MATHEMATICAL COMMUNICATIONS
Math. Commun. **20**(2015), 241–251

Generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials

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Received August 26, 2015; accepted October 22, 2015

Abstract. In this paper, generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials are introduced. The Binet form and some of their recursive features are given. Various families of multilinear and multilateral generating functions for these polynomials are derived. Furthermore, some special cases of the results are presented in this study.
AMS subject classifications: 11C08, 11B39, 33C45
Key words: generalized Vieta-Jacobsthal, generalized Vieta-Jacobsthal-Lucas polynomials, Vieta polynomials, multilinear and multilateral generating functions

1. Introduction

In [4], Horadam introduced a sequence of polynomials $\{J_n(x)\}$, Jacobsthal polynomials and $\{j_n(x)\}$, Jacobsthal-Lucas polynomials, defined by

$$\begin{aligned} J_{n+2}(x) &= J_{n+1}(x) + 2xJ_n(x), \\ j_{n+2}(x) &= j_{n+1}(x) + 2xj_n(x), \end{aligned}$$

respectively, where $J_0(x) = 0, J_1(x) = 1$ and $j_0(x) = 2,\ j_1(x) = 1$.

In [5], Horadam examined many recursive properties of these polynomials and defined augmented Jacobsthal representation polynomials and augmented Jacobsthal-Lucas representation polynomials. In [8], Shannon and Horadam gave some important relationships between Jacobsthal, Morgan-Voyce and Vieta polynomials. In [6], Horadam defined Vieta-Fibonacci and Vieta-Lucas polynomials by

$$\begin{aligned} V_n(x) &= xV_{n-1}(x) - V_{n-2}(x), \quad n \geq 2 \\ v_n(x) &= xv_{n-1}(x) - v_{n-2}(x), \quad n \geq 2, \end{aligned}$$

respectively, where $V_0(x) = 0$, $V_1(x) = 1$ and $v_0(x) = 2$, $v_1(x) = x$. Vieta-Pell and Vieta-Pell-Lucas polynomials are studied by Taşcı and Yalçın in [9]. The authors defined these polynomials for $|x| > 1$ by

$$\begin{aligned} t_n(x) &= 2xt_{n-1}(x) - t_{n-2}(x), \quad n \geq 2, \\ s_n(x) &= 2xs_{n-1}(x) - s_{n-2}(x), \quad n \geq 2, \end{aligned}$$

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respectively, with initial conditions $t_0(x) = 0$, $t_1(x) = 1$ and $s_0(x) = 2$, $s_1(x) = 2x$. Moreover, some recursive features for these polynomials are given in [9].

In this paper, we introduce Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials and a generalization of them. Exclusively, we deal with generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials. We present the Binet form and related properties. Furthermore, we also obtain various families of multilateral and multilinear generating functions and give their special cases for these polynomials.

2. Generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials

We can define Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials by

$$G_n(x) = G_{n-1}(x) - 2xG_{n-2}(x),$$
$$g_n(x) = g_{n-1}(x) - 2xg_{n-2}(x),$$

respectively, where $G_0(x) = 0$, $G_1(x) = 1$ and $g_0(x) = 2$, $g_1(x) = 1$.

Note that $G_n\left(-\frac{1}{2}\right) = F_n$ and $g_n\left(-\frac{1}{2}\right) = L_n$, where F_n and L_n are the nth Fibonacci and Lucas number, respectively. Let k be a nonnegative integer and assume that $1 - 2^{k+2}x \neq 0$. We consider generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials by the following recurrence relations

$$G_{k,n}(x) = G_{k,n-1}(x) - 2^k xG_{k,n-2}(x), \tag{1}$$
$$g_{k,n}(x) = g_{k,n-1}(x) - 2^k xg_{k,n-2}(x), \tag{2}$$

respectively, where $G_{k,0}(x) = 0$, $G_{k,1}(x) = 1$ and $g_{k,0}(x) = 2$, $g_{k,1}(x) = 1$.

If we set $k = 1$ in (1) and (2), respectively, we have $G_{1,n}(x) = G_n(x)$ and $g_{1,n}(x) = g_n(x)$.

