The h-vector of a standard determinantal scheme

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Abstract

In this dissertation we study the h-vector of a standard determinantal scheme $X \subseteq \mathbb{P}^n$ via the corresponding degree matrix. We find simple formulae for the length and the last entries of the h-vector, as well as an explicit formula for the h-polynomial. We also describe a recursive formula for the h-vector in terms of h-vectors corresponding to submatrices of the degree matrix of X. In codimension three we show that when the largest entry in the degree matrix of X is sufficiently large and the first subdiagonal is entirely positive the h-vector of X is of decreasing type.

We prove that if a standard determinantal scheme is level, then its h-vector is a log-concave pure O-sequence, and conjecture that the converse also holds. Among other cases, we prove the conjecture in codimension two, or when the entries of the corresponding degree matrix are positive.

We further investigate the combinatorial structure of the poset $\mathcal{H}_s^{(t,c)}$ consisting of h-vectors of length s, of codimension c standard determinantal schemes, having degree matrices of size $t \times (t+c-1)$ for some $t \geq 1$. We show that $\mathcal{H}_s^{(t,c)}$ obtains a natural stratification, where each strata contains a maximum h-vector. We prove furthermore, that the only strata in which there exists also a minimum h-vector is the one consisting of h-vectors of level standard determinantal schemes.

We also study posets of h-vectors of standard determinantal ideals, which arise from a matrix M, where the entries in each row have the same degree, and show the existence of a minimum and a maximum h-vector.

To my parents and my sister,
and to Maria

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Contents

1 Introduction

Classical determinantal rings have made their way from algebraic geometry to commutative algebra more than fifty years ago and have been an active research topic ever since. Over the years, the study has been extended to pfaffian ideals of generic skew-symmetric matrices and to determinantal ideals of ladders, of symmetric matrices and of homogeneous polynomial matrices.

An ideal of height c, which is defined by the maximal minors of a homogeneous, polynomial, $t \times (t+c-1)$ matrix M is called standard determinantal. As such an ideal is saturated, it defines a projective scheme $X \subseteq \mathbb{P}^n$, which we call standard determinantal. Classical examples of such objects are rational normal curves, rational normal scrolls and some Segre varieties.

To the defining matrix M of a standard determinantal ideal I we can assign another matrix A, whose entries are the degrees of the entries of M. In the literature the matrix A is referred to as the degree matrix of the ideal I or of the scheme defined by I. Since the shifts in the minimal free resolution of I, which is given by the Eagon-Northcott complex (see [12]), can be written in terms of the entries of the degree matrix A of I, a great piece of the numerical data about the standard determinantal ideal I is encoded in his degree matrix A. Using this fact we study the Hilbert function of a standard determinantal ideal via the corresponding degree matrix.

Among many others, Hilbert functions of standard determinantal ideals have been studied by S. Abhyankar [1], W. Bruns, A. Conca and J. Herzog [5, 10], S. Ghorpade [18, 17], N. Budur, M. Casanellas and E. Gorla [8].

In this work we are primarily interested in the following problems: firstly, when is the h-vector of a codimension c standard determinantal scheme of decreasing type, that is it is of the form $(h_0 < \cdots < h_i = \cdots = h_j > \cdots > h_s)$, and secondly, is it possible to characterize the standard determinantal schemes, whose h-vectors are pure O-sequences (i.e. the h-vector of some artinian monomial level algebra) via the corresponding degree matrix.

Next to the first two problems we study also the combinatorial structure of the poset $\mathcal{H}_s^{(t,c)}$, consisting of h-vectors of fixed length s and codimension c, and corresponding to degree matrices of size $t \times (t+c-1)$.

This work is organized as follows. In Chapter 2 we provide the necessary background results that we will need in the subsequent chapters. We fix some terminology and notation as well.

The starting point of Chapter 3 is Proposition 3.1. This result provides the key to many of our proofs. Using a basic double link from Gorenstein liaison theory, we describe a recursive formula for the h-vector of a standard determinantal scheme X with defining matrix M (that is the maximal minors of M generate the defining ideal I_X of X), in terms of h-vectors corresponding to submatrices of M. As a direct consequence we obtain a "cancelation" result, which states that any two standard determinantal schemes X and Y, with degree matrices A, and respectively B, have the same h-vector, if A has some zero entry $a_{k,l} = 0$, and B is obtained from A by deleting the k-th row and l-th column. This means in particular that studying properties of h-vectors of standard determinantal schemes, we may assume that none of the degree matrices contain zero entries.

Using Proposition 3.1 we obtain simple formulae for the length and the last entries of the h-vector (Proposition 3.11 and Proposition 4.15) as well as an explicit formula for the h-vector of any standard determinantal ring (Proposition 3.6). Motivated by Proposition 3.1 we derive (Lemma 3.19 and Lemma 3.20) numerical criteria for decreasing type of an O-sequence, which can be written as the component-wise sum of two other O-sequences of decreasing type. A well known result, proved by A. Geramita and J. Migliore (see [14]), states that the h-vector of a codimension 2 standard determinantal scheme is of decreasing type, if the first subdiagonal of the corresponding degree matrix is entirely positive. The criteria obtained in Lemma 3.19 and Lemma 3.20 in combination with Proposition 3.1 allow us to obtain a new simple proof for the result of A. Geramita and J. Migliore and to compute explicitly the place where the h-vector stops to increase and the place where it starts to decrease.

Finally we show that the h-vector of a standard determinantal scheme of codimension 3 is of decreasing type if the largest entry in the corresponding degree matrix is sufficiently large and the first subdiagonal is entirely positive (Theorem 3.38), or if all its entries are equal (Proposition 3.40) .

The main result in Chapter 4 (Theorem 4.9) states that the h-vector of a standard determinantal scheme $X \subseteq \mathbb{P}^n$ is a log-concave pure O-sequence if the degree matrix of X has equal rows, i.e. the polynomials in each column of the defining matrix M of I_X have the same degree.

We conjecture that the converse of this theorem also holds, namely if the h-vector of a standard determinantal ideal is a pure O-sequence, then all the degrees of the elements in a column of its defining matrix must be equal (Conjecture 4.12).

Besides beeing the Hilbert function of some monomial, artinian level algebra, pure O-sequences have a purely combinatorial description as they are the f-vector of a pure multicomplex, or of a pure order ideal. In [26], T. Hibi proved that if $h = (h_0, \ldots, h_s)$ is a pure O-sequence, then h is flawless, i.e. it holds $h_i \leq h_{s-i}$ for all $i = 0, \ldots, \lfloor s/2 \rfloor$. Other than the Hibi inequalities and some *ad hoc* methods, we are not aware of any criteria which imply non-purity for an O-sequence. In most specific examples, an exhaustive computer listing of all pure O-sequences with some fixed parameters is needed to check non-purity. Notice that the problem of giving a complete characterization for pure O-sequences is far from beeing solved. In fact such a task is considered to be nearly impossible by several experts (see M. Boij, J. Migliore, R.M. Miró-Roig, U. Nagel, F. Zanello [4]). The validity of Conjecture 4.12, together with the computational formulae we found, would provide a fast way to construct (for fixed codimension, socle degree and type) large families of O-sequences which are not pure.

Using the Eagon-Northcott complex, we show (in Proposition 4.13) that a standard determinantal ideal is level (i.e. its socle is concentrated in one degree) if and only if the polynomials in each column of the defining matrix M have the same degree. In the last part of the chapter we prove several cases of Conjecture 4.12. We prove in particular that the statement is true for any standard determinantal ideal whose degree matrix contains only positive entries or in which the entries in the first row are strictly bigger that the entries in the second row.

In Chapter 5 we consider the set $\mathcal{H}_s^{(t,c)}$ of all h-vectors of fixed codimension c and length s, corresponding to degree matrices of fixed size $t \times (t+c-1)$. We study the combinatorial structure of this poset. Grouping the degree matrices by the number of equal rows counted from top to bottom and considering the posets consisting of the corresponding h-vectors, we obtain a natural stratification on the poset $\mathcal{H}_s^{(t,c)}$. We prove that each strata and $\mathcal{H}_s^{(t,c)}$ itself contains a maximum, which we construct explicitly (Proposition 5.5, Proposition 5.14 and Corollary 5.20). We also show that in the strata consisting of h-vectors of level standard determinantal schemes , i.e. corresponding to degree matrices with equal rows, there exists a minimum h-vector and we construct it explicitly (Proposition 5.5). Furthermore, we prove that the h-vector of any standard determinantal scheme is bounded from above by the h-vector of a level standard determinantal scheme of the same codimension.

In the last part of Chapter 5 we study posets of h-vectors $h = (h_0, \dots, h_s)$ of standard determinantal ideals of height c, which arise from a matrix M, where the entries in each row have the same degree. In particular, we prove that this poset contains a minimum and maximum h-vector (Lemma 5.23 and Corollary 5.32). Moreover we show that posets of h-vectors obtained in this way have a natural stratification, where each strata contains a minimum (Corollary 5.28) and in addition the minimum h-vectors in the different strata are comparable (Proposition 5.29).

Many of the results in this work have been suggested and double-checked using intensive computer experiments done with CoCoA (see [9]) .

2 Preliminaries

In this chapter we will recall most of the algebraic and geometric notions that will be used through the work. For general considerations and further results on the topics presented here we refer the reader to the books of W. Bruns and U.Vetter [7], of R. M. Miro-Roig [33], and of C. Baetica [2] .

Let $S = K[X_0, \dots, X_n]$ be a polynomial ring over an infinite field K. For any two integers $t, c \geq 1$, a matrix M of size $t \times (t+c-1)$, with polynomial entries, is called *homogeneous* if and only if all its minors are homogeneous polynomials (if and only if all its entries and 2×2 minors are homogeneous). An ideal $I \subseteq S$ of height c is *standard determinantal* if it is generated by the maximal minors of a $t \times (t+c-1)$ homogeneous matrix $M = [f_{i,j}]$, where $f_{i,j} \in S$ are homogeneous polynomials of degree $a_j - b_i$. We will use the notation $I = I_{\max}(M)$. The matrix M is called *the defining matrix of* I and it defines a graded homomorphism of degree zero

$$\varphi : F = \bigoplus_{i=1}^{t} S(b_i) \longrightarrow \bigoplus_{i=1}^{t+c-1} S(a_j) = G,\ v \longmapsto vM$$

where $v = (v_1, \dots, v_t) \in F$. This homomorphism is often referred to as *the associated homomorphism* to I and is used in the computation of a minimal free resolution of I. More precisely, a minimal free resolution of a standard determinantal Ideal I with associated graded homomorphism $\varphi : F \to G$ is given by the Eagon-Northcott complex (see [12]):

$$0 \longrightarrow \bigwedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \bigwedge^{t} F \longrightarrow \bigwedge^{t+c-2} G^* \otimes S_{c-2}(F) \otimes \bigwedge^{t} F \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigwedge^{t} G^* \otimes S_0(F) \otimes \bigwedge^{t} F \longrightarrow S \longrightarrow S/I \longrightarrow 0,$$

where

$$G^* = \bigoplus_{j=1}^{t+c-1} S(-a_j), \qquad \bigwedge^{d} G^* = \bigoplus_{1 \leq j_1 < \cdots < j_d \leq t+c-1} S(-\sum_{i=1}^{d} a_{j_i})$$

$$\bigwedge^{t} F = S(\sum_{i=1}^{t} b_i), \qquad S_k(F) = \bigoplus_{1 \leq i_1 \leq \cdots \leq i_k \leq t} S(\sum_{j=1}^{k} b_{i_j}).$$

Without loss of generality we can assume that the defining matrix M of I does not contain invertible elements i.e. $f_{i,j} = 0$ for all i, j with $a_j = b_i$. Clearly whenever $a_j < b_i$ we have $f_{i,j} = 0$. To the matrix M we assign a matrix of integers $A = [a_{i,j}] \in \mathbb{Z}^{t \times (t+c-1)}$, where $a_{i,j} = a_j - b_i$, which is called *the degree* *matrix of the Ideal I*. We will assume that $a_1 \leq \cdots \leq a_{t+c-1}$ and $b_1 \leq \cdots \leq b_t$, so the entries of A increase from left to right and from bottom to top, i.e. $i \geq k$ and $j \leq l$ implies that $a_{i,j} \leq a_{k,l}$. If $r = \max\{i \mid a_{1,1} = \cdots = a_{i,1}\}$, we will say that A has r equal maximal rows. In the special case $r = t$ we say that A has equal rows. Similarly if $a_{1,1} = \cdots = a_{1,t}$ we will say that A has equal columns.

Remark 2.1. *The degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ of the standard determinantal ideal I determines its graded Betti numbers. More precisely, if we denote by H_i the graded free modules in the minimal free resolution of I, then for $i = 0, \ldots, c-1$:*

$$
\begin{aligned}
H_{i+1} &= \bigoplus_{\substack{1\leq j_1<\cdots<j_{t+i}\leq t+c-1 \\ 1\leq k_1\leq\cdots\leq k_i\leq t}} S(\sum_{j=1}^{i} b_{k_j} - \sum_{l=1}^{t+i} a_{j_l} + \sum_{j=1}^{t} b_j) \\
&= \bigoplus_{\substack{1\leq j_1<\cdots<j_{t+i}\leq t+c-1 \\ 1\leq k_1\leq\cdots\leq k_i\leq t}} S(-a_{k_1,j_1} - \cdots - a_{k_i,j_i} - a_{1,j_{i+1}} - \cdots - a_{t,j_{t+i}}).
\end{aligned}
$$

Furthermore, if we denote by m_{i+1} the minimal and by $M_{i+1}$the maximal shift in H_{i+1}, then it is not difficult to see that for all $i = 0, \ldots, c-1$:

$$
\begin{aligned}
m_{i+1} &= a_{t,1} + \cdots + a_{t,i} + a_{1,i+1} + \cdots + a_{t,t+i}, \\
M_{i+1} &= a_{1,c-i} + \cdots + a_{1,c-1} + a_{1,c} + \cdots + a_{t,t+c-1}.
\end{aligned}
$$

Notice that using the above notation we have $M_{i+1} = M_i + a_{1,c-i}$.

As we have seen the degree matrix of a standard determinantal ideal determines its graded Betti numbers. In fact even more is true, the degree matrix of I tells us whether S/I is componentwise linear or not.

Recall that a graded S-module M is called d-linear if $\beta^S_{i,j}(M) \neq 0$, if and only if $j = i + d$. Let $M_{<d>}$ be the S-module generated by M_d, then M is called *componentwise linear* if and only if $M_{<d>}$ is d-linear for any $d \in \mathbb{N}$. According to [35, Theorem 4.1] we have then

Theorem 2.2. *Let $I \subseteq S$ be a codimension c standard determinantal ideal with degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$. The ideal I is then componentwise linear if and only if one of the following statements holds:*

(1) $c = 1$,

(2) $c = 2$ *and* $a_{i,i} = 1$, *for all* $i = 1, \ldots, t$,

(3) $c \geq 3$ *and the entries in all rows of A except possibly the first one are equal to one.*

Abusing language we will call any matrix of integers $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ a *degree matrix* if it is the degree matrix of some standard determinantal ideal. The matrices of integers that are also degree matrices can be characterized in the following way:

Proposition 2.3. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a matrix of integers. Then A is a degree matrix if and only if it is homogeneous (i.e. $a_{i,j} + a_{k,l} = a_{i,l} + a_{k,j}$ for all $i, k = 1, \dots, t$ and $j, l = 1, \dots, t + c - 1$) and $a_{i,i} > 0$, for all $i = 1, \dots, t$.*

For the proof see e.g. [19, Proposition 2.4].

Definition 2.4. *A subscheme $X \subseteq \mathbb{P}^n$ is said to be arithmetically Cohen Macaulay (shortly aCM) if its homogeneous coordinate ring S/I_X is a Cohen Macaulay ring, i.e.* $\operatorname{depth}(S/I_X) = \dim(S/I_X)$. *By the graded version of the Auslander Buchsbaum formula we have* $\operatorname{pdim}(S/I_X) = \operatorname{codim}(X)$.

A standard determinantal scheme $X \subseteq \mathbb{P}^n$ of codimension c is a scheme whose defining ideal I_X is standard determinantal. Every standard determinantal scheme is arithmetically Cohen-Macaulay. More precisely in codimension 1 or 2 the family of standard determinantal schemes is equal to the family of arithmetically Cohen-Macaulay schemes. In codimension 3 or higher the inclusion is strict, i.e. there are aCM schemes that are not standard determinantal.

In codimension 2 (see [39, Proposition 2]), if $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+1)}$ is a degree matrix such that also the first subdiagonal is positive (i.e. $a_{i+1,i} > 0$, for any $i = 0, \dots t-1$), then there exists a smooth aCM curve $C \subset \mathbb{P}^3 = \mathbb{P}^3_{\mathbb{C}}$ with degree matrix A.

For any subscheme $X \subseteq \mathbb{P}^n$ we will use the notation

$$HF_X = HF_{S/I_X}(i) = \dim_K([S/I_X]_i)$$

for the Hilbert function of X. For an S-module N, we denote by

$$NZD_S(N) = \{f \in S \mid f \cdot g \neq 0, \forall g \in N \setminus \{0\}\}$$

the set of non-zero divisors of N in S.

Definition 2.5. **(A)** *Let $X \subseteq \mathbb{P}^n$ be an aCM projective scheme of dimension d with defining ideal I_X. Let $\mathfrak{a}_X = (I_X + (L_1, \dots, L_{d+1})) \subseteq S$, where $L_i \in S_1$ is a linear form such that $L_i \in NZD_S(S/(I_X + (L_1, \dots, L_{i-1})))$ for all $i = 1, \dots, d+1$. The ring $S/\mathfrak{a}_X \cong R/J_X$, where $R = K[X_1, \dots, X_c] \cong S/(L_1, \dots, L_{d+1})$ and $J_X \cong I_X(S/(L_1, \dots, L_{d+1}))$, is called the artinian reduction of X (or of its coordinate ring S/I_X). It has Krull dimension* 0 *and for his Hilbert function holds:*

$$HF_{R/J_X}(i) = \triangle^{d+1} HF_{S/I_X}(i).$$

Furthermore, as $[R/J_X]_n = 0$ for $n \gg 0$ the Hilbert function of R/J_X is a finite sequence of integers $1, h_1, h_2, \dots, h_s, 0$. The sequence
$h^X = (1, h_1, \dots, h_s)$ is called the h-vector of X .

(B) *The series* $HS_X(z) = \sum_{i \geq 0} HF_X(i) z^i$ *is called the Hilbert series of* X. *It is well known, that it can be written in rational form as*

$$HS_X(z) = \frac{\mathrm{hp}(z)}{(1-z)^{d+1}},$$

where $\dim(S/I_X) = d+1$. *The numerator*

$$\mathrm{hp}(z) = 1 + h_1 z + h_2 z^2 + \cdots + h_s z^s,$$

with $h_s \neq 0$ *is called h-polynomial of* X *(or of* S/I_X*) and its coefficients form the h-vector of* X, $h^X = (1, h_1, \ldots, h_s)$.

Clearly $\mathrm{hp}(1) = h_0 + \cdots + h_s = \deg(X) = e_0(S/I_X)$, *where we set* $h_0 = 1$. *We denote by* $\tau(h^X)$ *the degree of the h-polynomial.*

We call X *non-degenerate, if* $\operatorname{codim}(X) = h_1$.

(C) *We define the first difference* $\triangle h^X$ *of* h^X *as*

$$\triangle h_i^X = h_i^X - h_{i-1}^X \text{ for } i = 0, \ldots, s.$$

As the degree matrix A of a standard determinantal scheme X determines the graded Betti numbers of S/I_X and thus h^X, we will write h^A and $\mathrm{hp}^A(z)$ instead of h^X respectively $\mathrm{hp}(z)$. Notice that with this notation $h^{(a_1,\ldots,a_c)}$ and $\mathrm{hp}^{(a_1,\ldots,a_c)}(z)$ denote the h-vector, respectively the h-polynomial of a homogeneous complete intersection ideal generated in degrees $(a_1, \ldots, a_c)$.

The following simple lemma will be frequently used through this work.

Lemma 2.6. *Let* $I = (f_1, \ldots, f_c) \subseteq S$ *be a homogeneous complete intersection ideal generated in degrees* $(a_1, \ldots, a_c)$. *For the h-polynomial of* I *it holds*

$$\mathrm{hp}^{(a_1,\ldots,a_c)}(z) = \prod_{i=1}^{c} (1 + z + \cdots + z^{a_i - 1}).$$

Proof. For any $i = 1, \ldots, c$, we have the following short exact sequence

$$0 \longrightarrow S/(f_1, \ldots, f_{i-1})(-a_i) \xrightarrow{\times f_i} S/(f_1, \ldots, f_{i-1}) \longrightarrow S/(f_1, \ldots, f_i) \longrightarrow 0\ .$$

For $c = 1$ we obtain therefore

$$HS_{S/f_1 S}(z) = \frac{\mathrm{hp}^{(a_1)}(z)}{(1-z)^n} = \frac{(1-z^{a_1})}{(1-z)^{n+1}},$$

so $\mathrm{hp}^{(a_1)}(z) = (1 + \cdots + z^{a_1-1})$. The claim follows now easily by induction on the number of generators c.

□

Definition 2.7. *The Castelnuovo-Mumford regularity of a finitely generated S-module M is defined by*

$$reg_S(M) = \max\left\{j \mid \beta^S_{i,i+j}(M) \neq 0 \, for \ some \ i\right\}.$$

If $X \subseteq \mathbb{P}^n$ is a subscheme, we define the Castelnuovo-Mumford regularity of X as the Castelnuovo-Mumford regularity of its homogeneous ideal I_X and we will write $reg(I_X)$ or $reg(X)$ for it. Note that $reg(I_X) = reg(S/I_X) + 1$.

Remark 2.8. *If $X \subseteq \mathbb{P}^n$ is Cohen-Macaulay, then the Castelnuovo-Mumford regularity of X can be read off the h-vector $h^X = (h_0, \ldots, h_s)$ of X. More precisely $reg(I_X) = s + 1$.*

Next, we will summarize the main results on the behavior of Hilbert functions.

Let $d \in \mathbb{N}$. Any positive integer n can be written in the form

$$n = n_{(d)} = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_j}{j},$$

where $m_d > m_{d-1} > \cdots > m_j \geq j \geq 1$. This is called the *$d$-binomial expansion of n*. For any integers a, b we define

$$(n_{(d)})^a_b = \binom{m_d+a}{d+b} + \binom{m_{d-1}+a}{d-1+b} + \cdots + \binom{m_j+a}{j+b}.$$

Theorem 2.9. *Let $I \subseteq S$ be a homogeneous ideal, $A = S/I$ and $L \in A_1$ a general linear form. Then:*

(1) *Macaulay: $HF_A(d+1) \leq (HF_A(d)_{(d)})^1_1$, for any d,*

(2) *Gotzmann: If $HF_A(d+1) = (HF_A(d)_{(d)})^1_1$ and I is generated in degree $\leq d$, then $HF_A(d+i) = (HF_A(d)_{(d)})^i_i$, for all $i \geq 1$,*

(3) *Green: $HF_{A/LA}(d) \leq (HF_A(d)_{(d)})^{-1}_0$, for any d.*

Proof. For (i) and (ii) see [6, Theorem 4.2.10] and [6, Theorem 4.3.3].
(iii) See [22, Theorem 1].

□

Definition 2.10. *Let $h = (h_0, \ldots, h_s)$ be a sequence of positive integers. Then:*

(A) *h is called an O-sequence if $h_0 = 1$ and it satisfies Macaulay's bound $h_{d+1} \leq ((h_d)_{(d)})^1_1$ for all $1 \leq d \leq s-1$,*

(B) *h is called unimodal if $h_0 \leq h_1 \leq \ldots \leq h_i \geq \ldots \geq h_s$ for some i,*

(C) *h is called of decreasing type if $h_0 < \ldots < h_i = \cdots = h_j > \ldots > h_s$ for some i, j,*

(D) *h is called log-concave if for all* $0 < i < s$, $h_i^2 \geq h_{i-1}h_{i+1}$.

Let $h = (h_0, \ldots, h_s)$ be an O-sequence. For any integer $a > 0$ we define $h(a)$ to be the sequence

$$h(a) = (\underbrace{0, \ldots, 0}_{a}, h_0, \ldots, h_s)$$

and call it the *a-shift* of h.

Remark 2.11. *Notice that if* $h = (h_0, \ldots, h_s)$ *is a sequence of positive integers, then:*

(A) *h is of decreasing type* $\iff \triangle h_i < 0$ *implies* $\triangle h_{i+1} < 0$, *for all* $0 < i < s$.

In other words, once h has started decreasing it keeps decreasing.

(B) *If h is log-concave, then h is of decreasing type.*

We recall in the following some basic facts about level algebras.

Let $I \subseteq S$ be a homogeneous artinian ideal and $S/I = A = K \oplus A_1 \oplus \cdots \oplus A_s$, $A_s \neq 0$. The *socle of* A is denoted by $soc(A)$ and defined by

$$soc(A) = (0 \underset{A}{:} A_+),$$

where $A_+ = A_1 \oplus \cdots \oplus A_s$. Since $soc(A)$ is a homogeneous ideal of A, we can write $soc(A) = U_1 \oplus \cdots \oplus U_s$. Obviously, $A_s \subseteq soc(A)$ and therefore $A_s = U_s$. To the algebra A we can assign the vector $s(A) = (a_1, \ldots, a_s)$, where $a_i = \dim_K(U_i)$. This is referred to be *the socle vector* of A. The number s is called *the socle degree of* A. Artinian algebras with socle degree s and socle vector $s(A) = (0, \ldots, 0, a)$, $a > 0$ are called *level algebras of type a*.

An integer sequence $h = (h_0, \ldots, h_s)$ is called *level sequence* if there is a level artinian algebra, whose h-vector is equal h.

It is well known that the Betti numbers of A and the socle vector of A are related in the following way: if

$$0 \longrightarrow F_{n+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

is a minimal free resolution of A and $s(A) = (a_1, \ldots, a_s)$ its socle vector, then

$$F_{n+1} = \bigoplus_{j=1}^{s} S^{a_j}(-j-(n+1)).$$

In other words

$$\beta^S_{n+1,n+1+j} = a_j, \; \forall j = 1, \ldots, s.$$

In particular

$$A \text{ is level} \iff \beta^S_{n+1,n+1+j} = 0, \, \forall j \neq s.$$

We will recall now quickly a very useful method for constructing Artinian level algebras called Macaulay's Inverse System. For more details on this subject we refer to [15] or [29].

