ik2nd,Itau5,Ik1st,Itau4,Inv9,Fktau4,
Fdelta3rd,Tetakkk,Fdelta2nd,Inv10,Iktau4,Idelta3rd,Fdeltatau4,Last11,
Ideltatau4};
In[]:=mon[md_,j_]:= mon[md,j]=If[md=={0,0},1,If[md[[1]]<0||md[[2]]<0,{},
      Flatten[Table[listgen[[i]]mon[md-listdeg[[i]],i],{i,1,j}]]]]
```

Here we show that the 69 covariants generate $\mathcal{C}(V_8)_{d,m}$ with $d \leq 13$ and $m \leq 18$:

```
In[]:=Timing[
 For[deg=2,deg<14,deg++,
  For[j=0,j<10,j++,ord=2j;
   symm=SymTensor[8,deg]/.Table[v[i]->0,{i,19,1000}];
   kk=Coefficient[symm,v[ord]]+1;
   If[kk!=1,
    Print[{deg,ord},": the vector space of covs has dimension",kk-1]];
    ss=mon[{deg,ord},69];
    If[ss!={},
     tt=Intersection[ss,listgen];
      If[tt!={},
       ss1 = Complement[ss, tt];
       eval1=Table[ss1/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
         {j,1,Length[Matr[[k]]]}],{k,1,kk}];
       rk1=If[ss1=={},0,MatrixRank[eval1,Modulus->32003]];
      Print["-- the subpace spanned by covs of degree < ", deg,
       " has dim >=", rk1];
      Print["-- there are ", Length[tt], " gens of degree ", deg];
      eval=Table[ss/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
        {j,1,Length[Matr[[k]]]}],{k,1,kk}];
      rk = MatrixRank[eval, Modulus -> 32003];
      Print["-- the subpace spanned by covs of degree <= ", deg,
       " has dim >=", rk],
      eval=Table[ss/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
        {j,1,Length[Matr[[k]]]}],{k,1,kk}];
```

```
      rk=MatrixRank[eval,Modulus->32003];
      Print["-- the subpace spanned by covs of degree < ", deg,
       " has dim >=", rk]]]
 ]]]
```
{2,0}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,4}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,8}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,12}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=0
-- there are 1 gens of degree 2
-- the subpace spanned by covs of degree <= 2 has dim >=1
{2,16}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 2 has dim >=1
{3,0}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,4}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,6}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,8}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 3 has dim >=1
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=2
{3,10}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,12}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 3 has dim >=1
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=2
{3,14}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{3,16}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=1
{3,18}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 3 has dim >=0
-- there are 1 gens of degree 3
-- the subpace spanned by covs of degree <= 3 has dim >=1
{4,0}: the vector space of covs has dimension 2

```
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{4,4}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 2 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=3
{4,6}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 4 has dim >=0
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=1
{4,8}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 4 has dim >=3
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=4
{4,10}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 4 has dim >=0
-- there are 2 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{4,12}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 4 has dim >=3
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=4
{4,14}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{4,16}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 4 has dim >=4
{4,18}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 4 has dim >=1
-- there are 1 gens of degree 4
-- the subpace spanned by covs of degree <= 4 has dim >=2
{5,0}: the vector space of covs has dimension 2
-- the subpace spanned by covs of degree < 5 has dim >=1
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=2
{5,2}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 5 has dim >=0
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=1
{5,4}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 5 has dim >=2
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=4
{5,6}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 5 has dim >=1
-- there are 2 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=3
{5,8}: the vector space of covs has dimension 6
-- the subpace spanned by covs of degree < 5 has dim >=5
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=6
{5,10}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 5 has dim >=2
-- there are 3 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=5
```

```
{5,12}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 5 has dim >=7
{5,14}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 5 has dim >=4
-- there are 1 gens of degree 5
-- the subpace spanned by covs of degree <= 5 has dim >=5
{5,16}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 5 has dim >=7
{5,18}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 5 has dim >=5
{6,0}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 6 has dim >=3
-- there are 1 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=4
{6,2}: the vector space of covs has dimension 1
-- the subpace spanned by covs of degree < 6 has dim >=0
-- there are 1 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=1
{6,4}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 6 has dim >=5
-- there are 2 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=7
{6,6}: the vector space of covs has dimension 5
-- the subpace spanned by covs of degree < 6 has dim >=2
-- there are 3 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=5
{6,8}: the vector space of covs has dimension 11
-- the subpace spanned by covs of degree < 6 has dim >=10
-- there are 1 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=11
{6,10}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 6 has dim >=6
-- there are 1 gens of degree 6
-- the subpace spanned by covs of degree <= 6 has dim >=7
{6,12}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 6 has dim >=13
{6,14}: the vector space of covs has dimension 9
-- the subpace spanned by covs of degree < 6 has dim >=9
{6,16}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 6 has dim >=13
{6,18}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 6 has dim >=10
{7,0}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 7 has dim >=3
-- there are 1 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=4
{7,2}: the vector space of covs has dimension 3
-- the subpace spanned by covs of degree < 7 has dim >=1
-- there are 2 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=3
{7,4}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 7 has dim >=8
-- there are 2 gens of degree 7
-- the subpace spanned by covs of degree <= 7 has dim >=10
{7,6}: the vector space of covs has dimension 9
-- the subpace spanned by covs of degree < 7 has dim >=6
-- there are 3 gens of degree 7
```

```
-- the subpace spanned by covs of degree <= 7 has dim >=9
{7,8}: the vector space of covs has dimension 16
-- the subpace spanned by covs of degree < 7 has dim >=16
{7,10}: the vector space of covs has dimension 14
-- the subpace spanned by covs of degree < 7 has dim >=14
{7,12}: the vector space of covs has dimension 19
-- the subpace spanned by covs of degree < 7 has dim >=19
{7,14}: the vector space of covs has dimension 17
-- the subpace spanned by covs of degree < 7 has dim >=17
{7,16}: the vector space of covs has dimension 21
-- the subpace spanned by covs of degree < 7 has dim >=21
{7,18}: the vector space of covs has dimension 18
-- the subpace spanned by covs of degree < 7 has dim >=18
{8,0}: the vector space of covs has dimension 7
-- the subpace spanned by covs of degree < 8 has dim >=6
-- there are 1 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=7
{8,2}: the vector space of covs has dimension 4
-- the subpace spanned by covs of degree < 8 has dim >=2
-- there are 2 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=4
{8,4}: the vector space of covs has dimension 16
-- the subpace spanned by covs of degree < 8 has dim >=14
-- there are 2 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=16
{8,6}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 8 has dim >=11
-- there are 2 gens of degree 8
-- the subpace spanned by covs of degree <= 8 has dim >=13
{8,8}: the vector space of covs has dimension 25
-- the subpace spanned by covs of degree < 8 has dim >=25
{8,10}: the vector space of covs has dimension 21
-- the subpace spanned by covs of degree < 8 has dim >=21
{8,12}: the vector space of covs has dimension 31
-- the subpace spanned by covs of degree < 8 has dim >=31
{8,14}: the vector space of covs has dimension 26
-- the subpace spanned by covs of degree < 8 has dim >=26
{8,16}: the vector space of covs has dimension 35
-- the subpace spanned by covs of degree < 8 has dim >=35
{8,18}: the vector space of covs has dimension 29
-- the subpace spanned by covs of degree < 8 has dim >=29
{9,0}: the vector space of covs has dimension 8
-- the subpace spanned by covs of degree < 9 has dim >=7
-- there are 1 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=8
{9,2}: the vector space of covs has dimension 8
-- the subpace spanned by covs of degree < 9 has dim >=5
-- there are 3 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=8
{9,4}: the vector space of covs has dimension 21
-- the subpace spanned by covs of degree < 9 has dim >=20
-- there are 1 gens of degree 9
-- the subpace spanned by covs of degree <= 9 has dim >=21
{9,6}: the vector space of covs has dimension 22
-- the subpace spanned by covs of degree < 9 has dim >=22
{9,8}: the vector space of covs has dimension 35
-- the subpace spanned by covs of degree < 9 has dim >=35
```

```
{9,10}: the vector space of covs has dimension 33
-- the subpace spanned by covs of degree < 9 has dim >=33
{9,12}: the vector space of covs has dimension 45
-- the subpace spanned by covs of degree < 9 has dim >=45
{9,14}: the vector space of covs has dimension 42
-- the subpace spanned by covs of degree < 9 has dim >=42
{9,16}: the vector space of covs has dimension 51
-- the subpace spanned by covs of degree < 9 has dim >=51
{9,18}: the vector space of covs has dimension 48
-- the subpace spanned by covs of degree < 9 has dim >=48
{10,0}: the vector space of covs has dimension 12
-- the subpace spanned by covs of degree < 10 has dim >=11
-- there are 1 gens of degree 10
-- the subpace spanned by covs of degree <= 10 has dim >=12
{10,2}: the vector space of covs has dimension 10
-- the subpace spanned by covs of degree < 10 has dim >=8
-- there are 2 gens of degree 10
-- the subpace spanned by covs of degree <= 10 has dim >=10
{10,4}: the vector space of covs has dimension 32
-- the subpace spanned by covs of degree < 10 has dim >=32
{10,6}: the vector space of covs has dimension 30
-- the subpace spanned by covs of degree < 10 has dim >=30
{10,8}: the vector space of covs has dimension 51
-- the subpace spanned by covs of degree < 10 has dim >=51
{10,10}: the vector space of covs has dimension 48
-- the subpace spanned by covs of degree < 10 has dim >=48
{10,12}: the vector space of covs has dimension 66
-- the subpace spanned by covs of degree < 10 has dim >=66
{10,14}: the vector space of covs has dimension 61
-- the subpace spanned by covs of degree < 10 has dim >=61
{10,16}: the vector space of covs has dimension 77
-- the subpace spanned by covs of degree < 10 has dim >=77
{10,18}: the vector space of covs has dimension 70
-- the subpace spanned by covs of degree < 10 has dim >=70
{11,0}: the vector space of covs has dimension 13
-- the subpace spanned by covs of degree < 11 has dim >=13
{11,2}: the vector space of covs has dimension 17
-- the subpace spanned by covs of degree < 11 has dim >=15
-- there are 2 gens of degree 11
-- the subpace spanned by covs of degree <= 11 has dim >=17
{11,4}: the vector space of covs has dimension 42
-- the subpace spanned by covs of degree < 11 has dim >=42
{11,6}: the vector space of covs has dimension 45
-- the subpace spanned by covs of degree < 11 has dim >=45
{11,8}: the vector space of covs has dimension 69
-- the subpace spanned by covs of degree < 11 has dim >=69
{11,10}: the vector space of covs has dimension 70
-- the subpace spanned by covs of degree < 11 has dim >=70
{11,12}: the vector space of covs has dimension 91
-- the subpace spanned by covs of degree < 11 has dim >=91
{11,14}: the vector space of covs has dimension 90
-- the subpace spanned by covs of degree < 11 has dim >=90
{11,16}: the vector space of covs has dimension 108
-- the subpace spanned by covs of degree < 11 has dim >=108
{11,18}: the vector space of covs has dimension 105
-- the subpace spanned by covs of degree < 11 has dim >=105
{12,0}: the vector space of covs has dimension 20
```