The first few terms of the polynomials $G_{k,n}(x)$ and $g_{k,n}(x)$ are tabulated below.

n	$G_{k,n}(x)$	$g_{k,n}(x)$
0	0	2
1	1	1
2	1	$1 - 2^{k+1}x$
3	$1 - 2^k x$	$1 - 3.2^k x$
4	$1 - 2^{k+1}x$	$1 - 2^{k+2}x + 2^{2k+1}x^2$
5	$1 - 3.2^k x + 2^{2k}x^2$	$1 - 5.2^k x + 5.2^{2k}x^2$
6	$1 - 2^{k+2}x + 3.2^{2k}x^2$	$1 - 6.2^k x + 9.2^{2k}x^2 - 2^{3k+1}x^3$
7	$1 - 5.2^k x + 6.2^{2k}x^2 - 2^{3k}x^3$	$1 - 7.2^k x + 14.2^{2k}x^2 - 7.2^{3k}x^3$

Table 1: *The first few terms of $G_{k,n}(x)$ and $g_{k,n}(x)$*

The main recursive properties of the generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials are determined in this section. Initially, we can begin with the characteristic equation, that is,

$$\lambda^2 - \lambda + 2^k x = 0,$$

with roots

$$\alpha = \frac{1+\sqrt{1-2^{k+2}x}}{2},$$
$$\beta = \frac{1-\sqrt{1-2^{k+2}x}}{2},$$

so that

$$\begin{aligned} \alpha+\beta &= 1, \\ \alpha\beta &= 2^k x, \\ \alpha-\beta &= \Delta = \sqrt{1-2^{k+2}x}. \end{aligned} \tag{3}$$

The Binet forms of $G_{k,n}(x)$ and $g_{k,n}(x)$ are as follows:

$$G_{k,n}(x) = \frac{\alpha^n - \beta^n}{\Delta}, \tag{4}$$
$$g_{k,n}(x) = \alpha^n + \beta^n. \tag{5}$$

Moreover, we have

$$\alpha^2 + 2^k x = \alpha,$$
$$\beta^2 + 2^k x = \beta.$$

Explicit formula

The polynomials $G_{k,n}(x)$ and $g_{k,n}(x)$ can be stated explicitly by

$$G_{k,n}(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} \left(-2^k x\right)^j, \tag{6}$$
$$g_{k,n}(x) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-j} \binom{n-j}{j} \left(-2^k x\right)^j. \tag{7}$$

Generating function

The generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials are defined by the following generating functions

$$\sum_{i=0}^{\infty} G_{k,i+1}(x)\, y^i = \left(1 - y + 2^k x y^2\right)^{-1}, \tag{8}$$
$$\sum_{i=0}^{\infty} g_{k,i+1}(x)\, y^i = \left(1 - 2^{k+1} x y\right)\left(1 - y + 2^k x y^2\right)^{-1}. \tag{9}$$

Negative subscript

$$G_{k,-n}(x) = -\left(2^k x\right)^{-n} G_{k,n}(x),$$
$$g_{k,-n}(x) = \left(2^k x\right)^{-n} g_{k,n}(x),$$

Simson's formulas

$$G_{k,n+1}(x)\, G_{k,n-1}(x) - G_{k,n}^2(x) = -\left(2^k x\right)^{n-1},$$
$$g_{k,n+1}(x)\, g_{k,n-1}(x) - g_{k,n}^2(x) = \left(2^k x\right)^{n-1} \Delta^2.$$

Summation formulas

$$\sum_{j=1}^{n} G_{k,j}(x) = \frac{1 - G_{k,n+2}(x)}{2^k x}, \tag{10}$$
$$\sum_{j=1}^{n} g_{k,j}(x) = \frac{1 - g_{k,n+2}(x)}{2^k x}. \tag{11}$$

Proof. We will prove (10) by using the Binet form of $G_{k,n}(x)$ in (4) and (3).

$$\begin{aligned}
\sum_{j=1}^{n} G_{k,j}(x) &= \sum_{j=1}^{n} \frac{1}{\Delta}\left(\alpha^j - \beta^j\right) \\
&= \frac{1}{\Delta}\left[\frac{\alpha(1-\alpha^n)}{1-\alpha} - \frac{\beta(1-\beta^n)}{1-\beta}\right] \\
&= \frac{1}{\Delta}\left[\frac{(\alpha - \alpha^{n+1})(1-\beta) - (\beta - \beta^{n+1})(1-\alpha)}{1-(\alpha+\beta)+\alpha\beta}\right] \\
&= \frac{1}{\Delta}\left[\frac{(\alpha-\beta) - \left(\alpha^{n+1} - \beta^{n+1}\right) + \alpha\beta\left(\alpha^n - \beta^n\right)}{2^k x}\right] \\
&= \frac{1}{\Delta}\left(\frac{\Delta - \Delta G_{k,n+1}(x) + 2^k x \Delta G_{k,n}(x)}{2^k x}\right) \\
&= \frac{1 - \left[G_{k,n+1}(x) - 2^k x G_{k,n}(x)\right]}{2^k x}
\end{aligned}$$

By using (1), the proof is complete.