Let $R = K[X_1, \ldots, X_n]$ and $R' = K[Y_1, \ldots, Y_n]$ be two polynomial rings. The ring R' can be regarded as a R-module via the operation $X_i \circ f = \left(\dfrac{d}{dY_i}\right) f$, for any $f \in R'_j$. We have the following $1-1$ correspondence

$$\{\text{Ideals of}\, R\} \longleftrightarrow \{R\text{-submodules of}\, R'\},$$

given by $I \longmapsto I^{-1} = (0 \underset{R'}{:} I) = \{f \in R' |\, g \circ f = 0, \, \forall g \in I\}$ and $M \longmapsto (0 \underset{R}{:} M)$, where I is an ideal of R and M an R-submodule of R'.

The R-submodule I^{-1} is called the inverse system to I. Macaulay observed that

$$\dim_K(I_j^{-1}) = \dim_K(R_j) - \dim_K(I_j) = HF_{R/I}(j).$$

It holds then in particular:

$$I^{-1} \text{ is a finitely generated } R\text{-submodule of } R' \iff R/I \text{ is artinian.}$$

For a finite subset $\mathcal{M} \subset R'$, we will use the notation $d\mathcal{M} = \{df |\, f \in \mathcal{M}\}$, where $df := \left\{\dfrac{df}{dY_i}\right\}_{i=1,\ldots,n}$ denotes the set of all partial derivatives of f. For any integer $k \in \mathbb{N}$ and any polynomial $f \in R'$ we define inductively $d^k f := d^{k-1}(df)$.

If I^{-1} is generated as a R-module by $\{f_1, \ldots, f_s\}$,where $f_i \in R'_{d_i}$, then

$$I^{-1} = \left\langle d^k f_i |\, i = 1, \ldots, s \,\text{and}\, k = 0, \ldots, d_i \right\rangle_K$$

so that

$$\dim_K(I_j^{-1}) = \dim_K \left\langle d^{d_i - j} f_i |\, i = 1, \ldots, s \right\rangle_K.$$

The following theorem of Macaulay gives the connection between the socle vector of an artinian algebra $A = R/I$ and the inverse system I^{-1}.

Theorem 2.12. *Let $I \subseteq R$ be an artinian ideal. Then I^{-1} has exactly a_j minimal generators in degree j if and only if* $\dim_K(soc(R/I)_j) = a_j$.

An O-sequence is called *pure* if it is the Hilbert function of some artinian monomial level algebra. Pure O-sequences have also a purely combinatorial interpretation as follows. We will write $Mon(S)$ for the collection of all monomials of S. An *order ideal on* $Mon(S)$ is a finite subset $\Gamma \subseteq Mon(S)$ closed under division, i.e. if $M \in \Gamma$ and $N|M$, then $N \in \Gamma$. The partial order given by the divisibility of monomials gives Γ a poset structure. An order ideal is called *pure* if all maximal monomials have the same degree. We write

$$\Gamma = \langle M \in \Gamma |\, M \text{ is maximal with respect to division}\rangle.$$

To every order ideal Γ we associate its *f-vector* $f(\Gamma) = (f_0, \ldots, f_s)$, where $f_i(\Gamma) = |\,\{M \in \Gamma|\ \deg(M) = i\}\,|$. It is not difficult to check (using Macaulay's Inverse System) that a vector $h = (h_0, \ldots, h_s)$ is a pure O-sequence if and only if it is the f-vector of some pure order ideal. Therefore it follows in particular, that any pure O-sequence satisfies $h_{s-1} \leq h_1 \cdot h_s$.

We will finish the chapter recalling briefly the notion of basic double link from liaison theory.

Definition 2.13. *If $\mathfrak{b} \subseteq \mathfrak{a} \subseteq S$ are two homogeneous ideals such that $S/\mathfrak{b}$ is Cohen-Macaulay, $ht(\mathfrak{a}) = ht(\mathfrak{b}) + 1$ and $f \in NZD_S(S/\mathfrak{b})$ is a form of degree d, then the ideal $I = f \cdot \mathfrak{a} + \mathfrak{b}$ is called a basic double link of $\mathfrak{a}$.*

If $I = f \cdot \mathfrak{a} + \mathfrak{b}$ is a basic double link, then by [30, Theorem 3.5] I can be Gorenstein linked to $\mathfrak{a}$ in two steps if $\mathfrak{a}$ is unmixed and $S/\mathfrak{b}$ is generically Gorenstein (see also [24, Theorem 3.5]).

3 h-vectors of decreasing type

3.1 The h-polynomial of a standard determinantal scheme

In [20] Gorla constructed basic double links in which all involved ideals are standard determinantal (see also [30]). We will use this construction to obtain a recursive formula for the h-polynomial of a standard determinantal scheme.

For any matrix A we use the following notation: $A^{(k,l)}$ is the matrix obtained from A by deleting the k-th row and l-th column. By convention, $A^{(k,0)}$ (resp. $A^{(0,l)}$) means that only the k-th row (resp. the l-th column) has been deleted.

Proposition 3.1. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix. For any $k = 1, \dots, t$ and $l = 1, \dots, t+c-1$ such that $a_{k,l} \geq 0$, we have:*

$$\mathrm{hp}^{A}(z) = z^{a_{k,l}}\mathrm{hp}^{A^{(k,l)}}(z) + (1 + \cdots + z^{a_{k,l}-1})\mathrm{hp}^{A^{(0,l)}}(z).$$

Proof. We distinguish two cases.
Case 1: $a_{k,l} > 0$. Without loss of generality we can assume that $k, l = 1$. Consider the homogeneous matrix:

$$M = \begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,t+c-1} \\ 0 & f_{2,2} & \cdots & f_{2,t+c-1} \\ \vdots & \vdots & & \vdots \\ 0 & f_{t,2} & \cdots & f_{t,t+c-1} \end{bmatrix},$$

where the $f_{i,j}$'s are generically chosen forms in $S = K[X_0, \dots, X_n]$, with $n \geq c-1$ and $\deg(f_{i,j}) = a_{i,j}$. Such forms exist because the field K is infinite. Let $\mathfrak{a} = I_{\max}(M^{(1,1)})$ and $\mathfrak{b} = I_{\max}(M^{(0,1)})$ be two ideals which by the generic choice of the forms $f_{i,j}$ are standard determinantal. Thus, by construction we have $ht(\mathfrak{b}) = ht(\mathfrak{a}) - 1$ and $f_{1,1} \in NZD_S(S/\mathfrak{b})$. If $I := I_{\max}(M)$, then by direct computation on the generators we obtain that

$$I = f_{1,1}\mathfrak{a} + \mathfrak{b},$$

so I is a basic double link of $\mathfrak{a}$. By [20, Theorem 3.1], the ideal I is also standard determinantal. Notice that the corresponding degree matrices of $I, \mathfrak{a}$ and $\mathfrak{b}$ are $A, A^{(1,1)}$, respectively $A^{(0,1)}$. From the short exact sequence

$$0 \longrightarrow \mathfrak{b}(-a_{1,1}) \longrightarrow \mathfrak{a}(-a_{1,1}) \oplus \mathfrak{b} \longrightarrow I \longrightarrow 0$$

where the first map is given by the assignment $g \mapsto (g, f_{1,1} \cdot g)$ and the second by $(g, h) \mapsto gf_{1,1} - h$, it follows that, if $d = \dim_S(S/\mathfrak{a}) = n + 1 - c$, then

$$\begin{aligned}\mathrm{HS}_{S/I}(z) &= z^{a_{1,1}}\mathrm{HS}_{S/\mathfrak{a}}(z) + (1-z^{a_{1,1}})\mathrm{HS}_{S/\mathfrak{b}}(z)\\ &= \frac{z^{a_{1,1}}\mathrm{hp}^{A^{(1,1)}}(z)}{(1-z)^d} + \frac{(1-z^{a_{1,1}})\mathrm{hp}^{A^{(0,1)}}(z)}{(1-z)^{d+1}}\\ &= \frac{z^{a_{1,1}}\mathrm{hp}^{A^{(1,1)}}(z) + (1+\cdots+z^{a_{1,1}-1})\mathrm{hp}^{A^{(0,1)}}(z)}{(1-z)^d}\end{aligned}$$

and we conclude.

Case 2: $a_{k,l}=0$. By induction on t and c we will show that

$$\mathrm{hp}^{A}(z) = \mathrm{hp}^{A^{(k,l)}}(z).$$

Notice that it must be $t \geq 2$.

By the ordering of the entries in A, and because $a_{i,i} > 0$ for all i, if $a_{k,l}=0$, then $k > l$ (i.e. $a_{k,l}$ lies below the diagonal).

When $c=1$, the h-vector corresponding to A is just a sequence of ones, of length $tr(A) = \sum_{i=1}^{t} a_{i,i}$, so the only thing that has to be shown is $tr(A) = tr(A^{(k,l)})$. This follows easily observing that

$$tr(A^{(k,l)}) = \sum_{i=1}^{l-1} a_{i,i} + \sum_{i=l}^{k-1} a_{i,i+1} + \sum_{i=k+1}^{t} a_{i,i},$$

since using the homogeneity of A we have

$$tr(A^{(k,l)}) = tr(A^{(k,l)}) + a_{k,l} = tr(A).$$

Let $c > 1$. For $t=2$, since $a_{2,1}=0$, from Case 1 applied to the indices $(2, c+1)$, it follows that

$$\mathrm{hp}^{A}(z) = z^{a_{2,c+1}}\mathrm{hp}^{(a_{1,1},\ldots,a_{1,c})}(z) + (1+\cdots+z^{a_{2,c+1}-1})\mathrm{hp}^{A^{(0,c+1)}}(z).$$

The h-polynomial of a 1-row degree matrix is the h-polynomial of the corresponding complete intersection, namely

$$\mathrm{hp}^{(a_{1,1},\ldots,a_{1,c})}(z) = \prod_{i=1}^{c}(1+\cdots+z^{a_{1,i}-1}).$$

By induction on c we have

$$\mathrm{hp}^{A^{(0,c+1)}}(z) = \mathrm{hp}^{(a_{1,2},\ldots,a_{1,c})}(z) = \prod_{i=2}^{c}(1+\cdots+z^{a_{1,i}-1}),$$

so we obtain

$$\mathrm{hp}^{A}(z) = \left(1+\cdots+z^{a_{2,c+1}}+\cdots+z^{a_{1,1}+a_{2,c+1}-1}\right)\prod_{i=2}^{c}\left(1+\cdots+z^{a_{1,i}-1}\right).$$

As A is a homogeneous matrix, it holds that $a_{1,1} + a_{2,c+1} = a_{2,1} + a_{1,c+1}$ and we conclude.

When $t > 2$, there exists some positive entry $a_{i,i}$, with $i \neq k, l$. The matrices $A^{(i,i)}$ and $A^{(0,i)}$ contain $a_{k,l} = 0$. Therefore applying Case 1 for $a_{i,i}$ and using the induction hypothesis on t and c we obtain

$$\begin{aligned} \mathrm{hp}^{A}(z) &= z^{a_{i,i}} \mathrm{hp}^{A^{(i,i)}}(z) + (1 + \cdots + z^{a_{i,i}-1}) \mathrm{hp}^{A^{(0,i)}}(z) \\ &= z^{a_{i,i}} \mathrm{hp}^{(A^{(i,i)})^{(k,l)}}(z) + (1 + \cdots + z^{a_{i,i}-1}) \mathrm{hp}^{(A^{(0,i)})^{(k,l)}}(z) \\ &= \mathrm{hp}^{A^{(k,l)}}(z) \end{aligned}$$

□

Remark 3.2. *Proposition 3.1 implies the following recursive formula for the h-vector of A:*

$$h_i^A = h_{i-a_{k,l}}^{A^{(k,l)}} + \sum_{k=0}^{a_{k,l}-1} h_{i-k}^{A^{(0,l)}}.$$

In particular, if A has some entry $a_{k,l} = 0$, then $h^A = h^{A^{(k,l)}}$.

Remark 3.3. *As we are interested in studying the h-vectors of standard determinantal schemes, by Remark 3.2 we may assume from now on that none of the degree matrices contain zero entries.*

For any O-sequence $h = (h_0, \dots, h_s)$ we make the convention $h_i = 0$ if $i < 0$ or $i > s$.

Lemma 3.4. *Let $Q(z) \in \mathbb{Z}_{>0}[z]$ and $P(z) = \left(1 + z + \cdots + z^{a-1}\right) Q(z)$, $a \geq 1$ be polynomials, whose coefficients $h = (h_0, \dots, h_s)$, and respectively $H = (H_0, \dots, H_{s+a-1})$ form an O-sequence. If h is of decreasing type, then H is also of decreasing type.*

Proof. Since $H_i = \sum_{k=0}^{a-1} h_{i-k}$ for $i = 0, \dots, s+a-1$, we have $\triangle H_i = h_i - h_{i-a}$.

Assume that H is not of decreasing type. Then there exists an index i, with $1 \leq i \leq s + a - 1$, such that $\triangle H_i < 0$ and $\triangle H_{i+1} \geq 0$. Denote by t the least integer such that $h_t > h_{t+1}$. As h is of decreasing type and $h_i < h_{i-a}$ we have $i > t$. From the sequence of inequalities $h_{i-a} > h_i > h_{i+1} \geq h_{i+1-a}$ it follows that $i - a \geq t$. Therefore, we obtain

$$h_{i-a} > h_{i+1-a} > h_{i+2-a} > \cdots > h_i > h_{i+1},$$

which contradicts $\triangle H_{i+1} \geq 0$.

□

Next, using Proposition 3.1 we establish a combinatorial formula for the h-polynomial of a standard determinantal scheme.

For any degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ we define the vectors $d = (d_1, \ldots, d_{t+c-1})$ and $e = (e_1, \ldots, e_t)$ as follows:

$$e_i = a_{1,1} - a_{i,1} \text{ and } d_j = a_{1,j} - a_{1,1},$$

for all $i = 1, \ldots, t$, and $j = 1, \ldots, t + c - 1$. As the entries in A increase from left to right and from bottom to the top we have $0 = d_1 \le d_2 \le \cdots \le d_{t+c-1}$ and $0 = e_1 \le e_2 \le \cdots \le e_t$. Notice that $a_{i,j} = a_{1,1} + d_j - e_i$.

For any increasing sequence of integers $0 < i_1 < \cdots < i_{c-1} < t + c - 1$ and any matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$, we define two ordered sets of integers:

$$\begin{aligned} \{j_1, \ldots, j_{i_{c-1}-(c-1)}\} &= \{1, \ldots, i_{c-1}\} \setminus \{i_1, \ldots, i_{c-1}\}, \\ \mathrm{g}_A(i_1, \ldots, i_{c-1}) &= \{a_{i_1,i_1},\ a_{i_2-1,i_2},\ \ldots,\ a_{i_{c-1}-(c-2),i_{c-1}},\ \sum_{i=i_{c-1}-(c-2)}^{t} a_{i,i+c-1}\}. \end{aligned}$$

To the first set we associate a nonnegative number and to the second a polynomial in one variable :

$$\begin{aligned} e_A(i_1, \ldots, i_{c-1}) &= \sum_{i=1}^{i_{c-1}-(c-1)} a_{i,j_i}, \\ \mathrm{hci}_A(i_1, \ldots, i_{c-1}) &= \mathrm{hp}^{(\mathrm{g}_A(i_1,\ldots,i_{c-1}))}(z). \end{aligned}$$

For $c = 1$ we have by convention

$$\begin{aligned} &\mathrm{g}_A(i_1, \ldots, i_{c-1}) = \left\{ \sum_{i=1}^{t} a_{i,i} \right\}, \\ &e_A(i_1, \ldots, i_{c-1}) = 0 \end{aligned}$$

and in particular

$$\mathrm{hp}^A(z) = \mathrm{hci}_A(i_1, \ldots, i_{c-1}) = \mathrm{hp}^{(\sum_{i=1}^{t} a_{i,i})}(z).$$

Remark 3.5. *For any degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$, with the above notation, it holds*

$$e_A(i_1, \ldots, i_{c-1}) = \sum_{i=1}^{i_{c-1}-(c-1)} a_{i,j_i} = a_{1,1}(i_{c-1} - (c-1)) + \sum_{i=1}^{i_{c-1}-(c-1)} (d_{j_i} - e_i).$$

Proposition 3.6. *The h-polynomial of any degree matrix $A \in \mathbb{Z}^{t\times(t+c-1)}$ is given by*

$$\mathrm{hp}^A(z) = \sum_{0<i_1<\cdots<i_{c-1}<t+c-1} z^{e_A(i_1,\ldots,i_{c-1})} \cdot \mathrm{hci}_A(i_1,\ldots,i_{c-1}).$$

Proof. We will prove the statement by induction on c and t. For $t = 1$ or $c = 1$ we obtain only one summand and the equality clearly holds. So let $t, c > 1$.

According to Proposition 3.1 we have

$$\mathrm{hp}^A(z) = z^{a_{1,1}}\mathrm{hp}^{A^{(1,1)}}(z) + (1 + \cdots + z^{a_{1,1}-1})\mathrm{hp}^{A^{(0,1)}}(z).$$

Let us denote the entries of the matrix A by $(a_{i,j})$, the entries of the matrix $A^{(1,1)}$ by $(a'_{i,j})$ and the entries of $A^{(0,1)}$ by $(a''_{i,j})$. By definition $a'_{i,j} = a_{i+1,j+1}$ and $a''_{i,j} = a_{i,j+1}$. By the inductive hypothesis on t we have

$$\mathrm{hp}^{A^{(1,1)}}(z) = \sum_{0<i_1<\cdots<i_{c-1}<t+c-2} z^{e_{A^{(1,1)}}(i_1,\ldots,i_{c-1})} \cdot \mathrm{hci}_{A^{(1,1)}}(i_1,\ldots,i_{c-1}).$$

For a sequence $0 < k_1 < \cdots < k_{c-1} < t + c - 2$, using Remark 3.5 we obtain

$$\begin{aligned}\sum_{i=1}^{k_{c-1}-(c-1)} a'_{i,j_i} &= a_{1,1} + a_{2,2}(k_{c-1} - (c-1)) + \sum_{i=1}^{k_{c-1}-(c-1)} (d'_{j_i} - e'_i) - a_{1,1}\\ &= a_{1,1} + a_{2,2} + \cdots + a_{k_1,k_1} + a_{k_1+1,k_1+2} + \cdots + a_{k_{c-1}-(c-2),k_{c-1}} - a_{1,1}.\end{aligned}$$

On the other hand for a sequence $0 < i_1 < \cdots < i_{c-1} < t + c - 1$ given by $(i_1, \ldots, i_{c-1}) = (k_1 + 1, \ldots, k_{c-1} + 1)$ we have

$$\begin{aligned}\{j_1, j_2, \ldots, j_{k_{c-1}-(c-2)}\} &= \{1, \ldots, k_{c-1} + 1\} \setminus \{k_1 + 1, \ldots, k_{c-1} + 1\}\\ &= \{1, \ldots, k_1, k_1 + 2, \ldots, k_{c-1}\}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{k_{c-1}-(c-2)} a_{i,j_i} &= a_{1,1}(k_{c-1} - (c-2)) + \sum_{i=1}^{k_{c-1}-(c-2)} (d_{j_i} - e_i)\\ &= a_{1,1} + \cdots + a_{k_1,k_1} + a_{k_1+1,k_1+2} + \cdots + a_{k_{c-1}-(c-2),k_{c-1}}.\end{aligned}$$

In particular it follows that

$$e_{A^{(1,1)}}(k_1, \ldots, k_{c-1}) = e_A(k_1 + 1, \ldots, k_{c-1} + 1) - a_{1,1}.$$

It is easy to check that this implies

$$\mathrm{hp}^{A^{(1,1)}}(z) = \sum_{1<i_1<\cdots<i_{c-1}<t+c-1} z^{e_A(i_1,\ldots,i_{c-1})-a_{1,1}} \cdot \mathrm{hci}_A(i_1,\ldots,i_{c-1}).$$

By the inductive hypothesis on c we obtain

$$\mathrm{hp}^{A^{(0,1)}}(z) = \sum_{0<i_1<\cdots<i_{c-2}<t+c-2} z^{e_{A^{(0,1)}}(i_1,\ldots,i_{c-2})} \cdot \mathrm{hci}_{A^{(0,1)}}(i_1,\ldots,i_{c-2}).$$

It is not difficult to check as above that

$$\mathrm{g}_{A^{(0,1)}}(k_1,\ldots,k_{c-2}) = \mathrm{g}_A(1,k_1+1,\ldots,k_{c-2}+1) \setminus \{a_{1,1}\},$$

and that

$$e_{A^{(0,1)}}(k_1,\ldots,k_{c-2}) = e_A(1,k_1+1,\ldots,k_{c-2}+1).$$

This implies that

$$\mathrm{hp}^{A^{(0,1)}}(z) = \sum_{1<i_2<\cdots<i_{c-1}<t+c-1} z^{e_A(1,i_2,\ldots,i_{c-1})} \cdot \frac{\mathrm{hci}_A(1,i_2,\ldots,i_{c-1})}{(1+\cdots+z^{a_{1,1}-1})},$$

and we conclude.

□

Example 3.7. *Let $t=2$, $c=3$, so that $A=[a_{i,j}] \in \mathbb{Z}^{2\times 4}$. Using the vectors $e=(0,e_2)$ and $d=(0,d_2,d_3,d_4)$ we can write A in the form:*

$$\begin{bmatrix} a & a+d_2 & a+d_2+d_3 & a+d_2+d_3+d_4 \\ a-e_2 & a-e_2+d_2 & a-e_2+d_2+d_3 & a-e_2+d_2+d_3+d_4 \end{bmatrix}$$

There are the following possibilities for (i_1,i_2):

- $(i_1,i_2)=(1,2) \Longrightarrow \{1,2\}\setminus\{1,2\}=\{\varnothing\} \Longrightarrow e_A(1,2)=0,$
- $(i_1,i_2)=(1,3) \Longrightarrow \{1,2,3\}\setminus\{1,3\}=\{2\}=\{j_1\} \Longrightarrow e_A(1,3)=a_{1,2},$
- $(i_1,i_2)=(2,3) \Longrightarrow \{1,2,3\}\setminus\{2,3\}=\{1\}=\{j_1\} \Longrightarrow e_A(2,3)=a_{1,1}.$

It holds also

$$\begin{aligned}
\mathrm{hci}_A(1,2) &= \mathrm{hp}^{(a_{1,1},a_{1,2},a_{1,3}+a_{2,4})},\\
\mathrm{hci}_A(1,3) &= \mathrm{hp}^{(a_{1,1},a_{2,3},a_{2,4})},\\
\mathrm{hci}_A(2,3) &= \mathrm{hp}^{(a_{2,2},a_{2,3},a_{2,4})}.
\end{aligned}$$

and therefore

$$\mathrm{hp}^A(z) = z^{a_{1,1}}\mathrm{hp}^{(a_{2,2},a_{2,3},a_{2,4})}(z) + z^{a_{1,2}}\mathrm{hp}^{(a_{1,1},a_{2,3},a_{2,4})}(z) + \mathrm{hp}^{(a_{1,1},a_{1,2},a_{1,3}+a_{2,4})}(z).$$

For instance, if $A = \begin{bmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{bmatrix}$, *then* h^A *is computed via the componentwise sum:*

$$\begin{array}{ccccccccccccccccc} & 0 & 0 & 0 & 0 & 1 & 3 & 6 & 9 & 11 & 11 & 9 & 6 & 3 & 1 & 0 \\ + & 0 & 0 & 0 & 1 & 3 & 6 & 9 & 11 & 11 & 9 & 6 & 3 & 1 & 0 & 0 \\ & 1 & 3 & 6 & 9 & 11 & 12 & 12 & 12 & 12 & 12 & 11 & 9 & 6 & 3 & 1 \\ \hline & 1 & 3 & 6 & 10 & 15 & 21 & 27 & 32 & 34 & 32 & 26 & 18 & 10 & 4 & 1 \end{array}$$

As a direct consequence of Proposition 3.6 we obtain the following:

Corollary 3.8. *Let* $X \subseteq \mathbb{P}^n$ *be a standard determinantal scheme of codimension* c *with degree matrix* $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ *and assume that* $a = a_{i,j}$ *for all* i *and* j. *Then:*

(1) $\mathrm{hp}^A(z) = \sum_{i=1}^{t} \binom{t+c-2-i}{c-2} z^{a(t-i)} \mathrm{hp}^{(a,\dots,a,ia)}(z)$, *for* $c \geq 2$,

(2) $\deg(X) = a^c \left(\sum_{i=1}^{t} i \binom{t+c-2-i}{c-2} \right)$, *for* $c \geq 2$,

(3) $reg(I_X) = (t+c-1)a - (c-1)$.

For $c = 1$ *we have by convention* $\mathrm{hp}^A(z) = \mathrm{hp}^{(ta)}(z)$ *and* $\deg(X) = ta$.

Proof. (1) For any sequence $0 < i_1 < \cdots < i_{c-1} < t+c-1$ we have

$$e_A(i_1, \dots, i_{c-1}) = (i_{c-1} - (c-1))a$$

and

$$\mathrm{hci}_A(i_1, \dots, i_{c-1}) = \mathrm{hp}^{(a,\dots,a,(t-(i_{c-1}-(c-1)))a)}.$$

Therefore, to prove the claim it is enough to compute the number of summands appearing in the formula for $\mathrm{hp}^A(z)$. For any fixed $c-1 \leq i_{c-1} \leq t+c-2$ there are $\binom{i_{c-1}-1}{c-2}$ sequences $(i_1, \dots, i_{c-2}, i_{c-1})$, so there are $\sum_{i=1}^{t} \binom{t+c-2-i}{c-2}$ summands in $\mathrm{hp}^A(z)$ and the claim follows.
(2) Since $\deg(X) = \mathrm{hp}^A(1)$ and $\mathrm{hp}^{(a,\dots,a,ka)}(1) = ka^c$ for any k the claim follows from (1).
(3) $reg(I_X) = \deg(\mathrm{hp}^A(z)) + 1 = (t+c-1)a - (c-1)$.

