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On the Metrics Induced by Icosidodecahedron and Rhombic Triacontahedron

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ABSTRACT

The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. If every points of a line segment that connects any two points of the set are in the set, then it is convex. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. In this study, we give two new metrics to be their spheres an archimedean solid icosidodecahedron and its archimedean dual rhombic triacontahedron.

Key words: Archimedean solids, Catalan solids, metric, Chinese Checkers metric, Icosidodecahedron, Rhombic triacontahedron

MSC2010: 51K05, 51K99, 51M20

O metrici induciranoj ikosadodekaedrom i trijakontaedrom

SAŽETAK

Teorija konveksnih skupova je vitalno i klasično područje moderne matematike s bogatom primjenom. Ako se sve točke dužine, koja spaja bilo koje dvije točke skupa, nalaze u tom skupu, tada je taj skup konveksan. Sve se više geometrijskih aspekata o konveksnim skupovima razvija uvodeći neke pojmove, ponajprije poliedre. Konveksni poliedar je iznimno važno posebno tijelo u $\mathbb{R}^n$. Neki primjeri konveksnih podskupova euklidskog trodimenzionalnog prostora su Platonova tijela, Arhimedova tijela, tijela dualna Arhimedovim tijelima i Catalanova tijela. U ovom članku prikazujemo dvije metrike koje su sfere Arhimedovom tijelu ikosadodekaedru i njemu dualnom tijelu, trijakontaedru.

Ključne riječi: Arhimedova tijela, Catalanova tijela, metrika kineskog šaha, ikosadodekaedar, trijakontaedar

1 Introduction

Some mathematicians studied on metrics and improved metric geometry (some of these are [2], [3], [6], [7], [8], [9]). Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The maximum metric $d_M : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ is defined by

$$d_M(P_1,P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}.$$

Taxicab metric $d_T : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ is defined by

$$d_T(P_1,P_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

Then E. Krause asked the question of how to develop a metric which would be similar to movement made by playing Chinese Checkers [11]. An answer was given by G. Chen for plane [1]. In [5], Ö. Gelişgen, R. Kaya and M. Özcan extended Chinese-Checkers metric to $3-$dimensional space. The $CC-$metric $d_{CC} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ is defined by

$$d_{CC}(P_1,P_2) = d_L(P_1,P_2) + (\sqrt{2} - 1) d_S(P_1,P_2)$$

where $d_L(P_1,P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$ and $d_S(P_1,P_2) = \min\{|x_1 - x_2| + |y_1 - y_2|, |x_1 - x_2| + |z_1 - z_2|, |y_1 - y_2| + |z_1 - z_2|\}$.
Each of geometries induced by these metrics is a Minkowski geometry. Minkowski geometry is a non-euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). In a Minkowski geometry, the linear structure is just like the

Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set [13].

A polyhedron is a solid in three dimensions with flat faces, straight edges and vertices. A regular polyhedron is a polyhedron with congruent faces and identical vertices. There are only five regular convex polyhedra which are called platonic solids. Archimedes discovered the semiregular convex solids. However, several centuries passed before their rediscovery by the renaissance mathematicians. Finally, Kepler completed the work in 1620 by introducing prisms and antiprisms as well as four regular nonconvex polyhedra, now known as the Kepler–Poinsot polyhedra. Construction of the dual solids of the Archimedean solids was completed in 1865 by Catalan nearly two centuries after Kepler (see [10]). A convex polyhedron is said to be semiregular if its faces have a similar configuration of nonintersecting regular plane convex polygons of two or more different types about each vertex. These solids are commonly called the Archimedean solids. The duals are known as the Catalan solids. The Catalan solids are all convex. They are face-transitive when all its faces are the same but not vertex-transitive. Unlike Platonic solids and Archimedean solids, the face of Catalan solids are not regular polygons.