```
-- the subpace spanned by covs of degree < 12 has dim >=20
{12,2}: the vector space of covs has dimension 22
-- the subpace spanned by covs of degree < 12 has dim >=21
-- there are 1 gens of degree 12
-- the subpace spanned by covs of degree <= 12 has dim >=22
{12,4}: the vector space of covs has dimension 58
-- the subpace spanned by covs of degree < 12 has dim >=58
{12,6}: the vector space of covs has dimension 61
-- the subpace spanned by covs of degree < 12 has dim >=61
{12,8}: the vector space of covs has dimension 96
-- the subpace spanned by covs of degree < 12 has dim >=96
{12,10}: the vector space of covs has dimension 95
-- the subpace spanned by covs of degree < 12 has dim >=95
{12,12}: the vector space of covs has dimension 128
-- the subpace spanned by covs of degree < 12 has dim >=128
{12,14}: the vector space of covs has dimension 124
-- the subpace spanned by covs of degree < 12 has dim >=124
{12,16}: the vector space of covs has dimension 152
-- the subpace spanned by covs of degree < 12 has dim >=152
{12,18}: the vector space of covs has dimension 147
-- the subpace spanned by covs of degree < 12 has dim >=147
{13,0}: the vector space of covs has dimension 22
-- the subpace spanned by covs of degree < 13 has dim >=22
{13,2}: the vector space of covs has dimension 33
-- the subpace spanned by covs of degree < 13 has dim >=33
{13,4}: the vector space of covs has dimension 75
-- the subpace spanned by covs of degree < 13 has dim >=75
{13,6}: the vector space of covs has dimension 85
-- the subpace spanned by covs of degree < 13 has dim >=85
{13,8}: the vector space of covs has dimension 126
-- the subpace spanned by covs of degree < 13 has dim >=126
{13,10}: the vector space of covs has dimension 133
-- the subpace spanned by covs of degree < 13 has dim >=133
{13,12}: the vector space of covs has dimension 169
-- the subpace spanned by covs of degree < 13 has dim >=169
{13,14}: the vector space of covs has dimension 173
-- the subpace spanned by covs of degree < 13 has dim >=173
{13,16}: the vector space of covs has dimension 205
-- the subpace spanned by covs of degree < 13 has dim >=205
{13,18}: the vector space of covs has dimension 205
-- the subpace spanned by covs of degree < 13 has dim >=205
Out[63]= {48.459, Null}
```

Further we show that no generating covariants of V_8 occur in degree ≤ 16 and orders 2, 4, or 6:

```
In[]:=Timing[
 For[deg=14,deg<17,deg++,
  For[j=1,j<4,j++,ord=2j;
   symm=SymTensor[8,deg]/.Table[v[i]->0,{i,19,1000}];
   kk=Coefficient[symm,v[ord]]+1;
   If[kk!=1,
    Print[{deg,ord},": the vector space of covs has dimension",kk-1]];
   ss=mon[{deg,ord},69];
   eval=Table[ss/.Table[listgen[[j]]->Matr[[k]][[j]][[1]],
     {j,1,Length[Matr[[k]]]}],{k,1,kk}];
   rk=MatrixRank[eval,Modulus->32003];
```

```
    Print["-- the subpace spanned by covs of degree <= ", deg,
     " has dim >=", rk];
 ]]]
{14,2}: the vector space of covs has dimension 42
-- the subpace spanned by covs of degree < 14 has dim >=42
{14,4}: the vector space of covs has dimension 101
-- the subpace spanned by covs of degree < 14 has dim >=101
{14,6}: the vector space of covs has dimension 111
-- the subpace spanned by covs of degree < 14 has dim >=111
{15,2}: the vector space of covs has dimension 59
-- the subpace spanned by covs of degree < 15 has dim >=59
{15,4}: the vector space of covs has dimension 126
-- the subpace spanned by covs of degree < 15 has dim >=126
{15,6}: the vector space of covs has dimension 150
-- the subpace spanned by covs of degree < 15 has dim >=150
{16,2}: the vector space of covs has dimension 74
-- the subpace spanned by covs of degree < 16 has dim >=74
{16,4}: the vector space of covs has dimension 165
-- the subpace spanned by covs of degree < 16 has dim >=165
{16,6}: the vector space of covs has dimension 190
-- the subpace spanned by covs of degree < 16 has dim >=190
Out[]= {10.3206, Null}
```

A.9 The invariants of $V_2 \oplus V_3$

Here are the Mathematica computations supporting the proof of Proposition
5.9.2. We first compute the invariants j_2, j_3, j_4, j_5, j_7 of $V_2 \oplus V_3$:

```
qq=Table[a[2-i]Binomial[2,i],{i,0,2}];
cc=Table[b[3-i]Binomial[3,i],{i,0,3}];
cc1=Transvectant[cc,cc,2];cc2=Transvectant[cc1,cc,1];
j21=Transvectant[qq,qq,2].{1};j31=Transvectant[qq,cc1,2].{1};
j41=Transvectant[cc1,cc1,2].{1};
j51=Transvectant[exp[cc,2],exp[qq,3],6].{1};
j71=Transvectant[exp[qq,3],prod[cc,cc2],6].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j31},{j41},{j51}};degrees={2,3,4,5};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,3}];
MatrixRank[mat]
Out[]=2
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j51}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,3}];
```

```
MatrixRank[mat]
Out[]=2
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=3
In[]:=mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,4}];
MatrixRank[mat]
Out[]=3
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j71}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];
MatrixRank[mat]
Out[]=4
```

A.10   The invariants of $V_2 \oplus V_4$

Here are the Mathematica computations supporting the proof of Proposition 5.10.2. We first compute the invariants $j_{2,1}, j_{2,2}, j_{3,1}, j_{3,2}, j_4, j_6$ of $V_2 \oplus V_4$:

```
qq=Table[a[2-i]Binomial[2,i],{i,0,2}];
ff=Table[b[4-i]Binomial[4,i],{i,0,4}];
cc1=Transvectant[ff,ff,2];cc2=Transvectant[cc1,ff,1];
cc3=Transvectant[qq,ff,2];j21=Transvectant[qq,qq,2].{1};
j22=Transvectant[ff,ff,4].{1};j31=Transvectant[ff,cc1,4].{1};
j32=Transvectant[ff,exp[qq,2],4].{1};
j41=Transvectant[cc1,exp[qq,2],4].{1};
j61=Transvectant[cc2,exp[qq,3],6].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21,j22},{j31,j32},{j41}};degrees={2,3,4};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j41}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];
MatrixRank[mat]
Out[]=4
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=4
In[]:=mat=Table[mon1/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];
MatrixRank[mat]
Out[]=4
=== computations in degree 6 ===
```

```
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=9
In[]:=mon=Union[mon1,{j61}];
In[]:=mat=Table[mon/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,11}];
MatrixRank[mat]
Out[]=10
```

A.11   The invariants of $V_2 \oplus V_5$

Here are the computations supporting the proof of Proposition 5.11.2. We first
compute the invariants $j_2, j_3, j_4, j_5, j_6, j_{7,1}, \ldots, j_{7,3}, j_{8,1}, \ldots, j_{8,3}, j_{9,1} \ldots, j_{9,3}, j_{10,1}$,
$j_{10,2}, j_{11,1}, \ldots, j_{11,3}, j_{12,1}, \ldots, j_{12,3}, j_{13,1}, \ldots, j_{13,3}, j_{14}, j_{15}, j_{16}, j_{17}, j_{18}$ of $V_2 \oplus V_5$:

```
qq=Table[a[2-i]Binomial[2,i],{i,0,2}];
ff=Table[b[5-i]Binomial[5,i],{i,0,5}];
cc1=Transvectant[ff,ff,4];cc2=Transvectant[ff,cc1,2];
cc3=Transvectant[ff,ff,2];cc4=Transvectant[ff,cc1,1];
cc5=Transvectant[ff,cc3,1];j21=Transvectant[qq,qq,2].{1};
j31=Transvectant[cc1,qq,2].{1};j41=Transvectant[cc1,cc1,2].{1};
j51=Transvectant[cc3,exp[qq,3],6].{1};
j61=Transvectant[Transvectant[cc1,cc3,2],exp[qq,2],4].{1};
j71=Transvectant[exp[qq,5],exp[ff,2],10].{1};
j72=Transvectant[Transvectant[cc4,ff,2],exp[qq,3],6].{1};
j73=Transvectant[Transvectant[exp[cc1,2],cc3,4],qq,2].{1};
j81=Transvectant[Transvectant[cc2,cc2,2],cc1,2].{1};
j82=Transvectant[Transvectant[exp[cc1,2],cc3,3],exp[qq,2],4].{1};
j83=Transvectant[prod[ff,cc2],exp[qq,4],8].{1};
j91=Transvectant[prod[ff,Transvectant[exp[cc1,2],ff,4]],exp[qq,3],6].{1};
j92=Transvectant[Transvectant[exp[cc1,3],cc3,5],qq,2].{1};
j93=Transvectant[exp[qq,5],prod[cc4,ff],10].{1};
j101=Transvectant[prod[Transvectant[exp[cc1,2],ff,4],cc2],
 exp[qq,2],4].{1};
j102=Transvectant[prod[cc4,cc2],exp[qq,4],8].{1};
j111=Transvectant[prod[ff,cc5],exp[qq,7],14].{1};
j112=Transvectant[prod[ff,Transvectant[exp[cc1,3],ff,5]],exp[qq,3],6].{1};
j113=Transvectant[exp[Transvectant[exp[cc1,2],ff,4],2],qq,2].{1};
j121=Transvectant[Transvectant[cc2,cc2,2],
 Transvectant[cc2,cc2,2],2].{1};
j122=Transvectant[prod[Transvectant[exp[cc1,3],ff,5],cc2],
 exp[qq,2],4].{1};
j123=Transvectant[prod[Transvectant[cc1,cc5,2],ff],exp[qq,6],12].{1};
j131=Transvectant[prod[Transvectant[exp[cc1,3],ff,5],
 Transvectant[exp[cc1,2],ff,4]],qq,2].{1};
j132=Transvectant[prod[ff,Transvectant[exp[cc1,2],cc5,4]],
 exp[qq,5],10].{1};
j141=Transvectant[prod[ff,Transvectant[exp[cc1,3],cc5,6]],
 exp[qq,4],8].{1};
j151=Transvectant[prod[ff,Transvectant[exp[cc1,4],cc5,8]],
exp[qq,3],6].{1};
j161=Transvectant[prod[cc2,Transvectant[exp[cc1,4],cc5,8]],
 exp[qq,2],4].{1};
j171=Transvectant[prod[Transvectant[exp[cc1,2],ff,4],
 Transvectant[exp[cc1,4],cc5,8]],qq,2].{1};
```