$\square$

More generally it holds:

Corollary 3.9. *If $X \subseteq \mathbb{P}^n$ is a codimension c standard determinantal scheme, whose degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ has $a_{i,j} = a_j$ for all i and j then:*

$$\mathrm{hp}^A(z) = \sum_{0<i_1<\cdots<i_{c-1}<t+c-1} z^{\sum_{i=1}^{i_{c-1}-(c-1)} a_{j_i}} \mathrm{hp}^{(a_{i_1},\ldots,a_{i_{c-1}},\sum_{i=i_{c-1}-(c-2)}^{t} a_{i+c-1})}.$$

It is sometimes useful to have a more explicit formula for the h-polynomial of a standard determinantal scheme. In codimension 2 or 3 we have

Lemma 3.10. *For the h-polynomial $\mathrm{hp}^A(z)$ of a degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ we have*

(1) *For $c = 2$ and if:*

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,t+1} \\ a_{2,1} & & a_{2,t+1} \\ \vdots & & \vdots \\ a_{t,1} & \cdots & a_{t,t+1} \end{bmatrix} = \begin{bmatrix} u_t & d_t & & & & \\ & u_{t-1} & d_{t-1} & & & \\ & & \ddots & \ddots & & \\ & & & & u_1 & d_1 \end{bmatrix}$$

it holds

$$\mathrm{hp}^A(z) = \sum_{i=1}^{t} z^{\sum_{j=i+1}^{t} u_j} \mathrm{hp}^{(u_i,\sum_{j=1}^{i} d_j)}.$$

(2) *For $c = 3$ and if*

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,t+2} \\ \vdots & & \vdots \\ a_{t,1} & \cdots & a_{t,t+2} \end{bmatrix} = \begin{bmatrix} u_t & d_t & v_t & & & & \\ & u_{t-1} & d_{t-1} & v_{t-1} & & & \\ & & & \ddots & \ddots & & \\ & & & & u_1 & d_1 & v_1 \end{bmatrix}$$

it holds

$$\mathrm{hp}^A(z) = \sum_{k=1}^{t}\sum_{j=k}^{t} z^{\alpha_{k,j}+\sum_{i=j+1}^{t} u_i} \mathrm{hp}^{(u_j,d_k,\sum_{j=1}^{k} v_j)},$$

where $\alpha_{k,j}$ are the entries of the matrix

$$[\alpha_{k,j}] = \begin{bmatrix} 0 & d_2 & d_2+d_3 & \cdots & \sum_{i=2}^{t} d_i \\ 0 & 0 & d_3 & & \sum_{i=3}^{t} d_i \\ \vdots & & & \ddots & \vdots \\ & & & & d_t \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}.$$

Proof. Analogous to the proof of Proposition 3.6. ∎

We now focus on the degree and the leading coefficient of the h-polynomial.

Proposition 3.11. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and let $h^A = (h_0, \dots, h_{\tau(h^A)})$. Then:*

(1) $\tau(h^A) = a_{1,1} + \cdots + a_{1,c} + a_{2,c+1} + \cdots + a_{t,t+c-1} - c,$

(2) $h_{\tau(h^A)} = \binom{r+c-2}{c-1}$, *where* $r = \max\{i|\, a_{1,1} = \cdots = a_{i,1}\}$,

(3) *if $X \subseteq \mathbb{P}^n$ is a standard determinantal scheme with degree matrix A then*

$$\deg(X) = \sum_{0<i_1<\cdots<i_{c-1}<t+c-1} a_{i_1,i_1}\cdots a_{i_{c-1}-(c-2),i_{c-1}}\cdot(\textstyle\sum_{i=i_{c-1}-(c-2)}^{t} a_{i,i+c-1}).$$

For $c = 1$, we have $\deg(X) = \sum_{i=1}^{t} a_{i,i}.$

Proof. We will prove the claim by induction on t and c. For $t, c = 1$ statements (1),(2) and (3) are clear, so let $t, c > 1$.
(1) By Remark 3.2 applied to the indices $(t, t+c-1)$ we have

$$h_i^A = h_{i-a_{t,t+c-1}}^{A^{(t,t+c-1)}} + \sum_{k=0}^{a_{t,t+c-1}-1} h_{i-k}^{A^{(0,t+c-1)}} .$$

Thus by induction

$$\begin{aligned}\tau(h^A) &= \max\{\tau(h^{A^{(t,t+c-1)}}) + a_{t,t+c-1}, \tau(h^{A^{(0,t+c-1)}}) + a_{t,t+c-1} - 1\}\\ &= \max\left\{\sum_{i=1}^{c} a_{1,i} + \sum_{i=2}^{t} a_{i,i+c-1} - c, \sum_{i=1}^{c-1} a_{1,i} + \sum_{i=2}^{t} a_{i,i+c-2} + a_{t,t+c-1} - c\right\}\end{aligned}$$

and the statement in (1) follows.
(2) From the proof of (1), as

$$\tau(h^{A^{(t,t+c-1)}}) + a_{t,t+c-1} \geq \tau(h^{A^{(0,t+c-1)}}) + a_{t,t+c-1} - 1,$$

using the homogeneity of A we deduce in particular that

$$\begin{aligned}h_{\tau(h^A)}^A = h_{\tau(h^{A^{(t,t+c-1)}})}^{A^{(t,t+c-1)}} &\iff \tau(h^{A^{(t,t+c-1)}}) > \tau(h^{A^{(0,t+c-1)}}) - 1\\ &\iff a_{1,t+c-2} > a_{t,t+c-2} \iff a_{1,t+c-1} > a_{t,t+c-1}\\ &\iff A \text{ does not have equal rows.}\end{aligned}$$

Therefore it is enough to prove the second statement for matrices with equal rows (i.e. with $r = t$). We have by induction:

$$\begin{aligned} h_{\tau(h^A)} &= h_{\tau(h^{A^{(t,t+c-1)}})} + h_{\tau(h^{A^{(0,t+c-1)}})} = \binom{t+c-3}{c-1} + \binom{t+c-3}{c-2} \\ &= \binom{t+c-2}{c-1}. \end{aligned}$$

(3) follows directly from $\deg(X) = \mathrm{hp}^A(1)$.

□

Remark 3.12. **(A)** *Let $A \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and let $h^A = (h_0, \dots, h_s)$. We denote by $h' = (h'_0, \dots, h'_{s'})$ the h-vector of $A^{(t,t+c-1)}$ and by $h'' = (h''_0, \dots, h''_{s''})$ the h-vector given by*

$$h''_i = \sum_{k=0}^{a_{t,t+c-1}-1} h_{i-k}^{A^{(0,t+c-1)}},$$

where $h_{i-k}^{A^{(0,t+c-1)}} = 0$ if $i < k$. By Proposition 3.11 we have $s' - s'' = e_t$ and by Proposition 3.1, h^A is computed by component-wise addition:

$$\begin{array}{ccccccccccc} 0 & \dots & 0 & h'_0 & \dots & h'_{s-e_t} & h'_{s-e_t+1} & \dots & h'_s & + \\ h''_0 & \dots & h''_{a_{t,t+c-1}-1} & h''_{a_{t,t+c-1}} & \dots & h''_{s''} & 0 & \dots & 0 & \\ \hline h^A_0 & \dots & h^A_{a_{t,t+c-1}-1} & h^A_{a_{t,t+c-1}} & \dots & h^A_{s-e_t} & h^A_{s-e_t+1} & \dots & h^A_s & \end{array}$$

In particular, since $0 = e_r \le e_{r+1} \le \dots \le e_t$, the last e_{r+1} entries of h^A are equal to the last e_{r+1} entries of $h^{\bar{A}}$, where $\bar{A}$ is the $r \times (r+c-1)$ upper-left block of A.

(B) *Notice that if $X \subseteq \mathbb{P}^n$ is a standard determinantal scheme, with Artinian reduction R/J_X of the coordinate ring S/I_X, then by Proposition 3.11 the last entry $\dim_K(soc(R/J_X)_{\tau(h^A)})$ of the socle vector of R/J_X depends only on the codimension of X and the number of equal maximal rows in the degree matrix of X.*

(C) *Since a standard determinantal scheme X is Cohen-Macaulay, Proposition 3.11 gives us an easy way to read the Castelnuovo-Mumford regularity of X directly from the corresponding degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$. More precisely*

$$reg(X) = a_{1,1} + \dots + a_{1,c} + a_{2,c+1} + \dots + a_{t,t+c-1} - (c-1).$$

3.2 Criteria for decreasing type

For any homogeneous ideal $I \subseteq S$ we will write $beg(I)$ for the least integer α, such that $I_\alpha \neq 0$.

Definition 3.13. *Let $h = (h_0, \ldots, h_s)$ be an O-sequence. We define:*

(1) $s(h) := \max\{i|\, h_i > h_{i-1}\} = \max\{i|\, \triangle h_i > 0\}$,

(2) $t(h) := \min\{i|\, h_i > h_{i+1}\} = \min\{i|\, \triangle h_{i+1} < 0\}$,

(3) $d(h) := \min\{i|\, \triangle h_i \geq \triangle h_{i+1}\} = \min\left\{i|\, \triangle^2 h_{i+1} \leq 0\right\}$,

(4) $e(h) := \max\left\{j|\, \triangle^2 h_{i+1} \leq 0, \forall\, d(h) \leq i \leq j\right\}$,

(5) *If $t(h) - s(h) + 1 = l$, we will say that h has a flat of length l.*

Example 3.14. *We have for instance:*

	0	1	2	3	4	5	6	7	8	9	10
h	1	3	6	10	14	17	17	14	9	4	1
$\triangle h$	1	2	3	4	4	3	0	-3	-5	-5	-3
$\triangle^2 h$	1	1	1	1	0	-1	-3	-3	-2	0	2

Therefore, $s(h) = 5, t(h) = 6, d(h) = 3$ and $e(h) = 8$. Moreover h has a flat of length 2.

Remark 3.15. *Notice that:*

(A) *For any O-sequence h we have $d(h) \leq t(h)$ and $d(h) \leq e(h)$.*

(B) *For an O-sequence h, which is obtained by component-wise addition of the O-sequences h' and h'', such that $\max\{d(h'), d(h'')\} \leq \min\{e(h'), e(h'')\}$, holds*

$$d(h) \leq \max\{d(h'), d(h'')\} \leq \min\{e(h'), e(h'')\} \leq e(h).$$

A bound on the first difference of the h-vector of a standard determinantal scheme can be easily obtained.

Lemma 3.16. *For the h-vector $h^X = (h_0, \ldots, h_s)$ of a codimension c standard determinantal scheme $X \subseteq \mathbb{P}^n$ it holds:*

$$\triangle h_i^X \leq \left(\left(\triangle h_2^X\right)_{(2)}\right)_{i-2}^{i-2}, \text{ for all } i = 2, \ldots, s\ .$$

Proof. We will show the claim using the same proof techniques as in [4, Proposition 3.6].

Since the claim is trivial for $s = 2$ we can assume that $s \geq 3$. Let R/J_X be the artinian reduction of S/I_X and let $L \in (R/J_X)_1$ be a general linear form. The short exact sequence

$$0 \longrightarrow \frac{(J_X \underset{R}{:} L)}{J_X}(-1) \longrightarrow \frac{R}{J_X}(-1) \xrightarrow{\times L} \frac{R}{J_X} \longrightarrow \frac{R}{(J_X, L)} \longrightarrow 0$$

shows that $\triangle h_i^X \leq HF_{R/(J_X,L)}(i)$, where the equality holds if and only if the kernel of the multiplication map is trivial. Therefore, by [25, Theorem 6.2]

$$\triangle h_i^X = HF_{R/(J_X,L)}(i), \, \forall i \leq \left\lfloor \frac{s+1}{2} \right\rfloor.$$

Repeated application of Theorem 2.9 (1) and of the relations $(n_{(d)})_b^a = (n_{(d-1)})_{b+1}^{a+1}$ shows then

$$\triangle h_i \leq HF_{R/(J_X,L)}(i) \leq \Big(\big(HF_{R/(J_X,L)}(2)\big)_{(2)} \Big)_{i-2}^{i-2} = \Big((\triangle h_2)_{(2)} \Big)_{i-2}^{i-2}.$$

□

The first difference of the h-vector of a standard determinantal codimension 2 scheme X can be expressed also via the minimal free resolution of the defining ideals.

Lemma 3.17. *Let $X \subseteq \mathbb{P}^n$ be a codimension 2 standard determinantal scheme with minimal free resolution*

$$0 \longrightarrow \bigoplus_{j=1}^{t} S(-b_j) \longrightarrow \bigoplus_{i=1}^{t+1} S(-a_i) \longrightarrow I_X \longrightarrow 0\,.$$

For any positive integer d let

$$\lambda(d) = |\Lambda(d)| = |\,\{i |\, a_i \leq d\}\,|,\; \mu(d) = |M(d)| = |\,\{j |\, b_j \leq d\}\,|.$$

We have then $\triangle h_d^X = 1 - \lambda(d) + \mu(d)$.

Proof. Let R/J_X be the artinian reduction of S/I_X, where $R = K[X_0, X_1]$.

The minimal free resolution of R/J_X is then given by

$$0 \longrightarrow \bigoplus_{j=1}^{t} R(-b_j) \longrightarrow \bigoplus_{i=1}^{t+1} R(-a_i) \longrightarrow R \longrightarrow R/J_X \longrightarrow 0\,.$$

Thus

$$h_d^X = HF_{R/J}(d) = d + 1 - \sum_{i\in\Lambda(d)} HF_R(d-a_i) + \sum_{j\in M(d)} HF_R(d-b_j)$$

and therefore

$$\begin{aligned}\triangle h_d^X &= 1 - \sum_{i\in\Lambda(d)} (d+1-a_i) + \sum_{i\in\Lambda(d-1)} (d-a_i)\\ &+ \sum_{j\in M(d)} (d+1-b_j) - \sum_{j\in M(d-1)} (d-b_j)\\ &= 1 - (\sum_{i\in\Lambda(d)} ((d+1-a_i)-(d-a_i)))\\ &+ \sum_{i\in M(d)} (d+1-b_j-(d-b_j))\\ &= 1-\lambda(d)+\mu(d).\end{aligned}$$

□

Remark 3.18. *Notice that by Lemma 3.17 we have:*

(A) *For any $d \geq a_{t+1}$ it holds $\triangle h_d^X \leq 0$. In particular $a_{t+1} > s(h^X)$.*

(B) *$\triangle^2 h_d^X = \triangle\mu(d) - \triangle\lambda(d)$ and therefore for any $d \geq a_{t+1}+1$ it holds $\triangle^2 h_d^X \geq 0$.*

Motivated by Remark 3.2, we now establish numerical conditions, which ensure that an O-sequence, which can be written as a component-wise sum of two other O-sequences, is of decreasing type.

Lemma 3.19. *Let h and h' be two O-sequences of decreasing type and let H be another O-sequence obtained by the componentwise sum of $h(a)$ and h' for some $a \in \mathbb{N}$, i.e. $H_i = h_{i-a} + h'_i$ for all i. Assume further that $\tau(h) + a \geq t(h') - 1$. If one of the following conditions holds:*

(1) *$s(h') \leq s(h) + a \leq t(h')$,*

(2) *$s(h') \leq t(h) + a \leq t(h')$,*

then H is of decreasing type.

Proof. Since $H_i = h_{i-a} + h'_i$ for all i, we have $\tau(H) = \max\{\tau(h)+a, \tau(h')\}$. By assumption $s(h') \leq s(h) + a \leq t(h')$, therefore it holds $\triangle H_i > 0$ for any $i = 0, \dots, s(h)+a$. Since we either have $t(h)+a \leq t(h')$ or $t(h)+a > t(h')$ and $\triangle H_i = 0$, $\forall i = s(h)+a, \dots, t(h)+a$, $\triangle H_i < 0$, $\forall i = t(h)+a+1, \dots, \tau(H)$ in the first case, and $\triangle H_i = 0$, $\forall i = s(h)+a, \dots, t(h')$, respectivelly $\triangle H_i < 0$, $\forall i = t(h')+1, \dots, \tau(H)$ in the second, we conclude.
The second statement follows in a similar manner.

□

Lemma 3.20. *Let h and h' be two O-sequences of decreasing type and let H be an O-sequence such that $H_i = h_{i-a} + {h'}_i$ for some $a \in \mathbb{N}$. Assume that:*

(1) $d(h) + a \le t(h') \le e(h) + a$,

(2) $d(h') \le t(h) + a \le e(h')$,

(3) $d(h) \le t(h) \le e(h)$ *and* $d(h') \le t(h') \le e(h')$.

Then H is of decreasing type.

Proof. To prove the statement we have to show, that if there is some index $i = 0, \ldots, \tau(H) - 1$, such that $\triangle H_i < 0$, then $\triangle H_{i+1} < 0$. Assume that

$$\triangle H_i = \triangle h_{i-a} + \triangle h'_i < 0$$

We distinguish three cases:
Case 1: $\triangle h_{i-a} < 0$ and $\triangle h'_i < 0$.
As h and h' are of decreasing type we have that $\triangle h_{i-a+1}$ and $\triangle h'_{i+1}$ are both strictly smaller than 0 or one of them is zero. Thus $\triangle H_{i+1} < 0$.
Case 2: $\triangle h_{i-a} < 0$ and $\triangle h'_i \ge 0$.
We have then $i - a > t(h)$ and $i \le t(h')$. We obtain therefore the following inequalities:

$$d(h') + 1 \le t(h) + a + 1 \le i \le t(h') \le e(h) + a,$$

so that in particular it is true that

$$t(h) + 1 \le i - a \le e(h) \text{ and } d(h') + 1 \le i \le t(h'),$$

and thus

$$\triangle h_{i-a} \ge \triangle h_{i-a+1} \text{ and } \triangle h'_i \ge \triangle h'_{i+1}.$$

It follows then

$$\begin{aligned} \triangle H_{i+1} &= \triangle h_{i+1-a} + \triangle h'_{i+1} \\ &\le \triangle h_{i-a} + \triangle h'_i = \triangle H_i < 0. \end{aligned}$$

Case 3: $\triangle h_{i-a} \ge 0$ and $\triangle h'_i < 0$.
From the above assumption follows $i - a \le t(h)$ and $i > t(h')$. Thus we obtain in this case:

$$d(h) + a + 1 \le t(h') + 1 \le i \le t(h) + a \le e(h'),$$

so it holds in particular

$$d(h) + 1 \le i - a \le t(h) \text{ and } t(h') + 1 \le i \le e(h')$$

and therefore

$$\triangle h_{i-a} \geq \triangle h_{i+1-a} \text{ and } \triangle h'_i \geq \triangle h'_{i+1}.$$

As in the previous case it follows

$$\begin{aligned}\triangle H_{i+1} &= \triangle h_{i+1-a} + \triangle h'_{i+1} \\ &\leq \triangle h_{i-a} + \triangle h'_i = \triangle H_i < 0.\end{aligned}$$

□

Motivated by Lemma 3.19 and Lemma 3.20 we study in the following whether the integers $d(h), s(h), t(h)$ and $e(h)$ can be explicitly computed, or at least bounded.

Proposition 3.21. *Let $X \subseteq \mathbb{P}^n$ be a standard determinantal codimension 3 scheme with degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+2)}$. It holds then:*

$$d(h^A) = beg(I_X) - 1 = \sum_{i=1}^{t} a_{i,i} - 1.$$

Proof. Let R/J_X be the artinian reduction of S/I_X, where $R = K[X_1, X_2, X_3]$. We have then by definition

$$b = beg(I_X) = beg(J_X) = \min\left\{d \mid HF_R(d) > HF_{R/J_X}(d)\right\}.$$

Since $h_i = HF_{R/J_X}(i)$,

$$h_i = \frac{(i+1)(i+2)}{2}, \forall i = 0, \ldots, b-1 \text{ and } h_b < \frac{(b+1)(b+2)}{2}$$

we obtain

$$\triangle h_i = i+1, \forall i = 0, \ldots, b-1 \text{ and } \triangle h_b \leq b$$

so that

$$\triangle^2 h_i = 1, \forall i = 0, \ldots, b-1 \text{ and } \triangle^2 h_b \leq 0.$$

We have therefore by definition $d(h^A) = b - 1 = \sum_{i=1}^{t} a_{i,i} - 1$.

□

Remark 3.22. *Notice that for $codim(X) = 2$ the same proof as above shows that $d(h^A) = 0$. Unfortunately it does not easily generalize to any codimension, since for $c > 3$ it only shows that $\triangle^2 h_b \leq \binom{c-3+b}{c-3} - 1$.*

Next we will consider h-vectors corresponding to standard determinantal schemes of codimension 2, whose degree matrix has entirely positive subdiagonal. In their joint work (see [14, Proposition 1.3]) A. Geramita and J. Migliore showed that such h-vectors are of decreasing type (see also [19, Remark 4.10]).

We would like to point out that the first results in this direction were obtained by J. Harris and by R. Maggioni and A. Ragusa. J. Harris showed (see [23]) that the general hyperplane section of reduced and irreducible curve in $\mathbb{P}^n_K$ (where $char(K) = 0$) is a set X of points in $\mathbb{P}^{n-1}_K$ with the *uniform position property* (*UPP*), i.e. all subsets of X having the same cardinality have the same Hilbert-function. Later, R. Maggioni and A. Ragusa (see [31] and [32]) proved that the h-vector of any set of points with the UPP is of decreasing type.

Applying Lemma 3.19 and Remark 3.2 we will give a new simple proof for the result of A. Geramita and J. Migliore, which also allows us to compute the integers $s(h)$ and $t(h)$ using the corresponding degree matrix.

Remark 3.23. *It is not difficult to see, that if $I \subseteq S$ is a homogeneous complete intersection generated in degrees a and b, then the h-vector of I is of the form*

$$h = (h_0 < h_1 < \ldots < h_{a-1} = \ldots = h_{b-1} > \ldots > h_{a+b-2}),$$

where $h_{a-1} = \cdots = h_{b-1} = a$ and the first difference satisfies:

- $\triangle h(i) = 1 \, \forall \, i = 0, \ldots, a-1,$
- $\triangle h(i) = 0 \, \forall \, i = a, \ldots, b-1,$
- $\triangle h(i) = -1 \, \forall \, i = b, \ldots, a+b-2.$

In particular one has that $d(h) = 0$, $s(h) = a-1$, $t(h) = b-1$ and $e(h) = a+b-3$.

Proposition 3.24. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+1)}$ be a degree matrix with positive subdiagonal, i.e. $a_{i+1,i} > 0$ for all $i = 1, \ldots, t-1$. Then h^A is of decreasing type and it holds:*

(1) $s\left(h^A\right) = \sum_{i=1}^{t-1} a_{i+1,i} + a_{1,t} - 1 = \sum_{i=1}^{t} a_{i,i} - 1,$

(2) $t\left(h^A\right) = \sum_{i=1}^{t-1} a_{i+1,i} + a_{1,t+1} - 1 = \sum_{i=1}^{t-1} a_{i,i} + a_{t,t+1} - 1,$

(3) $\tau\left(h^A\right) = \sum_{i=1}^{t} a_{i,i} + a_{1,t+1} - 2.$

For the first difference we have in particular:

- $\triangle h^A_i = 1 \, \forall \, i = 0, \ldots, s(h^A),$
- $\triangle h^A_i = 0 \, \forall \, i = s(h^A) + 1, \ldots, t(h^A),$

- $\triangle h^A < 0 \, \forall \, i = t(h^A)+1, \ldots, \tau(h^A)$,

and $d(h^A) \leq t(h^A) \leq e(h^A)$.

Proof. We will prove the statement by induction on t. For $t = 1$ the claim follows from Remark 3.23.

Let $t > 1$. We modify the matrix A by moving the first row after the last one, obtaining the matrix

$$B = \begin{bmatrix} a_{2,1} & a_{2,2} & \cdots & a_{2,1} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,t+1} \\ \vdots & \vdots & & \vdots \\ a_{t,1} & a_{t,2} & \cdots & a_{t,t+1} \\ a_{1,1} & a_{1,2} & \cdots & a_{1,t+1} \end{bmatrix}.$$

By Remark 3.2 (see also Lemma 3.10) we have

$$h_i^A = h_i^B = h_{i-a_{2,1}}^{B^{(1,1)}} + h_i^{(a_{2,1}, \sum_{i=2}^t a_{i,i} + a_{1,t+1})}.$$

For simplicity we will use the following notation

$$h' = h^{B^{(1,1)}} \text{ and } h'' = h^{(a_{2,1}, \sum_{i=2}^t a_{i,i} + a_{1,t+1})}.$$

It holds by induction

$$s(h') = \sum_{i=2}^{t-1} a_{i+1,i} + a_{1,t} - 1,$$
$$t(h') = \sum_{i=2}^{t-1} a_{i+1,i} + a_{1,t+1} - 1.$$

By Remark 3.23 it follows

$$s(h'') = a_{2,1} - 1 \text{ and } t(h'') = \sum_{i=2}^{t} a_{i,i} + a_{1,t+1} - 1.$$

Using the homogeneity of A it is easy to see that

$$a_{2,1} + a_{1,2} + a_{1,3} + \textstyle\sum_{i=3}^t a_{i,i+1} \geq \sum_{i=2}^t a_{i,i} + a_{1,t+1},$$

so $\tau(h') + a_{2,1} \geq t(h'') - 1$. Similarly $s(h') + a_{2,1} \leq t(h'')$, wherefrom by Lemma 3.19, h^A is of decreasing type. Since it also holds $s(h') + a_{2,1} \leq t(h'')$ we have

$$s(h^A) = s(h') + a_{2,1} \text{ and } t(h^A) = t(h') + a_{2,1}.$$

The statement about the first difference follows easily from the induction hypothesis together with Remark 3.23.

$\square$

We ask next whether the h-vector h^A of a standard determinantal scheme of codimension 3 satisfies the conditions of Lemma 3.20. Since by Remark 3.2 $h^A = h'_{i-a} + H_i$, where $H_i = \sum_{k=0}^{a-1} h_{i-k}$ and h is a codimension 2 standard determinantal h-vector, we would like to know in particular how $d(H), s(H), t(H)$ and $e(H)$ are related to $d(h), s(h), t(h)$ and $e(h)$.