As the proof of (11) is similar, we omit it. ∎

Some interrelationships

The following relationships between $G_{k,n}(x)$ and $g_{k,n}(x)$ can be given as

$$\begin{aligned} G_{k,n+1}(x) - 2^k x G_{k,n-1}(x) &= g_{k,n}(x), \qquad (12)\\ g_{k,n+1}(x) - 2^k x g_{k,n-1}(x) &= \Delta^2 G_{k,n}(x), \\ G_{k,n}(x) + g_{k,n}(x) &= 2G_{k,n+1}(x), \\ \Delta^2 G_{k,n}(x) + g_{k,n}(x) &= 2g_{k,n+1}(x), \\ G_{k,n}(x)\, g_{k,n}(x) &= G_{k,2n}(x), \\ \Delta G_{k,n}(x) + g_{k,n}(x) &= 2\alpha^n, \\ \Delta G_{k,n}(x) - g_{k,n}(x) &= -2\beta^n, \\ G_{k,m}(x)\, g_{k,n}(x) + G_{k,n}(x)\, g_{k,m}(x) &= 2G_{k,m+n}(x), \\ g_{k,m}(x)\, g_{k,n}(x) + \Delta^2 G_{k,m}(x)\, G_{k,n}(x) &= 2g_{k,m+n}(x). \end{aligned}$$

Proof. The properties given above can be seen by using the Binet form and suitable operations. So, we only prove (12). From (4), we have

$$\begin{aligned} G_{k,n+1}(x) - 2^k x G_{k,n-1}(x) &= \frac{\alpha^{n+1} - \beta^{n+1}}{\Delta} - 2^k x \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\Delta} \right) \\ &= \frac{\alpha^n \left(\alpha - 2^k x \alpha^{-1} \right) - \beta^n \left(\beta - 2^k x \beta^{-1} \right)}{\Delta} \\ &= \frac{\alpha^n (\alpha - \beta) - \beta^n (\beta - \alpha)}{\Delta} \\ &= \frac{(\alpha - \beta)(\alpha^n + \beta^n)}{\Delta} \\ &= \alpha^n + \beta^n. \end{aligned}$$

By using (5), the proof is complete. □

Differentiation formulas

$$\begin{aligned} \frac{dg_{k,n}(x)}{dx} &= -2^k n G_{k,n-1}(x), \\ \Delta^2 \frac{dG_{k,n}(x)}{dx} &= -2^k n g_{k,n-1}(x) + 2^{k+1} G_{k,n}(x). \end{aligned}$$

Diagonal functions

We denote rising diagonal functions of generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials by $R_{k,n}(x)$ and $r_{k,n}(x)$, respectively.

It is obvious that $R_{k,n}(x)$ and $r_{k,n}(x)$ hold the following recurrence relations.

$$\begin{aligned} R_{k,n}(x) &= R_{k,n-1}(x) - 2^k x R_{k,n-3}(x), \quad n \geqslant 3, \\ r_{k,n}(x) &= r_{k,n-1}(x) - 2^k x r_{k,n-3}(x), \quad n \geqslant 3. \end{aligned}$$

n	$R_{k,n}(x)$	$r_{k,n}(x)$
0	0	2
1	1	1
2	1	1
3	1	$1-2^{k+1}x$
4	$1-2^kx$	$1-3.2^kx$
5	$1-2^{k+1}x$	$1-2^{k+2}x$
6	$1-3.2^kx$	$1-5.2^kx+2^{2k+1}x^2$
7	$1-2^{k+2}x+2^{2k}x^2$	$1-6.2^kx+5.2^{2k}x^2$

Table 2: *The first few terms of $R_{k,n}(x)$ and $r_{k,n}(x)$*

So, $R_{k,n}(x)$ and $r_{k,n}(x)$ have the generating functions as follows:

$$\sum_{i=1}^{\infty} R_{k,i}(x)\, y^{i-1} = \left(1-y+2^kxy^3\right)^{-1}, \tag{13}$$

$$\sum_{i=0}^{\infty} r_{k,i}(x)\, y^{i} = (2-y)\left(1-y+2^kxy^3\right)^{-1}. \tag{14}$$