```
j181=Transvectant[exp[cc1,7],prod[ff,cc5],14].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j31},{j41},{j51},{j61},{j71,j72,j73},
{j81,j82,j83},{j91,j92,j93},{j101,j102},{j111,j112,j113},
{j121,j122,j123},{j131,j132},{j141},{j151},{j161},{j171}};
degrees={2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,3}];
MatrixRank[mat]
Out[]=2
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j51}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,3}];
MatrixRank[mat]
Out[]=2
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j61}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];
MatrixRank[mat]
Out[]=4
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j71,j72,j73}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,7}];
MatrixRank[mat]
Out[]=6
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=6
In[]:=mon=Union[mon1,{j81,j82,j83}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,10}];
MatrixRank[mat]
Out[]=9
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=9
In[]:=mon=Union[mon1,{j91,j92,j93}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,13}];
MatrixRank[mat]
Out[]=12
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
```

```
Out[]=15
In[]:=mon=Union[mon1,{j101,j102}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,18}];
MatrixRank[mat]
Out[]=17
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=21
In[]:=mon=Union[mon1,{j111,j112,j113}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,25}];
MatrixRank[mat]
Out[]=24
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=31
In[]:=mon=Union[mon1,{j121,j122,j123}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,34}];
MatrixRank[mat]]
Out[]={3.33221,33}  //the rank is 33;
                    //the computation took 3.3 seconds
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=42
In[]:=mon=Union[mon1,{j131,j132}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,43}];
MatrixRank[mat]]
Out[]={6.14438,42}  //the rank is 42;
                    //the computation took 6.1 seconds
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=62
In[]:=mon=Union[mon1,{j141}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,57}];
MatrixRank[mat]]
Out[]={12.4968,56}  //the rank is 56;
                    //the computation took 12.4 seconds
=== computations in degree 15 ===
In[]:=mon1=listmonomials[15,gens,degrees];Length[mon1]
Out[]=83
In[]:=mon=Union[mon1,{j151}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,74}];
MatrixRank[mat]]
Out[]={30.7419,73}  //the rank is 73;
                    //the computation took 30.7 seconds
=== computations in degree 16 ===
In[]:=mon1=listmonomials[16,gens,degrees];Length[mon1]
Out[]=115
In[]:=mon=Union[mon1,{j161}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,95}];
MatrixRank[mat]]
```

```
Out[]={65.2281,94}  //the rank is 94;
                      //the computation took 65.2 seconds
=== computations in degree 17 ===
In[]:=mon1=listmonomials[17,gens,degrees];Length[mon1]
Out[]=150
In[]:=mon=Union[mon1,{j171}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,118}];
MatrixRank[mat]]
Out[]={127.956,117}  //the rank is 117;
                      //the computation took 127.9 seconds
=== computations in degree 18 ===
In[]:=mon1=listmonomials[18,gens,degrees];Length[mon1]
Out[]=206
In[]:=mon=Union[mon1,{j181}];
In[]:=Timing[mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,149}];
MatrixRank[mat]]
Out[]={335.941,148}  //the rank is 148;
                      //the computation took 335.9 seconds
```

For the computations in degrees ≥ 19, we write a matrix `eval` with 1240 evaluations of our 29 invariants. This saves computation time.

```
In[]:=gens={{jj21},{jj31},{jj41},{jj51},{jj61},{jj71,jj72,jj73},
{jj81,jj82,jj83},{jj91,jj92,jj93},{jj101,jj102},
{jj111,jj112,jj113},{jj121,jj122,jj123},{jj131,jj132},{jj141},
{jj151},{jj161},{jj171},{jj181}};
degrees={2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18};
gens1=Flatten[gens];
=== computations in degree 19 ===
In[]:=mon=listmonomials[19,gens,degrees];Length[mon]
Out[]=269
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,184}];
MatrixRank[mat,Modulus->32003]]
Out[]={1.00006,183}  //the rank is 183;
                      //the computation took 1.0 seconds
=== computations in degree 20 ===
In[]:=mon=listmonomials[20,gens,degrees];Length[mon]
Out[239]=360
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,227}];
MatrixRank[mat,Modulus->32003]]
Out[]={1.46809,226}  //the rank is 226;
                      //the computation took 1.4 seconds
=== computations in degree 21 ===
In[]:=mon=listmonomials[21,gens,degrees];Length[mon]
Out[]=467
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,275}];
MatrixRank[mat,Modulus->32003]]
Out[]={2.32415,274}  //the rank is 274;
                      //the computation took 2.3 seconds
=== computations in degree 22 ===
In[]:=mon=listmonomials[22,gens,degrees];Length[mon]
Out[]=615
```

```
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,333}];
MatrixRank[mat,Modulus->32003]]
Out[]={4.42828,332}  //the rank is 332;
                     //the computation took 4.4 seconds
=== computations in degree 23 ===
In[]:=mon=listmonomials[23,gens,degrees];Length[mon]
Out[]=796
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,401}];
MatrixRank[mat,Modulus->32003]]
Out[]={7.07644,400}  //the rank is 400;
                     //the computation took 7.0 seconds
=== computations in degree 24 ===
In[]:=mon=listmonomials[24,gens,degrees];Length[mon]
Out[]=1039
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,480}];
MatrixRank[mat,Modulus->32003]]
Out[]={11.7567,479}  //the rank is 479;
                     //the computation took 11.7 seconds
=== computations in degree 26 ===
In[]:=mon=listmonomials[26,gens,degrees];Length[mon]
Out[]=1721
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,670}];
MatrixRank[mat,Modulus->32003]]
Out[]={32.9141,669}  //the rank is 669;
                     //the computation took 32.9 seconds
=== computations in degree 28 ===
In[]:=mon=listmonomials[28,gens,degrees];Length[mon]
Out[]=2815
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,921}];
MatrixRank[mat,Modulus->32003]]
Out[]={89.5536,920}  //the rank is 920;
                     //the computation took 89.5 seconds
=== computations in degree 30 ===
In[]:=mon=listmonomials[30,gens,degrees];Length[mon]
Out[]=4543
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,1239}];
MatrixRank[mat,Modulus->32003]]
Out[]={226.434,1238}  //the rank is 1238;
                      //the computation took 226.4 seconds
```

A.12   The invariants of $V_2 \oplus V_6$

Here are the Mathematica computations supporting the proof of Proposition 5.12.2. We first compute the invariants $j_{2,1}, j_{2,2}, j_{4,1}, \ldots, j_{4,4}, j_{6,1}, \ldots, j_{6,5}, j_{7,1}, \ldots, j_{7,3}, j_8, j_{9,1}, \ldots, j_{9,7}, j_{10}, j_{11,1}, j_{11,2}, j_{13}, j_{15}$ of $V_2 \oplus V_6$:

```
qq=Table[a[2-i]Binomial[2,i],{i,0,2}];
ss=Table[b[6-i]Binomial[6,i],{i,0,6}];
cc1=Transvectant[ss,ss,4];cc2=Transvectant[ss,ss,2];
```

```
cc3=Transvectant[ss,cc1,4];cc4=Transvectant[cc1,cc1,2];
cc5=Transvectant[ss,cc1,1];j21=Transvectant[ss,ss,6].{1};
j22=Transvectant[qq,qq,2].{1};j41=Transvectant[cc1,cc1,4].{1};
j42=Transvectant[cc1,exp[qq,2],4].{1};j43=Transvectant[ss,exp[qq,3],6].{1};
j44=Transvectant[cc3,qq,2].{1};j61=Transvectant[cc1,cc4,4].{1};
j62=Transvectant[cc2,exp[qq,4],8].{1};
j63=Transvectant[Transvectant[cc1,cc3,2],qq,2].{1};
j64=Transvectant[Transvectant[ss,cc1,2],exp[qq,3],6].{1};
j65=Transvectant[Transvectant[ss,cc3,2],exp[qq,2],4].{1};
j71=Transvectant[Transvectant[cc1,cc3,1],exp[qq,2],4].{1};
j72=Transvectant[Transvectant[ss,cc3,1],exp[qq,3],6].{1};
j73=Transvectant[Transvectant[ss,cc1,1],exp[qq,4],8].{1};
j81=Transvectant[Transvectant[ss,exp[cc3,2],4],qq,2].{1};
j91=Transvectant[Transvectant[cc1,exp[cc3,2],3],qq,2].{1};
j92=Transvectant[Transvectant[ss,exp[cc3,2],3],exp[qq,2],4].{1};
j93=Transvectant[Transvectant[Transvectant[ss,cc1,2],cc3,1],
 exp[qq,3],6].{1};
j94=Transvectant[Transvectant[Transvectant[ss,cc1,1],cc3,2],
 exp[qq,3],6].{1};
j95=Transvectant[Transvectant[cc2,cc3,1],exp[qq,4],8].{1};
j96=Transvectant[Transvectant[cc2,cc1,1],exp[qq,5],10].{1};
j97=Transvectant[Transvectant[cc2,ss,1],exp[qq,6],12].{1};
j101=Transvectant[cc4,exp[cc3,2],4].{1};
j111=Transvectant[Transvectant[ss,exp[cc3,3],5],qq,2].{1};
j112=Transvectant[Transvectant[Transvectant[ss,cc1,1],
 exp[cc3,2],4],exp[qq,2],4].{1};
j131=Transvectant[Transvectant[Transvectant[ss,cc1,1],
 exp[cc3,3],6],qq,2].{1};
j151=Transvectant[cc5,exp[cc3,4],8].{1};
```

Here are the Singular [DGPS] computations supporting the proof of Proposition 5.12.1:

```
> ring r=0,(a0,a1,a2,b0,b1,b2,b3,b4,b5,b6),dp;
   //here we import the invariants we computed with Mathematica:
> <"j21_26.csv";
> <"j41_26.csv";
> <"j42_26.csv";
> <"j43_26.csv";
> <"j44_26.csv";
> <"j61_26.csv";
> <"j62_26.csv";
> <"j101_26.csv";
  //here we show that if q=x^2 and b6=1, then
  (j21,j41+j42-2*j44,j42-j43+j44,j61,j101) is the unit ideal:
> ideal ii=a0-1,a1,a2,b6-1,b4-b5^2,j21,j41+j42-2*j44,
j42-j43+j44,j61,j101;
> ideal jj=std(ii);jj;
jj[1]=1
```

Here are the rank computations supporting the proof of Proposition 5.12.2:

```
In[]:=gens={{j21,j22},{j41,j42,j43,j44},{j61,j62,j63,j64,j65},
{j71,j72,j73},{j81},{j91,j92,j93,j94,j95,j96,j97},{j101},
{j111,j112},{j131}};degrees={2,4,6,7,8,9,10,11,13};
=== computations in degree 2 ===
In[]:=mon={j21,j22};
```

```
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,3}];MatrixRank[mat]
Out[]=2
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j41,j42,j43,j44}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,8}];MatrixRank[mat]
Out[]=7
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=12
In[]:=mon=Union[mon1,{j61,j62,j63,j64,j65}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,18}];MatrixRank[mat]
Out[]=17
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=0
In[]:=mon=Union[mon1,{j71,j72,j73}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,18}];MatrixRank[mat]
Out[]=3
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=37
In[]:=mon=Union[mon1,{j81}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,39}];MatrixRank[mat]
Out[]=38
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=6
In[]:=mon=Union[mon1,{j91,j92,j93,j94,j95,j96,j97}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,14}];MatrixRank[mat]
Out[]=13
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=79
In[]:=mon=Union[mon1,{j101}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,79}];MatrixRank[mat]
Out[]=78
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=35
In[]:=mon=Union[mon1,{j111,j112}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,37}];MatrixRank[mat]
Out[]=36
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=161
In[]:=mon=Union[mon1,{}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
```

```
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,152}];MatrixRank[mat]
Out[]=151
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=104
In[]:=mon=Union[mon1,{j131}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,87}];MatrixRank[mat]
Out[]=86
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=307
In[]:=mon=Union[mon1,{}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,272}];MatrixRank[mat]
Out[]=271
=== computations in degree 15 ===
In[]:=mon1=listmonomials[15,gens,degrees];Length[mon1]
Out[]=249
In[]:=mon=Union[mon1,{j151}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,6}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,181}];MatrixRank[mat]
Out[]=180
```

For the computations in degrees ≥ 16, we write a matrix `eval` with 2860 evaluations of our 27 invariants. This saves computation time.

```
=== computations in degree 16 ===
In[]:=mon=listmonomials[16,gens,degrees];Length[mon]
Out[]=585
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,470}];
MatrixRank[mat,Modulus->32003]]
Out[]={14.1249,469}  //the rank is 469;
                     //the computation took 14.1 seconds
=== computations in degree 17 ===
In[]:=mon=listmonomials[17,gens,degrees];Length[mon]
Out[]=549
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,342}];
MatrixRank[mat,Modulus->32003]]
Out[]={6.4084,341}   //the rank is 341;
                     //the computation took 6.4 seconds
=== computations in degree 18 ===
In[]:=mon=listmonomials[18,gens,degrees];Length[mon]
Out[]=1087
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,775}];
MatrixRank[mat,Modulus->32003]]
Out[]={45.0068,774}   //the rank is 774;
                      //the computation took 45.0 seconds
=== computations in degree 20 ===
In[]:=mon=listmonomials[20,gens,degrees];Length[mon]
Out[]=1986
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,1234}];
MatrixRank[mat,Modulus->32003]]
```

```
Out[]={162.37,1233}   //the rank is 1233;
                      //the computation took 162.3 seconds
=== computations in degree 24 ===
In[]:=mon=listmonomials[24,gens,degrees];Length[mon]
Out[]=6465
In[]:=Timing[mat=Table[mon/.
Table[gens1[[j]]->eval[[i]][[j]],{j,1,Length[gens1]}],{i,1,2858}];
MatrixRank[mat,Modulus->32003]]
Out[]={1836.,2857}    //the rank is 2857;
                      //the computation took 1836 seconds
```

A.13   The invariants of $V_3 \oplus V_4$

Here are the Mathematica computations supporting the proof of Proposition 5.13.2. We first compute the invariants $j_2, j_3, j_4, j_{5,1}, j_{5,2}, j_{6,1}, j_{6,2}, j_{7,1}, \ldots, , j_{7,3}, j_{8,1}, \ldots, j_{8,3}, j_{9,1}, \ldots, j_{9,4}, j_{10,1}, j_{10,2}, j_{11}$ of $V_3 \oplus V_4$:

```
cc=Table[a[3-i]Binomial[3,i],{i,0,3}];
qq=Table[b[4-i]Binomial[4,i],{i,0,4}];
cc1=Transvectant[cc,cc,2];cc2=Transvectant[qq,qq,2];
j21=Transvectant[qq,qq,4].{1};j31=Transvectant[cc2,qq,4].{1};
j41=Transvectant[cc1,cc1,2].{1};
j51=Transvectant[Transvectant[qq,cc2,1],exp[cc,2],6].{1};
j52=Transvectant[Transvectant[qq,exp[cc,2],2],exp[cc,2],6].{1};
j61=Transvectant[exp[cc1,2],cc2,4].{1};
j62=Transvectant[Transvectant[Transvectant[qq,cc1,2],exp[cc,2],2],
 qq,4].{1};
j71=Transvectant[exp[cc,4],exp[qq,3],12].{1};
j72=Transvectant[
 Transvectant[Transvectant[qq,cc1,2],exp[cc,2],2],cc2,4].{1};
j73=Transvectant[Transvectant[qq,cc2,1],
 prod[cc,Transvectant[cc,cc1,1]],6].{1};
j81=Transvectant[Transvectant[Transvectant[qq,cc1,2],
 prod[cc,Transvectant[cc,cc1,1]],2],qq,4].{1};
j82=Transvectant[exp[cc,4],prod[exp[qq,2],cc2],12].{1};
j83=Transvectant[
 Transvectant[Transvectant[cc2,cc1,2],exp[cc,2],2],cc2,4].{1};
j91=Transvectant[Transvectant[Transvectant[cc2,cc1,2],
 prod[cc,Transvectant[cc,cc1,1]],2],qq,4].{1};
j92=Transvectant[exp[cc,4],prod[prod[qq,cc2],cc2],12].{1};
j93=Transvectant[prod[exp[cc,3],Transvectant[cc,cc1,1]],
 exp[qq,3],12].{1};
j94=Transvectant[Transvectant[Transvectant[qq,cc1,2],
 prod[cc,Transvectant[cc,cc1,1]],2],cc2,4].{1};
j101=Transvectant[Transvectant[Transvectant[cc2,cc1,2],
 prod[cc,Transvectant[cc,cc1,1]],2],cc2,4].{1};
j102=Transvectant[Transvectant[prod[exp[cc,3],
 Transvectant[cc,cc1,1]],exp[qq,3],10],qq,4].{1};
j111=Transvectant[Transvectant[Transvectant[prod[exp[cc,3],
 Transvectant[cc,cc1,1]],exp[qq,3],10],qq,2],4].{1};
```

   Here are the rank computations:

```
In[]:=gens={{j21},{j31},{j41},{j51,j52},{j61,j62},{j71,j72,j73},
{j81,j82,j83},{j91,j92,j93,j94},{j101,j102},{j111}};
```

```
degrees={2,3,4,5,6,7,8,9,10,11};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,3}];MatrixRank[mat]
Out[]=2
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j51,j52}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,4}];MatrixRank[mat]
Out[]=3
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j61,j62}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,6}];MatrixRank[mat]
Out[]=5
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=4
In[]:=mon=Union[mon1,{j71,j72,j73}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,8}];MatrixRank[mat]
Out[]=7
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=8
In[]:=mon=Union[mon1,{j81,j82,j83}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,12}];MatrixRank[mat]
Out[]=11
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=12
In[]:=mon=Union[mon1,{j91,j92,j93,j94}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,17}];MatrixRank[mat]
Out[]=16
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=20
In[]:=mon=Union[mon1,{j101,j102}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,23}];MatrixRank[mat]
Out[]=22
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=29
In[]:=mon=Union[mon1,{j111}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,31}];MatrixRank[mat]
Out[]=30
```

```
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=44
In[]:=mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,43}];MatrixRank[mat]
Out[]=42
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=59
In[]:=mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,56}];MatrixRank[mat]
Out[]=55
=== computations in degree 18 ===
In[]:=mon1=listmonomials[18,gens,degrees];Length[mon1]
Out[]=283
In[]:=mat=Table[mon1/.Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,194}];
MatrixRank[mat,Modulus->32003]
Out[]=193
```

A.14 The invariants of $V_1 \oplus V_2 \oplus V_3$

Here are the Mathematica computations supporting the proof of Proposition 5.14.2. We compute the invariants $j_2, j_{3,1}, \ldots, j_{3,3}, j_{4,1}, \ldots, j_{4,4}, j_{5,1}, \ldots, j_{5,4}, j_{6,1}, j_{6,2}, j_7$ of $V_1 \oplus V_2 \oplus V_3$:

```
ll=Table[a[1-i]Binomial[1,i],{i,0,1}];
qq=Table[b[2-i]Binomial[2,i],{i,0,2}];
cc=Table[c[3-i]Binomial[3,i],{i,0,3}];
cc1=Transvectant[cc,cc,2];cc2=Transvectant[cc1,cc,1];
j21=Transvectant[qq,qq,2].{1};j31=Transvectant[qq,exp[ll,2],2].{1};
j32=Transvectant[qq,cc1,2].{1};j33=Transvectant[cc,prod[qq,ll],3].{1};
j41=Transvectant[cc1,cc1,2].{1};j42=Transvectant[cc,exp[ll,3],3].{1};
j43=Transvectant[cc1,exp[ll,2],2].{1};
j44=Transvectant[prod[cc,ll],exp[qq,2],4].{1};
j51=Transvectant[exp[cc,2],exp[qq,3],6].{1};
j52=Transvectant[Transvectant[Transvectant[cc,ll,1],qq,1],exp[ll,2],2].{1};
j53=Transvectant[qq,Transvectant[cc2,ll,1],2].{1};
j54=Transvectant[Transvectant[cc1,qq,1],exp[ll,2],2].{1};
j61=Transvectant[prod[ll,cc],Transvectant[prod[ll,cc],prod[ll,cc],2],4].{1};
j62=Transvectant[Transvectant[cc1,qq,1],
 Transvectant[prod[ll,qq],cc,2],2].{1};
j71=Transvectant[exp[qq,3],prod[cc,cc2],6].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j31,j32,j33},{j41,j42,j43,j44},{j51,j52,j53,j54},
{j61,j62},{j71}};degrees={2,3,4,5,6,7};
=== computations in degree 3 ===
In[]:=mon1=listmonomials[3,gens,degrees];
In[]:=mon=Union[mon1,{j31,j32,j33}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,3}];MatrixRank[mat]
Out[]=3
```