Lemma 3.25. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+1)}$ be a degree matrix with corresponding h-vector $h^A = (h_0, \ldots, h_s)$. For any $a = 1, \ldots, s(h^A)$ let $H = (H_0, \ldots, H_{s+a-1})$ be the O-sequence given by $H_i = \sum_{k=0}^{a-1} h_{i-k}$. It holds then:*

(1) $d(H) = a - 1$,

(2) $s(h^A) \leq s(H)$ *and* $t(h^A) \leq t(H) \leq t(h^A) + a - 1$,

(3) $t(h^A) + a - 1 \leq e(H)$,

(4) $e(h) \leq e(H)$,

(5) $d(H) \leq t(H) \leq e(H)$.

Proof. (1) Since $H_i = \displaystyle\sum_{k=0}^{a-1} h_{i-k}$, it holds $\triangle^2 H_{i+1} = \triangle h_{i+1} - \triangle h_{i+1-a}$. Therefore, for $i = a - 1$ we have $\triangle^2 H_a = \triangle h_a - \triangle h_0 = 1 - 1 = 0$ and for any $0 \leq i < a - 1$, $\triangle^2 H_{i+1} = \triangle h_{i+1} = 1$ so that by definition $d(H) = a - 1$.
(2) The first inequality is obvious, since as h^A is of decreasing type we have

$$\triangle H_{s(h^A)} = h_{s(h^A)} - h_{s(h^A)-a} > 0.$$

As h is of decreasing type, for any integer i such that $\triangle H_{i+1} < 0$ it follows that $i + 1 > t(h^A)$ and thus $t(h^A) \leq t(H)$. On the other hand, for $i = t(h^A) + a - 1$ we have

$$\triangle H_{i+1} = h_{t(h^A)+a} - h_{t(h^A)} < 0$$

and therefore $t(H) \leq t(h^A) + a - 1$.
(3) By (1) we have $d(H) = a - 1$. We distinguish three cases:
Case 1: $a - 1 \leq i \leq s(h^A) - 1$. We obtain

$$\triangle^2 H_i = \triangle h_i - \triangle h_{i-a} = 1 - 1 = 0.$$

Case 2: $s(h^A) - 1 \leq i \leq t(h^A) - 1$.
As $\triangle h_{i+1} = 0$, $\triangle h_{i+1-a} \in \{0, 1\}$ it holds

$$\triangle^2 H_{i+1} = \triangle h_{i+1} - \triangle h_{i+1-a} \leq 0.$$

Case 3: $t(h^A) \leq i \leq t(h^A) + a - 1$.
Then $\triangle h_{i+1} < 0$ and $\triangle h_{i+1-a} \in \{0, 1\}$, so

$$\triangle^2 H_{i+1} = \triangle h_{i+1} - \triangle h_{i+1-a} \leq 0.$$

(4)Let $e = e(h)$. By (1) $d(H) = a - 1$ and so for any $a - 1 \leq i \leq e$ we have $0 \leq i + 1 - a \leq e - (a - 1)$ and thus

$$\triangle h_{i+1} \leq \triangle h_i \leq \cdots \leq \triangle h_{i+1-a}.$$

The inequalities show in particular that $\triangle^2 H_{i+1} \leq 0$ and we conclude.
(5) Follows directly from (2) and (3).

$\square$

Corollary 3.26. *With the same assumptions as in Lemma 3.25 and requiring in addition that $t(h^A) - s(h^A) \geq a - 1$ we obtain:*

$$t(H) = t(h^A).$$

Proof. Let $t = t(h^A)$ and $s = s(h^A)$. Then by assumption $t - a + 1 \geq s$. As h^A is of decreasing type it holds $h_{t+1} < h_t = h_{t-a+1}$. Thus $t = t(h^A) \geq t(H)$ and the claim follows from Lemma 3.25.

$\square$

Lemma 3.27. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+1)}$ be a codimension 2 degree matrix. Then:*

(1) *If A has equal rows, then $e(h^A) = \tau(h^A) - 1$.*

(2) *If A does not have equal rows, it holds $e(h^A) = \sum_{i=1}^{t} a_{i,i} + a_{t,t+1} - 2$.*

Proof. (1) We proceed by induction on t. For $t = 1$ the claim follows from Remark 3.23. When $t > 1$, by Remark 3.2 we have

$$\triangle^2 h^A_{i+1} = \triangle^2 h^{A^{(t,t+1)}}_{i+1-a_{t,t+1}} + \triangle^2 h^{(a_{t,t+1}, \sum_{i=1}^{t} a_{i,i})}_{i+1}.$$

Since

$$\triangle^2 h_i^{(a_{t,t+1}, \sum_{i=1}^{t} a_{i,i})} = \begin{cases} 0, & i \neq a_{t,t+1}, \sum_{i=1}^{t} a_{i,i} \\ -1, & i = a_{t,t+1}, \sum_{i=1}^{t} a_{i,i} \end{cases}$$

and as by induction $e = e(h^{A^{(t,t+1)}}) = \tau(h^{A^{(t,t+1)}}) - 1$, we obtain

$$\triangle^2 h^A_{e+1+a_{t,t+1}} = \triangle^2 h^A_{\tau(h^A)} = \triangle^2 h^{A^{(t,t+1)}}_{e+1} \leq 0$$

and the claim follows.
(2) We use induction on t. For $t = 2$ by Remark 3.12, h^A is computed in the following way

$$\begin{array}{cccccccccc} + & 0 & \cdots & 0 & h'_0 & \cdots & h'_{s-e_2} & h'_{s-e_2+1} & h'_{s-e_2+2} & \cdots & h'_s \\ & h''_0 & \cdots & h''_{a_{2,3}-1} & h''_{a_{2,3}} & \cdots & h''_{\tau(h'')} & 0 & 0 & \cdots & 0 \\ \hline & h^A_0 & \cdots & h^A_{a_{2,3}-1} & h^A_{a_{2,3}} & \cdots & h^A_{s-e_2} & h^A_{s-e_2+1} & h^A_{s-e_2+2} & \cdots & h^A_s \end{array}$$

Therefore, $\triangle^2 h^A$ is obtained via:

$$\begin{array}{ccccccccccc}
+ & 0 & \cdots & 0 & \triangle^2 h'_0 & \cdots & \triangle^2 h'_{s-e_2} & \triangle^2 h'_{s-e_2+1} & \triangle^2 h'_{s-e_2+2} & \cdots & \triangle^2 h'_s \\
 & \triangle^2 h''_0 & \cdots & \triangle^2 h''_{a_{2,3}-1} & \triangle^2 h''_{a_{2,3}} & \cdots & \triangle^2 h''_{\tau(h'')} & 0 & 1 & \cdots & 0 \\
\hline
 & \triangle^2 h^A_0 & \cdots & \triangle^2 h^A_{a_{2,3}-1} & \triangle^2 h^A_{a_{2,3}} & \cdots & \triangle^2 h^A_{s-e_2} & \triangle^2 h^A_{s-e_2+1} & \triangle^2 h^A_{s-e_2+2} & \cdots & \triangle^2 h^A_s
\end{array}$$

where $h' = h^{(a_{1,1},a_{1,2})}$ and $h'' = h^{(a_{2,3},a_{1,1}+a_{1,2})}$. Since $\triangle^2 h'_i = -1$ only for $i = a_{1,1}, a_{1,2}$ and otherwise $\triangle^2 h'_i = 0$, and as $s - e_2 + 2 = a_{1,1} + a_{1,2} + a_{2,3} - (a_{1,2} - a_{2,2}) + 2 = a_{1,1} + a_{2,2} + a_{2,3} + 2 > a_{1,2}$, we have $\triangle^2 h^A_{s-e_2+2} = 1$ and $\triangle^2 h^A_{s-e_2+1} \leq 0$. Therefore $e(h^A) = s - e_2$ as claimed.

Let $t > 2$. Using Remark 3.12 we can write as above h^A as the componentwise sum of $h' = h^{A^{(t,t+1)}}$ and $h'' = h^{(a_{t,t+1},\sum_{i=1}^t a_{i,i})}$. Then

$$e(h^A) = \min\{e(h') + a_{t,t+1}, s - e_t\}.$$

By induction

$$e(h') + a_{t,t+1} = \sum_{i=1}^{t-1} a_{i,i} + a_{t-1,t} + a_{t,t+1} - 2.$$

Since

$$\begin{aligned} s - e_t &= a_{1,1} + a_{1,2} + a_{2,3} + \cdots + a_{t,t+1} - (a_{1,t-1} - a_{t,t-1}) - 2 \\ &= \sum_{i=1}^{t-1} a_{i,i} + a_{t,t} + a_{t,t+1} - 2 \end{aligned}$$

we conclude.

□

Example 3.28. *Consider the codimension 3 degree matrix* $A = \begin{bmatrix} 4 & 5 & 6 & 6 \\ 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 \end{bmatrix}$.

By Remark 3.2 we have then

$$h^A = h^{A^{(3,4)}}_{i-3} + \sum_{k=0}^{2} h^{A^{(0,4)}}_{i-k} = h^{A^{(3,4)}}_{i-3} + H_i$$

It holds in particular:

	0	1	2	3	4	5	6	7	8	9	10	11
$h^{A^{(0,4)}}$	1	2	3	4	5	6	7	7	6	4	2	1
$\triangle h^{A^{(0,4)}}$	1	1	1	1	1	1	1	0	-1	-2	-2	-1
$\triangle^2 h^{A^{(0,4)}}$	1	0	0	0	0	0	0	-1	-1	-1	0	1

and

$$\begin{aligned} d(h^{A^{(0,4)}}) &= 0, \\ s(h^{A^{(0,4)}}) &= 2+5-1 = 6, \\ t(h^{A^{(0,4)}}) &= 2+6-1 = 7, \\ e(h^{A^{(0,4)}}) &= 4+3+4-2 = 9. \end{aligned}$$

For H we have

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
H	1	3	6	9	12	15	18	20	20	17	12	7	3	1
$\triangle H$	1	2	3	3	3	3	3	2	0	-3	-4	-5	-4	-2
$\triangle^2 H$	1	1	1	0	0	0	0	-1	-2	-3	-1	-1	1	2

Therefore

$$\begin{aligned} &d(H) = 2, \\ &s(h^{A^{(0,4)}}) < s(H) = 7 \\ &t(h^{A^{(0,4)}}) < t(H) = 8 \\ &t(h^{A^{(0,4)}}) < t(H) < t(h^{A^{(0,4)}}) + 2 \\ &e(h^{A^{(0,4)}}) < e(H) = 10 \end{aligned}$$

and

$$t(h^{A^{(0,4)}}) + 2 < e(H)$$

as expected.

Let $A \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix with positive subdiagonal and let B be the matrix obtained from A by moving the first row after the last one as in the proof of Proposition 3.24. By Remark 3.2 we have then

$$h_i^A = h_{i-a_{2,1}}^{B^{(1,1)}} + \sum_{k=0}^{a_{2,1}-1} h_{i-k}^{B^{(0,1)}}.$$

For brevity we use the notation $h' = h^{B^{(1,1)}}$, $h'' = h^{B^{(0,1)}}$ and H will be the h-vector given by $H_i = \sum_{k=0}^{a_{2,1}-1} h_{i-k}^{B^{(0,1)}}$. We have then:

Lemma 3.29. *For any degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ it is true that*

$$d(h^A) \le t(h^A) \le e(h^A).$$

Proof. Since the first inequality is trivial, we show only the second. We use the abbreviation $a = a_{2,1}$. As $h^A_{i+1} = h'_{i+1-a} + \sum_{k=0}^{a-1} h''_{i+1-k}$, we have

$$\begin{aligned}
\triangle h^A_{i+1} &= \triangle h'_{i+1-a} + h''_{i+1} - h''_{i+1-a}\\
\triangle^2 h^A_{i+1} &= \triangle^2 h'_{i+1-a} + \triangle h''_{i+1} - \triangle h''_{i+1-a}.
\end{aligned}$$

Let $t = t(h^A)$. Then by definition it follows

$$\begin{aligned}
\triangle h^A_{t+1} < 0 &\Longleftrightarrow h''_{t+1-a} > \triangle h'_{t+1-a} + h''_{t+1},\\
\triangle h^A_{t} \geq 0 &\Longleftrightarrow h''_{t-a} \leq \triangle h'_{t-a} + h''_{t}.
\end{aligned}$$

From those inequalities it follows in particular

$$\begin{aligned}
\triangle h''_{t+1-a} &= h''_{t+1-a} - h''_{t-a} > \triangle h'_{t+1-a} + h''_{t+1} - h''_{t-a}\\
&\geq \triangle h'_{t+1-a} + h''_{t+1} - \triangle h'_{t-a} - h''_{t}\\
&= \triangle^2 h'_{t+1-a} + \triangle h''_{t+1}.
\end{aligned}$$

This shows that $\triangle^2 h^A_{t+1} \leq 0$ and the claim follows.

□

Remark 3.30. *The following example shows that the second condition $d(H) \leq t(h') + a_{2,1} \leq e(H)$ of Lemma 3.20 does not hold in general. Let*

$$A = \begin{bmatrix} 9 & 10 & 10 & 11 \\ 1 & 2 & 2 & 3 \end{bmatrix}.$$

We have then

$$h' = h^{(10,10,11)} = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 64, 71, 76, 79, 80, 79, 76, 71, 64, 55, 45, 36, 28, 21, 15, 10, 6, 3, 1).$$

Since $a_{2,1} = 1$, it follows

$$H = h'' = h^{A^{(0,1)}} = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 12, 10, 8, 7, 6, 5, 4, 3, 2, 1).$$

We obtain therefore

$$t(h') + a_{2,1} = 15 > 13 = e(h'').$$

A more involved computation shows that if A is a degree matrix given by

$$A = \begin{bmatrix} a+b+c+d+1 & 2b+c+d+1 & b+2c+d+1 & b+c+2d+1 \\ a & b & c & d \end{bmatrix},$$

where a, b, c, d are positive integers such that $2a + b \geq c + 2$, then $t(h') + a_{2,1} > e(H)$.

Since the answer to the question of whether any codimension 3 degree matrix satisfies the first two conditions of Lemma 3.20 is negative, we study whether there are numerical conditions on the entries of the matrix which ensure that the inequalities (1) and (2) in Lemma 3.20 hold. Keeping the same notation as above we obtain the following result:

Lemma 3.31. *For any degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+2)}$ with $a_{2,1} = a_{t,1}$ we have*

$$d(h') + a_{2,1} \leq t(H) \leq e(h') + a_{2,1}.$$

Proof. By Proposition 3.21 we have $d(h')+a_{2,1} = a_{1,1}+\sum_{i=2}^{t} a_{i,i}$. As by Lemma 3.25 $t(h'') \leq t(H)$ the first inequality follows directly from Proposition 3.24.

For the second inequality it is enough to show that $t(h'') - 1 \leq e(h')$. We proceed by induction on t.

Let $t = 2$, by Proposition 3.24 we have $t(h'') = a_{2,2} + a_{1,4} - 1$, on the other hand, as $h' = h^{(a_{1,2},a_{1,3},a_{1,4})}$, it holds that $e(h') = a_{1,4}+a_{1,3}-2$ (see also Remark 3.32) and the claim follows.

For $t > 2$ we want to show that $e(h') \geq t(h'') - 1 = \sum_{i=2}^{t} a_{i,i} + a_{1,t+2} - 2$. We denote by

$$B' = B^{(1,1)} = \begin{bmatrix} a_{3,2} & a_{3,3} & \cdots & a_{3,t+2} \\ a_{4,2} & a_{4,3} & \cdots & a_{4,t+2} \\ \vdots & \vdots & & \vdots \\ a_{t,2} & a_{t,3} & \cdots & a_{t,t+2} \\ a_{1,2} & a_{1,3} & \cdots & a_{1,t+2} \end{bmatrix}.$$

According to our notation $h' = h^{B'}$. By Remark 3.2 we have then:

$$h'_i = h^{B'^{(1,1)}}_{i-a_{3,2}} + \sum_{k=0}^{a_{3,2}-1} h^{B'^{(0,1)}}_{i-k} = g'_{i-a_{3,2}} + \sum_{k=0}^{a_{3,2}-1} g''_{i-k}.$$

We will denote by G the sequence defined by $G_i = \sum_{k=0}^{a_{3,2}-1} g''_{i-k}$. By induction

$$e(g') + a_{3,2} \geq t(g'') + a_{3,2} - 1 = \sum_{i=3}^{t} a_{i,i} + a_{3,2} + a_{1,t+2} - 2 = t(h'') - 1.$$

On the other hand by Lemma 3.25 it is true that $e(G) \geq t(g'') + a_{3,2} - 1$. Since

$$\max\{d(g') + a_{3,2}, d(G)\} \leq \min\{e(g') + a_{3,2}, e(G)\},$$

it holds that $e(h') \geq \min\{e(g') + a_{3,2}, e(G)\}$ and the claim follows.

$\square$

Remark 3.32. *Let $I \subseteq S$ be a complete intersection ideal with generators in degrees a, b and c. Using Remark 3.23 and the fact that the entries of the corresponding h-vector $h^{(a,b,c)}$ are computed via $h_i^{(a,b,c)} = \sum_{k=0}^{a-1} h_{i-k}^{(b,c)}$ it is not difficult to verify that:*

(A) *$h^{(a,b,c)}$ has a flat of length $\geq 3 \iff (c-b+1)-(a-1) \geq 3$*

(i.e. $\iff c \geq a+b+1$). It holds in particular that

$$s(h^{(a,b,c)}) = a+b-2 \text{ and } t(h^{(a,b,c)}) = c-1.$$

(B) *$h^{(a,b,c)}$ does not have a flat of length $\geq 3 \iff c \leq a+b$ and we have*

(1) *$c = a+b \Longrightarrow s(h^{(a,b,c)}) = c-2$ and $t(h^{(a,b,c)}) = c-1$,*

(2) *$c = a+b-1 \Longrightarrow s(h^{(a,b,c)}) = t(h^{(a,b,c)}) = c-1$,*

(3) *$c = a+b-2 \Longrightarrow s(h^{(a,b,c)}) = c-1$ and $t(h^{(a,b,c)}) = c$,*

Note that in all of the three cases above we have $h_{t(h^{(a,b,c)})} = ab$.

(4) *If $(c-b+1)-(a-1) = k < 0$, we have the following cases:*

- *If k is odd , $s(h^{(a,b,c)}) = t(h^{(a,b,c)}) = \dfrac{a+b+c-3}{2}$,*
- *If k is even, $t(h^{(a,b,c)}) = \dfrac{a+b+c-2}{2}$ and*

  $s(h^{(a,b,c)}) = \dfrac{a+b+c-4}{2}$.

Notice also that we have $d(h^{(a,b,c)}) = a-1$ and $e(h^{(a,b,c)}) = b+c-2$.

From Remark 3.32 by induction it easily follows:

Remark 3.33. *Let $I \subseteq S$ be a codimension $c \geq 3$ ideal generated in degrees $(a_1, \ldots, a_c)$ and assume that $a_c \geq a_1 + \cdots + a_{c-1} - (c-2)$. Then:*

$$s(h^{(a_1,\ldots,a_c)}) = a_1 + \cdots + a_{c-1} - (c-1)$$
$$t(h^{(a_1,\ldots,a_c)}) = a_c - 1.$$

3.3 Conditions for decreasing type in codimension 3

In this section we will show that any degree matrix $A \in \mathbb{Z}^{t\times(t+2)}$, whose first subdiagonal is entirely positive and whose largest entry is "big enough", has an h-vector of decreasing type. We will also prove that the h-vector of a standard determinantal ideal of codimension 3 with defining matrix M is of decreasing type, if all entries in M have the same degree (i.e. all entries in the degree matrix of I are equal).

We have seen in Proposition 3.6 that the h-vector of any standard determinantal scheme can be obtained as a component-wise sum of complete intersections. We ask therefore whether the entries of any h-vector of a complete intersection can be explicitly computed. In codimension three using the fact $\mathrm{hp}^{(a_1,a_2,a_3)}(z) = \prod_{i=1}^{3}(1+\cdots+z^{a_i-1})$ (see Lemma 2.6), a more involved computation (see [38, Lemma 2.9]) shows:

Lemma 3.34. *Let $2 \le a_1 \le a_2 \le a_3$ and $h^{(a_1,a_2,a_3)} = (h_0,\ldots,h_{a_1+a_2+a_3-3})$ be the h-vector of a homogeneous complete intersection ideal $I \subseteq K[X_1,X_2,X_3]$, generated in degrees a_1, a_2 and a_3. Then:*

(A) *If $a_1+a_2-1 \le a_3$, we have:*

- **(1)** $h_i = \binom{i+2}{2}$, *for* $i = 0,\ldots,a_1-2$,
- **(2)** $h_i = \binom{a_1+1}{2} + j\cdot a_1$, *for* $i = a_1-1,\ldots,a_2-1$ *and* $j = i-(a_1-1)$,
- **(3)** $h_i = \binom{a_1+1}{2} + a_1(a_2-a_1) + \sum_{k=1}^{j}(a_1-k)$, *for* $i = a_2,\ldots,a_1+a_2-3$ *and* $j = i-(a_2-1)$,
- **(4)** $h_i = a_1\cdot a_2$, *for* $i = a_1+a_2-2,\ldots,a_3-1$,
- **(5)** $h_i = h_{a_1+a_2+a_3-3-i}$, *for* $i \ge a_3$.

(B) *If $a_1+a_2-1 > a_3$, then:*

- **(1)** $h_i = \binom{i+2}{2}$, *for* $i = 0,\ldots,a_1-2$,
- **(2)** $h_i = \binom{a_1+1}{2} + j\cdot a_1$, *for* $i = a_1-1,\ldots,a_2-1$ *and* $j = i-(a_1-1)$,
- **(3)** $h_i = \binom{a_1+1}{2} + a_1(a_2-a_1) + \sum_{k=1}^{j}(a_1-k)$, *for* $i = a_2-1,\ldots,a_3-1$ *and* $j = i-(a_2-1)$,
- **(4)** $h_i = h_{a_3-1} + \sum_{k=1}^{j}(a_1+a_2-a_3-2k)$,
  for $i = a_3-1,\ldots,a_3-1+\left\lfloor\frac{a_1+a_2-a_3-1}{2}\right\rfloor$ *and* $j = i-(a_3-1)$,
- **(5)** $h_i = h_{a_1+a_2+a_3-3-i}$, *for* $i > a_3-1+\left\lfloor\frac{a_1+a_2-a_3-1}{2}\right\rfloor$.

Using Remark 3.32 we can easily establish numerical conditions on the entries of a degree matrix $A \in \mathbb{Z}^{2\times 4}$, which ensure that h^A is of decreasing type.

Lemma 3.35. *Let $A = [a_{i,j}] \in \mathbb{Z}^{2\times 4}$ be a degree matrix, such that $a_{2,1} > 0$. Assume that one of the following conditions holds:*

(1) $a_{1,4} \geq a_{1,2} + a_{1,3} - 1$,

(2) $a_{1,4} = a_{1,2} + a_{1,3} - 2$ *and* $a_{2,1} \leq a_{2,2} + a_{2,3} - 2$,

(3) $a_{1,4} < a_{1,2} + a_{1,3} - 2$ *and* $a_{1,2} \leq a_{2,3} + a_{2,4} + 2(a_{2,2} - 1)$,

then h^A is of decreasing type.

Proof. By Remark 3.2 we have $h_i^A = h_{i-a_{2,1}}^{(a_{1,2},a_{1,3},a_{1,3})} + \sum_{k=0}^{a_{2,1}-1} h_{i-k}^{A^{(0,1)}}$.

We write for simplicity $h' = h^{(a_{1,2},a_{1,3},a_{1,3})}$, $h'' = h^{A^{(0,1)}}$ and H denotes the O-sequence given by $H_i = \sum_{k=0}^{a_{2,1}-1} h_{i-k}^{A^{(0,1)}}$. By Lemma 3.29 and Lemma 3.25 we have $d(h') \leq t(h') \leq e(h')$ and $d(H) \leq t(H) \leq e(H)$. On the other hand, since $d(h') = a_{1,2} - 1$, $e(h') = a_{1,3} + a_{1,4} - 2$ and $t(H) \geq t(h'') = a_{2,2} + a_{1,4} - 2$ we obtain $d(h') + a_{2,1} \leq t(H) \leq e(h') + a_{2,1}$.

Since $d(H) = a_{2,1} - 1 \leq t(h') + a_{2,1}$ and $e(H) \geq e(h'') = a_{1,3} + a_{2,3} + a_{2,4} - 2$, using Remark 3.32, it is easy to check that if one of the conditions (1)-(3) holds, then $d(H) \leq t(h') + a_{2,1} \leq e(H)$, so by Lemma 3.20 we can conclude the proof. $\square$

Example 3.36. *Consider the degree matrices*

$$A_1 = \begin{bmatrix} 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 5 \end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 \end{bmatrix},$$

which satisfy the first, second and respectively the third condition. The corresponding h-vectors are:

$$\begin{aligned} h^{A_1} &= (1,3,6,10,14,17,18,17,14,9,4,1), \\ h^{A_2} &= (1,3,6,10,14,17,17,14,9,4,1), \\ h^{A_3} &= (1,3,6,10,14,16,14,9,4,1) \end{aligned}$$

and as expected they are of decreasing type.

Next, we will generalize Lemma 3.35 (1).

Remark 3.37. *If A is a degree matrix, whose first subdiagonal has negative entries, then h^A is in general not of decreasing type and not even unimodal as the following example shows. For*

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 3 & 3 & 3 & 3 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

we have $h^A = (1,3,6,10,9,11,5,3)$*, which is not of decreasing type and not unimodal.*

Theorem 3.38. *Let* $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+2)}$ *be a degree matrix with positive subdiagonal, i.e.* $a_{i+1,i} > 0$ *for all* $i = 1,\dots,t-1$*. Assume that* $a_{1,t+2} \geq a_{1,t+1} + a_{1,t} - 1$*. Then* h^A *is of decreasing type and* $t(h^A) = a_{1,t+2} + \sum_{i=1}^{t-1} a_{i+1,i} - 1$*.*

Proof. We will prove the claim by induction on t. Since for $t = 1$ there is nothing to show, let $t > 1$.