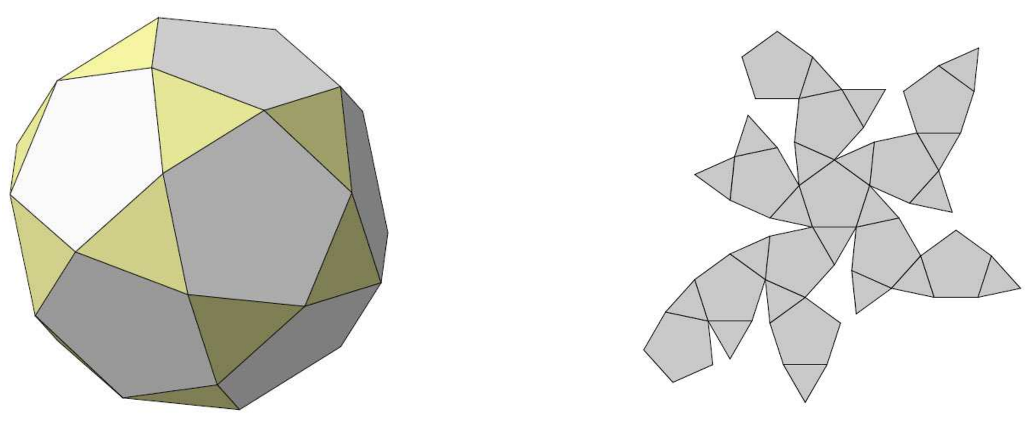
According to studies of mentioned researches unit spheres of Minkowski geometries which are furnished by these metrics are associated with convex solids. For example, unit spheres of maximum space and taxicab space are cubes and octahedrons, respectively, which are Platonic Solids [4], [6]. And unit sphere of CC-space is a deltoidal icositetrahedron which is a Catalan solid [5]. Therefore, there are some metrics in which unit spheres of space furnished by them are convex polyhedra. That is, convex polyhedra are associated with some metrics. When a metric is given we can find its unit sphere. On the contrary a question can be asked; “Is it possible to find the metric when a convex polyhedron is given?”. In this study we find the metrics of which unit spheres are an icosidodecahedron, one of the Archimedean Solids and a rhombic triacontahedron which is Archimedean dual (a catalan solid) of icosidodecahedron.

2 Icosidodecahedron Metric

One type of convex polyhedrons is the Archimedean solids. The fifth book of the “Synagoge” or “Collection” of the Greek mathematician Pappus of Alexandria, who lived in the beginning of the fourth century AD, gives the first known mention of the thirteen “Archimedean solids”. Although, Archimedes makes no mention of these solids in any of his extant works, Pappus lists this solids and attributes to Archimedes in his book [16].

An Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other.

One of the Archimedean solids is the icosidodecahedron. An icosidodecahedron is a polyhedron which has 32 faces, 60 edges and 30 vertices. Twelve of its faces are regular pentagons and twenty of them are equilateral triangles [14].



(a) (b)

Figure 1: (a) *Icosidodecahedron*, (b) *Net of icosidodecahedron*

To find the metrics of which unit spheres are convex polyhedrons, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface. Therefore we describe the metric which unit sphere is an icosidodecahedron as following:

Definition 1 *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$.*
The distance function $d_{ID} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ icosidodecahedron distance between P_1 and P_2 is defined by

$$d_{ID}(P_1,P_2) = \max \left\{ \begin{array}{l} u + (\varphi - 1)\max\{v, (\varphi-1)w, (1-\varphi)u + v + w\}, \\ v + (\varphi - 1)\max\{w, (\varphi-1)u, u + (1-\varphi)v + w\}, \\ w + (\varphi - 1)\max\{u, (\varphi-1)v, u + v + (1-\varphi)w\} \end{array} \right\}$$

where $u = |x_1 - x_2|$, $v = |y_1 - y_2|$, $w = |z_1 - z_2|$ and $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

According to icosidodecahedron distance, there are three different paths from P_1 to P_2. These paths are
i) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{\sqrt{5}}{2})$ angle with another coordinate axis.
ii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{1}{2})$ angle with another coordinate axis.
iii) union of three line segments each of which is parallel to a coordinate axis.