If we use (13) and (14) jointly, we get

$$r_{k,n}(x)+R_{k,n}(x)=2R_{k,n+1}(x).$$

Now, we consider descending diagonal functions of the generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials by $D_{k,n}(x)$ and $d_{k,n}(x)$, respectively.

n	$D_{k,n}(x)$	$d_{k,n}(x)$
0	0	2
1	1	$1-2^{k+1}x$
2	$1-2^kx$	$1-3.2^kx+2^{2k+1}x^2$
3	$1-2^{k+1}x+2^{2k}x^2$	$1-2^{k+2}x+5.2^{2k}x^2-2^{3k+1}x^3$
4	$1-3.2^kx+3.2^{2k}x^2-2^{3k}x^3$	$1-5.2^kx+9.2^{2k}x^2-7.2^{3k}x^3+2^{4k+1}x^4$

Table 3: *The first few terms of $D_{k,n}(x)$ and $d_{k,n}(x)$*

For $n \geqslant 1$, we have

$$D_{k,n}(x) = \left(1-2^kx\right)^{n-1}, \tag{15}$$

$$d_{k,n}(x) = \left(1-2^{k+1}x\right)\left(1-2^kx\right)^{n-1}. \tag{16}$$

Descending diagonal functions in (15) and (16) have the following generating functions for $\left|t\left(1-2^kx\right)\right|<1$ as

$$\sum_{i=1}^{\infty} D_{k,i}(x)\, y^{i-1} = \left[1-\left(1-2^kx\right)t\right]^{-1},$$

$$\sum_{i=1}^{\infty} d_{k,i}(x)\, y^{i-1} = \left(1-2^{k+1}x\right)\left[1-\left(1-2^kx\right)t\right]^{-1}.$$

From (15) and (16), for $n \geqslant 2$, we get

$$\frac{D_{k,n}(x)}{D_{k,n-1}(x)} = \frac{d_{k,n}(x)}{d_{k,n-1}(x)} = \left(1 - 2^k x\right),$$

and we also have

$$\begin{aligned} d_{k,n}(x) &= D_{k,n+1}(x) - 2^k x D_{k,n}(x), \\ d_{k,n+1}(x) - 2^k x d_{k,n}(x) &= \left(1 - 2^{k+1} x\right)^2 D_{k,n}(x). \end{aligned}$$

3. Multilinear and multilateral generating functions

In this section, we firstly derive several families of bilinear and bilateral generating functions for the generalized Vieta-Jacobsthal polynomials $G_{k,n}(x)$ generated by (8) and given explicitly by (6) by using a similar method considered in [1, 2, 3].

Theorem 1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s complex variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum_{i=0}^{\infty} a_i \Omega_{\mu+\nu i}(y_1, ..., y_s) z^i, \tag{17}$$

where $a_i \neq 0$, $\mu, \nu \in \mathbb{C}$ and

$$\Theta_{k,n,p}^{\mu,\nu}(x; y_1, ..., y_s; \tau) := \sum_{i=0}^{[n/p]} a_i G_{k,n-pi+1}(x)\, \Omega_{\mu+\nu i}(y_1, ..., y_s) \tau^i, \tag{18}$$

where $n, p \in \mathbb{N}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{k,n,p}^{\mu,\nu}\left(x; y_1, ..., y_s; \frac{\eta}{t^p}\right) t^n = \left(1 - t + 2^k x t^2\right)^{-1} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta), \tag{19}$$

provided that each member of (19) exists.

Proof. If we denote the left-hand side of (19) by S and use (18), then we obtain

$$S = \sum_{n=0}^{\infty} \sum_{i=0}^{[n/p]} a_i G_{k,n-pi+1}(x)\, \Omega_{\mu+\nu i}(y_1, ..., y_s) \eta^i t^{n-pi}.$$

Replacing n by $n + pi$ we may write

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} a_i\, G_{k,n+1}(x)\, \Omega_{\mu+\nu i}(y_1, ..., y_s) \eta^i t^n \\ &= \sum_{n=0}^{\infty} G_{k,n+1}(x)\, t^n \sum_{i=0}^{\infty} a_i \Omega_{\mu+\nu i}(y_1, ..., y_s) \eta^i \\ &= \left(1 - t + 2^k x t^2\right)^{-1} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta), \end{aligned}$$

which completes the proof. □

Now we derive several families of bilinear and bilateral generating functions for the generalized Vieta-Jacobsthal-Lucas polynomials $g_{k,n}(x)$ generated by (9) and given explicitly by (7).