By Remark 3.2 we have $h_i^A = h'_{i-a_{2,1}} + \sum_{k=0}^{a_{2,1}-1} h''_{i-k}$, where $h' = h^{A^{(2,1)}}$, $h'' = h^{A^{(0,1)}}$ and H is given by $H_i = \sum_{k=0}^{a_{2,1}-1} h''_{i-k}$. By induction h' is of decreasing type and

$$t(h') = \sum_{i=2}^{t-1} a_{i+1,i} + a_{1,t+2} - 1.$$

According to Proposition 3.24 and Lemma 3.25, H is of decreasing type and

$$t(H) \geq t(h'') = \sum_{i=2}^{t} a_{i,i} + a_{1,t+2} - 1.$$

It holds then

$$t(h') + a_{2,1} \leq t(h'') \leq t(H).$$

For $i = t(h') + a_{2,1}$ we have by definition $\triangle h'_{i+1-a_{2,1}} = \triangle h'_{t(h')+1} < 0$.
Using the homogeneity of the degree matrix A we obtain $t(h') + 1 \geq s(h'')$.
This shows

$$\triangle H_{t(h')+1+a_{2,1}} = h''_{t(h')+1+a_{2,1}} - h''_{t(h')+1} \leq 0$$

and therefore

$$\triangle h^A_{i+1} = \triangle h'_{i+1-a_{2,1}} + \triangle H_{i+1} < 0.$$

On the other hand for any $i < t(h') + a_{2,1}$ (as $i + 1 \leq t(h') + a_{2,1} \leq t(H)$ and $i + 1 - a_{2,1} \leq t(h')$) we have $\triangle H_{i+1} \geq 0$ and $\triangle h'_{i+1-a_{2,1}} \geq 0$. This implies $\triangle h^A_{i+1} \geq 0$ and it holds in particular that $t(h^A) = \sum_{i=1}^{t-1} a_{i+1,i} + a_{1,t+2} - 1$ as claimed.

Next we show that for any $t(h^A) + 1 \leq i \leq \tau(h^A)$ we have $\triangle h_i^A < 0$.
Let $i \geq t(h^A) + 1 = t(h') + a_{2,1} + 1$, we have then $\triangle h'_{i-a_{2,1}} < 0$ and as $t(h') + a_{2,1} + 1 \geq s(h'') + a_{2,1}$ it holds also that $\triangle H_i = h''_i - h''_{i-a_{2,1}} \leq 0$. It follows therefore that $\triangle h_i^A = \triangle h'_{i-a_{2,1}} + \triangle H_i < 0$ and we conclude. $\square$

Remark 3.39. *Notice that if the first subdiagonal is not entirely positive, then the condition $a_{1,t+2} \geq a_{1,t+1} + a_{1,t} - 1$ has no influence on the type of the h-vector. For*

$$A_1 = \begin{bmatrix} 2 & 5 & 5 & 10 \\ -1 & 2 & 2 & 7 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & 4 & 4 & 7 \\ -1 & 1 & 1 & 4 \end{bmatrix},$$

we have $h^{A_1} = (1,3,6,10,13,15,17,18,18,17,15,14,12,8,5,3,1)$ is of decreasing type, while $h^{A_2} = (1,3,6,8,10,11,10,10,8,5,3,1)$ is not.

The simplest case where the numerical condition $a_{1,t+2} \geq a_{1,t+1} + a_{1,t} - 1$ does not hold is given by a degree matrix $A \in \mathbb{Z}^{t\times(t+2)}$ whose entries are all equal. The next result shows that also in this case h^A is of decreasing type.

Proposition 3.40. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+2)}$ be a degree matrix with equal entries,i.e. $a_{i,j} = a$ for all i and j. Then h^A is of decreasing type and it holds:*

$$d(h^A) = at - 1 \text{ and } e(h^A) = (t+1)a - 2.$$

Proof. Applying Corollary 3.8 we can write the h-polynomial of A in the following way

$$\mathrm{hp}^A(z) = (1 + z + \cdots + z^{a-1}) \sum_{i=1}^{t} (t+1-i) z^{(t-i)a} \mathrm{hp}^{(a,ia)}(z).$$

If we denote by $h' = (h'_0, \ldots, h'_{(t+1)a-2})$ the coefficients of the polynomial $\sum_{i=1}^{t}(t+1-i)z^{(t-i)a}\mathrm{hp}^{(a,ia)}(z)$, then by Remark 3.23 we have in particular

$$\begin{aligned}
\triangle h'_i &= 1,\ \forall i = 1, \ldots, a-1 \\
\triangle h'_i &= 2,\ \forall i = a, \ldots, 2a-1 \\
\triangle h'_i &= 3,\ \forall i = 2a, \ldots, 3a-1 \\
&\vdots \\
\triangle h'_i &= t,\ \forall i = (t-1)a, \ldots, ta-1 \\
\triangle h'_i &= -\frac{t(t+1)}{2},\ \forall i = ta, \ldots, (t+1)a-2.
\end{aligned}$$

Notice that $\triangle h'_{(t+1)a-1} = -\frac{t(t+1)}{2}$.

Obviously h' is a sequence of decreasing type, so by Lemma 3.4 h^A is also of decreasing type. Since $h^A_i = \sum_{k=0}^{a-1} h'_{i-k}$, it holds that

$$\triangle^2 h^A_{i+1} = \triangle h'_{i+1} - \triangle h'_{i+1-a}.$$

By Lemma 3.21 we have $d(h^A) = ta - 1$. Since for any $ta - 1 \leq i \leq (t+1)a - 2$, $\triangle^2 h^A_{i+1} \leq 0$ we conclude $e(h^A) = (t+1)a - 2$.

□

Remark 3.41. *For a degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+2)}$, which satisfies the condition $a_{1,t+2} \geq a_{1,t+1} + a_{1,t} - 1$, Theorem 3.38 provides a simple formula for the first place $t(h^A)$, where the h-vector h^A start decreasing. In general it appears to be very difficult to compute explicitly $t(h^A)$, even if we assume that the entries of A are all equal to a. In the settings of Proposition 3.40, based on our computations with the computer algebra system CoCoA, we conjecture that $t(h^A) = at + \left\lfloor \dfrac{2a-1}{t+3} \right\rfloor$.*

Example 3.42. *Consider the matrix $A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$. The formula:*

$$\mathrm{hp}^A(z) = (1+z+z^2)\sum_{i=1}^{3}(4-i)z^{(3-i)3}\mathrm{hp}^{(3,i3)}(z)$$

obtained in the proof of Proposition 3.40 suggests the following easy way for computing h^A. First compute the component-wise sum of complete intersection h-vectors

$$\begin{array}{ccccccccccc}
 & & & & & & 1 & 2 & 3 & 2 & 1 \\
 & & & & & & 1 & 2 & 3 & 2 & 1 \\
+ & & & & & & 1 & 2 & 3 & 2 & 1 \\
 & & & 1 & 2 & 3 & 3 & 3 & 3 & 2 & 1 \\
 & & & 1 & 2 & 3 & 3 & 3 & 3 & 2 & 1 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 1 \\
\hline
1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 & 12 & 6
\end{array}$$

and then, in order to obtain h^A, simply shift the result $a = 3$ times by one and add the entries component-wise as follows:

$$\begin{array}{ccccccccccccc}
 & & 1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 & 12 & 6 \\
+ & 1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 & 12 & 6 & \\
1 & 2 & 3 & 5 & 7 & 9 & 12 & 15 & 18 & 12 & 6 & & \\
\hline
1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 45 & 36 & 18 & 6
\end{array}$$

We obtain as expected

$$\begin{aligned}
d(h^A) &= 3\cdot 3 - 1 = 8, \\
e(h^A) &= 4\cdot 3 - 2 = 10, \\
t(h^A) &= 3\cdot 3 + \left\lfloor \frac{2.3-1}{3+3} \right\rfloor = 9.
\end{aligned}$$

Using Matroid theory we will show in the next chapter, that a more general result than the one given in Proposition 3.40 is true.

4 Standard determinantal schemes and pure O-sequences

Our aim in this chapter is to characterize the degree matrices whose h-vectors are pure O-sequences and to answer the question whether each pure O-sequence can be obtained as the h-vector of some standard determinantal scheme.
The content of this chapter is joint work with A. Constantinescu.

We will first recall some of the algebraic and combinatorial notions that will be used through this section. Since in codimension one all h-vectors are finite sequences of ones and thus pure O-sequences, we will assume from now on that the codimension c is greater than two.

For a positive integer n we will write $[n]$ for the set $\{1, \ldots, n\}$. A *simplicial complex* Δ on *the vertex set* $[n]$ is a collection of subsets on $[n]$, which is closed under the operation of taking subsets i.e. if $G \subseteq F \in \Delta$, then $G \in \Delta$. An element $F \in \Delta$ is called a *face of* Δ. The faces of Δ which are maximal with respect to inclusion will be called *facets* . We use the notation

$$\mathcal{F}(\Delta) := \{F \in \Delta | \, F \, \text{is a facet}\}.$$

The *dimension of a face* $F \in \Delta$ is defined to be $|F| - 1$ and *the dimension of* Δ is given by

$$\dim(\Delta) := \max \{\dim(F) | \, F \in \Delta\}.$$

A simplicial complex is called *pure* if all facets have the same cardinality.
If $F_1, \ldots, F_m$ are subsets of $[n]$, then we denote by $\langle F_1, \ldots, F_m \rangle$ *the smallest simplicial complex on* $[n]$ *that contains them.* More precisely

$$\langle F_1, \ldots, F_m \rangle := \{F \subset [n] \, | \, \exists i \in \{1, \ldots, m\} : \, F \subset F_i\}.$$

Observe that a simplicial complex Δ is determined by $\mathcal{F}(\Delta)$, we have

$$\Delta = \langle F | \, F \in \mathcal{F}(\Delta) \rangle.$$

The dual complex of Δ is the simplicial complex Δ^c on $[n]$ with facets

$$\mathcal{F}(\Delta^c) = \{[n] \setminus F | \, F \in \mathcal{F}(\Delta)\}.$$

For a vertex $v \in \Delta$ *the link of* v *in* Δ is the following simplicial complex:

$$\text{link}_\Delta(v) = \{F \in \Delta | \, v \notin F \, \text{and} \, F \cup \{v\} \in \Delta\}.$$

We will write $\Delta \setminus v$ for the simplicial complex

$$\Delta \setminus v = \{F \in \Delta | \, v \notin F\}$$

and call $\Delta \setminus v$ *the deletion of* v. A vertex $v \in \Delta$ with $v \in F$ for any $F \in \mathcal{F}(\Delta)$ is called a *cone point of* Δ.

Denote by $S = K[x_1, \ldots, x_n]$ the polynomial ring over K in n variables. For each subset $F \subset [n]$ we define the monomial x_F and the prime ideal $\mathfrak{p}_F$ as follows

$$x_F = \prod_{i \in F} x_i,\ \mathfrak{p}_F = (x_i|\, i \in F).$$

The Stanley-Reisner ideal of Δ is defined by $I_\Delta = (x_F|\, F \notin \Delta)$, so it is generated by the monomials x_F corresponding to the minimal nonfaces $F \in \Delta$.
It is well known that the prime decomposition of the Stanley-Reisner ideal is given by:

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} \mathfrak{p}_{[n] \setminus F} = \bigcap_{G \in \mathcal{F}(\Delta^c)} \mathfrak{p}_G.$$

A collection of vertices $F \subseteq [n]$ is called a *vertex cover of* Δ if $F \cap G \neq \varnothing$ for all $G \in \mathcal{F}(\Delta)$. A vertex cover F is called *basic* if there is no proper subset of F which is again a basic cover. The ideal

$$J(\Delta) = I_{\Delta^c} = \bigcap_{F \in \mathcal{F}(\Delta)} \mathfrak{p}_F$$

is called *the cover ideal of* Δ. The name comes from the following equality

$$J(\Delta) = (x_F|\, F \text{ is a basic vertex cover of } \Delta).$$

We denote by $K[\Delta] = S/I_\Delta$ *the Stanley-Reisner ring of* Δ and by $HS_{K[\Delta]}(z)$ the Hilbert-Series of $K[\Delta]$. A general result states that $HS_{K[\Delta]}(z)$ can be written in a rational form:

$$HS_{K[\Delta]} = \frac{\mathrm{hp}_{K[\Delta]}(z)}{(1-z)^d},$$

where $d = \dim(K[\Delta]) = \dim(\Delta) + 1$. The numerator $\mathrm{hp}_{K[\Delta]}(z) = 1 + h_1 z + \cdots + h_s z^s$ is called *the h-polynomial of the Stanley-Reisner ring* $K[\Delta]$ and its coefficients are the entries of the h-vector of $k[\Delta]$, $h_{K[\Delta]} = (1, h_1, \ldots, h_s)$.

Remark 4.1. *In the classical terminology h^Δ denotes the h-vector of the dual simplicial complex Δ^c. We will adopt this notation throughout this chapter, thus*

$$h^\Delta := h_{K[\Delta^c]}.$$

Definition 4.2. *A simplicial complex Δ is called matroid complex (or just matroid) if one of the following properties holds:*

(1) The augmentation axiom*: For any two faces $F, G \in \Delta$ with $|F| < |G|$ there exists $i \in G$ such that $F \cup \{i\} \in \Delta$,*

(2) The exchange property*: For any two facets $F, G \in \mathcal{F}(\Delta)$ and for any $i \in F$ there exists a $j \in G$ such that $(F \setminus \{i\}) \cup \{j\} \in \Delta$,*

(3) *For any subset $F \subset [n]$ the restriction $\Delta_{\restriction F} = \{G \in \Delta | \, G \subseteq F\}$ is pure.*

Example 4.3. *If $A \in \mathbb{F}^{m \times n}$ is a matrix over a field $\mathbb{F}$ and $c_i(A)$ denotes the i-th column of A, then*

$$\Delta = \{F \subset [n] | \, F = \{i_1, \ldots, i_k\} : \, c_{i_1}(A), \ldots, c_{i_k}(A) \text{ lin. independent over} \mathbb{F}\}$$

is easily seen to be a matroid (see e.g. [36, Proposition 1.1.1]). This matroid is called the vector matroid of A *and denoted by $M[A]$.*

More generally the concept of matroid is an "abstraction of linear independence".

In this work we will use matroids to obtain a connection between h-vectors of standard determinantal ideals and pure O-sequences.

Remark 4.4. *According to [11, Remark 1.7] (see also [34, Remark 2.4]), if Δ is a matroid and $v \in \Delta$ not a cone point, then*

$$h_i^{\Delta} = h_{i-1}^{\Delta \setminus v} + h_i^{\mathrm{link}_{\Delta}(v)}.$$

Notice that since Δ is a matroid, then both $\Delta \setminus v$ and $\mathrm{link}_{\Delta}(v)$ are matroids as well.

Definition 4.5. *Let Δ and Γ be a simplicial complexes and $\varphi : \Delta \longrightarrow \Gamma$ a function. Then φ is an isomorphism if φ is bijective and whenever $F \subseteq G \in \Delta$, we have $\varphi(G) \subseteq \varphi(F)$.*

Definition 4.6. *A matroid Δ is called representable over a field $\mathbb{F}$ (or $\mathbb{F}$-representable) if there exists a matrix $A \in \mathbb{F}^{m \times n}$ such that $\Delta \cong M[A]$.*

The following duality results are well known:

Theorem 4.7. *([36, Theorem 2.1.1] and [36, Corollary 2.2.9])*
For a simplicial complex Δ on $[n]$ holds:

(1) *Δ is matroid $\Longleftrightarrow$ Δ^c is a matroid.*

(2) *If Δ is a matroid, then:*

$$\Delta \text{ is } \mathbb{F}\text{-representable} \Longleftrightarrow \Delta^c \text{ is } \mathbb{F}\text{-representable.}$$

Huh recently showed (see [27]) that the h-vector of any matroid representable over a field of characteristic zero is log-concave (i.e. its entries satisfy the inequality $h_i^2 \geq h_{i-1} \cdot h_i$ for any i).

The problem of characterizing pure O-sequences is far from being solved. One of the main results on this topic is due to Hibi:

Theorem 4.8. *([26, Theorem 1.1])*
Let $h = (h_0, \ldots, h_s)$ be a pure O-sequence. Then:

$$h_i \le h_j \text{ for all } 0 \le i \le j \le s - i.$$

This has the following consequences:

(1) *h is flawless i.e. $h_i \le h_{s-i}$ for all $0 \le i \le \left\lfloor \frac{s}{2} \right\rfloor$,*

(2) *the first half of h is non-decreasing i.e. $h_0 \le h_1 \le \cdots \le h_{\left\lfloor \frac{s}{2} \right\rfloor}$.*

We are now ready to prove the first result in this section, namely a characterization of the h-vectors corresponding to a degree matrix with equal rows. More precisely:

Theorem 4.9. *Let $X \subseteq \mathbb{P}^n$ be a codimension c standard determinantal scheme. If the degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ of X has equal rows, i.e. $a_{i,j} = a_j$, $\forall i = 1, \ldots, t$, $\forall j = 1, \ldots, t+c-1$, then h^A is a log-concave pure O-sequence. In particular, it holds $h^A = f(\Gamma)$, where Γ is the order ideal of $Mon(K[y_1, \ldots, y_c])$ given by*

$$\left\langle \left\{ y_1^{(\sum_{i=1}^{l_1-1} a_i)-1} \cdot y_2^{(\sum_{i=l_1}^{l_2-1} a_i)-1} \cdots y_c^{(\sum_{i=l_{c-1}}^{t+c-1} a_i)-1} \mid \forall 1 = l_0 < l_1 < \cdots < l_{c-1} \le t+c-1 \right\} \right\rangle.$$

Proof. We will write for brevity $m = t + c - 1$. For $i = 1, \ldots, m$ let $A_i = \{v_{i,1}, \ldots, v_{i,a_i}\}$ be a set of vertices of cardinality a_i. As in [11], we define the simplicial complex $\Delta_0(c, m, (a_1, \ldots, a_m))$ on $\cup_{i=1}^m A_i$ as

$$\big\{\{v_{i_1}, \ldots, v_{i_c}\} \mid 1 \le i_1 < \cdots < i_c \le m \,\text{and}\, v_{i_j} \in A_{i_j} \text{ for every } v_{i_j}\big\}.$$

One can easily check that $\Delta_0(c, m, (a_1, \ldots, a_m))$ is a matroid. We will show by induction on c and t that the h-vectors h^A and $h^{\Delta_0(c,m,(a_1,\ldots,a_m))}$ coincide.

For $t = 1$ or $c = 1$ the claim is trivial. Let $t, c > 1$. By Remark 3.2 applied for a_m we have

$$h^A = h'_{i-a_m} + \sum_{k=0}^{a_m-1} h''_{i-k},$$

where h' and h'' are the h-vectors corresponding to the degree matrices $A' = A^{(t,m)}$ and $A'' = A^{(0,m)}$ respectively. On the other side, applying a_m-times the formula in Remark 4.4 , once for every vertex in A_m we obtain

$$h_i^{\Delta_0(c,m,(a_1,\ldots,a_m))} = h_{i-a_m}^{\Delta_0(c,m-1,(a_1,\ldots,a_{m-1}))} + \sum_{k=0}^{a_m-1} h_{i-k}^{\Delta_0(c-1,m-1,(a_1,\ldots,a_{m-1}))},$$

and we conclude by induction. In particular by [11, Theorem 3.5] we have $h^{\Delta_0(c,m,(a_1,\dots,a_m))} = f(\Gamma)$, so that h^A is a pure O-sequence as claimed.

Furthermore $\Delta_0(c,m,(a_1,\dots,a_m))$ is representable over any infinite field $\mathbb{F}$ of characteristic zero. A presentation matrix D can be constructed as follows: choose m generic vectors $w_1,\dots,w_m \in \mathbb{F}^c$, that is any c of them are linearly independent. Let the first a_1 columns of D be w_1, the next a_2 be equal to w_2 and so on. We clearly have $\Delta_0(c,m,(a_1,\dots,a_m)) \cong M[D]$. Since by [36, Corollary 2.2.9] a matroid is representable over $\mathbb{F}$ if and only if its dual is representable over $\mathbb{F}$ we obtain by [27, Theorem 3] that h^A is log-concave.

□

Remark 4.10. *As the following example shows the restriction to degree matrices with equal rows in Theorem 4.9 cannot be omitted. If A is a degree matrix, which does not have equal rows, then the corresponding h-vector h^A is in general neither log-concave nor a pure O-sequence. For*

$$A = \begin{bmatrix} 5 & 5 & 6 \\ 2 & 2 & 3 \end{bmatrix},$$

we have $h^A = (1,2,3,4,5,6,7,7,5,3,2,1)$. Since any pure O-sequence, whose last entry is one can be regarded as the h-vector of some complete intersection ideal, we clearly have that h^A cannot be a pure O-sequence, as it is not symmetric. Moreover h^A is not log-concave, since $h_9^2 < h_8 \cdot h_{10}$.

Remark 4.11. *One could also ask whether the proof of Theorem 4.9 applies also to any degree matrix A with equal columns and especially whether the corresponding h-vector h^A is log-concave. Unfortunately this is not the case. For example the h-vector corresponding to the matrix*

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$h^A = (1,3,6,10,15,21,13,9,3,1)$ is not log-concave (since $h_6^2 < h_5 \cdot h_7$), so in particular there is no $\mathbb{F}$-representable matroid Δ with $h^\Delta = h^A$.

Next, we study whether the converse of Theorem 4.9 is also true. We are able to answer this question in some particular cases and we conjecture that a more general statement is true, namely:

Conjecture 4.12. *For any degree matrix $A \in \mathbb{Z}^{t\times(t+c-1)}$, which does not have zero entries we have:*

$$h^A \text{ is a pure O-sequence} \iff A \text{ has equal rows.}$$

Recall that a subscheme $X \subseteq \mathbb{P}^n$ is called level, if the artinian reduction of its coordinate ring S/I_X is level, i.e. if the last free module in the minimal free resolution of S/I_X is of the form $S^b(-d)$.

The next result shows that any standard determinantal scheme can be characterized in terms of the corresponding degree matrix.

Theorem 4.13. *Let $X \subseteq \mathbb{P}^n$ be a standard determinantal scheme of codimension c with degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$. It holds then:*

$$X \text{ is level} \Longleftrightarrow A \text{ has equal rows.}$$

Proof. Since the defining ideal I_X of X is standard determinantal it is generated by the maximal minors of a homogeneous matrix $M = [f_{i,j}]$, where the $f_{i,j}$'s are forms of degree $a_{i,j} = a_j - b_i$. The matrix M defines a graded homomomorphism of degree 0

$$\varphi : F = \bigoplus_{i=1}^{t} S(b_i) \longrightarrow \bigoplus_{j=1}^{t+c-1} S(a_j) = G.$$

The minimal free resolution of S/I_X is given by the Eagon-Northcott complex with respect to φ (see e.g. [7],[33]). Therefore the last free module in it is of the form

$$F_c = \bigwedge^{t+c-1} G^* \otimes S_{c-1}(F) \otimes \bigwedge^{t} F,$$

where

$$G^* = \bigoplus_{j=1}^{t+c-1} S(-a_j), \quad \bigwedge^{t+c-1} G^* = S(-\sum_{j=1}^{t+c-1} a_j), \quad \bigwedge^{t} F = S(\sum_{i=1}^{t} b_i)$$

and

$$S_{c-1}(F) = \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S(\sum_{j=1}^{c-1} b_{k_j}) \ .$$

We can rewrite the shifts in F_c in terms of the entries of A. More precisely

$$\begin{aligned} F_c = & \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S(-a_1 - \cdots - a_{t+c-1} + b_1 + \cdots + b_t + b_{k_1} + \cdots + b_{k_{c-1}}) \\ = & \bigoplus_{1\le k_1\le\cdots\le k_{c-1}\le t} S(-a_{k_1,1} - \cdots - a_{k_{c-1},c-1} - a_{1,c} - \cdots - a_{t,t+c-1}). \end{aligned}$$

Since X is level if and only if $F_c = S^b(-d)$, we have in particular that the summation indices $(1, t, \dots, t), \dots, (t, t, \dots, t)$ are all equal, that is

$$a_{1,1} + a_{t,2} + \cdots + a_{t,c-1} = \cdots = a_{t,1} + a_{t,2} + \cdots + a_{t,c-1}.$$

This implies that $a_{1,1} = \cdots = a_{t,1}$ and therefore the rows of A are equal. ∎

In the case of standard determinantal schemes of codimension 2 we can actually say more, as we show in the following result.

Our Conjecture 4.12 says, that the restriction on the codimension of the scheme is not necessary.

Proposition 4.14. *Let $X \subseteq \mathbb{P}^n$ be a codimension 2 standard determinantal scheme whose degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+1)}$ does not have a zero entry. Then*

$$X \text{ is level} \Longleftrightarrow A \text{ has equal rows} \Longleftrightarrow h^A \text{is a pure O-sequence.}$$

Proof. The first equivalence follows directly from Theorem 4.13. If the rows of A are equal (i.e. the entries in each column of A are equal), then by Theorem 4.9 h^A is a pure O-sequence. So we only have to show the " $\Leftarrow$ " implication in the second equivalence. Assume that h^A is a pure O-sequence and let R/J_X be the artinian reduction of S/I_X, where $R = K[x_1, x_2]$. Then there exists an artinian monomial level algebra $R/\mathfrak{a}$, such that $h^A = HF_{R/J_X} = HF_{R/\mathfrak{a}}$. Since A has no zero entries and we are in codimension 2, by the Theorem of Hilbert-Burch (see for instance [13, Theorem 20.15]) HF_{R/J_X} determines uniquely the minimal free resolution of R/J_X (and also of S/I_X). Thus $R/\mathfrak{a}$ level implies that also R/J_X is level and we conclude.

□

We now prove a Proposition which describes the last part of the h-vector of a degree matrix with equal rows. In what follows, we use the convention that $\binom{a}{b} = 0$, if $b < 0$ or $a < b$.