Thus icosidodecahedron distance between P_1 and P_2 is the sum of Euclidean lengths of these two line segments or $\frac{\sqrt{5}-1}{2}$ times the sum of Euclidean lengths of these three line segments.
Figure 2 illustrates icosidodecahedron way from P_1 to P_2 if maximum value is $|x_1 - x_2| + (\frac{\sqrt{5}-1}{2})|y_1 - y_2|$, $|x_1 - x_2| + (\frac{\sqrt{5}-1}{2})^2|z_1 - z_2|$ or $(\frac{\sqrt{5}-1}{2})(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)$.

Lemma 1 *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. u, v, w denote $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$, respectively. Then*

$$d_{ID}(P_1,P_2) \geq u + (\varphi-1)\max\{v, (\varphi-1)w, (1-\varphi)u + v + w\},$$
$$d_{ID}(P_1,P_2) \geq v + (\varphi-1)\max\{w, (\varphi-1)u, u + (1-\varphi)v + w\},$$
$$d_{ID}(P_1,P_2) \geq w + (\varphi-1)\max\{u, (\varphi-1)v, u + v + (1-\varphi)w\}.$$

Proof. Proof is trivial by the definition of maximum function. □

Theorem 1 *The distance function d_{ID} is a metric. Also according to d_{ID}, the unit sphere is an icosidodecahedron in $\mathbb{R}^3$.*

Proof. Let $d_{ID} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ be the icosidodecahedron distance function and $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ and $P_3 = (x_3, y_3, z_3)$ are distinct three points in $\mathbb{R}^3$. u, v, w denote $|x_1 - x_2|$, $|y_1 - y_2|$, $|z_1 - z_2|$, respectively. To show that d_{ID} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