```
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41,j42,j43,j44}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,6}];MatrixRank[mat]
Out[]=5
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=3
In[]:=mon=Union[mon1,{j51,j52,j53,j54}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,8}];MatrixRank[mat]
Out[]=7
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=11
In[]:=mon=Union[mon1,{j61,j62}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,14}];MatrixRank[mat]
Out[]=13
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=19
In[]:=mon=Union[mon1,{j71}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,21}];MatrixRank[mat]
Out[]=20
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=35
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,32}];MatrixRank[mat]
Out[]=31
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=52
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,45}];MatrixRank[mat]
Out[]=44
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=80
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,64}];MatrixRank[mat]
Out[]=63
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=118
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
```

```
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,89}];MatrixRank[mat]]
Out[]={5.22033,88}      //the rank is 88;
                        //the computation took 5.2 seconds
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=182
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,124}];MatrixRank[mat]]
Out[]={22.3214,123}     //the rank is 123;
                        //the computation took 22.3 seconds
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=379
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,214}];MatrixRank[mat]]
Out[]={25.9656,213}     //the rank is 213;
                        //the computation took 25.9 seconds
```

A.15 The invariants of $V_1 \oplus V_2 \oplus V_4$

Here are the Mathematica computations supporting the proof of Proposition 5.15.2. We first compute the invariants $j_{2,1}, j_{2,2}, j_{3,1}, \ldots, j_{3,3}, j_{4,1}, j_{4,2}, j_{5,1}, \ldots,$ $j_{5,3}, j_{6,1}, \ldots, j_{6,4}, j_{7,1}, j_{7,2}, j_8, j_9$ of $V_1 \oplus V_2 \oplus V_4$:

```
ll=Table[a[1-i]Binomial[1,i],{i,0,1}];
qq=Table[b[2-i]Binomial[2,i],{i,0,2}];
ff=Table[c[4-i]Binomial[4,i],{i,0,4}];
cc1=Transvectant[ff,ff,2];cc2=Transvectant[cc1,ff,1];
cc3=Transvectant[qq,ff,2];j21=Transvectant[qq,qq,2].{1};
j22=Factor[Transvectant[ff,ff,4].{1}];j31=Transvectant[ff,cc1,4].{1};
j32=Transvectant[ff,exp[qq,2],4].{1};j33=Transvectant[qq,exp[ll,2],2].{1};
j41=Transvectant[cc1,exp[qq,2],4].{1};j42=Transvectant[cc3,exp[ll,2],2].{1};
j51=Transvectant[ff,exp[ll,4],4].{1};
j52=Transvectant[cc1,prod[qq,exp[ll,2]],4].{1};
j53=Transvectant[Transvectant[qq,exp[ll,2],1],cc3,2].{1};
j61=Transvectant[cc1,exp[ll,4],4].{1};j62=Transvectant[cc2,exp[qq,3],6].{1};
j63=Transvectant[Transvectant[qq,exp[ll,2],1],Transvectant[qq,cc1,2],2].{1};
j64=Transvectant[Transvectant[qq,ff,1],exp[ll,4],4].{1};
j71=Transvectant[Transvectant[qq,cc1,1],exp[ll,4],4].{1};
j72=Transvectant[Transvectant[cc2,exp[qq,2],4],exp[ll,2],2].{1};
j8=Transvectant[Transvectant[Transvectant[qq,cc1,2],ff,1],exp[ll,4],4].{1};
j9=Transvectant[cc2,exp[ll,6],6].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21,j22},{j31,j32,j33},{j41,j42},{j51,j52,j53},
      {j61,j62,j63,j64},{j71,j72},{j8},{j9}};
      degrees={2,3,4,5,6,7,8,9};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];
      mon=Union[mon1,{j41,j42}];Length[mon]
Out[]=5
```

```
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,6}];MatrixRank[mat]
Out[]=5
=== computations in degree 5
In[]:=mon1=listmonomials[5,gens,degrees];
      mon=Union[mon1,{j51,j52,j53}];Length[mon]
Out[]=9
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,10}];MatrixRank[mat]
Out[]=9
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];
      mon=Union[mon1,{j61,j62,j63,j64}];Length[mon]
Out[]=18
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,19}];MatrixRank[mat]
Out[]=18
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];
      mon=Union[mon1,{j71,j72}];Length[mon]
Out[]=23
In[]:=mat=Table[mon/.
        Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,24}];
      MatrixRank[mat]
Out[]=23
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];
      mon=Union[mon1,{j8}];Length[mon]
Out[]=44
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,44}];MatrixRank[mat]
Out[]=43
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];
      mon=Union[mon1,{j9}];Length[mon]
Out[]=66
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,64}];MatrixRank[mat]
Out[]=63
=== computations in degree 10 ===
In[]:=mon=listmonomials[10,gens,degrees];Length[mon]
Out[]=102
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,94}];MatrixRank[mat]
Out[]=93
=== computations in degree 11 ===
In[]:=mon=listmonomials[11,gens,degrees];Length[mon]
Out[]=155
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
```

```
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,137}];MatrixRank[mat]
Out[]=136
=== computations in degree 12 ===
In[]:=mon=listmonomials[12,gens,degrees];Length[mon]
Out[]=242
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,201}];
      MatrixRank[mat,Modulus->32003]
Out[]=200
=== computations in degree 15 ===
In[]:=mon=listmonomials[15,gens,degrees];Length[mon]
Out[]=740
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
        Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
        Table[a[j]->Random[Integer,{0,10}],{j,0,1}],{i,1,511}];
      MatrixRank[mat,Modulus->32003]
Out[]=510
```

A.16   The invariants of $V_1 \oplus V_2 \oplus V_5$

We compute the generating invariants with Mathematica:

```
qq=Table[a[2-i]Binomial[2,i],{i,0,2}];
ff=Table[b[5-i]Binomial[5,i],{i,0,5}];
ll=Table[c[1-i]Binomial[1,i],{i,0,1}];
cc1=Transvectant[ff,ff,4];cc2=Transvectant[ff,cc1,2];
cc3=Transvectant[ff,ff,2];cc4=Transvectant[ff,cc1,1];
cc5=Transvectant[qq,ff,2];cc6=Transvectant[ff,cc3,1];
cc7=Transvectant[cc1,cc3,2];cc8=Transvectant[cc1,cc3,1];
cc9=Transvectant[cc6,cc1,2];cc10=Transvectant[cc6,exp[cc1,2],4];
cc11=Transvectant[exp[cc1,2],cc3,3];j21=Transvectant[qq,qq,2].{1};
j31=Transvectant[cc1,qq,2].{1};j32=Transvectant[qq,exp[ll,2],2].{1};
j41=Transvectant[cc1,cc1,2].{1};
j42=Transvectant[Transvectant[ff,exp[qq,2],4],ll,1].{1};
j43=Transvectant[cc1,exp[ll,2],2].{1};
j51=Transvectant[cc3,exp[qq,3],6].{1};
j52=Transvectant[Transvectant[cc2,qq,2],ll,1].{1};
j53=Transvectant[Transvectant[ff,exp[qq,3],5],ll,1].{1};
j54=Transvectant[Transvectant[cc1,qq,1],exp[ll,2],2].{1};
j55=Transvectant[Transvectant[ff,qq,2],exp[ll,3],3].{1};
j61=Transvectant[cc7,exp[qq,2],4].{1};
j62=Transvectant[Transvectant[cc1,cc2,2],ll,1].{1};
j63=Transvectant[Transvectant[cc4,exp[qq,2],4],ll,1].{1};
j64=Transvectant[Transvectant[cc3,exp[qq,2],4],exp[ll,2],2].{1};
j65=Transvectant[cc2,exp[ll,3],3].{1};
j66=Transvectant[Transvectant[ff,exp[qq,2],3],exp[ll,3],3].{1};
j67=Transvectant[ff,exp[ll,5],5].{1};
j68=Transvectant[Transvectant[cc2,exp[qq,2],3],ll,1].{1};
j71=Transvectant[exp[qq,5],exp[ff,2],10].{1};
j72=Transvectant[Transvectant[cc4,ff,2],exp[qq,3],6].{1};
j73=Transvectant[Transvectant[exp[cc1,2],cc3,4],qq,2].{1};
j74=Transvectant[Transvectant[ff,qq,1],exp[ll,5],5].{1};
j75=Transvectant[Transvectant[cc3,qq,2],exp[ll,4],4].{1};
j76=Transvectant[Transvectant[cc4,qq,2],exp[ll,3],3].{1};
```

```
j77=Transvectant[Transvectant[cc2,qq,1],exp[ll,3],3].{1};
j78=Transvectant[Transvectant[cc7,qq,2],exp[ll,2],2].{1};
j79=Transvectant[Transvectant[cc3,exp[qq,3],5],exp[ll,2],2].{1};
j710=Transvectant[Transvectant[Transvectant[cc1,cc2,1],qq,2],
 ll,1].{1};
j711=Transvectant[Transvectant[cc4,exp[qq,3],5],ll,1].{1};
j712=Transvectant[Transvectant[Transvectant[cc1,cc2,2],qq,1],
 ll,1].{1};
j81=Transvectant[Transvectant[cc2,cc2,2],cc1,2].{1};
j82=Transvectant[cc11,exp[qq,2],4].{1};
j83=Transvectant[prod[ff,cc2],exp[qq,4],8].{1};
j84=Transvectant[Transvectant[exp[cc1,3],ff,5],ll,1].{1};
j85=Transvectant[Transvectant[exp[qq,4],cc6,8],ll,1].{1};
j86=Transvectant[Transvectant[Transvectant[cc1,cc2,1],exp[qq,2],3],
 ll,1].{1};
j87=Transvectant[Transvectant[exp[cc1,2],cc3,4],exp[ll,2],2].{1};
j88=Transvectant[Transvectant[cc8,exp[qq,2],4],
exp[ll,2],2].{1};
j89=Transvectant[Transvectant[cc7,exp[qq,2],3],
exp[ll,2],2].{1};
j810=Transvectant[Transvectant[cc1,cc2,1],exp[ll,3],3].{1};
j811=Transvectant[cc7,exp[ll,4],4].{1};
j812=Transvectant[Transvectant[cc3,exp[qq,2],3],exp[ll,4],4].{1};
j813=Transvectant[cc4,exp[ll,5],5].{1};
j814=Transvectant[cc3,exp[ll,6],6].{1};
j91=Transvectant[prod[ff,Transvectant[exp[cc1,2],ff,4]],
 exp[qq,3],6].{1};
j92=Transvectant[Transvectant[exp[cc1,3],cc3,5],qq,2].{1};
j93=Transvectant[prod[cc4,ff],exp[qq,5],10].{1};
j94=Transvectant[Transvectant[cc3,qq,1],exp[ll,6],6].{1};
j95=Transvectant[Transvectant[cc8,qq,2],exp[ll,4],4].{1};
j96=Transvectant[Transvectant[cc7,qq,1],exp[ll,4],4].{1};
j97=Transvectant[Transvectant[cc6,exp[qq,3],6],exp[ll,3],3].{1};
j98=Transvectant[Transvectant[Transvectant[cc3,exp[cc1,2],3],qq,2],
 exp[ll,2],2].{1};
j99=Transvectant[Transvectant[Transvectant[ff,exp[cc1,3],5],qq,1],
 ll,1].{1};
j910=Transvectant[Transvectant[cc9,exp[qq,3],6],
ll,1].{1};
j911=Transvectant[Transvectant[cc6,exp[qq,5],9],ll,1].{1};
j912=Transvectant[Transvectant[Transvectant[exp[cc1,2],cc3,4],qq,1],
 exp[ll,2],2].{1};
j101=Transvectant[prod[Transvectant[exp[cc1,2],ff,4],cc2],exp[qq,2],
 4].{1};
j102=Transvectant[prod[cc4,cc2],exp[qq,4],8].{1};
j103=Transvectant[cc8,exp[ll,6],6].{1};
j104=Transvectant[Transvectant[cc6,exp[qq,2],4],exp[ll,5],5].{1};
j105=Transvectant[Transvectant[cc3,exp[cc1,2],3],exp[ll,4],4].{1};
j106=Transvectant[Transvectant[Transvectant[cc1,cc6,2],exp[qq,2],4],
 exp[ll,3],3].{1};
j107=Transvectant[Transvectant[cc3,exp[cc1,3],5],exp[ll,2],2].{1};
j108=Transvectant[
 Transvectant[Transvectant[exp[cc1,2],cc6,4],exp[qq,2],4],ll,1].{1};
j109=Transvectant[Transvectant[Transvectant[cc1,cc6,2],exp[qq,4],7],
 ll,1].{1};
j111=Transvectant[prod[ff,cc6],exp[qq,7],14].{1};
j112=Transvectant[prod[ff,Transvectant[exp[cc1,3],ff,5]],exp[qq,3],
```