Proposition 4.15. *Let $A \in \mathbb{Z}^{r\times(r+c-1)}$ be a degree matrix with equal rows and let $a_{l,j} = a_j$ for all l and j. For any $i = 0, \dots, a_{r+1} - 1$ we have:*

$$h^A_{s-i} = \binom{r+c-2}{c-1} \cdot \binom{c+i-1}{c-1} + \\ + \sum_{\alpha=1}^{c-1} (-1)^\alpha \binom{r-\alpha+c-2}{c-1-\alpha} \sum_{1\le j_1<\dots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\dots-a_{j_\alpha}}{c-1}.$$

Proof. We will prove the claim by induction on r and c using the binomial formula

$$\sum_{i=0}^{a-1} \binom{d-i}{b} = \binom{d+1}{b+1} - \binom{d-a+1}{b+1}.$$

For $r, c = 1$ the claim is trivial. Let $c > 1$ and denote by $h' = (h'_0, \dots, h'_{s'})$ the h-vector of a homogeneous complete intersection ideal generated in degrees $(a_2, \dots, a_c)$. For $i = 0, \dots, a_2 - 1$ it follows by induction that:

$$\begin{aligned} h_{s-i} &= \sum_{k=0}^{a_1-1} h'_{s'-(i-k)} \\ &= \sum_{k=0}^{a_1-1} \binom{c-2+i-k}{c-2} + \sum_{k=0}^{a_1-1} \binom{c-2+i-k-a_2}{c-2} \\ &= \binom{c-1+i}{c-2} - \binom{c-1+i-a_1}{c-1}. \end{aligned}$$

Let $r > 1$. For $c = 1$ there is nothing to show so let $c > 1$. We will write for brevity $h' = h^{A^{(1,1)}}$, $h'' = h^{A^{(0,1)}}$, $s = \tau(h^A)$, $s' = \tau(h^{A^{(1,1)}})$, $s'' = \tau(h^{A^{(0,1)}})$. Since $s = s' + a_1$ and $s = s'' + (a_1 - 1)$ for any $i = 0, \dots, a_{r+1} - 1$, by Remark 3.2 we have

$$h^A_{s-i} = h'_{s'-i} + \sum_{j=0}^{a_1-1} h''_{s''-(i-j)} \ .$$

By induction, using the binomial formula, keeping track of the correspondence between the indices in A, A' and A'' , and taking into account that if $j_\alpha = a_{r+1}$, then for any $i = 0, \dots, a_{r+1} - 1$ it holds that $\binom{c-1+i-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} = 0$, we have

$$\begin{aligned} h'_{s'-i} = &\binom{r+c-3}{c-1}\binom{c-1+i}{c-1} + \\ &+ \sum_{\alpha=1}^{c-1} (-1)^\alpha \binom{r-\alpha+c-3}{c-1-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=0}^{a_1-1} h''_{s''-(i-j)} = &\binom{r+c-3}{c-2}\binom{c-1+i}{c-1} - \binom{r+c-3}{c-2}\binom{c-1+i-a_1}{c-1} + \\ &+ \sum_{\alpha=1}^{c-1} (-1)^\alpha \binom{r-\alpha+c-3}{c-2-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1} - \\ &- \sum_{\alpha=1}^{c-2} (-1)^\alpha \binom{r-\alpha+c-3}{c-2-\alpha} \sum_{2\le j_1<\cdots<j_\alpha\le r} \binom{c+i-1-a_1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1}. \end{aligned}$$

Putting the above results together we obtain

$$\begin{aligned} h^A_{s-i} = &\binom{r+c-2}{c-1}\binom{c-1+i}{c-1} \\ &+\sum_{\alpha=1}^{c-1}(-1)^{\alpha}\binom{r-\alpha+c-2}{c-1-\alpha}\sum_{2\leq j_1<\cdots<j_\alpha\leq r}\binom{c+i-1-a_{j_1}-\cdots-a_{j_\alpha}}{c-1}+ \\ &+\sum_{\alpha=2}^{c-1}(-1)^{\alpha}\binom{r-\alpha+c-2}{c-1-\alpha}\sum_{2\leq j_1<\cdots<j_{\alpha-1}\leq r}\binom{c+i-1-a_1-a_{j_1}-\cdots-a_{j_{\alpha-1}}}{c-1}- \\ &-\binom{r+c-3}{c-2}\binom{c-1+i-a_1}{c-1} \end{aligned}$$

and the assertion follows.

□

Remark 4.16. *Notice that by Remark 3.12, the formula from Proposition 4.15 together with Lemma 3.11 provides a lower bound for the last* $e_{r+1}+1 = a_{1,1} - a_{r+1,1} + 1$ *entries of* h^A *for any degree matrix* $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ *with* r *equal maximal rows.*

It is not difficult to check (see e.g. [16, Proposition 5.4]) that the entries of a pure O-sequence can be bounded as follows:

Lemma 4.17. *If* $h = (1, c, h_2, \ldots, h_s)$ *is a pure O-sequence, then*

$$h_i \leq \min\left\{\binom{c-1+i}{c-1}, h_s\cdot\binom{c-1+s-i}{c-1}\right\}, \quad \forall i = 0,\ldots,s.$$

For the sake of completeness we add here a proof of this fact.

Proof. Since h is a pure O-sequence, there is an artinian monomial level algebra R/I, where $R = K[X_1,\ldots,X_c]$, such that $h_i = HF_{R/I}(i)$. It follows then obviously

$$h_i = HF_{R/I}(i) \leq HF_R(i) = \binom{c-1+i}{c-1} \text{ for every } i.$$

Let $F_1,\ldots,F_{h_s} \in R_s$ be the generators of the inverse system I^{-1}. Since $\dim_K(\langle d^{s-j}F_i\rangle_K) = \dim_K(\langle d^jF_i\rangle_K)$ for any i we have

$$\begin{aligned} h_i = HF_{R/I}(i) = \dim_K(I^{-1}_i) &= \dim_K\left\langle d^{s-i}F_j \,|\, j=1,\ldots,h_s\right\rangle_K \\ &\leq h_s\cdot\binom{c-1+s-i}{c-1}. \end{aligned}$$

□

The next result shows that h^A is not a pure O-sequence, when the second-largest entry in the first column of A is positive.

Proposition 4.18. *Let $X \subseteq \mathbb{P}^n$ be a codimension c standard determinantal scheme, whose degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ has $r < t$ equal maximal rows (i.e. $a_{1,1} = \cdots = a_{r,1}$) and no zero entries. If $a_{r+1,1} > 0$, then h^A is not a pure O-sequence.*

Proof. By Remark 3.12 the last e_{r+1} entries of h^A are equal to the last e_{r+1} entries of $h^{\bar{A}}$, where $\bar{A}$ is the $r \times (r+c-1)$ upper-left block of A.

As $a_{r+1,1} > 0$, we have that $e_{r+1} = a_{1,1} - a_{r+1,1} < a_{1,1}$ and by Proposition 4.15

$$h^{\bar{A}}_{s-e_{r+1}} = \binom{r+c-2}{c-1} \cdot \binom{c-1+e_{r+1}}{c-1}.$$

Starting from the lower right corner of A, repeated application of Remark 3.12 shows that

$$h^{A}_{s-e_{r+1}} \geq \binom{r+c-2}{c-1} \cdot \binom{c-1+e_{r+1}}{c-1} + \binom{r+c-3}{c-2}.$$

According to Proposition 3.11 the last entry of h^A is $h_s = \binom{r+c-2}{c-1}$, Lemma 4.17 implies therefore that h^A is not a pure O-sequence.

□

Proposition 4.19. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and assume that $a_{2,1} < 0$. Let $h^A = (h_0, \dots, h_s)$. Then for all $i_0 \in \{a_{1,1}, \dots, a_{1,1} - a_{2,1} - 1\}$ we have $h_{i_0} > h_{s-i_0}$.*

Proof. According to Remark 3.12,

$$h^A_{s-i} = h^{(a_{1,1},\dots,a_{1,c})}_{s-i}, \text{ for } i = 0, \dots, (a_{1,1} - a_{2,1} - 1).$$

By Proposition 4.15 , for all $i = 0, \dots, a_{1,2} - 1$ we have

$$h^{(a_{1,1},\dots,a_{1,c})}_{s-i} = \binom{c-1+i}{c-1} - \binom{c-1+i-a_{1,1}}{c-1}.$$

In particular, as $a_{1,2} - 1 = a_{1,1} + a_{2,2} - a_{2,1} - 1 > a_{1,1} - a_{2,1} - 1 \geq a_{1,1}$, we obtain

$$h^{(a_{1,1},\dots,a_{1,c})}_{s-i} < \binom{c-1+i}{c-1}, \qquad \text{for every } i = a_{1,1}, \dots, (a_{1,1} - a_{2,1} - 1).$$

Thus, as $h^A_i = \binom{c-1+i}{c-1}$ for all $i = 0, \dots, \sum_{j=1}^{t} a_{j,j} - 1$, every index $i_0 \in \{a_{1,1}, \dots, a_{1,1} - a_{2,1} - 1\}$ satisfies $h_{i_0} > h_{s-i_0}$.

□

Hibi proved in [26] that all pure O-sequences are flawless i.e. $h_i \leq h_{s-i}$ for $i = 0, \dots, \lfloor s/2 \rfloor$. For this reason Proposition 4.18 and Proposition 4.19 have the following direct consequence:

Corollary 4.20. *Conjecture 4.12 holds for any degree matrix $A \in \mathbb{Z}^{t\times(t+c-1)}$, which has positive entries or one maximal row, i.e. $a_{1,1} > a_{2,1}$.*

The following examples show that Proposition 4.19 has no easy generalization to matrices with two or more maximal rows.

Example 4.21. *The matrices A, B below and their upper left 3×4 submatrices $A^{(4,5)}$, $B^{(4,5)}$ show that the conditions $a_{r+1,r} < 0$ and $a_{t,t-1} < 0$ do not influence the fulfillment of the Hibi inequalities. Clearly h^A and $h^{A^{(4,5)}}$ satisfy Hibi's inequalities, while h^B and $h^{B^{(4,5)}}$ do not. A quick exhaustive computer search shows that none of the four is a pure O-sequence.*

$$A = \begin{bmatrix} 2 & 2 & 5 & 5 & 5 \\ 2 & 2 & 5 & 5 & 5 \\ -2 & -2 & 1 & 1 & 1 \\ -2 & -2 & 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 5 & 5 & 5 \\ 1 & 2 & 5 & 5 & 5 \\ -3 & -2 & 1 & 1 & 1 \\ -3 & -2 & 1 & 1 & 1 \end{bmatrix}$$

$$h^A = (1, 2, 3, 4, 5, 6, 4, 4, 4, 2) \quad h^B = (1, 2, 3, 4, 5, 3, 3, 3, 2)$$

$$h^{A^{(4,5)}} = (1, 2, 3, 4, 5, 4, 4, 4, 2) \quad h^{B^{(4,5)}} = (1, 2, 3, 4, 3, 3, 3, 2)$$

The matrices C and D below show that for one maximal row and all entries positive both situations may appear, namely h^C does not satisfy Hibi's inequalities, while h^D does. By Proposition 4.18 none of them is a pure O-sequence.

$$C = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$h^C = (1, 3, 6, 10, 9, 7, 3, 1) \qquad h^D = (1, 3, 6, 4, 1)$$

For a degree matrix $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ which does not satisfy the conditions of Proposition 4.18 and Proposition 4.19 it is often difficult to show explicitly that h^A is not a pure O-sequence, especially when c and $\tau(h^A)$ are large. In this case also the exhaustive computer search in the corresponding list of pure O-sequences is not feasible. Using general theory, we will present in the following two methods, which in some cases can be successfully used to show that a given h-vector h^A is not a pure O-sequence.

Remark 4.22. **(A)** *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix with corresponding h-vector $h^A = (h_0, \dots, h_s)$ and let $J \subseteq R = K[X_1, \dots, X_c]$ be an artinian ideal such that $h_i = HF_{R/J}(i)$ for all i. If $L \subseteq R$ is the* *lex-segment ideal associated to h^A (that is the ideal generated by all monomials remaining after deleting the smallest h_i monomials of degree i for all $i \geq 0$), then a well-known result (see [3, 28]), proved by Bigatti, Hulett and Pardue, states that*

$$\beta^R_{i,j}(R/J) \leq \beta^R_{i,j}(R/L) \text{ for all } i \text{ and } j.$$

According to a result obtained from Peeva in [37], the graded Betti-numbers of R/J can be obtained from those of R/L by a sequence of consecutive cancellations. This shows in particular, that R/J could be level, only if any entry in the last column (except of course the one in the last row) in the Betti-diagram of R/L can be canceled with something in the next last column and one row lower. Clearly, if such cancellation is not possible R/J is not level. This shows in particular, that there is no artinian monomial level algebra whose Hilbert-function coincide with h^A and therefore h^A can not be a pure O-sequence.

For example if

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 8 & 12 \\ 1 & 1 & 3 & 3 & 8 & 12 \\ -1 & -1 & 1 & 1 & 6 & 10 \\ -1 & -1 & 1 & 1 & 6 & 10 \end{bmatrix},$$

then $h^A = (1, 3, 6, 10, 14, 18, 22, 26, 30, 32, 34, 34, 34, 32, 30, 26, 22, 18, 13, 11, 5, 3)$.

The Betti-diagram of the lex-segment ideal corresponding to h^A is

	0	1	2	3
8	0	2	3	1
9	0	1	2	1
10	0	3	6	3
11	0	3	5	2
12	0	4	8	4
13	0	4	8	4
14	0	6	11	5
15	0	5	10	5
16	0	5	10	5
17	0	6	11	5
18	0	2	4	2
19	0	6	12	6
20	0	2	4	2
21	0	3	6	3

Since $\beta_{17,20} = 5 > 4 = \beta_{18,20}$ and $\beta_{19,22} = 6 > 4 = \beta_{20,22}$ the cancellation of the 5 with 4 and of the 6 with 4 is not possible and so h^A is not a pure O-sequence.

(B) *Another useful tool for showing that a given h-vector is not a pure O-sequence is provided by the Macaulay-inverse system. Consider for example the degree matrix*

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & 3 & 6 \\ 1 & 1 & 3 & 3 & 3 & 6 \\ -1 & -1 & 1 & 1 & 1 & 4 \end{bmatrix},$$

with corresponding h-vector $h^A = (1, 4, 10, 17, 26, 31, 33, 33, 27, 21, 10, 4)$.

Notice that the Betti-diagram of the lexsegment ideal associated to h^A *is of the form*

	0	1	2	3	4
2	0	3	3	1	0
3	0	0	0	0	0
4	0	6	14	11	3
5	0	7	19	17	5
6	0	8	21	19	6
7	0	12	33	30	9
8	0	9	26	25	8
9	0	13	37	35	11
10	0	6	18	18	6
11	0	4	12	12	4

Thus, since consecutive cancellation is theoretically possible, the arguments in (A) can not be used to show that h^A *is not a pure O-sequence.*

Let $I \subseteq K[x, y, z, w]$ *be a standard determinantal ideal with degree matrix* A. *The h-vector* h^A *has maximal growth in degrees* 3 *and* 4, *and* $\dim_K(I_3) = 3$. *This shows in particular that the generators of the ideal in degree* 3 *have a gcd of degree 2. We can assume without loss of generality that they are of the form* Qx, Qy *and* Qz. *Suppose now that* h^A *is a pure O-sequence, then there are 4 monomial generators,* F_1, F_2, F_3 *and* F_4, *of the inverse system* I^{-1}. *The generators* Qx, Qy *and* Qz *of* I *annihilate all the monomials* F_1, F_2, F_3 *and* F_4. *This shows that after taking the derivatives with respect to* Q *the result is* w^9 *i.e.* $F_i = Qw^9$ *for all* i. *Since this is impossible,* h^A *is not a pure O-sequence.*

We finish the chapter by answering the second question we asked at the beginning.

Remark 4.23. *Beside of the characterization of the degree matrices whose h-vector is a pure O-sequence, we ask whether each pure O-sequence can be obtained as the h-vector of some degree matrix. Using the computer algebra system CoCoA we are able to answer this question negatively.*

Computing all possible h-vectors $h^A = (1, 3, h_2, \ldots, h_7)$ *of length* $s = 7$ *with* $h_s = 3$ *(notice that by Proposition 3.11 A has to be a degree matrix with two equal maximal rows) we have:*

$$(1,3,6,10,15,21,13,3),(1,3,6,10,14,14,9,3),(1,3,6,10,12,12,7,3),$$
$$(1,3,6,10,12,12,9,3),(1,3,6,9,11,10,7,3),(1,3,5,7,9,7,5,3),$$
$$(1,3,6,8,8,8,7,3),(1,3,5,7,7,7,5,3),(1,3,5,5,5,5,5,3),$$
$$(1,3,3,3,3,3,3,3),(1,3,6,10,15,17,11,3),(1,3,6,8,9,9,5,3),$$
$$(1,3,6,10,9,11,5,3).$$

One can see now that the pure O-sequence $(1, 3, 4, 4, 4, 4, 4, 3)$ *does not appear in the above list.*

5 Posets of h-vectors

In this section we will show that the set of all h-vectors of fixed codimension and length, corresponding to degree matrices of fixed size, has a natural stratification. We will prove in particular, that each strata contains a maximum, which we will construct explicitly. Furthermore, we will show that the only strata which has also a minimum is the one consisting of h-vectors of level standard determinantal schemes.

We would like to stress that the degree matrices we will deal with during this chapter are allowed to have zero entries.

A *poset* $(P, \leq)$ (short for *partially ordered set*) is a set P equipped with a binary relation " $\leq$ " that is reflexive (i.e. $a \leq a$ for all $a \in P$), antisymmetric ($a \leq b \leq a$ implies $a = b$) and transitive ($a \leq b \leq c$ implies $a \leq c$).

For two degree matrices A and B we will write $h^A \leq h^B$ if $h_i^A \leq h_i^B$ for all i. If $h^A \leq h^B$ we will write also $A \leq_h B$. With this order, the set

$$M^{(c)} := \bigcup_{t \geq 1} M^{(t,c)},$$

where

$$M^{(t,c)} := \big\{A \in \mathbb{Z}^{t \times (t+c-1)} |\, A \,\text{is a degree matrix}\big\},$$

becomes a poset for any fixed integer $c \geq 1$. For an integer $s \geq 1$ we define

$$N_s^{(c)} := \big\{A \in M^{(c)} |\, \tau(h^A) = s\big\}.$$

To $N_s^{(c)}$ we assign the poset $\mathcal{H}_s^{(c)} := \Big\{h^A |\, A \in N_s^{(c)}\Big\}$.

Notice that the degree matrices in $N_s^{(c)}$ are not of fixed size. This implies in particular together with Remark 3.2 that the map

$$N_s^{(c)} \longrightarrow \mathcal{H}_s^{(c)},\ A \longmapsto h^A$$

is surjective but certainly not bijective.

Definition 5.1. *Let $(P, \leq)$ be a poset.*

(1) *An element $x \in P$ is called a maximal element in P if there exist no $z \in P \setminus \{x\}$ such that $x \leq z$.*

(2) *An element $y \in P$ is called a minimal element in P if there exist no $z \in P \setminus \{y\}$such that $z \leq y$.*

(3) *A maximal element $x \in P$ which satisfies $x \geq y$ for any $y \in P$ is called maximum.*

(4) *A minimal element $x \in P$ which satisfies $x \leq y$ for any $y \in P$ is called minimum.*

For totally ordered sets, the notions of maximal element and maximum on one hand and minimal element and minimum on the other hand coincide.

The existence of a minimum and maximum h-vector in the poset $\mathcal{H}_s^{(c)}$ can be easily shown.

Lemma 5.2. *There exist h-vectors $h^{\min}, h^{\max} \in \mathcal{H}_s^{(c)}$ such that*

$$h^{\min} \leq h \leq h^{\max}, \text{ for all } h \in \mathcal{H}_s^{(c)}.$$

Proof. Let A be the degree matrix $A = [1, \dots, 1, s] \in \mathbb{Z}^{1 \times c}$. Then clearly $h^A \leq h^B$, for all $B \in N_s^{(c)}$, so that $h^{\min} = h^A = (1, \dots, 1)$.

Let $C = [c_{i,j}] \in \mathbb{Z}^{(s+1)\times(s+c)}$ be a degree matrix with $c_{i,j} = 1, \forall i, j$. We claim that $h^{\max} = h^C$. Choose $A \in N_s^{(c)}$ and let $X \subseteq \mathbb{P}^n$ be a standard determinantal scheme with degree matrix A. Let $J_X \subseteq R = K[X_1, \dots, X_c]$ be the artinian reduction of the defining ideal I_X of X. Since $h^A = (h_0, \dots, h_s)$ and $h_i^A = HF_{R/J_X}(i)$ for all i, we have $[J_X]_i = R_i, \forall i \geq s+1$, so that $J_X \supseteq R_+^{s+1}$. On the other hand the ideal R_+^{s+1} is standard determinantal with defining matrix

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_c & 0 & \cdots & 0 \\ 0 & & & & & & \\ \vdots & & & \ddots & & \ddots & \vdots \\ 0 & & \cdots & 0 & X_1 & \cdots & X_c \end{bmatrix} \in R^{(s+1)\times(s+c)}.$$

and degree matrix C. We obtain therefore

$$h_i^A = HF_{R/J_X}(i) \leq HF_{R/R_+^{s+1}}(i) = h_i^C, \text{ for all } i.$$

and the assertion follows.

□

As we just have seen it is not difficult to determine the minimum and the maximum in $\mathcal{H}_s^{(c)}$. The situation changes quickly if we study only subsets of $M^{(c)}$ and $N_s^{(c)}$ consisting of degree matrices of fixed size.

Consider the following subset of $M^{(t,c)}$:

$$N_s^{(t,c)} := \{A \in M^{(t,c)} | \, \tau(h^A) = s\},$$

for an integer $s \geq t-1$. We denote by

$$\mathcal{H}_s^{(t,c)} := \left\{h^A | \, A \in N_s^{(t,c)}\right\}$$

the corresponding set of h-vectors. For an integer $1 \leq r \leq t$ we define

$$N_s^{(t,r,c)} = \left\{ A \in N_s^{(t,c)} |\, a_{1,1} = \cdots = a_{r,1} > a_{r+1,1} \right\}$$

and

$$\mathcal{H}_s^{(t,r,c)} := \left\{ h^A |\, A \in N_s^{(t,r,c)} \right\}.$$

We obtain a natural stratification on $N_s^{(t,c)}$ and on $\mathcal{H}_s^{(t,c)}$, namely

$$N_s^{(t,c)} {=} N_s^{(t,1,c)} \cup \ldots \cup N_s^{(t,r,c)} \cup \ldots \cup N_s^{(t,t,c)}$$

and

$$\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,r,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}.$$

5.1 Posets of h-vectors of level standard determinantal schemes

By Theorem 4.13 any element in the poset $\mathcal{H}_s^{(t,t,c)}$ is the h-vector of some codimension c, level, standard determinantal scheme. We will show first that in $\mathcal{H}_s^{(t,t,c)}$ there are a minimum and a maximum. In order to show this we introduce the following notation:

$$\mathbb{N}^{\deg}_{(t,c,s)} := \left\{ (a_1, \ldots, a_{t+c-1}) \in \mathbb{N}^{t+c-1} |\, \sum_{i=1}^{t+c-1} a_i = s + c, a_1 \leq \cdots \leq a_{t+c-1} \right\}.$$

Thus an element $\mathbf{a} \in \mathbb{N}^{\deg}_{(t,c,s)}$ is a partition of $s+c$ ordered in an increasing way. Obviously, there exists one to one correspondence $\mathbb{N}^{\deg}_{(t,c,s)} \longleftrightarrow N_s^{(t,t,c)}$, given by $\mathbf{a} \longmapsto A$, where each row of A is equal to $\mathbf{a}$.

Definition 5.3. *Let $(P, \leq)$ be a poset and let $x, y \in P$. We say that x covers y (in the poset P) if $x \neq y$ and $y \leq x$, and there does not exist $z \in P \setminus \{x, y\}$ such that $y \leq z \leq x$.*

For two elements $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$ we will write $\mathbf{a} \lhd \mathbf{b}$ if and only if $\mathbf{a} = \mathbf{b}$ or there exist $i < j \in \mathbb{N}$ such that

$$\mathbf{a} = (b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_{j-1}, b_j + 1, b_{j+1}, \ldots, b_{t+c-1}).$$

If there exist elements $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{N}^{\deg}_{(t,c,s)}$ such that $\mathbf{a} = \mathbf{a}_1 \lhd \cdots \lhd \mathbf{a}_m = \mathbf{b}$, we will use the notation $\mathbf{a} < \mathbf{b}$.

Obviously the relation $\lhd$ does not define a partial order on $\mathbb{N}^{\deg}_{(t,c,s)}$, since it is not transitive. If $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c}$ and $\mathbf{a} \lhd \mathbf{b} \lhd \mathbf{c}$, it does not hold $\mathbf{a} \lhd \mathbf{c}$. So, in order to make $\mathbb{N}^{\deg}_{(t,c,s)}$ to a poset, we have to take the transitive closure of $\lhd$, which is given by $<$. It is easy to verify that $\mathbf{a} < \mathbf{b} < \mathbf{a}$ implies $\mathbf{a} = \mathbf{b}$.

We will show next, that the correspondence $\mathbb{N}^{\deg}_{(t,c,s)} \longleftrightarrow N_s^{(t,t,c)}$ preserves the partial order.

Lemma 5.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$ *and let* $A, B \in N_s^{(t,t,c)}$ *be the corresponding degree matrices with rows equal to* $\mathbf{a}$*, respectively* $\mathbf{b}$*. If* $\mathbf{a} \lhd \mathbf{b}$*, then* $h^A \leq h^B$*.*

Proof. We may assume that $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a} \lhd \mathbf{b}$, hence that $\mathbf{b} = (b_1, \ldots, b_{t+c-1})$ and $\mathbf{a} = (b_1, \ldots, b_i - 1, \ldots, b_j + 1, \ldots, b_{t+c-1})$. We will prove the claim by induction on t and c. For $c = 1$ the claim is trivial.

Let $c > 1$. For $t = 1$, it holds by Lemma 2.6:

$$\begin{aligned}\mathrm{hp}^A(z) &= \mathrm{hp}^{(b_1,\ldots,b_i-1,\ldots,b_j+1,\ldots,b_c)}(z)\\ &= \mathrm{hp}^{(b_1,\ldots,b_{i-1},b_{i+1},\ldots,b_{j-1},b_{j+1},\ldots,b_c)}(z) \cdot \mathrm{hp}^{(b_i-1,b_j+1)}(z).\end{aligned}$$

Since for any $c, d \in \mathbb{N}$ we have

$$\begin{aligned}h^{(c,d)} &= (1, \ldots, \underbrace{c-1, c, \ldots, c, c-1}_{d-c+3}, \ldots, 1),\\ h^{(c-1,d+1)} &= (1, \ldots, \underbrace{c-1, \ldots, c-1}_{d-c+3}, \ldots, 1)\end{aligned}$$

and $\tau(h^{(c,d)}) = \tau(h^{(c-1,d+1)})$ (i.e. $\deg(\mathrm{hp}^{(c,d)}(z)) = \deg(\mathrm{hp}^{(c-1,d+1)})(z))$, it clearly holds $h^{(c,d)} \geq h^{(c-1,d+1)}$, and therefore $h^A \leq h^B$ as claimed.