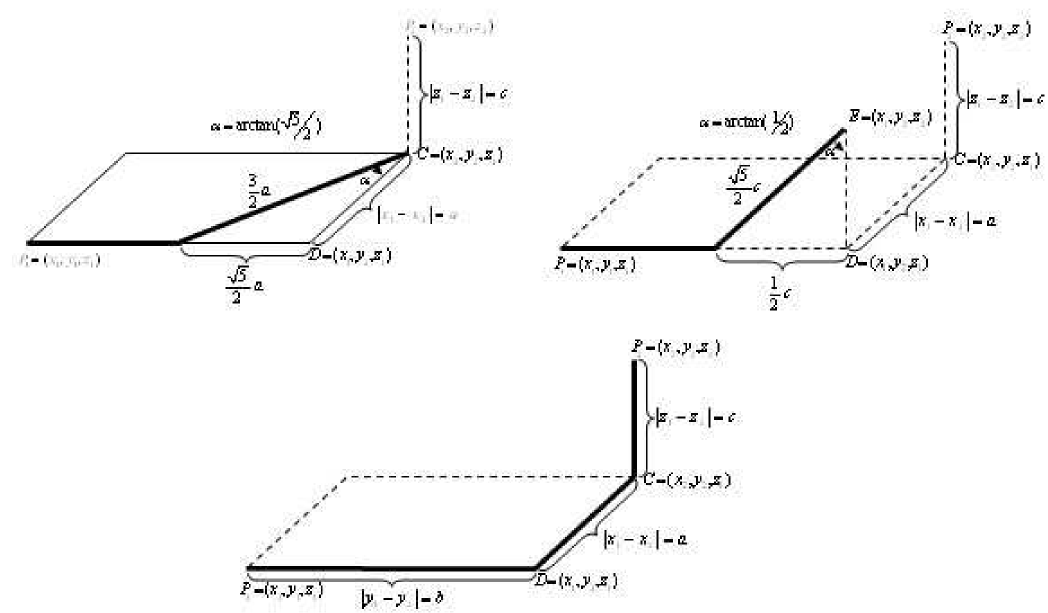


Figure 2: *ID way from P_1 to P_2*

M1) $d_{ID}(P_1,P_2)\geq 0$ and $d_{ID}(P_1,P_2)=0$ iff $P_1=P_2$
M2) $d_{ID}(P_1,P_2)=d_{ID}(P_2,P_1)$
M3) $d_{ID}(P_1,P_3)\leq d_{ID}(P_1,P_2)+d_{ID}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{ID}(P_1,P_2)\geq 0$. If $d_{ID}(P_1,P_2)=0$ then there are possible three cases. These cases are

Case I:
$d_{ID}(P_1,P_2)=u+(\varphi-1)\max\{v,(\varphi-1)w,(1-\varphi)u+v+w\}$

Case II:
$d_{ID}(P_1,P_2)=v+(\varphi-1)\max\{w,(\varphi-1)u,u+(1-\varphi)v+w\}$

Case III:
$d_{ID}(P_1,P_2)=w+(\varphi-1)\max\{u,(\varphi-1)v,u+v+(1-\varphi)w\}$.

Case I: If $d_{ID}(P_1,P_2)=u+(\varphi-1)\max\{v,(\varphi-1)w,(1-\varphi)u+v+w\}$, then

$$\begin{aligned}
&u+(\varphi-1)\max\{v,(\varphi-1)w,(1-\varphi)u+v+w\}=0\\
&\Leftrightarrow u=0 \text{ and } (\varphi-1)\max\{v,(\varphi-1)w,(1-\varphi)u+v+w\}=0\\
&\Leftrightarrow x_1=x_2,\ y_1=y_2,\ z_1=z_2\\
&\Leftrightarrow (x_1,y_1,z_1)=(x_2,y_2,z_2)\\
&\Leftrightarrow P_1=P_2
\end{aligned}$$

The other cases can be shown by similar way in Case I. Thus we get $d_{ID}(P_1,P_2)=0$ iff $P_1=P_2$.

Since $|x_1-x_2|=|x_2-x_1|$, $|y_1-y_2|=|y_2-y_1|$ and $|z_1-z_2|=|z_2-z_1|$, obviously $d_{ID}(P_1,P_2)=d_{ID}(P_2,P_1)$. That is, d_{ID} is symmetric.

Let $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ are distinct three points in $\mathbb{R}^3$. u_1, v_1, w_1, u_2, v_2, w_2 denote $|x_1-x_3|$, $|y_1-y_3|$, $|z_1-z_3|$, $|x_2-x_3|$, $|y_2-y_3|$, $|z_2-z_3|$, respectively.

Then by using the property $|a-b+b-c|\leq|a-b|+|b-c|$ we get

$$\begin{aligned}
&d_{ID}(P_1,P_3)\\
&=\max\left\{\begin{array}{l}u_1+(\varphi-1)\max\{v_1,\ (\varphi-1)w_1,(1-\varphi)u_1+v_1+w_1\},\\ v_1+(\varphi-1)\max\{w_1,\ (\varphi-1)u_1,u_1+(1-\varphi)v_1+w_1\},\\ w_1+(\varphi-1)\max\{u_1,\ (\varphi-1)v_1,u_1+v_1+(1-\varphi)w_1\}\end{array}\right\}\\
&\leq\max\left\{\begin{array}{l}u_1+u_2+(\varphi-1)\max\left\{\begin{array}{l}v_1+v_2,\ (\varphi-1)(w_1+w_2),\\(1-\varphi)(u_1+u_2)\\ \ +v_1+v_2+w_1+w_2\end{array}\right\},\\ v_1+v_2+(\varphi-1)\max\left\{\begin{array}{l}w_1+w_2,\ (\varphi-1)(u_1+u_2),\\u_1+u_2+(1-\varphi)(v_1+v_2)\\ \ +w_1+w_2\end{array}\right\},\\ w_1+w_2+(\varphi-1)\max\left\{\begin{array}{l}u_1+u_2,\ (\varphi-1)(v_1+v_2),\\u_1+u_2+v_1+v_2\\ \ +(1-\varphi)(w_1+w_2)\end{array}\right\}\end{array}\right\}\\
&=I.
\end{aligned}$$