```
6].{1};
j113=Transvectant[exp[Transvectant[exp[cc1,2],ff,4],2],qq,2].{1};
j114=Transvectant[Transvectant[cc6,qq,2],exp[ll,7],7].{1};
j115=Transvectant[Transvectant[Transvectant[cc1,cc6,2],qq,2],
 exp[ll,5],5].{1};
j116=Transvectant[Transvectant[cc10,qq,2],
exp[ll,3],3].{1};
j117=Transvectant[Transvectant[Transvectant[cc6,exp[cc1,3],6],qq,2],
 ll,1].{1};
j118=Transvectant[
 Transvectant[cc10,exp[qq,3],5],ll,1].{1};
j121=Transvectant[Transvectant[cc2,cc2,2],Transvectant[cc2,cc2,2],
 2].{1};
j122=Transvectant[prod[Transvectant[exp[cc1,3],ff,5],cc2],exp[qq,2],
 4].{1};
j123=Transvectant[prod[Transvectant[cc1,cc6,2],ff],exp[qq,6],12].{1};
j124=Transvectant[cc6,exp[ll,9],9].{1};
j125=Transvectant[cc9,exp[ll,7],7].{1};
j126=Transvectant[cc10,exp[ll,5],5].{1};
j127=Transvectant[Transvectant[cc6,exp[cc1,3],6],exp[ll,3],3].{1};
j128=Transvectant[Transvectant[cc6,exp[cc1,4],8],ll,1].{1};
j129=Transvectant[
Transvectant[Transvectant[cc6,exp[cc1,3],6],exp[qq,2],3],ll,1].{1};
j131=Transvectant[prod[Transvectant[exp[cc1,3],ff,5],
 Transvectant[exp[cc1,2],ff,4]],qq,2].{1};
j132=Transvectant[prod[ff,Transvectant[exp[cc1,2],cc6,4]],exp[qq,5],
 10].{1};
j133=Transvectant[Transvectant[Transvectant[cc6,exp[cc1,4],8],qq,1],
 ll,1].{1};
j141=Transvectant[prod[ff,Transvectant[exp[cc1,3],cc6,6]],exp[qq,4],
 8].{1};
j142=Transvectant[Transvectant[cc6,exp[cc1,5],9],ll,1].{1};
j15=Transvectant[prod[ff,Transvectant[exp[cc1,4],cc6,8]],
 exp[qq,3],6].{1};
j16=Transvectant[prod[cc2,Transvectant[exp[cc1,4],cc6,8]],
 exp[qq,2],4].{1};
j17=Transvectant[prod[Transvectant[exp[cc1,2],ff,4],
 Transvectant[exp[cc1,4],cc6,8]],qq,2].{1};
j18=Transvectant[exp[cc1,7],prod[ff,cc6],14].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j31,j32},{j41,j42,j43},{j51,j52,j53,j54,j55},
{j61,j62,j63,j64,j65,j66,j67,j68},{j71,j72,j73,j74,j75,j76,j77,j78,
j79,j710,j711,j712},{j81,j82,j83,j84,j85,j86,j87,j88,j89,j810,j811,
j812,j813,j814},{j91,j92,j93,j94,j95,j96,j97,j98,j99,j910,j911,j912},
{j101,j102,j103,j104,j105,j106,j107,j108,j109},{j111,j112,j113,j114,
j115,j116,j117,j118},{j121,j122,j123,j124,j125,j126,j127,j128,j129},
{j131,j132,j133},{j141,j142},{j15},{j16},{j17},{j18}};
degrees={2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
In[]:=mon=Union[mon1,{j41,j42,j43}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];
 MatrixRank[mat]
```

```
Out[]=4
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=2
In[]:=mon=Union[mon1,{j51,j52,j53,j54,j55}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,7}];
 MatrixRank[mat]
Out[]=7
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=7
In[]:=mon=Union[mon1,{j61,j62,j63,j64,j65,j66,j67,j68}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,18}];
 MatrixRank[mat]
Out[]=15
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=13
In[]:=mon=Union[mon1,{j71,j72,j73,j74,j75,j76,j77,j78,j79,j710,
 j711,j712}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,26}];
 MatrixRank[mat]
Out[]=25
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=31
In[]:=mon=Union[mon1,{j81,j82,j83,j84,j85,j86,j87,j88,j89,j810,
 j811,j812,j813,j814}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,46}];
 MatrixRank[mat]
Out[]=45
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=60
In[]:=mon=Union[mon1,{j91,j92,j93,j94,j95,j96,j97,j98,j99,j910,
 j911,j912}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,71}];
 MatrixRank[mat,Modulus->32003]
Out[]=70
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=117
In[]:=mon=Union[mon1,{j101,j102,j103,j104,j105,j106,j107,j108,j109}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,113}];
 MatrixRank[mat,Modulus->32003]
```

```
Out[]=112
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=203
In[]:=mon=Union[mon1,{j111,j112,j113,j114,j115,j116,j117,j118}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,176}];
 MatrixRank[mat,Modulus->32003]
Out[]=175
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=357
In[]:=mon=Union[mon1,{j121,j122,j123,j124,j125,j126,j127,j128,j129}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,271}];
 MatrixRank[mat,Modulus->32003]
Out[]=270
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=587
In[]:=mon=Union[mon1,{j131,j132,j133}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,386}];
 MatrixRank[mat,Modulus->32003]
Out[]=385
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=984
In[]:=mon=Union[mon1,{j141,j142}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,567}];
 MatrixRank[mat,Modulus->32003]
Out[]=566
=== computations in degree 15 ===
In[]:=mon1=listmonomials[15,gens,degrees];Length[mon1]
Out[]=1583
In[]:=mon=Union[mon1,{j15}];
In[]:=mat=Table[mon/.Table[b[j]->Random[Integer,{0,10}],{j,0,5}]/.
 Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
 Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,801}];
 MatrixRank[mat,Modulus->32003]
Out[]=800
```

A.17   The invariants of $2V_2 \oplus V_3$

Here are the `Mathematica` computations supporting the proof of Proposition 5.17.2. We first compute the invariants $j_{2,1} \ldots, j_{2,3}, j_{3,1}, j_{3,2}, j_{4,1}, j_{4,2}, j_{5,1}, \ldots, j_{5,4}, j_{6,1}, \ldots, j_{6,3}, j_{7,1} \ldots, j_{7,4}$ of $2V_2 \oplus V_3$:

```
qq1=Table[a[2-i]Binomial[2,i],{i,0,2}];
qq2=Table[b[2-i]Binomial[2,i],{i,0,2}];
```

```
cc=Table[c[3-i]Binomial[3,i],{i,0,3}];
cc1=Transvectant[cc,cc,2];cc2=Transvectant[cc,cc1,1];
j21=Transvectant[qq1,qq1,2].{1};j22=Transvectant[qq1,qq2,2].{1};
j23=Transvectant[qq2,qq2,2].{1};j31=Transvectant[cc1,qq1,2].{1};
j32=Transvectant[cc1,qq2,2].{1};j41=Transvectant[cc1,cc1,2].{1};
j42=Transvectant[Transvectant[qq1,cc1,1],qq2,2].{1};
j51=Transvectant[exp[qq1,3],exp[cc,2],6].{1};
j52=Transvectant[exp[qq2,3],exp[cc,2],6].{1};
j53=Transvectant[prod[exp[qq2,2],qq1],exp[cc,2],6].{1};
j54=Transvectant[prod[exp[qq1,2],qq2],exp[cc,2],6].{1};
j61=Transvectant[prod[Transvectant[qq1,cc,2],
 Transvectant[exp[qq1,2],cc,3]],qq2,2].{1};
j62=Transvectant[prod[Transvectant[qq1,cc,2],
 Transvectant[qq1,cc,1]],exp[qq2,2],4].{1};
j63=Transvectant[prod[cc,Transvectant[qq1,cc,1]],exp[qq2,3],6].{1};
j71=Transvectant[exp[qq1,3],prod[cc,cc2],6].{1};
j72=Transvectant[exp[qq2,3],prod[cc,cc2],6].{1};
j73=Transvectant[prod[exp[qq2,2],qq1],prod[cc,cc2],6].{1};
j74=Transvectant[prod[exp[qq1,2],qq2],prod[cc,cc2],6].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21,j22,j23},{j31,j32},{j41,j42},{j51,j52,j53,j54},
{j61,j62,j63},{j71,j72,j73,j74}};degrees={2,3,4,5,6,7};
=== computations in degree 2 ===
In[]:=mon={j21,j22,j23};
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,4}];MatrixRank[mat]
Out[]=3
=== computations in degree 3 ===
In[]:=mon={j31,j32};
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,4}];MatrixRank[mat]
Out[]=2
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=6
In[]:=mon=Union[mon1,{j41,j42}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,9}];MatrixRank[mat]
Out[]=8
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=6
In[]:=mon=Union[mon1,{j51,j52,j53,j54}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,11}];MatrixRank[mat]
Out[]=10
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=19
In[]:=mon=Union[mon1,{j61,j62,j63}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
```