Let $t > 1$. We assume that $j < t + c - 1$. The case $j = t + c - 1$ is proved similarly. Applying Remark 3.1 for b_{t+c-1} on B and A we have:

$$h^B = h^{B^{(t,t+c-1)}} + \sum_{k=0}^{b_{t+c-1}-1} h_{i-k}^{B^{(0,t+c-1)}} \text{ and } h^A = h^{A^{(t,t+c-1)}} + \sum_{k=0}^{b_{t+c-1}-1} h_{i-k}^{A^{(0,t+c-1)}}.$$

As by induction it holds $h^{B^{(t,t+c-1)}} \geq h^{A^{(t,t+c-1)}}$ and $h^{B^{(0,t+c-1)}} \geq h^{A^{(0,t+c-1)}}$, we conclude.

∎

With Lemma 5.4 we are now ready to determine explicitly the minimum and the maximum in the poset $\mathcal{H}_s^{(t,t,c)}$.

Proposition 5.5. *For any integer* $s \geq t-1$ *there exist h-vectors* $h^{\min}$ *and* $h^{\max}$ *in* $\mathcal{H}_s^{(t,t,c)}$ *such that* $h^{\min} \leq h \leq h^{\max}$*, for all* $h \in \mathcal{H}_s^{(t,t,c)}$*.*

Proof. Fix $s \geq t - 1$ and let $\mathbf{a} = (1, \ldots, 1, s - t + 2) \in \mathbb{N}^{\deg}_{(t,c,s)}$. To prove that $h^A = h^{\min}$, by Lemma 5.4 it suffices to show that $\mathbf{a} < \mathbf{b}$ $\forall \mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Let $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}, \mathbf{b} \neq \mathbf{a}$. We can find integers i and j such that $b_i > 1$ and $b_j < s-t+2$. It follows that

$$\mathbf{b} \triangleright \mathbf{b}' = (b_1, \dots, b_k - 1, \dots, b_l + 1, \dots, b_{t+c-1}),$$

where $k = \min\{i | \, b_i > 1\}$ and $l = \max\{j | \, b_j < s - t + 2\}$. If $\mathbf{a} = \mathbf{b}'$, we are done, otherwise we can repeat the process with $\mathbf{b}'$ instead of $\mathbf{b}$. Clearly after finitely many steps we will reach $\mathbf{a}$, since the result of each step is a non-decreasing partition of $s + c$, where the difference between the entries at the positions k and l increase by 2.

Let $\mathbf{c} = (c_1, \dots, c_{t+c-1}) \in \mathbb{N}^{\deg}_{(t,c,s)}$, where $c_1 = \cdots = c_k = d$ and $c_{k+1} = \cdots = c_{t+c-1} = d + 1$, for some $d \in \mathbb{N}$. According to Lemma 5.4 in order to show that $h^C = h^{\min}$ it is enough to show that $\mathbf{c} > \mathbf{b}$ for any $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Let $\mathbf{b} \in \mathbb{N}^{\deg}_{(t,c,s)}$. Since $\mathbf{b} \neq \mathbf{c}$ there exist indexes $i < j$, such that $b_j - b_i \geq 2$. Therefore

$$\mathbf{b} \triangleleft \mathbf{b}' = (b_1, \dots, b_k + 1, \dots, b_l - 1, \dots, b_{t+c-1}),$$

where $k = \max\{m | \, b_m = b_i\}$ and $l = \min\{n | \, b_n = b_j\}$. If $\mathbf{b}' = \mathbf{c}$ we are done, otherwise we repeat the process with $\mathbf{b}'$ instead of $\mathbf{b}$. After finitely many steps $\mathbf{c}$ will be reached, since the result of each step is a non-decreasing partition of $s+c$, where the difference between the entries at the positions k and l decreases by 2.

□

Remark 5.6. *We would like to point out that the existence of a minimum and a maximum in a poset of h-vectors is a very rare and unexpected property.*

O. Greco, M. Mateev and C. Söger showed (see [21]) that a similar result holds also for the poset of h-vectors of the union of two sets of points in $\mathbb{P}^2$. More precisely the existence of a minimum in this poset was proved and the existence of a maximum, conjectured.

A useful tool for dealing with finite posets is the Hasse diagram.

Definition 5.7. *Starting with a poset $(P, \leq)$, we define a directed graph with vertex set P by the rule that (x, y) is an edge if x covers y in P. The digraph H is called a Hasse digraph for P. When it is drawn in the plane with edges as straight lines going from the lower endpoint to the upper endpoint it is called a Hasse diagram.*

Using Hasse diagrams we can easily visualize the structure of the posets $\mathcal{H}_s^{(t,r,c)}$, as illustrated in the following example:

Example 5.8. *Consider the poset $\mathcal{H}_7^{(2,2,3)}$. Computing the possible partitions of 10, we can obtain the Hasse diagram of $\mathbb{N}^{\deg}_{(2,3,7)}$ by drawing an edge for any $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{\deg}_{(2,3,7)}$ with $\mathbf{a} \lhd \mathbf{b}$.*

$$\begin{tikzcd}
 & (2,2,3,3) = \mathbf{a_1} \arrow[dl] \arrow[dr] & \\
\mathbf{a_2} = (1,3,3,3) \arrow[d] & & (2,2,2,4) = \mathbf{a_3} \arrow[dll] \arrow[d] \\
\mathbf{a_4} = (1,2,3,4) \arrow[rr] \arrow[d] \arrow[dr] & & (1,2,2,5) = \mathbf{a_5} \arrow[dl] \arrow[d] \\
\mathbf{a_6} = (1,1,4,4) \arrow[r] & \mathbf{a_7} = (1,1,3,5) \arrow[r] & (1,1,2,6) = \mathbf{a_8} \arrow[d] \\
 & & (1,1,1,7) = \mathbf{a_9}
\end{tikzcd}$$

In the notation of Lemma 5.4 the corresponding h-vectors are

$$\begin{aligned}
h^{A_1} &= (1,3,6,10,14,14,9,3),\\
h^{A_2} &= (1,3,6,10,12,12,7,3),\\
h^{A_3} &= (1,3,6,10,12,12,9,3),\\
h^{A_4} &= (1,3,6,9,11,10,7,3),\\
h^{A_5} &= (1,3,6,8,8,8,7,3),\\
h^{A_6} &= (1,3,5,7,9,7,5,3),\\
h^{A_7} &= (1,3,5,7,7,7,5,3),\\
h^{A_8} &= (1,3,5,5,5,5,5,3),\\
h^{A_9} &= (1,3,3,3,3,3,3,3).
\end{aligned}$$

We can easily see that $\mathcal{H}_7^{(2,2,3)}$ has the same Hasse diagram as $\mathbb{N}^{\deg}_{(2,3,7)}$ and the minimum, respectively maximum h-vector corresponds to $\mathbf{a_9}$, respectively $\mathbf{a_1}$.

Notice also that $h^{A_3} \geq h^{A_2}$ but $\mathbf{a}_3$ and $\mathbf{a_2}$ are incomparable. This shows in particular, that in general from h^A covers h^B does not follow A covers B.

5.2 h-vectors of degree matrices with r-maximal rows

Having seen that $\mathcal{H}_s^{(t,t,c)}$ contains a minimum and a maximum h-vector, it is natural to ask whether the same is also true for any $\mathcal{H}_s^{(t,r,c)}$, $1 \leq r \leq t-1$. As the following example shows if $r \leq t-1$ the existence of a minimum is in general not granted.

Example 5.9. *Consider the set $N_7^{(4,3,3)}$. It consists of the following degree matrices*

$$A_1 = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 3 & 3 & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

In particular it follows that $\mathcal{H}_7^{(4,3,3)}$ consists of the h-vectors

$$\begin{aligned} h^{A_1} &= (1,3,6,10,15,21,18,6), \\ h^{A_2} &= (1,3,6,10,15,18,15,6), \\ h^{A_3} &= (1,3,6,10,13,13,12,6), \\ h^{A_4} &= (1,3,6,10,14,16,12,6), \\ h^{A_5} &= (1,3,6,10,12,15,9,6). \end{aligned}$$

The corresponding Hasse diagram is given by:

$$\begin{tikzcd} & h^{A_1} \arrow[d] & \\ & h^{A_2} \arrow[d] & \\ & h^{A_4} \arrow[dl] \arrow[dr] & \\ h^{A_5} & & h^{A_3} \end{tikzcd}$$

and shows that in $\mathcal{H}_7^{(4,3,3)}$ there is no minimum.

We will show next that for any $1 \leq r \leq t-1$, the poset $\mathcal{H}_s^{(t,r,c)}$ contains a maximum h-vector, which correspond to a matrix of the form

$$A = \begin{bmatrix} a_1 & \cdots & a_{t+c-1} \\ \vdots & & \vdots \\ a_1 & \cdots & a_{t+c-1} \\ a_1 - 1 & \cdots & a_{t+c-1} - 1 \\ \vdots & & \vdots \\ a_1 - 1 & \cdots & a_{t+c-1} - 1 \end{bmatrix} \tag{5.1}$$

where $a_1 + \cdots + a_{t+c-1} = s + c + (t-r)$ and the matrix $A' \in \mathbb{Z}^{r \times (t+c-1)}$ consisting of the first r equal rows of A satisfies $h^{A'} = h^{\max} \in \mathcal{H}_s^{(r,r,c+(t-r))}$.

By Proposition 5.5 we can write A more precisely in the the following way:

$$A = \begin{bmatrix} a & \cdots & a & a+1 & \cdots & a+1 \\ \vdots & & \vdots & \vdots & & \vdots \\ a & \cdots & a & a+1 & \cdots & a+1 \\ a-1 & \cdots & a-1 & a & \cdots & a \\ \vdots & & \vdots & \vdots & & \vdots \\ a-1 & \cdots & a-1 & a & \cdots & a \end{bmatrix},$$

for some $a \geq 1$, such that the sum of the entries in the first row is equal to $s + c + (t-r)$.

Remark 5.10. *Notice that for any matrix $B \in N_s^{(t,r,c)}$, we have*

$$b_{1,1} + \cdots + b_{1,t+c-1} \geq s + c + (t-r).$$

This follows directly from Proposition 3.11 since

$$s + c = b_{1,1} + \cdots + b_{1,c} + b_{2,c} + b_{r,c+(r-1)} + b_{r+1,c+r} + \cdots + b_{t,t+c-1}$$

and

$$\begin{aligned} b_{i,c+(i-1)} &= b_{1,c+(i-1)}, \quad \text{for all} \quad i = 2, \ldots, r \\ b_{i,c+(i-1)} &\leq b_{1,c+(i-1)} - 1, \quad \text{for all} \quad i = r+1, \ldots, t. \end{aligned}$$

Before proving that $h^A = h^{\max}$ for the matrix defined in (5.1), we introduce the following notation

$$\begin{aligned} L_s^{(t,r,c)} &= \left\{ B \in N_s^{(t,r,c)} | a_{1,1} - a_{i,1} = 1, \, \forall i \geq r+1 \right\} \\ R_s^{(t,r,c)} &= \{ B \in N_s^{(t,r,c)} | \, B \notin L_s^{(t,r,c)} \} = N_s^{(t,r,c)} \setminus L_s^{(t,r,c)}. \end{aligned}$$

Obviously, it holds $N_s^{(t,r,c)} = L_s^{(t,r,c)} \cup R_s^{(t,r,c)}$. Furthermore, by definition any $B = [b_{i,j}] \in L_s^{(t,r,c)}$ is of the form

$$B = \begin{bmatrix} b_1 & \cdots & b_{t+c-1} \\ \vdots & & \vdots \\ b_1 & \cdots & b_{t+c-1} \\ b_1 - 1 & \cdots & b_{t+c-1} - 1 \\ \vdots & & \vdots \\ b_1 - 1 & \cdots & b_{t+c-1} - 1 \end{bmatrix}, \tag{5.2}$$

and it holds $b_1 + \cdots + b_{t+c-1} = s + c + (t - r)$.

Lemma 5.11. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix and assume that there exist indexes $1 \leq i < j \leq t + c - 1$ such that $a_{1,j} - a_{1,i} \geq 2$. Let $k = \max\{m|\, a_{1,i} = \cdots = a_{1,m}\}$ and $l = \min\{n|\, a_{1,n} = \cdots = a_{1,j}\}$. If B is the degree matrix obtained from A by adding 1 to the k-th column and subtracting 1 from the l-th column, then $h^A \leq h^B$.*

Proof. We prove the claim by induction on t and c. For $t = 1$ the claim follows from Lemma 5.4. Let $t > 1$, for $c = 1$ the claim is trivial, so let $c > 1$.

Without loss of generality we may apply Proposition 3.1 for $a_{t,t+c-1}$, assuming that $B^{(t,t+c-1)}$ and $B^{(0,t+c-1)}$ contain the modified columns of A. It follows then by induction that $h^{A^{(t,t+c-1)}} \leq h^{B^{(t,t+c-1)}}$ and $h^{A^{(0,t+c-1)}} \leq h^{B^{(0,t+c-1)}}$, so by Remark 3.2 we conclude.

□

Lemma 5.12. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be a degree matrix which satisfies the following conditions:*

(1) *A has $r \leq t - 1$ maximal rows, i.e. $a_{1,1} = \cdots = a_{r,1}$,*

(2) *there exists an index $1 \leq j \leq t+c-1$ such that $a_{1,j} \geq 2$ and $a_{1,j} - a_{1,j-1} \geq 1$,*

(3) *there exists an index $r+1 \leq i \leq t$ such that $a_{i-1,1} - a_{i,1} \geq 1$ and if $i = r+1$, then $a_{r,1} - a_{r+1,1} \geq 2$.*

Let B be the matrix obtained from A by adding 1 to the i-th row and subtracting 1 from the j-th column. Then $h^A \leq h^B$.

Proof. We proceed by induction on t and c. For $c = 1$ and $t \geq 1$ the claim is trivial. Let $c > 1$, $t = 2$ and let

$$B = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} - 1 & a_{1,j+1} & \cdots & a_{1,t+c-1} \\ a_{2,1} + 1 & \cdots & a_{2,j-1} + 1 & a_{2,j} & a_{2,j+1} + 1 & \cdots & a_{2,t+c-1} + 1 \end{bmatrix}$$

be the matrix obtained from A by adding 1 to the second row and subtracting 1 from the j-th column. We assume that $j < t + c - 1$. The computation for $j = t + c - 1$ is analogous. Applying Remark 3.2 for $a_{1,t+c-1}$ we have

$$h_i^B = h_{i-a_{1,t+c-1}}^{(a_{2,1}+1,\dots,a_{2,j},\dots,a_{2,t+c-2}+1)} + \sum_{k=0}^{a_{1,t+c-1}-1} h_{i-k}^{B^{(0,t+c-1)}}.$$

By the inductive hypothesis on c it holds $h^{B^{(0,t+c-1)}} \geq h^{A^{(0,t+c-1)}}$. Since we obviously have $h^{(a_{2,1}+1,\dots,a_{2,j},\dots,a_{2,t+c-2}+1)} \geq h^{(a_{2,1},\dots,a_{2,j},\dots,a_{2,t+c-2})}$, the claim follows.

Let $t > 2$. Obviously there is an entry $a_{k,l} > 0$ which remains unchanged by performing the operation described in the statement and such that $B^{(k,l)}$ and $B^{(0,l)}$ contain the modified row and column of A. By the inductive hypothesis on t and c we have $h^{B^{(k,l)}} \geq h^{A^{(k,l)}}$, and $h^{B^{(0,l)}} \geq h^{A^{(0,l)}}$. The assertion follows therefore from Remark 3.2 applied for the indexes (k,l).

□

Remark 5.13. *Notice that the operation defined in Lemma 5.12 does not change the number of equal rows or the length of the corresponding h-vector.*

Proposition 5.14. *Let r, t, c be positive integers, where $t \geq 2$ and $r \leq t-1$. There exists a h-vector $h^{\max} \in \mathcal{H}_s^{(t,r,c)}$, such that $h \leq h^{\max}$ for all $h \in \mathcal{H}_s^{(t,r,c)}$. Moreover, it holds $h^{\max} = h^A$, where A is the degree matrix described in* (5.1).

Proof. Let $C = [c_{i,j}] \in N_s^{(t,r,c)}$. We can assume that $C \in L_s^{(t,r,c)}$, as for any $C \in R_s^{(t,r,c)}$ repeated application of Lemma 5.12, will produce a matrix $B = [b_{i,j}] \in L_s^{(t,r,c)}$, which has the form described in (5.2). Furthermore, for the corresponding h-vectors it holds $h^C \leq h^B$. Notice that since by each step the entries in a certain non-maximal row increase by one, only finitely many steps are needed to obtain B. If $C \in L_s^{(t,r,c)}$ is not equal to the matrix A defined in (5.1), then there have to be entries $c_{1,i} < c_{1,j}$, such that $c_{1,j} - c_{1,i} \geq 2$. Applying Lemma 5.11 on C will produce a degree matrix $A' \in L_s^{(t,r,c)}$, such that $h^{A'} \geq h^C$. If $A = A'$ we have found the maximal h-vector, otherwise we apply Lemma 5.11 on A'. Since each time we lower the difference between a pair of columns of the matrix, after finitely many steps we will reach the matrix A.

□

The next example illustrates the operations described in Lemma 5.11 and Lemma 5.12.

Example 5.15. *Consider again the set $N_7^{(4,3,3)}$ from Example 5.9. We have*

$$L_7^{(4,3,3)} = \{A_1, A_2, A_3, A_4\} \text{ and } R_7^{(4,3,3)} = \{A_5\}.$$

Writing $A \xrightarrow[(i,j)]{(+,-)} B$, respectively $A \xrightarrow[(i,j)]{\binom{+}{-}} B$ for the degree matrix B obtained from A by applying Lemma 5.11 on the columns i and j or respectively applying Lemma 5.12 on the i-th row and j-th column of A, we have

$$
\begin{tikzcd}
A_5 \arrow[r, "\binom{+}{-}", "{(4,4)}"'] & A_4 \arrow[r, "{(+,-)}", "{(3,4)}"'] & A_2 \arrow[r, "{(+,-)}", "{(2,6)}"'] & A_1 \\
& A_3 \arrow[u, "{(5,6)}", "{(+,-)}"'] & &
\end{tikzcd}
$$

5.3 Maximum h-vector

The next problem we will approach is whether there exists a maximum h-vector in the poset $\mathcal{H}_s^{(t,c)}$. We will show in this section that there is one and it is equal to the maximum h-vector $h^{\max}$ in $\mathcal{H}_s^{(t,t,c)}$. We start with some preparatory lemmas.

Lemma 5.16. *Given* $\mathbf{a} = (a_1, \ldots . a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *two integer sequences, such that* $a_i \leq b_i$ *for all* $i = 1, \ldots, n$. *Let* $\mathbf{c} = (c_1, \ldots, c_n)$ *and* $\mathbf{d} = (d_1, \ldots, d_n)$ *be two permutations of* $\mathbf{a}$*, respectively* $\mathbf{b}$ *such that* $c_1 \leq \cdots \leq c_n$ *and* $d_1 \leq \cdots \leq d_n$. *Then* $c_i \leq d_i$ *for all* $i = 1, \ldots, n$.

Proof. We will prove the claim by induction on n. For $n = 2$ we have the following possibilities

- $a_1 \leq a_2$ and $b_1 \leq b_2$,
- $a_1 \geq a_2$ and $b_1 \geq b_2$,
- $a_1 \leq a_2$ and $b_1 \geq b_2$,
- $a_1 \geq a_2$ and $b_1 \leq b_2$.

Obviously in the first two cases there is nothing to show. The inequalities in the third and the fourth case imply $a_1 \leq a_2 \leq b_2 \leq b_1$ respectively $a_2 \leq a_1 \leq b_1 \leq b_2$ and the claim follows.

Let $n \geq 2$ and

$$a_i = \min \{a_k | \, k = 1, \ldots, n\}, \, b_j = \min \{b_k | \, k = 1, \ldots, n\}.$$

We have the sequences

$$(a_i, a_1, \ldots, a_{i-1}, \widehat{a_i}, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$(b_j, b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_{j-1}\widehat{b_j}, b_{j+1}, \ldots, b_n),$$

where $a_i \leq a_j \leq b_j \leq b_i$. After reordering the subsequences obtained from $\mathbf{a}$ and $\mathbf{b}$ by removing a_i and b_j, we have

$$\widetilde{a} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

and

$$\widetilde{b} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{j-1}, b_i, b_{j+1}, \dots, b_n).$$

As $a_j \le b_j \le b_i$, it holds by induction $\widetilde{a_i} \le \widetilde{b_i}$ for any i and the assertion follows. $\square$

The next result gives us a direct way how to compare h-vectors corresponding to degree matrices with equal rows.

Lemma 5.17. *Let $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ and $B = [b_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ be two degree matrices with equal rows, such that $a_{i,j} \le b_{i,j}$ for all $i = 1, \dots, t$ and $j = 1, \dots, t+c-1$. Then $h^A \le h^B$.*

Proof. The claim follows directly from Theorem 4.9. $\square$

Next, using Lemma 5.16 and Lemma 5.17 we show that the h-vector of any standard determinantal scheme is bounded from above by the h-vector of a level standard determinantal scheme.

Theorem 5.18. *To any standard determinantal scheme $X \subseteq \mathbb{P}^n$ there exists a level standard determinantal scheme $Y \subseteq \mathbb{P}^n$ of the same codimension such that*

$$h^X \le h^Y \text{ and } \tau(h^X) = \tau(h^Y).$$

Proof. Let A be the degree matrix of X. Without loss of generality we can assume that $\tau(h^X) = s$ and $A \in N_s^{(t,c)}$. To prove the claim it suffices to show that there is a degree matrix $B \in N_s^{(t,t,c)}$ such that $h^A \le h^B$. We will show this by induction on t and c.

When $t = 1$ the claim is trivial, so let $t > 1$ and $c = 1$.
Let $B \in N_s^{(t,t,1)}$ be the degree matrix, whose rows are equal to a nondecreasing reordering $a_{\sigma(1),\sigma(1)} \le \dots \le a_{\sigma(t),\sigma(t)}$ of the diagonal elements of A. We have then obviously $h^A = h^B$.
Let $c > 1$ and $a_{i_0,j_0} = \min\{a_{1,1}, \dots, a_{1,c}, a_{2,c+1}, \dots, a_{t,t+c-1}\}$. If $(b_1, \dots, b_{t+c-1})$ is a nondecreasing reordering of $(a_{1,1}, \dots, a_{1,c}, a_{2,c+1}, \dots, a_{t,t+c-1})$ and $B \in N_s^{(t,t,c)}$ is the matrix whose rows are equal to $(b_1, \dots, b_{t+c-1})$, then $b_1 = a_{i_0,j_0}$ and by Remark 3.2, applied on A for the indices (i_0, j_0) and on B for $(1,1)$, we have

$$h_i^A = h_{i-a_{i_0,j_0}}^{A^{(i_0,j_0)}} + \sum_{k=0}^{a_{i_0,j_0}-1} h_{i-k}^{A^{(0,j_0)}} \text{ and } h_i^B = h_{i-b_1}^{B^{(1,1)}} + \sum_{k=0}^{b_1-1} h_{i-k}^{B^{(0,1)}}.$$

We distinguish the following cases:

Case 1: $i_0 \in \{1, t\}$. Since the proof of the claim for $i_0 = 1$ is the same as for $i_0 = t$, we will show it only for $i_0 = 1$ (notice that $i_0 = 1$ implies $j_0 = 1$). Consider first $A^{(1,1)}$. As

$$\begin{array}{cccccc} a_{1,2} & \cdots & a_{1,c} & a_{2,c+1} & \cdots & a_{t,t+c-1} \\ \vee\!| & & \vee\!| & \| & & \| \\ a_{2,2} & \cdots & a_{2,c} & a_{2,c+1} & \cdots & a_{t,t+c-1} \end{array}$$

by Lemma 5.16 we have

$$\begin{array}{ccccc} b_2 & \leq & \cdots & \leq & b_{t+c-1} \\ \vee\!| & & & & \vee\!| \\ d_2 & \leq & \cdots & \leq & d_{t+c-1} \end{array},$$

where $(d_2, \cdots, d_{t+c-1})$ is the nondecreasing reordering of $(a_{2,2}, \ldots, a_{2,c}, a_{2,c+1}, \ldots, a_{t,t+c-1})$. If D is the matrix with rows equal to $(d_2, \cdots, d_{t+c-1})$, then Lemma 5.17 together with the inductive hypothesis shows that $h^{A^{(1,1)}} \leq h^D \leq h^{B^{(1,1)}}$. On the other hand, for $A^{(0,1)}$, as $(b_2, \ldots, b_{t+c-1})$ is the nondecreasing reordering of $(a_{1,2}, \ldots, a_{1,c}, a_{2,c+1}, \ldots a_{t,t+c-1})$, it holds by induction that $h^{A^{(0,1)}} \leq h^{B^{(0,1)}}$and we can conclude.