Therefore one can easily find that $I\leq d_{ID}(P_1,P_2)+d_{ID}(P_2,P_3)$ from Lemma 1. So, $d_{ID}(P_1,P_3)\leq d_{ID}(P_1,P_2)+d_{ID}(P_2,P_3)$. Consequently, icosidodecahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that icosidodecahedron distance is 1 from $O=(0,0,0)$ is

$$S_{ID}=\left\{(x,y,z):\max\left\{\begin{array}{l}|x|+(\varphi\text{-}1)\max\left\{\begin{array}{l}|y|,(\varphi\text{-}1)|z|,\\(1\text{-}\varphi)|x|+|y|+|z|\end{array}\right\},\\ |y|+(\varphi\text{-}1)\max\left\{\begin{array}{l}|z|,(\varphi\text{-}1)|x|,\\|x|+(1\text{-}\varphi)|y|+|z|\end{array}\right\},\\ |z|+(\varphi\text{-}1)\max\left\{\begin{array}{l}|x|,(\varphi\text{-}1)|y|,\\|x|+|y|+(1\text{-}\varphi)|z|\end{array}\right\}\end{array}\right\}=1\right\}.$$

Thus the graph of S_{ID} is as in Figure 3. □

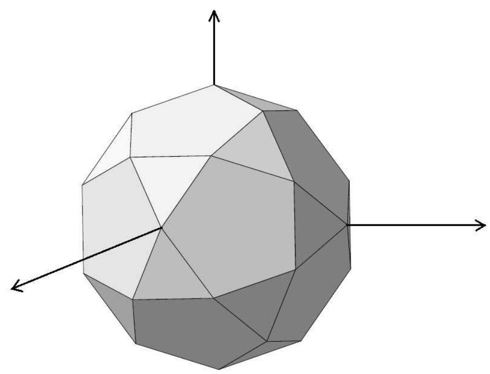


Figure 3: *Icosidodecahedron*

Corollary 1 *The equation of the icosidodecahedron with center (x_0,y_0,z_0) and radius r is*

$$\max\left\{\begin{array}{l}|x-x_0|+(\varphi-1)\max\left\{\begin{array}{l}|y-y_0|,(\varphi-1)|z-z_0|,\\(1-\varphi)|x-x_0|\\+|y-y_0|+|z-z_0|\end{array}\right\},\\ |y-y_0|+(\varphi-1)\max\left\{\begin{array}{l}|z-z_0|,(\varphi-1)|x-x_0|,\\(1-\varphi)|y-y_0|\\+|x-x_0|+|z-z_0|\end{array}\right\},\\ |z-z_0|+(\varphi-1)\max\left\{\begin{array}{l}|x-x_0|,(\varphi-1)|y-y_0|,\\(1-\varphi)|z-z_0|\\+|x-x_0|+|y-y_0|\end{array}\right\}\end{array}\right\}=r$$

which is a polyhedron which has 32 faces with vertices; such that all permutations of the three axis components and all posible +/- sign changes of each axis component of $(0,0,r)$, and $(\frac{\varphi-1}{2}r,\frac{1}{2}r,\frac{\varphi}{2}r)$, where $\varphi=\frac{1+\sqrt{5}}{2}$ the golden ratio.