```
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,23}];MatrixRank[mat]
Out[]=22
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=28
In[]:=mon=Union[mon1,{j71,j72,j73,j74}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,33}];MatrixRank[mat]
Out[]=32
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=56
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,56}];MatrixRank[mat]
Out[]=55
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=86
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,81}];
MatrixRank[mat,Modulus->32003]
Out[]=80
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=140
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,129}];
MatrixRank[mat,Modulus->32003]
Out[]=128
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=210
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,179}];
MatrixRank[mat,Modulus->32003]]
Out[]={19.3252,178}   //the rank is 178;
                      //the computation took 19.3 seconds
=== computations in degree 12 ===
Computationsindegree12
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=330
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,269}];
MatrixRank[mat,Modulus->32003]]
Out[]={43.8027,268}   //the rank is 268;
                      //the computation took 43.8 seconds
=== computations in degree 13 ===
In[]:=mon1=listmonomials[13,gens,degrees];Length[mon1]
Out[]=480
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
```

```
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,363}];
MatrixRank[mat,Modulus->32003]]
Out[]={103.666,362}   //the rank is 362;
                      //the computation took 103.6 seconds
=== computations in degree 14 ===
In[]:=mon1=listmonomials[14,gens,degrees];Length[mon1]
Out[]=730
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,516}];
MatrixRank[mat,Modulus->32003]]
Out[]={223.402,515}   //the rank is 515;
                      //the computation took 223.4 seconds
=== computations in degree 15 ===
In[]:=mon1=listmonomials[15,gens,degrees];Length[mon1]
Out[]=1046
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,687}];
MatrixRank[mat,Modulus->32003]]
Out[]={514.496,686}   //the rank is 686;
                      //the computation took 514.4 seconds
=== computations in degree 17 ===
In[]:=mon1=listmonomials[17,gens,degrees];Length[mon1]
Out[]=2182
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,3}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,1219}];
MatrixRank[mat,Modulus->32003]]
Out[]={2072.15,1218}   //the rank is 1218;
                       //the computation took 2072.1 seconds
```

A.18   The invariants of $2V_2 \oplus V_4$

Here are the computations supporting the proofs of Propositions 5.18.1 and 5.18.2. We first compute the invariants $j_{2,1} \ldots, j_{2,4}, j_{3,1}, \ldots, j_{3,4}, j_{4,1}, \ldots, j_{4,5}, j_{5,1}, j_{5,2}, j_{6,1}, \ldots, j_{6,4}$ of $2V_2 \oplus V_4$, using `Mathematica`:

```
qq1=Table[a[2-i]Binomial[2,i],{i,0,2}];
qq2=Table[b[2-i]Binomial[2,i],{i,0,2}];
ff=Table[c[4-i]Binomial[4,i],{i,0,4}];
cc1=Transvectant[ff,ff,2];cc2=Transvectant[cc1,ff,1];
cc3=Transvectant[qq1,ff,2];cc4=Transvectant[qq2,ff,2];
j21=Transvectant[qq1,qq1,2].{1};j22=Transvectant[qq2,qq2,2].{1};
j23=Transvectant[qq1,qq2,2].{1};j24=Transvectant[ff,ff,4].{1};
j31=Transvectant[ff,cc1,4].{1};j32=Transvectant[exp[qq1,2],ff,4].{1};
j33=Transvectant[exp[qq2,2],ff,4].{1};
j34=Transvectant[prod[qq1,qq2],ff,4].{1};
j41=Transvectant[exp[qq1,2],cc1,4].{1};
j42=Transvectant[exp[qq2,2],cc1,4].{1};
j43=Transvectant[prod[qq1,qq2],cc1,4].{1};
j44=Transvectant[Transvectant[cc3,qq1,1],qq2,2].{1};
j45=Transvectant[Transvectant[cc4,qq2,1],qq1,2].{1};
j51=Transvectant[Transvectant[cc1,prod[qq1,qq2],3],qq1,2].{1};
j52=Transvectant[Transvectant[cc1,prod[qq1,qq2],3],qq2,2].{1};
```

```
j61=Transvectant[cc2,exp[qq1,3],6].{1};
j62=Transvectant[cc2,exp[qq2,3],6].{1};
j63=Transvectant[cc2,prod[exp[qq1,2],qq2],6].{1};
j64=Transvectant[cc2,prod[exp[qq2,2],qq1],6].{1};
```

Here are the Singular [DGPS] computations that support the proof of Proposition 5.18.1:

```
> ring r=0,(a0,a1,a2,b0,b1,b2,c0,c1,c2,c3,c4),dp;
    //here we import the invariants we computed with Mathematica:
> <"j21_224.csv";
> <"j22_224.csv";
> <"j23_224.csv";
> <"j24_224.csv";
> <"j31_224.csv";
> <"j32_224.csv";
> <"j33_224.csv";
> <"j41_224.csv";
> <"j42_224.csv";
> ideal ii=j21+j23,j22+j24,j21-j24,j31,j32,j33,j41,j42;
> ideal jj=std(ii);
> reduce(j21^7,jj);
r
0
> reduce(j22^7,jj);
r
0
> reduce(j23^7,jj);
r
0
> reduce(j24^7,jj);
r
0
```

Here are the rank computations, performed in Mathematica:

```
In[]:=gens={{j21,j22,j23,j24},{j31,j32,j33,j34},
{j41,j42,j43,j44,j45},{j51,j52},{j61,j62,j63,j64}};
degrees={2,3,4,5,6};
=== computations in degree 2 ===
In[]:=mon={j21,j22,j23};
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,4}];MatrixRank[mat]
Out[]=3
=== computations in degree 3 ===
In[]:=mon={j31,j32,j33,j34};
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,5}];MatrixRank[mat]
Out[]=4
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=10
In[]:=mon=Union[mon1,{j41,j42,j43,j44,j45}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
```

```
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,16}];MatrixRank[mat]
Out[]=15
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=16
In[]:=mon=Union[mon1,{j51,j52}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,19}];MatrixRank[mat]
Out[]=18
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=50
In[]:=mon=Union[mon1,{j61,j62,j63,j64}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,54}];MatrixRank[mat]
Out[]=53
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=68
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,66}];MatrixRank[mat]
Out[]=65
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=164
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,149}];
MatrixRank[mat,Modulus->32003]
Out[]=148
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=226
In[]:=mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,199}];
MatrixRank[mat,Modulus->32003]
Out[]=198
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=461
In[]:=Timing[mat=Table[mon1/.Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,372}];
MatrixRank[mat,Modulus->32003]]
Out[]={59.0917,371}   //the rank is 371;
                      //the computation took 59.0 seconds
=== computations in degree 12 ===
In[]:=mon1=listmonomials[12,gens,degrees];Length[mon1]
Out[]=1221
In[]:=Timing[mat=Table[mon1/.
Table[c[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,2}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,2}],{i,1,854}];
```

```
MatrixRank[mat,Modulus->32003]]
Out[]={477.706, 853}  //the rank is 853;
                      //the computation took 477.7 seconds
```

A.19 The invariants of $V_1 \oplus V_3 \oplus V_4$

Here are the Mathematica computations supporting the proof of Proposition 5.19.2. We first compute the 63 generators:

```
cc=Table[a[3-i]Binomial[3,i],{i,0,3}];
qq=Table[b[4-i]Binomial[4,i],{i,0,4}];
ll=Table[c[1-i]Binomial[1,i],{i,0,1}];
cc1=Transvectant[cc,cc,2];cc2=Transvectant[qq,qq,2];
cc3=Transvectant[cc,qq,2];j21=Transvectant[qq,qq,4].{1};
j31=Transvectant[cc2,qq,4].{1};j32=Transvectant[qq,prod[ll,cc],4].{1};
j41=Transvectant[cc1,cc1,2].{1};
j42=Transvectant[prod[ll,cc],prod[ll,cc],4].{1};
j43=Transvectant[cc,exp[ll,3],3].{1};
j44=Transvectant[cc2,prod[ll,cc],4].{1};
j51=Transvectant[Transvectant[qq,cc2,1],exp[cc,2],6].{1};
j52=Transvectant[Transvectant[qq,exp[cc,2],2],exp[cc,2],6].{1};
j53=Transvectant[qq,exp[ll,4],4].{1};
j54=Transvectant[prod[qq,exp[ll,2]],exp[cc,2],6].{1};
j55=Transvectant[Transvectant[qq,cc1,2],exp[ll,2],2].{1};
j56=Transvectant[cc3,exp[ll,3],3].{1};
j57=Transvectant[prod[ll,Transvectant[cc,cc1,1]],qq,4].{1};
j58=Transvectant[Transvectant[ll,cc1,1],Transvectant[qq,cc,3],1].{1};
j61=Transvectant[exp[cc1,2],cc2,4].{1};
j62=Transvectant[Transvectant[prod[ll,cc],prod[ll,cc],2],prod[ll,cc],4].{1};
j63=Transvectant[cc2,exp[ll,4],4].{1};
j65=Transvectant[Transvectant[Transvectant[qq,cc1,2],exp[cc,2],2],qq,4].{1};
j64=Transvectant[cc2,prod[ll,Transvectant[cc,cc1,1]],4].{1};
j66=Transvectant[cc2,prod[cc,Transvectant[ll,cc1,1]],4].{1};
j67=Transvectant[prod[exp[ll,2],cc2],exp[cc,2],6].{1};
j68=Transvectant[Transvectant[cc1,cc2,2],exp[ll,2],2].{1};
j69=Transvectant[Transvectant[cc,cc2,2],exp[ll,3],3].{1};
j610=Transvectant[Transvectant[exp[cc,2],exp[qq,2],6],
 Transvectant[cc,ll,1],2].{1};
j71=Transvectant[exp[cc,4],exp[qq,3],12].{1};
j72=Transvectant[Transvectant[Transvectant[qq,cc1,2],exp[cc,2],2],cc2,4].{1};
j73=Transvectant[Transvectant[qq,cc2,1],
 prod[cc,Transvectant[cc,cc1,1]],6].{1};
j74=Transvectant[Transvectant[cc1,ll,1],
 Transvectant[qq,Transvectant[cc1,cc,1],3],1].{1};
j75=Transvectant[Transvectant[exp[cc,2],prod[qq,cc2],6],
 Transvectant[cc,ll,1],2].{1};
j76=Transvectant[Transvectant[Transvectant[cc1,cc,1],qq,2],exp[ll,3],3].{1};
j77=Transvectant[prod[qq,exp[ll,2]],prod[cc,Transvectant[cc1,cc,1]],6].{1};
j78=Transvectant[Transvectant[qq,Transvectant[cc1,exp[ll,2],1],2],
 exp[ll,2],2].{1};
j79=Transvectant[exp[ll,5],Transvectant[cc,qq,1],5].{1};
j710=Transvectant[Transvectant[Transvectant[cc3,
 Transvectant[qq,ll,1],2],cc3,1],cc,3].{1};
j711=Transvectant[Transvectant[Transvectant[cc3,cc,2],
 Transvectant[qq,ll,1],1],Transvectant[qq,ll,1],3].{1};
j712=Transvectant[prod[Transvectant[qq,cc1,1],exp[ll,2]],exp[cc,2],6].{1};
```