Theorem 2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,...,y_s;z):=\sum_{i=0}^{\infty}a_i\Omega_{\mu+\nu i}(y_1,...,y_s)z^i,\quad a_i\neq 0,\mu,\nu\in\mathbb{C}$$

and

$$\Theta^{\mu,\nu}_{k,n,p}(x;y_1,...,y_s;\tau):=\sum_{i=0}^{[n/p]}a_i g_{k,n-pi+1}(x)\,\Omega_{\mu+\nu i}(y_1,...,y_s)\tau^i,$$

where $n,p\in\mathbb{N}$. Then we have

$$\sum_{n=0}^{\infty}\Theta^{\mu,\nu}_{k,n,p}\left(x;y_1,...,y_s;\frac{\eta}{t^p}\right)t^n=\left(1-2^{k+1}xt\right)\left(1-t+2^kxt^2\right)^{-1}\Lambda_{\mu,\nu}(y_1,...,y_s;\eta),\tag{20}$$

provided that each member of (20) exists.

Proof. In precisely the same manner as described the proof of Theorem 1 and using the generating function (9) we can prove Theorem 2. □

By expressing the multivariable function

$$\Omega_{\mu+\nu i}(y_1,...,y_s),\quad i\in\mathbb{N}_0,s\in\mathbb{N}$$

in terms of a simpler function of one and more variables, it is possible to give many applications of Theorems 1 and 2.

For example, set

$$s=1\ \text{ and }\ \Omega_{\mu+\nu i}(y)=P^{(\alpha,\beta)}_{\mu+\nu i}(y)$$

in (17), where Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$ are defined by

$$P^{(\alpha,\beta)}_n(x)=\frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n; & \dfrac{1-x}{2}\\ 1+\alpha; & \end{matrix}\right],$$

where ${}_2F_1$ denotes Gauss's hypergeometric series whose natural generalization of an arbitrary number of p numerator and q denominator parameters $(p,q\in\mathbb{N}_0)$ is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p; & z\\ \beta_1,...,\beta q; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ \cdots\ (\alpha_p)_n}{(\beta_1)_n\ \cdots\ (\beta_q)_n}\frac{z^n}{n!}.$$

Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of gamma function) by

$$\begin{aligned}(\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^- \\ &= \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases}\end{aligned}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function. Jacobi polynomials are generated by [7]

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n \\ &\quad = (1-t)^{-1-\alpha-\beta}\ {}_2F_1\left[\begin{matrix} \frac{1}{2}(1+\alpha+\beta),\ \frac{1}{2}(2+\alpha+\beta);\ \frac{2t(x-1)}{(1-t)^2} \\ 1+\alpha\ ; \end{matrix}\right].\end{aligned} \tag{21}$$

Then we obtain the following result which provides a class of bilateral generating functions for the Jacobi polynomials and the generalized Vieta-Jacobsthal polynomials.

Corollary 1. *If* $\Lambda_{\mu,\nu}(y;z) := \sum\limits_{i=0}^{\infty} a_i P_{\mu+\nu i}^{(\alpha,\beta)}(y) z^i$, *where* $a_i \neq 0$, $\nu, \mu \in \mathbb{C}$ *and*

$$\Theta_{k,n,p}^{\mu,\nu}(x; y_1, ..., y_s; \tau) := \sum_{i=0}^{[n/p]} a_i G_{k,n-pi+1}(x) P_{\mu+\nu i}^{(\alpha,\beta)}(y) \tau^i,$$

where $n, p \in \mathbb{N}$. *Then we have*

$$\sum_{n=0}^{\infty} \Theta_{k,n,p}^{\mu,\nu}\left(x; y_1, ..., y_s; \frac{\eta}{t^p}\right) t^n = \left(1-t+2^k x t^2\right)^{-1} \Lambda_{\mu,\nu}(y;\eta), \tag{22}$$

provided that each member of (22) exists.

Remark 1. *Using the generating relation (21) for the Jacobi polynomials and taking* $a_i = \frac{(1+\alpha+\beta)_i}{(1+\alpha)_i}$, $\mu = 0$, $\nu = 1$, *we have*

$$\begin{aligned}&\sum_{n=0}^{\infty} \sum_{i=0}^{[n/p]} \frac{(1+\alpha+\beta)_i}{(1+\alpha)_i} G_{k,n-pi+1}(x) P_i^{(\alpha,\beta)}(y) \eta^i t^{n-pi} \\ &\quad = \left(1-t+2^k x t^2\right)^{-1} (1-\eta)^{-1-\alpha-\beta} \\ &\quad\quad \times {}_2F_1\left[\begin{matrix} \frac{1}{2}(1+\alpha+\beta),\ \frac{1}{2}(2+\alpha+\beta);\ \frac{2\eta(y-1)}{(1-\eta)^2} \\ 1+\alpha\ ; \end{matrix}\right].\end{aligned}$$

Choosing $s = 1$ and $\Omega_{\mu+\nu i}(y) = G_{k,\mu+\nu i+1}(y)$, $(\mu, \nu \in \mathbb{N}_0)$ in Theorem 1, we obtain the following class of bilinear generating functions for the generalized Vieta-Jacobsthal polynomials $G_{k,n}(x)$.