Lemma 2 *Let l be the line through the points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector(p,q,r), then*

$$d_{ID}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where $\mu(P_1P_2)=$

$$\frac{\max\left\{\begin{array}{l}|p|+(\varphi-1)\max\{|q|,(\varphi-1)|r|,(1-\varphi)|p|+|q|+|r|\},\\|q|+(\varphi-1)\max\{|r|,(\varphi-1)|p|,|p|+(1-\varphi)|q|+|r|\},\\|r|+(\varphi-1)\max\{|p|,(\varphi-1)|q|,|p|+|q|+(1-\varphi)|r|\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $r \in \mathbb{R}$. Thus, $d_{ID}(P_1,P_2)$ is equal to

$$|\lambda| \left(\max \left\{ \begin{array}{l} |p| + (\varphi - 1)\max \left\{ \begin{array}{l} |q|, (\varphi-1)|r|, \\ (1-\varphi)|p| + |q| + |r| \end{array} \right\}, \\ |q| + (\varphi - 1)\max \left\{ \begin{array}{l} |r|, (\varphi-1)|p|, \\ |p| + (1-\varphi)|q| + |r| \end{array} \right\}, \\ |r| + (\varphi - 1)\max \left\{ \begin{array}{l} |p|, (\varphi-1)|q|, \\ |p| + |q| + (1-\varphi)|r| \end{array} \right\} \end{array} \right\} \right)$$

and $d_E(A,B) = |\lambda| \sqrt{p^2+q^2+r^2}$ which implies the required result. □

The lemma above says that $d_{ID}-$distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries.

Corollary 2 *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1,X) = d_E(P_2,X)$ if and only if $d_{ID}(P_1,X) = d_{ID}(P_2,X)$.*

Corollary 3 *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{ID}(X,P_1) \ / \ d_{ID}(X,P_2) = d_E(X,P_1) \ / \ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{ID}-$distances along a line are the same.

3 Rhombic Triacontahedron Metric

The duals of thirteen Archimedean solids are known as Catalan solids. Unlike Platonic and Archimedean solids, faces of Catalan solids are not regular polygons. Rhombic triacontahedron is one of the Catalan solids with 30 faces, 32 vertices and 60 edges. Its faces are rhombuses. The ratio of the long diagonal to the short diagonal of each face is exactly equal to $\varphi = \frac{1+\sqrt{5}}{2}$, which is the golden ratio [15].

Definition 2 *Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{RT} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ Rhombic triacontahedron distance between P_1 and P_2 is defined by $d_{RT}(P_1,P_2) =$*

$$\frac{\varphi}{2}\max \left\{ \begin{array}{l} |x_1-x_2| + (2\varphi-3)\max \left\{ \begin{array}{l} (\varphi+1)|z_1-z_2| + \varphi|y_1-y_2|, \\ |x_1-x_2| \end{array} \right\}, \\ |y_1-y_2| + (2\varphi-3)\max \left\{ \begin{array}{l} (\varphi+1)|x_1-x_2| + \varphi|z_1-z_2|, \\ |y_1-y_2| \end{array} \right\}, \\ |z_1-z_2| + (2\varphi-3)\max \left\{ \begin{array}{l} (\varphi+1)|y_1-y_2| + \varphi|x_1-x_2|, \\ |z_1-z_2| \end{array} \right\} \end{array} \right\}$$

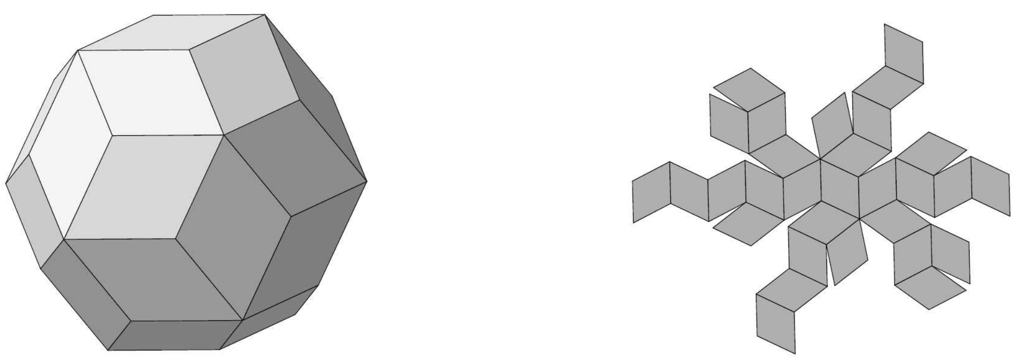
where $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio.