```
j713=Transvectant[Transvectant[cc3,Transvectant[qq,ll,1],2],
 Transvectant[qq,exp[ll,2],2],2].{1};
j81=Transvectant[Transvectant[Transvectant[qq,cc1,2],prod[cc,
 Transvectant[cc,cc1,1]],2],qq,4].{1};
j82=Transvectant[exp[cc,4],prod[exp[qq,2],cc2],12].{1};
j83=Transvectant[Transvectant[Transvectant[cc2,cc1,2],exp[cc,2],2],
 cc2,4].{1};
j84=Transvectant[Transvectant[exp[cc,2],prod[cc2,qq],6],
 Transvectant[cc3,ll,1],2].{1};
j85=Transvectant[Transvectant[cc1,ll,1],Transvectant[qq,
 Transvectant[cc1,Transvectant[qq,ll,1],1],3],1].{1};
j86=Transvectant[Transvectant[Transvectant[cc1,cc3,1],qq,2],exp[ll,3],3].{1};
j87=Transvectant[Transvectant[Transvectant[cc1,
 Transvectant[qq,ll,1],1],qq,2],exp[ll,3],3].{1};
j88=Transvectant[prod[qq,exp[ll,2]],prod[cc,Transvectant[cc1,cc3,1]],6].{1};
j89=Transvectant[Transvectant[qq,Transvectant[Transvectant[cc,
 Transvectant[qq,ll,1],2],exp[ll,2],1],2],exp[ll,2],2].{1};
j810=Transvectant[Transvectant[exp[cc,2],exp[qq,2],6],
 Transvectant[Transvectant[cc,cc1,1],ll,1],2].{1};
j811=Transvectant[prod[qq,prod[ll,Transvectant[cc,qq,3]]],
 prod[cc,Transvectant[cc1,cc,1]],6].{1};
j91=Transvectant[Transvectant[Transvectant[cc2,cc1,2],prod[cc,
 Transvectant[cc,cc1,1]],2],qq,4].{1};
j92=Transvectant[exp[cc,4],prod[prod[qq,cc2],cc2],12].{1};
j93=Transvectant[prod[exp[cc,3],Transvectant[cc,cc1,1]],exp[qq,3],12].{1};
j94=Transvectant[Transvectant[Transvectant[qq,cc1,2],prod[cc,
 Transvectant[cc,cc1,1]],2],cc2,4].{1};
j95=Transvectant[Transvectant[Transvectant[qq,cc1,2],prod[cc,
 Transvectant[Transvectant[qq,ll,1],cc1,1]],2],qq,4].{1};
j96=Transvectant[Transvectant[exp[cc,2],prod[cc2,prod[cc,ll]],6],
 Transvectant[cc3,ll,1],2].{1};
j97=Transvectant[Transvectant[exp[cc,2],prod[cc2,prod[cc,ll]],6],
 Transvectant[Transvectant[qq,ll,1],ll,1],2].{1};
j98=Transvectant[Transvectant[Transvectant[qq,
 Transvectant[qq,prod[cc,ll],2],2],cc1,2],exp[cc,2],2],qq,4].{1};
j99=Transvectant[Transvectant[Transvectant[qq,ll,1],cc2,1],exp[ll,5],5].{1};
j910=Transvectant[Transvectant[Transvectant[qq,cc2,1],exp[cc,2],4],
 exp[ll,4],4].{1};
j101=Transvectant[Transvectant[Transvectant[cc2,cc1,2],prod[cc,
 Transvectant[cc,cc1,1]],2],cc2,4].{1};
j102=Transvectant[Transvectant[prod[exp[cc,3],Transvectant[cc,cc1,1]],
 exp[qq,3],10],qq,4].{1};
j103=Transvectant[Transvectant[cc,prod[Transvectant[cc,Transvectant[cc,
 Transvectant[cc,Transvectant[qq,cc2,1],3],1],3],
 Transvectant[cc,qq,2]],3],ll,1].{1};
j111=Transvectant[Transvectant[Transvectant[prod[exp[cc,3],
 Transvectant[cc,cc1,1]],exp[qq,3],10],qq,2],qq,4].{1};
```

Here are the rank computations:

```
In[]:=gens={{j21},{j31,j32},{j41,j42,j43,j44},{j51,j52,j53,j54,j55,j56,
j57,j58},{j61,j62,j63,j64,j65,j66,j67,j68,j69,j610},{j71,j72,j73,j74,j75,
j76,j77,j78,j79,j710,j711,j712,j713},{j81,j82,j83,j84,j85,j86,j87,j88,
j89,j810,j811},{j91,j92,j93,j94,j95,j96,j97,j98,j99,j910},{j101,j102,j103},
{j111}};degrees={2,3,4,5,6,7,8,9,10,11};
=== computations in degree 4 ===
In[]:=mon1=listmonomials[4,gens,degrees];Length[mon1]
Out[]=1
```

```
In[]:=mon=Union[mon1,{j41,j42,j43,j44}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,6}];MatrixRank[mat]
Out[]=5
=== computations in degree 5 ===
In[]:=mon1=listmonomials[5,gens,degrees];Length[mon1]
Out[]=2
In[]:=mon=Union[mon1,{j51,j52,j53,j54,j55,j56,j57,j58}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,11}];MatrixRank[mat]
Out[]=10
=== computations in degree 6 ===
In[]:=mon1=listmonomials[6,gens,degrees];Length[mon1]
Out[]=8
In[]:=mon=Union[mon1,{j61,j62,j63,j64,j65,j66,j67,j68,j69,j610}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,19}];MatrixRank[mat]
Out[]=18
=== computations in degree 7 ===
In[]:=mon1=listmonomials[7,gens,degrees];Length[mon1]
Out[]=18
In[]:=mon=Union[mon1,{j71,j72,j73,j74,j75,j76,j77,j78,j79,j710,j711,j712,
 j713}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,32}];MatrixRank[mat]
Out[]=31
=== computations in degree 8 ===
In[]:=mon1=listmonomials[8,gens,degrees];Length[mon1]
Out[]=44
In[]:=mon=Union[mon1,{j81,j82,j83,j84,j85,j86,j87,j88,j89,j810,j811}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,56}];MatrixRank[mat]
Out[]=55
=== computations in degree 9 ===
In[]:=mon1=listmonomials[9,gens,degrees];Length[mon1]
Out[]=87
In[]:=mon=Union[mon1,{j91,j92,j93,j94,j95,j96,j97,j98,j99,j910}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,93}];MatrixRank[mat]
Out[]=92
=== computations in degree 10 ===
In[]:=mon1=listmonomials[10,gens,degrees];Length[mon1]
Out[]=169
In[]:=mon=Union[mon1,{j101,j102,j103}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,145}];MatrixRank[mat]
Out[]=144
=== computations in degree 11 ===
In[]:=mon1=listmonomials[11,gens,degrees];Length[mon1]
Out[]=295
```

```
In[]:=mon=Union[mon1,{j111}];
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,224}];
MatrixRank[mat,Modulus->32003]
Out[]=223
=== computations in degree 12 ===
In[]:=mon=listmonomials[12,gens,degrees];Length[mon]
Out[]=514
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,342}];
MatrixRank[mat,Modulus->32003]
Out[]=341
=== computations in degree 13 ===
In[]:=mon=listmonomials[13,gens,degrees];Length[mon]
Out[]=847
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,500}];
MatrixRank[mat,Modulus->32003]
Out[]=499
=== computations in degree 14 ===
In[]:=mon=listmonomials[14,gens,degrees];Length[mon]
Out[]=1412
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,726}];
MatrixRank[mat,Modulus->32003]
Out[]=725
=== computations in degree 15 ===
In[]:=mon=listmonomials[15,gens,degrees];Length[mon]
Out[]=2306
In[]:=mat=Table[mon/.Table[c[j]->Random[Integer,{0,10}],{j,0,1}]/.
Table[b[j]->Random[Integer,{0,10}],{j,0,4}]/.
Table[a[j]->Random[Integer,{0,10}],{j,0,3}],{i,1,1032}];
MatrixRank[mat,Modulus->32003]
Out[]=1031
```

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Appendix B

Curriculum Vitae

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Education

**Ph.D. Mathematics (2013),
University of Basel, Switzerland**
Thesis: *Invariants of binary forms*
Publications: two in the *Journal of Symbolic Computations*,
one in *Transformation Groups*, one in *SIGMA*

First-grade qualification as Mathematics Teacher (2011)
(*eerstegraads bevoegdheid*)

**M.Sc. Mathematics (2004),
University of Kaiserslautern,Germany**
Grade: sehr gut (1.5, on a scale from 5 to 1)
Thesis: *Infinitesimal deformations of plane curve singularities*
Specialisation: Computer Algebra and Algebraic Geometry

**Graduate Studies Diploma, Mathematics (1998),
University Babes Bolyai, Cluj Napoca, Romania**
Grade: 10 (on a scale from 1 to 10)
Thesis: *Pontrjagin classes*
Specialisation: Topology and Function Theory

**Bachelor Diploma, Mathematics (1997)
University Babes Bolyai, Cluj Napoca, Romania**
Grade: 9.5 (on a scale from 1 to 10)

Work   Mathematics Teacher (present), Erasmus College, Eindhoven, NL

Mathematics Teacher $(2011 - 2012)$, Metameer, Stevensbeek, NL

Research Mathematician $(2007 - 2010)$, Shell, Rijswijk, NL

Research Assistant (2005), University of Basel, CH

Teaching Assistant $(2002 - 2004)$, University of Kaiserslautern, DE

Mathematics Teacher$(2001 - 2002)$, High School Unirea, Cluj Napoca, RO

Journalist $(2000 - 2001)$, Transylvania Journal, Cluj Napoca, RO

Journalist $(1997 - 2000)$, Radio Sonic, Cluj Napoca, RO