Case 2: $2 \leq i_0 \leq t-1$. Looking at the matrix $A^{(0,j_0)}$ we obtain the inequalities

$$\begin{array}{cccccccc} a_{1,1} & \cdots & a_{1,c-1} & a_{1,c} & \cdots & a_{i_0-1,j_0-1} & a_{i_0+1,j_0+1} & \cdots & a_{t,t+c-1} \\ \| & & \| & \vee\!| & & \vee\!| & \| & & \| \\ a_{1,1} & \cdots & a_{1,c-1} & a_{2,c} & \cdots & a_{i_0,j_0-1} & a_{i_0+1,j_0+1} & \cdots & a_{t,t+c-1} \end{array},$$

which according to Lemma 5.16 imply the following inequalities on the corresponding nondecreasing reorderings:

$$\begin{array}{ccccc} b_2 & \leq & \cdots & \leq & b_{t+c-1} \\ \vee\!| & & & & \vee\!| \\ f_2 & \leq & \cdots & \leq & f_{t+c-1} \end{array}.$$

By the induction hypothesis and Lemma 5.17 we have $h^{A^{(0,j_0)}} \leq h^F \leq h^{B^{(0,1)}}$, where F is the matrix whose rows are equal to $(f_2, \ldots, f_{t+c-1})$.
Considering the matrix $A^{(i_0,j_0)}$ and using the fact that the nondecreasing reordering of $(a_{1,1}, \ldots, a_{1,c}, a_{2,c+1}, \ldots, a_{i_0-1,j_0-1}, a_{i_0+1,j_0+1}, \ldots a_{t,t+c-1})$ is $(b_2, \ldots, b_{t+c-1})$ we have by induction $h^{A^{(i_0,j_0)}} \leq h^{B^{(1,1)}}$ and the claim follows from Remark 3.2 applied on A and B for the indices (i_0, j_0), respectively $(1,1)$. $\square$

Example 5.19. *Consider the matrix* $A = \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & 1 & 2 & 3 & 3 \\ -1 & 0 & 1 & 2 & 2 \end{bmatrix} \in N_8^{(3,3)}$*, with corresponding h-vector* $h^A = (1,3,6,9,10,9,6,3,1)$*. The nondecreasing reordering of* $(1,2,3,3,2)$ *is* $(1,2,2,3,3)$*, therefore we obtain the matrix*

$$B = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \end{bmatrix},$$

whose corresponding h-vector is $h^B = (1,3,6,10,15,20,21,15,6)$ *and* $h^A \leq h^B$*.*

Theorem 5.18 provides the tool needed for showing the existence of the maximum in $\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}$.

Corollary 5.20. *For any positive integers $t, c, s \in \mathbb{N}$ and $s \geq t-1$ there exists a h-vector $H^{\max} \in \mathcal{H}_s^{(t,c)}$, such that $h \leq H^{\max}$ for all $h \in \mathcal{H}_s^{(t,c)}$. Furthermore $H^{\max} = h^{\max} \in \mathcal{H}_s^{(t,t,c)}$.*

Proof. Let $A \in N_s^{(t,c)}$. By Theorem 5.18 there is a matrix $B \in N_s^{(t,t,c)}$ such that $h^A \leq h^B$. By Proposition 5.5 there exists a degree matrix $C \in N_s^{(t,t,c)}$ such that $h^C = h^{\min} \in \mathcal{H}_s^{(t,t,c)}$. We have therefore $h^A \leq h^B \leq h^C$ and the claim follows. □

According to Corollary 5.20, $\mathcal{H}_s^{(t,c)} = \mathcal{H}_s^{(t,1,c)} \cup \ldots \cup \mathcal{H}_s^{(t,t,c)}$ contains a maximum h-vector, which is the maximum in the stratum $\mathcal{H}_s^{(t,t,c)}$. Therefore it is natural to ask whether there exists a minimum h-vector in $\mathcal{H}_s^{(t,c)}$. Obviously if there exist one, then by Proposition 3.11 it has to come from $\mathcal{H}_s^{(t,1,c)}$. As we have seen in the previous section the poset $\mathcal{H}_s^{(t,1,c)}$ does not have a minimum in general (see also Example 5.21). It turns out that the same is true also for $\mathcal{H}_s^{(t,c)}$. The following example is a good illustration for this fact.

Example 5.21. *Consider the poset $\mathcal{H}_7^{(2,3)} = \mathcal{H}_7^{(2,1,3)} \cup \mathcal{H}_7^{(2,2,3)}$. The strata $\mathcal{H}_7^{(2,1,3)}$ consist of the h-vectors:*

$$
\begin{aligned}
h^{A_1} &= (1,3,6,10,12,9,4,1), & h^{A_2} &= (1,3,6,9,10,8,4,1),\\
h^{A_3} &= (1,3,6,8,8,6,3,1), & h^{A_4} &= (1,3,6,7,7,7,4,1),\\
h^{A_5} &= (1,3,5,7,7,5,3,1), & h^{A_6} &= (1,3,4,5,4,4,2,1),\\
h^{A_7} &= (1,3,4,4,4,4,3,1).
\end{aligned}
$$

For $\mathcal{H}_7^{(2,2,3)}$ we have

$$
\begin{aligned}
h^{B_1} &= (1,3,6,10,14,14,9,3), & h^{B_2} &= (1,3,6,10,12,12,9,3),\\
h^{B_3} &= (1,3,6,10,12,12,7,3), & h^{B_4} &= (1,3,6,9,11,10,7,3),\\
h^{B_5} &= (1,3,5,7,9,7,5,3), & h^{B_6} &= (1,3,6,8,8,8,7,3),\\
h^{B_7} &= (1,3,5,7,7,7,5,3), & h^{B_8} &= (1,3,5,5,5,5,5,3),\\
h^{B_9} &= (1,3,3,3,3,3,3,3).
\end{aligned}
$$

The Hasse diagram corresponding to $\mathcal{H}_7^{(2,3)}$ is then:

$$
\begin{tikzcd}
 & h^{B_1} \arrow[d] & & \\
 & h^{B_2} \arrow[d] & & \\
 & h^{B_3} \arrow[dl] \arrow[dr] & & \\
h^{B_4} \arrow[d] \arrow[dr] & & h^{A_1} \arrow[d] & \\
h^{B_5} \arrow[d] \arrow[dr] & h^{B_6} \arrow[d] \arrow[dr] \arrow[drr] & h^{A_2} \arrow[d] \arrow[dr] & \\
h^{B_7} \arrow[ddr] \arrow[ddrr] \arrow[ddrrr] & h^{B_8} \arrow[dd] \arrow[ddr] \arrow[ddrr] & h^{A_4} \arrow[dr] & h^{A_3} \arrow[d] \\
 & & & h^{A_5} \arrow[dl] \arrow[d] \\
 & h^{B_9} & h^{A_6} & h^{A_7}
\end{tikzcd}
$$

hence there exist more than one minimal h-vector.

Notice that in the poset $\mathcal{H}_7^{(2,1,3)}$ there are two minimal elements (so, there is no minimum in $\mathcal{H}_7^{(2,1,3)}$) and non of them is comparable to the minimum h-vector in $\mathcal{H}_7^{(2,2,3)}$.

5.4 h-vectors of degree matrices with equal columns

As we have already seen the existence of a minimum in a poset of h-vectors is a rare property. Therefore it is natural to ask, whether there are other posets, besides the one defined in Proposition 5.5, where the existence of both minimum and maximum h-vector is ensured. Inspired by Proposition 5.5, the natural guess for such a poset is the set of h-vectors of length s corresponding to degree matrices with equal columns.

In the following we will use the notation

$$Q^{(c)} := \bigcup_{t\geq 1} \big\{A \in \mathbb{Z}^{t\times(t+c-1)} \,|\, A \text{ is a degree matrix with equal columns}\big\},$$

$$P^{(c,s)} := \big\{A \in Q^{(c)} \,|\, \tau(h^A) = s\big\} \text{ and } \mathcal{G}^{(c,s)} := \big\{h^A \,|\, A \in P^{(c,s)}\big\}.$$

Remark 5.22. *By Lemma 3.11, if $A = [a_{i,j}] \in \mathbb{Z}^{t\times(t+c-1)}$ is a degree matrix, then the length of the corresponding h-vector is given by*

$$\tau(h^A) = a_{1,1} + \cdots + a_{1,c} + a_{2,c+1} + \cdots + a_{t,t+c-1} - c.$$

Therefore, in order to obtain all matrices A with equal columns corresponding to a h-vector of fixed length $s = \tau(h^A)$, one has to consider all partitions $(a_1, \ldots, a_c, a_{c+1}, \ldots, a_m)$ of $s + c$, such that

$$a = a_1 = \cdots = a_c \text{ and } a_{c+1} \geq \cdots \geq a_m.$$

Any matrix $A \in P^{(c,s)}$ is therefore of the form

$$A = \begin{bmatrix} a & \cdots & a & & \cdots & a \\ a_{c+1} & & & a_{c+1} & & \\ \vdots & & & & \ddots & \vdots \\ a_m & \cdots & & & \cdots & a_m \end{bmatrix} \in \mathbb{Z}^{m-(c-1)\times m}.$$

With Remark 5.22 in mind, the existence of a maximum in $\mathcal{G}^{(c,s)}$ is easily seen:

Lemma 5.23. *There exists a h-vector $h^{\max} \in \mathcal{G}^{(c,s)}$, such that $h \leq h^{\max}$ for all $h \in \mathcal{G}^{(c,s)}$.*

Proof. Let $B \in P^{(c,s)}$ be the degree matrix with all entries equal 1, corresponding to the partition $(1, \cdots, 1)$ of $s + c$. The same proof as in Lemma 5.2 shows that $h^A \leq h^B$, for all $A \in P^{(c,s)}$.

□

Remark 5.24. *Let $A, B \in P^{(c,s)}$ and assume that I_A and I_B are standard determinantal ideals with degree matrices A, respectively B. If $h^A \leq h^B$, then obviously $beg(I_A) \leq beg(I_B)$, i.e.*

$$beg(I_A) = a + a_{c+1} + \cdots + a_m \leq b + b_{c+1} + \cdots + b_k = beg(I_B),$$

where $(a_1, \ldots, a_c, a_{c+1}, \ldots, a_m)$ and $(b_1, \ldots, b_c, b_{c+1}, \ldots, b_k)$, are the partitions corresponding to A, respectively B, and $a = a_1 = \ldots = a_c$, $b = b_1 = \ldots = b_c$. This inequality implies in particular $s + c - (c-1)a \leq s + c - (c-1)b$, so that

$$a \geq b \text{ and } \underbrace{a_{c+1} + \cdots + a_m}_{q_a} \leq \underbrace{b_{c+1} + \cdots + b_k}_{q_b}.$$

This shows that if there exists a minimum h-vector in the poset $\mathcal{G}^{(c,s)}$, then it will correspond to a matrix $A \in P^{(c,s)}$ which is obtained from a partition of $s+c$ of the form $(a_1, \ldots, a_c, a_{c+1}, \ldots, a_m)$, where $a = a_1 = \cdots = a_c$ has the largest possible value and consequently $q_a = a_{c+1} + \cdots + a_m$ the smallest possible value.

We will prove in the following that there exists a minimum $h^{\min}$ in $\mathcal{G}^{(c,s)}$ and it holds that $h^{\min} = h^A$, where A is the matrix corresponding to a partition of the form

$$(a_1,\ldots,a_c,a_{c+1},\ldots,a_m)=(\underbrace{a,\ldots,a}_{c},\underbrace{1,\ldots,1}_{q_a}),$$

where a takes the maximal possible value such that $ca+q_a=s+c$. Notice that

$$q_a=0 \Longleftrightarrow (a-1)c=s.$$

In particular, if $q_a=0$, then h^A is the h-vector of a homogeneous codimension c complete intersection ideal generated in a single degree a, i.e. $h^A=h^{(a,\ldots,a)}$.

Fix integers c and s, and let a and q_a be defined as in Remark 5.24. We introduce the notation: for any $1\le b\le a$

$$P^{(c,s,b)}=\left\{A\in P^{(c,s)}|\,\text{all entries in the first row of A are equal to b}\right\}$$

and

$$\mathcal{G}^{(c,s,b)}=\left\{h^A|\,A\in P^{(c,s,b)}\right\}.$$

We can write therefore

$$P^{(c,s)}=P^{(c,s,1)}\cup\ldots\cup P^{(c,s,a)} \text{ and } \mathcal{G}^{(c,s)}=\mathcal{G}^{(c,s,1)}\cup\ldots\cup\mathcal{G}^{(c,s,a)}.$$

We will see in the following that each of those sets contains a minimum. In order to prove it we will use the following lemma.

Lemma 5.25. *The following inequalities for h-vectors are true:*

(1) *Let $A=\begin{bmatrix} a-1 & \cdots & a-1\\ 1 & \cdots & 1\end{bmatrix}\in\mathbb{Z}^{2\times(c+1)}$, $a\ge 2$ be a degree matrix and $h^{(a,\ldots,a)}$ the h-vector of a homogeneous codimension c complete intersection ideal generated in degree a. Then $h^A\le h^{(a,\ldots,a)}$.*

(2) *Let $A=\begin{bmatrix} a & \cdots & a\\ b & \cdots & b\end{bmatrix}\in\mathbb{Z}^{2\times(c+1)}$ and $B=\begin{bmatrix} a & \cdots & a\\ b-1 & \cdots & b-1\\ 1 & \cdots & 1\end{bmatrix}\in\mathbb{Z}^{3\times(c+2)}$ be two degree matrices, where $2\le b\le a$. Then $h^A\ge h^B$.*

(3) *For two degree matrices*

$$A=\begin{bmatrix} a_1 & \cdots & a_1\\ \vdots & & \vdots\\ a_t & \cdots & a_t\end{bmatrix} \text{ and } B=\begin{bmatrix} a_1 & \cdots & a_1\\ \vdots & & \vdots\\ a_t-1 & \cdots & a_t-1\\ 1 & \cdots & 1\end{bmatrix}$$

of size $t\times(t+c-1)$, respectively $(t+1)\times(t+c)$, where $a_t\ge 2$, we have $h^A\ge h^B$.

(4) *For two degree matrices*

$$A = \begin{bmatrix} a_1 & \cdots & a_1 \\ \vdots & & \vdots \\ a_i & \cdots & a_i \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_1 & \cdots & a_1 \\ \vdots & & \vdots \\ a_i - 1 & \cdots & a_i - 1 \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

of size $t \times (t+c-1)$, respectively $(t+1) \times (t+c)$, where $a_i \geq 2$, we have $h^A \geq h^B$.

Proof. (1) We will show the assertion by induction on c. For $c = 1$, we have $h^A = \underbrace{(1, \dots, 1)}_{a} = h^{(a)}$.

Let $c > 1$. By Remark 3.1 and the inductive hypothesis it follows:

$$\begin{aligned} h^A &= h_{i-1}^{(a-1,\dots,a-1)} + h_i^{A^{(0,c+1)}} \\ &\leq h_{i-1}^{(a-1,\dots,a-1)} + h_i^{(a,\dots,a)}, \end{aligned}$$

where the first h-vector involved in the sum is "of codimension c" and the second, "of codimension $c-1$". The entries of the h-vector of a codimension c homogeneous complete intersection ideal generated in degree a can be computed as follows:

$$\begin{aligned} h_i^{(a,\dots,a)} &= \sum_{k=0}^{a-1} h_{i-k}^{(a,\dots,a)} = h_i^{(a,\dots,a)} + \sum_{k=0}^{a-2} h_{(i-1)-k}^{(a,\dots,a)} \\ &= h_i^{(a,\dots,a)} + h_{i-1}^{(a-1,a,\dots,a)}. \end{aligned}$$

As by Lemma 2.6 the inequality $h_i^{(a-1,\dots,a-1)} \leq h_i^{(a-1,a,\dots,a)}$ is true for all i, the claim follows.

(2) We proceed by induction on c. For $c = 1$ the claim is trivial.

Let $c > 1$. Since by (1) we have $h^{A^{(1,1)}} \geq h^{B^{(1,1)}}$ and by the induction hypothesis $h^{A^{(0,1)}} \geq h^{B^{(0,1)}}$, we can conclude by Remark 3.1.

(3) We will prove the claim by induction on t and c. For $c = 1$ there is nothing to show and when $t = 2$ the claim follows from (2).

Let $c > 1$ and $t > 2$. Applying Remark 3.1 for a_1, as by induction it holds $h^{A^{(1,1)}} \geq h^{B^{(1,1)}}$ and $h^{A^{(0,1)}} \geq h^{B^{(0,1)}}$, we can conclude.

(4) We use induction on c and on the number k of rows, whose entries are all equal one. For $c = 1$ the claim is trivial and for $k = 1$, according to (3), we have $h^{A^{(t,t+c-1)}} \geq h^{B^{(t+1,t+c)}}$ and $h^{A^{(0,t+c-1)}} \geq h^{B^{(0,t+c)}}$. Thus by Remark 3.1 we can

conclude. Let $k, c > 1$, we have then by induction $h^{A^{(t,t+c-1)}} \geq h^{B^{(t+1,t+c)}}$ and $h^{A^{(0,t+c-1)}} \geq h^{B^{(0,t+c)}}$ and the claim follows again from Remark 3.1.

□

Applying the lemma we have just proved, allows us to show the following statement:

Proposition 5.26. *For any $1 \leq b \leq a$ let $B = [b_{i,j}] \in P^{(c,s,b)}$ be a degree matrix, such that all entries of B, except those in the first row, are equal to one. Then $h^B \leq h^{B'}$ for all $B' \in P^{(c,s,b)}$.*

Proof. Let $B' \in P^{(c,s,b)}$ and assume that the number of rows, whose entries are all equal to one is $k \geq 2$ and let $i = \max\{j | \, a_j > 1\}$. Applying Lemma 5.25 on B' will produce a degree matrix $B'' \in P^{(c,s,b)}$ with at least $k+1$ rows whose entries are all equal to one. Furthermore the entries in the i-th row of B'' are smaller by one than the entries in the i-th row of B'. As $h^{B''} \leq h^{B'}$, if B'' is equal to B we are done, otherwise we repeat the process with B'' instead of B'. Since each time the number of rows of ones increases and the entries in some row decrease, while the length s of the corresponding h-vectors remains unchanged, after finitely many steps we will obtain the matrix B.

□

Remark 5.27. *If $X \subseteq \mathbb{P}^n$ is a standard determinantal scheme of codimension $c \geq 3$ whose degree matrix has all entries equal to one except those in the first row, then by Theorem 2.2 its defining ideal I_X is componentwise linear.*

Corollary 5.28. *Let $1 \leq b \leq a$. There exists a h-vector $h^{\min} \in \mathcal{G}^{(c,s,b)}$, such that $h^{\min} \leq h$ for all $h \in \mathcal{G}^{(c,s,b)}$. Moreover if $B = [b_{i,j}] \in \mathcal{G}^{(c,s,b)}$ is defined by*

$$b_{i,j} = \begin{cases} b, & i = 1 \\ 1, & i \neq 1 \end{cases},$$

then

$$h^{\min} = \begin{cases} h^B, & a \neq b \text{ or } (a-1)c \neq s \\ h^{(a,\ldots,a)}, & a = b \text{ and } (a-1)c = s \end{cases}.$$

Proof. We distinguish the following cases:

Case 1: $a = b$ and $(a-1)c = s$.
We have then $q_a = 0$ and in particular $\mathcal{G}^{(c,s,a)} = \{[a, \ldots, a]\}$, so that the only minimal h-vector is given by $h^{\min} = h^{(a,\ldots,a)}$.

Case 2: $a = b$ and $(a-1)c \neq q_a$.
As $q_a > 0$ by Proposition 5.26 it holds that $h^{\min} = h^B \leq h^{B'}$ for all $B' \in \mathcal{G}^{(c,s,a)}$.

Case 3: $a \neq b$.
Since $a > b$ and $q_b > q_a \geq 0$ we have by Proposition 5.26 $h^{\min} = h^B \leq h^{B'}$ for all $B' \in \mathcal{G}^{(c,s,b)}$.

□

We show next, that the minimum h-vectors in the posets $\mathcal{G}^{(c,s,b)}, 1 \leq b \leq a$ are comparable.

Proposition 5.29. *Let h and h' be the minimum h-vectors of $\mathcal{G}^{(c,s,b)}$, respectively $\mathcal{G}^{(c,s,d)}$. If $b > d$, then $h \leq h'$.*

Before starting with the proof of Proposition 5.29 we need the following lemma:

Lemma 5.30. *Let* $B = \begin{bmatrix} b-1 & \cdots & b-1 \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{Z}^{(c+1)\times 2c}$, *$b \geq 3$ be a degree matrix and let $h^{(b,\dots,b)}$ be the h-vector of a homogeneous complete intersection ideal of codimension c generated in degree b.Then $h^{(b,\dots,b)} \leq h^B$.*

Proof. In order to avoid technicalities and to shorten the proof, we will show the claim for $c = 3$. It is easy to check, that the same proof applies also for $c > 3$.

Using Remark 3.1 repeatedly, starting from the lower right corner of the matrix, we can easily see, that h^B is computed via componentwise addition in the following way:

$$\begin{array}{c|c|c|c|ccc|}
 & 0 & 0 & 0 & & h^{(b-1,b-1,b-1)} & \\
 & 0 & 0 & 0 & & h^{(b-1,b-1)} & \\
 & 0 & 0 & 0 & & h^{(b-1,b-1)} & \\
 & 0 & 0 & 0 & & h^{(b-1,b-1)} & \\
+ & 0 & 0 & 1 & & \cdots & 1 \\
 & 0 & 0 & 1 & & \cdots & 1 \\
 & 0 & 0 & 1 & & \cdots & 1 \\
 & 0 & 1 & 1 & & \cdots & 1 \\
 & 0 & 1 & 1 & & \cdots & 1 \\
 & 1 & 1 & 1 & & \cdots & 1
\end{array} \qquad (1)$$

where in the first row (counted from bottom to the top) there are $(b-1)+3$ entries. Since $h^{(b,b,b)}$ is computed as the componentwise sum of b copies of $h^{(b,b)}$

$$\begin{array}{cccccccccc}
+ & & & & h_0 & & \cdots & \cdots & & h_s \\
 & & & \iddots & & & & & \iddots & \\
 & & h_0 & \cdots & \cdots & h_{s-2} & h_{s-1} & h_s & & \\
 & h_0 & h_1 & \cdots & \cdots & h_{s-1} & h_s & & & \\
h_0 & h_1 & h_2 & \cdots & \cdots & h_s & & & &
\end{array} \qquad (2)$$

and $h^{(b,b)}$ can be obtained in the same way as the componentwise sum of b copies of $h^{(b)} = (\underbrace{1,\dots,1}_{b})$, it can be easily seen, that the componentwise sum (2) is a part of the componentwise sum (1) and therefore $h^B \geq h^{(b,b,b)}$.

□

Remark 5.31. *The key point in Lemma 5.30 is that the number of rows of* 1*'s is sufficiently large. If this number is smaller than the codimension, then the statement in Lemma 5.30 is in general no longer true, as the following example shows:*

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix},\ h^A = (1,2,3,1),$$

$$B = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ h^B = (1,2,3,4,1)$$

Since $h^{(3,3)} = (1,2,3,2,1)$*, we obviously have* $h^A \leq h^{(3,3)} \leq h^B$*.*

We are now ready to prove Proposition 5.29.

Proof. Obviously it suffices to prove the assertion for $d = b-1$.

Let $B \in \mathbb{Z}^{t\times(t+c-1)}$ and $D \in \mathbb{Z}^{k\times(k+c-1)}$ be two degree matrices with equal columns, such that $h = h^A \in \mathcal{G}^{(c,s,b)}$ and $h' = h^D \in \mathcal{G}^{(c,s,b-1)}$. According to Lemma 3.11 and Corollary 5.28 it holds $k = t+c$. We will show by induction on t and c that $h^B \leq h^D$. When $c = 1$, we have $h^B = (\underbrace{1,\dots,1}_{s}) = h^D$ and there is nothing to show.

Let $c > 1$. As for $t = 1$ the claim follows from Lemma 5.30, let $t > 1$. Since by induction $h^{D^{(k,k+c-1)}} \geq h^{B^{(t,t+c-1)}}$ and $h^{D^{(0,k+c-1)}} \geq h^{B^{(0,t+c-1)}}$, applying Remark 3.1 on D and B for the indices $(k, k+c-1)$, respectively $(t, t+c-1)$ allows us to finish the proof.

$\square$

Corollary 5.32. *There exists a h-vector* $h^{\min} \in \mathcal{G}^{(c,s)}$ *such that* $h^{\min} \leq h$ *for all* $h \in \mathcal{G}^{(c,s)}$*. In particular* $h^{\min}$ *is the minimum h-vector in the poset* $\mathcal{G}^{(c,s,a)}$*.*

Proof. Since $\mathcal{G}^{(c,s)} = \mathcal{G}^{(c,s,1)} \cup \ldots \cup \mathcal{G}^{(c,s,a)}$, the claim follows directly from Corollary 5.28 and Proposition 5.29.

$\square$

Example 5.33. *For* $s = 7$ *and* $c = 3$ *we have* $a = 3$*, and* $q_a = 1$*. Therefore* $P^{(3,7)} = P^{(3,7,1)} \cup P^{(3,7,2)} \cup P^{(3,7,3)} = \{A_1\} \cup \{A_2, A_3, A_4\} \cup \{A_5\}$*, where*

$$A_1 = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{Z}^{8\times 10},\ A_5 = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The corresponding h-vectors are

$$\begin{aligned} h^{A_1} &= (1,3,6,10,15,21,28,36), \\ h^{A_2} &= (1,3,6,10,15,21,18,6), \\ h^{A_3} &= (1,3,6,10,15,21,13,3), \\ h^{A_4} &= (1,3,6,10,15,21,7,1), \\ h^{A_5} &= (1,3,6,10,9,7,3,1). \end{aligned}$$

The Hasse diagram for $\mathcal{G}^{(3,7)}$ *is then of the form:*

$$\begin{tikzcd} h^{A_1} \arrow[d] \\ h^{A_2} \arrow[d] \\ h^{A_3} \arrow[d] \\ h^{A_4} \arrow[d] \\ h^{A_5} \end{tikzcd}$$

In particular it holds that $h^{\max} = h^{A_1}$ *and* $h^{\min} = h^{A_5}$.

Remark 5.34. *The Hasse diagram in Example 5.33 shows that the poset* $\mathcal{G}^{(3,7)}$ *is totally ordered, i.e. any two h-vectors are comparable. Based on our computations with the computer algebra system CoCoA, we conjecture that this is always the case, i.e. the poset* $\mathcal{G}^{(c,s)}$ *is a totally ordered set for any two positive integers* c, s.

As we have seen in this final section, it appears that the poset $\mathcal{G}^{(c,s)}$ has a very interesting structure. It has a natural stratification $\mathcal{G}^{(c,s)} = \mathcal{G}^{(c,s,1)} \cup \ldots \cup \mathcal{G}^{(c,s,a)}$ and in each stratum there is a minimum h-vector. Furthermore all minima are comparable. Thus for given integers $c, s \geq 1$, the poset $\mathcal{G}^{(c,s)}$ provides a number of examples of non pure O-sequences, which are h-vectors of standard determinantal schemes with fixed socle degree and codimension, and which have the additional extremal property of being the minimum in the set of h-vectors corresponding to degree matrices with equal columns and fixed first row.

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Curriculum Vitae

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