According to the rhombic triacontahedron distance, there are two types path from P_1 to P_2. These paths are:

i) union of three line segments which one is parallel to a coordinate axis and other line segments are made $\arctan(\frac{1}{2})$ and $\arctan(\frac{\sqrt{5}}{2})$ angle with other coordinate axes.

ii) a line segment which is parallel to a coordinate axis.

Thus rhombic triacontahedron distance between P_1 and P_2 is the Euclidean length of line segment which is parallel to a coordinate axis or $\frac{\sqrt{5}+1}{4}$ times the sum of Euclidean lengths of three line segments. Figure 5 shows that the path between P_1 and P_2 in case of the maximum is $|y_1-y_2| + \frac{3-\sqrt{5}}{2}|z_1-z_2| + \frac{\sqrt{5}-1}{2}|x_1-x_2|$.



(a) (b)

Figure 4: (a) *Rhombic triacontahedron*, (b) *Net of rhombic triacontahedron*

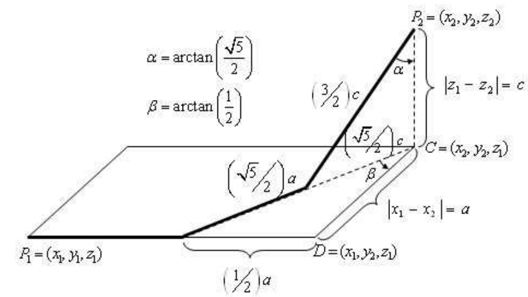


Figure 5: *RT way from P_1 to P_2 in the case $|y_1-y_2| \geq |x_1-x_2| \geq |z_1-z_2|$*

Lemma 3 *Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. Then*

$$d_{RT}(P_1,P_2) \geq \frac{\varphi}{2}\left(|x_1-x_2|+(2\varphi-3)\max\{(\varphi+1)|z_1-z_2|+\varphi|y_1-y_2|,|x_1-x_2|\}\right)$$
$$d_{RT}(P_1,P_2) \geq \frac{\varphi}{2}\left(|y_1-y_2|+(2\varphi-3)\max\{(\varphi+1)|x_1-x_2|+\varphi|z_1-z_2|,|y_1-y_2|\}\right)$$
$$d_{RT}(P_1,P_2) \geq \frac{\varphi}{2}\left(|z_1-z_2|+(2\varphi-3)\max\{(\varphi+1)|y_1-y_2|+\varphi|x_1-x_2|,|z_1-z_2|\}\right).$$

where $\varphi=\frac{1+\sqrt{5}}{2}$.

Proof. Proof is trivial by the definition of maximum function. □

Theorem 2 *The distance function d_{RT} is a metric. Also according to d_{RT}, unit sphere is a rhombic triacontahedron in $\mathbb{R}^3$.*

Proof. Let $d_{RT}:\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ be the rhombic triacontahedron distance function and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ are distinct three points in $\mathbb{R}^3$. To show that d_{RT} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3\in\mathbb{R}^3$.
M1) $d_{RT}(P_1,P_2)\geq 0$ and $d_{RT}(P_1,P_2)=0$ iff $P_1=P_2$
M2) $d_{RT}(P_1,P_2)=d_{RT}(P_2,P_1)$
M3) $d_{RT}(P_1,P_3)\leq d_{RT}(P_1,P_2)+d_{RT}(P_2,P_3)$.
One can easily show that the rhombic triacontahedron distance function satisfies above axioms by similar way in Theorem 1.
Consequently, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that rhombic triacontahedron distance is 1 from $O=(0,0,0)$ is $S_{RT}=$

$$\left\{(x,y,z):\frac{\varphi}{2}\max\left\{\begin{array}{l}|x|+(2\varphi-3)\max\{(\varphi+1)|z|+\varphi|y|,|x|\},\\|y|+(2\varphi-3)\max\{(\varphi+1)|x|+\varphi|z|,|y|\},\\|z|+(2\varphi-3)\max\{(\varphi+1)|y|+\varphi|x|,|z|\}\end{array}\right\}=1\right\}.$$