Corollary 2. *If* $\Lambda_{\mu,\nu}(y;z) := \sum\limits_{i=0}^{\infty} a_i G_{k,\mu+\nu i+1}(y)\, z^i$*, where* $a_i \neq 0\,,\ \ \nu,\mu \in \mathbb{C}$ *and*

$$\Theta_{k,n,p}^{\mu,\nu}(x;y_1,...,y_s;\tau) := \sum_{i=0}^{[n/p]} a_i G_{k,n-pi+1}(x)\, G_{k,\mu+\nu i+1}(y)\, \tau^i,$$

where $n,p \in \mathbb{N}$*. Then we have*

$$\sum_{n=0}^{\infty} \Theta_{k,n,p}^{\mu,\nu}\left(x;y_1,...,y_s;\frac{\eta}{t^p}\right) t^n = \left(1-t+2^k x t^2\right)^{-1} \Lambda_{\mu,\nu}(y;\eta), \tag{23}$$

provided that each member of (23) exists.

Remark 2. *Using the generating relation (8) for the generalized Vieta-Jacobsthal polynomials and taking* $a_i = 1$*,* $\mu = 0$*,* $\nu = 1$*, we have*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{[n/p]} G_{k,n-pi+1}(x)\, G_{k,i+1}(y) \eta^i t^{n-pi} = \frac{1}{(1-t+2^k x t^2)\,(1-\eta+2^k y \eta^2)}.$$

If we set

$$s=1 \ \text{ and } \ \Omega_{\mu+\nu k}(y) = G_{k,\mu+\nu i+1}(y)$$

in Theorem 2, then we obtain the following result which provides a class of bilateral generating functions for the generalized Vieta-Jacobsthal polynomials $G_{k,n}(x)$ and the generalized Vieta-Jacobsthal-Lucas polynomials $g_{k,n}(x)$.

Corollary 3. *If* $\Lambda_{\mu,\nu}(y;z) := \sum\limits_{i=0}^{\infty} a_i G_{k,\mu+\nu i+1}(y)\, z^i$*, where* $a_i \neq 0\,,\ \ \nu,\mu \in \mathbb{C}$ *and*

$$\Theta_{k,n,p}^{\mu,\nu}(x;y_1,...,y_s;\tau) := \sum_{i=0}^{[n/p]} a_i g_{k,n-pi+1}(x)\, G_{k,\mu+\nu i+1}(y)\, \tau^i,$$

where $n,p \in \mathbb{N}$*. Then we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{i=0}^{[n/p]} g_{k,n-pi+1}(x)\, G_{k,i+1}(y)\, \eta^i t^{n-pi} \\ &\qquad = \left(1-2^{k+1}xt\right)\left(1-t+2^k x t^2\right)^{-1} \Lambda_{\mu,\nu}(y;\eta), \end{aligned} \tag{24}$$

provided that each member of (24) exists.

Remark 3. *Using the generating relation (8) for the generalized Vieta-Jacobsthal polynomials and taking* $a_i = 1$*,* $\mu = 0$*,* $\nu = 1$*, we have*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{[n/p]} g_{k,n-pi+1}(x)\, G_{k,i+1}(y)\, \eta^i t^{n-pi} = \frac{\left(1-2^{k+1}xt\right)}{(1-t+2^k x t^2)\,(1-\eta+2^k y \eta^2)}.$$

Furthermore, for every suitable choice of the coefficients a_i $(i \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1, ..., y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, such as the product of Jacobi and generalized Vieta-Jacobsthal polynomials, the assertions of Theorems 1 and 2 can be applied in order to derive various families of multilinear and multilateral generating functions for the generalized Vieta-Jacobsthal polynomials and the generalized Vieta-Jacobsthal-Lucas polynomials.

Acknowledgement

The authors would like to thank the referees for their valuable suggestions and comments which helped to improve the quality and readability of the paper.

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