Thus the graph of S_{RT} is as in Figure 6. □

Corollary 4 *The equation of the rhombic triacontahedron with center (x_0,y_0,z_0) and radius r is*

$$\frac{\varphi}{2}\max\left\{\begin{array}{l}|x-x_0|+(2\varphi-3)\max\left\{\begin{array}{l}(\varphi+1)|z-z_0|+\varphi|y-y_0|,\\|x-x_0|\end{array}\right\},\\|y-y_0|+(2\varphi-3)\max\left\{\begin{array}{l}(\varphi+1)|x-x_0|+\varphi|z-z_0|,\\|y-y_0|\end{array}\right\},\\|z-z_0|+(2\varphi-3)\max\left\{\begin{array}{l}(\varphi+1)|y-y_0|+\varphi|x-x_0|,\\|z-z_0|\end{array}\right\}\end{array}\right\}$$
$$=r.$$

which is a polyhedron which has 30 faces with vertices; such that all permutations of the three axis components and all posible +/- sign changes of each axis component of $(\mu r,0,r)$, $(0,\delta r,r)$ and $(\mu r,\mu r,\mu r)$, where $\mu=\frac{\sqrt{5}-1}{2}$ and $\delta=\frac{3-\sqrt{5}}{2}$.

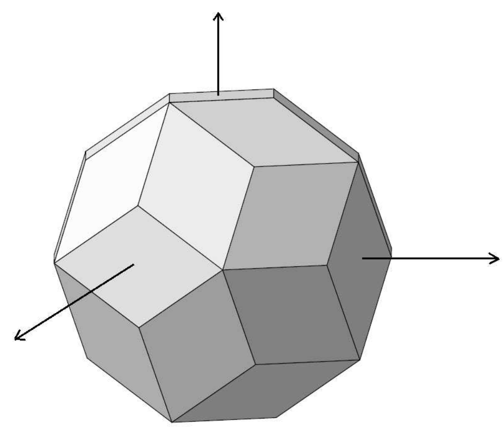


Figure 6: *Rhombic Triacontahedron*

Lemma 4 *Let l be the line through the points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p,q,r), then*

$$d_{RT}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\frac{\varphi}{2}\max\left\{\begin{array}{l}|p|+(2\varphi-3)\max\{(\varphi+1)|r|+\varphi|q|,|p|\},\\|q|+(2\varphi-3)\max\{(\varphi+1)|p|+\varphi|r|,|q|\},\\|r|+(2\varphi-3)\max\{(\varphi+1)|q|+\varphi|p|,|r|\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $r\in\mathbb{R}$. Thus,
$d_{RT}(P_1,P_2)=$

$$|\lambda|\left(\frac{\varphi}{2}\max\left\{\begin{array}{l}|p|+(2\varphi-3)\max\{(\varphi+1)|r|+\varphi|q|,|p|\},\\|q|+(2\varphi-3)\max\{(\varphi+1)|p|+\varphi|r|,|q|\},\\|r|+(2\varphi-3)\max\{(\varphi+1)|q|+\varphi|p|,|r|\}\end{array}\right\}\right)$$

and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result. □

The previous lemma says that $d_{RT}-$distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 5 *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1,X)=d_E(P_2,X)$ if and only if $d_{RT}(P_1,X)=d_{RT}(P_2,X)$.*

Corollary 6 *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{RT}(X,P_1)\ /\ d_{RT}(X,P_2)=d_E(X,P_1)\ /\ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{RT}-$distances along a line are the same.

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